

Four-loop splitting functions at small x

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ABSTRACT: We consider the expansion of small- x resummed DGLAP splitting functions at next-to-leading logarithmic (NLL) accuracy to four-loop order, namely next-to-next-to-next-to-leading order (N³LO). From this, we extract the exact LL and NLL small- x contributions to the yet unknown N³LO splitting functions, both in the standard $\overline{\text{MS}}$ scheme and in the $Q_0\overline{\text{MS}}$ scheme usually considered in small- x literature. We show that the impact of unknown subleading logarithmic contributions (NNLL and beyond) at N³LO is significant, thus motivating future work towards their computation. Our results will be also needed in future to match NLL resummation to N³LO evolution. In turn, we propose an improved implementation of the small- x resummation and therefore release a new version of the resummation code (HELL 3.0) which contains these changes.

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1 Introduction

The data thus far collected by the experiments at the CERN Large Hadron Collider (LHC) have reaffirmed the Standard Model as a remarkably successful theory of fundamental particles and their interactions. Thus, in absence of striking signature of new physics phenomena, the theoretical community is compelled to perform calculations with ever smaller uncertainties so that predictions with ever increasing accuracy and precision can be compared to data of outstanding quality, thereby exposing subtle differences and discrepancies that may reveal the presence of physics beyond the Standard Model.

In the context of strong interactions, accuracy is usually achieved by computing predictions that include an increasing number of terms of the perturbative expansion in the strong coupling α_s (henceforth, the fixed-order expansion). Leading-order (LO) cross-sections in QCD can be computed for an essentially arbitrary number of external particles. Automation has been achieved in recent years also for NLO calculations and an increasing number of NNLO calculations is now available in computer programs. Moreover, for hadron-collider processes with simple topologies, recent milestone calculations have achieved N³LO accuracy [1, 2]. This is particularly important because the main production channel of the Higgs

boson, i.e. gluon-gluon fusion [3–5], falls under this category. Furthermore, precise theoretical predictions for LHC processes also require precise and reliable parton distribution functions (PDFs). In particular, the lack of a N³LO determination of PDFs is an important source of uncertainty on the Higgs cross-section [6]. Although a global determination of such PDF set cannot be foreseen in the near future, several ingredients are either already available, or focus of current research. For instance, deep-inelastic scattering (DIS) coefficient functions with massless quarks have been known at three loops for a long time [7], and a lot of progress has been done in the context of heavy quarks, e.g. [8–12]. Another important ingredient of this rather ambitious task is the determination of the DGLAP kernels, which control the scale dependence of the PDFs, at N³LO. Recent progress with four-loop splitting functions [13, 14] suggests that this calculation could be completed rather soon.

A complementary approach to the fixed-order expansion consists of exploiting all-order resummation. In the context of PDF determination, small- x (or high-energy) resummation is of particular relevance. Small- x resummation of DGLAP evolution is known to next-to-leading logarithmic accuracy (NLL) and it is based on the BFKL equation [15–20]. However, the proper inclusion of LL and NLL corrections is far from trivial, due to the perturbative instability of the BFKL evolution kernel. This problem has been tackled by more than one group in the 1990s, see for instance refs. [21–24], refs. [25–30] and refs. [31–34], and resulted in resummed anomalous dimensions for PDF evolution. These techniques have recently been applied in the context of PDF determination in refs. [35, 36], where small- x resummation at NLL accuracy has been included in the PDF evolution and in the computation of DIS structure functions through the public code HELL [37, 38].

It has been found that small- x resummation stabilises the perturbative behaviour of both evolution kernels and partonic coefficient functions, thereby improving the description of structure-function data at low x . In particular, it is well-known that potentially large logarithms at small- x are absent in NLO splitting functions due to accidental cancellations, while they start to contribute at NNLO. As a consequence, PDFs determined with NNLO theory improved by NLL small- x resummation differ rather significantly from the ones determined with NNLO alone. Furthermore, while at NNLO the most singular term in the gluon splitting function is of order $\frac{\alpha_s^3}{x} \log \frac{1}{x}$ (the term with two logarithms being again accidentally zero), at N³LO the most singular term is of order $\frac{\alpha_s^4}{x} \log^3 \frac{1}{x}$. Hence, the aforementioned instability at low x is very likely to become rather worse at N³LO. Small- x resummation would then be mandatory for improved precision. We note that in order to resum all small- x logarithms that appear at N³LO, one would have to consider NNLL resummation, which would be based on the three-loop BFKL kernel, which despite a lot of recent progress [39–45] is not yet fully known.

In this work we examine in some detail the fixed-order expansion of the NLL resummed splitting functions up to four loops. This exercise is interesting for several reasons. First, it enables us to predict the coefficients of the leading and next-to-leading small- x contributions to the yet-unknown N³LO splitting functions, thus offering either a strong check or a way of complementing the fixed-order result at small x . Second, because the resummation also includes subleading effects, mostly related to the running of the strong coupling, we

are able to assess the impact of unknown NNLL (or higher) contributions on the four-loop result. Third, although we predict that N³LO splitting functions will be unstable at small- x (much more than the NNLO ones), their inclusion will be most likely beneficial at moderate and large x , and therefore we conclude that the most reliable result in future will be obtained by using N³LO evolution provided it is supplemented by small- x resummation. The expansion of the resummation to $\mathcal{O}(\alpha_s^4)$ presented here is also a crucial ingredient for the N³LO+NLL matching procedure. Finally, by explicitly studying the behaviour of subleading contributions up to forth order in perturbation theory, we are able to identify a potential source of instability in our previous implementation of the resummation. We propose here an improved way of dealing with this class of subleading contributions and consequently we release a new version of the resummation code HELL 3.0, where these changes are implemented.

2 DGLAP evolution at small x

In this section we summarise small- x resummation of the DGLAP splitting functions. Small- x logarithms appear in the singlet sector and we have therefore to consider a 2×2 evolution matrix that couples together the quark singlet and the gluon. Currently, small- x resummation is known to NLL and we find convenient to express resummed and matched results as

$$P_{ij}^{\text{N}^k\text{LO+NLL}}(x, \alpha_s) = P_{ij}^{\text{N}^k\text{LO}}(x, \alpha_s) + \Delta_{k+1} P_{ij}^{\text{NLL}}(x, \alpha_s), \quad i, j = g, q, \quad (2.1)$$

where $P_{ij}^{\text{N}^k\text{LO}}$ are the $(k+1)$ -loop splitting functions and $\Delta_{k+1} P_{ij}^{\text{NLL}}$ represent the resummed predictions P_{ij}^{NLL} minus their expansion up to order α_s^{k+1} , namely

$$\Delta_{k+1} P_{ij}^{\text{NLL}}(x, \alpha_s) = P_{ij}^{\text{NLL}}(x, \alpha_s) - \sum_{j=0}^k \alpha_s^{j+1} P_{ij}^{\text{NLL}(j)}(x), \quad (2.2)$$

where $P_{ij}^{\text{NLL}(j)}(x)$ is the $\mathcal{O}(\alpha_s^{j+1})$ contribution to $P_{ij}^{\text{NLL}}(x, \alpha_s)$. Eq. (2.1) is valid, in principle, for any value of k . Matching of the resummation to NNLO ($k = 2$ in the above notation) was achieved in ref. [38] and later applied in refs. [35, 36] for PDF determination. In this work we instead focus on the matching to the next perturbative order, namely N³LO ($k = 3$). We note, however, that in order to really improve the quality of the result, one should also increase the logarithmic accuracy of the resummation contribution so that no potentially large logarithm is left unresummed. Therefore, one would like to reach at least N³LO+NNLL: we will leave this rather ambitious goal to future work.

Small- x resummation of DGLAP evolution is usually performed in a conjugate (Mellin) space. Therefore, we define the entries of the anomalous dimension matrix in the singlet sectors as

$$\gamma_{ij}(N, \alpha_s) = \int_0^1 dx x^N P_{ij}(x, \alpha_s). \quad (2.3)$$

In this non-standard notation, usually adopted in the small- x resummation literature, the leading small- x logarithms of the form $\frac{1}{x} \log^k \frac{1}{x}$ are mapped into poles in $N = 0$.

2.1 Brief recap of small- x resummation of DGLAP evolution

We now recall how the resummation of DGLAP splitting functions is constructed, mainly following ref. [38]. First, one considers the plus eigenvalue $\gamma_+(N, \alpha_s)$ of the singlet anomalous dimension matrix eq. (2.3). This is resummed by first exploiting the duality between DGLAP evolution and BFKL evolution, and then supplementing the result by the resummation of a class of subleading contributions originating from the running of the strong coupling. Additionally, requiring the symmetry of the resummed BFKL kernel and imposing momentum conservation leads to perturbatively stable results. Since the knowledge of the BFKL kernel at N^k LO allows the resummation of γ_+ at N^k LL, at the moment we can only reach NLL accuracy, γ_+^{NLL} .

Once the eigenvalue γ_+ is resummed, one proceeds with the resummation of γ_{qg} . Its all-order behaviour at NLL is described by the equation [46, 47]

$$\gamma_{qg}^{\text{NLL}}(N, \alpha_s) = \alpha_s \sum_{k \geq 0} h_k \left[\gamma_+^k(N, \alpha_s) \right], \quad (2.4)$$

where the square-bracket notation is defined by the recursion [30, 48]

$$\left[\gamma_+^{k+1}(N, \alpha_s) \right] = \left(\gamma_+(N, \alpha_s) - k r(N, \alpha_s) \right) \left[\gamma_+^k(N, \alpha_s) \right], \quad (2.5)$$

with

$$r(N, \alpha_s) = \alpha_s^2 \beta_0 \frac{d}{d\alpha_s} \log(\gamma_+(N, \alpha_s)). \quad (2.6)$$

Note that in refs. [37, 38] a variant of the resummation where $r(N, \alpha_s) \rightarrow \alpha_s \beta_0$, which corresponds to a limit in which γ_+ is assumed to be proportional to α_s , was used to infer an uncertainty on γ_{qg} . Because only a finite number of coefficients h_k are known, the implementation of the resummation, described in ref. [37], is only approximate. However, for not-too-large values of α_s , the implementation is numerically stable and reliable.

Simple power-counting at small- N shows that the quark anomalous dimension γ_{qg} starts at NLL. Therefore, at this accuracy we have some freedom in how we choose the logarithmic accuracy of γ_+ appearing in eq. (2.4). In ref. [37] a dedicated anomalous dimension, denoted LL', was constructed specifically for this purpose. This LL' anomalous dimension is essentially a LL anomalous dimension, but its singular structure, which at resummed level is encoded in the position of the rightmost pole in N space, is taken from the NLL result. The reason for using this hybrid object can be summarised as follows:

- on the one hand, it is preferable to use the γ_+^{NLL} anomalous dimension in order to avoid singularity mismatches between different entries of the anomalous dimension matrix;
- on the other hand, since using γ_+ at LL in eq. (2.4) is formally sufficient to achieve NLL accuracy in γ_{qg} , it was convenient from a numerical point of view to use as much of the LL result as possible, because of its better numerical stability.

However, recently in ref. [38] various improvements in the construction of the resummed anomalous dimensions have been proposed and implemented in the numerical code HELL. With these developments, the computation of the full NLL anomalous dimension is faster and much more stable and reliable, and it is therefore now possible to either use the hybrid

LL' result or the full NLL one in eq. (2.4). In particular, the latter choice corresponds to the original approach of ref. [30]. We will explore the effects of both options in the following.

All the other entries of the singlet anomalous dimension matrix can be derived from the plus eigenvalue and γ_{qg} . In particular, the results can be written in a rather simple form if we consider the resummed contributions $\Delta_k \gamma_{ij}$, defined according to eq. (2.2):

$$\Delta_k \gamma_{gg}^{\text{NLL}}(N, \alpha_s) = \Delta_k \gamma_+^{\text{NLL}}(N, \alpha_s) - \frac{C_F}{C_A} \Delta_k \gamma_{qg}^{\text{NLL}}(N, \alpha_s), \quad (2.7a)$$

$$\Delta_k \gamma_{gq}^{\text{NLL}}(N, \alpha_s) = \frac{C_F}{C_A} \Delta_k \gamma_{gg}^{\text{NLL}}(N, \alpha_s), \quad (2.7b)$$

$$\Delta_k \gamma_{qq}^{\text{NLL}}(N, \alpha_s) = \frac{C_F}{C_A} \Delta_k \gamma_{qg}^{\text{NLL}}(N, \alpha_s). \quad (2.7c)$$

We recall that these relations are able to predict only the LL part of γ_{qg} , while we do not have enough knowledge to predict its NLL part, which is then only approximate in the equation above. Having all the anomalous dimensions in the singlet sector, we can construct the splitting functions by Mellin inversion.¹

2.2 Perturbative expansion of the resummation

We now consider the perturbative expansion of the resummed result presented in the previous section. The goal is twofold. On the one hand, the expansion of the resummation is needed to construct the resummed contributions $\Delta_k \gamma_{ij}^{\text{NLL}}$, $\Delta_k P_{ij}^{\text{NLL}}$, eq. (2.2), namely for the matching of the resummed result to fixed order. On the other hand, the α_s expansion of the resummed results provides a prediction for the small- x behaviour of the fixed-order splitting functions.

In ref. [38] we have already determined the expansion of the NLL resummed splitting functions to $\mathcal{O}(\alpha_s^3)$, which was needed to match resummation to NNLO. In that case, there was no point in using the result of the expansion to predict the NNLO behaviour at small- x , as the three-loop splitting functions are known [49, 50]. In this work we push the expansion to one extra order, $\mathcal{O}(\alpha_s^4)$. These results would be needed to match resummation to N³LO, and specifically to construct N³LO+NLL resummed results. The four-loop splitting functions, however, are not yet fully known [13, 14]. Therefore, at the moment our results can be used to construct approximations, valid at small x , of the unknown N³LO splitting functions, or simply to supplement the ongoing computation with the knowledge of the exact small- x behaviour. In future, when the four-loop splitting functions will be computed, it will also serve as a cross check.

In order to obtain the expansion of the resummed entries of the anomalous dimension matrix, we have to expand both γ_+^{NLL} and γ_{qg}^{NLL} to the desired accuracy; the other anomalous dimensions are recovered using eqs. (2.7). Let us first introduce a generic notation for the expansion of the plus anomalous dimension,

$$\gamma_+(N, \alpha_s) = \alpha_s \gamma_0 + \alpha_s^2 \gamma_1 + \alpha_s^3 \gamma_2 + \alpha_s^4 \gamma_3 + \mathcal{O}(\alpha_s^5), \quad (2.8)$$

¹In phenomenological applications, a damping at large x is added to make the transition from the small- x region (where the resummation is relevant) to the large- x region (where the fixed-order description is appropriate) as smooth as possible, and finally momentum conservation is reimposed. These details have been described in ref. [38] and are not repeated here.

which is valid both for the NLL anomalous dimension and for the auxiliary LL' one. The expansion of the qg anomalous dimension, according to eqs. (2.4), (2.5) and (2.6), is given by

$$\begin{aligned} \gamma_{qg}^{\text{NLL}}(N, \alpha_s) &= \alpha_s h_0 + \alpha_s^2 h_1 \gamma_0 + \alpha_s^3 [h_2 \gamma_0 (\gamma_0 - \beta_0) + h_1 \gamma_1] \\ &\quad + \alpha_s^4 [h_3 \gamma_0 (\gamma_0 - \beta_0) (\gamma_0 - 2\beta_0) + h_2 \gamma_1 (N) (2\gamma_0 - T\beta_0) + h_1 \gamma_2] \\ &\quad + \mathcal{O}(\alpha_s^5), \end{aligned} \tag{2.9}$$

where the anomalous dimensions $\gamma_{0,1,2}$ are the coefficients eq. (2.8) of the expansion of either γ_+^{NLL} or $\gamma_+^{\text{LL}'}$, depending on the choice adopted in eq. (2.4). Note that the expansion eq. (2.9) depends on a parameter T . This parameter has been introduced to account for the two variants of the resummation of running coupling contributions described above. In particular, when $r(N, \alpha_s)$ as given by eq. (2.6) is used then $T = 2$, while for the variation $r(N, \alpha_s) \rightarrow \alpha_s \beta_0$ then $T = 1$. The first h_k coefficients appearing in eq. (2.9) are [47]

$$h_0 = \frac{n_f}{3\pi}, \quad h_1 = \frac{n_f}{3\pi} \frac{5}{3}, \quad h_2 = \frac{n_f}{3\pi} \frac{14}{9}, \quad h_3 = \frac{n_f}{3\pi} \left(\frac{82}{81} + 2\zeta_3 \right). \tag{2.10}$$

Note that the knowledge of γ_{qg}^{NLL} to $\mathcal{O}(\alpha_s^4)$ requires the expansion of the plus anomalous dimension up to $\mathcal{O}(\alpha_s^3)$, as it does not depend on γ_3 . This is due to the fact that γ_{qg} is a pure NLL quantity, as it is clear from the factor of α_s in front of eq. (2.4).

We thus need to compute the first four orders of γ_+^{NLL} , while we just need the first three orders of $\gamma_+^{\text{LL}'}$ as it only possibly enters in the expansion of γ_{qg} . The precise construction of these resummed anomalous dimensions was presented in detail in ref. [38], and we do not repeat it here (some details are given in appendix A). We just recall that due to the actual construction of the plus eigenvalue, which is based on the duality between DGLAP and BFKL evolutions, the LL and LL' resummed anomalous dimensions automatically contain the fixed LO anomalous dimension, while the NLL anomalous dimension contains the NLO one.² However, the qg anomalous dimension, eq. (2.4), requires a purely resummed anomalous dimension, which goes to zero at large N , in order to avoid producing spurious large- N terms (see discussion in appendix B.2 of ref. [37]). Therefore, we first define the resummed contributions $\Delta_1 \gamma_+^{\text{LL}'}$ and $\Delta_2 \gamma_+^{\text{NLL}}$ to be the resummed results at LL' and NLL minus the LO and NLO anomalous dimensions, respectively (the notation is the same as in ref. [38]). Then, we construct the purely resummed LL' and NLL anomalous dimensions as

$$\gamma_+^{\text{LL}'}(N, \alpha_s) = \alpha_s \gamma_0^{\text{LL}'}(N) + \Delta_1 \gamma_+^{\text{LL}'}(N, \alpha_s), \tag{2.11a}$$

$$\gamma_+^{\text{NLL}}(N, \alpha_s) = \alpha_s \gamma_0^{\text{NLL}}(N) + \alpha_s^2 \gamma_1^{\text{NLL}}(N) + \Delta_2 \gamma_+^{\text{NLL}}(N, \alpha_s). \tag{2.11b}$$

The functions $\gamma_0^{\text{LL}'}$, γ_0^{NLL} and γ_1^{NLL} are not fixed by the resummation, and we thus have a degree of arbitrariness in how to define them. Instead, the expansions of the resummed contributions

$$\Delta_1 \gamma_+^{\text{LL}'}(N, \alpha_s) = \alpha_s^2 \gamma_1^{\text{LL}'}(N) + \alpha_s^3 \gamma_2^{\text{LL}'}(N) + \mathcal{O}(\alpha_s^4), \tag{2.12a}$$

$$\Delta_2 \gamma_+^{\text{NLL}}(N, \alpha_s) = \alpha_s^3 \gamma_2^{\text{NLL}}(N) + \alpha_s^4 \gamma_3^{\text{NLL}}(N) + \mathcal{O}(\alpha_s^5) \tag{2.12b}$$

²These fixed-order anomalous dimensions used in the construction of resummation are actually approximated, as explained in ref. [38]. This fact is however immaterial for the present discussion.

can be derived from the resummed results of ref. [38]. Their computation is presented in appendix A. Part of them, specifically $\gamma_1^{\text{LL}'}$ and γ_2^{NLL} , have been already computed and presented in ref. [38]. The next terms, $\gamma_2^{\text{LL}'}$ and γ_3^{NLL} , are reported here for the first time.

Starting from LL' resummation, the first order of the anomalous dimension was chosen in ref. [37] to include the LL and NLL contributions of the LO anomalous dimension. The NLL term, being it a constant at this order, is further multiplied by a function $1/(N+1)$ to make it vanish at large N . The expression, which we adopt also here, is

$$\gamma_0^{\text{LL}'} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1}, \tag{2.13}$$

where a_{ij} are defined in eq. (A.3). The next orders, as predicted by the resummation, are given by (see appendix A)

$$\gamma_1^{\text{LL}'} = \beta_0 \left(\frac{3}{32} \kappa_0 - c_0 \right) \left(\frac{1}{N} - \frac{4N}{(N+1)^2} \right) \tag{2.14}$$

$$\begin{aligned} \gamma_2^{\text{LL}'} = & \frac{\lambda_2}{N^2} + \frac{\lambda_1}{N} - (\lambda_2 + \lambda_1) \frac{4N}{(N+1)^2} \\ & + \left(\frac{a_{11}}{N^2} + \frac{2(a_{11} + a_{10})}{(N+1)^2} \right) \left[\frac{a_{11}a_{10}}{(N+1)^2} - \frac{a_{11}a_{10}}{4} \frac{4N}{(N+1)^2} \right. \\ & \left. + a_{11} \left(\frac{a_{11}}{N} + a_{10} - \frac{2(a_{11} + a_{10})N}{N+1} \right) [\psi_1(N+1) - \zeta_2] \right], \end{aligned} \tag{2.15}$$

where all the coefficients are defined in appendix A. Eq. (2.15) is a new result. Note that, by construction, both $\gamma_1^{\text{LL}'}$ and $\gamma_2^{\text{LL}'}$ vanish in $N=1$, as the resummation is built to preserve momentum conservation.

Moving to the NLL resummation, we need to choose both the LO and NLO contributions. We decide to adopt the same strategy as in the LL', i.e. keeping only the LL and NLL contributions from the fixed orders. In particular, for the LO term we use exactly the same approximation used for the LL', and at NLO, due to the fact that the LL term is accidentally zero, we simply have a NLL term,

$$\gamma_0^{\text{NLL}} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1}, \tag{2.16}$$

$$\gamma_1^{\text{NLL}} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1}, \tag{2.17}$$

where a_{21} , the NLL coefficient of the NLO, is defined in eq. (A.3). In γ_1^{NLL} we have also included a subtraction term of the same form as the NLL term in γ_0^{NLL} , which restores momentum conservation, i.e. $\gamma_1^{\text{NLL}}(1) = 0$. We have decided to add this feature as the effect is formally NNLL, and it makes it more in line with the LL' case where the $\mathcal{O}(\alpha_s^2)$ term vanishes in $N=1$. We stress that we have played with variants of both γ_0^{NLL} and γ_1^{NLL} , adding momentum conservation to the first, relaxing it in the second, varying the way it is implemented, and so on: the effect at the level of the splitting functions is moderate. The third and fourth coefficients are instead found expanding the resummation, and their form

is (see appendix A)

$$\begin{aligned} \gamma_2^{\text{NLL}} = & \beta_0^2 \frac{\kappa_0}{16} \left(\frac{1}{N} - \frac{4N}{(N+1)^2} \right) \\ & + \left(\frac{a_{11}}{N^2} + \frac{2(a_{11}+a_{10})}{(N+1)^2} \right) \left[\rho + \frac{a_{21}}{N+1} + \frac{a_{11}a_{10}}{(N+1)^2} - \left(\rho + \frac{a_{21}}{2} + \frac{a_{11}a_{10}}{4} - \beta_0 a_{11} \right) \frac{4N}{(N+1)^2} \right. \\ & \left. + a_{11} \left(\frac{a_{11}}{N} + a_{10} - \frac{2(a_{11}+a_{10})N}{N+1} + \beta_0 \right) [\psi_1(N+1) - \zeta_2] \right], \end{aligned} \quad (2.18)$$

$$\begin{aligned} \gamma_3^{\text{NLL}} = & \frac{\rho_3}{N^3} + \frac{\rho_2}{N^2} + \frac{\rho_1}{N} - (\rho_3 + \rho_2 + \rho_1) \frac{4N}{(N+1)^2} \\ & + \left(\frac{a_{21}}{N^2} + \frac{2(a_{21}+a_{20})}{(N+1)^2} \right) \left[\rho + \frac{a_{21}}{N+1} + \frac{a_{11}a_{10}}{(N+1)^2} - \left(\rho + \frac{a_{21}}{2} + \frac{a_{11}a_{10}}{4} - \beta_0 a_{11} \right) \frac{4N}{(N+1)^2} \right. \\ & \left. + a_{11} \left(\frac{a_{11}}{N} + a_{10} - \frac{2(a_{11}+a_{10})N}{N+1} + \beta_0 \right) [\psi_1(N+1) - \zeta_2] \right] \\ & + \left(\frac{a_{11}^2}{N^2} + \frac{2a_{11}(a_{11}+a_{10})}{(N+1)^2} \right) \left\{ \left(\frac{a_{10}^2}{2} - a_{10}(a_{11} - \beta_0) - a_{11}^2 \right) \left(\frac{2}{(N+1)^3} - \frac{N}{(N+1)^2} \right) \right. \\ & + \left(\frac{a_{21}a_{10}}{a_{11}} + a_{20} \right) \left(\frac{1}{(N+1)^2} - \frac{N}{(N+1)^2} \right) \\ & + \left(\frac{a_{21}}{N} + a_{20} - \frac{2(a_{21}+a_{20})N}{N+1} \right) [\psi_1(N+1) - \zeta_2] \\ & + \left(\frac{a_{11}}{N} + a_{10} - \frac{2(a_{11}+a_{10})N}{N+1} \right)^2 \left(\zeta_3 - \frac{1}{2} \psi_2(N+1) \right) \\ & + \left(\frac{a_{11}}{N} + a_{10} - \frac{2(a_{11}+a_{10})N}{N+1} \right) \left[\left(3\zeta_3 - \frac{1}{2} \psi_2(N+1) \right) \beta_0 + \frac{\tilde{\chi}'_1(0, N)}{a_{11}} \right. \\ & \left. \left. + \frac{2a_{10}}{(N+1)^3} + \frac{a_{21}/a_{11}}{(N+1)^2} \right] \right\}, \end{aligned} \quad (2.19)$$

where again the various coefficients are defined in appendix A, and we have left implicit the function $\tilde{\chi}'_1(0, N)$, eq. (A.49). Eq. (2.18) was already presented in ref. [38],³ while eq. (2.19) is a new result of this study.

Before moving further, some comments are in order. We observe that, due to accidental zeros of the LL singularity both at NLO and NNLO, the leading singularity at these two orders is the NLL one, namely $1/N$ and $1/N^2$, respectively. These NLL poles are predicted correctly by NLL resummation, so in particular γ_2^{NLL} has the exact leading singularity (γ_1^{NLL} has it by construction). In contrast, the leading singularity of both $\gamma_1^{\text{LL}'}$ and $\gamma_2^{\text{LL}'}$ is predicted by the resummation and thus, being the resummation just accurate at LL, is not exact. While the full LL' anomalous dimension, being an all-order result, is reliable, each term of its perturbative expansion may not be. In particular, these two terms, $\gamma_1^{\text{LL}'}$ and $\gamma_2^{\text{LL}'}$, do not contain anything of the exact result, because they are zero at LL. Since the impact of these two orders in the expansion of resummed splitting functions (in particular P_{qg} and P_{qq}) may be substantial, we may expect that using LL' resummation may give rise

³Note that eq. (2.18) differs from the analogous result of ref. [38] by the subleading (NNLL) β_0 terms appearing in the second and third lines. Their origin is discussed in appendix B.

to somewhat unreliable resummed contributions when matched to high orders, e.g. NNLO or N³LO. More precisely, we may expect NLL resummation, which has the exact leading contributions at these two orders, to lead to more reliable matched results to high orders.

These considerations suggest that the use of the NLL anomalous dimension in the construction of γ_{gg} (as originally suggested in ref. [30]) is preferred. Thus, from now on we will adopt the NLL anomalous dimension as default ingredient for the resummation of anomalous dimension matrix, and possibly consider LL' resummation as a variant to estimate the impact of subleading logarithmic contributions. Both options are now available in the new 3.0 version of the HELL code.

2.3 The four-loop splitting functions at small- x in the $Q_0\overline{\text{MS}}$ scheme

The results of the previous section allow to construct all $\Delta_4\gamma_{ij}^{\text{NLL}}$, and thus by Mellin inversion all $\Delta_4P_{ij}^{\text{NLL}}$, needed to match NLL resummation to N³LO evolution. While these results will be in practice of no use until the computation of the N³LO splitting functions will be completed, it is interesting to extract from them the small- x terms at LL and NLL of the yet unknown four-loop splitting functions. These may be also useful to construct approximate predictions of the four-loop splitting functions while waiting for their full computation.

We remind the reader that the resummation procedure previously described lead to all-order results which are not in the traditional $\overline{\text{MS}}$ scheme, but rather in a related factorization scheme called $Q_0\overline{\text{MS}}$ [39, 47, 51, 52], which is particularly suitable for small- x resummation. Indeed, in the $\overline{\text{MS}}$ scheme there are some cancellations of large small- x contributions taking place between parton evolution and coefficient functions. The Q_0 variant of the scheme automatically removes these large contributions from both objects, leading to resummed predictions which are perturbatively much more stable. For this reason, here as well as in previous studies, we always use the $Q_0\overline{\text{MS}}$ scheme for results including small- x resummation. The difference between the $\overline{\text{MS}}$ and $Q_0\overline{\text{MS}}$ factorization schemes influences the resummation of the anomalous dimensions beyond the leading logarithmic accuracy, as well as the resummation of the coefficient functions. We first concentrate on $Q_0\overline{\text{MS}}$, while we present results for $\overline{\text{MS}}$ in the next section.

The small- N expansion of the expansion terms of γ_+^{NLL} is given in appendix A. With those results, we can construct all the other entries using eqs. (2.9) and (2.7). Denoting with $\gamma_{ij}^{(k)}$ the exact $\mathcal{O}(\alpha_s^{k+1})$ anomalous dimension, we have

$$\begin{aligned} \gamma_{gg}^{(3)}(N) &= \frac{1}{N^4} \frac{C_A^4}{\pi^4} 2\zeta_3 \\ &+ \frac{1}{N^3} \frac{1}{\pi^4} \left[C_A^4 \left(-\frac{1205}{162} + \frac{67}{36}\zeta_2 + \frac{1}{4}\zeta_2^2 - \frac{77}{6}\zeta_3 \right) + n_f C_A^3 \left(-\frac{233}{162} + \frac{13}{36}\zeta_2 + \zeta_3 \right) \right. \\ &\quad \left. + n_f C_A^2 C_F \left(\frac{617}{243} - \frac{13}{18}\zeta_2 + \frac{2}{3}\zeta_3 \right) \right] \\ &+ \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned} \tag{2.20a}$$

$$\gamma_{gg}^{(3)}(N) = \frac{1}{N^3} \frac{C_A^3 n_f}{3\pi^4} \left(\frac{82}{81} + 2\zeta_3 \right) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{2.20b}$$

The gq and qq anomalous dimensions are obtained by simply multiplying the gg and qq ones by C_F/C_A , even though we stress again that only the $1/N^4$ pole of the resulting gq anomalous dimension is correct. Hence we have

$$\gamma_{gq}^{(3)}(N) = \frac{1}{N^4} \frac{C_A^3 C_F}{\pi^4} 2\zeta_3 + \mathcal{O}\left(\frac{1}{N^3}\right), \quad (2.21a)$$

$$\gamma_{qq}^{(3)}(N) = \frac{1}{N^3} \frac{C_A^2 C_F n_f}{3\pi^4} \left(\frac{82}{81} + 2\zeta_3\right) + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (2.21b)$$

The corresponding small- x logarithms in the four-loop splitting functions are easily obtained by Mellin inversion of eqs. (2.20), according to

$$\mathcal{M}^{-1}\left[\frac{1}{N^4}\right] = \frac{1}{6} \frac{\log^3 \frac{1}{x}}{x}, \quad \mathcal{M}^{-1}\left[\frac{1}{N^3}\right] = \frac{1}{2} \frac{\log^2 \frac{1}{x}}{x}, \quad (2.22)$$

where \mathcal{M}^{-1} denotes the inverse Mellin transform. Thus, four-loop splitting functions exhibit a much stronger growth at small- x than those at a previous order, which behave like $\alpha_s^3 x^{-1} \log x$. To our knowledge, the NLL contribution to P_{gg} is explicitly presented here for the first time.

2.4 The four-loop splitting functions at small- x in the $\overline{\text{MS}}$ scheme

The effect of scheme change between $Q_0\overline{\text{MS}}$ and $\overline{\text{MS}}$ turns out to be of relative order α_s^3 , and thus all the fixed-order results considered in previous studies on small x resummation happened to be identical in either scheme. However, in this work we are considering resummation matched (or expanded) to N³LO, thus becoming sensitive to the scheme choice even at fixed order. It is thus important to recall how the conversion is performed. The goal of this subsection is also to provide the small- x contributions of the N³LO splitting functions in the $\overline{\text{MS}}$ scheme, namely in the scheme in which the full four-loop computation will likely be carried out.

A factorization scheme change is a multiplicative redefinition of the PDFs f and coefficient functions C . Focussing on $\overline{\text{MS}}$ and $Q_0\overline{\text{MS}}$, and considering both processes with one or two hadrons in the initial state (i.e. coefficient functions with one or two flavour indices), we have

$$f_i^{\overline{\text{MS}}}(N, Q^2) = \Lambda_{ij}^{-1}(N, \alpha_s) f_j(N, Q^2), \quad (2.23)$$

$$C_i^{\overline{\text{MS}}}(N, \alpha_s) = C_j(N, \alpha_s) \Lambda_{ji}(N, \alpha_s), \quad (2.24)$$

$$C_{ij}^{\overline{\text{MS}}}(N, \alpha_s) = C_{kl}(N, \alpha_s) \Lambda_{ik}(N, \alpha_s) \Lambda_{jl}(N, \alpha_s), \quad (2.25)$$

where $\alpha_s = \alpha_s(Q^2)$ and we denoted with a $\overline{\text{MS}}$ label quantities in that scheme and without label quantities in the $Q_0\overline{\text{MS}}$ scheme. Accordingly, the anomalous dimensions change as

$$\gamma_{ij}^{\overline{\text{MS}}}(N, \alpha_s) = \Lambda_{ik}^{-1}(N, \alpha_s) \gamma_{kl}(N, \alpha_s) \Lambda_{lj}(N, \alpha_s) - \Lambda_{ik}^{-1}(N, \alpha_s) Q^2 \frac{d\Lambda_{kj}(N, \alpha_s)}{dQ^2}. \quad (2.26)$$

The function Λ_{ij} is a matrix in flavour space implementing the scheme change. As far as small- x scheme changes are concerned, this matrix is trivial in its non-singlet part, so we

focus only on the singlet. Up to NLL, its form is given by [30]⁴

$$\begin{pmatrix} \Lambda_{gg} & \Lambda_{gq} \\ \Lambda_{qg} & \Lambda_{qq} \end{pmatrix} = \begin{pmatrix} R & \frac{C_F}{C_A}(R-1) \\ 0 & 1 \end{pmatrix} + \alpha_s \begin{pmatrix} \cdot & \cdot \\ v & \cdot \end{pmatrix} + \text{NNLL}, \quad (2.27)$$

where both R and v are LL functions, i.e. functions of α_s/N to all orders. The form of the LL part of the matrix is such that the scheme change has no effect on the LL part of the anomalous dimension matrix. The three empty slots in the NLL part of the matrix have an effect only on the γ_{gq} entry at NLL, which is however not determined by NLL resummation (as we already stressed), and are thus of no relevance. Furthermore, they also affect the resummation of partonic coefficient functions but this effect is beyond the accuracy currently achieved in the context of small- x resummation. Thus, at the currently available logarithmic accuracy, the three empty slots can be any LL function of α_s/N .

The scheme-change function was calculated long ago [47]

$$R(M) = \sqrt{\frac{-1 \Gamma(1-M)\chi_0(M)}{M \Gamma(1+M)\chi'_0(M)}} \exp\left\{ M\psi(1) + \int_0^M dc \frac{\psi'(1) - \psi'(1-c)}{2\psi(1) - \psi(c) - \psi(1-c)} \right\}, \quad (2.28)$$

$$v(M) = \frac{R(M) - 1}{M} h(M), \quad (2.29)$$

where $h(M) = \sum_{k \geq 0} h_k M^k$ is the function used for resumming γ_{qg} , eq. (2.4), $\chi_0(M)$ is the BFKL kernel at LO (eq. (A.39) in $N = 0$), $\Gamma(M)$ and $\psi(M)$ the gamma and digamma functions respectively. The functions $R(M)$ and $v(M)$ have to be evaluated in $M = \gamma_+(N, \alpha_s)$ in eq. (2.27), where γ_+ is the resummed one (in either scheme, as only its LL part needs to be correct, and at LL the scheme change is ineffective on the anomalous dimensions). The form of v , eq. (2.29), is such that at NLL both γ_{qg} and γ_{qq} are the same in $\overline{\text{MS}}$ and $Q_0\overline{\text{MS}}$. Additionally, we have already noted that the matrix structure of Λ_{ij} at LL is such that it has no effect on the LL anomalous dimensions. Thus, the only anomalous dimension which is sensitive at NLL to the scheme change is γ_{gg} , and we have

$$\overline{\gamma_{gg}}(N, \alpha_s) = \gamma_{gg}(N, \alpha_s) + \alpha_s^4 \left[\beta_0 8 \zeta_3 \gamma_0^3(N) + \mathcal{O}\left(\frac{1}{N^2}\right) \right] + \mathcal{O}(\alpha_s^5), \quad (2.30)$$

having used the expansion of the function R in powers of M ,

$$R(M) = 1 + \frac{8}{3} \zeta_3 M^3 + \mathcal{O}(M^4). \quad (2.31)$$

The scheme change contribution is entirely due to the derivative term in eq. (2.26). Of course, also the NLL part of γ_{qg} changes by the scheme change, but as we already repeated several times NLL resummation is not able to predict it. To conclude, we report the actual

⁴Note that there is a typo in ref. [30] that we correct here.

expansion to NLL of the gg anomalous dimension in the $\overline{\text{MS}}$ scheme:

$$\begin{aligned}
 \gamma_{gg}^{\overline{\text{MS}}(3)}(N) &= \frac{1}{N^4} \frac{C_A^4}{\pi^4} 2\zeta_3 \\
 &+ \frac{1}{N^3} \frac{1}{\pi^4} \left[C_A^4 \left(-\frac{1205}{162} + \frac{67}{36}\zeta_2 + \frac{1}{4}\zeta_2^2 - \frac{11}{2}\zeta_3 \right) + n_f C_A^3 \left(-\frac{233}{162} + \frac{13}{36}\zeta_2 - \frac{1}{3}\zeta_3 \right) \right. \\
 &\quad \left. + n_f C_A^2 C_F \left(\frac{617}{243} - \frac{13}{18}\zeta_2 + \frac{2}{3}\zeta_3 \right) \right] \\
 &+ \mathcal{O}\left(\frac{1}{N^2}\right).
 \end{aligned}
 \tag{2.32}$$

At this accuracy, all the other entries are identical to their $Q_0\overline{\text{MS}}$ counterparts given in section 2.3. To our knowledge, the NLL contributions to P_{gg} in the $\overline{\text{MS}}$ scheme are explicitly presented here for the first time.

3 Numerical results and discussion

Thus far we have presented analytical results. We now concentrate on numerics and we illustrate the difference between the two variants of the resummation discussed above. We also present approximate results for the four-loop splitting functions which are based on the expansion of the resummation and we critically assess the trustworthiness of this construction.

3.1 Resummed splitting functions at NNLO+NLL

First, we consider the four singlet splitting functions at fixed order and with resummation using the NLL anomalous dimension, which is the new default in HELL 3.0. In figure 1 we show P_{gg} (blue) and P_{qg} (orange) in the left plot, and P_{gq} (blue) and P_{qq} (orange) in the right plot, at LO (dotted), NLO (dashed), NNLO (dot-dot-dashed) and NNLO+NLL (solid). The resummed result is supplemented with an uncertainty band, which aims to estimate the impact of unknown subleading logarithmic contributions. Following ref. [38], this band is obtained by considering variations of the way RC resummation of γ_+ is implemented and of the way the resummation of γ_{qg} is performed, and summing in quadrature the two effects.⁵ The qualitative aspect of these results is the same of those obtained with the HELL 2.0 settings of ref. [38], i.e. using the LL' anomalous dimension.⁶

To better appreciate similarities and differences, we compare the two variants of the resummation in figure 2, focussing on P_{qg} on the left and on P_{gq} on the right. The current default, denoted with “NNLO+NLL” (solid red) is compared to the choice we made in ref. [38], which has been labelled “NNLO+NLL (LL')” (dot-dashed green). For completeness, we also show fixed-order results (gray). We see that the difference for P_{gg} are very

⁵In fact, in ref. [38] we considered only the second variation for P_{qg} and P_{gq} ; we now use a more symmetric approach and use both for all the splitting functions.

⁶We warn the reader that we have discovered a bug in the implementation of our NLL results. The numerical impact is not dramatic and it is discussed in detail in appendix B. All numerical results presented here, including the ones with HELL 2.0 settings (i.e., using the LL' anomalous dimension), have been obtained with the corrected implementation of the resummation.

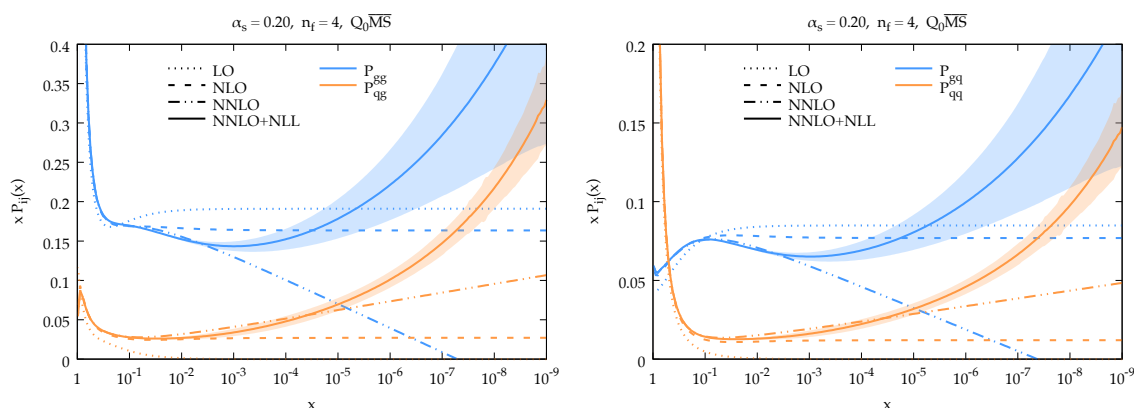


Figure 1. The resummed and matched splitting functions at NNLO+NLL (solid) accuracy compared with the fixed-order results at LO (dotted), NLO (dashed) and NNLO (dot-dot-dashed). The left plot shows P_{gg} (blue) and P_{qg} (orange), and the right plot P_{gg} (blue) and P_{qq} (orange). The plots are for $\alpha_s = 0.2$ and $n_f = 4$ in the $Q_0\overline{\text{MS}}$ scheme.

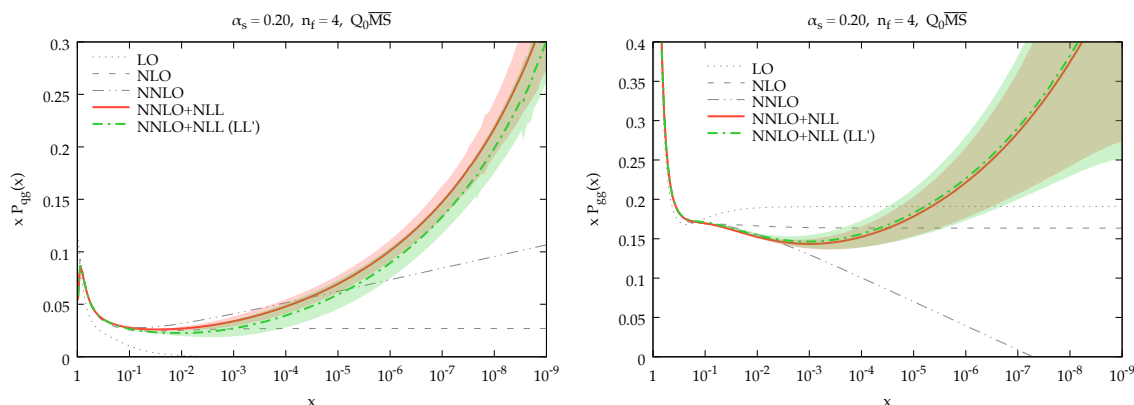


Figure 2. Comparison between resummed and matched splitting functions in two variants of small- x resummation.

small and well within the uncertainty band. The difference for P_{qg} is not large either, even though the uncertainty bands are smaller and comparable in size with such a difference. Similar considerations hold for the other splitting functions because, as it is clear from figure 1, P_{qq} and P_{gq} behave similarly to P_{gg} and P_{gg} at small x , due to the colour-charge relations, eqs. (2.7). We have checked that the comparison between the predictions obtained using the NLL anomalous dimension versus the LL' remains equivalent also at NLO+NLL accuracy.

A striking feature of the result is the small size of the uncertainty band that we obtain for P_{qg} . This is rather counterintuitive because P_{qg} (and P_{qq}) start at NLL and they are therefore known only at their leading non-vanishing logarithmic accuracy. Perhaps this signals the limitation in our way of estimated theoretical uncertainties, which is currently purely based on the variation of subleading contributions related to the running of the

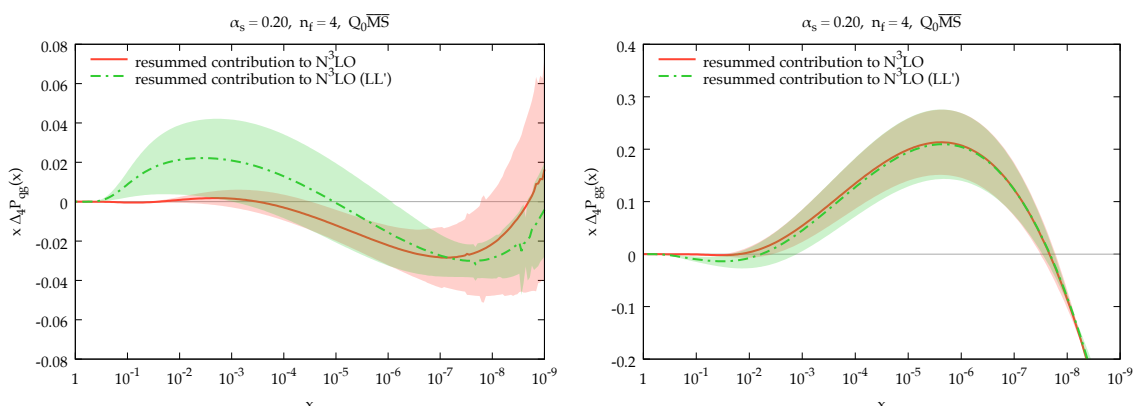


Figure 3. Comparison between the resummed contributions $\Delta_4 P_{qg}$ (left) and $\Delta_4 P_{gg}$ (right) to be added to N³LO splitting functions (when available) in two variants of small- x resummation.

strong coupling. Thus, in order to better assess the impact of subleading logarithms, it would be important to push the accuracy of the resummation, at least for the quark-initiated splitting functions, one logarithmic order higher.

We now consider the resummation matched to one order higher, in view a future combination with N³LO splitting functions. Also in this case, we compare the two variants of resummation, which led to very similar results when matched to NNLO. We have already argued from theoretical grounds that usage of the LL' variant is less favourable because one has less control over the subleading poles that appear in the expansion of the resummation. In figure 3 we see that this worry is indeed justified. In these plots we show the resummed contributions $x\Delta_4 P_{qg}$ (on the left) and $x\Delta_4 P_{gg}$ (on the right), which would have to be added to the N³LO splitting functions according to eq. (2.1). The solid red curve denotes the default resummation in HELL 3.0 based on the NLL anomalous dimension, while in dot-dashed green we show the LL' variant. Both plots show an issue of the LL', which was absent when matching at lower orders: the resummed contributions give rise to a (most likely) spurious contribution at moderate x , which is instead absent if full NLL is employed in the resummation. Indeed, for $x \gtrsim 10^{-3} \div 10^{-2}$ we expect to be outside the resummation region, and the effect of resummation should be smaller compared to the fixed-order contribution, which is more reliable in this region. This behaviour is violated in $\Delta_4 P_{ij}$ when using the LL' anomalous dimension, due a large contribution of $\gamma_2^{LL'}$, eq. (2.15), entering in eq. (2.9), which makes $\Delta_4 P_{ij}$ even larger than $\Delta_3 P_{ij}$ in this region, despite it being of higher order in α_s . Consistently, the green curve also has a rather large uncertainty band in that region, which makes it almost compatible with the red curve, which has instead a smaller uncertainty, as one would expect. We interpret this behaviour as further confirmation of the aforementioned theoretical arguments in favour of our new default. However, it should be stressed once again that both curves feature the same logarithmic accuracy, and hence this discrepancy contributes to our theoretical uncertainty.

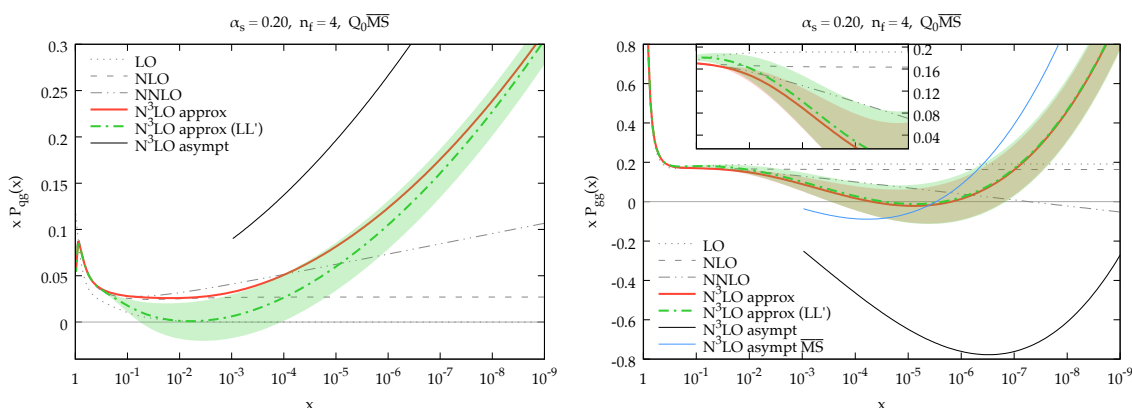


Figure 4. Approximate N³LO prediction (solid) of the P_{qg} (left) and P_{gg} (right) splitting functions as obtained by the $\mathcal{O}(\alpha_s^4)$ expansion of the resummation. The NLL asymptotic behaviour is also shown (light solid).

3.2 Approximate N³LO

We can use the expansion of the resummed splitting functions to $\mathcal{O}(\alpha_s^4)$ to make an approximate prediction of the N³LO splitting functions, simply by adding it to the exact NNLO ones. This is shown in solid red in figure 4 for P_{qg} (left plot) and P_{gg} (right plot), and in dot-dashed green for the LL' variant. According to the discussion in the previous section, the latter curve is not expected to be accurate in the region of moderately large x , where it has an unphysically large effect. These predictions are further supplemented by the same uncertainty band that appears on resummed results, which happens to be invisible for P_{qg} when using our new default implementation⁷ (and thus confirms that our uncertainty band underestimates the actual uncertainty on P_{qg}). Additionally, we also show in light solid black the asymptotic small- x behaviour at N³LO, as obtained by adding to the exact NNLO the pure NLL contributions, eq. (2.20), without any subleading effects. Note that these results are obtained in the $Q_0\overline{\text{MS}}$ scheme. While we have not implemented the resummation (and hence its expansion) in $\overline{\text{MS}}$, we can easily plot the asymptotic behaviour of the splitting functions in this scheme, exploiting the results of section 2.4. The quark splitting function P_{qg} is unaffected, while the $\overline{\text{MS}}$ asymptotic result for P_{gg} , eq. (2.32), is shown in solid cyan.

In the small- x limit, approximate predictions behave as their asymptotic expansions, by construction. The difference is due to subleading NNLL contributions, behaving as $\frac{1}{x} \log \frac{1}{x}$ at this order (a straight line in the plots). While these NNLL contributions are subleading at asymptotically small x , their effect is sizeable for all the x range shown in the plots, which is rather large, reaching $x = 10^{-9}$. This is true in particular for P_{gg} ,

⁷Indeed, at this order, our uncertainty band originates from the parameter T appearing in eq. (2.9), and on the potential dependence of the anomalous dimensions $\gamma_{0,1,2}$ on the parameter T' defined in appendix A. Since the NLL anomalous dimension is more precise than the LL' one, none of the $\gamma_{0,1,2}^{\text{NLL}}$ depends on T' , while $\gamma_2^{\text{LL}'}$ does (and also γ_3^{NLL} , which contributes to P_{gg}). It is the latter (T') dependence that generates the uncertainty bands in figure 4, while the T parameter variation has no appreciable effect at this order on any of the two variants.

where the pure NLL asymptotic curve is very different from the approximate N³LO, so much that in order to display the asymptotic behaviour we had to plot $xP_{gg}(x)$ in a rather extended range. For this reason, we also added an inset which zooms in to the region $10^{-5} < x < 0.1$, most relevant for HERA and LHC physics, to better appreciate the perturbative behaviour of the fixed-order splitting functions. This shows once again that subleading contributions have a very important role at intermediate and moderately small values of x . Similar conclusions were reached some time ago in ref. [53] (see also ref. [54] for a similar study in the non-singlet case). This also suggests that the approximate N³LO prediction that we plotted has a huge uncertainty, likely larger than what we estimate with our uncertainty bands.

By comparing resummed results (figure 1) and their expansions (figure 2), we can conclude that, while the small- x contributions to P_{gg} , as obtained from the expansion of the resummation, behave in a perturbative way, the prediction of the N³LO contribution to P_{gg} , which is more directly sensitive to the (perturbatively unstable) BFKL kernel, is very different from its all-order counterpart, as it was the NNLO contribution. We must conclude that these approximate N³LO predictions cannot be regarded as a faithful estimate of the actual N³LO (especially for P_{gg} and P_{gq}) due to potentially underestimated subleading contributions. However, what we can certainly conclude is that exact N³LO evolution (when available) will be unreliable at small x , and thus it will necessarily have to be supplemented with the all-order resummation of small- x contributions.

As we have argued before, because of the sensitivity of the N³LO splitting functions to subleading logarithmic contributions, it would be important to push the resummation to NNLL accuracy. However, NNLL resummation requires at least the knowledge of the NNLO BFKL kernel, which is so far only known in a collinear approximation [39]. It will be important in the future to explore the possibility of computing the BFKL kernel to NNLO [40–45], and perhaps to consider the option of using its collinear approximation.

Finally, it is interesting to note that the asymptotic behaviour in $\overline{\text{MS}}$ appears to be closer to the all-order result (albeit computed in a different scheme). This suggests that a future study of the resummation in $\overline{\text{MS}}$ may reveal interesting properties in terms of the size of subleading contributions, despite the fact that in this scheme we expect stronger cancellations between coefficient functions and parton evolution.

4 Conclusions

In this paper we have discussed the role of higher-order corrections to the splitting functions, which govern the evolution of the parton distribution functions. In particular, we have exploited results in small- x resummation to study the behaviour of the yet-unknown four-loop splitting functions in the singlet sector. Our results stress once again the fact that small- x singularities lead to loss of perturbative stability, when higher orders are considered. This has been masked so far by accidental cancellations at NLO but it becomes apparent at NNLO, even though also there the strongest singularity is accidentally zero, thus slightly mitigating the perturbative deterioration. Instead, at the next orders there are no accidental cancellations, so the perturbative instability is no longer moderated, and

we estimate it to be rather severe at N³LO. Thanks to this work, this potential instability can be solved by adding the resummation to the full four-loop splitting functions, when these will become available.

We have also investigated the possibility of using the expansion of the resummation to construct approximate N³LO splitting functions. Unfortunately, we have found that subleading corrections, which are only partially included in our approach, have a sizeable impact at moderate x , thus rendering the construction of approximate fixed-order splitting functions rather uncertain. However, our asymptotic results can be used as a check on the full four-loop calculation, or for complementing an approximate computation based on integer Mellin moments.

While performing these studies we have encountered a potential source of instability in the way the resummation of the quark anomalous dimension γ_{qq} was implemented in HELL 2.0 in ref. [38], which was based on a hybrid resummation formula denoted LL'. Therefore, we have adopted as a new default a resummation fully based on NLL and consequently released a new version of the resummation code HELL 3.0:

www.ge.infn.it/~bonvini/hell

As the distinction between the two choices is beyond the accuracy of the calculation, the old option can be, and should be, still used to estimate theoretical uncertainties. Furthermore, we anticipate that an analogous situation appears in the resummation of partonic coefficient functions. This issue will be discussed in a forthcoming study [55]. Finally, these results have been recently exploited in a double-resummed calculation of the Higgs production cross section [56].

Acknowledgments

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A Perturbative expansion of resummed anomalous dimensions to N³LO

In this appendix we derive the expansion in powers of α_s of the resummed plus eigenvalue of the singlet anomalous dimension matrix presented in ref. [38]. Specifically, we provide the detailed computation of the expansion of the NLL resummed anomalous dimensions up to $\mathcal{O}(\alpha_s^4)$, and of the LL' anomalous dimension up to $\mathcal{O}(\alpha_s^3)$, as needed for matching NLL resummation in DGLAP evolution to N³LO. We recall that the anomalous dimensions at LL' and NLL are constructed as [38]

$$\begin{aligned} \gamma_+^{\text{resLL}'}(N, \alpha_s) &= \gamma_{\text{DL-LO}}(N, \alpha_s) + \Delta_{\text{DL-LO}} \gamma_{\text{rc}}^{\text{NLL}}(N, \alpha_s) + \gamma_{\text{match}}^{\text{LO+LL}'}(N, \alpha_s) \\ &\quad - \left[\Delta_{\text{DL-LO}} \gamma_{\text{rc}}^{\text{NLL}}(1, \alpha_s) + \gamma_{\text{match}}^{\text{LO+LL}'}(1, \alpha_s) \right] f_{\text{mom}}(N), \end{aligned} \quad (\text{A.1a})$$

$$\begin{aligned} \gamma_+^{\text{resNLL}}(N, \alpha_s) &= \gamma_{\text{DL-NLO}}(N, \alpha_s) + \Delta_{\text{DL-NLO}} \gamma_{\text{rc}}^{\text{NLL}}(N, \alpha_s) + \Delta \gamma_{\text{ss}}^{\text{rc}}(N, \alpha_s) + \gamma_{\text{match}}^{\text{ss}}(N, \alpha_s) \\ &\quad - \left[\Delta_{\text{DL-LO}} \gamma_{\text{rc}}^{\text{NLL}}(1, \alpha_s) + \Delta \gamma_{\text{ss}}^{\text{rc}}(1, \alpha_s) + \gamma_{\text{match}}^{\text{ss}}(1, \alpha_s) \right] f_{\text{mom}}(N), \end{aligned} \quad (\text{A.1b})$$

where $\gamma_{\text{DL-(N)LO}}$ is the double-leading (DL) anomalous dimension, $\Delta_{\text{DL-(N)LO}}\gamma_{\text{rc}}^{\text{NLL}}$ is the contribution (to be added to the DL) coming from the resummation of running-coupling (RC) effects, $\Delta\gamma_{ss}^{\text{rc}}$ is a running-coupling correction to the fixed-coupling DL construction at NLL, $\gamma_{\text{match}}^{\text{LO+LL}'}$ and $\gamma_{\text{match}}^{ss}$ are matching functions to cancel mismatched singularities, and the second line of each equation restores momentum conservation, i.e. the constraint $\gamma_+^{\text{res LL}'}(1, \alpha_s) = \gamma_+^{\text{res NLL}}(1, \alpha_s) = 0$, through a subleading function f_{mom} defined in eq. (A.8). All these ingredients have been presented in ref. [38] and will be used in the following.

Before starting, we recall that the DL anomalous dimension is constructed starting from the fixed-order BFKL kernel matched to the fixed-order anomalous dimension. Thus, one of the ingredients of (N)LL resummation is the (N)LO anomalous dimension. In ref. [38] an approximate form for the input anomalous dimension was suggested to facilitate the numerical implementation and to solve a potential issue. The approximation does not represent any loss of accuracy, as the only requirements needed for the input anomalous dimension are to be accurate at NLL and to conserve momentum, both of which are satisfied in the approximation of ref. [38]. At LO and NLO they are given by

$$\hat{\gamma}_0 = \frac{a_{11}}{N} + a_{10} - \frac{2(a_{11} + a_{10})N}{N + 1}, \tag{A.2a}$$

$$\hat{\gamma}_1 = \frac{a_{21}}{N} + a_{20} - \frac{2(a_{21} + a_{20})N}{N + 1}, \tag{A.2b}$$

with

$$a_{11} = \frac{C_A}{\pi}, \tag{A.3a}$$

$$a_{21} = n_f \frac{26C_F - 23C_A}{36\pi^2}, \tag{A.3b}$$

$$a_{10} = -\frac{11C_A + 2n_f(1 - 2C_F/C_A)}{12\pi}, \tag{A.3c}$$

$$a_{20} = \frac{1}{\pi^2} \left[\frac{1643}{24} - \frac{33}{2}\zeta_2 - 18\zeta_3 + n_f \left(\frac{4}{9}\zeta_2 - \frac{68}{81} \right) + n_f^2 \frac{13}{2187} \right]. \tag{A.3d}$$

In the next, we start from LL' resummation, and then move to NLL. The computation follows closely the one presented in section 3 and 4 of ref. [38], extending it to one extra order.

A.1 Expansion of the LL' anomalous dimension

We start expanding the LL' anomalous dimension up to $\mathcal{O}(\alpha_s^3)$. The first ingredient for resummation is the DL resummed anomalous dimension γ_{DL} , which is obtained from the implicit equation

$$\chi_{\Sigma}(\gamma_{\text{DL}}(N, \alpha_s), N, \alpha_s) = N. \tag{A.4}$$

The function $\chi_{\Sigma}(M, N, \alpha_s)$ is the so-called off-shell BFKL kernel [29, 37]. For LL resummation it is given by [29]

$$\chi_{\Sigma}^{\text{LO}}(M, N, \alpha_s) = \chi_s \left(\frac{\alpha_s}{M} \right) + \chi_s \left(\frac{\alpha_s}{1 - M + N} \right) + \alpha_s \tilde{\chi}_0(M, N) + c_{\text{mom}}^{\text{LO}}(\alpha_s) f_{\text{mom}}(N), \tag{A.5}$$

where the function $\chi_s(\alpha_s/M)$ is the dual of the LO anomalous dimension $\alpha_s\hat{\gamma}_0$,

$$\alpha_s\hat{\gamma}_0\left(\chi_s\left(\frac{\alpha_s}{M}\right)\right) = M \quad \Leftrightarrow \quad \chi_s\left(\frac{1}{\hat{\gamma}_0(N)}\right) = N, \quad (\text{A.6})$$

and

$$\tilde{\chi}_0(M, N) = \chi_{01}\left[\psi(1) + \psi(1+N) - \psi(1+M) - \psi(2-M+N)\right] \quad (\text{A.7})$$

is the off-shell extension of the LO BFKL kernel after subtracting double counting with χ_s . The last term restores the momentum conservation constraint $\gamma_{\text{DL}}(1, \alpha_s) = 0$, namely by duality $\chi_{\Sigma}(0, 1, \alpha_s) = 1$, through a function

$$f_{\text{mom}}(N) = \frac{4N}{(N+1)^2}, \quad (\text{A.8})$$

and with the coefficient

$$c_{\text{mom}}^{\text{LO}}(\alpha_s) = -\chi_s\left(\frac{\alpha_s}{2}\right) - \alpha_s\tilde{\chi}_0(0, 1). \quad (\text{A.9})$$

The coefficient χ_{01} appearing in eq. (A.7) is the first of the expansion of χ_s ,

$$\chi_s\left(\frac{\alpha_s}{M}\right) = \sum_{k=1}^{\infty} \chi_{0k}\left(\frac{\alpha_s}{M}\right)^k. \quad (\text{A.10})$$

All χ_{0k} coefficients are determined in terms of a_{11} and a_{10} , eq. (A.3), through eq. (A.6) and eq. (A.2). In particular, the first three coefficients are given by

$$\chi_{01} = a_{11}, \quad \chi_{02} = a_{11}a_{10}, \quad \chi_{03} = a_{11}(a_{10}^2 - 2a_{11}a_{10} - 2a_{11}^2). \quad (\text{A.11})$$

Following ref. [38], we write the α_s -expansion of the DL anomalous dimension

$$\gamma_{\text{DL-LO}}(N, \alpha_s) = \alpha_s\hat{\gamma}_0(N) + \alpha_s^2\tilde{\gamma}_1(N) + \alpha_s^3\tilde{\gamma}_2(N) + \mathcal{O}(\alpha_s^4), \quad (\text{A.12})$$

where $\hat{\gamma}_0$ is the input LO anomalous dimension eq. (A.2) used in the definition of χ_s , eq. (A.6), while $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the predictions of the resummation that we aim to find. Then, we substitute it into eq. (A.4) with χ_{Σ} given in eq. (A.5), and expand the equation in powers of α_s . The most delicate function to expand is the collinear χ_s in eq. (A.5), for which we find (omitting arguments to facilitate reading)

$$\begin{aligned} \chi_s\left(\frac{\alpha_s}{\gamma_{\text{DL-LO}}}\right) &= \chi_s\left(\frac{1}{\hat{\gamma}_0}\left[1 - \alpha_s\frac{\tilde{\gamma}_1}{\hat{\gamma}_0} + \alpha_s^2\frac{\tilde{\gamma}_1^2 - \hat{\gamma}_0\tilde{\gamma}_2}{\hat{\gamma}_0^2} + \mathcal{O}(\alpha_s^3)\right]\right) \\ &= \chi_s\left(\frac{1}{\hat{\gamma}_0}\right) - \alpha_s\frac{\tilde{\gamma}_1}{\hat{\gamma}_0^2}\chi_s'\left(\frac{1}{\hat{\gamma}_0}\right) + \alpha_s^2\left[\frac{\tilde{\gamma}_1^2 - \hat{\gamma}_0\tilde{\gamma}_2}{\hat{\gamma}_0^3}\chi_s'\left(\frac{1}{\hat{\gamma}_0}\right) + \frac{\tilde{\gamma}_1^2}{2\hat{\gamma}_0^4}\chi_s''\left(\frac{1}{\hat{\gamma}_0}\right)\right] + \mathcal{O}(\alpha_s^3) \\ &= N + \alpha_s\frac{\tilde{\gamma}_1}{\hat{\gamma}_0'} + \alpha_s^2\left[\frac{\tilde{\gamma}_2}{\hat{\gamma}_0'} - \frac{\tilde{\gamma}_1^2\hat{\gamma}_0''}{2\hat{\gamma}_0'^3}\right] + \mathcal{O}(\alpha_s^3), \end{aligned} \quad (\text{A.13})$$

where in the last equality we have used the definition eq. (A.6), and the formulae for the derivatives

$$\chi_s'\left(\frac{1}{\hat{\gamma}_0}\right) = -\frac{\hat{\gamma}_0^2}{\hat{\gamma}_0'}, \quad \chi_s''\left(\frac{1}{\hat{\gamma}_0}\right) = \frac{\hat{\gamma}_0^2}{\hat{\gamma}_0'}\left[2\hat{\gamma}_0 - \frac{\hat{\gamma}_0^2\hat{\gamma}_0''}{\hat{\gamma}_0'^2}\right], \quad (\text{A.14})$$

which can be derived from the very same definition. The prime \prime denotes a derivative with respect to the argument of the function, so $\chi'_s(1/\hat{\gamma}_0)$ is a derivative with respect to $1/\hat{\gamma}_0$, and $\hat{\gamma}''_0$ is a double derivative with respect to N . The anticollinear χ_s gives instead

$$\chi_s\left(\frac{\alpha_s}{1 - \gamma_{\text{DL-LO}} + N}\right) = \alpha_s \frac{\chi_{01}}{1 + N} + \alpha_s^2 \frac{\chi_{02} + \chi_{01}\hat{\gamma}_0}{(1 + N)^2} + \mathcal{O}(\alpha_s^3). \quad (\text{A.15})$$

The kernel eq. (A.5) expands as

$$\tilde{\chi}_0(\gamma_{\text{DL-LO}}(N, \alpha_s), N) = \tilde{\chi}_0(0, N) + \alpha_s \hat{\gamma}_0 \tilde{\chi}'_0(0, N) + \mathcal{O}(\alpha_s^2), \quad (\text{A.16})$$

where the derivative is with respect to M , i.e. the first argument. Putting everything together eq. (A.4) brings to the expanded equality

$$\begin{aligned} N = N + \alpha_s & \left[\frac{\tilde{\gamma}_1}{\hat{\gamma}'_0} + \frac{\chi_{01}}{1 + N} + \tilde{\chi}_0(0, N) - \left(\frac{\chi_{01}}{2} + \tilde{\chi}_0(0, 1) \right) f_{\text{mom}}(N) \right] \\ & + \alpha_s^2 \left[\frac{\tilde{\gamma}_2}{\hat{\gamma}'_0} - \frac{\tilde{\gamma}_1^2 \hat{\gamma}''_0}{2 \hat{\gamma}_0'^3} + \frac{\chi_{02} + \chi_{01}\hat{\gamma}_0}{(1 + N)^2} + \hat{\gamma}_0 \tilde{\chi}'_0(0, N) - \frac{\chi_{02}}{4} f_{\text{mom}}(N) \right] \\ & + \mathcal{O}(\alpha_s^3), \end{aligned} \quad (\text{A.17})$$

from which it immediately follows

$$\tilde{\gamma}_1(N) = -\hat{\gamma}'_0(N) \left[\frac{\chi_{01}}{1 + N} + \tilde{\chi}_0(0, N) - \left(\frac{\chi_{01}}{2} + \tilde{\chi}_0(0, 1) \right) f_{\text{mom}}(N) \right], \quad (\text{A.18})$$

$$\tilde{\gamma}_2(N) = \frac{\tilde{\gamma}_1^2(N) \hat{\gamma}''_0(N)}{2 \hat{\gamma}_0'^2(N)} - \hat{\gamma}'_0(N) \left[\frac{\chi_{02} + \chi_{01}\hat{\gamma}_0(N)}{(1 + N)^2} + \hat{\gamma}_0(N) \tilde{\chi}'_0(0, N) - \frac{\chi_{02}}{4} f_{\text{mom}}(N) \right]. \quad (\text{A.19})$$

Note that the $\mathcal{O}(\alpha_s^0)$ term cancels automatically, because $\hat{\gamma}_0$ in eq. (A.12) is the one used in the definition of χ_s , eq. (A.6). The expansion terms of the off-shell kernel $\tilde{\chi}_0(M, N)$, eq. (A.7), are given by

$$\tilde{\chi}_0(0, N) = -\frac{\chi_{01}}{1 + N}, \quad \tilde{\chi}'_0(0, N) = -\frac{\chi_{01}}{(1 + N)^2} + \chi_{01}[\psi_1(1 + N) - \zeta_2], \quad (\text{A.20})$$

which lead to the predictions

$$\tilde{\gamma}_1(N) = 0, \quad (\text{A.21})$$

$$\tilde{\gamma}_2(N) = -\hat{\gamma}'_0(N) \left[\frac{\chi_{02}}{(1 + N)^2} + \chi_{01}[\psi_1(1 + N) - \zeta_2] \hat{\gamma}_0(N) - \frac{\chi_{02}}{4} f_{\text{mom}}(N) \right]. \quad (\text{A.22})$$

As already noted in ref. [38], the fact that $\tilde{\gamma}_1$ vanishes is not surprising: indeed, the LL pole of the exact NLO $\gamma_+^{(1)}$ and NNLO $\gamma_+^{(2)}$ are accidentally zero, so the only part which is supposed to be predicted correctly by LL resummation was indeed expected to vanish.

Having computed the expansion of the DL part, we now move to the RC contributions. The function that resums the running-coupling effects is given by [38]

$$\gamma_{\text{rc}}(N, \alpha_s) = M_{\text{min}} + \beta_0 \bar{\alpha}_s \left[z \frac{k'_\nu(z)}{k_\nu(z)} - 1 \right], \quad (\text{A.23})$$

where $k_\nu(z)$ is a Bateman function, with

$$\frac{1}{\bar{\alpha}_s} = \frac{1}{\alpha_s} + \frac{\kappa' - 2c'/M_{\text{min}}^2}{\bar{\kappa} + 2(N - \bar{c})/M_{\text{min}}^2} \quad (\text{A.24a})$$

$$z = \frac{1}{\beta_0 \bar{\alpha}_s} \sqrt{\frac{N - \bar{c}}{\bar{\kappa}/2 + (N - \bar{c})/M_{\text{min}}^2}} \quad (\text{A.24b})$$

$$\nu = \left(\frac{c'}{N - \bar{c}} + \frac{\kappa' - 2c'/M_{\text{min}}^2}{\bar{\kappa} + 2(N - \bar{c})/M_{\text{min}}^2} \right) \bar{\alpha}_s z, \quad (\text{A.24c})$$

$$\bar{c}(\alpha_s) = c(\alpha_s) - \alpha_s c'(\alpha_s), \quad \bar{\kappa}(\alpha_s) = \kappa(\alpha_s) - \alpha_s \kappa'(\alpha_s). \quad (\text{A.24d})$$

In the equations above, $M_{\text{min}}(\alpha_s)$ is the position of the minimum of the BFKL kernel,⁸ and $c(\alpha_s)$ and $\kappa(\alpha_s)$ are the value and curvature of the kernel at such minimum. In deriving eq. (A.23), the α_s -dependence of the BFKL kernel has been approximated linearly, keeping the value of the kernel and its α_s -derivative correct. In ref. [38] a variant of this approximation in which the kernel is assumed to be proportional to α_s (i.e., as if it was a purely LO kernel) was considered to study the impact of subleading logarithmic contributions. This variant is recovered by letting $c' \rightarrow c/\alpha_s$, $\kappa' \rightarrow \kappa/\alpha_s$, and represents an equally valid alternative. The RC contribution to be added to the DL-LO result is given by

$$\begin{aligned} \Delta_{\text{DL-LO}} \gamma_{\text{rc}}(N, \alpha_s) &\equiv \gamma_{\text{rc}}(N, \alpha_s) - \left[M_{\text{min}} - \sqrt{\frac{N - c}{\kappa/2 + (N - c)/M_{\text{min}}^2}} - \beta_0 \alpha_s \right] \\ &= \beta_0 \alpha_s^2 \frac{3\kappa_0/32 - c_0}{N} \\ &\quad + \beta_0 \alpha_s^3 \left[c_0 \frac{3\kappa_0/32 - c_0}{N^2} + T' \frac{3\kappa_1/32 - c_1}{N} + \kappa_0 \frac{\beta_0 + 6m_1}{16N} + \kappa_0 \frac{24c_0 - 3\kappa_0}{256N^2} \right] \\ &\quad + \mathcal{O}(\alpha_s^4), \end{aligned} \quad (\text{A.25})$$

where $\kappa_{0,1}$ and $c_{0,1}$ are the $\mathcal{O}(\alpha_s^{1,2})$ terms of κ and c , and m_1 is determined from $M_{\text{min}}(\alpha_s) = 1/2 + \alpha_s m_1 + \mathcal{O}(\alpha_s^2)$. To cover both α_s -dependence approximations, we have introduced a parameter T' which equals 2 in the default approximation and equals 1 in the limit $c' \rightarrow c/\alpha_s$, $\kappa' \rightarrow \kappa/\alpha_s$. The values of κ_1 , c_1 and m_1 depend on the actual kernel used. For LL' resummation, the DL-NLO is used for RC resummation, differently from pure LL resummation which uses the DL-LO kernel. Thus, the second order coefficients in eq. (A.25) are m_1^{NLO} , c_1^{NLO} and κ_1^{NLO} , explicitly given in section A.3. Because in LL' resummation the RC contributions, computed from the NLO BFKL kernel, are matched to the DL-LO

⁸Note that in ref. [38] we suggested to compute the M_{min} from the kernel in symmetric variables, while we now decided to use the one in DIS variables, as we discuss in greater detail in section A.3.

anomalous dimension, there is a mismatch in the singularities at small x which are cured by the matching function $\gamma_{\text{match}}^{\text{LO+LL}'}$ defined in ref. [38]. Its expansion gives

$$\gamma_{\text{match}}^{\text{LO+LL}'}(N, \alpha_s) = \alpha_s^3 \frac{16c_0(\delta\kappa_1 + 6\kappa_0\delta m_1) + \kappa_0(16\delta c_1 - 3\delta\kappa_1 - 15\kappa_0\delta m_1)}{512N^2} + \mathcal{O}(\alpha_s^4), \quad (\text{A.26})$$

where

$$\delta m_1 = m_1^{\text{NLO}} - m_1^{\text{LO}}, \quad (\text{A.27})$$

$$\delta c_1 = c_1^{\text{NLO}} - c_1^{\text{LO}}, \quad (\text{A.28})$$

$$\delta\kappa_1 = \kappa_1^{\text{NLO}} - \kappa_1^{\text{LO}}, \quad (\text{A.29})$$

are the differences between the coefficients computed with the DL-NLO and the DL-LO BFKL kernels. Explicit results are given in section A.3.

Putting everything together, the expansion in powers of α_s of the full LL' anomalous dimension is given by

$$\begin{aligned} \gamma^{\text{resLL}'}(N, \alpha_s) &= \alpha_s \hat{\gamma}_0(N) + \alpha_s^2 \beta_0 \left(\frac{3}{32} \kappa_0 - c_0 \right) \left(\frac{1}{N} - f_{\text{mom}}(N) \right) \\ &+ \alpha_s^3 \left\{ \frac{\lambda_2}{N^2} + \frac{\lambda_1}{N} - (\lambda_2 + \lambda_1) f_{\text{mom}}(N) \right. \\ &\quad \left. - \hat{\gamma}'_0(N) \left[\frac{\chi_{02}}{(1+N)^2} + \chi_{01} [\psi_1(1+N) - \zeta_2] \hat{\gamma}_0(N) - \frac{\chi_{02}}{4} f_{\text{mom}}(N) \right] \right\} \\ &+ \mathcal{O}(\alpha_s^4), \end{aligned} \quad (\text{A.30})$$

with

$$\lambda_1 = \beta_0 \left[\frac{3}{32} T' \kappa_1^{\text{NLO}} - T' c_1^{\text{NLO}} + \frac{\beta_0 + 6m_1^{\text{NLO}}}{16} \kappa_0 \right] \quad (\text{A.31a})$$

$$\lambda_2 = \beta_0 \frac{48c_0\kappa_0 - 3\kappa_0^2 - 256c_0^2}{256} + \frac{16c_0(\delta\kappa_1 + 6\kappa_0\delta m_1) + \kappa_0(16\delta c_1 - 3\delta\kappa_1 - 15\kappa_0\delta m_1)}{512}. \quad (\text{A.31b})$$

Using eq. (A.11) and replacing the explicit form of $\hat{\gamma}_0(N)$, eq. (A.2), we obtain the results presented in eq. (2.14) and eq. (2.15).

A.2 Expansion of the NLL anomalous dimensions

We now move to the NLL anomalous dimension. This time, we need to push our expansion to one order higher. The off-shell kernel needed for NLL resummation is

$$\begin{aligned} \chi_{\Sigma}^{\text{NLO}}(M, N, \alpha_s) &= \chi_{s,\text{NLO}}(M, \alpha_s) + \chi_{s,\text{NLO}}(1 - M + N, \alpha_s) \\ &+ \alpha_s \tilde{\chi}_0(M, N) + \alpha_s^2 \tilde{\chi}_1(M, N) + \alpha_s^2 \chi_1^{\text{corr}}(M, N, \alpha_s) \\ &+ c_{\text{mom}}^{\text{NLO}}(\alpha_s) f_{\text{mom}}(N). \end{aligned} \quad (\text{A.32})$$

The function $\chi_{s,\text{NLO}}(M, \alpha_s)$ is the generalization of χ_s to the next order, which is obtained as the exact dual of the NLO anomalous dimension,

$$\chi_{s,\text{NLO}}(\alpha_s \hat{\gamma}_0(N) + \alpha_s^2 \hat{\gamma}_1(N), \alpha_s) = N. \quad (\text{A.33})$$

This kernel can be expanded as

$$\chi_{s,\text{NLO}}(M, \alpha_s) = \sum_{j=0}^{\infty} \alpha_s^j \sum_{k=1}^{\infty} \chi_{jk} \left(\frac{\alpha_s}{M} \right)^k, \quad (\text{A.34})$$

which generalizes eq. (A.10), which is just the $j = 0$ part of this function. All these coefficients are given in terms of a_{11} , a_{10} , a_{21} and a_{20} ; the relevant ones for what follows are

$$\chi_{11} = a_{21}, \quad \chi_{12} = a_{21}a_{10} + a_{11}a_{20}, \quad \chi_{21} = 0. \quad (\text{A.35})$$

The kernel $\tilde{\chi}_1(M, N)$ was given in eqs. (A.23)–(A.29) of ref. [37], and can be written as

$$\tilde{\chi}_1(M, N) = \tilde{\chi}_1^u(M, N) - \tilde{\chi}_1^u(0, N) + \tilde{\chi}_1^u(0, 0), \quad (\text{A.36})$$

where the function

$$\tilde{\chi}_1^u(M, N) = \check{\chi}_1(M, N) - \chi_{02} \left(\frac{1}{M^2} + \frac{1}{(1-M+N)^2} \right) - \chi_{11} \left(\frac{1}{M} + \frac{1}{1-M+N} \right) \quad (\text{A.37})$$

is regular in $M = 0$ and $1 - M + N = 0$. The function $\check{\chi}_1$ is given by [37]

$$\begin{aligned} \check{\chi}_1(M, N) = & -\frac{1}{2} \chi_{00}(M, N) \chi_{01} \left[2\psi_1(1+N) - \psi_1(M) - \psi_1(1-M+N) \right] \\ & - \frac{1}{2} \beta_0 \chi_{01} \left[\left(\frac{\chi_0(M, N)}{\chi_{01}} \right)^2 - \psi_1(M) - \psi_1(1-M+N) \right] \\ & + \frac{\chi_{01}^2}{4} \left\{ \left(\frac{12\chi_{11} - 26\chi_{02}}{3\chi_{01}^2} - \frac{1}{2} - 2\zeta_2 \right) \left[\psi(1) - \psi(M) \right] \right. \\ & + 3\zeta(3) + \psi_2(M) + \zeta_2 \left[\psi\left(\frac{1+M}{2}\right) - \psi\left(\frac{M}{2}\right) \right] + 4\phi_L^+(M) \\ & + \frac{3}{4(1-2M)} \left[\psi_1\left(\frac{1+M}{2}\right) - \psi_1\left(\frac{M}{2}\right) + \psi_1\left(\frac{1}{4}\right) - \psi_1\left(\frac{3}{4}\right) \right] \\ & - \left(\frac{9}{2} + \frac{6\chi_{02}}{\chi_{01}^2} \right) \frac{2+3M(1-M)}{16} \left[\frac{\psi_1\left(\frac{1+M}{2}\right) - \psi_1\left(\frac{M}{2}\right) + \psi_1\left(\frac{1}{4}\right) - \psi_1\left(\frac{3}{4}\right)}{1-2M} \right. \\ & \quad \left. + \frac{\psi_1\left(\frac{1+M}{2}\right) - \psi_1\left(\frac{M}{2}\right) + \psi_1\left(-\frac{1}{4}\right) - \psi_1\left(\frac{1}{4}\right)}{2(1+2M)} \right. \\ & \quad \left. - \frac{\psi_1\left(\frac{1+M}{2}\right) - \psi_1\left(\frac{M}{2}\right) + \psi_1\left(\frac{3}{4}\right) - \psi_1\left(\frac{5}{4}\right)}{2(3-2M)} \right] \\ & \left. + (M \leftrightarrow 1-M+N) \right\}, \quad (\text{A.38}) \end{aligned}$$

with

$$\chi_0(M, N) = \chi_{01} \left[\psi(1) + \psi(1+N) - \psi(M) - \psi(1-M+N) \right], \quad (\text{A.39})$$

$$\phi_L^+(M) = \int_0^1 dx x^{M-1} \frac{\text{Li}_2(x)}{1+x}. \quad (\text{A.40})$$

The extra term $\chi_1^{\text{corr}}(M, N)$ was corrected in ref. [38]; however, there was another issue, which we have discovered only now. This issue has an effect at NLL, namely the claimed accuracy, but it manifests itself (at NLL) only in terms of $\mathcal{O}(\alpha_s^4)$ and beyond of the anomalous dimension (for this reason, the comparison of the expansion of the anomalous dimension with the exact one at three loops was successful). The origin of the issue and our solution are discussed in detail in section B. The actual expression that we use here is given by

$$\chi_1^{\text{corr}}(M, N, \alpha_s) = \beta_0 \left[-\chi_{01}\psi_1(1-M+N) + \frac{\chi_0(M, N)}{M} + \chi'_0(M, N) - \frac{(1-M+N)^{-1}}{1+N} \left(\chi_{01} - \chi'_s \left(\frac{\alpha_s}{1-M+N} \right) \right) \right]. \quad (\text{A.41})$$

Finally, the momentum conservation coefficient is given by

$$c_{\text{mom}}^{\text{NLO}} = -\chi_{s,\text{NLO}}(2, \alpha_s) - \alpha_s \tilde{\chi}_0(0, 1) - \alpha_s^2 \tilde{\chi}_1(0, 1) - \alpha_s^2 \chi_1^{\text{corr}}(0, 1, \alpha_s). \quad (\text{A.42})$$

We now consider the expansion of the DL-NLO anomalous dimension

$$\gamma_{\text{DL-NLO}}(N, \alpha_s) = \alpha_s \hat{\gamma}_0(N) + \alpha_s^2 \hat{\gamma}_1(N) + \alpha_s^3 \tilde{\gamma}_2(N) + \alpha_s^4 \tilde{\gamma}_3(N) + \mathcal{O}(\alpha_s^5), \quad (\text{A.43})$$

where both $\hat{\gamma}_0(N)$ and $\hat{\gamma}_1(N)$ are now an input, eq. (A.2), and $\tilde{\gamma}_2(N)$ and $\tilde{\gamma}_3(N)$ are the objects that we aim to compute. The expansion of the collinear $\chi_{s,\text{NLO}}$ proceeds as in eq. (A.13), and gives

$$\chi_{s,\text{NLO}}(\gamma_{\text{DL-NLO}}, \alpha_s) = N + \alpha_s^2 \frac{\tilde{\gamma}_2}{\hat{\gamma}'_0} + \alpha_s^3 \left[\frac{\tilde{\gamma}_3}{\hat{\gamma}'_0} - \frac{\tilde{\gamma}_2 \hat{\gamma}'_1}{\hat{\gamma}_0'^2} \right] + \mathcal{O}(\alpha_s^4). \quad (\text{A.44})$$

The anticollinear $\chi_{s,\text{NLO}}$ expands as

$$\begin{aligned} \chi_{s,\text{NLO}}(1 - \gamma_{\text{DL-NLO}} + N, \alpha_s) &= \alpha_s \frac{\chi_{01}}{1+N} + \alpha_s^2 \left[\frac{\chi_{02} + \chi_{01} \hat{\gamma}_0}{(1+N)^2} + \frac{\chi_{11}}{1+N} \right] \\ &+ \alpha_s^3 \left[\frac{\chi_{03} + 2\chi_{02} \hat{\gamma}_0 + \chi_{01} \hat{\gamma}_0^2}{(1+N)^3} + \frac{\chi_{12} + \chi_{11} \hat{\gamma}_0 + \chi_{01} \hat{\gamma}_1}{(1+N)^2} + \frac{\chi_{21}}{1+N} \right] \\ &+ \mathcal{O}(\alpha_s^4). \end{aligned} \quad (\text{A.45})$$

The kernels give instead

$$\begin{aligned} \tilde{\chi}_0(\gamma_{\text{DL-NLO}}, N) &= \tilde{\chi}_0(0, N) + \alpha_s \hat{\gamma}_0 \tilde{\chi}'_0(0, N) + \alpha_s^2 \left[\hat{\gamma}_1 \tilde{\chi}'_0(0, N) + \frac{1}{2} \hat{\gamma}_0^2 \tilde{\chi}''_0(0, N) \right] \\ &+ \mathcal{O}(\alpha_s^3), \end{aligned} \quad (\text{A.46})$$

$$\tilde{\chi}_1(\gamma_{\text{DL-NLO}}, N) = \tilde{\chi}_1(0, N) + \alpha_s \hat{\gamma}_0 \tilde{\chi}'_1(0, N) + \mathcal{O}(\alpha_s^2), \quad (\text{A.47})$$

$$\begin{aligned} \chi_1^{\text{corr}}(\gamma_{\text{DL-NLO}}, N, \alpha_s) &= \chi_1^{\text{corr}}(0, N, 0) + \alpha_s [\hat{\gamma}_0 \partial_M \chi_1^{\text{corr}}(0, N, 0) + \partial_{\alpha_s} \chi_1^{\text{corr}}(0, N, 0)] \\ &+ \mathcal{O}(\alpha_s^2), \end{aligned} \quad (\text{A.48})$$

where implicit derivatives denoted with a \prime are with respect to M , i.e. the first argument.

The new functions appearing in the equations above are given by

$$\frac{1}{2}\tilde{\chi}_0''(0, N) = -\frac{\chi_{01}}{(1+N)^3} + \chi_{01} \left[\zeta_3 - \frac{1}{2}\psi_2(1+N) \right], \quad (\text{A.49a})$$

$$\begin{aligned} \tilde{\chi}_1(0, N) &= \tilde{\chi}_1^u(0, 0) \\ &= \chi_{01}^2 \left[\frac{5}{2}\zeta_3 - \frac{1}{24} \right] + \chi_{01}\beta_0\zeta_2 + \chi_{02} \left[\frac{53}{18} - \zeta_2 \right] - \chi_{11}, \end{aligned} \quad (\text{A.49b})$$

$$\begin{aligned} \tilde{\chi}_1'(0, N) &= \tilde{\chi}_1^{u'}(0, N) \\ &= \left(\frac{3\chi_{01}^2}{4} + \chi_{02} \right) \left(\frac{1}{(3+2N)^2} + \frac{1}{(1-2N)^2} \right) \left[\frac{9\zeta_2}{32} - \frac{3}{128} \left(\Phi_1(N) - 16 + 2\psi_1\left(\frac{1}{4}\right) \right) \right] \\ &\quad + \left(\frac{\chi_{01}^2}{4} + 11\chi_{02} \right) \frac{2}{(1+2N)^2} \left[-\frac{9\zeta_2}{32} - \frac{3}{128} \left(\Phi_1(N) - 2\psi_1\left(\frac{1}{4}\right) \right) \right] \\ &\quad + \left[\left(\frac{9\chi_{01}^2}{2048} + \frac{3\chi_{02}}{512} \right) \left(\frac{1}{3+2N} - \frac{1}{1-2N} \right) + \left(\frac{3\chi_{01}^2}{1024} + \frac{33\chi_{02}}{256} \right) \frac{1}{1+2N} \right] \Phi_2(N) \\ &\quad + \chi_{11} [\psi_2(2+N) - \zeta_2] - 2\beta_2\chi_{01}\zeta_3 \\ &\quad + \chi_{02} \left[\frac{887}{108} + \frac{167}{24}\zeta_2 + \frac{3}{2}\zeta_3 - \frac{2}{(1+N)^3} - \frac{13}{6}\psi_1(1+N) - \frac{47}{48}\psi_1\left(\frac{1}{4}\right) \right] \\ &\quad + \chi_{01}^2 \left[\frac{23}{144} - \frac{\zeta_2}{32} + \frac{37}{40}\zeta_2^2 + \frac{\zeta_2}{8}\Phi_1(N) + \frac{1}{64}\psi_1\left(\frac{1}{4}\right) + \left(\frac{\zeta_2}{2} - \frac{1}{8} \right) \psi_1(1+N) \right. \\ &\quad \quad \left. + \frac{1}{2}\psi_1^2(1+N) + \frac{1}{12}\psi_3(1+N) - \phi_L^{+'}(1+N) \right], \end{aligned} \quad (\text{A.49c})$$

$$\Phi_n(N) = \psi_n\left(\frac{1+N}{2}\right) - \psi_n\left(1 + \frac{N}{2}\right), \quad (\text{A.49d})$$

$$\chi_1^{\text{corr}}(0, N, 0) = \beta_0\chi_{01}[\psi_1(1+N) - 2\zeta_2], \quad (\text{A.49e})$$

$$\partial_M \chi_1^{\text{corr}}(0, N, 0) = \beta_0\chi_{01} \left[3\zeta_3 - \frac{1}{2}\psi_2(1+N) \right], \quad (\text{A.49f})$$

$$\partial_{\alpha_s} \chi_1^{\text{corr}}(0, N, 0) = \beta_0 \frac{2\chi_{02}}{(1+N)^3}. \quad (\text{A.49g})$$

Plugging each expansion in eq. (A.4) we find

$$\begin{aligned} N &= N + \alpha_s \left[\frac{\chi_{01}}{1+N} + \tilde{\chi}_0(0, N) - \left(\frac{\chi_{01}}{2} + \tilde{\chi}_0(0, 1) \right) f_{\text{mom}}(N) \right] \\ &\quad + \alpha_s^2 \left[\frac{\tilde{\gamma}_2}{\tilde{\gamma}_0'} + \frac{\chi_{02} + \chi_{01}\hat{\gamma}_0}{(1+N)^2} + \frac{\chi_{11}}{1+N} + \hat{\gamma}_0\tilde{\chi}_0'(0, N) + \tilde{\chi}_1(0, N) + \chi_1^{\text{corr}}(0, N, 0) \right. \\ &\quad \quad \left. - \left(\frac{\chi_{02}}{4} + \frac{\chi_{11}}{2} + \tilde{\chi}_1(0, 1) + \chi_1^{\text{corr}}(0, 1, 0) \right) f_{\text{mom}}(N) \right] \\ &\quad + \alpha_s^3 \left[\frac{\tilde{\gamma}_3}{\tilde{\gamma}_0'} - \frac{\tilde{\gamma}_2\tilde{\gamma}_1'}{\tilde{\gamma}_0'^2} + \frac{\chi_{03} + 2\chi_{02}\hat{\gamma}_0 + \chi_{01}\hat{\gamma}_0^2}{(1+N)^3} + \frac{\chi_{12} + \chi_{11}\hat{\gamma}_0 + \chi_{01}\hat{\gamma}_1}{(1+N)^2} + \frac{\chi_{21}}{1+N} \right. \\ &\quad \quad + \hat{\gamma}_1\tilde{\chi}_0'(0, N) + \frac{1}{2}\hat{\gamma}_0^2\tilde{\chi}_0''(0, N) + \hat{\gamma}_0\tilde{\chi}_1'(0, N) + \hat{\gamma}_0\partial_M \chi_1^{\text{corr}}(0, N, 0) + \partial_{\alpha_s} \chi_1^{\text{corr}}(0, N, 0) \\ &\quad \quad \left. - \left(\frac{\chi_{03}}{8} + \frac{\chi_{12}}{4} + \frac{\chi_{21}}{2} + \partial_{\alpha_s} \chi_1^{\text{corr}}(0, 1, 0) \right) f_{\text{mom}}(N) \right] \\ &\quad + \mathcal{O}(\alpha_s^4). \end{aligned} \quad (\text{A.50})$$

Using eq. (A.20), we find that both the $\mathcal{O}(\alpha_s^0)$ and $\mathcal{O}(\alpha_s)$ contributions vanish automatically. From the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ terms it immediately follows

$$\tilde{\gamma}_2(N) = -\hat{\gamma}'_0 \left[\frac{\chi_{02}}{(1+N)^2} + \frac{\chi_{11}}{1+N} + \tilde{\chi}_1(0, N) + \chi_1^{\text{corr}}(0, N, 0) + \chi_{01}[\psi_1(1+N) - \zeta_2] \hat{\gamma}_0 \right. \\ \left. - \left(\frac{\chi_{02}}{4} + \frac{\chi_{11}}{2} + \tilde{\chi}_1(0, 1) + \chi_1^{\text{corr}}(0, 1, 0) \right) f_{\text{mom}}(N) \right]. \quad (\text{A.51})$$

$$\tilde{\gamma}_3(N) = \frac{\tilde{\gamma}_2 \hat{\gamma}'_1}{\hat{\gamma}'_0} \\ - \hat{\gamma}'_0 \left[\frac{\chi_{03} + 2\chi_{02}(\hat{\gamma}_0 + \beta_0)}{(1+N)^3} + \frac{\chi_{12} + \chi_{11}\hat{\gamma}_0}{(1+N)^2} + \frac{\chi_{21}}{1+N} + \chi_{01} \left[\zeta_3 - \frac{1}{2}\psi_2(1+N) \right] \hat{\gamma}_0^2 \right. \\ \left. + \chi_{01}[\psi_1(1+N) - \zeta_2] \hat{\gamma}_1 + \hat{\gamma}_0 \tilde{\chi}'_1(0, N) + \beta_0 \chi_{01} \left[3\zeta_3 - \frac{1}{2}\psi_2(1+N) \right] \hat{\gamma}_0 \right. \\ \left. - \left(\frac{\chi_{03} + 2\beta_0 \chi_{02}}{8} + \frac{\chi_{12}}{4} + \frac{\chi_{21}}{2} \right) f_{\text{mom}}(N) \right], \quad (\text{A.52})$$

where we have partially used information from eqs. (A.20) and (A.49). Further using eq. (A.49) we can rewrite eq. (A.51) as⁹

$$\tilde{\gamma}_2(N) = -\hat{\gamma}'_0 \left[\frac{\chi_{02}}{(1+N)^2} + \frac{\chi_{11}}{1+N} + \rho + \chi_{01}[\psi_1(1+N) - \zeta_2](\hat{\gamma}_0 + \beta_0) \right. \\ \left. - \left(\frac{\chi_{02}}{4} + \frac{\chi_{11}}{2} + \rho - \beta_0 \chi_{01} \right) f_{\text{mom}}(N) \right] \quad (\text{A.53})$$

with

$$\rho = \chi_{01}^2 \left[\frac{5}{2}\zeta_3 - \frac{1}{24} \right] + \chi_{02} \left[\frac{53}{18} - \zeta_2 \right] - \chi_{11} \\ = \frac{1}{\pi^2} \left[C_A^2 \left(-\frac{74}{27} + \frac{11}{12}\zeta_2 + \frac{5}{2}\zeta_3 \right) + n_f C_A \left(\frac{4}{27} + \frac{1}{6}\zeta_2 \right) + n_f C_F \left(\frac{7}{27} - \frac{1}{3}\zeta_2 \right) \right]. \quad (\text{A.54})$$

The N³LO function $\tilde{\gamma}_3(N)$ cannot be simplified significantly, so we do not manipulate it further.

We now move to the running-coupling contributions. The RC correction to the duality is implemented through the function

$$\Delta\gamma_{ss}^{\text{rc}}(N, \alpha_s) = -\beta_0 \alpha_s \left[\frac{\chi_0''(M)\chi_0(M)}{2\chi_0'^2(M)} - 1 \right]_{M=\gamma_s(\alpha_s/N)} \\ = -\alpha_s^4 \beta_0 \frac{\chi_{01}^3}{N^3} 12\zeta_3 + \mathcal{O}(\alpha_s^5), \quad (\text{A.55})$$

⁹Note that this expression differs from the analogous in ref. [38] by NNLL terms, due to the correction to the function χ_1^{corr} .

which contributes at NLL. Further RC corrections from RC resummation, eq. (A.23), are NNLL corrections. They amount to

$$\begin{aligned}
\Delta_{\text{DL-NLO}}\gamma_{\text{rc}}(N, \alpha_s) &\equiv \gamma_{\text{rc}}(N, \alpha_s) - \left[M_{\text{min}} - \sqrt{\frac{N-c}{\kappa/2 + (N-c)/M_{\text{min}}^2}} - \beta_0\alpha_s \right. \\
&\quad \left. + \frac{1}{4}\beta_0\alpha_s^2 \left(3\frac{\kappa' - 2c'/M_{\text{min}}^2}{\kappa + 2(N-c)/M_{\text{min}}^2} - \frac{c'}{N-c} \right) \right] \\
&= \beta_0^2\alpha_s^3 \frac{\kappa_0}{16N} \\
&\quad + \beta_0^2\alpha_s^4 \left[\frac{\kappa_0(192c_0 - 7\kappa_0)}{1024N^2} + \frac{\kappa_0(2m_1^{\text{NLO}} - 3\beta_0) + T'\kappa_1^{\text{NLO}}}{16N} \right] \\
&\quad + \mathcal{O}(\alpha_s^5), \tag{A.56}
\end{aligned}$$

where as before T' is either 2 or 1 depending on the approximate α_s dependence chosen. To cancel the singularity mismatch between eq. (A.56) and eq. (A.55) we further need the matching function

$$\begin{aligned}
\gamma_{\text{match}}^{\text{ss}}(N, \alpha_s) &= \frac{1}{4}\beta_0\alpha_s^2 \left[\frac{c_0}{N - \alpha_s c_0} - \frac{c'}{N-c} + \frac{c' - c_0}{N} \right] \\
&= -\alpha_s^4\beta_0 c_0 \frac{(1 + T')c_1^{\text{NLO}}}{4N^2} + \mathcal{O}(\alpha_s^5). \tag{A.57}
\end{aligned}$$

The coefficients above are the ones obtained from the NLO kernel, given in section A.3.

Putting everything together according to eq. (A.1), we obtain

$$\begin{aligned}
\gamma^{\text{resNLL}}(N, \alpha_s) &= \alpha_s\hat{\gamma}_0(N) + \alpha_s^2\hat{\gamma}_1(N) \\
&\quad + \alpha_s^3 \left\{ \beta_0^2 \frac{\kappa_0}{16} \left(\frac{1}{N} - f_{\text{mom}}(N) \right) \right. \\
&\quad \quad - \hat{\gamma}'_0 \left[\frac{\chi_{02}}{(1+N)^2} + \frac{\chi_{11}}{1+N} + \rho + \chi_{01}[\psi_1(1+N) - \zeta_2](\hat{\gamma}_0 + \beta_0) \right. \\
&\quad \quad \quad \left. \left. - \left(\frac{\chi_{02}}{4} + \frac{\chi_{11}}{2} + \rho - \beta_0\chi_{01} \right) f_{\text{mom}}(N) \right] \right\} \\
&\quad + \alpha_s^4 \left\{ \frac{\rho_3}{N^3} + \frac{\rho_2}{N^2} + \frac{\rho_1}{N} - (\rho_3 + \rho_2 + \rho_1)f_{\text{mom}}(N) \right. \\
&\quad \quad - \hat{\gamma}'_1 \left[\frac{\chi_{02}}{(1+N)^2} + \frac{\chi_{11}}{1+N} + \rho + \chi_{01}[\psi_1(1+N) - \zeta_2](\hat{\gamma}_0 + \beta_0) \right. \\
&\quad \quad \quad \left. - \left(\frac{\chi_{02}}{4} + \frac{\chi_{11}}{2} + \rho - \beta_0\chi_{01} \right) f_{\text{mom}}(N) \right] \\
&\quad \quad - \hat{\gamma}'_0 \left[\frac{\chi_{03} + 2\chi_{02}(\hat{\gamma}_0 + \beta_0)}{(1+N)^3} + \frac{\chi_{12} + \chi_{11}\hat{\gamma}_0}{(1+N)^2} + \frac{\chi_{21}}{1+N} \right. \\
&\quad \quad \quad \left. - \left(\frac{\chi_{03} + 2\beta_0\chi_{02}}{8} + \frac{\chi_{12}}{4} + \frac{\chi_{21}}{2} \right) f_{\text{mom}}(N) \right. \\
&\quad \quad \quad + \chi_{01} \left[\zeta_3 - \frac{1}{2}\psi_2(1+N) \right] \hat{\gamma}_0(\hat{\gamma}_0 + \beta_0) + \chi_{01}2\zeta_3\beta_0\hat{\gamma}_0 \\
&\quad \quad \quad \left. \left. + \chi_{01}[\psi_1(1+N) - \zeta_2]\hat{\gamma}_1 + \hat{\gamma}_0\tilde{\chi}'_1(0, N) \right] \right\} \\
&\quad + \mathcal{O}(\alpha_s^5), \tag{A.58}
\end{aligned}$$

where

$$\rho_3 = -\beta_0 \chi_{01}^3 12\zeta_3, \tag{A.59a}$$

$$\rho_2 = \beta_0^2 \kappa_0 \left[\frac{3}{16} c_0 - \frac{7}{1024} \kappa_0 \right] - \beta_0 c_0 c_1^{\text{NLO}} \frac{1+T'}{4}, \tag{A.59b}$$

$$\rho_1 = \beta_0^2 \frac{\kappa_0 (2m_1^{\text{NLO}} - 3\beta_0) + T' \kappa_1^{\text{NLO}}}{16}, \tag{A.59c}$$

and with $\tilde{\chi}'_1(0, N)$ given in eq. (A.49). The $\mathcal{O}(\alpha_s^4)$ contribution in eq. (A.58) is a new result.

Eq. (A.58) is rather complex and not optimal to be used in a numerical code. In particular, we also need to compute the inverse Mellin transform of this object in order to provide the expansion of the resummed splitting functions, and it is clearly very complicated (if not impossible) to obtain analytic expressions for such inverse. Thus, we consider a pole expansion of the $\mathcal{O}(\alpha_s^4)$ term of eq. (A.58), γ_3^{NLL} . Specifically, we compute the residues of all the poles in $N = 0, -1, -2$, and construct an approximation based only on these terms,

$$\gamma_3^{\text{NLL}}(N) \simeq \sum_{j=0}^3 \frac{g_{0j}}{N^{j+1}} + \sum_{j=0}^6 \frac{g_{1j}}{(N+1)^{j+1}} + \sum_{j=0}^2 \frac{g_{2j}}{(N+2)^{j+1}}. \tag{A.60}$$

This approximation has the advantage of describing in x space all contributions behaving as $x^k \log^j x$ for all non-vanishing terms with $j \geq 0$ and for $k = -1, 0, 1$, namely all non-vanishing terms at small x plus the leading corrections to them. Of course this approximation will not be accurate at large x , but since the final results will be damped at large x the inaccuracy should be negligible. To verify the quality of our approximation, we have compared it with a simpler approximation which does not include the contributions from the poles in $N = -2$, i.e. those terms behaving as $x \log^j x$ in x space. The difference between the two results is almost imperceptible.

Before concluding, it is useful to extract from eq. (A.58) its small- x behaviour up to NLL, which provides a prediction for the yet unknown four-loop anomalous dimensions. Expanding the anomalous dimensions eq. (A.2) in $N = 0$,

$$\hat{\gamma}_0(N) = \frac{a_{11}}{N} + a_{10} + \mathcal{O}(N), \quad \hat{\gamma}'_0(N) = -\frac{a_{11}}{N^2} + \mathcal{O}(N^0), \tag{A.61}$$

$$\hat{\gamma}_1(N) = \frac{a_{21}}{N} + \mathcal{O}(N^0), \quad \hat{\gamma}'_1(N) = -\frac{a_{21}}{N^2} + \mathcal{O}(N^0), \tag{A.62}$$

and knowing that $\tilde{\chi}'_1(0, N)$ is finite in $N = 0$, we can easily compute the $N = 0$ expansion of eq. (A.58). For this result, we also need the function $\phi_L^+(M)$ and its derivative evaluated in $M = 0, 1$,¹⁰

$$\phi_L^+(0) = -\zeta_2 \log 2 + \frac{13}{8} \zeta_3, \quad \phi_L^+(1) = \zeta_2 \log 2 - \frac{5}{8} \zeta_3, \tag{A.63}$$

$$\phi_L^{+'}(0) = -\frac{13}{16} \zeta_4, \quad \phi_L^{+'}(1) = -\frac{3}{16} \zeta_4. \tag{A.64}$$

¹⁰We could not compute the derivatives analytically, so we have used the PSLQ algorithm to find the results of the integrals.

Denoting with $\gamma_+^{(n)}$ the exact anomalous dimensions at $\mathcal{O}(\alpha_s^{n+1})$, we find

$$\gamma_+^{(2)}(N) = \frac{C_A}{\pi^3 N^2} \frac{C_A^2(54\zeta_3 + 99\zeta_2 - 395) + n_f(C_A - 2C_F)(18\zeta_2 - 71)}{108} + \mathcal{O}\left(\frac{1}{N}\right), \quad (\text{A.65})$$

$$\begin{aligned} \gamma_+^{(3)}(N) &= \frac{C_A^4}{\pi^4 N^4} 2\zeta_3 \\ &+ \frac{C_A^2}{\pi^4 N^3} \left[C_A^2 \left(-\frac{1205}{162} + \frac{67}{36}\zeta_2 + \frac{1}{4}\zeta_2^2 - \frac{77}{6}\zeta_3 \right) + n_f C_A \left(-\frac{233}{162} + \frac{13}{36}\zeta_2 + \zeta_3 \right) \right. \\ &\quad \left. + n_f C_F \left(\frac{233}{81} - \frac{13}{18}\zeta_2 + \frac{4}{3}\zeta_3 \right) \right] \\ &+ \mathcal{O}\left(\frac{1}{N^2}\right) \end{aligned} \quad (\text{A.66})$$

To our knowledge, the NLL term at N³LO was never explicitly presented in the literature. Note that the scheme is $Q_0\overline{\text{MS}}$. The conversion to $\overline{\text{MS}}$ was presented in section 2.4.

A.3 Computation of the coefficients describing the minimum of the BFKL kernel

In this section we compute the expansion coefficients of $c(\alpha_s)$ and $\kappa(\alpha_s)$ of the BFKL kernel in symmetric variables. We will assume that the minimum is not in $M = 1/2$, but in

$$\tilde{M}_{\min}(\alpha_s) = \frac{1}{2} + \alpha_s \tilde{m}_1 + \alpha_s^2 \tilde{m}_2 + \dots \quad (\text{A.67})$$

The expansion coefficients of the minimum can be found by solving perturbatively the minimum condition

$$\partial_M \chi(M, \alpha_s)|_{M=\tilde{M}_{\min}(\alpha_s)} = 0, \quad (\text{A.68})$$

where this $\chi(M, \alpha_s)$ is the on-shell kernel in symmetric variables, obtained by the on-shell condition

$$\chi(M, \alpha_s) = \chi^\sigma(M, \chi(M, \alpha_s), \alpha_s), \quad (\text{A.69})$$

where $\chi^\sigma(M, N, \alpha_s)$ is the off-shell kernel in symmetric variables, related to the DL kernel via [29, 38]

$$\chi^\sigma(M, N, \alpha_s) = \chi_\Sigma(M + N/2, N, \alpha_s). \quad (\text{A.70})$$

Using eq. (A.69) in eq. (A.68) we get (here M -partial derivatives act on the first argument, which is then set to the written value)

$$\partial_M \chi^\sigma\left(\tilde{M}_{\min}(\alpha_s), c(\alpha_s), \alpha_s\right) = 0, \quad (\text{A.71})$$

having used the definition $c(\alpha_s) = \chi(\tilde{M}_{\min}(\alpha_s), \alpha_s)$ which, in terms of the off-shell kernel, leads to the implicit equation

$$c(\alpha_s) = \chi^\sigma(\tilde{M}_{\min}(\alpha_s), c(\alpha_s), \alpha_s). \quad (\text{A.72})$$

Equations (A.71) and (A.72) can be iteratively solved order by order in perturbation theory. Introducing the expansions

$$c(\alpha_s) = \alpha_s c_0 + \alpha_s^2 c_1 + \dots \quad (\text{A.73})$$

$$\chi^\sigma(M, N, \alpha_s) = \alpha_s \chi_0^\sigma(M, N) + \alpha_s^2 \chi_1^\sigma(M, N) + \dots \quad (\text{A.74})$$

we have

$$0 = \alpha_s \partial_M \chi_0^\sigma \left(\frac{1}{2} + \alpha_s \tilde{m}_1 + \dots, \alpha_s c_0 + \dots \right) + \alpha_s^2 \partial_M \chi_1^\sigma \left(\frac{1}{2} + \alpha_s \tilde{m}_1 + \dots, \alpha_s c_0 + \dots \right) + \dots \quad (\text{A.75})$$

$$\alpha_s c_0 + \alpha_s^2 c_1 + \dots = \alpha_s \chi_0^\sigma \left(\frac{1}{2} + \alpha_s \tilde{m}_1 + \dots, \alpha_s c_0 + \dots \right) + \alpha_s^2 \chi_1^\sigma \left(\frac{1}{2} + \alpha_s \tilde{m}_1 + \dots, \alpha_s c_0 + \dots \right) + \dots \quad (\text{A.76})$$

and further expanding we get

$$0 = \alpha_s \partial_M \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + \alpha_s^2 \left[\tilde{m}_1 \partial_M^2 \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + c_0 \partial_N \partial_M \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + \partial_M \chi_1^\sigma \left(\frac{1}{2}, 0 \right) \right] + \dots \quad (\text{A.77})$$

$$\alpha_s c_0 + \alpha_s^2 c_1 + \dots = \alpha_s \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + \alpha_s^2 \left[\tilde{m}_1 \partial_M \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + c_0 \partial_N \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + \chi_1^\sigma \left(\frac{1}{2}, 0 \right) \right] + \dots \quad (\text{A.78})$$

From these equations we find that $\partial_M \chi_0^\sigma \left(\frac{1}{2}, 0 \right) = 0$, which was our assumption in eq. (A.67). Note that, more in general, we have $\partial_M \chi_0^\sigma \left(\frac{1}{2}, N \right) = 0$ for any N , which implies that $\partial_N \partial_M \chi_0^\sigma \left(\frac{1}{2}, 0 \right) = 0$. Before listing the results, let us also consider the curvature κ , with expansion

$$\kappa(\alpha_s) = \alpha_s \kappa_0 + \alpha_s^2 \kappa_1 + \dots \quad (\text{A.79})$$

which is given by $\kappa(\alpha_s) = \partial_M^2 \chi(\tilde{M}_{\min}, \alpha_s)$ and hence, in terms of off-shell kernel [57],

$$\kappa(\alpha_s) = \frac{\partial_M^2 \chi^\sigma}{1 - \partial_N \chi^\sigma} \Big|_{M=\tilde{M}_{\min}(\alpha_s), N=c(\alpha_s)}. \quad (\text{A.80})$$

Expanding this equation we find

$$\begin{aligned} \kappa(\alpha_s) &= \left[\alpha_s \partial_M^2 \chi_0^\sigma \left(\frac{1}{2} + \alpha_s \tilde{m}_1 + \dots, \alpha_s c_0 + \dots \right) + \alpha_s^2 \partial_M^2 \chi_1^\sigma \left(\frac{1}{2} + \dots, 0 + \dots \right) + \dots \right] \\ &\quad \times \left[1 + \alpha_s \partial_N \chi_0^\sigma \left(\frac{1}{2} + \dots, 0 + \dots \right) + \dots \right] \\ &= \alpha_s \partial_M^2 \chi_0^\sigma \left(\frac{1}{2}, 0 \right) \\ &\quad + \alpha_s^2 \left[\partial_M^2 \chi_1^\sigma \left(\frac{1}{2}, 0 \right) + \partial_M^2 \chi_0^\sigma \left(\frac{1}{2}, 0 \right) \partial_N \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + c_0 \partial_N \partial_M^2 \chi_0^\sigma \left(\frac{1}{2}, 0 \right) + \tilde{m}_1 \partial_M^3 \chi_0^\sigma \left(\frac{1}{2}, 0 \right) \right] \\ &\quad + \dots \end{aligned} \quad (\text{A.81})$$

Note that the last term vanishes due to the fact that the LO kernel is symmetric, and hence all its odd M -derivatives are zero. Putting everything together, we then obtain

$$c_0 = \chi_0^\sigma, \tag{A.82}$$

$$\kappa_0 = \partial_M^2 \chi_0^\sigma, \tag{A.83}$$

$$\tilde{m}_1 = -\frac{\partial_M \chi_1^\sigma}{\kappa_0}, \tag{A.84}$$

$$c_1 = \chi_1^\sigma + c_0 \partial_N \chi_0^\sigma, \tag{A.85}$$

$$\kappa_1 = \partial_M^2 \chi_1^\sigma + \kappa_0 \partial_N \chi_0^\sigma + c_0 \partial_N \partial_M^2 \chi_0^\sigma, \tag{A.86}$$

where every off-shell kernel is implicitly assumed to be computed in $M = 1/2$, $N = 0$. Explicitly, we have at lowest order

$$c_0 = \chi_{01} 4 \log 2, \tag{A.87a}$$

$$\kappa_0 = \chi_{01} 28 \zeta_3, \tag{A.87b}$$

while for the higher order coefficients the result depends on the actual kernel considered. For DL-LO kernel we have

$$\tilde{m}_1^{\text{LO}} = 0, \tag{A.88a}$$

$$\begin{aligned} c_1^{\text{LO}} &= 8\chi_{02} - 8\chi_{01}^2 \zeta_2 \log 2 \\ &= -15.00496429 - 0.04503163717 n_f, \end{aligned} \tag{A.88b}$$

$$\begin{aligned} \kappa_1^{\text{LO}} &= 192\chi_{02} - \chi_{01}^2 (144\zeta_2^2 \log 2 + 56\zeta_2 \zeta_3) \\ &= -507.744719 - 1.080759292 n_f, \end{aligned} \tag{A.88c}$$

while for DL-NLO kernel we have

$$\tilde{m}_1^{\text{NLO}} = \beta_0 \left(\frac{4 \log 2}{7\zeta_3} - \frac{1}{2} \right), \tag{A.89a}$$

$$\begin{aligned} c_1^{\text{NLO}} &= \chi_{02} \left[\frac{33\pi^3}{64} - \frac{26}{3} \log 2 \right] + 8\beta_0 \chi_{01} [\log 2 - \log^2 2] + \chi_{11} 4 \log 2 \\ &\quad + \chi_{01}^2 \left[\frac{73\pi^3}{786} - \frac{\log 2}{2} - 2\zeta_2 \log 2 - \frac{11}{2} \zeta_3 + 2\phi_L^+ \left(\frac{1}{2} \right) \right] \\ &= -11.696833425 - 0.4102968810 n_f, \end{aligned} \tag{A.89b}$$

$$\begin{aligned} \kappa_1^{\text{NLO}} &= \chi_{02} \left[-\frac{3\pi^3}{32} + \frac{55\pi^5}{64} - \frac{182}{3} \zeta_3 \right] + \chi_{11} 28 \zeta_3 + \beta_0 \chi_{01} [64 \log 2 + 56 \zeta_3 - 112 \log 2 \zeta_3] \\ &\quad + \chi_{01}^2 \left[-\frac{9\pi^3}{128} + \frac{79\pi^5}{786} - \frac{7}{2} \zeta_3 - 14 \zeta_2 \zeta_3 - 372 \zeta_5 + 2\phi_L^{+\prime\prime} \left(\frac{1}{2} \right) \right] \\ &= -494.250393369 - 5.23585215538 n_f. \end{aligned} \tag{A.89c}$$

Unfortunately, we could not be able to express the function $\phi_L^+(M)$, eq. (A.40), and its second derivative computed in $M = 1/2$ in terms of elementary constants. Finally, the differences of the coefficients needed for the LL' result have the following numerical values:

$$\delta m_1 = -0.1492429211 + 0.00904502552 n_f, \tag{A.90a}$$

$$\delta c_1 = 3.308130862 - 0.3652652438 n_f, \tag{A.90b}$$

$$\delta \kappa_1 = 13.49432608 - 4.155092863 n_f. \tag{A.90c}$$

Note that ultimately one wants to construct the anomalous dimensions which are dual to the BFKL kernel in asymmetric (or DIS) variables. In ref. [38] we argued that $\Delta_{\text{DL-(N)LO}}\gamma_{\text{rc}}^{(\text{N})\text{LL}}$, which contains the RC resummation corrections to be added to the DL result, could be computed directly from the kernel in symmetric variables, as the conversion from symmetric to asymmetric amounts to adding $+N/2$ to the anomalous dimension, which is then subtracted again by the matching. However, the argument was superficial, and induced by the collinear approximation of the kernel used for computing the resummation of RC effects. Indeed, the actual DL BFKL kernel in symmetric variables behaves asymptotically as $-2M$ in the collinear region, which by duality (either fixed-coupling or running-coupling) reproduces the $-N/2$ behaviour which is then removed by the conversion from symmetric to asymmetric variables. Therefore, using a collinear approximation to the kernel is more appropriate if the kernel is the one in asymmetric variables. Thus, the RC parameters (c , κ and M_{min}) should be the ones of such kernel. While the curvature and the value of the kernel at the minimum are not sensitive to the conversion, the position of the minimum is, and thus in the 3.0 version of the HELL code we use the results of ref. [38] with

$$M_{\text{min}}(\alpha_s) = \tilde{M}_{\text{min}}(\alpha_s) + c(\alpha_s)/2. \tag{A.91}$$

Note that in the expansion of this appendix this implies that $m_1 = \tilde{m}_1 + c_0/2$, while δm_1 , eq. (A.90a), remains unchanged.

B Running coupling corrections to the DGLAP-BFKL duality

The BFKL kernel (in symmetric variables) is not fully symmetric for the exchange $M \leftrightarrow 1 - M$ because of a number of effects, all induced by the running of the strong coupling. The origin of these effects can be traced back to (see e.g. [29]):

- the scale at which α_s is computed in different pieces of the fixed-order kernel;
- the conversion from the unintegrated (k_t -dependent) PDFs, to which the original BFKL equation refers, to the integrated (collinear) PDFs, to which the DGLAP equation refers;
- the so-called running coupling corrections to the duality, namely the correct derivation of the duality between DGLAP and BFKL taking into account the scale dependence of the strong coupling.

These effects have been extensively discussed in the literature and they will not be re-derived; here we limit ourselves to present them. All these effects can be implemented as corrections to the BFKL kernel, which then gives by naive (i.e., fixed-coupling) duality the correct NLL contributions to the resummed anomalous dimension. These corrections amount to the following expression to be added to the symmetric kernel,

$$\Delta\chi(M, \alpha_s) = \alpha_s^2 \beta_0 \left[-\psi_1(1 - M) + \frac{\chi_0(M)}{M} + \frac{\chi_0(M)\chi_0''(M)}{2\chi_0'(M)} \right] + \mathcal{O}(\alpha_s^3), \tag{B.1}$$

where the three contributions in square brackets correspond to the three items above, respectively. The last term contains a $\chi'_0(M)$ in the denominator, which is singular in $M = 1/2$, and thus produces a new singularity that makes the result perturbatively unstable. It has been realised long ago [28, 29] that these singular contributions can be removed (resummed) if the running coupling corrections to duality are accounted for to all orders rather than included perturbatively. This is the goal of the RC resummation, which provides an anomalous dimension where these contributions are resummed by solving the BFKL equation with running coupling (in a given approximation). Thus, it is convenient to translate the last correction to the kernel in a correction to the anomalous dimension, which is then regulated when the RC resummed result is added. This is always possible, and a simple computation reveals that one can pour the last term of eq. (B.1) into

$$\Delta\gamma(N, \alpha_s) = -\alpha_s\beta_0 \left[\frac{\chi_0(M)\chi_0''(M)}{2\chi_0'^2(M)} \right]_{M=\gamma_s(\alpha_s/N)}, \quad (\text{B.2})$$

up to subleading logarithms. This expression, however, is not yet optimal for numerical implementation. Indeed, expanding this contribution we get

$$\Delta\gamma(N, \alpha_s) = -\alpha_s\beta_0 + \mathcal{O}(\alpha_s^4). \quad (\text{B.3})$$

This implies that the naive dual of the kernel (the symmetric kernel plus $\Delta\chi$, eq. (B.1), without the last term) does not coincide with the high-energy limit of the exact anomalous dimension at $\mathcal{O}(\alpha_s)$. This is problematic when one wants to resum the collinear singularities of the kernel using χ_s , the dual of the LO anomalous dimension, as the singularities would not cancel. One could in principle cure this problem by computing χ_s from a modified fixed-order anomalous dimension to which the $-\alpha_s\beta_0$ term, eq. (B.3), has been subtracted, namely $\hat{\gamma}_0 \rightarrow \hat{\gamma}_0 + \alpha_s\beta_0$. However, this becomes too artificial and not very physical. A better way to solve the issue is to pour back the lowest order term of the expansion of $\Delta\gamma$ into $\Delta\chi$. We would thus have (using the same names for the modified objects)

$$\Delta\gamma(N, \alpha_s) = -\alpha_s\beta_0 \left[\frac{\chi_0(M)\chi_0''(M)}{2\chi_0'^2(M)} - 1 \right]_{M=\gamma_s(\alpha_s/N)}, \quad (\text{B.4})$$

$$\Delta\chi(M, \alpha_s) = \alpha_s^2\beta_0 \left[-\psi_1(1-M) + \frac{\chi_0(M)}{M} + \chi_0'(M) \right] + \mathcal{O}(\alpha_s^3). \quad (\text{B.5})$$

In this way, the singularities of the kernel can be resummed with the standard χ_s , and the dual anomalous dimension receives a correction of order $\mathcal{O}(\alpha_s^4)$, which is then accounted for to all orders by RC resummation. This $\Delta\gamma$, eq. (B.4), is indeed the function $\Delta\gamma_{ss}^{\text{rc}}$, eq. (A.55), used in our construction. The $\Delta\chi$ contribution, instead, generates the χ_1^{corr} term appearing in the off-shell kernel, eq. (A.32). Indeed, the naive off-shell extension of $\Delta\chi$ gives

$$\Delta\chi(M, \alpha_s) \rightarrow \alpha_s^2\beta_0 \left[-\psi_1(1-M+N) + \frac{\chi_0(M, N)}{M} + \chi_0'(M, N) \right] + \mathcal{O}(\alpha_s^3). \quad (\text{B.6})$$

The actual χ_1^{corr} that we use, eq. (A.41), further adds a higher order contribution to resum the singularity in $M = 1 + N$, thus stabilizing the resulting kernel without affecting the logarithmic accuracy of the resummed anomalous dimension.

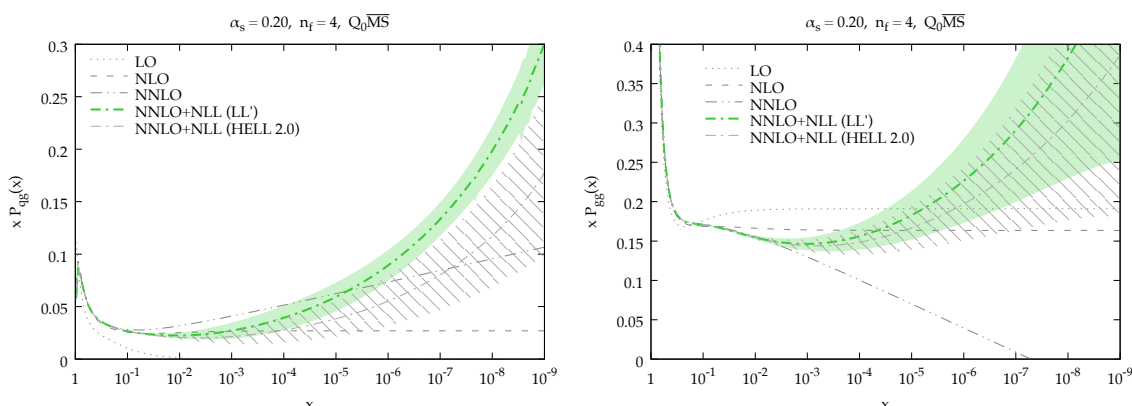


Figure 5. Comparison between resummed and matched splitting functions in HELL 3.0 with LL' anomalous dimension and HELL 2.0, which contained a bug.

In our previous implementation of the resummation, ref. [38], we used a different (wrong) form of χ_1^{corr} , taken from ref. [29],

$$\chi_1^{\text{corr, HELL 2.0}}(M, N, \alpha_s) = \alpha_s^2 \beta_0 \left[-\psi_1(1 - M + N) + \frac{\chi_0(M, N)}{M} - \frac{\chi_{01}}{M^2} \right] + \mathcal{O}(\alpha_s^3), \quad (\text{B.7})$$

which differs from the correct eq. (B.6) by the last term. The singularity of the last term is the correct one, and thus leads to a proper resummation through χ_s , which is the argument used by ref. [29] to introduce such a “subtraction” term. However, the missing M -dependent contributions at $\mathcal{O}(\alpha_s^2)$ produce spurious NLL contributions in the anomalous dimension. This NLL effect starts to appear at $\mathcal{O}(\alpha_s^4)$, which explains why the comparison of the three-loop anomalous dimension obtained from the resummation to the exact result was successful. For this reason, the full resummed result after the correction is not very different from the previous bugged one.

To conclude, we show in figure 5 the comparison between the P_{qg} and P_{gg} splitting functions obtained using the LL' anomalous dimension before (dot-dashed gray) and after (dot-dashed green) the correction of the bug, and the new definition of M_{min} , eq. (A.91). Both splitting functions appear to be harder after bug corrections, which is due to both γ_+^{NLL} and $\gamma_+^{\text{LL'}}$ being indeed harder. We stress however that much of this effect is induced by the change in M_{min} , rather than to the correction in χ_1^{corr} . The uncertainty band on P_{qg} is significantly reduced, again mostly due to the different M_{min} used. Overall, however, the effect is not dramatic, and likely comparable to (if not smaller than) unknown subleading corrections, which in P_{qg} are likely underestimated by our uncertainty band, as we already commented in section 3.

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