HÖLDER REGULARITY FOR MAXWELL'S EQUATIONS UNDER MINIMAL ASSUMPTIONS ON THE COEFFICIENTS

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Abstract. We prove global Hölder regularity for the solutions to the time-harmonic anisotropic Maxwell's equations, under the assumptions of Hölder continuous coefficients. The regularity hypotheses on the coefficients are minimal. The same estimates hold also in the case of bianisotropic material parameters.

1. Introduction

This paper focuses on the Hölder regularity of the solutions $E, H \in H(\mathrm{curl}, \Omega) := \{F \in L^2(\Omega; \mathbb{C}^3) : \mathrm{curl} F \in L^2(\Omega; \mathbb{C}^3)\}$ to the time-harmonic Maxwell's equations [17]

$$\begin{cases} \mathrm{curl} H = \mathrm{i}\omega\varepsilon E + J_e & \text{in } \Omega, \\ \mathrm{curl} E = -\mathrm{i}\omega\mu H + J_m & \text{in } \Omega, \\ E \times \nu = G \times \nu & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain of class $C^{1,1}$ and the coefficients ε and μ belong to $L^\infty\left(\Omega; \mathbb{C}^{3\times 3}\right)$ and are such that for every $\eta \in \mathbb{C}^3$

$$\Lambda^{-1} |\eta|^2 \le \overline{\eta} \cdot \left(\varepsilon + \overline{\varepsilon}^T\right) \eta,\ \Lambda^{-1} |\eta|^2 \le \overline{\eta} \cdot \left(\mu + \overline{\mu}^T\right) \eta \text{ and } |\mu| + |\varepsilon| \le \Lambda \text{ a.e. in } \Omega \tag{2}$$

for some $\Lambda > 0$. The 3×3 matrix ε represents the electric permittivity and μ the magnetic permeability. The current sources J_e and J_m are in $L^2\left(\Omega; \mathbb{C}^3\right)$, the boundary value G belongs to $H(\mathrm{curl}, \Omega)$ and the frequency ω is in $\mathbb{C} \setminus \{0\}$. We are interested in finding (minimal) conditions on the parameters and on the sources such that the electric field E and/or the magnetic field H are Hölder continuous. The study of the minimal regularity of $\partial\Omega$ needed goes beyond the scopes of this work; domains with rougher boundaries are considered in [4, 13, 7, 8, 9, 10].

Let us mention the main known results concerning this problem. The Hölder continuity of the solutions under the assumption of Lipschitz coefficients was proven in [23]. The needed regularity of the coefficients was reduced from $W^{1,\infty}$ to $W^{1,3+\delta}$ for some $\delta > 0$ in [3]. The case of bianisotropic materials was treated in [14, 3], with similar hypotheses and results. For related recent papers, see [24, 21, 19, 16]. The arguments of all these works are based on the H^1 regularity of the electromagnetic fields, which was first obtained in [22] for Lipschitz coefficients, and then in [3] for $W^{1,3+\delta}$ coefficients. Thus, the coefficients were always required to belong to some Sobolev space.

The purpose of this work is to show that it is sufficient to assume that the coefficients are Hölder continuous. Due to the terms εE and μH in (1), this is

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the most natural hypothesis on ε and/or μ, and turns out to be minimal (see Remark 3 below). Our approach is very different from that of [3], and is based on the Helmholtz decomposition of the electromagnetic fields, as in [22, 23] and several related works. However, the argument used is new, and allows to avoid any additional differentiability of E and H. As far as the differentiability of the fields is concerned, it is worth mentioning that ideas similar to those used in this work may be applied to prove the H^1 regularity of the fields with $W^{1,3}$ coefficients [2].

Before stating the main results of this work, we need to define the weak solutions of the Maxwell system. We say that $(E,H) \in H(\mathrm{curl},\Omega)^2$ is a weak solution of (1) if

$$\int_\Omega H \cdot \mathrm{curl}\Phi_1 \, dx = \int_\Omega (\mathrm{i}\omega\varepsilon E + J_e) \cdot \Phi_1 \, dx, \tag{3a}$$

$$\int_\Omega E \cdot \mathrm{curl}\Phi_2 \, dx = \int_\Omega G \cdot \mathrm{curl}\Phi_2 - (\mathrm{curl}G + \mathrm{i}\omega\mu H - J_m) \cdot \Phi_2 \, dx, \tag{3b}$$

for every $(\Phi_1,\Phi_2) \in H_0(\mathrm{curl},\Omega)\times H(\mathrm{curl},\Omega)$, where $H_0(\mathrm{curl},\Omega) = \{F \in H(\mathrm{curl},\Omega) : F \times \nu = 0 \text{ on } \partial\Omega\}$. These identities are formally equivalent to (1) thanks to an integration by parts [17, Theorem 3.29]. Note that for $F \in H(\mathrm{curl},\Omega)$, the tangential trace $F\times\nu$ on $\partial\Omega$ belongs to $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{C}^3)$ and has to be interpreted in the weak sense.

The main result of this paper regarding the joint regularity of E and H, under the assumptions that both ε and μ are Hölder continuous, reads as follows.

Theorem 1. *Assume that* (2) *holds true and that*

$$\varepsilon \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}), \qquad \|\varepsilon\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3})} \le \Lambda, \tag{4}$$

$$\mu \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}), \qquad \|\mu\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3})} \le \Lambda, \tag{5}$$

for some $\alpha \in (0,\frac{1}{2}]$. *Take* $J_e, J_m \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ *and* $G \in C^{1,\alpha}(\mathrm{curl},\Omega)$, *where*

$$C^{N+1,\alpha}(\mathrm{curl},\Omega) = \{F \in C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3) : \mathrm{curl}F \in C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3)\}, \qquad N \in \mathbb{N},$$

equipped with the canonical norms. Let $(E,H) \in H(\mathrm{curl},\Omega)^2$ *be a weak solution of* (1). *Then* $E, H \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ *and*

$$\|(E,H)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2} \le C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e,J_m)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big)$$

for some constant C *depending only on* Ω, Λ *and* ω.

The higher regularity version is given below in Theorem 7. This result can be easily extended to treat the case of bianisotropic materials, see Theorem 8 below.

If only one of the parameters is $C^{0,\alpha}$, for instance ε, the corresponding field E will be Hölder continuous, provided that μ is real. (The Campanato spaces $L^{2,\lambda}$ are defined in Section 4.)

Theorem 2. *Assume that* (2) *and* (4) *hold true and that* $\Im\mu \equiv 0$. *Take* $J_e, G \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ *with* $\mathrm{curl}G \in L^{2,\lambda}(\Omega;\mathbb{C}^3)$ *for some* $\lambda > 1$ *and* $J_m \in L^{2,\lambda}(\Omega;\mathbb{C}^3)$. *Let* $(E,H) \in H(\mathrm{curl},\Omega)^2$ *be a weak solution of* (1). *Then* $E \in C^{0,\beta}(\overline{\Omega};\mathbb{C}^3)$, *where*

$\beta = \min(\frac{\tilde\lambda-1}{2}, \frac{\lambda-1}{2}, \alpha)$ *for some* $\tilde\lambda \in (1,2)$ *depending only on* Ω *and* Λ*, and*

$$\|E\|_{C^{0,\beta}(\overline{\Omega};\mathbb{C}^3)} \le C\big(\|E\|_{L^2(\Omega;\mathbb{C}^3)} + \|(G,J_e)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2} + \|(\mathrm{curl} G, J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}\big)$$

for some constant C *depending only on* Ω, Λ *and* ω.

The corresponding result for the Hölder regularity of H, assuming ε real and (5), is completely analogous; the details are omitted.

It is worth mentioning that the regularity assumptions on the coefficients given in the above theorems are indeed minimal.

Remark 3. *Let* $\Omega = B(0,1)$ be the unit ball and take $\alpha \in (0,1)$. Let $f \in L^\infty((-1,1);\mathbb{R}) \setminus C^\alpha((-1,1);\mathbb{R})$ such that $\Lambda^{-1} \le f \le \Lambda$ in $(-1,1)$. Let ε be defined by $\varepsilon(x) = f(x_1)$. Choosing $J_e = (-\mathrm{i}\omega, 0, 0) \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$, observe that $E(x) = (f(x_1)^{-1}, 0, 0)$ and $H \equiv 0$ are weak solutions in $H(\mathrm{curl},\Omega)^2$ to

$$\mathrm{curl} H = \mathrm{i}\omega\varepsilon E + J_e \quad \text{in } \Omega, \qquad \mathrm{curl} E = -\mathrm{i}\omega H \quad \text{in } \Omega,$$

such that $E \notin C^{0,\alpha}(\Omega;\mathbb{C}^3)$. This shows that interior Hölder regularity cannot hold if ε is not Hölder continuous, even in the simplified case where ε depends only on one variable.

The proofs of these results are based on the use of the scalar and vector potentials of the fields E and H obtained with the Helmholtz decomposition. We show that the study of the original system substantially reduces to the study of two elliptic problems for the scalar potentials, which may be treated with classical elliptic methods. This simple, and yet very powerful, idea allows to treat the Maxwell's system as if it were elliptic. The technique developed in this paper has then proven useful for a variety of problems, e.g. to state a version for Maxwell's equations of Meyers's higher integrability theorem [2], to give asymptotic expansions of the solutions in presence of defects in the material parameters [2] and for the spectral analysis of the Maxwell operator in unbounded domains [1], and has great potential for the study of other aspects of Maxwell's equations.

This paper is structured as follows. In Section 2 we prove Theorem 1 and discuss the corresponding higher regularity result. Section 3 is devoted to the study of bianisotropic materials. Finally, in Section 4 we prove Theorem 2, by using standard elliptic estimates in Campanato spaces.

2. Joint Hölder regularity of E and H

Since regularity properties are local, without loss of generality in the rest of the paper we assume that Ω is connected and simply connected and that its boundary $\partial\Omega$ is connected.

2.1. Preliminary results.

We start by recalling the Helmholtz decomposition of a vector field.

Lemma 4 ([5, Theorem 6.1], [4, Section 3.5])**.** *Take* $F \in L^2(\Omega;\mathbb{C}^3)$.

(1) There exist $q \in H^1_0(\Omega;\mathbb{C})$ *and* $\Phi \in H^1(\Omega;\mathbb{C}^3)$ *such that*

$$F = \nabla q + \mathrm{curl}\Phi \quad \textit{in } \Omega,$$

$\mathrm{div}\Phi = 0$ *in* Ω *and* $\Phi \cdot \nu = 0$ *on* $\partial\Omega$.

(2) There exist $q \in H^1(\Omega;\mathbb{C})$ *and* $\Phi \in H^1(\Omega;\mathbb{C}^3)$ *such that*

$$F = \nabla q + \mathrm{curl}\Phi \quad \text{in } \Omega,$$

$\mathrm{div}\Phi = 0$ *in* Ω *and* $\Phi \times \nu = 0$ *on* $\partial\Omega$.

In both cases, there exists $C > 0$ *depending only on* Ω *such that*

$$\|\Phi\|_{H^1(\Omega;\mathbb{C}^3)} \le C \|F\|_{L^2(\Omega;\mathbb{C}^3)} .$$

We shall need the following key estimate.

Lemma 5 ([5])**.** *Take* $p \in (1,\infty)$ *and* $F \in L^p(\Omega;\mathbb{C}^3)$ *such that* $\mathrm{curl}F \in L^p(\Omega;\mathbb{C}^3)$, $\mathrm{div}F \in L^p(\Omega;\mathbb{C})$ *and either* $F \cdot \nu = 0$ *or* $F \times \nu = 0$ *on* $\partial\Omega$*. Then* $F \in W^{1,p}(\Omega;\mathbb{C}^3)$ *and*

$$\|F\|_{W^{1,p}(\Omega;\mathbb{C}^3)} \le C\big(\|\mathrm{curl}F\|_{L^p(\Omega;\mathbb{C}^3)} + \|\mathrm{div}F\|_{L^p(\Omega;\mathbb{C})}\big),$$

for some $C > 0$ *depending only on* Ω *and* p*.*

2.2. **Proof of Theorem 1.** With an abuse of notation, several positive constants depending only on Ω, Λ and ω will be denoted by the same letter C.

First, we express $E - G$ and H by means of scalar and vector potentials by using Lemma 4: there exist $q_E \in H^1_0(\Omega;\mathbb{C})$, $q_H \in H^1(\Omega;\mathbb{C})$ and $\Phi_E, \Phi_H \in H^1(\Omega;\mathbb{C}^3)$ such that

$$E - G = \nabla q_E + \mathrm{curl}\Phi_E \quad \text{in } \Omega, \qquad H = \nabla q_H + \mathrm{curl}\Phi_H \quad \text{in } \Omega, \tag{6}$$

and

$$\begin{cases} \mathrm{div}\Phi_E = 0 & \text{in } \Omega, \\ \Phi_E \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \mathrm{div}\Phi_H = 0 & \text{in } \Omega, \\ \Phi_H \times \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Moreover, there exists $C > 0$ depending only on Ω such that

$$\|\Phi_E\|_{H^1(\Omega;\mathbb{C}^3)} \le C \|(E,G)\|_{L^2(\Omega;\mathbb{C}^3)^2} , \qquad \|\Phi_H\|_{H^1(\Omega;\mathbb{C}^3)} \le C \|H\|_{L^2(\Omega;\mathbb{C}^3)} . \tag{8}$$

By Lemma 5, the vector potentials enjoy additional regularity.

Lemma 6. *Assume that* (2) *holds true and take* $p \in [2,\infty)$*. Take* $J_e, J_m \in L^p(\Omega;\mathbb{C}^3)$ *and* $G \in W^{1,p}(\mathrm{curl},\Omega)$*. Let* $(E,H) \in W^{1,p}(\mathrm{curl},\Omega)^2$ *be a weak solution of* (1)*, where*

$$W^{1,p}(\mathrm{curl},\Omega) := \{F \in L^p(\Omega;\mathbb{C}^3) : \mathrm{curl}F \in L^p(\Omega;\mathbb{C}^3)\},$$

equipped with the canonical norm. Then $\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H \in W^{1,p}(\Omega;\mathbb{C}^3)$ *and*

$$\begin{aligned} \|\mathrm{curl}\Phi_E\|_{W^{1,p}(\Omega;\mathbb{C}^3)} &\le C \|(H, J_m, \mathrm{curl}G)\|_{L^p(\Omega;\mathbb{C}^3)^3} , \\ \|\mathrm{curl}\Phi_H\|_{W^{1,p}(\Omega;\mathbb{C}^3)} &\le C \|(E, J_e)\|_{L^p(\Omega;\mathbb{C}^3)^2} , \end{aligned}$$

for some constant C *depending only on* Ω*,* Λ *and* ω*.*

Proof. Set $\Psi_E := \mathrm{curl}\Phi_E$. Observe that for every test function $\Phi \in C^\infty(\overline{\Omega};\mathbb{C}^3)$ we have

$$\begin{aligned} \int_\Omega \mathrm{curl}(\nabla q_E) \cdot \Phi - \nabla q_E \cdot \mathrm{curl}\Phi \, dx &= \int_\Omega q_E \mathrm{div}(\mathrm{curl}\Phi) \, dx - \int_{\partial\Omega} q_E \mathrm{curl}\Phi \cdot \nu \, ds \\ &= 0, \end{aligned}$$

which implies that $\nabla q_E \times \nu = 0$ on $\partial\Omega$ ([17, Theorem 3.33]). Hence, by (6) and the third equation of (1) we obtain

$$\Psi_E \times \nu = (\mathrm{curl}\Phi_E) \times \nu = (E - G) \times \nu - \nabla q_E \times \nu = 0 \quad \text{on } \partial\Omega,$$

Thus, using the first equation of (1) and the identities $\mathrm{curl}\nabla = 0$ and $\mathrm{div}\,\mathrm{curl} = 0$ we obtain

$$
\begin{cases} \mathrm{curl}\Psi_E = -\mathrm{i}\omega\mu H + J_m - \mathrm{curl}G & \text{in } \Omega, \\ \mathrm{div}\Psi_E = 0 & \text{in } \Omega, \\ \Psi_E \times \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}
$$

Therefore, by Lemma 5 we have that $\mathrm{curl}\Phi_E \in W^{1,p}(\Omega;\mathbb{C}^3)$ and

$$
\|\mathrm{curl}\Phi_E\|_{W^{1,p}(\Omega;\mathbb{C}^3)} \leq C \|(H, J_m, \mathrm{curl}G)\|_{L^p(\Omega;\mathbb{C}^3)^3}.
$$

The proof for Φ_H is similar, only the boundary conditions have to be handled in a different way. As above, set $\Psi_H := \mathrm{curl}\Phi_H$. For every test function $\varphi \in C^\infty(\overline{\Omega};\mathbb{C})$ we have

$$
\int_\Omega \Psi_H \cdot \nabla\varphi - \varphi\,\mathrm{div}\Psi_H \, dx = \int_\Omega \mathrm{curl}\Phi_H \cdot \nabla\varphi \, dx = 0
$$

since $\mathrm{curl}\nabla = 0$ and $\Phi_H \times \nu = 0$ on $\partial\Omega$. This identity implies that

$$
\Psi_H \cdot \nu = 0 \quad \text{on } \partial\Omega.
$$

Moreover $\mathrm{div}\Psi_H = 0$ in Ω and using the second equation of (1) we obtain $\mathrm{curl}\Psi_H = \mathrm{i}\omega\varepsilon E + J_e \in L^p(\Omega;\mathbb{C}^3)$. Therefore, by Lemma 5 we have that $\mathrm{curl}\Phi_H \in W^{1,p}(\Omega;\mathbb{C}^3)$ and

$$
\|\mathrm{curl}\Phi_H\|_{W^{1,p}(\Omega;\mathbb{C}^3)} \leq C \|(E, J_e)\|_{L^p(\Omega;\mathbb{C}^3)^2}.
$$

This concludes the proof. □ □

We are now in a position to prove Theorem 1.

Proof of Theorem 1 The proof is divided into two steps.

Step 1. $W^{1,6}$-regularity of the scalar potentials. By Lemma 6 with $p = 2$ and the Sobolev embedding theorem, we have that $\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H \in L^6(\Omega;\mathbb{C}^3)$ and

$$
\|(\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H)\|_{L^6(\Omega;\mathbb{C}^3)^2} \leq C \|(E, H, \mathrm{curl}G, J_e, J_m)\|_{L^2(\Omega;\mathbb{C}^3)^5}. \tag{10}
$$

By (6) and (3a) with $\Phi_1 = \nabla\varphi$ for $\varphi \in H_0^1(\Omega;\mathbb{C})$ (arguing as in the first part of the proof of Lemma 6, we have $\Phi_1 \in H_0(\mathrm{curl}, \Omega)$) we obtain

$$
\int_\Omega (\mathrm{i}\omega\varepsilon(G + \mathrm{curl}\Phi_E + \nabla q_E) + J_e) \cdot \nabla\varphi \, dx = 0, \qquad \varphi \in H_0^1(\Omega;\mathbb{C}).
$$

In other words, q_E is a weak solution of

$$
\begin{cases} -\mathrm{div}(\varepsilon\nabla q_E) = \mathrm{div}(\varepsilon G + \varepsilon\mathrm{curl}\Phi_E - \mathrm{i}\omega^{-1}J_e) & \text{in } \Omega, \\ q_E = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}
$$

Similarly, using (6) and (3b) with $\Phi_2 = \nabla\varphi$ for $\varphi \in H^1(\Omega;\mathbb{C})$ we have

$$
\int_\Omega (\mathrm{curl}G + \mathrm{i}\omega\mu(\nabla q_H + \mathrm{curl}\Phi_H) - J_m) \cdot \nabla\varphi \, dx = 0, \qquad \varphi \in H^1(\Omega;\mathbb{C}).
$$

In other words, q_H is a weak solution of

$$
\begin{cases} -\mathrm{div}(\mu\nabla q_H) = \mathrm{div}(\mu\mathrm{curl}\Phi_H + \mathrm{i}\omega^{-1}J_m - \mathrm{i}\omega^{-1}\mathrm{curl}G) & \text{in } \Omega, \\ -(\mu\nabla q_H) \cdot \nu = (\mu\mathrm{curl}\Phi_H + \mathrm{i}\omega^{-1}J_m - \mathrm{i}\omega^{-1}\mathrm{curl}G) \cdot \nu & \text{on } \partial\Omega. \end{cases} \tag{12}
$$

Therefore, by the L^p theory for elliptic equations with complex coefficients (see, e.g., [6, Theorem 1]) applied to the above boundary value problems, we obtain $\nabla q_E, \nabla q_H \in L^6(\Omega;\mathbb{C}^3)$ and

$$\|\nabla q_E, \nabla q_H)\|_{L^6(\Omega;\mathbb{C}^3)^2} \leq C\big(\|(\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H)\|_{L^6(\Omega;\mathbb{C}^3)^2} + \|G\|_{W^{1,6}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{L^6(\Omega;\mathbb{C}^3)^2}\big). \tag{13}$$

Step 2. $C^{1,\alpha}$-regularity of the scalar potentials. Combining (10) and (13) we have $E, H \in L^6(\Omega;\mathbb{C}^3)$ and

$$\|(E,H)\|_{L^6(\Omega;\mathbb{C}^3)^2} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{W^{1,6}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{L^6(\Omega;\mathbb{C}^3)^2}\big).$$

Thus, by Lemma 6 with $p = 6$ we obtain $\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H \in W^{1,6}(\Omega;\mathbb{C}^3)$ and

$$\|(\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H)\|_{W^{1,6}(\Omega;\mathbb{C}^3)^2} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{W^{1,6}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{L^6(\Omega;\mathbb{C}^3)^2}\big).$$

By the Sobolev embedding theorem, this implies $\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H \in C^{0,\frac{1}{2}}(\overline{\Omega};\mathbb{C}^3)$ and

$$\|(\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H)\|_{C^{0,\frac{1}{2}}(\overline{\Omega};\mathbb{C}^3)^2} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{W^{1,6}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{L^6(\Omega;\mathbb{C}^3)^2}\big).$$

In view of (4)-(5), by applying classical Schauder estimates for elliptic systems [15, 18] to (11) and (12) we obtain

$$\|(q_E, q_H)\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})^2} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big).$$

Finally, the result follows from (6) and the last two estimates. □

2.3. **Higher regularity.** The proof of Theorem 1 is based on the regularity of the scalar and vector potentials of the electric and magnetic fields. In particular, the regularity of Φ_E and Φ_H follows from Lemma 5, while the regularity of q_E and q_H follows from standard L^p and Schauder estimates for elliptic systems. Since all these estimates admit higher regularity generalisations [5, 18], by following the argument outlined above we immediately obtain the corresponding higher regularity result.

Theorem 7. *Assume that* (2) *holds true, that $\partial\Omega$ is of class $C^{N+1,1}$ and that*

$$\varepsilon, \mu \in C^{N,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}), \qquad \|(\varepsilon,\mu)\|_{C^{N,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3})^2} \leq \Lambda,$$

for $\alpha \in (0, \frac{1}{2}]$ and $N \in \mathbb{N}$. Take $J_e, J_m \in C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3)$ and $G \in C^{N+1,\alpha}(\mathrm{curl},\Omega)$. Let $(E,H) \in H(\mathrm{curl},\Omega)^2$ be a weak solution of (1)*. Then $E, H \in C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3)$ and*

$$\|(E,H)\|_{C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3)} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{N+1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{C^{N,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big)$$

for some constant C depending only on Ω, Λ, ω and N.

3. The case of bianisotropic materials

In this section, we investigate the Hölder regularity of the solutions of the following problem

$$\begin{cases} \mathrm{curl} H = \mathrm{i}\omega\,(\varepsilon E + \xi H) + J_e & \text{in } \Omega, \\ \mathrm{curl} E = -\mathrm{i}\omega\,(\zeta E + \mu H) + J_m & \text{in } \Omega, \\ E \times \nu = G \times \nu & \text{on } \partial\Omega. \end{cases} \tag{14}$$

In this general case, (2) is not sufficient to ensure ellipticity. As we will see, the leading order coefficient of the coupled elliptic system corresponding to (11)-(12) is

$$A = A_{ij}^{\alpha\beta} = \begin{bmatrix} \Re\varepsilon & -\Im\varepsilon & \Re\xi & -\Im\xi \\ \Im\varepsilon & \Re\varepsilon & \Im\xi & \Re\xi \\ \Re\zeta & -\Im\zeta & \Re\mu & -\Im\mu \\ \Im\zeta & \Re\zeta & \Im\mu & \Re\mu \end{bmatrix},$$

where the Latin indices $i, j = 1, \ldots, 4$ identify the different 3×3 block sub-matrices, whereas the Greek letters $\alpha, \beta = 1, 2, 3$ span each of these 3×3 block sub-matrices. We assume that A is in $L^\infty(\Omega;\mathbb{R})^{12\times 12}$ and that satisfies a strong Legendre condition (as in [12, 15]), namely

$$A_{ij}^{\alpha\beta}\eta_\alpha^i\eta_\beta^j \geq \Lambda^{-1}\left|\eta\right|^2,\ \eta\in\mathbb{R}^{12} \quad \text{and} \quad \left|A_{ij}^{\alpha\beta}\right| \leq \Lambda \qquad \text{a.e. in } \Omega \tag{15}$$

for some $\Lambda > 0$. This condition is satisfied by a large class of materials, including chiral materials and all natural materials [3, Lemma 10 and Remark 11]. Moreover, generalising the regularity assumptions given in (4)-(5), we suppose that

$$\varepsilon, \xi, \zeta, \mu \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}), \qquad \|(\varepsilon,\xi,\zeta,\mu)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3})^4} \leq \Lambda \tag{16}$$

for some $\alpha \in (0, \frac{1}{2}]$.

The main result of this section reads as follows.

Theorem 8. *Assume that* (15) *and* (16) *hold true. Take* $J_e, J_m \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ *and* $G \in C^{1,\alpha}(\mathrm{curl},\Omega)$. *Let* $(E,H) \in H(\mathrm{curl},\Omega)^2$ *be a weak solution of* (14). *Then* $E, H \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ *and*

$$\|(E,H)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2} \leq C\big(\|(E,H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e,J_m)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big)$$

for some constant C *depending only on* Ω, Λ *and* ω.

Proof. The main ingredients are the same used for the proof of Theorem 1. In particular, the regularity result on the vector potentials Φ_E and Φ_H of $E - G$ and H given in Lemma 6 holds true also in this case. The only difference lies in the fact that, since the bianisotropy mixes the electric and magnetic properties, the corresponding estimates will be

$$\|(\mathrm{curl}\Phi_E, \mathrm{curl}\Phi_H)\|_{W^{1,p}(\Omega;\mathbb{C}^3)^2} \leq C\,\|(E,H,J_e,J_m,\mathrm{curl}G)\|_{L^p(\Omega;\mathbb{C}^3)^5}. \tag{17}$$

Similarly, as far as the scalar potentials are concerned, the two equations (11)-(12) become a fully coupled elliptic system, namely

$$-\mathrm{div}(\varepsilon\nabla q_E + \xi\nabla q_H) = \mathrm{div}(\varepsilon G + \varepsilon\mathrm{curl}\Phi_E + \xi\mathrm{curl}\Phi_H - \mathrm{i}\omega^{-1}J_e),$$
$$-\mathrm{div}(\zeta\nabla q_E + \mu\nabla q_H) = \mathrm{div}(\zeta G + \zeta\mathrm{curl}\Phi_E + \mu\mathrm{curl}\Phi_H + \mathrm{i}\omega^{-1}(J_m - \mathrm{curl}G)),$$

in Ω, augmented with the boundary conditions

$$\begin{aligned}&q_E = 0,\\ &-(\zeta\nabla q_E + \mu\nabla q_H)\cdot\nu = (\zeta G + \zeta\mathrm{curl}\Phi_E + \mu\mathrm{curl}\Phi_H + \mathrm{i}\omega^{-1}(J_m - \mathrm{curl}G))\cdot\nu.\end{aligned}$$

on $\partial\Omega$. More precisely, the weak form of this system reads

$$\begin{aligned}&\int_\Omega(\varepsilon\nabla q_E + \xi\nabla q_H)\cdot\nabla\varphi_1\,dx = \int_\Omega(\varepsilon G + \varepsilon\mathrm{curl}\Phi_E + \xi\mathrm{curl}\Phi_H - \mathrm{i}\omega^{-1}J_e)\cdot\nabla\varphi_1\,dx,\\ &\int_\Omega(\zeta\nabla q_E + \mu\nabla q_H)\cdot\nabla\varphi_2\,dx = \int_\Omega\left(\zeta G + \zeta\mathrm{curl}\Phi_E + \mu\mathrm{curl}\Phi_H + \frac{\mathrm{curl}G - J_m}{\mathrm{i}\omega}\right)\cdot\nabla\varphi_2\,dx,\end{aligned}$$

for every $(\varphi_1,\varphi_2)\in H^1_0(\Omega;\mathbb{C})\times H^1(\Omega;\mathbb{C})$. By (15), this system is strongly elliptic, and since the coefficients are Hölder continuous, both the L^p theory and the Schauder theory are applicable [18, Theorem 6.4.8].

We now present a quick sketch of the proof, which follows exactly the same structure of the proof of Theorem 1. By (17) with $p = 2$ we first deduce that $\mathrm{curl}\Phi_E$ and $\mathrm{curl}\Phi_H$ belong to L^6. Thus, by applying the L^p theory to the elliptic system above, we deduce that the scalar potentials are in $W^{1,6}$. By (6), this implies that E and H are in L^6. Using again (17) with $p = 6$ we deduce that $\mathrm{curl}\Phi_E$ and $\mathrm{curl}\Phi_H$ are Hölder continuous. Finally, by the Schauder estimates we deduce that ∇q_E and ∇q_H are Hölder continuous. The corresponding norm estimate follows as in the proof of Theorem 1. □ □

4. Hölder regularity of the electric field E

The proof of Theorem 2 is based on standard elliptic estimates in Campanato spaces [11], which we now introduce. For $\lambda \geq 0$, let $L^{2,\lambda}(\Omega;\mathbb{C})$ be the Banach space of functions $u \in L^2(\Omega;\mathbb{C})$ such that

$$[u]^2_{2,\lambda;\Omega} := \sup_{x\in\Omega, 0<\rho<\mathrm{diam}\Omega}\rho^{-\lambda}\int_{\Omega(x,\rho)}\left|u(y) - \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}u(z)\,dz\right|^2 dy < \infty,$$

where $\Omega(x,\rho) = \Omega\cap\{y\in\mathbb{R}^3 : |y-x| < \rho\}$. The space $L^{2,\lambda}(\Omega;\mathbb{C})$ is naturally equipped with the norm

$$\|u\|_{L^{2,\lambda}(\Omega;\mathbb{C})} = \|u\|_{L^2(\Omega;\mathbb{C})} + [u]_{2,\lambda;\Omega}.$$

We shall use the following standard properties.

Lemma 9 ([20, Chapter 1]). *Take $\lambda \geq 0$ and $p\in[2,\infty)$.*

(1) If $\lambda\in(3,5)$ then $L^{2,\lambda}(\Omega;\mathbb{C}) \cong C^{0,\frac{\lambda-3}{2}}\left(\overline{\Omega};\mathbb{C}\right)$.
(2) If $\lambda < 3$, $u \in L^2(\Omega;\mathbb{C})$ and $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ then $u\in L^{2,2+\lambda}(\Omega;\mathbb{C})$, and the embedding is continuous.
(3) The embedding $L^p(\Omega;\mathbb{C}) \hookrightarrow L^{2,3\frac{p-2}{p}}(\Omega;\mathbb{C})$ is continuous.

We now state the regularity result regarding Campanato estimates we will use.

Lemma 10 ([20, Theorem 2.19]). *Assume that (2) holds and that $\Im\mu\equiv 0$. There exists $\tilde\lambda\in(1,2)$ depending only on Ω and Λ such that if $F\in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ for some $\lambda\in[0,\tilde\lambda]$, and $u\in H^1(\Omega;\mathbb{C})$ satisfies*

$$\begin{cases}\mathrm{div}(\mu\nabla u) = \mathrm{div}F & \text{in } \Omega,\\ \mu\nabla u\cdot\nu = F\cdot\nu & \text{on } \partial\Omega,\end{cases}$$

then $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ *and*

$$\|\nabla u\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\,\|F\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \tag{18}$$

for some constant C *depending only on* Ω *and* Λ.

We shall need the following generalisation of Lemma 5 to the case of Campanato estimates. For a proof, see the second part of the proof of [23, Theorem 3.4].

Lemma 11. *Take* $\lambda \in [0,2)$ *and* $F \in L^2(\Omega;\mathbb{C}^3)$ *such that* $\mathrm{curl}F \in L^{2,\lambda}(\Omega;\mathbb{C}^3)$, $\mathrm{div}F \in L^{2,\lambda}(\Omega;\mathbb{C})$ *and* $F\times\nu = 0$ *on* $\partial\Omega$. *Then* $\nabla F \in L^{2,\lambda}(\Omega;\mathbb{C}^3)$ *and*

$$\|\nabla F\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\big(\|\mathrm{curl}F\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} + \|\mathrm{div}F\|_{L^{2,\lambda}(\Omega;\mathbb{C})}\big),$$

for some $C>0$ *depending only on* Ω *and* λ.

We are now in a position to prove Theorem 2.

Proof of Theorem 2 With an abuse of notation, several positive constants depending only on Ω, Λ and ω will be denoted by the same letter C.

Write $E-G$ and H in terms of scalar and vector potentials (q_E,Φ_E) and (q_H,Φ_H), as in (6). By Lemma 6 and the Sobolev embedding theorem $\mathrm{curl}\Phi_H \in L^6(\Omega;\mathbb{C}^3)$ and $\|\mathrm{curl}\Phi_H\|_{L^6(\Omega;\mathbb{C}^3)} \leq C\,\|(E,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^2}$. Thus, by Lemma 9, part (3), we have that $\mathrm{curl}\Phi_H \in L^{2,2}(\Omega;\mathbb{C}^3)$ and

$$\|\mathrm{curl}\Phi_H\|_{L^{2,2}(\Omega;\mathbb{C}^3)} \leq C\,\|(E,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^2}\,.$$

Therefore, applying Lemma 10 to (12) we obtain that $\nabla q_H \in L^{2,\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)$ and

$$\|\nabla q_H\|_{L^{2,\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)} \leq C(\|(E,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|(\mathrm{curl}G,J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}).$$

Combining the last two inequalities we obtain the estimate

$$\|H\|_{L^{2,\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)} \leq C(\|(E,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|(\mathrm{curl}G,J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}).$$

As a consequence, applying Lemma 11 to $\Psi_E = \mathrm{curl}\Phi_E$, by (9) and the fact that L^∞ is a multiplier space for $L^{2,\min(\tilde\lambda,\lambda)}$, we obtain that $\nabla\mathrm{curl}\Phi_E \in L^{2,\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)$ and

$$\|\nabla\mathrm{curl}\Phi_E\|_{L^{2,\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)} \leq C(\|(E,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|(\mathrm{curl}G,J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}).$$

Hence, by Lemma 9, part (2), and (8) we have that $\mathrm{curl}\Phi_E \in L^{2,2+\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)$ and

$$\begin{aligned}\|\mathrm{curl}\Phi_E\|_{L^{2,2+\min(\tilde\lambda,\lambda)}(\Omega;\mathbb{C}^3)}&\\ &\leq C(\|(E,G,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^3} + \|(\mathrm{curl}G,J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}).\end{aligned}$$

Then, by Lemma 9, part (1), we obtain that $\mathrm{curl}\Phi_E \in C^{0,\frac{\min(\tilde\lambda,\lambda)-1}{2}}(\overline\Omega;\mathbb{C}^3)$ and

$$\begin{aligned}\|\mathrm{curl}\Phi_E\|_{C^{0,\frac{\min(\tilde\lambda,\lambda)-1}{2}}(\overline\Omega;\mathbb{C}^3)}&\\ &\leq C(\|(E,G,J_e)\|_{L^2(\Omega;\mathbb{C}^3)^3} + \|(\mathrm{curl}G,J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2}).\end{aligned}$$

By (4), classical Schauder estimates applied to (11) yield $\nabla q_E \in C^{0,\beta}(\overline{\Omega};\mathbb{C}^3)$, where $\beta = \min(\frac{\tilde{\lambda}-1}{2}, \frac{\lambda-1}{2}, \alpha)$, and

$$\|\nabla q_E\|_{C^{0,\beta}(\overline{\Omega};\mathbb{C}^3)} \leq C(\|E\|_{L^2(\Omega;\mathbb{C}^3)} + \|(\mathrm{curl} G, J_m)\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)^2} + \|(G, J_e)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}).$$

Finally, combining the last two estimates yields the result. □

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