

A note on surfaces with $p_g = q = 2$ and an irrational fibration

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Abstract

We study several examples of surfaces with $p_g = q = 2$ and maximal Albanese dimension that are endowed with an irrational fibration.

0 Introduction

The classification of surfaces S of general type with $\chi(\mathcal{O}_S) = 1$, i.e. $p_g(S) = q(S)$, is currently an active area of research, see for instance [BCP06]. In this case some well-known results imply $p_g \leq 4$. While surfaces with $p_g = q = 4$ and $p_g = q = 3$ have been completely described in [D82], [CCML98], [HP02], [Pi02], the classification of those with $p_g = q = 2$ is still missing, albeit some new interesting examples were recently discovered ([CH06, Pe11, PP14, PP13a, PP13b]). For a recent survey on this topic we refer the reader to [Pe13].

One of the most useful techniques used to understand the geometry of an algebraic surface S is the study of its fibrations $f: S \rightarrow C$, where C is a smooth curve. If $g(C) \geq 1$ we say that the f is *irrational*, and if all its smooth fibres are isomorphic we say that f is *isotrivial*. The study of irrational fibrations on surfaces with $p_g = q = 2$ was started by Zucconi in [Z03a]. Later on, however, it was found that Zucconi's results were incomplete: see [Pe11], where the first author deals with the isotrivial case.

If the image of the Albanese map of S is a curve then everything is known: S is a so-called generalized hyperelliptic surface, i.e. a quotient $S = (C_1 \times C_2)/G$ by the diagonal action of a finite group G on $C_1 \times C_2$ such that the Galois morphism $C_1 \rightarrow C_1/G$ is unramified and $C_2/G \cong \mathbb{P}^1$, see [Ca00, Z03b, Pe11]. Therefore it only remains to investigate the case where S has *maximal Albanese dimension*, i.e. its Albanese map $\alpha: S \rightarrow \text{Alb}(S)$ is generically finite; in this situation the base of any irrational fibration on S is an elliptic curve E (see Proposition 1.3). An important invariant of the fibration f is the push-forward of the relative canonical bundle $f_*\omega_{S/E}$ (see for instance [BHPV03, Chapter III]), and the aim of this paper is to explicitly calculate this invariant in many different examples related to our previous work on the subject, see [PP13a, PP13b, PP14].

This article is organized as follows. In Section 1 we fix the notation and the terminology and we state some technical results needed in the sequel of the paper. Moreover, by using methods borrowed from [Fu78a, Fu78b, Ba00, CD13], we deduce the following structure result, see Propositions 1.6 and 1.7.

Theorem. *Let S be a minimal surface of general type with $p_g = q = 2$ and maximal Albanese dimension. Let $f: S \rightarrow E$ be an irrational fibration whose general fibre F has*

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genus g . Then $g(E) = 1$ and there exist $p \in E$ and $r \geq 1$ such that

$$f_* \omega_{S/E} = \mathcal{O}_E \oplus E_p(r, 1) \oplus \bigoplus_{i=2}^{g-r} \mathcal{Q}_i.$$

Here $E_p(r, 1)$ denotes the unique indecomposable vector bundle on E of rank r and determinant $\mathcal{O}_E(p)$, whereas the \mathcal{Q}_i are pairwise non-isomorphic, non-trivial line bundles in $\text{Tors}(\text{Pic}^0 E)$.

Finally, $r = 1$ if and only if the divisor $K_S - F_p$ is effective.

This theorem corrects and extends Zucconi's results quoted above. For instance, in [Z03a] only the case where $r = 1$ is considered, and the existence of non-isotrivial irrational fibrations is overlooked. See Remark 1.9 for more details.

In Section 2 we provide several examples with $r = 1$ and $r \geq 2$, both in the isotrivial case (Examples 2.2, 2.3, 2.4, 2.5) and in the non-isotrivial one (Examples 2.6, 2.10, 2.11, 2.12). In particular, we show the existence of surfaces S with $p_g = q = 2$ and $(K_S^2, \deg \alpha) \in \{(4, 2)\}, \{(5, 3), (6, 2), (6, 4)\}$, such that S contains an infinite family $f_n: S \rightarrow E$ of non-isotrivial, irrational fibrations whose fibre genera form an unbounded sequence. In the case $(K_S^2, \deg \alpha) = (4, 2)$ the integer $r(f_n)$ can be arbitrarily large, too: in fact, $r(f_n) = n^2 + 1$.

In the discussion of the non-isotrivial case we need some explicit computations on $(1, 2)$ -polarized abelian surfaces. For the reader's convenience, we put them in the Appendix.

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Notation and conventions. We work over the field \mathbb{C} of complex numbers. Throughout the paper we use italic letters for line bundles and capital letters for the corresponding Cartier divisors, so we write for instance $\mathcal{L} = \mathcal{O}_S(L)$.

By *surface* we mean a projective, non-singular surface S , and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic*.

If A is an abelian variety and $\widehat{A} := \text{Pic}^0(A)$ is its dual variety, $A[2]$ and $\widehat{A}[2]$ stand for the corresponding subgroups of 2-division points. If $x \in A$, we write $t_x: A \rightarrow A$ for the translation by x . Given any line bundle \mathcal{L} on A we denote by $\phi_{\mathcal{L}}$ the morphism $\phi_{\mathcal{L}}: A \rightarrow \widehat{A}$ given by $x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$. If $c_1(\mathcal{L})$ is non-degenerate then $\phi_{\mathcal{L}}$ is an isogeny, and we write $K(\mathcal{L})$ for its kernel.

1 Preliminaries

Given any fibration $f: S \rightarrow C$ with general fibre F , we define the *slope* of f as the ratio

$$\lambda(f) = K_{S/C}^2 / \Delta(f),$$

where

$$K_{S/C}^2 = K_S^2 - 8(g(C) - 1)(g(F) - 1),$$

$$\Delta(f) = \chi(\mathcal{O}_S) - (g(C) - 1)(g(F) - 1)$$

and $g(C)$ and $g(F)$ are the genera of C and F , respectively.

Proposition 1.1. *Let $f : S \rightarrow C$ be a relatively minimal fibration with $\lambda(f) = 4$ and $q(S) > g(C)$. Then necessarily $q(S) = g(C) + 1$ and moreover $f_*\omega_{S/C}$ is the direct sum of \mathcal{O}_C and a semistable sheaf of rank $g(F) - 1$.*

Proof. See [Xi87, Theorem 3 p. 462]. \square

Recall that a fibration $f : S \rightarrow C$ is said to be *isotrivial* if all its smooth fibres are isomorphic.

Proposition 1.2. *Let $f : S \rightarrow C$ be a relatively minimal isotrivial fibration, with S non ruled and $g(C) \geq 1$. If S is not isogenous to an unmixed product (see Subsection 2.1 for the definition) we have the sharp inequality*

$$K_S^2 \leq 8\chi(\mathcal{O}_S) - 2$$

and if equality holds then S is a minimal surface of general type whose canonical model has precisely two ordinary double points as singularities. Moreover, under the further assumption that K_S is ample, we have the sharp inequality

$$K_S^2 \leq 8\chi(\mathcal{O}_S) - 5.$$

In particular, if K_S is ample and

$$8\chi(\mathcal{O}_S) - 5 < K_S^2 < 8\chi(\mathcal{O}_S)$$

then f is not isotrivial.

Proof. See [Pol10]. \square

In the sequel, S will be a smooth, minimal surface with $p_g = q = 2$. We denote by $A := \text{Alb}(S)$ the Albanese variety of S and by $\alpha : S \rightarrow A$ the corresponding Albanese map. We also assume that S has maximal Albanese dimension, i.e. that α is generically finite onto the abelian surface A .

Proposition 1.3. *If $f : S \rightarrow E$ is an irrational fibration, then E is an elliptic curve.*

Proof. We must have $1 \leq g(E) \leq 2$, because $q(S) = 2$ and the fibration is irrational. If $g(E) = 2$, using the embedding $E \hookrightarrow \text{Jac}(E)$ and the universal property of the Albanese map, we obtain a morphism $\beta : A \rightarrow \text{Jac}(E)$, whose image is isomorphic to the curve E . On the other hand, the image of β must be a translated of an abelian subvariety of A , hence $g(E) = 1$, contradiction. \square

Remark 1.4. If S admits an irrational fibration $f : S \rightarrow E$, then A is a non-simple abelian surface. In fact, since E is an elliptic curve (Proposition 1.3), the universal property of the Albanese map yields a surjective morphism $A \rightarrow E$, whose kernel is a 1-dimensional subtorus of A .

If E is an elliptic curve and $p \in E$, we set $E_p(1, 1) := \mathcal{O}_E(p)$ and for all $r \geq 2$ we denote by $E_p(r, 1)$ the unique indecomposable vector bundle rank r on E defined recursively by the short exact sequence

$$0 \rightarrow \mathcal{O}_E \rightarrow E_p(r, 1) \rightarrow E_p(r-1, 1) \rightarrow 0.$$

For any $\mathcal{Q} \in \text{Pic}^0 E$ we have $h^0(E, E_p(r, 1) \otimes \mathcal{Q}) = 1$ and $h^1(E, E_p(r, 1) \otimes \mathcal{Q}) = 0$, see [At57, Lemma 8 and 15].

Lemma 1.5. *Let \mathcal{A} be an indecomposable vector bundle over an elliptic curve E . If $\deg \mathcal{A} = d < 0$, then $H^0(E, \mathcal{A}) = 0$.*

Proof. We work by induction on $r := \text{rank } \mathcal{A}$. If $r = 1$ the result is clear. Assume now that the claim is true for any vector bundle of rank $r - 1$. If $h^0(E, \mathcal{A}) > 0$, then there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0,$$

where \mathcal{B} is a vector bundle on E of degree d and rank $r - 1$. Hence, by Serre duality and the inductive hypothesis, we infer

$$\text{Ext}^1(\mathcal{B}, \mathcal{O}_E) = H^1(E, \mathcal{B}^*) = H^0(E, \mathcal{B}) = 0,$$

a contradiction because \mathcal{A} is indecomposable. \square

Proposition 1.6. *Let $f: S \longrightarrow E$ be an irrational fibration on a minimal surface with $p_g = q = 2$ and maximal Albanese dimension. Then*

$$f_* \omega_{S/E} = \mathcal{O}_E \oplus E_p(r, 1) \oplus \bigoplus_{i=2}^{g-r} \mathcal{Q}_i,$$

where $p \in E$, $r \geq 1$ and the \mathcal{Q}_i are pairwise non-isomorphic, non-trivial torsion line bundles on E .

Proof. By [Fu78a, Theorem 2.7 and Theorem 3.1] we can write

$$f_* \omega_{S/E} = \mathcal{O}_E^{\oplus h} \oplus \mathcal{F},$$

where $h = h^1(E, f_* \omega_S)$ and \mathcal{F} is a locally free sheaf such that $H^1(E, \mathcal{F} \otimes \omega_E) = 0$.

From [Ba00, Proposition 1.8 (i)] it follows $h = q(S) - g(E) = 1$. Moreover, by [Fu78b] and [CD13, Corollary 21] we have

$$\mathcal{F} = \mathcal{A} \oplus \bigoplus \mathcal{Q}_i, \tag{1}$$

where \mathcal{A} is an ample vector bundle on E and each \mathcal{Q}_i is a non-trivial, torsion line bundle.

Since \mathcal{A} is ample, each indecomposable direct summand \mathcal{A}_i of \mathcal{A} has degree > 0 , hence $h^0(E, \mathcal{A}_i) > 0$ ([H71, Lemma 1.1 and Proposition 1.2]). But $h^0(E, \mathcal{A}) = 1$, so \mathcal{A} is indecomposable and by [At57, p. 434] we infer that there exists $p \in E$ such that $\mathcal{A} = E_p(r, 1)$, where $r = \text{rank } \mathcal{A}$.

It remains to show that the line bundles \mathcal{Q}_i are pairwise non-isomorphic. By contradiction, and without loss of generality, assume $\mathcal{Q}_1 = \mathcal{Q}_2$. Then

$$f_* \omega_S \otimes \mathcal{Q}_1^{-1} = \mathcal{Q}_1^{-1} \oplus (E_p(r, 1) \otimes \mathcal{Q}_1^{-1}) \oplus \mathcal{O}_E \oplus \mathcal{O}_E \oplus \bigoplus_{i=3}^{g-r} (\mathcal{Q}_i \otimes \mathcal{Q}_1^{-1}),$$

which implies, by projection formula, $h^0(S, \omega_S \otimes f^* \mathcal{Q}_1^{-1}) \geq 3$. Now Serre duality and projection formula yield

$$h^2(S, \omega_S \otimes f^* \mathcal{Q}_1^{-1}) = h^0(S, f^* \mathcal{Q}_1) = h^0(E, \mathcal{Q}_1) = 0,$$

hence $h^1(S, \omega_S \otimes f^* \mathcal{Q}_1^{-1}) \geq 2$. On the other hand, a direct calculation using the Leray spectral sequence of $f: S \longrightarrow E$ shows that, given $\mathcal{Q} \in \text{Pic}^0 E$, we have

$$h^1(S, \omega_S \otimes f^* \mathcal{Q}) = \begin{cases} 2 & \text{if } \mathcal{Q} = \mathcal{O}_E \\ 1 & \text{if } \mathcal{Q} = \mathcal{Q}_i^{-1} \text{ for some } i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This is a contradiction. \square

We now denote the integer r which appears in Proposition 1.6 by $r(f)$. If $p \in E$, we write F_p for the fibre of $f : S \rightarrow E$ over p , namely $F_p = f^{-1}(p)$. The following result shows that $r(f)$ is related to the existence in paracanonical systems of reducible curves containing fibres of f (we refer the reader to [Be88] and [MPP13] for generalities and results about paracanonical systems on surfaces).

Proposition 1.7. *Let $f : S \rightarrow E$ be an irrational fibration on a minimal surface with $p_g = q = 2$ and maximal Albanese dimension. Then the following are equivalent:*

- (1) $r(f) = 1$;
- (2) for any $\eta \in \text{Pic}^0 E$ there exists a (unique) point $p_\eta \in E$ such that the linear system $|K_S + f^*\eta - F_{p_\eta}|$ is not empty. More precisely, $h^0(S, K_S + f^*\eta - F_{p_\eta}) = 1$;
- (3) there exists a (unique) point $p \in E$ such that the linear system $|K_S - F_p|$ is not empty. More precisely, $h^0(S, K_S - F_p) = 1$.

Proof. (1) \Rightarrow (2) If $r(f) = 1$, there exists $p \in E$ such that $f_*\omega_S = \mathcal{O}_E \oplus \mathcal{O}_E(p) \oplus \bigoplus_{i=2}^{g-1} \mathcal{Q}_i$, so projection formula yields, for any $q \in E$,

$$\begin{aligned} H^0(S, K_S + f^*\eta - F_q) &= H^0(E, f_*(\omega_S \otimes f^*\eta \otimes f^*\mathcal{O}_E(-q))) \\ &= H^0(E, \mathcal{O}_E(-q) \otimes \eta) \oplus H^0(E, \mathcal{O}_E(p-q) \otimes \eta) \oplus \bigoplus_{i=2}^{g-1} H^0(E, \mathcal{Q}_i \otimes \mathcal{O}_E(-q) \otimes \eta) \\ &= H^0(E, \mathcal{O}_E(p-q) \otimes \eta). \end{aligned}$$

Now it suffices to choose as p_η the unique point q such that $\mathcal{O}_E(q) = \mathcal{O}_E(p) \otimes \eta$.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) Assume that there exists $p \in E$ such that $|K_S - F_p|$ is not empty. Using Proposition 1.6 and projection formula we can write

$$\begin{aligned} H^0(S, K_S - F_p) &= H^0(E, f_*(\omega_S \otimes f^*\mathcal{O}_E(-p))) \\ &= H^0(E, \mathcal{O}_E(-p)) \oplus H^0(E, E_p(r, 1) \otimes \mathcal{O}_E(-p)) \oplus \bigoplus_{i=2}^{g-r} H^0(E, \mathcal{Q}_i \otimes \mathcal{O}_E(-p)) \\ &= H^0(E, E_p(r, 1) \otimes \mathcal{O}_E(-p)). \end{aligned}$$

The indecomposable vector bundle $E_p(r, 1) \otimes \mathcal{O}_E(-p)$ has degree $1 - r$, which is a negative integer for $r > 1$. Therefore Lemma 1.5 implies that it has a global section if and only if $r = 1$, i.e. when $E_p(r, 1) = \mathcal{O}_E(p)$, and this completes the proof. \square

Corollary 1.8. *Let $f : S \rightarrow E$ be an irrational fibration on a minimal surface with $p_g = q = 2$ and maximal Albanese dimension. If $r(f) = 1$, then $2 \leq g(F) \leq 5$.*

Proof. If $r(f) = 1$ then $K_S - F$ is effective (Proposition 1.7). Since K_S is nef it follows $K_S(K_S - F) \geq 0$, that is $K_S F \leq K_S^2$. By Bogomolov-Miyaoka-Yau inequality ([BHPV03, Chapter VII]) we have $K_S^2 \leq 9\chi(\mathcal{O}_S) = 9$, hence $g(F) \leq 5$. \square

Remark 1.9. In [Z03a, Corollary 2.4 and Theorem 2.8] it is stated that all irrational fibrations on surfaces with $p_g = q = 2$ and maximal Albanese dimension satisfy $r(f) = 1$ and $2 \leq g(F) \leq 5$, and that they are isotrivial as soon as $g(F) > 2$. We will see in the next section that this is not true: for example, we will show the existence of non-isotrivial, irrational fibrations with $r(f) \geq 2$ and $g(F)$ arbitrarily large, see Examples 2.6, 2.10 and 2.11. We found that some of the arguments in [Z03a] are actually incomplete: for instance, the analysis of the case $\mathcal{L} = \mathcal{O}_E$ is missing in [Z03a, proof of Lemma 2.3].

2 Examples

2.1 Isotrivial examples

Isotrivial, irrational fibrations on surfaces with $p_g = q = 2$ were classified in [Pe11]. The aim of this section is to compute the integer $r(f)$ for some of them. Before doing this we introduce some notation and terminology, referring the reader to [Pol10] for further details.

A smooth surface S is called a *standard isotrivial fibration* if there exists a finite group G , acting faithfully on two smooth projective curves C_1 and C_2 and diagonally on their product, so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)/G$. We denote such a desingularization by $\lambda: S \rightarrow T$. In particular, if the action of G is free then T is smooth and we call $S = T$ a surface *isogenous to an unmixed product*.

If $\lambda: S \rightarrow T = (C_1 \times C_2)/G$ is any standard isotrivial fibration, composing the two morphism $T \rightarrow C_1/G$ and $T \rightarrow C_2/G$ with λ one obtains two fibrations $f_1: S \rightarrow C_1/G$ and $f_2: S \rightarrow C_2/G$, whose smooth fibres F_1 and F_2 are isomorphic to C_2 and C_1 , respectively. Moreover $F_1 F_2 = |G|$.

We denote by \mathbf{m}_1 and \mathbf{m}_2 the vectors of branching data of f_1 and f_2 , respectively. For instance $\mathbf{m}_1 = (2, 2)$ means that f_1 has two branching points, both with branching order 2. The irregularity of S is $q(S) = g(C_1/G) + g(C_2/G)$, see [Fre71]. Moreover, if $g(C_1)$ and $g(C_2)$ are both strictly positive, the surface S is necessarily a minimal model.

Lemma 2.1. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to an unmixed product, and assume that G is an abelian group. Then $f_{i*}\omega_S$ splits into a direct sum of line bundles.*

Proof. Roughly speaking, this follows from the fact that the irreducible representations of a finite, abelian group are 1-dimensional, together with the structure results for abelian covers given in [Pa91]. Indeed, let us consider the commutative diagram

$$\begin{array}{ccc} C_1 \times C_2 & \xrightarrow{\Phi} & S \\ \pi_i \downarrow & & \downarrow f_i \\ C_i & \xrightarrow{\phi_i} & C_i/G. \end{array} \quad (3)$$

Since G is abelian, there exists a direct sum decomposition

$$\Phi_*\omega_{C_1 \times C_2} = \omega_S \oplus \bigoplus_{\chi \in \widehat{G}^*} (\omega_S \otimes \mathcal{L}_\chi), \quad (4)$$

where \widehat{G}^* is the subset of non-trivial characters of G and the \mathcal{L}_χ are line bundles on S . Applying f_{i*} to both sides of (4) it follows that $f_{i*}\omega_S$ is a direct summand of $f_{i*}\Phi_*\omega_{C_1 \times C_2}$. Using the commutativity of (3), we obtain

$$\begin{aligned} f_{i*}\Phi_*\omega_{C_1 \times C_2} &= \phi_{i*}\pi_{i*}\omega_{C_1 \times C_2} \\ &= (\phi_{i*}\omega_{C_i})^{\oplus g(C_{i+1})} = \omega_{C_i/G}^{\oplus g(C_{i+1})} \oplus \bigoplus_{\chi \in \widehat{G}^*} (\omega_{C_i/G} \otimes \mathcal{M}_\chi)^{\oplus g(C_{i+1})}, \end{aligned}$$

where the integer $i+1$ has to be intended mod 2 and the \mathcal{M}_χ are line bundles on C_i/G . Therefore we are done, because the right-hand side is a direct sum of line bundles and the decomposition of a vector bundle into indecomposable ones is unique up to reordering of the summands ([At56]). \square

We are now ready to compute $r(f_i)$ in some of isotrivial examples. We only give the construction data and we refer the reader to [Pe11, Sections 3 and 4] for the details. We write E_i for the elliptic curve C_i/G .

Example 2.2. $g(C_1) = 3$, $g(C_2) = 3$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $K_S^2 = 8$.
 $m_1 = (2, 2)$, $m_2 = (2, 2)$.

Since in this case G is abelian, Lemma 2.1 yields

$$r(f_1) = 1, \quad r(f_2) = 1.$$

Example 2.3. $g(C_1) = 2$, $g(C_2) = 2$, $G = \mathbb{Z}/2\mathbb{Z}$, $K_S^2 = 4$.
 $m_1 = (2, 2)$, $m_2 = (2, 2)$.

In this case

$$r(f_1) = 1, \quad r(f_2) = 1,$$

because $g(C_i) = 2$.

Example 2.4. $g(C_1) = 3$, $g(C_2) = 3$, $G = Q_8$ or $G = D_8$, $K_S^2 = 4$.
 $m_1 = (2)$, $m_2 = (2)$.

In this case

$$r(f_1) = 2, \quad r(f_2) = 2.$$

Indeed, $\lambda(f_i) = 4$ so by Proposition 1.1 it follows that $f_{i*}\omega_{S/E_i}$ is the direct sum of \mathcal{O}_{E_i} with a semistable vector bundle of rank 2 and degree 1; such a bundle is necessarily of the form $E_p(2, 1)$ for some $p \in E$, see Proposition 1.6.

Example 2.5. $g(C_1) = 3$, $g(C_2) = 3$, $G = S_3$, $K_S^2 = 5$.
 $m_1 = (3)$, $m_2 = (3)$.

In this case

$$r(f_1) = 2, \quad r(f_2) = 2.$$

In fact, we have

$$\text{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2),$$

so S contains a (-3) -curve W and two (-2) -curves Z_1, Z_2 such that $WZ_1 = WZ_2 = 0$ and $Z_1Z_2 = 1$. Using the results in [Pol10, Section 2] one checks that the linear equivalence classes of the fibres F_i of $f_i: S \rightarrow E_i$ are

$$F_1 = 3Y_1 + W + 2Z_1 + Z_2, \quad F_2 = 3Y_2 + W + Z_1 + 2Z_2,$$

where the Y_i satisfy $Y_i^2 = -1$ and $K_S Y_i = 1$. Moreover $F_1 F_2 = |G| = 6$, hence we infer $Y_1 Y_2 = 0$, see Figure 1.

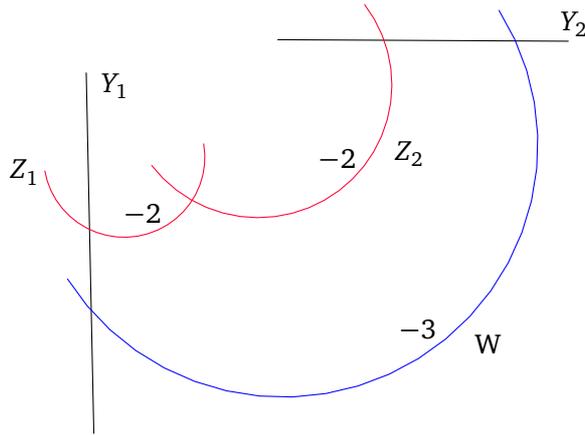


Figure 1: Configuration of the curves Y_i, Z_j, W in Example 2.5

By applying Serrano's canonical bundle formula ([Se96, Theorem 4.1]) we obtain

$$K_S = 2Y_1 + Z_1 + 2Y_2 + Z_2 + W + Z_1 + Z_2 = F_1 + (2Y_2 - Y_1 + Z_2).$$

If $r(f_1) = 1$ then $K_S - F_1$ would be numerically equivalent to an effective divisor (Proposition 1.7), hence $2Y_2 - Y_1 + Z_2$ would be numerically equivalent to an effective divisor. On the other hand, we have

$$(2Y_2 - Y_1 + Z_2)F_2 = -Y_1F_2 = -2 < 0,$$

and this is a contradiction because F_2 is nef. It follows $r(f_1) = 2$. The proof of $r(f_2) = 2$ is completely similar.

2.2 Non-isotrivial examples

Example 2.6. We give examples of non-isotrivial, irrational fibrations on surfaces with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2. Such surfaces were constructed and classified in [PP13b]. Here we only state the main result, referring the reader to that paper for more details. Let (A, \mathcal{L}) be a $(1, 2)$ -polarized abelian surface and denote by $\phi_2: A[2] \rightarrow \widehat{A}[2]$ the restriction of the canonical homomorphism $\phi_{\mathcal{L}}: A \rightarrow \widehat{A}$ to the subgroup of 2-division points. Then $\ker \phi_2 = K(\mathcal{L}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $\text{im } \phi_2$ consists of four line bundles $\{\mathcal{O}_A, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$; the set $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$ will be denoted by $\text{im } \phi_2^\times$. The surfaces we are interested in can be constructed by using the following

Proposition 2.7. *Given an abelian surface A with a symmetric polarization \mathcal{L} of type $(1, 2)$, not of product type, for any $\mathcal{Q} \in \text{im } \phi_2$ there exists a curve $D \in |\mathcal{L}^2 \otimes \mathcal{Q}|$ whose unique non-negligible singularity is an ordinary quadruple point at the origin $0 \in A$. Let $\mathcal{Q}^{1/2}$ be a square root of \mathcal{Q} , and if $\mathcal{Q} = \mathcal{O}_A$ assume moreover $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$. Then the minimal desingularization S of the double cover of A branched over D and defined by $\mathcal{L} \otimes \mathcal{Q}^{1/2}$ is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and whose Albanese map $\alpha: S \rightarrow A$ is a generically finite double cover.*

Conversely, every minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2 can be constructed in this way.

Finally, the moduli space of these surfaces is a disjoint union of three connected, irreducible components

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II}$$

of dimension 4, 4, 3, respectively, where

- surfaces of type Ia satisfy $\mathcal{Q} = \mathcal{O}_A$ and $\mathcal{Q}^{1/2} \notin \text{im } \phi_2^\times$;
- surfaces of type Ib satisfy $\mathcal{Q} = \mathcal{O}_A$ and $\mathcal{Q}^{1/2} \in \text{im } \phi_2^\times$;
- surfaces of type II satisfy $\mathcal{Q} \in \text{im } \phi_2^\times$.

Proof. See [PP13b, Theorems 2.6 and 3.7]. □

Now assume that the polarization $\mathcal{L} = \mathcal{O}_A(L)$ on A is of *special type*, i.e. the linear system $|L|$ contains a member of the form $E'_1 + E'_2$, where the E'_i are two elliptic curves such that $E'_1 E'_2 = 2$, see [Ba87, p. 46] or [PP13a, Section 1]. In particular A is not simple, see Remark 1.4. The branch locus D of $\alpha: S \rightarrow A$ intersect each E'_i in four generically distinct points, in fact $D(E'_1 + E'_2) = 2L^2 = 8$. An explicit construction of such a pair (A, \mathcal{L}) is given in the Appendix, by taking a degree 2 isogeny $\psi: A \rightarrow B$, where $B = E_1 \times E_2$ is the product of two elliptic curves. We now have two exact sequences of complex tori

$$0 \rightarrow E'_i \rightarrow A \xrightarrow{\pi_i} E_i \rightarrow 0, \quad i = 1, 2$$

and the composition $\pi_i \circ \alpha$ gives an irrational fibration $f_i: S \rightarrow E_i$, whose fibre F_i has genus 3. Hence the surface S admits two irrational fibrations f_1 and f_2 , both of genus 3, whose fibres F_1 and F_2 satisfy $F_1 F_2 = 2E'_1 E'_2 = 4$. By [PP13b, Remark 2.12] the general

surface S constructed in such a way has ample canonical class, hence by Proposition 1.2 it follows that $f_i: S \rightarrow E_i$ is not isotrivial.

We want now to compute $r(f_i)$. Let $\sigma: \tilde{A} \rightarrow A$ be the blow-up at the point $0 \in A$; we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\tilde{\alpha}} & \tilde{A} \\ & \searrow \alpha & \downarrow \sigma \\ & & A, \end{array}$$

where $\tilde{\alpha}: S \rightarrow \tilde{A}$ is a flat double cover. Denote by $\Lambda \subset \tilde{A}$ the exceptional divisor of σ ; then the preimage of Λ in S is an elliptic curve Z such that $Z^2 = -2$. The branch locus of $\tilde{\alpha}$ is a smooth curve $\tilde{D} \in |\sigma^*(2L + Q) - 4\Lambda|$ and the square root of \tilde{D} determining the double cover is $\tilde{L} := \sigma^*(L + Q^{1/2}) - 2\Lambda$. Hence we have

$$\begin{aligned} \tilde{\alpha}_* \omega_S &= \omega_{\tilde{A}} \oplus (\omega_{\tilde{A}} \otimes \mathcal{O}_{\tilde{A}}(\tilde{L})) \\ &= \mathcal{O}_{\tilde{A}}(\Lambda) \oplus \mathcal{O}_{\tilde{A}}(\sigma^*(L + Q^{1/2}) - \Lambda). \end{aligned} \quad (5)$$

The smooth elliptic fibration $\pi_i: A \rightarrow E_i$ induces, by composition with $\sigma: \tilde{A} \rightarrow A$, an elliptic fibration $\tilde{\pi}_i: \tilde{A} \rightarrow E_i$ with a unique singular fibre (the one containing the exceptional divisor Λ). We define Γ_{ip} to be the fibre of $\tilde{\pi}_i$ over $p \in E_i$.

Clearly $f_i = \tilde{\pi}_i \circ \tilde{\alpha}$, so by using (5) we can write

$$\begin{aligned} f_{i*} \omega_{S/E_i} &= f_{i*} \omega_S = \tilde{\pi}_{i*} \tilde{\alpha}_* \omega_S \\ &= \tilde{\pi}_{i*} \mathcal{O}_{\tilde{A}}(\Lambda) \oplus \tilde{\pi}_{i*} \mathcal{O}_{\tilde{A}}(\sigma^*(L + Q^{1/2}) - \Lambda) \\ &= \mathcal{O}_{E_i} \oplus \tilde{\pi}_{i*} \mathcal{O}_{\tilde{A}}(\sigma^*(L + Q^{1/2}) - \Lambda). \end{aligned} \quad (6)$$

Proposition 2.8. *We have $r(f_i) = 1$ if and only if there exists a point $p \in E_i$ such that $H^0(\tilde{A}, \sigma^*(L + Q^{1/2}) - \Lambda - \Gamma_{ip}) \neq 0$. Otherwise $r(f_i) = 2$.*

Proof. Since $g(F_i) = 3$ we have either $r(f_i) = 1$ or $r(f_i) = 2$. Looking at (6) and arguing as in the proof of Proposition 1.7, we see that $r(f_i) = 1$ if and only if there exists $p \in E_i$ such that $H^0(E_i, \tilde{\pi}_{i*} \mathcal{O}_{\tilde{A}}(\sigma^*(L + Q^{1/2}) - \Lambda) \otimes \mathcal{O}_{E_i}(-p)) \neq 0$. By projection formula this vector space has the same dimension as $H^0(\tilde{A}, \sigma^*(L + Q^{1/2}) - \Lambda - \Gamma_{ip})$, so we are done. \square

The pencil $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ contains in general exactly two reducible divisors ([PP13a, Remark 1.16]). On the other hand, the point 0 is not in the base locus of $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ (because $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$), so there is at most one reducible divisor in the pencil containing 0 . Corollary 2.9 below shows how the existence of such a divisor determines the integers $r(f_i)$.

Corollary 2.9. *The set $\{r(f_1), r(f_2)\}$ is as follows.*

- (1) *if no reducible curve in $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ contains 0 , then $\{r(f_1), r(f_2)\} = \{2\}$;*
- (2) *if there exists a reducible curve in $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ having a node at 0 , then $\{r(f_1), r(f_2)\} = \{1\}$;*
- (3) *if there exists a reducible curve in $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ which is smooth at 0 , then $\{r(f_1), r(f_2)\} = \{1, 2\}$.*

Proof. If no reducible curve in $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ contains 0 , then $H^0(\tilde{A}, \sigma^*(L + Q^{1/2}) - \Lambda - \Gamma_{ip}) = 0$ for $i = 1, 2$, so Proposition 2.8 shows that $r(f_1) = r(f_2) = 2$. This is case (1).

Therefore we can assume that there is a reducible curve $C \in |\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ containing 0 . Using a slight abuse of notation we write $C = E'_1 + E'_2$ (actually, C is a translate of $E'_1 + E'_2 \in |\mathcal{L}|$). Let \tilde{E}'_i be the strict transform of E'_i via $\sigma: \tilde{A} \rightarrow A$. There are two possibilities.

- 0 is an ordinary double point for C . Then

$$\sigma^*C - \Lambda = \sigma^*E'_1 + \sigma^*E'_2 - \Lambda = \tilde{E}'_1 + \tilde{E}'_2 + \Lambda.$$

Since $\tilde{E}'_i + \Lambda$ is the (unique) reducible fibre of $\tilde{\pi}_i: \tilde{A} \rightarrow E_i$, $i = 1, 2$, by Proposition 2.8 it follows $r(f_1) = r(f_2) = 1$. This is case **(2)**.

- 0 is a smooth point for C . Without loss of generality, we can assume that 0 belongs to E'_1 but not to E'_2 . Then

$$\sigma^*C - \Lambda = \sigma^*E'_1 + \sigma^*E'_2 - \Lambda = \tilde{E}'_1 + \sigma^*E'_2.$$

Since $\sigma^*E'_2$ is a fibre of $\tilde{\pi}_2$ but \tilde{E}'_1 is not a fibre of $\tilde{\pi}_1$, it follows $r(f_2) = 1$, $r(f_1) = 2$. This is case **(3)**.

The proof is now complete. \square

All the three cases in Corollary 2.9 actually occur. We give explicit examples in the Appendix, see in particular Proposition 2.16, Proposition 2.17 and Remark 2.18.

Example 2.10. The same construction used in Example 2.6 can be used in other situations. We describe a couple of examples, leaving the details to the reader.

- In [PP13a] we studied some surfaces S (originally constructed in [CH06]) with $p_g = q = 2$ and $K^2 = 5$. Their Albanese map $\alpha: S \rightarrow A$ is a generically finite triple cover of a $(1, 2)$ -polarized abelian surface (A, \mathcal{L}) , branched over a divisor $D \in |2\mathcal{L}|$ with an ordinary quadruple point. We can choose A such that there is a degree 2 isogeny $\psi: A \rightarrow B$, where $B = E_1 \times E_2$ is the product of two elliptic curves. Then S admits two non isotrivial, irrational fibrations $f_i: S \rightarrow E_i$, both with the general fibre of genus 3.
- In [PP14] we constructed some new surfaces S with $p_g = q = 2$ and $K^2 = 6$. Their Albanese map $\alpha: S \rightarrow A$ is a generically finite quadruple cover of a $(1, 3)$ -polarized abelian surface (A, \mathcal{L}) , branched over a divisor $D \in |2\mathcal{L}|$ with six ordinary cusps. We can choose A such that there is a degree 3 isogeny $\psi: A \rightarrow B$, where $B = E_1 \times E_2$ is the product of two elliptic curves. Then S admits two non isotrivial, irrational fibrations $f_i: S \rightarrow E_i$, both with the general fibre of genus 4.

Example 2.11. We can further specialize the construction described in Examples 2.6 and 2.10, assuming that $E_1 = E_2$, so that the abelian surface A is isogenous to the product $B = E \times E$ of an elliptic curve E with itself.

This allows us to obtain examples with *infinitely many* irrational fibrations whose fibre genera are arbitrarily large. In fact, for any $n \geq 1$ let us consider the elliptic fibration $g_n: B \rightarrow E$ defined by $(x, y) \mapsto x \oplus ny$, where \oplus is the group law on E . Composing with a degree 2 (resp. a degree 3) isogeny $\psi: A \rightarrow B$, we obtain an elliptic fibration $h_n: A \rightarrow E$. Moreover, we can choose ψ in such a way that the induced $(1, 2)$ -polarization (resp. $(1, 3)$ -polarization) $\mathcal{L} = \mathcal{O}_A(L)$ on A is not of product type: see the Appendix, where the case $\deg \psi = 2$ is discussed in detail.

If E_n is the general fibre of h_n , we have $\lim_{n \rightarrow +\infty} E_n L = +\infty$. Therefore, repeating the previous constructions, we can build families of surfaces S with $p_g = q = 2$ and $(K_S^2, \deg \alpha) \in \{(5, 3), (6, 2), (6, 4)\}$, such that S contains an infinite family $f_n: S \rightarrow E$ of non-isotrivial, irrational fibrations. Moreover the fibre of f_n has genus strictly increasing with n , hence Corollary 1.8 implies $r(f_n) \geq 2$ for almost all n .

These examples demonstrate that the (still incomplete) classification of irrational fibrations on surfaces with $p_g = q = 2$ and maximal Albanese dimension needs to be much subtler than the one attempted in [Z03a].

Example 2.12. The same idea of Example 2.11 allows us to produce examples with $(K_S^2, \deg \alpha) = (4, 2)$ and $r(f)$ arbitrarily large. In fact, let $B = E \times E$ and let $\mathcal{L} = \mathcal{O}_B(L)$ be a principal product polarization on B . Take the fibration $g_n: B \rightarrow E$ defined as above and let E_n be its fibre; then $E_n L = n^2 + 1$. The linear system $|2L|$ is base-point free, see [Ke91, Lemma 2.8], so the general curve $D \in |2L|$ is smooth. Therefore the double cover $S \rightarrow B$ branched over D is a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 4$. Moreover the infinite family of fibrations g_n induce a family of irrational fibrations $f_n: S \rightarrow E$, whose fibre F_n has genus $g(F_n) = n^2 + 2$. Hence all the fibrations f_n are distinct and by Propositions 1.1 and 1.6 we have

$$(f_n)_* \omega_{S/E} = \mathcal{O}_E \oplus E_p(n^2 + 1, 1),$$

which gives $r(f_n) = n^2 + 1$.

Remark 2.13. It would be interesting to compute the integer $r(f)$ for all the surfaces in Examples 2.10 and 2.11.

Remark 2.14. The situation described in Examples 2.11 and 2.12 can only occur for irrational pencils over an elliptic base. In fact, a classical result of Severi states that a surface of general type has at most finitely many pencils over curves of genus ≥ 2 , see [MLP11, Section 2].

Appendix: explicit computations on abelian surfaces with $(1, 2)$ -polarization of special type

We start with a principally polarized abelian surface B which is the product of two elliptic curves, i.e. $B := E_1 \times E_2$. Then the period matrix of B is

$$\begin{pmatrix} \tau_1 & 0 & 1 & 0 \\ 0 & \tau_2 & 0 & 1 \end{pmatrix},$$

where $\text{Im}(\tau_i) > 0$ for $i = 1, 2$; hence $B = \mathbb{C}^2 / \Lambda_B$, the lattice Λ_B being spanned by the four column vectors

$$\lambda_1 := \begin{pmatrix} \tau_1 \\ 0 \end{pmatrix}, \quad \lambda_2 := \begin{pmatrix} 0 \\ \tau_2 \end{pmatrix}, \quad \mu_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notice that

$$E_1 = \mathbb{C} / (\tau_1 \mathbb{Z} \oplus \mathbb{Z}), \quad E_2 = \mathbb{C} / (\tau_2 \mathbb{Z} \oplus \mathbb{Z}).$$

There is a natural principal polarization of product type on B , which is induced by the alternating form $E_B: \Lambda_B \times \Lambda_B \rightarrow \mathbb{Z}$, where

$$E_B(\lambda_1, \mu_1) = 1, \quad E_B(\mu_1, \lambda_1) = -1, \quad E_B(\lambda_2, \mu_2) = 1, \quad E_B(\mu_2, \lambda_2) = -1$$

and all the other values are zero. Let $\widehat{B} := \text{Pic}^0(B)$ be the dual abelian variety of B . By the Appell-Humbert Theorem its elements can be identified with the characters $\Lambda_B \rightarrow \mathbb{C}^*$; we will indicate such a character χ^B by the vector

$$(\chi^B(\lambda_1), \chi^B(\lambda_2), \chi^B(\mu_1), \chi^B(\mu_2)).$$

The principal polarization yields an isomorphism $B \rightarrow \widehat{B}$, sending the point $x \in B$ to the character $\exp(2\pi i E_B(\cdot, x))$.

The finite subgroup $\widehat{B}[2]$ of \widehat{B} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and corresponds to the 16 characters $\Lambda_B \rightarrow \mathbb{C}^*$ with values in $\{\pm 1\}$. The four characters giving the 2-torsion line

bundles which are pullback from E_1 are $\exp(2\pi i E_B(\cdot, x))$, where x is one of the points $0, \frac{\lambda_1}{2}, \frac{\mu_1}{2}, \frac{\lambda_1 + \mu_1}{2}$, so they can be written as

$$\begin{aligned}\chi_0^B &= (1, 1, 1, 1), & \chi_1^B &= (1, 1, -1, 1), \\ \chi_2^B &= (-1, 1, 1, 1), & \chi_3^B &= (-1, 1, -1, 1).\end{aligned}\tag{7}$$

Analogously, the four characters giving the 2-torsion line bundles which are pullback from E_2 are $\exp(2\pi i E_B(\cdot, x))$, where x is one of the points $0, \frac{\lambda_2}{2}, \frac{\mu_2}{2}, \frac{\lambda_2 + \mu_2}{2}$, so they can be written as

$$\begin{aligned}\chi_0^B &= (1, 1, 1, 1), & \chi_4^B &= (1, 1, 1, -1), \\ \chi_5^B &= (1, -1, 1, 1), & \chi_6^B &= (1, -1, 1, -1).\end{aligned}\tag{8}$$

Multiplying the four characters in (7) with those in (8) we obtain all the 16 characters in $\widehat{B}[2]$.

Now we construct an abelian surface A with a symmetric $(1, 2)$ -polarization of special type (and which is not a product polarization) as a double cover of B . We start by a sublattice $\Lambda_A \subset \Lambda_B$ of index 2, and then we take $A := \mathbb{C}^2 / \Lambda_A$. For instance we may consider

$$\Lambda_A := \lambda_1 \mathbb{Z} \oplus (\lambda_1 + \lambda_2) \mathbb{Z} \oplus (\mu_1 - \mu_2) \mathbb{Z} \oplus 2\mu_2 \mathbb{Z}.$$

The alternating form E_B induces an alternating form $E_A: \Lambda_A \times \Lambda_A \rightarrow \mathbb{Z}$ given by

$$E_A(\lambda_1, \mu_1 - \mu_2) = 1, \quad E_A(\mu_1 - \mu_2, \lambda_1) = -1, \quad E_A(\lambda_1 + \lambda_2, 2\mu_2) = 2, \quad E_A(2\mu_2, \lambda_1 + \lambda_2) = -2$$

and all the other values are zero. This defines a symmetric $(1, 2)$ -polarization $\mathcal{L} = \mathcal{O}_A(L)$ on A , such that

$$K(\mathcal{L}) = \left\langle \mu_2, \frac{\lambda_1 + \lambda_2}{2} \right\rangle.$$

We indicate a character $\chi^A: \Lambda_A \rightarrow \mathbb{C}^*$ by the vector

$$(\chi^A(\lambda_1), \chi^A(\lambda_1 + \lambda_2), \chi^A(\mu_1 - \mu_2), \chi^A(2\mu_2)).$$

The degree 2 isogeny $\psi: A \rightarrow B$ induces a degree 2 isogeny $\widehat{\psi}: \widehat{B} \rightarrow \widehat{A}$, obtained by restriction of the characters $\chi^B: \Lambda_B \rightarrow \mathbb{C}^*$ to the sublattice Λ_A . In other words, $\widehat{\psi}(\chi^B) = \chi^A$, where χ^A is defined by

$$\begin{aligned}\chi^A(\lambda_1) &= \chi^B(\lambda_1), \\ \chi^A(\lambda_1 + \lambda_2) &= \chi^B(\lambda_1 + \lambda_2) = \chi^B(\lambda_1)\chi^B(\lambda_2), \\ \chi^A(\mu_1 - \mu_2) &= \chi^B(\mu_1 - \mu_2) = \chi^B(\mu_1)\chi^B(\mu_2)^{-1}, \\ \chi^A(2\mu_2) &= \chi^B(2\mu_2) = \chi^B(\mu_2)^2.\end{aligned}\tag{9}$$

By (9) it follows immediately that $\ker \widehat{\psi}$ is the group of order 2 generated by $\chi_1^B \chi_4^B = (1, 1, -1, -1)$. Notice that $\chi_1^B \chi_4^B$ is the character $\exp(2\pi i E_B(\cdot, \frac{\lambda_1 + \lambda_2}{2}))$, that is the image of $\frac{\lambda_1 + \lambda_2}{2}$ via the isomorphism $B \rightarrow \widehat{B}$. On the other hand, the generator of $\ker \widehat{\psi}$ corresponds to the 2-torsion line bundle on B inducing the étale double cover $A \rightarrow B$; since it is not a pullback of a line bundle from E_1 or from E_2 , it follows that the $(1, 2)$ -polarization \mathcal{L} on A is not of product type.

We can also give the following interpretation of the 16 characters $\Lambda_A \rightarrow \{\pm 1\}$ corresponding to the 2-torsion line bundles on A : eight of them arise from the 2-torsion line bundles on B , namely

$$\begin{aligned}\chi_0^A &= (1, 1, 1, 1), & \chi_1^A &= (1, 1, -1, 1) \\ \chi_2^A &= (-1, -1, 1, 1), & \chi_3^A &= (-1, -1, -1, 1), \\ \chi_5^A &= (1, -1, 1, 1), & \chi_1^A \chi_5^A &= (1, -1, -1, 1), \\ \chi_2^A \chi_5^A &= (-1, 1, 1, 1), & \chi_3^A \chi_5^A &= (-1, 1, -1, 1),\end{aligned}$$

whereas the remaining eight are the images in \widehat{A} of the 16 square roots of the generator of $\ker \widehat{\psi}$:

$$\begin{aligned} \varepsilon_1 &= (1, 1, 1, -1), & \varepsilon_2 &= (1, 1, -1, -1) \\ \varepsilon_3 &= (1, -1, 1, -1), & \varepsilon_4 &= (1, -1, -1, -1), \\ \varepsilon_5 &= (-1, -1, 1, -1), & \varepsilon_6 &= (-1, -1, -1, -1), \\ \varepsilon_7 &= (-1, 1, 1, -1), & \varepsilon_8 &= (-1, 1, -1, -1). \end{aligned}$$

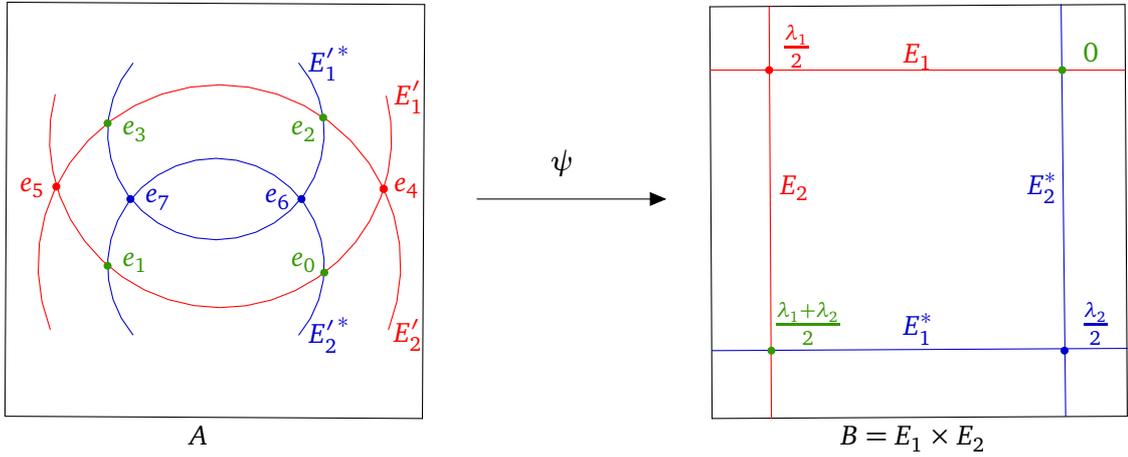
Consider now the isogeny induced by \mathcal{L} , namely

$$\phi = \phi_{\mathcal{L}}: A \longrightarrow \widehat{A}, \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then the restricted map $\phi_2: A[2] \longrightarrow \widehat{A}[2]$ satisfies $\ker \phi_2 = K(\mathcal{L})$. The image of ϕ_2 is generated by $\phi_2(\frac{\lambda_1}{2})$ and $\phi_2(\frac{\mu_1 - \mu_2}{2})$. Since the line bundle $\phi(x)$ corresponds to the character $\exp(2\pi i E_A(\cdot, x))$, by simple computations we obtain

$$\text{im } \phi_2 = \{\chi_0^A, \chi_1^A, \chi_2^A \chi_5^A, \chi_3^A \chi_5^A\}.$$

The complete linear system $|\mathcal{L}|$ contains exactly two reducible divisors $E'_1 + E'_2$ and $E_1^* + E_2^*$, which are the pullback via ψ of two reducible divisors $E_1 + E_2$ and $E_1^* + E_2^*$ on B ; moreover $E_1^* + E_2^*$ is the translated of $E'_1 + E'_2$ by the point $\frac{\lambda_1 + \lambda_2}{2}$, see Figure 2.



$$\begin{aligned} e_0 &= 0, \quad e_1 = \mu_2, \quad e_2 = \frac{\lambda_1 + \lambda_2}{2}, \quad e_3 = \mu_2 + \frac{\lambda_1 + \lambda_2}{2}, \\ e_4 &= \frac{\lambda_1}{2}, \quad e_5 = \mu_2 + \frac{\lambda_1}{2}, \\ e_6 &= \frac{\lambda_2}{2}, \quad e_7 = \mu_2 + \frac{\lambda_2}{2}. \end{aligned}$$

Figure 2: The degree 2 isogeny $\psi: A \longrightarrow B$.

Lemma 2.15. Fix $x \in A$. Then the linear system $|t_x^* \mathcal{L}|$ contains a reducible element D passing through 0 if and only if we are in one of the following cases:

- (1) $\psi(x) = (y_1, 0)$;
- (2) $\psi(x) = (y_1, \frac{\tau_2}{2})$;
- (3) $\psi(x) = (0, y_2)$;
- (4) $\psi(x) = (\frac{\tau_1}{2}, y_2)$,

where we use the notation $(y_1, y_2) \in B = E_1 \times E_2$.

Proof. We refer again to Figure 2. The curve $\psi(D) \subset B$ is a translate of $E_1 + E_2$ containing 0. This means that either E_1 or E_2^* must be a component of $\psi(D)$. In the former case, either E_1 is fixed by the translation by $\psi(x)$ (case (1)) or such a translation sends E_1^* to E_1 (case (2)). In the latter case, either E_2^* is fixed by the translation by $\psi(x)$ (case (3)) or such a translation sends E_2 to E_2^* (case (4)). This completes the proof. \square

Proposition 2.16. *There exists a reducible curve $D \in |\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ such that 0 is an ordinary double point for D if and only if $\mathcal{Q} = \mathcal{O}_A$ and $\mathcal{Q}^{1/2}$ corresponds to the character χ_1^A .*

Proof. If 0 is an ordinary double point for D , looking at Figure 2 we see that $|\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ has to be in the linear system containing the translate of $E_1' + E_2'$ by $\frac{\lambda_1}{2}$ (or, which is the same, the translate of $E_1'^* + E_2'^*$ by $\frac{\lambda_2}{2}$). In other words we must have $\mathcal{L} \otimes \mathcal{Q}^{1/2} = t_{\lambda_1/2}^* \mathcal{L}$. By [BL04, Lemma 2.3.2] it follows that $\mathcal{Q}^{1/2}$ corresponds to the character $\exp(E_A(\cdot, \frac{\lambda_2}{2}))$, which is precisely χ_1^A . Notice that, in the notation of Lemma 2.15, this situation corresponds to either case (1) with $\psi(x) = (\frac{\tau_1}{2}, 0)$ or to case (3) with $\psi(x) = (0, \frac{\tau_2}{2})$. \square

Proposition 2.17. *Take $\mathcal{Q} \in \text{im } \phi_2$. Then there exists a reducible curve $D \in |\mathcal{L} \otimes \mathcal{Q}^{1/2}|$ such that 0 is a smooth point of D if and only if we are in one of the following two cases:*

- (a) $\mathcal{Q} = \mathcal{O}_A$ and $\mathcal{Q}^{1/2}$ corresponds to one of the four characters $\chi_2^A, \chi_3^A, \chi_5^A, \chi_1^A \chi_5^A$;
- (b) \mathcal{Q} corresponds to the character $\chi_1^A \in \text{im } \phi_2^\times$.

Proof. If a curve as in the statement exists, by Lemma 2.15 we see that $\mathcal{L} \otimes \mathcal{Q}^{1/2} = t_x^* \mathcal{L}$, where either $\psi(x) = (\frac{\tau_1}{2}, y_2)$ or $\psi(x) = (y_1, \frac{\tau_2}{2})$ and y_i has either order 2 or order 4. Now a tedious but straightforward computation shows that if y_i has order 2 then $x \in \{\frac{\mu_1}{2}, \frac{\lambda_1 + \mu_1}{2}, \frac{\mu_2}{2}, \frac{\lambda_2 + \mu_2}{2}\}$ and we are in case (a), whereas if y_i has order 4 we are in case (b). \square

Remark 2.18. Using the terminology of [PP13b] (explained in Proposition 2.7), the cases in Proposition 2.16 belong to *surfaces of type Ib*, those in Proposition 2.17, (a) belong to *surfaces of type Ia* and those in Proposition 2.17, (b) to *surfaces of type II*.

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