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A methodology for modelling the flow regime in unsaturated infinite slopes

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ABSTRACT: The paper presents a novel methodology to calculate the flow regime in unsaturated infinite slopes by means of horizontal ground infiltration models. This is achieved by decomposing the flow regime of the infinite slope into a symmetric part and an antisymmetric part, whose respective solutions are then combined to determine the actual seepage. The solution of the antisymmetric part is trivial and does not even need integration of the continuity equation while the symmetric part reduces to the case of one-dimensional vertical infiltration into a horizontal soil deposit, for which analytical/numerical solutions exist in the literature. The idealised geometry implies that all surface infiltration crosses the slope along the shortest path perpendicular to the ground, while the flux parallel to the slope is fed by an upstream source at infinite distance. The paper also calculates the expression of the piezometric head gradient parallel to the slope, which is the Neumann boundary condition to be imposed on the upstream and downstream sides of any soil slice perpendicular to the inclined ground. This avoids the use of arbitrary boundary conditions in finite element/difference models as done in current practice. The proposed methodology is finally validated against analytical and numerical solutions of transient seepage across an infinite unsaturated slope.

Keywords: Groundwater; Slopes; Partial saturation; Finite Difference; FEM

1 INTRODUCTION

The calculation of the flow regime across unsaturated soils presents additional complexities compared to the saturated case because of the dependency of permeability and degree of saturation on pore water pressure through constitutive laws, which introduce a degree of non-linearity in the governing differential equations. These equations are solved via closed-form derivations or, if this is not possible, via approximated numerical models.

Rigorous analytical solutions of the monodimensional vertical flow across a horizontal unsaturated soil deposit have been obtained for different constitutive laws and boundary conditions, e.g. Huang and Wu, 2012; Lu and Griffiths, 2004; Tracy, 2011; Wang et al., 2009. However, the same is not true for the bidimensional flow across an infinite unsaturated slope, which has been rigorously solved only in few specific cases (Lu and Godt, 2008; Travis et al., 2010, Zhan et al., 2013). In those cases where a rigorous solution is unavailable, the unsaturated flow regime is usually evaluated via finite element or finite difference models of a relatively long segment of the infinite slope (El Shamy, 2007), whose upstream and downstream sides are subjected to arbitrary boundary conditions (typically hydrostatic or impermeable). The inaccuracies introduced by this artefact are minimised by calculating the solution at the mid-section of the slope segment, which is the farthest section from the extremities where the arbitrary boundary conditions have been imposed.

To overcome some of the above limitations, this paper presents a general methodology that enables the rigorous calculation of the bidimensional flow across an unsaturated infinite slope, provided that a rigorous solution already exists for the parent case of a horizontal soil deposit. If such parent solution is unavailable and a numerical model must therefore be created, the present paper defines the Neumann boundary condition to be imposed on the upstream and downstream sides of a slope slice perpendicular to the ground.

The proposed methodology relies on the decomposition of the actual seepage problem into two, namely antisymmetric and symmetric, sub-problems (Bianchi et al., 2022). The antisymmetric problem is immediately solved, without even integrating the governing partial differential equations, as the pore pressure field must be zero over the entire slope domain. Conversely, the symmetric problem reduces to the case of monodimensional flow across a horizontal soil deposit, for which several analytical and numerical solutions are already available. The antisymmetric and symmetric problems are then combined to obtain the actual flow field.

The infinite geometry of the slope also allows a physical interpretation of the flow regime by distinguishing between the seepage components parallel and perpendicular to the ground. All surface infiltration crosses the slope along the shortest path, that is the path perpendicular to the ground, while the seepage parallel to the ground is fed by an upstream source located at infinite distance.

2 DECOMPOSITION OF THE FLOW REGIME

Bianchi et al. (2022) have demonstrated that the flow regime across an infinite slope can be decomposed into antisymmetric and symmetric parts, which are separately solved and then combined to compute the actual hydraulic field. Bianchi et al. (2022) have also validated this decomposition methodology for the case of steady state flow while the present paper validates it for the case of transient flow.

It is useful to recall first the main aspects of the proposed decomposition methodology. Figure 1 provides a schematic representation of an infinite homogeneous unsaturated slope of thickness L (measured normal to the surface) and angle β with respect to the horizontal. The infinite slope is subjected to a constant pore water pressure u^b at the bottom and to either a constant pore water pressure u^t or a constant infiltration rate normal to the ground q^t at the top.

The unsaturated soil permeability K is defined as:

$$K = \kappa_r \, K^{sat} \tag{1}$$

where κ_r is the relative permeability function, accounting for the dependency of the unsaturated hydraulic conductivity on the pressure head u/γ_w (where u is the pore water pressure and $\gamma_w = 10\,kN/m^3$ is the specific water weight) and K^{sat} is the constant saturated permeability.

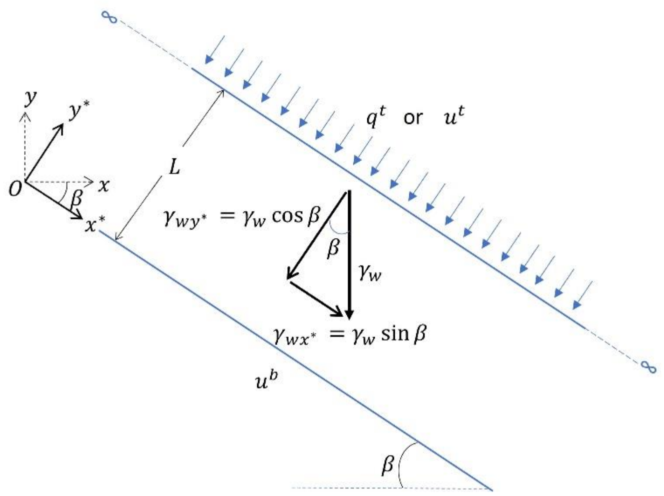

Figure 1. Unsaturated infinite slope and hydraulic boundary conditions

The saturated permeability is further expressed in terms of the constant intrinsic permeability κ, the specific water weight γ_w and the dynamic water viscosity μ as:

$$K^{sat} = \frac{\kappa\,\gamma_w}{\mu} \tag{2}$$

The decomposition of the specific water weight γ_w into two components parallel $\gamma_{w_{x^*}}$ and perpendicular $\gamma_{w_{y^*}}$ to the slope (Figure 1) enables the separation of the flow regime into antisymmetric and symmetric parts. These two parts can be individually solved and subsequently combined to calculate the overall flow regime, as explained in the following.

2.1 *Antisymmetric flux components*

The antisymmetric seepage is governed by the specific water weight component parallel to the slope $\gamma_{w_{x^*}}$ shown in Figure 1 (see also Bianchi et al., 2022). In this case, every section perpendicular to the slope constitutes an antisymmetry axis, which implies that the flow field is always parallel to the ground and the corresponding pore water pressure u^{asym} is zero everywhere. Consequently, the antisymmetric piezometric head h^{asym} is defined as:

$$h^{asym} = -x^* + \frac{u^{asym}}{\gamma_{wx^*}} = -x^* \tag{3}$$

where the minus sign is introduced because the component of specific water weight parallel to the slope $\gamma_{w_{x^*}}$ has the same direction of the x^* axis.

The antisymmetric permeability K^{asym} is subsequently evaluated from Equation (1) by introducing inside Equation (2) the component of the specific water weight parallel to the slope $\gamma_{w_{x^*}}$ as:

$$K^{asym} = \kappa_r \frac{\kappa\,\gamma_{wx^*}}{\mu} = \kappa_r \frac{\kappa\,\gamma_w}{\mu} \sin\beta = K \sin\beta \tag{4}$$

Finally, based on Equations (3-4), the two flux components $q_{x^*}^{asym}$ and $q_{y^*}^{asym}$ are calculated according to Darcy's law as:

$$q_{x^*}^{asym} = -K^{asym}\frac{\partial h^{asym}}{\partial x^*} = -K\sin\beta\,\frac{\partial h^{asym}}{\partial x^*} = = K\sin\beta \tag{5}$$

$$q_{y^*}^{asym} = -K^{asym}\frac{\partial h^{asym}}{\partial y^*} = -K\sin\beta\,\frac{\partial h^{asym}}{\partial y^*} = 0 \tag{6}$$

2.2 *Symmetric flux components*

The symmetric seepage is governed by the component of the specific water weight perpendicular to the slope $\gamma_{w_{y^*}}$ shown in Figure 1 (see also Bianchi et al., 2022).

The problem is therefore equivalent to that of vertical infiltration across a horizontal soil deposit of thickness L and infinite extension in the x^* direction. Therefore, every section perpendicular to the slope is a symmetry axis, which implies that the flow field is perpendicular to the ground and the pore water pressure u^{sym} depends only on the y^* coordinate. Therefore, the symmetric piezometric head h^{sym} is evaluated as:

$$h^{sym} = y^* + \frac{u^{sym}}{\gamma_{wy^*}} = y^* + \frac{u^{sym}}{\gamma_w \cos\beta} \tag{7}$$

As mentioned above, the symmetric permeability K^{sym} is evaluated from Equation (1) by introducing inside Equation (2) the component of the specific water weight perpendicular to the slope $\gamma_{w_{y^*}}$ as:

$$K^{sym} = \kappa_r \frac{\kappa\,\gamma_{wy^*}}{\mu} = \kappa_r \frac{\kappa\,\gamma_w}{\mu} \cos\beta = K\cos\beta \tag{8}$$

Finally, based on Equations (7-8), the flux components $q_{x^*}^{sym}$ and $q_{y^*}^{sym}$ are calculated according to Darcy's law as:

$$q_{x^*}^{sym} = -K^{sym}\,\frac{\partial h^{sym}}{\partial x^*} = -K\cos\beta\,\frac{\partial h^{sym}}{\partial x^*} = 0 \tag{9}$$

$$q_{y^*}^{sym} = -K^{sym}\,\frac{\partial h^{sym}}{\partial y^*} = -K\cos\beta\,\frac{\partial h^{sym}}{\partial y^*} = \\ = -K\cos\beta\left(1 + \frac{\partial\left(\frac{u^{sym}}{\gamma_w \cos\beta}\right)}{\partial y^*}\right) \tag{10}$$

2.3 *Flow regime in the unsaturated infinite slope*

The flux parallel to the slope q_{x^*} coincides with the antisymmetric part $q_{x^*}^{asym}$ (Equation 5) because the symmetric part $q_{x^*}^{sym}$ (Equation 9) is zero:

$$q_{x^*} = q_{x^*}^{asym} = K\sin\beta \tag{11}$$

Similarly, the flux perpendicular to the slope q_{y^*} coincides with the symmetric part $q_{y^*}^{sym}$ (Equation 10) because the antisymmetric part $q_{y^*}^{asym}$ (Equation 6) is zero:

$$q_{y^*} = q_{y^*}^{sym} = -K\cos\beta\left(1 + \frac{\partial\left(\frac{u^{sym}}{\gamma_w \cos\beta}\right)}{\partial y^*}\right) \tag{12}$$

Moreover, the fluxes parallel and perpendicular to the slope, q_{x^*} and q_{y^*}, can be alternatively calculated according to Darcy's law from the corresponding gradients of the actual piezometric head h as:

$$q_{x^*} = -K\frac{\partial h}{\partial x^*} \tag{13}$$

$$q_{y^*} = -K\frac{\partial h}{\partial y^*} \tag{14}$$

By equating Equation (11) to Equation (13) and Equation (12) to Equation (14), the components of the gradient of the actual piezometric head h in the direction parallel and perpendicular to the slope are calculated as:

$$\frac{\partial h}{\partial x^*} = -\sin\beta \tag{15}$$

$$\frac{\partial h}{\partial y^*} = \cos\beta\left(1 + \frac{1}{\cos\beta}\,\frac{\partial\left(\frac{u^{sym}}{\gamma_w}\right)}{\partial y^*}\right) \tag{16}$$

Equation (15) corresponds to the hydraulic constraint acting on any section perpendicular to the slope and constitutes a Neumann boundary condition for such sections. It is easily shown that the same boundary condition can be alternatively expressed in terms of the pressure head u/γ_w as:

$$\frac{\partial\left(\frac{u}{\gamma_w}\right)}{\partial x^*} = 0 \tag{17}$$

Integration of Equations (15-16) leads to the following expression of the actual piezometric head h:

$$h = -x^*\sin\beta + y^*\cos\beta + \frac{u^{sym}}{\gamma_w} \tag{18}$$

Given that the geometric component of the piezometric head is $y = -x^*\sin\beta + y^*\cos\beta$, Equation (18) implies that the symmetric pore pressure field coincides with the actual pore pressure field, i.e. $u^{sym} = u$.

By considering Equations (11-12) and recalling that $u^{sym} = u$, the water mass balance is expressed in the reference system (O, x^*, y^*) as:

$$\frac{\partial\theta}{\partial t} = -\frac{\partial q_{x^*}}{\partial x^*} - \frac{\partial q_{y^*}}{\partial y^*} = \\ = -\frac{\partial}{\partial x^*}[K\sin\beta] + \frac{\partial}{\partial y^*}\left[K\cos\beta\left(1 + \frac{\partial\left(\frac{u}{\gamma_w \cos\beta}\right)}{\partial y^*}\right)\right] \tag{19}$$

where θ is the volumetric water content.

The first term on the right-hand side of Equation (19) is, however, zero because the flux parallel to the slope q_{x^*} coincides with the antisymmetric part $q_{x^*}^{asym}$ which does not vary along the x^* coordinate. Equation (19) can therefore be simplified to:

$$\frac{\partial\theta}{\partial t} = -\frac{\partial q_{y^*}}{\partial y^*} = \frac{\partial}{\partial y^*}\left[K\cos\beta\left(1 + \frac{\partial\left(\frac{u}{\gamma_w \cos\beta}\right)}{\partial y^*}\right)\right] \tag{20}$$

Equation (20) defines the monodimensional variation of the pore water pressure along the y^* direction and coincides with Richards' vertical infiltration equation across a horizontal soil deposit with the permeability and specific water weight equal to the symmetric components $K\cos\beta$ and $\gamma_w \cos\beta$, respectively. It is therefore possible to exploit published numerical or analytical solutions of the vertical infiltration across a horizontal soil deposit for calculating the seepage across an infinite slope by replacing the permeability and specific water weight with their symmetric counterparts. The validity of this approach has already been demonstrated by Bianchi et al. (2022) for the case of steady state flow across an unsaturated infinite slope. The present work extends the validation to the case of transient seepage, as discussed in the following.

3 VALIDATION OF DECOMPOSITION METHODOLOGY FOR TRANSIENT FLOW

In order to solve the water balance of Equation (19) or Equation (20) in a closed form, the following simple constitutive laws, which assume an exponential decay of permeability K and volumetric water content θ with decreasing pore pressure head u/γ_w, are introduced (recall that, under partly saturated conditions, the tensile pore water pressure has negative sign):

$$K = K^{sat}\,\kappa_r = K^{sat}\,e^{\alpha\,\gamma_w \frac{u}{\gamma_w}} \tag{21}$$

$$\theta = \theta^d + (\theta^s - \theta^d)\,e^{\alpha\,\gamma_w \frac{u}{\gamma_w}} \tag{22}$$

In Equation (22), θ^d is the residual volumetric water content corresponding to the driest conditions at an infinite negative pore water pressure while θ^s is the saturated volumetric water content at zero pore water pressure. For simplicity, the exponential decay of both permeability (Equation (21)) and volumetric water content (Equation (22)) with decreasing pressure head is governed by a single material parameter α.

The proposed decomposition methodology is then validated for the case of transient seepage across an unsaturated infinite slope of thickness $L = 5\,m$ and inclination $\beta = 30°$. The top and bottom boundaries are subjected to a constant infiltration rate $q^t = 0.5 \cdot 10^{-6}\,m/s$ and a fixed pore water pressure head $u^b/\gamma_w = 0\,m$, respectively, while the initial pressure head across the domain is calculated as $u/\gamma_w = -y^*$. The corresponding initial values of permeability K and volumetric water content θ are evaluated by Equation (21) and Equation (22), respectively, with parameter values α=$10^{-2}kPa^{-1}$, $K^{sat} = 10^{-6}\,m/s$, $\theta^s = 0.4$ and $\theta^d = 0.04$.

The problem of transient seepage has been solved via a monodimensional finite difference model of Richards' equation where the permeability and specific water weight have been replaced with the symmetric components $K\cos\beta$ and $\gamma_w \cos\beta$, respectively. The finite difference model adopts a spatial discretization of $\Delta y^* = 0.01\,m$ (Figure 2) and a time discretization of $\Delta t = 0.0028\,h$ (Celia et al., 1990). The same problem has also been solved via a bidimensional finite element model of Richards' equation where the permeability and specific water weight coincide with the actual values K and γ_w. The bidimensional finite element model has been created with the commercial software COMSOL Multiphysics and consists of a slope slice of unit width perpendicular to the ground (Figure 3). In this case, an automatic adaptive time step has been chosen, which is based on a local error estimate with a tolerance of 10^{-8}. As depicted in Figure 3, the finite element model has been discretized with a very fine triangular mesh consisting of 5448 elements and 2850 nodes. The Neumann boundary condition of Equation (15) has been imposed on the upstream and downstream sides of the discretised slope slice.

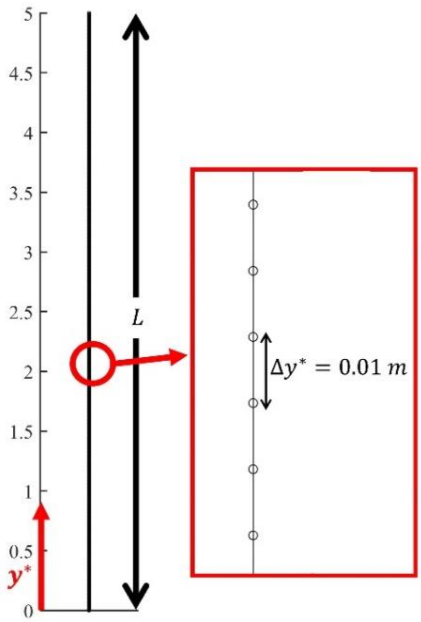

Figure 2. Monodimensional finite difference model

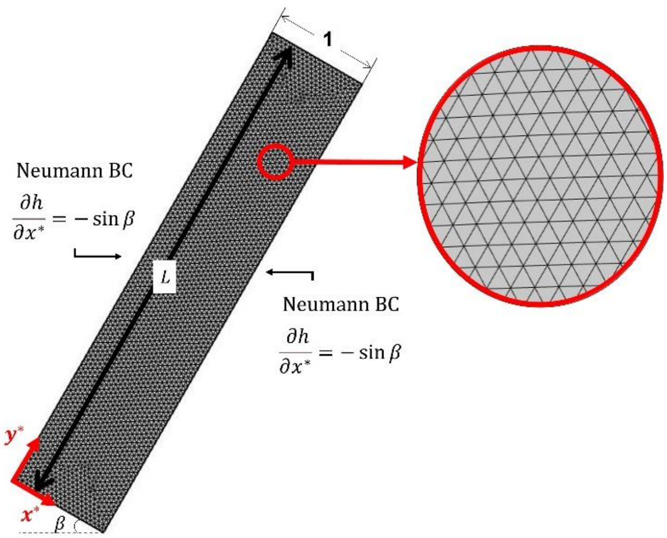

Figure 3. Bidimensional finite element model

Figure 4 shows a perfect match between the pore water pressure profiles evaluated along the direction y^* perpendicular to the ground at different times by both the monodimensional finite difference scheme (red lines) and the bidimensional finite element scheme (yellow diamonds). The two analyses agree at all times up until the final steady state, when they both match the analytical solution (white circle) presented in Bianchi et al. (2022). This confirms the validity of the proposed decomposition methodology, including the correctness of the Neumann boundary condition of Equation (15), for calculating the transient seepage across an infinite slope.

Moreover, Figure 5 compares the flux vectors calculated by the monodimensional and bidimensional models at two different times. The monodimensional model only calculates the flux component perpendicular to the slope, which is then added to the flux component parallel to the slope, as obtained from Equation (11), to define the flux vector. Inspection of Figure 5 indicates that the direction and intensity of the flux vectors calculated by the two models are identical, thus further corroborating the validity of the proposed decomposition methodology.

Inspection of Figure 5 also indicates that, consistently with Equation (12), the magnitude of the flux vector is lowest around the mid-depth of the slope where both the pressure head gradient and the permeability, which reduces with decreasing values of pressure head, are smallest as shown in Figure 4.

Near the surface, the flux perpendicular to the slope dominates the seepage, which is therefore mostly fed by ground infiltration. However, as depth increases, the flux vectors progressively rotate becoming nearly parallel to the slope at the bottom boundary, where seepage is therefore mostly fed by the upstream source at infinite distance.

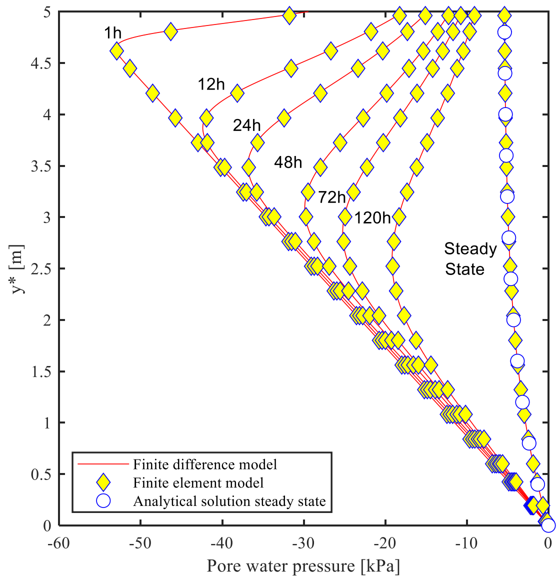

Figure 4. Comparison of pore water pressure profiles perpendicular to the ground calculated at different times by the monodimensional finite difference model and the bidimensional finite element model

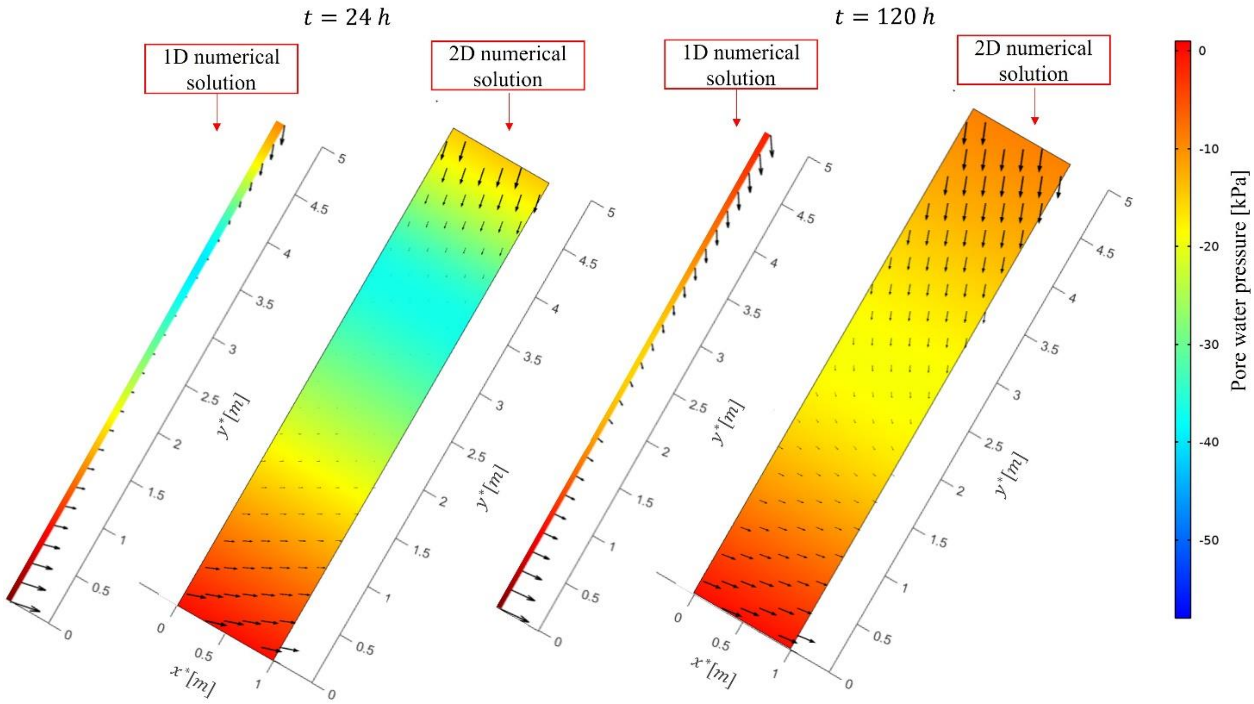

Figure 5. Comparison of the flux vector fields calculated at two different times by the monodimensional finite difference model and the bidimensional finite element model

The above results confirm, once again, that the flow regime across an infinite slope can be alternatively evaluated by either a simplified monodimensional model with values of specific water weight and permeability scaled by $\cos\beta$ (where β is the slope angle) or by a full bidimensional model subjected to appropriate upstream and downstream boundary conditions.

4 CONCLUSIONS

This paper has presented a simplified methodology for evaluating the seepage across an unsaturated infinite slope under both transient and stationary conditions. It is demonstrated that the flow perpendicular to the slope can be calculated from Richards' equation of vertical infiltration into a horizontal soil layer where the specific water weight and permeability have been scaled by a factor of $\cos\beta$ (where β is the slope angle). The problem of vertical infiltration into a horizontal soil layer has already been rigorously solved for various soil constitutive laws and boundary conditions. It is therefore possible to adapt these analyses to the evaluation of the flow regime across an infinite slope by simply replacing, inside the corresponding solutions, the specific water weight and permeability with their scaled counterparts.

If a rigorous solution is unviable, the seepage across an infinite unsaturated slope can be calculated via approximated numerical models, typically finite difference or finite element models. Either a monodimensional model incorporating the scaled values of specific water weight and permeability or a bidimensional model incorporating the actual values of these two parameters may be employed. Bidimensional models of a slope segment require, however, the imposition of appropriate boundary conditions on the upstream and downstream sides of the discretised domain. To this end, the present paper has defined the gradient of the piezometric head in the direction parallel to the slope, which is the Neumann boundary condition acting on the two sides perpendicular to the ground. The rigorous definition of this boundary condition overcomes the current need of discretizing large slope segments to minimise the inaccuracies due to the imposition of arbitrary boundary conditions on the upstream and downstream sides of the model.

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