Interphase zone effect on the spherically symmetric elastic response of a composite material reinforced by spherical inclusions.

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Abstract

This work deals with the problem of modelling the effect due to an interphase zone between inclusion and matrix in particulate composites to better estimate the bulk modulus of materials with inclusions. To this end, in this paper the problem of a body containing a hollow or solid spherical inclusion subjected to a spherically symmetric loading is investigated in the framework of the elasticity theory. The interphase zone around the inclusion is modeled by considering the elastic properties varying with the radius moving away from the interface with inclusion and, asymptotically approaching the value of the homogeneous matrix. The explicit solutions are obtained in closed form by using hypergeometric functions and numerical investigations are performed to highlight the localised effects of the graded interphase in the stress transfer between inclusion and matrix. Finally, the exact solutions are used to estimate the effective bulk modulus of a material containing a dispersion of hollow or solid spherical inclusions with graded interphase zone.

Keywords: Elasticity; Particle reinforced-composites; Bulk modulus; Spherical inclusions.

1 Introduction

In many particulate-reinforced composites the interface between the matrix and the inclusion plays an important role to estimate their effective elastic properties. From the first papers in which inclusion/matrix interface are assumed perfectly bonded together, a number of researchers have attempted to account for interface effects by using analytical and numerical approaches. Furthermore, in some cases, the inclusions are surrounded by thin interface layers whose thickness is usually much smaller than the inclusion sizes and, as a consequence, the properties of the interface layer do not significantly affect the elastic constant of the composite; on the contrary, if the interface thickness is comparable with the inclusion size, the effect of the interface zone properties may be substantial on the evaluation of the elastic properties of the composite: i.e. nanoparticle reinforced materials (Sevostianov and Kachanov, 2007), hollow particle filled composites (Tagliavia et al., 2011), as well as concrete (Lutz and Zimmerman, 1996a).

In 1964 Hashin and Rosen developed a model to take into account the interphase effects, by considering an homogenous interphase zone around the inclusion with different elastic properties from those of matrix or inclusion; but, starting from 1990, some authors presented models in which a distinct inhomogeneous interphase zone with step by step (Herve and Zaoui 1993) or smooth variation of the elastic moduli is introduced (Jayaraman and Reifsnider, 1992). Indeed, Lutz and Zimmerman (1996a) investigated the transfer of the stress between solid spherical inclusion and matrix to predict the bulk modulus of the composite, modelling the interphase as a layer with elastic properties variable in the radial direction but with a smooth transition between interphase and matrix. A similar approach is also used by authors to estimate the thermal/electrical conductivity of particulate composites that contain a dispersion of solid spherical inclusions (Lutz and Zimmerman, 1996b; Lutz and Zimmerman, 2005).

Though great of attention has been paid to the study of solid inclusions with interphase zone (Wang and Jasiuk, 1998; Shen and Li, 2003, 2005); starting from 1970, the demand for light weight and high strength materials have increased the use of micro hollow spheres (Lee and Westmann, 1970; Huang and Gibson, 1993). Some papers are devoted to estimating the elastic properties of hollow-sphere-reinforced composite with perfect interface (Bardella and Genna, 2001; Marur, 2005; Porfiri and Gupta, 2009) or imperfect interface (Tagliavia et al. 2011; Marur, 2014). But, experimental results and application to nanocomposites also suggest investigating the effects of an inhomogeneous interphase zone around inclusion which, when the inclusion geometry is comparable with the thickness of the interphase, allows to a stress concentration around the inclusion that may affect the elastic properties of the composite (Shen and Li, 2005; Sevostianov and Kachanov, 2007). As far as the author knows, in the literature are not present elastic analytical solutions in closed form for spherical solid or hollow inclusions taking into account graded interphase effects.

This work deals with the problem introduced above. In particular, in this paper an elastic analytical solution is developed in closed form for the problem of hydrostatic pressure of a homogeneous body containing a hollow spherical inclusion, and the solution for solid inclusion is determined as a consequence. In section 2, assuming spherically symmetric loading, the mathematical model is formulated with the introduction of an inhomogeneous interphase around the inclusion to describe the transition zone between the inclusion and the matrix; in other words, this interphase is considered as a functionally graded material (FGM) with radial variation of the elastic moduli from the centre of the inclusion that asymptotically assumes the homogeneous elastic properties of the matrix. Similar elastic solutions are obtained by the author to study both the effects of a FGM thin layer on the inner surface of a cylinder under pressure (Sburlati, 2012) and how the stress concentration factor is altered by a layer around a hole in a plate subjected to uniaxial load (Sburlati, 2013). In section 3 the analytical solution is obtained in the framework of the elasticity theory by applying the theory of hypergeometric functions (Abramovitz and Stegun, 1964; Erdelyi, 1953). The explicit solution is written in closed form in section 4 and permit us to understand the manner in which the interphase affects the stress transfer between spherical inclusions and matrix. The solution is also written in detail for solid inclusion without interphase. In section 5 the in closed-form analytical solution is used to obtain effective bulk modulus of the equivalent composite, by adopting the energy approach (Willis, 1981). Numerical examples are performed in section 6 to highlight the effects of the geometric and physical properties of the interphase on the stress transfer and the bulk modulus.

The results obtained in the paper may be the starting point to predict other elastic properties of the equivalent homogeneous composite as the thermal expansion coefficient and the conductivity. Furthermore, we remark that the interphase region may describe phenomena occurring during the material processing stage but the understanding of the role of the interphase may also permit us to tailor the interphase properties around the inclusions in order to enhance the mechanical properties of composites (i.e. Paskaramoorthy et al. 2009; Batra, 2011; Yao et al. 2013; Zhang Y. et al. 2013).

2 Mathematical model

In this section a two-phase model is introduced to evaluate the effects of a graded interphase on the stress transfer mechanism between a spherical inclusion and a matrix in a particulate composite subjected to a spherically symmetric loading. We start to analyse the case of hollow spherical inclusion and then we obtain the solution for solid spherical inclusion.

The three-dimensional problem studied for the hollow spherical inclusion case is shown in Figure 1. A single, homogeneous, isotropic, spherical hollow inclusion of inner radius a and outer radius b is embedded in a matrix and subjected to a remote hydrostatic load p. The

matrix is modelled as an isotropic graded material in radial direction in order to describe an interphase zone of thickness t around the inclusion in which the elastic properties, moving away from the interface with the inclusion, asymptotically approach the constant values of the elastic properties of the matrix without inclusion. The interface between inclusion and interphase is assumed still distinct since the inhomogeneous region is restricted to the matrix region around the inclusion. Again, the graded interphase zone and the matrix are described with a power variation law in radial direction.

By using a spherical coordinate system $(0; r, \theta, \phi)$, the load symmetry permits us to reduce the problem to determine the radial displacement u and the radial and hoop stresses: $\sigma_r, \sigma_\theta = \sigma_\phi$. The stresses and the displacement in inclusion and in matrix will be index with the superscript (i) and (m) respectively. Then, the power laws for the Lamé moduli in the isotropic graded interphase/matrix region (m) are assumed as

$$\lambda(r) = \lambda_m + \bar{\lambda} \left(\frac{b}{r}\right)^{\beta}, \qquad \mu(r) = \mu_m + \bar{\mu} \left(\frac{b}{r}\right)^{\beta}, \qquad (2.1)$$

where

$$\bar{\lambda} = \lambda_{ip} - \lambda_m, \qquad \bar{\mu} = \mu_{ip} - \mu_m. \tag{2.2}$$

and the Lamé constants λ_m, μ_m are the asymptotic values of the matrix and λ_{ip}, μ_{ip} the values of the interphase elastic constants at the interface between inclusion and interphase (r = b). The parameter $\beta > 0$ is called the inhomogeneity parameter and permits us to control the interphase thickness, indeed high beta values correspond to small interphase thickness and viceversa. Furthermore, the sign of the quantities $\bar{\lambda}$ and $\bar{\mu}$ describe *hard* or *soft* interphases in which the elastic properties respectively decrease ($\bar{\lambda} > 0, \bar{\mu} > 0$) or increase ($\bar{\lambda} < 0, \bar{\mu} < 0$) away from inclusion/interphase interface.

To complete the mathematical problem, we write the boundary and the asymptotic conditions in the following form (Love, 1944):

$$\sigma_r^{(i)}(a) = 0, \tag{2.3}$$

$$\lim_{r \to \infty} \sigma_r^{(m)}(r) = -p.$$
(2.4)

where p > 0 is the remote hydrostatic pressure. Then, the interphase and the inclusion in r = b are assumed perfectly bonded together as

$$\sigma_r^{(i)}(b) = \sigma_r^{(m)}(b), \qquad u^{(i)}(b) = u^{(m)}(b).$$
(2.5)

3 Analytical solution

The field elasticity equation in terms of radial displacement, for inhomogeneous isotropic material with elastic properties described by equation (2.1) and (2.2) and subjected to a radially symmetric load, assumes the following form (Sokolnikoff, 1956):

$$\left(\left(\frac{r}{b}\right)^{\beta}L-1\right)u''(r)+\left(2\left(\frac{r}{b}\right)^{\beta}L+\beta-2\right)\frac{1}{r}u'(r)-\left(2\left(\frac{r}{b}\right)^{\beta}L-\bar{N}\right)\frac{1}{r^{2}}u(r)=0,$$
(3.1)

where

$$L = -\frac{2\mu_m + \lambda_m}{2\bar{\mu} + \bar{\lambda}}, \quad \bar{N} = \frac{2(2\bar{\mu} + \bar{\lambda}(1+\beta))}{2\bar{\mu} + \bar{\lambda}}.$$
(3.2)

and the quantity L may take positive (*soft* interphase) or negative (*hard* interphase) values while the quantity \bar{N} is always positive.

We remark that the second-order, linear, ordinary differential equation with variable coefficients assumes the same form as equation (10a) of Lutz and Zimmerman (1996a) and that these authors solved this equation for solid spherical inclusion by using the method of Frobenius series. Instead, in this paper we perform a suitable rewriting of (3.1) in order to obtain the solution in closed form using the hypergeometric functions (Erdelyi, 1953).

To this end, the following transformation

$$u(r) = \left(\frac{r}{b}\right)^{\frac{\beta-1}{2}} T(r), \qquad (3.3)$$

and the change of variable in the form

$$s(r) = \left(\frac{r}{b}\right)^{\beta} L, \qquad (3.4)$$

are introduced. Substituting equations (3.3) and (3.4), the equation (3.1) for L > 0 ($\mu_{ip} <$

 μ_m and $\lambda_{ip} < \lambda_m$), is written as

$$s(1-s)T''(s) + (1-2s)T'(s) - \left(\frac{1}{4} + \frac{G^2}{s} - \frac{9}{4\beta^2}\right)T(s) = 0,$$
(3.5)

while, for L < 0 ($\mu_{ip} > \mu_m$ and $\lambda_{ip} > \lambda_m$), instead it assumes the form

$$s(1+s)T''(s) + (1+2s)T'(s) + \left(\frac{1}{4} - \frac{G^2}{s} - \frac{9}{4\beta^2}\right)T(s) = 0,$$
(3.6)

where the always positive quantity is introduced as

$$G^{2} = \frac{1}{4} \left(1 - \frac{2}{\beta} + \frac{4\bar{N} + 1}{\beta^{2}} \right) > 0.$$
(3.7)

We observe that the equation (3.6) may be reduced to equation (3.5) by putting s = -s; in this way, the solution of the differential equations (3.5) and (3.6) may be written in a unique form as a linear combination of two hypergeometric functions, that we choose regular at infinity to be able to manage asymptotic boundary condition (Erdelyi, 1953, (see vol.1, p.75)). So we write the solution of (3.6) in the form of a linear combination of the following two functions:

$$T_{1}(s) = -s^{-\frac{3+\beta}{2\beta}} {}_{2}F_{1}\left(\frac{1}{2} - G + \frac{3}{2\beta}, \frac{1}{2} + G + \frac{3}{2\beta}; 1 + \frac{3}{\beta}; -s^{-1}\right),$$

$$T_{2}(s) = -s^{\frac{3-\beta}{2\beta}} {}_{2}F_{1}\left(\frac{1}{2} - G - \frac{3}{2\beta}, \frac{1}{2} + G - \frac{3}{2\beta}; 1 - \frac{3}{\beta}; -s^{-1}\right).$$
(3.8)

where ${}_{2}F_{1}(a, b; c; s)$ is the hypergeometric function (Abramovitz and Stegun, 1964 (see p.563 eqs.15.5.7-8)). Then, we introduce the following expressions

$$\Theta_{1}(r) = {}_{2}F_{1}\left(\frac{1}{2} - G + \frac{3}{2\beta}, \frac{1}{2} + G + \frac{3}{2\beta}; 1 + \frac{3}{\beta}; \frac{b^{\beta}}{r^{\beta}L}\right),
\Theta_{2}(r) = {}_{2}F_{1}\left(\frac{1}{2} - G - \frac{3}{2\beta}, \frac{1}{2} + G - \frac{3}{2\beta}; 1 - \frac{3}{\beta}; \frac{b^{\beta}}{r^{\beta}L}\right),
\Theta_{3}(r) = {}_{2}F_{1}\left(\frac{3}{2} - G + \frac{3}{2\beta}, \frac{3}{2} + G + \frac{3}{2\beta}; 2 + \frac{3}{\beta}; \frac{b^{\beta}}{r^{\beta}L}\right),
\Theta_{4}(r) = {}_{2}F_{1}\left(\frac{3}{2} - G - \frac{3}{2\beta}, \frac{3}{2} + G - \frac{3}{2\beta}; 2 - \frac{3}{\beta}; \frac{b^{\beta}}{r^{\beta}L}\right),$$
(3.9)

where, to help reading, we recall that

$$\Theta_3(r) = \frac{r^{\beta+1}}{b^\beta c_1 \beta} \Theta_1'(r), \qquad \Theta_4(r) = \frac{r^{\beta+1}}{b^\beta c_2 \beta} \Theta_2'(r), \qquad (3.10)$$

and, we also introduce the quantities

$$c_{1} = \frac{(2G\beta + \beta + 3)(2G\beta - \beta - 3)}{4\beta L(\beta + 3)},$$

$$c_{2} = \frac{(2G\beta + \beta - 3)(2G\beta - \beta + 3)}{4\beta L(\beta - 3)},$$
(3.11)

that can be rewritten in the following form

$$c_{1} = \frac{4(\mu_{ip} - \mu_{m})}{(\lambda_{m} + 2\,\mu_{m})\,(\beta + 3)},$$

$$c_{2} = -\frac{(3\,(\lambda_{ip} - \lambda_{m}) + 2\,(\mu_{ip} - \mu_{m}))}{(\lambda_{m} + 2\,\mu_{m})\,(\beta - 3)}.$$
(3.12)

We observe that these quantities are zero in absence of the interphase zone ($\lambda_{ip} = \lambda_m, \mu_{ip} = \mu_m$).

So, using (3.4) and (3.3) and introducing two integration constants A_1 and A_2 , we obtain the radial displacement in the graded material as

$$u(r) = \frac{A_1}{r^2} \Theta_1(r) + A_2 r \Theta_2(r).$$
(3.13)

Consequently, the stress field is

$$\sigma_r(r) = \frac{A_1}{r^3} f_1(r) + A_2 f_2(r),$$

$$\sigma_\theta(r) = \frac{A_1}{r^3} f_3(r) + A_2 f_4(r),$$
(3.14)

where

$$f_1(r) = \frac{\lambda(r) + 2\mu(r)}{r^{\beta}} b^{\beta} c_1 \beta \Theta_3(r) - 4\mu(r) \Theta_1(r),$$

$$f_{2}(r) = \frac{\lambda(r) + 2\mu(r)}{r^{\beta}} b^{\beta} c_{2} \Theta_{4}(r) + (3\lambda(r) + 2\mu(r)) \Theta_{2}(r),$$

$$f_{3}(r) = \frac{\lambda(r)}{r^{\beta}} b^{\beta} \beta c_{1} \Theta_{3}(r) + 2\mu(r) \Theta_{1}(r),$$

$$f_{4}(r) = \frac{\lambda(r)}{r^{\beta}} b^{\beta} \beta c_{2} \Theta_{4}(r) + (3\lambda(r) + 2\mu(r)) \Theta_{2}(r),$$
(3.15)

in which $\lambda(r)$ and $\mu(r)$ are the functions introduced in (2.1) (2.2).

The integration constants A_1, A_2 of (3.13) and (3.14), will be determined to find the explicit solution of the problem of Figure 1, by using interface and asymptotic conditions as detailed in the next section.

In the case of homogeneous material the field equation (3.1) becomes (Love, 1944):

$$u''(r) + \frac{2}{r}u'(r) - \frac{2}{r^2}u(r) = 0, \qquad (3.16)$$

whose solution is

$$u(r) = \frac{B_1}{r^2} + B_2 r, ag{3.17}$$

and the stresses are

$$\sigma_r(r) = -4\mu \frac{B_1}{r^3} + (3\lambda + 2\mu) B_2,$$

$$\sigma_\theta(r) = 4\mu \frac{B_1}{r^3} + (3\lambda + 2\mu) B_2.$$
(3.18)

We remark that as a consequence of the form of (3.12), in absence of interphase zone, $(\lambda_{ip} = \lambda_m, \mu_{ip} = \mu_m)$, we obtain: G = 1/2 and $\Theta_i(r) = 1$, for (i = 1, 2, 3, 4). In this way, the solution (3.13,14) assumes the form of equations (3.17,18).

This solution will be assumed to describe homogeneous inclusion in the problem of Figure 1 with $\lambda = \lambda_i$, $\mu = \mu_i$.

4 Explicit in closed-form solution

In this section the explicit solution of the problem studied is obtained by applying the equations (3.13) and (3.14) for the matrix (m), in terms of the constants A_1, A_2 , and the equations (3.17) and (3.18) for the inclusion (i), in terms of the constants B_1, B_2 . Using boundary condition (2.3) and the asymptotic condition (2.4), together with interface conditions (2.5), the four constants are explicitly determined. In the following subsections we explicitly write the constant values respectively for hollow and solid inclusions. But, first of all, we observe that to introduce the condition (2.4) due to the remote load on the matrix it may be convenient to write the asymptotic expressions for the graded solution (3.13,14) obtained in the previous section.

4.1 Asymptotic form of the solution for inhomogeneous material

For a generic hypergeometric function ${}_{2}\mathrm{F}_{1}(A,B;C;s)$ we write its asymptotic expansion (Abramovitz and Stegun, 1965) as

$$_{2}F_{1}(A,B;C;s) = 1 + \frac{BAs}{C} + O(s^{2}).$$
 (4.1)

Neglecting higher order terms in (4.1) and using expression (3.13) for the displacement in graded material, we obtain

$$u^{(a)}(r) = \left(1 - \frac{b^{\beta}c_1}{r^{\beta}}\right) \frac{A_1}{r^2} + \left(1 - \frac{b^{\beta}c_2}{r^{\beta}}\right) r A_2,$$
(4.2)

and for the stresses (3.14)

$$\sigma_r^{(a)}(r) = \frac{A_1}{r^3} \bar{f}_1(r) + A_2 \bar{f}_2(r),$$

$$\sigma_{\theta}^{(a)}(r) = \frac{A_1}{r^3} \bar{f}_3(r) + A_2 \bar{f}_4(r),$$
(4.3)

with

$$\bar{f}_{1}(r) = \frac{4\left(\mu_{m} - \mu_{ip}\right)b^{\beta}}{r^{\beta}} + \frac{\left(\left(\lambda_{m} + 2\,\mu_{m}\right)\beta + 4\,\mu_{m}\right)b^{\beta}}{r^{\beta}}c_{1} - 4\,\mu_{m},$$

$$\bar{f}_{2}(r) = \frac{\left(3\left(\lambda_{ip} - \lambda_{m}\right) + 2\left(\mu_{ip} - \mu_{m}\right)\right)b^{\beta}}{r^{\beta}} + \frac{\left(\left(\lambda_{m} + 2\,\mu_{m}\right)\beta - 3\,\lambda_{m} - 2\,\mu_{m}\right)b^{\beta}}{r^{\beta}}c_{2} + 3\,\lambda_{m} + 2\,\mu_{m},$$

$$\bar{f}_{3}(r) = \frac{2\left(\mu_{ip} - \mu_{m}\right)b^{\beta}}{r^{\beta}} + \frac{\left(\beta\lambda_{m} - 2\mu_{m}\right)b^{\beta}}{r^{\beta}}c_{1} + 2\mu_{m},$$

$$\bar{f}_{4}(r) = \frac{\left(3\left(\lambda_{ip} - \lambda_{m}\right) + 2\left(\mu_{ip} - \mu_{m}\right)\right)b^{\beta}}{r^{\beta}} + \frac{\left(\lambda_{m}\beta - 3\,\lambda_{m} - 2\,\mu_{m}\right)b^{\beta}}{r^{\beta}}c_{2} + 3\,\lambda_{m} + 2\,\mu_{m}.$$
(4.4)

where the superscript (a) denotes the asymptotic expressions of the displacement and stresses.

4.2 Explicit solution for a hollow inclusion with and without interphase

Now, we assume solution (3.13-14) for the matrix (m) and the solution (3.17-18) for the inclusion (i). To determine the constant values we impose the condition (2.3-5) of section 2. The condition (2.3) permits us to obtain

 $(2) + 0 \rightarrow 3$

$$B_1 = \frac{(3\lambda_i + 2\mu_1)a^3}{4\mu_i}B_2,\tag{4.5}$$

and the asymptotic form (4.3) used to apply the radial stress condition due to remote load (2.4) gives rise to

$$A_2 = -\frac{p}{3\lambda_m + 2\mu_m}.\tag{4.6}$$

The constants A_1 and B_2 are determined using interface conditions (2.5) at r = b on the displacement and the radial stress. So, we explicitly obtain the remaining constants for hollow spherical inclusion with interphase in the following form:

$$A_1 = \frac{N_A}{D}, \quad B_2 = \frac{N_B}{D},$$

where

$$N_{A} = p \, b^{3} \, \beta \, c_{2} \left(\lambda_{ip} + 2 \, \mu_{ip}\right) \left(\left(3\lambda_{i} + 2\mu_{i}\right) a^{3} + 4 \, b^{3} \mu_{i} \right) \Theta_{4} \left(b\right) + + p \, b^{3} \left(\left(3 \, \lambda_{ip} + 2 \, \mu_{ip} + 4 \, \mu_{i}\right) \left(3 \, \lambda_{i} + 2 \, \mu_{i}\right) a^{3} - 4 \, \mu_{i} \left(3 \, \lambda_{i} + 2 \, \mu_{i} - 3 \, \lambda_{ip} - 2 \, \mu_{ip}\right) b^{3} \right) \Theta_{2} \left(b\right), \\N_{B} = -4 \, p \, b^{3} \mu_{i} \left(\lambda_{ip} + 2 \, \mu_{ip}\right) \left(\left(\Theta_{3} \left(b\right) \, \beta \, c_{1} - 3 \, \Theta_{1} \left(b\right)\right) \Theta_{2} \left(b\right) - \Theta_{1} \left(b\right) \Theta_{4} \left(b\right) \, c_{2}\beta \right), \\\text{and} \\D = \beta \, c_{1} \left(3 \, \lambda_{m} + 2 \, \mu_{m}\right) \left(\lambda_{ip} + 2 \, \mu_{ip}\right) \left(\left(3 \, \lambda_{i} + 2 \, \mu_{i}\right) a^{3} + 4 \, b^{3} \mu_{i} \right) \Theta_{3} \left(b\right) +$$

$$D = \beta c_1 (3 \lambda_m + 2 \mu_m) (\lambda_{ip} + 2 \mu_{ip}) ((3 \lambda_i + 2 \mu_i) a^3 + 4 b^3 \mu_i) \Theta_3 (b) + +4 (3 \lambda_m + 2 \mu_m) ((\mu_i - \mu_{ip}) (3 \lambda_i + 2 \mu_i) a^3 - \mu_i (3 \lambda_i + 2 \mu_i + 4 \mu_{ip}) b^3) \Theta_1 (b),$$

$$(4.7)$$

where the quantities $\Theta_1(b), \Theta_2(b), \Theta_3(b), \Theta_4(b)$, are obtained by putting r = b in (3.9).

The constants A_1 and B_2 for hollow spherical inclusion without interphase assume the forms (4.5,6) while the (4.7) become

$$A_{1} = \frac{p \, b^{3} \, \left(\left(3 \, \lambda_{m} + 2 \, \mu_{m} + 4 \, \mu_{i}\right) \left(3 \, \lambda_{i} + 2 \, \mu_{i}\right) a^{3} - 4 \, \mu_{i} \left(3 \, \lambda_{i} + 2 \, \mu_{i} - 3 \, \lambda_{m} - 2 \, \mu_{m}\right) b^{3}\right)}{4 \, \left(2 \, \mu_{m} + 3 \, \lambda_{m}\right) \left(\left(\left(\mu_{i} - \mu_{m}\right) \left(2 \, \mu_{i} + 3 \, \lambda_{i}\right) a^{3} - \mu_{i} \left(4 \, \mu_{m} + 3 \, \lambda_{i} + 2 \, \mu_{i}\right) b^{3}\right)\right)},$$

$$B_{2} = \frac{3 \, p \, b^{3} \mu_{i} \left(\lambda_{m} + 2 \, \mu_{m}\right)}{\left(2 \, \mu_{m} + 3 \, \lambda_{m}\right) \left(\left(\mu_{i} - \mu_{m}\right) \left(2 \, \mu_{i} + 3 \, \lambda_{i}\right) a^{3} - \mu_{i} \left(4 \, \mu_{m} + 3 \, \lambda_{i} + 2 \, \mu_{i}\right) b^{3}\right)}.$$

(4.8)

Indeed, the hypergeometric functions for homogeneous matrix are equal to one and furthermore $c_1 = c_2 = 0$ from (3.12).

The solution for the problem shown in section 2 is thus solved in closed form substituting the constants (4.6,7 or 8) in the equations (3.13,14) and (3.17,18) respectively to explicitly obtain the elastic response in the matrix and in the inclusion.

4.3 Explicit solution for a solid inclusion with and without interphase

In this section we also explicitly write the constant forms for the case of spherical solid inclusion of radius b. In this case the condition (2.3) gives $B_1 = 0$ and the remote load condition is in the same form as (4.6). The remaining constants are obtained putting a = 0in (4.7) and rewritten to help reading as

$$A_{1} = \frac{p b^{3} ((3 \lambda_{i} + 2 \mu_{i} - 3 \lambda_{ip} - 2 \mu_{ip}) \Theta_{2}(b) - \beta c_{2} (\lambda_{ip} + 2 \mu_{ip}) \Theta_{4}(b))}{(3 \lambda_{m} + 2 \mu_{m}) ((4 \mu_{ip} + 3 \lambda_{i} + 2 \mu_{i}) \Theta_{1}(b) - \beta c_{1} (\lambda_{ip} + 2 \mu_{ip}) \Theta_{3}(b))},$$

$$A_{2} = -\frac{p}{3\lambda_{m} + 2\mu_{m}},$$

$$B_{1} = 0,$$

$$B_{2} = \frac{p (\lambda_{ip} + 2 \mu_{ip}) ((c_{1}\Theta_{3}(b) \Theta_{2}(b) - \Theta_{4}(b) \Theta_{1}(b) c_{2}) \beta - 3 \Theta_{2}(b) \Theta_{1}(b))}{(3 \lambda_{m} + 2 \mu_{m}) ((4 \mu_{ip} + 3 \lambda_{i} + 2 \mu_{i}) \Theta_{1}(b) - \beta c_{1} (\lambda_{ip} + 2 \mu_{ip}) \Theta_{3}(b))}.$$
(4.9)

The case without interphase is obtained from (4.9) for $\lambda_{ip} = \lambda_m$, $\mu_{ip} = \mu_m$, and $c_1 = c_2 = 0$, in the simple form

$$A_{1} = \frac{pb^{3} (3\lambda_{i} + 2\mu_{i} - 3\lambda_{m} - 2\mu_{m})}{(3\lambda_{m} + 2\mu_{m}) (3\lambda_{i} + 2\mu_{i} + 4\mu_{m})},$$

$$B_{2} = -\frac{3p (\lambda_{m} + 2\mu_{m})}{(3\lambda_{m} + 2\mu_{m}) (3\lambda_{i} + 2\mu_{i} + 4\mu_{m})}.$$
(4.10)

5 Effective bulk modulus of a material containing a dispersion of hollow or solid inclusions

The explicit solution obtained in the previous section permit us to determine the effective bulk modulus in a particulate composite that contains a random dispersion of spherical inclusions, taking into account the effect of an interphase zone. As in previous section 4, the hollow inclusion case with interphase is firstly studied by using an energy approach and the solid case is obtained as a consequence.

The strain energy of a body Ω can be computed as

$$U = \frac{1}{2} \int_{\Omega} \sigma \cdot \epsilon \, \mathrm{dV}. \tag{5.1}$$

This energy is written for two different spherical bodies of radius R and same hydrostatic pressure load in r = R. The first one is an inhomogeneous solid with a single, centered spherical, hollow inclusion of inner radius a (with $\sigma_r(a) = 0$) and outer radius b. By using the divergence theorem, the equation (5.1) assumes the form

$$U_{comp} = 2\pi u(R)\sigma_r(R)R^2.$$
(5.2)

In similar way, we write the strain energy for the second homogeneous solid (with 0 < r < R) in the form

$$U_{h} = \frac{2\pi R^{3} \left(\sigma_{r} \left(R\right)\right)^{2}}{3K_{eff}},$$
(5.3)

where K_{eff} is the effective bulk modulus of the equivalent homogeneous solid, determined assuming $U_{comp} = U_h$ (Willis, 1981). The comparison of the strain energy of two solids is performed by introduction the assumption of infinite bodies $(R \to \infty)$; so doing, we get

$$K_{eff} = \lim_{R \to \infty} \frac{\sigma_r(R)R}{3\,u(R)}.$$
(5.4)

Now, we consider the ratio $c = (b/R)^3$, that denotes the volumetric fraction of the inclusion in the matrix, and the ratio $\alpha = a/b$; furthermore, we assume that these parameters are fixed in this limit. In order to explicitly get the effective bulk modulus of equation (5.4), we write the radial stress and the displacement in r = R in terms of c:

$$\sigma (R) = 4 \left(\left(\mu_m - \mu_{ip} \right) c^{\frac{\beta}{3}} - \mu_m \right) p h_1 (c) Nc + + \left(c^{\frac{\beta}{3}} \left(\lambda_m + 2 \mu_m \right) + c^{\frac{2\beta}{3}} \left(2 \left(\mu_{ip} - \mu_m \right) + \lambda_{ip} - \lambda_m \right) \right) c_1 \beta p h_3 (c) Nc + - \frac{3 \lambda_m + 2 \mu_m + c^{\frac{\beta}{3}} \left(3 \left(\lambda_{ip} + \lambda_m \right) + 2 \left(\mu_{ip} - \mu_m \right) \right)}{3 \lambda_m + 2 \mu_m} p h_2 (c) + - \frac{c^{\frac{2\beta}{3}} \left(2 \left(\mu_{ip} - \mu_m \right) + \lambda_{ip} - \lambda_m \right) + c^{\frac{\beta}{3}} \left(\lambda_m + 2 \mu_m \right)}{3 \lambda_m + 2 \mu_m} c_2 \beta p h_4 (c) ,$$
(5.5)

$$u(R) = \left(h_1(c) Nc - \frac{h_2(c)}{3\lambda_m + 2\mu_m}\right) R p,$$

where we have set $h_i(c) = \Theta_i(b/c^{1/3}), i = 1, ..4$ (see equations (3.9)) and $N = \frac{A_1}{p b^3}$ is

written in terms of the ratio α as:

$$N = \frac{N_1}{N_2},$$

$$N_1 = (2\,\mu_i + 3\,\lambda_i)\,((2\,\mu_{ip} + \lambda_{ip})\,\Theta_4(b)\,\beta\,c_2 + (3\,\lambda_{ip} + 4\,\mu_i + 2\,\mu_{ip})\,\Theta_2(b))\,\alpha^3 + \\ + 4\,(2\,\mu_{ip} + \lambda_{ip})\,\mu_i\Theta_4(b)\,\beta\,c_2 - 4\,\mu_i\,(3\,\lambda_i + 2\,\mu_i - 3\,\lambda_{ip} - 2\,\mu_{ip})\,\Theta_2(b),$$

$$N_2 = (2\,\mu_i + 3\,\lambda_i)\,(2\,\mu_m + 3\,\lambda_m)\,((\lambda_{ip} + 2\,\mu_{ip})\,\Theta_3(b)\,\beta\,c_1 + 4\,(\mu_i - \mu_{ip})\,\Theta_1(b))\,\alpha^3 + \\ + 4\,(2\,\mu_m + 3\,\lambda_m)\,\mu_i\,((2\,\mu_{ip} + \lambda_{ip})\,\Theta_3(b)\,\beta\,c_1 - (3\,\lambda_i + 2\,\mu_i + 4\,\mu_{ip})\,\Theta_1(b))\,.$$
(5.6)

Substituting equations (5.5) in (5.4) we obtain an expression not depending from the radius R but only from c; in this way, the effective bulk mdulus in terms of elastic and geometric properties is obtained in closed form for the case of nondilute inclusions (Christensen, 2005).

Then, in order to obtain the bulk modulus for $c \ll 1$ (dilute inclusion case), we assume $\beta > 3$ (see section 6 for a discussion on this assumption) and neglect higher terms in c in equations (5.5). So doing, we obtain

$$\sigma_r (R) = -(4 \,\mu_m \, N \, c + 1) \, p,$$

$$u(R) = \left(N \, c - \frac{1}{2 \,\mu_m + 3 \,\lambda_m} \right) p \, R,$$
(5.7)

and substituting equations (5.7) in (5.4) we perform an expansion in c and consider only the first term in c. So the bulk modulus for a dilute dispersion of *hollow inclusions with interphase* becomes

$$\frac{K_{eff}}{K_m} = (3 K_m + 4 \mu_m) N c + 1,$$
(5.8)

where $K_m = (3\lambda_m + 2\mu_m)/3$.

Now, the bulk modulus for *hollow inclusion without interphase* is obtained by using equations (5.4-5) or (5.8) with

$$N = \frac{(2\,\mu_i + 3\,\lambda_i)\,(3\,\lambda_m + 4\,\mu_i + 2\,\mu_m)\,\alpha^3 - 4\,\mu_i\,(3\,\lambda_i + 2\,\mu_i - 3\,\lambda_m - 2\,\mu_m)}{(4\,(2\,\mu_i + 3\,\lambda_i)\,(\mu_i - \mu_m)\,\alpha^3 - 4\,\mu_i\,(3\,\lambda_i + 2\,\mu_i + 4\,\mu_m))\,(2\,\mu_m + 3\,\lambda_m)},\tag{5.9}$$

that, gives a result in agreement with the effective bulk modulus obtained by Porfiri and Gupta (2009).

The case of *solid inclusion with interphase* is obtained assuming

$$N = \frac{(2\,\mu_{ip} + \lambda_{ip})\,\Theta_4(b)\beta\,c_2 - (3\,\lambda_i + 2\,\mu_i - 3\,\lambda_{ip} - 2\,\mu_{ip})\,\Theta_2(b)}{((2\,\mu_{ip} + \lambda_{ip})\,\Theta_3(b)\beta\,c_1 - (3\,\lambda_i + 2\,\mu_i + 4\,\mu_{ip})\,\Theta_1(b))\,(2\mu_m + 3\lambda_m)}.$$
(5.10)

Finally, for solid inclusion without interphase case, assuming $\lambda_{ip} = \lambda_m$, $\mu_{ip} = \mu_m$ and $c_1 = c_2 = 0$ in (5.6), we get (i.e. Christensen (2005)):

$$N = \frac{3\lambda_i + 2\mu_i - 3\lambda_m - 2\mu_m}{(3\lambda_i + 2\mu_i + 4\mu_m)(2\mu_m + 3\lambda_m)}.$$
(5.11)

Furthermore, we remark that formula (5.8), in the case $1/L \ll 1$, corresponding to small differences between the elastic properties of the interphase ($\lambda_{ip} \cong \lambda_m, \mu_{ip} \cong \mu_m$), assumes a simplified form by using the following asymptotic expansions (4.1) for $\Theta_i(b)$:

$$\Theta_1(b) = 1 - c_1, \ \Theta_2(b) = 1 - c_2, \ \Theta_3(b) = 1, \ \Theta_4(b) = 1.$$
 (5.12)

6 Numerical results

In this section we numerically investigate the elastic solutions obtained in the previous sections by introducing three numerical examples. The first permits us to check the elastic solution of subsection (4.2.3) using the data of the paper of Lutz and Zimmerman (1996a), for the solid inclusion case of radius b with and without inclusion. The second example investigates the graded interphase effects in hollow inclusions for different inhomogeneity parameter values in comparison with the case without inclusion. The third example concerns the investigation of the interphase effects on the effective bulk modulus obtained in section 5.

Firstly, in this section, the behaviour of the distribution law in the interphase zone for the hard interphase case is shown in Figure 2 in terms of the β -parameter that describes the interphase thickness t; in particular, we assume that the elastic modulus value in r = b + t is obtained to less than d% of the asymptotic value of the matrix. So, for the hard interphase, we write

$$t = \left(\frac{100}{d}\right)^{1/\beta} \left(\frac{\lambda_{ip} - \lambda_m}{\lambda_m}\right)^{1/\beta} b - b, \qquad (6.1)$$

and so we explicitly obtain

$$\beta = \ln\left(\frac{100\left(\lambda_{ip} - \lambda_m\right)}{\lambda_m d}\right) \left(\ln\left(\frac{b+t}{b}\right)\right)^{-1} \tag{6.2}$$

and by solving with respect to β , we determine the value that gives the required interphase thickness in terms of the elastic properties λ_{ip} and λ_m ; in a similar way, for the other Lamé modulus. Larger values of β -parameter correspond to interphase zones that are more localized and viceversa. In Figure 2 the behaviour of the distribution law for different interphase thicknesses (t = b - a, t = b/2, t = a, t = b) is shown for hard interphases assuming respectively $\lambda_{ip} = 1.5\lambda_m, \mu_{ip} = 1.5\mu_m$. By equation (6.2), assuming d = 1, the values of the inhomogeneity parameters respectively become: $\beta = 21.46, 9.65, 6.65, 5.64$. In a similar way it is possible to obtain the distribution law behaviour for soft interphase assuming $\lambda_{ip} = 0.5\lambda_m, \mu_{ip} = 0.5\mu_m$ that permit us to determine the same corresponding β values.

Example 1. In this example a solid inclusion of radius b embedded in a matrix is considered assuming soft and hard interphase with elastic properties of Figure 2 and d = 1. The interphase thickness ratio t = 0.25 b is considered and corresponds, by equations (6.2), to $\beta = 17.53$. For the comparisons with the case without interphase we will put $\lambda_{ip} = \lambda_m$ and $\mu_{ip} = \mu_m$. We observe that the value $\beta = 10$, assumed in the paper of Lutz and Zimmerman (1996a), allow us to get: $(\lambda(b+t) - \lambda_m)/\lambda_m \approx 5\%$.

The normalized radial displacement is shown in Figure 3 for soft and hard interphases and the case without interphase; we observe that the effect of the interphase is localized around the interphase zone and the behaviour is similar to the case without interphase. The effect of interphase in matrix decreases quickly and disappears at r = 2a. In Figure 4 the normalized radial stress shows an increase in inclusion and matrix, for hard interphase, while a decrease for soft inclusion, with respect to the absence of interphase zone. The normalized hoop stress $\sigma_{\theta} = \sigma_{\phi}$ is shown in Figure 5. The gap due to the mismatch between the elastic properties of inclusion and interphase increases for soft interphase and the effects, also in this case, are confined to inside and around inclusion. The numerical results obtained by the solution from Section 4.2.2 are in perfect agreement with the results obtained by Lutz and Zimmerman (1996a).

Example 2. In this example hollow inclusions are investigated by assuming the numerical values of the elastic properties of the previous example in the cases of hard and soft interphase. The hollow inclusion is considered with ratio radius a/b = 0.8 and the responses to different interphase thicknesses are shown assuming the ratios: t/b = 0.25, 1, 2 corresponding, from equation (5.2) respectively at $\beta = 17.53, 5.64, 3.56$. Larger values of interphase thickness are not usual in application, however, these cases are presented to better highlight the effects of the interphase zone thickness.

In Figure 6 the normalized radial displacement is shown; we observe that the displacement in inclusion decreases for soft interphase and increases for hard interphase with respect to the case without interphase; the decrease is more significant for soft interphase and augments with the increase of the interphase thickness. The displacement in matrix increases or decreases respectively for soft or hard interphase. In Figure 7 the consequence of the graded interphase on the normalized radial stress is shown to be localized outside the inclusion where the stress increases, for hard interphase, or decreases, for soft interphase. Furthermore, the effects increase with larger interphase thickness. Instead, the hoop stress shown in Figure 8 presents different behaviour in which the gap at the interface increases, in the case of soft interphase with respect to the hard interphase case and the case without interphase. In addition, the stress inside inclusion is perturbed and, in the case of soft interphase, increases with a greater interphase thickness. We observe that the interphase effects decrease more quickly with respect to the radial stress and displacement.

Example 3. Now, the effective bulk modulus for hollow or solid inclusions obtained in section 5 is studied in order to shown how the interphase thickness alters the bulk modulus of the composite. In particular, in Figure 9 the normalized effective bulk modulus vs the volumetric fraction c is obtained for solid and hollow inclusions with and without interphase. By assuming numerical parameters of the previous examples with $\beta = 10$ (Lutz et al.(1997)), we observe the interphase sensitivity on bulk modulus in solid and hollow inclusions for

nondilute and dilute cases.

7 Conclusions

Explicit elastic solutions in closed forms are obtained for hollow and solid inclusions with graded interphase zone around the inclusion subjected to a radially symmetric load. The different effects of the interphase zone in solid or hollow inclusions are highlight with numerical investigations and shown how the thickness of the interphase alters the stress concentration around the inclusion and so may affects the bulk modulus of the composite. The presence of the interphase zone was found to have effect inside hollow inclusion for the radial displacement and hoop stress and in particular for the case of soft inclusion; on the contrary, for the radial stress no particular effect are present in inclusion and the maximum radial stress, that occurs at the interface between inclusion and interphase, is greater than the remote load. On the contrary, the maximum hoop stress occurs on the inner surface of the inclusion with values much greater than the load applied. Outside the inclusion the effect of interphase is localized around inclusion and decreases more quickly for the hoop stress compared to radial stress. The stresses in inclusion and the maximum stress value at the interface are quite sensitive at the interphase thickness for hard interphase; instead, for soft interphase the interphase thickness affects the maximum hoop stress and the displacement in the inclusion. Then, the explicit solutions determined are used to estimate the bulk modulus for a distribution of spherical hollow or solid inclusion with interphase; numerical results permit us to investigate also the variation of the bulk modulus with the volumetric fractions of the particulate composite.

Finally, we observe that the solution of section 4 may be the starting point to estimate other elastic properties, as the effective thermal expansion coefficient and the thermal-electric conductivity, in a particulate composite with a dispersion of hollow or solid spherical inclusions taking into account physical and geometric interphase properties and, will be the subject of a future paper. Figure 1. Sketch of the mathematical problem studied.

Figure 2. Distribution law of the graded matrix elastic properties in radial direction for different interphase thickness.

Figure 3. Normalized radial displacement for a solid inclusion with hard and soft interphase.

Figure 4. Normalized radial stress for a solid inclusion with hard and soft interphase.

Figure 5. Normalized hoop stress for a solid inclusion with hard and soft interphase.

Figure 6. Normalized radial displacement for a hollow inclusion with hard or soft interphase and different interphase thickness.

Figure 7. Normalized radial stress for a hollow inclusion with hard or soft interphase and different interphase thickness.

Figure 8. Normalized hoop stress for a hollow inclusion with hard or soft interphase and different interphase thickness.

Figure 9. Normalized bulk modulus vs volumetric fraction c for solid and hollow inclusions.

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Figure 9: Normalized bulk modulus vs volumetric fraction c for solid and hollow inclusions.