Proof analysis for Lewis counterfactuals

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Abstract

A deductive system for Lewis counterfactuals is presented, based directly on Lewis' influential generalisation of relational semantics that uses ternary similarity relations. This deductive system builds on a method of enriching the syntax of sequent calculus by labels for possible worlds. The resulting labelled sequent calculus is shown to be equivalent to the axiomatic system **VC** of Lewis. It is further shown to have the structural properties that are needed for an analytic proof system that supports root-first proof search. Completeness of the calculus is proved in a direct way, such that for any given sequent either a formal derivation or a countermodel is provided; it is also shown how finite countermodels for unprovable sequents can be extracted from failed proof search, by which the completeness proof turns into a proof of decidability.

1 Introduction

Kripke's introduction of the relational semantics for modal logic was a decisive turning point for philosophical logic: the earlier axiomatic studies were replaced by a solid semantical method that displayed the connections between modal axioms and conditions on the accessibility relation between possible worlds. It took a mere ten years after Kripke for David Lewis' semantical approach to conditionals to emerge, in a work that has become almost as classic as that of Kripke: the little book *Counterfactuals* of 1973.

The successes of the semantic methods have not been followed by equally powerful syntactic theories of modal and conditional concepts and reasoning: Concerning the former, the situation was so depicted by Melvin Fitting in his article (2007) in the Handbook of Modal Logic: "No proof procedure suffices for every normal modal logic determined by a class of frames"; Concerning the latter, as stated by Graham Priest "there are presently no known tableau systems" for Lewis' logic for counterfactuals (2008, p. 93).

In the article *Proof analysis in modal logic*, the first author showed in 2005 how Kripke's semantics can be exploited to enrich the syntax of systems of proof, especially sequent calculi, to answer to the challenge of developing analytic proof systems for standard systems of modal logic. In particular, it has turned out that a more expressive language, with a formal notation of labels that represent possible worlds, is the crucial component in answering to the challenge, set by the successful semantical methods, for the proof theory of philosophical logic. The approach has been extended to wider frame classes in later work (Negri 2014), and in Dyckhoff and Negri (2014) it was shown how the method can capture any nonclassical logic characterized by arbitrary first-order frame conditions in their relational semantics. Notably, in these calculi, all the rules are invertible and a strong form of completeness holds for them, with a simultaneous construction of formal proofs, for derivable sequents, or countermodels, for underivable ones (Negri 2014a).

In this paper, we show how, in perfect analogy to the method of proof analysis in modal logic, systems of proof can be introduced for Lewis' counterfactuals. The main novelty is that the basic relation to be expressed in the syntax is three dimensional. Ternary relations where proposed by Lewis himself as a formal account of the topological topological truth conditions for counterfactuals, in the setting of comparative similarity based on sphere semantics, a special form of neighbourhood semantics for the conditionals. In this semantics, there is an $\exists\forall$ nesting of quantifiers in the truth conditions for the counterfactual conditional, which makes the determination of the rules of the calculus an interesting and challenging task. Our solution uses indexed modalities, which allow to split the semantic clause in two separate parts, and correspondingly, the rules for the counterfactual conditional depend on rules for the indexed modality, which are standard modal labelled rules. The result is a system, called **G3LC** below, which is a sound and complete Gentzen-style sequent calculus for Lewis' original counterfactual. The system has all the structural rules (weakening, contraction, and cut) admissible, and all its rules are invertible.

We introduce the system **G3LC** in Section 2 and then briefly discuss its extension to a system provided with a non-trivial aletheic modality (in the case at hand, S4). In Section 3, we present some interesting structural properties of **G3LC** and in particular a cut elimination theorem. Then we show (Section 4) that Lewis' axioms and rules are, respectively, admissible and derivable, which allows us to show that the calculus is complete (by soundness and by Lewis' own proof of completeness). Finally, we prove directly completeness and decidability results in Section 5. Related literature and further work is discussed in the concluding Section.

2 A sequent calculus for Lewis conditional

In this Section, we shall present a labelled sequent calculus for Lewis conditional starting from Lewis' own semantics based on ternary relation (Lewis 1973, 1973a).

We recall that the strategy for internalising the semantics into the syntax of a good sequent calculus consists of the following stages: to start with, the language is extended by labelled formulas, of the form $x : A$, and by expressions of the form xRy. Labelled formulas $x : A$ correspond to the statement that A is true at node/possible world x; expressions of the form xRy corresponds to relations between nodes/possible worlds in a frame. Then the compositional semantic clauses that define the truth of a formula at a world are translated into natural deduction inference rules for labelled expressions; second, such rules are appropriately converted into sequent calculus rules; third, the characteristic frame properties that distinguish the modal system at hand are converted into rules for the relational part of the calculus following the method of translation of axioms into sequent calculus rules introduced in Negri and von Plato (1998) and further developed in Negri and von Plato (2011). In this way, the frame properties are carried over to the calculus by the addition of rules for binary accessibility relations regarded as binary atomic predicates with the labels as arguments. In the case of normal modal logics the method allows to obtain analytic sequent calculi for any system characterised by arbitrary first-order frame conditions (Dyckhoff and Negri 2014). All these stages from meaning to rules are detailed in the case of intuitionistic and standard modal logics in Negri and von Plato (2014).

The truth conditions for Lewis' conditional are spelled out in terms of a three-place similarity relation $\leq$ among worlds, with the intuitive meaning of "$x \leq_w y$" being "x is at least as similar to w as y is". The following properties are generally assumed for this ternary relation:

1. *Transitivity*: If $x \leq_w y$ and $y \leq_w z$ then $x \leq_w z$,
2. *Strong connectedness*: Either $x \leq_w y$ or $y \leq_w x$,
3. *L-Minimality*: If $x \leq_w w$ then $x = w$.

Through the method of conversion into rules (cf. Negri and von Plato 1998 or chapter 6 of Negri and von Plato 2001, or, for more generality, part III of Negri and von Plato 2011) these are converted into the following sequent calculus rules:

$$\frac{x \leq_w z, x \leq_w y, y \leq_w z, \Gamma \Rightarrow \Delta}{x \leq_w y, y \leq_w z, \Gamma \Rightarrow \Delta}\,{\scriptstyle Trans} \qquad \frac{x \leq_w y, , \Gamma \Rightarrow \Delta \quad y \leq_w x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle SConn}$$

$$\frac{x = w, \Gamma \Rightarrow \Delta}{x \leq_w w, \Gamma \Rightarrow \Delta}\,{\scriptstyle LMin}$$

Lewis' truth conditions for the counterfactual conditional are as follows:

$w \Vdash A \,\Box\!\!\rightarrow B$ iff either

1. There is no z such that $z \Vdash A$, or
2. there is x such that $x \Vdash A$ and for all y, if $y \leq_w x$ then $y \Vdash A \supset B$.

As previously anticipated, the truth condition for $A \,\Box\!\!\rightarrow B$ has a universal quantification in the scope of an existential, and thus it is not of a form that can be directly translated into rules following the method of generation of labelled sequent rules for intensional operators (as expounded in Negri 2005); a more complex formalism in the line of the method of *systems of rules* (cf. Negri 2014) would have to be invoked to maintain the primitive language.

We observe however that the rules for the labelled calculus for Lewis' conditional can be presented following the general method of embedding the neighbourhood semantics for non-normal modal logics into the standard relational semantics normal modal systems through the use of *indexed modalities* (cf. the method formulated in general terms in Gasquet and Herzig (1996) for classical modal logics and used in Giordano *et al.* (2003) for obtaining a tableau calculus for preference-based conditional logics). Specifically, the relation of similarity is used to define a ternary accessibility relation

$$xR_wy \equiv y \leq_w x$$

In turn, this relation defines an indexed necessity modality with the truth condition

$$x \Vdash \Box_w A \equiv \forall y.xR_wy \rightarrow y \Vdash A$$

Then the truth condition for the conditional gets replaced by the following

$w : A \,\Box\!\!\rightarrow B$ iff either

1. There is no z such that $z : A$, or

2. there is x such that $x : A$ and $x : \Box_w(A \supset B)$.

Observe that the presentation of a calculus formulated in terms of indexed modalities is directly faithful to Lewis' original idea of conditional implication as a variably strict conditional.

The sequent system is obtained as an extension of the propositional part of the contraction and cut-free sequent calculus G3K for basic modal logic introduced in Negri (2005). In addition there are rules for the similarity and the equality relation. For the latter, we have just two rules, reflexivity and the scheme of *replacement*, $Repl_{At}$, where $At(x)$ stands for an atomic labelled formula $x : P$ or a relation of the form $y = z$, yR_wz, with x one of y, w, z. Symmetry of equality follows as a special case of $Repl_{At}$ as well as *Euclidean transitivity* which, together with symmetry, gives the usual *transitivity*.[1]

The intuitive meaning of the rules for the similarity relation is obtained directly by rephrasing them in terms of the meaning of the ternary similarity relation. For example, L-Minimality asserts that every world is as close as possible to itself.

The following rule, corresponding to reflexivity of the relation R_w, will be useful

$$\frac{xR_wx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ref}\,(x, w \textit{ in } \Gamma, \Delta)$$

The rule need not be taken as a primitive rule of the calculus because it is derivable, simply by writing its premiss twice and applying *SConn*. This admissible rule is used to simplify the derivation of the third of the Lewis axioms below.

The rules for Lewis conditional are obtained from its truth condition following the general method of Negri (2005) for turning the truth conditions of alethic modalities into rules of a labelled sequent calculus: the right to left direction in the semantic explanation gives the right rule and the other direction gives the left rules. Quantification over world is replaced by the condition that certain variables in the rules (eigenvariables) should be fresh. Since the truth condition for the Lewis conditional is a disjunction of two conditions, there are two right rules and one left rule with two premisses. The system is presented in Table 1. We can now state a couple of important results that will be needed below. First we need a definition of weight of formulas:

Definition 2.1. *The weight* $\mathtt{w}(A)$ *of a formula A is defined inductively by the following:*
$\mathtt{w}(\gamma) = 1$ *for* γ *the constant* $\perp$*, an atomic formula, or a relational atom,*
$\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ *for* $\circ$ *conjunction, disjunction, or implication,*
$\mathtt{w}(\Box_x A) = \mathtt{w}(A) + 1$,
$\mathtt{w}(A \mathrel{\Box\!\!\rightarrow} B) = \mathtt{w}(A) + \mathtt{w}(B) + 3$.

Observe that since we have taken negation $\neg A$ as defined by $A \supset\perp$, we have $\mathtt{w}(\neg A) \equiv \mathtt{w}(A) + 2$. Observe also that $\mathtt{w}(\Box_x(A \supset B)) < \mathtt{w}(A \mathrel{\Box\!\!\rightarrow} B)$.

We can further prove the following lemma:

Lemma 2.2. *All the sequents of the form* $x : A, \Gamma \Rightarrow \Delta, x : A$ *are derivable in* **G3LC** .

Proof. By induction on the weight of A. The base cases, with weight 1 holds because $x : A, \Gamma \Rightarrow \Delta, x : A$ is either an initial sequent or conclusion of $L\perp$. For the inductive steps,

[1] The general reasons for the architecture behind the rules of equality are discussed in section 6.5 of Negri and von Plato (2001) for extensions of first-order systems and the equality rules for labelled systems in Negri (2005) and Negri and von Plato (2011).

Table1 : **Lewis basic counterfactual conditional sequent system (G3LC)** *based on ternary similarity*

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta} L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B} R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta} L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B} R\supset$$

$$\frac{}{x : \perp, \Gamma \Rightarrow \Delta} L\perp$$

Similarity rules:

$$\frac{xR_wz, xR_wy, yR_wz, \Gamma \Rightarrow \Delta}{xR_wy, yR_wz, \Gamma \Rightarrow \Delta} Trans \qquad \frac{xR_wy, \Gamma \Rightarrow \Delta \quad yR_wx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} SConn\ (x, y, w\ in\ \Gamma, \Delta)$$

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} Ref\ (x\ in\ \Gamma, \Delta) \qquad \frac{x = y, At(x), At(y), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta} Repl_{At}$$

$$\frac{x = w, wR_wx, \Gamma \Rightarrow \Delta}{wR_wx, \Gamma \Rightarrow \Delta} LMin$$

Conditional rules:

$$\frac{xR_wy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_w A} R\Box_w\ (y\ fresh) \qquad \frac{xR_wy, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta}{xR_wy, x : \Box_w A, \Gamma \Rightarrow \Delta} L\Box_w$$

$$\frac{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} R\Box\!\!\rightarrow_1\ (z\ fresh)$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : A \quad \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : \Box_w(A \supset B)}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} R\Box\!\!\rightarrow_2$$

$$\frac{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \quad x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta} L\Box\!\!\rightarrow\ (x\ fresh)$$

the propositional cases as well as the case of $A \equiv \Box_w A$ are obtained by root-first application of the right and left rules of the outermost connective of A and the inductive hypothesis. If A is $B \Box\!\!\rightarrow C$ we have the following derivation, with the leaves derivable by inductive hypothesis:

$$\dfrac{\dfrac{z : B, x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, z : B \qquad \dfrac{y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, y : B \quad y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, y : \Box_x(B \supset C)}{y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, R \Box\!\!\rightarrow_2}{z : B, x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, L \Box\!\!\rightarrow}{x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, R \Box\!\!\rightarrow_1$$

QED

2.1 Adding the alethic modality

We can consider also alethic modalities defined in terms of a suitable binary accessibility relation R defined in terms of the given ternary relation. Building on Lewis' idea[2] that worlds accessible from x are worlds in the union of the set of spheres around x, and using the relation between spheres and ternary relations, we establish the following equivalence:

$$xRy \text{ iff } zR_x y \text{ for some } z$$

By standard prenex conversions we thus have that the alethic modality defined by R is expressed, in terms of the ternary accessibility relations, as follows:

$$x \Vdash \Box A \text{ iff } \forall zy.zR_x y \supset y \Vdash A$$

The forcing condition gives the following rules for $\Box$ in terms of the ternary accessibility relation

$$\dfrac{y : A, x : \Box A, zR_x y, \Gamma \Rightarrow \Delta}{x : \Box A, zR_x y, \Gamma \Rightarrow \Delta}\, L\Box \qquad\qquad \dfrac{zR_x y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box \ (y, z \textit{ fresh})$$

As usual for systems with a classical base, $\Diamond A$ does not have to be treated separately since it can defined in terms of $\Box A$ and negation.

The definition of necessity in terms of conditional

$$\Box A \equiv \neg A \Box\!\!\rightarrow A$$

can now be formally derived as an equivalence in the calculus **G3LC** extended with the above two rules as follows (for brevity, we omit to copy the main formula in the premisses). In one direction we have

$$\dfrac{\dfrac{z : \neg A \Rightarrow z : \neg A \qquad \dfrac{\dfrac{\dfrac{zR_x y, y : \neg A, x : \Box A, y : A \Rightarrow y : A}{zR_x y, y : \neg A, x : \Box A \Rightarrow y : A}\, L\Box}{zR_x y, x : \Box A \Rightarrow y : \neg A \supset A}\, R\supset}{x : \Box A \Rightarrow z : \Box_x(\neg A \supset A)}\, R\Box_x}{z : \neg A, x : \Box A \Rightarrow x : \neg A \Box\!\!\rightarrow A}\, R \Box\!\!\rightarrow_2}{x : \Box A \Rightarrow x : \neg A \Box\!\!\rightarrow A}\, R \Box\!\!\rightarrow_1$$

where the topmost sequents are derivable by Lemma 2.2. For the converse direction we have

[2] Cf. *Counterfactuals*, pp. 22–23.

$$\dfrac{\dfrac{zR_xy, y : A, \Rightarrow y : A}{zR_xy \Rightarrow y : A, y : \neg A}\,R\neg \qquad \dfrac{\dfrac{\dfrac{zR_xy, wR_xw, w : \neg A, w : \neg A \supset A \Rightarrow y : A}{zR_xy, wR_xw, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow y : A}\,L\Box_x}{zR_xy, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow y : A}\,Ref}{\dfrac{zR_xy, x : \neg A \,\Box\!\!\rightarrow A \Rightarrow y : A}{x : \neg A \,\Box\!\!\rightarrow A \Rightarrow x : \Box A}\,R\Box}\,L\,\Box\!\!\rightarrow$$

where the topmost sequents are derivable, the left one by Lemma 2.2 and the right one by a step of $L \supset$ and Lemma 2.2. Similarly for the equivalence $\Diamond A \equiv \neg(A \,\Box\!\!\rightarrow \neg A)$ (the proof is left to the reader).

Observe that the rules of $\Box$ can be simplified if R_x is a reflexive relation because R then turns out to be total; the necessity modality then becomes the universal modality and the rules are simply obtained by removing the accessibility relation.

$$\dfrac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\,L\Box \qquad \dfrac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\,R\Box\ (y\ fresh)$$

If the analysis of counterfactuals is embedded in a Kripke frame for modalities, then the truth conditions are the following[3]:

$w : A \,\Box\!\!\rightarrow B$ iff either:

1. there is no z such that wRz and $z : A$, or
2. there is x such that wRx and $x : A$ and for all y, if $y \leq_w x$ then $y : A \supset B$.

Accordingly, we give a table for Lewis' counterfactual conditional with an S4-accessibility relation. This can be regarded as the "full" system (Table 2).

For the sake of simplicity, we shall refer below only to the version of the system without accessibility relation, but the same results can be obtained, *mutatis mutandis* for the full systems. To consider the system without an accessibility relation is equivalent to assuming that the accessibility relation is total, i.e. that all the worlds are accessible.

3 Structural properties

The proof of admissibility of the structural rules in **G3LC** follows the pattern presented in Negri and von Plato (2011), section 11.4. Likewise, some preliminary results are needed, namely height-preserving admissibility of substitution (in short, hp-substitution) and height-preserving invertibility (in short, hp-invertibility) of the rules. We recall that the *height* of a derivation is its height as a tree, i.e. the length of its longest branch, and that $\vdash_n$ denotes derivability with derivation height bounded by n in a given system. In what follows, the results are all referred to system **G3LC**. The definition of substitution of labels is obtained by extending in the obvious way the clauses given in the definition of substitution given in the aforementioned reference to the ternary relations and the indexed modalities of **G3LC**.

Proposition 3.1. *If* $\vdash_n \Gamma \Rightarrow \Delta$*, then* $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$.

Proof. By induction on the height of the derivation. If it is 0, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\perp$. Then the same is true of $\Gamma(y/x) \Rightarrow \Delta(y/x)$. If the derivation height is $n > 0$, we consider the last rule in the derivation. If $\Gamma \Rightarrow \Delta$ has been derived by

[3]Cf. *Countefactuals*, pp. 49.

Lewis counterfactual conditional sequent system (G3LC4)
based on comparative similarity in $S4$

Initial sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \perp, \Gamma \Rightarrow \Delta}\, L\perp$$

Similarity rules:

$$\frac{xR_wz, xR_wy, yR_wz, \Gamma \Rightarrow \Delta}{xR_wy, yR_wz, \Gamma \Rightarrow \Delta}\, Trans \qquad \frac{xR_wy, \Gamma \Rightarrow \Delta \quad yR_wx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, SConn\ (x, y, w\ in\ \Gamma, \Delta)$$

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref\ (x\ in\ \Gamma, \Delta) \qquad \frac{x = y, At(x), At(y), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\, Repl_{At}$$

$$\frac{x = w, wR_wx, \Gamma \Rightarrow \Delta}{wR_wx, \Gamma \Rightarrow \Delta}\, LMin$$

Modal rules:

$$\frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, RefR \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\, TransR$$

$$\frac{zR_xy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box(y,\ z\ fresh) \qquad \frac{zR_xy, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{zR_xy, x : \Box A, \Gamma \Rightarrow \Delta}\, L\Box$$

$$\frac{zR_xy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{zR_xy, \Gamma \Rightarrow \Delta, x : \Diamond A}\, R\Diamond \qquad \frac{zR_xy, y : A, x : \Diamond A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\, L\Diamond(y,\ z\ fresh)$$

Conditional rules:

$$\frac{xR_wy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_w A}\, R\Box_w\ (y\ fresh) \qquad \frac{xR_wy, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta}{xR_wy, x : \Box_w A, \Gamma \Rightarrow \Delta}\, L\Box_w$$

$$\frac{wRz, z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_1\ (z\ fresh)$$

$$\frac{wRx, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : A \quad wRx, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : \Box_w(A \supset B)}{wRx, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_2$$

$$\frac{wRz, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \quad wRz, wRx, x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{wRz, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\, L\Box\!\!\rightarrow\ (x\ fresh)$$

a propositional or a relational rule, or by $L\Box_w$ or $R \Box\!\!\rightarrow_2$, we apply the induction hypothesis and then the rule. Rules with variable conditions require that we avoid a clash of the substituted variable with the fresh variable in the premiss. This is the case of $R\Box_w$, $R\Box\!\!\rightarrow_1$, and $L\Box\!\!\rightarrow$. So, if $\Gamma \Rightarrow \Delta$ has been derived by any of these rules, we apply the inductive hypothesis twice to the premiss, first to replace the fresh variable with another fresh variable different, if necessary, from the one we want to substitute, then to make the substitution, and then apply the rule. QED

Proposition 3.2. *The rules of left and right weakening are hp-admissible in* **G3LC**.

Proof. Straightforward induction, with a similar proviso as in the above proof for rules with variable conditions. QED

Next, we prove *hp-invertibility* of the rules of **G3LC**, i.e. for every rule of the form $\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$, if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$, and for every rule of the form $\frac{\Gamma' \Rightarrow \Delta' \quad \Gamma'' \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta}$ if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$ and $\vdash_n \Gamma'' \Rightarrow \Delta''$.

Lemma 3.3. *All the propositional rules of* **G3LC** *are hp-invertible.*

Proof. Similar to the proof of Lemma 11.7 in Negri and von Plato (2011). QED

Lemma 3.4. *The following hold:*

(i) *If* $\vdash_n \Gamma \Rightarrow \Delta, x : \Box_w A$, *then* $\vdash_n xR_w y, \Gamma \Rightarrow \Delta, y : A$,

(ii) *If* $\vdash_n w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$, *then* $\vdash_n x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$.

Proof. Both proofs are by induction on n.

(i) *Base case*: suppose that $\Gamma \Rightarrow \Delta, x : \Box_w A$ is an initial sequent or conclusion of $L\perp$. Then, in the former case, $x : \Box_w A$ not being atomic, $xR_w y, \Gamma \Rightarrow \Delta, y : A$ is an initial sequent, in the latter it is a conclusion of $L\perp$. *Inductive step*: assume hp-invertibility up to n, and assume $\vdash_{n+1} \Gamma \Rightarrow \Delta, x : \Box_w A$. If $x : \Box_w A$ is principal, then the premiss $xR_w y, \Gamma \Rightarrow \Delta, y : A$ (possibly obtained through hp-substitution) has a derivation of height n. Otherwise, it has one or two premisses of the form $\Gamma' \Rightarrow \Delta', x : \Box_w A$ of derivation height $\leq n$. By induction hypothesis we have $xR_w y, \Gamma' \Rightarrow \Delta', y : A$ for each premiss, with derivation height at most n. Thus, $\vdash_{n+1} xR_w y, y : A, \Gamma \Rightarrow \Delta, y : A$.

(ii) *Base case*: Similar to the above. *Inductive step*: assume hp-invertibility up to n, and assume $\vdash_{n+1} w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$. If $w : A \Box\!\!\rightarrow B$ is principal in the last rule, then the the left premiss is, or gives by hp substitution on the eigenvariable, a derivation of height at most n of $x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$. If the last rule is a relational rule, or a logical rule in which $w : A \Box\!\!\rightarrow B$ is not principal, consider the one or two premisses of the form $w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'$ of derivation height $\leq n$. Then, by induction hypotesis, we have at most $\vdash_n x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'$ for each premiss. By application of the last rule, the conclusion $\vdash_{n+1} x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$ follows. QED

Observe that Lemma 3.4(ii) states hp-invertibility of $L\Box\!\!\rightarrow$ with respect to the second premiss; its hp-invertibility with respect to the first premiss is a special case of Proposition 3.2. Therefore, as a general result, we have:

Corollary 3.5. *All the rules of* **G3LC** *are hp-invertible.*

Proof. By Lemmas 3.3 and 3.4, and Proposition 3.2 for all the other cases. QED

The rules of contraction of **G3LC** have the following form, where ϕ is either a relational atom of the form xR_wy or a labelled formula $x : A$:

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi} RC$$

Since relational atoms never appear on the right, there are just three contraction rules to be considered. We do not need to give different names for these rules since we can prove that all of them are hp-admissible:

Theorem 3.6. *The rules of left and right contraction are hp-admissible in* **G3LC**.

Proof. By simultaneous induction on the height of derivation for left and right contraction.

If $n = 0$ the premiss is either an initial sequent or a conclusion of a zero-premiss rule. In each case, the contracted sequent is also an initial sequent or a conclusion of the same zero-premiss rule.

If $n > 0$, consider the last rule used to derive the premiss of contraction. If the contraction formula is not principal in it, both occurrences are found in the premisses of the rule and they have a smaller derivation height. By the induction hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premisses. If the contraction formula is principal in it, we distinguish three cases: 1. A rule in which the principal formulas appear also in the premiss (such as *Trans*, *SConn*, *LMin*, the left rule for $\Box_w$ or the rules for $\Box\!\!\rightarrow$). 2. A rule in which the active formulas are proper subformulas of the principal formula (such as the rules for &, $\vee$, $\supset$). 3. A rule in which active formulas are relational atoms and proper subformulas of the principal formula (like rule $R\Box_w$). 4. Rule $L\Box\!\!\rightarrow$.

In the first case we have, for instance,

$$\frac{xR_wy, x : \Box_w A, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta}{xR_wy, x : \Box_w A, x : \Box_w A, \Gamma \Rightarrow \Delta} L\Box_w$$

By the induction hypothesis applied to the premiss we obtain

$$xR_wy, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta$$

with height n, and by a step of $L\Box_w$ we have

$$xR_wy, x : \Box_w A, \Gamma \Rightarrow \Delta$$

with height $n + 1$.

In the second case, contraction is reduced to contraction on shorter derivations (and smaller formulas).

In the third case, a subformula of the contraction formula and a relational atom are found in the premiss, for instance

$$\frac{xR_wy, \Gamma \Rightarrow \Delta, x : \Box_w A, y : A}{\Gamma \Rightarrow \Delta, x : \Box_w A, x : \Box_w A} R\Box_w$$

By Lemma 3.4 applied to the premiss, we obtain a derivation of height $n - 1$ of

$$xR_wy, xR_wy, \Gamma \Rightarrow \Delta, y : A, y : A$$

that yields, by the induction hypothesis for left and right contraction, a derivation of height $n-1$ of

$$xR_w y, \Gamma \Rightarrow \Delta, y : A$$

and the conclusion $\Gamma \Rightarrow \Delta, x : \Box_w A$ follows in one more step by $R\Box_w$.

Finally, if the premiss of contraction is derived by $L\Box\!\!\rightarrow$ we have

$$\frac{w : A \Box\!\!\rightarrow B, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \quad x : A, x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}{w : A \Box\!\!\rightarrow B, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\, L\Box\!\!\rightarrow$$

By inductive hypothesis applied to the left premiss we have a derivation of height $n-1$ of $w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$; by Lemma 3.4 and inductive hypothesis applied to the second premiss we have a derivation of height $n-1$ of $x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$. Thus, by a step of $L\Box\!\!\rightarrow$ we obtain a derivation of height n of $w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$. QED

Theorem 3.7. *Cut is admissible in* **G3LC**.

Proof. The proof is by induction on the weight of the cut formula and subinduction on the sum of the heights of derivations of the premisses (cut-height). The cases pertaining initial sequents and the propositional rules of the calculus are dealt with as in Theorem 11.9 of Negri and von Plato (2011) and therefore omitted here. Also the cases with cut formula not principal in both premisses of cut are dealt in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. So, the only cases to focus on are those with cut formula of the form $\Box_w A$ or $A \Box\!\!\rightarrow B$ which is principal in both premisses of cut. The former case presents, apart from the indexing on the accessibility relation, no difference with respect to the case of a plain modality, so we proceed to analyse the latter. This case splits into two cases, depending on whether the left premiss is derived by $R\Box\!\!\rightarrow_1$ or $R\Box\!\!\rightarrow_2$.

In the first case we have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{y : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_1 \quad \dfrac{\overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_3}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\, L\Box\!\!\rightarrow}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

This is converted into a derivation with three cuts of reduced height as follows (we have to split the result of the conversion to fit it in the page): First, we have a derivation $\mathcal{D}_4$

$$\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} \quad \overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : A}\, Cut$$

Further, by application of hp-substitution, we have another derivation $\mathcal{D}_5$

$$\dfrac{\overset{\mathcal{D}_1(z/y)}{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} \quad \dfrac{\overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_3}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\, L\Box\!\!\rightarrow}{z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

The two derivations are then used as premisses of a third cut of reduced weight as follows

$$\dfrac{\dfrac{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : A \quad z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\Gamma^2, \Gamma'^2 \Rightarrow \Delta^2, \Delta'^2}\, Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*$$

In the second case we have a derivation of the form

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\!\rightarrow B, x : A} \quad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\!\rightarrow B, x : \Box_w(A \supset B)}}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\!\rightarrow B}\, R \Box\!\!\!\rightarrow_2 \quad \dfrac{\dfrac{\mathcal{D}_3}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \dfrac{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\, L \Box\!\!\!\rightarrow}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

The cut is converted into six cuts of reduced height or weight of cut formula as follows: First we have the derivation (call it $\mathcal{D}_5$)

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\!\rightarrow B, x : A} \quad \dfrac{\dfrac{\mathcal{D}_3}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \dfrac{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : A}\, Cut$$

with a cut of reduced height. We also have the derivation (call it $\mathcal{D}_6$)

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\!\rightarrow B, x : \Box_w(A \supset B)} \quad \dfrac{\dfrac{\mathcal{D}_3}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \dfrac{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\, L \Box\!\!\!\rightarrow}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : \Box_w(A \supset B)}\, Cut \quad \dfrac{\mathcal{D}_4(x/y)}{x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{x : A, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2}\, Cut$$

with two cuts, the upper of reduced height, and the lower of reduced weight; finally we obtain the derivation

$$\dfrac{\dfrac{\mathcal{D}_5 \quad \mathcal{D}_6}{\Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}\, Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*$$

with a cut or reduced weight and repeated applications of contraction. QED

To ensure the consequences of cut elimination we need to establishing another crucial property of our system. We say that a labelled system has the *subterm property* if every variable occurring in any derivarion is either an eigenvariable or occurs in the conclusion.[4] Clearly, the rules of **G3LC** do not, as they stand, satisfy the subterm property, but we can prove that we can, without loss of generality, restrict proof search to derivations that have the subterm property.

Proposition 3.8. *Every sequent derivable in* **G3LC** *is derivable by a derivation that satisfies the subterm property.*

Proof. By induction on the height of the derivation. For the inductive step, the conclusion is clear if the last step is one of the rules in which all the labels in the premisses satisfy the subterm property. For the other rules (in this specific calculus, rules *Ref* and $R \Box\!\!\!\rightarrow_1$), we consider the violating cases in which the premisses contain a label which is not in the conclusion. Using hp-substitution, we replace it to a label in the conclusion and obtain a derivation of the same height that satisfies the subterm property. QED

By the above result, in the following we shall always restrict attention to derivations with the subterm property.

[4] This property is called *analyticity* in Dyckhoff and Negri (2012).

4 Lewis' axioms and rules

In this Section we shall prove that Lewis' axiomatic system for counterfactuals **VC**, regarded as Lewis' *official logic of counterfactuals*, is captured by our system **G3LC** by showing that Lewis' axioms are provable in **G3LC** and that the inference rules of **VC** are admissible.[5] Together with a proof of soundness of our rules with respect to Lewis' semantics for counterfactuals based on a ternary accessibility relation, this result will give an indirect completeness proof for our system. We shall however present in Section 5 a direct completeness proof for **G3LC** with respect to the same semantics. For brevity, in the following proofs the repetition of the main formula in the premisses is not indicated.

Proposition 4.1. *The following rules are admissible in* **G3LC***:*

1. *Modus Ponens:* $\frac{\vdash A \quad \vdash A \supset B}{\vdash B}$
2. *Deduction within Conditionals: for any $n \geq 1$*

$$\frac{\vdash A_1 \& \ldots \& A_n \supset B}{\vdash ((D \,\Box\!\!\rightarrow A_1) \& \ldots \& (D \,\Box\!\!\rightarrow A_n)) \supset (D \,\Box\!\!\rightarrow B)}$$

3. *Interchange of logical equivalents: if $\vdash A \supset\subset B$ and $\vdash \Phi(A)$ then $\vdash \Phi(B)$, where Φ is an arbitrary formula in the language.*

For the proof, see the Appendix, on Section 8.1.

Next, we prove that all the axioms of **VC** are derivable in **G3LC**, i.e. for each axiom A the sequent $\Rightarrow x : A$ is derivable in the calculus where x is an arbitrary label.

Proposition 4.2. *The following axioms are derivable in* **G3LC***:*

1. *Propositonal tautologies,*
2. $A \,\Box\!\!\rightarrow A$*,*
3. $(\neg A \,\Box\!\!\rightarrow A) \supset (B \,\Box\!\!\rightarrow A)$*,*
4. $(A \,\Box\!\!\rightarrow \neg B) \vee (((A \& B) \,\Box\!\!\rightarrow C) \supset\subset (A \,\Box\!\!\rightarrow (B \supset C)))$*,*
5. $(A \,\Box\!\!\rightarrow B) \supset (A \supset B)$*,*
6. $(A \& B) \supset (A \,\Box\!\!\rightarrow B)$*.*

For the proof, see the Appendix, on Section 8.2.

4.1 Equivalence with L-SC

We can show that the rules of **G3LC** are interderivable with Lewis' defining semantic condition for the counterfactual, suitably formalized in a first-order language with variables ranging over possible worlds as follows:

$$w : A \,\Box\!\!\rightarrow B \leftrightarrow \forall z(z : \neg A) \vee \exists x(x : A \,\&\, x : \Box_w(A \supset B)) \qquad \textbf{L-SC}$$

[5]Cf. p. 132 of *Counterfactuals*.

For brevity, we will shorten the right-hand side of the biconditional **L-SC** with the letter K.

Observe that the proofs below use quantification over worlds that is not part of the calculus that we have introduced. The derivations have to be read as a formal account of the reasoning that can be done at the meta-level to justify the rules in view of the semantic explanation of the counterfactual, and vice versa. As possible worlds can be regarded as first-order entities, we take the usual rules for the quantifiers for that purpose.

Given the above constraints, we show:

Proposition 4.3. *The rules of* **G3LC** *are interderivable with* **L-SC**.

Proof. To save space, we shall not copy the main formula in the premisses. The following is a proof that $L\Box\!\!\rightarrow$ gives the left-to-right direction of **L-SC**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 2.2}}{z : A \Rightarrow z : A, \dots} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 2.2}}{x : A \Rightarrow x : A, \dots} \qquad \dfrac{\text{Lemma 2.2}}{x : A, x : \Box_w(A \supset B) \Rightarrow x : \Box_w(A \supset B)}}{x : A, x : \Box_w(A \supset B) \Rightarrow x : A \,\&\, x : \Box_w(A \supset B)}{\scriptstyle R\&}}{x : A, x : \Box_w(A \supset B) \Rightarrow \exists x(x : A \,\&\, x : \Box_w(A \supset B))}{\scriptstyle R\exists}}{x : A, x : \Box_w(A \supset B), z : A \Rightarrow \exists x(x : A \,\&\, x : \Box_w(A \supset B))}{\scriptstyle LW}}{w : A \Box\!\!\rightarrow B, z : A \Rightarrow \exists x(x : A \,\&\, x : \Box_w(A \supset B))}{\scriptstyle L\Box\!\rightarrow}}{w : A \Box\!\!\rightarrow B \Rightarrow z : \neg A, \exists x(x : A \,\&\, x : \Box_w(A \supset B))}{\scriptstyle R\neg}}{w : A \Box\!\!\rightarrow B \Rightarrow \forall z(z : \neg A), \exists x(x : A \,\&\, x : \Box_w(A \supset B))}{\scriptstyle R\forall}}{w : A \Box\!\!\rightarrow B \Rightarrow K}{\scriptstyle R\vee}}{\Rightarrow w : A \Box\!\!\rightarrow B \supset K}{\scriptstyle R\supset}$$

The following is a derivation of the right-to-left direction of **L-SC** that uses rules $R\Box\!\!\rightarrow_1$ and $R\Box\!\!\rightarrow_2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 2.2}}{y : A \Rightarrow y : A}}{y : \neg A, y : A \Rightarrow}{\scriptstyle L\neg}}{\forall z(z : \neg A), y : A \Rightarrow}{\scriptstyle L\forall}}{\forall z(z : \neg A) \Rightarrow w : A \Box\!\!\rightarrow B}{\scriptstyle R\Box\!\rightarrow_1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 2.2}}{x : A \Rightarrow x : A} \qquad \dfrac{\text{Lemma 2.2}}{\dots, x : \Box_w(A \supset B) \Rightarrow x : \Box_w(A \supset B)}}{x : A, x : \Box_w(A \supset B) \Rightarrow w : A \Box\!\!\rightarrow B}{\scriptstyle R\Box\!\rightarrow_2}}{x : A \,\&\, x : \Box_w(A \supset B) \Rightarrow w : A \Box\!\!\rightarrow B}{\scriptstyle L\&}}{\exists x(x : A \,\&\, x : \Box_w(A \supset B)) \Rightarrow w : A \Box\!\!\rightarrow B}{\scriptstyle L\exists}}{K \Rightarrow w : A \Box\!\!\rightarrow B}{\scriptstyle L\vee}}{\Rightarrow K \supset w : A \Box\!\!\rightarrow B}{\scriptstyle L\supset}$$

For the remaining derivations that prove the equivalence, we use the right-to-left direction of **L-SC** to get first a derivation of rule $R\Box\!\!\rightarrow_1$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, z : \neg A, w : A \Box\!\!\rightarrow B}{\scriptstyle R\neg}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), w : A \Box\!\!\rightarrow B}{\scriptstyle R\forall}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), \exists x(x : A \,\&\, x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}{\scriptstyle RW}}{\Gamma \Rightarrow \Delta, K, w : A \Box\!\!\rightarrow B}{\scriptstyle R\vee} \qquad K \Rightarrow w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\scriptstyle Cut}$$

and then a derivation of $R\Box\!\!\rightarrow_2$ by the second disjunct

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A, w : A \Box\!\!\rightarrow B \quad \Gamma \Rightarrow \Delta, x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, x : A \,\&\, x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B}{\scriptstyle R\&}}{\Gamma \Rightarrow \Delta, \exists x(x : A \,\&\, x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}{\scriptstyle R\exists}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), \exists x(x : A \,\&\, x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}{\scriptstyle RW}}{\Gamma \Rightarrow \Delta, K, w : A \Box\!\!\rightarrow B}{\scriptstyle R\vee} \qquad K \Rightarrow w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\scriptstyle Cut}$$

Finally we have a derivation of $L\Box\!\!\rightarrow$ using the left-to-right direction of **L-SC**

$$\dfrac{w : A \Box\!\!\rightarrow B \Rightarrow K \qquad \dfrac{\dfrac{\dfrac{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A}{w : A \Box\!\!\rightarrow B, z : \neg A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\neg}}{w : A \Box\!\!\rightarrow B, \forall z(z : \neg A), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\forall} \qquad \dfrac{\dfrac{x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{x : A \,\&\, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\&}}{\exists x(x : A \,\&\, x : \Box_w(A \supset B)), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\exists}}{w : A \Box\!\!\rightarrow B, K, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\vee}}{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}$$

QED

5 Completeness

In this Section we shall give a direct completeness proof for **G3LC** with respect to Lewis semantics. The proof has the overall structure of the completeness proof for labelled systems for modal and non-classical logics given in Negri (2009) and Negri (2014a), but the semantics is here based on comparative similarity systems rather than on Kripke models.

Definition 5.1. *Let $\mathcal{W}$ be the set of variables (labels) used in derivations in* **G3LC**. *A* **comparative similarity system** *$\mathcal{S}$ is an assignment to every $w \in \mathcal{W}$ of a two-place relation $\preceq_w$ with the aforementioned conditions:*

1. *Transitivity: If $x \preceq_w y$ and $y \preceq_w z$ then $x \preceq_w z$,*
2. *Strong connectedness: Either $x \preceq_w y$ or $y \preceq_w x$,*
3. *L-Minimality: If $x \preceq_w w$ then $x = w$.*

An **interpretation** *of the labels in $\mathcal{W}$ in $\mathcal{S}$ a map $[\![\cdot]\!] : \mathcal{W} \to \mathcal{S}$. A* **valuation** *of atomic formulas in $\mathcal{S}$ is a map $\mathcal{V} : AtFrm \to \mathcal{P}(\mathcal{S})$ that assigns to each atom P the set of elements of $\mathcal{W}$ in which P holds. Instead of writing $w \in \mathcal{V}(P)$, we adopt the standard notation $w \Vdash P$.*

Valuations are extended to arbitrary formulas by the following inductive clauses:
$\mathcal{V}_\bot$: $x \Vdash \bot$ for no x.
$\mathcal{V}_\&$: $x \Vdash A\&B$ iff $x \Vdash A$ and $x \Vdash B$.
$\mathcal{V}_\vee$: $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.
$\mathcal{V}_\supset$: $x \Vdash A \supset B$ iff if $x \Vdash A$ then $x \Vdash B$.
$\mathcal{V}_{\Box_w}$: $x \Vdash \Box_w A$ iff for all y, if $y \preceq_w x$ then $y \Vdash A$.
$\mathcal{V}_{\Box\!\rightarrow}$: $x \Vdash A \Box\!\!\rightarrow B$ iff either $z \Vdash A$ for no z, or $y \Vdash A$ and $y \Vdash \Box_x(A \supset B)$ for some y.

Definition 5.2. *A labelled formula $x : A$ (resp. a relational atom xR_wy) is* **true** *for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in a system $\mathcal{S}$ iff $[\![x]\!] \Vdash A$ (resp. $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$). A sequent $\Gamma \Rightarrow \Delta$ is true for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in a system $\mathcal{S}$ if, whenever for all labelled formulas $x : A$ and relational atom xR_wy in Γ it is the case that $[\![x]\!] \Vdash A$ and $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, then for some $w : B$ in Δ, $[\![w]\!] \Vdash B$. A sequent is* **valid** *in a system $\mathcal{S}$ iff it is true for every interpretation and valuation in $\mathcal{S}$.*

Theorem 5.3. (Soundness) *If a sequent is derivable in* **G3LC** *then it is valid in every comparative similarity system $\mathcal{S}$ where conditions (1)-(3) hold.*

For the proof, see the Appendix, on Section 8.3.

Theorem 5.4. (Completeness) *Let* $\Gamma \Rightarrow \Delta$ *be a sequent in the language of* **G3LC**. *If it is valid in every comparative similarity system, it is derivable in* **G3LC**.

Proof. Immediate by Proposition 4.1, Proposition 4.2, Theorem 5.3 and Lewis' own completeness proof (cf. *Countefactuals*, pp. 118-134). QED

Completeness can be established also as a corollary of the following:

Theorem 5.5. *Let* $\Gamma \Rightarrow \Delta$ *be a sequent in the language of* **G3LC**. *Then either it is derivable in* **G3LC** *or it has a countermodel in* $\mathcal{S}$.

For the proof, see Section 8.4 of the Appendix.

6 Decidability

In general cut elimination alone does not ensure terminating proof search in a given calculus. The exhaustive proof search used in the proof of Theorem 5.5 is not a decision method nor an effective method of finding countermodels when proof search fails, as it may produce infinite branches and therefore infinite countermodels. By way of example, consider the following branch in the search for a proof of the sequent $\Rightarrow w : \Box_x \neg \Box_x A \supset \Box_x B$ (this is analogous to the case for S4 discussed in Section 11.5 of Negri and von Plato 2011):

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{wR_xy, yR_xz, wR_xz, zR_xt, w : \Box_x\neg\Box_xA \Rightarrow t : A, z : A, y : B}}{wR_xy, yR_xz, wR_xz, w : \Box_x\neg\Box_xA \Rightarrow z : \Box_xA, z : A, y : B}\,R\Box_x}{wR_xy, yR_xz, wR_xz, w : \Box_x\neg\Box_xA, z : \neg\Box_xA \Rightarrow z : A, y : B}\,L\supset}{wR_xy, yR_xz, wR_xz, w : \Box_x\neg\Box_xA \Rightarrow z : A, y : B}\,L\Box_x}{wR_xy, yR_xz, w : \Box_x\neg\Box_xA \Rightarrow z : A, y : B}\,Trans}{wR_xy, w : \Box_x\neg\Box_xA \Rightarrow y : \Box_xA, y : B}\,R\Box_x}{wR_xy, w : \Box_x\neg\Box_xA, y : \neg\Box_xA \Rightarrow y : B}\,L\supset}{wR_xy, w : \Box_x\neg\Box_xA \Rightarrow y : B}\,L\Box_x}{\cfrac{w : \Box_x\neg\Box_xA \Rightarrow w : \Box_xB}{\Rightarrow w : \Box_x\neg\Box_xA \supset \Box_xB}\,R\supset}\,R\Box_x
$$

Clearly the search goes on forever because of the new accessibility relations that are created by the right rules for the indexed modalities, $R\Box_x$, together with *Trans*. To see that this search does not end up in a derivation, we may nevertheless exhibit a finite countermodel by a suitable truncation of the otherwise infinite countermodel provided by the completeness proof.

Following the method of finitization of countermodels generated by proof search in a labelled calculus, presented for intuitionistic propositional logic in Negri (2014a) and for multi-modal logics in Garg et al. (2012), we define a *saturation* condition for branches on a reduction tree. Intuitively, a branch is saturated when its leaf is not an initial sequent nor a conclusion of $L\bot$, and when it is closed under all the rules except for $R\Box_x$ in case it generates a loop modulo new labelling. To obtain the finite countermodel, we define a partial order through the reflexive and transitive closure of the similarity relation together with a relation that witnesses such loops. Let $\downarrow \Gamma(\downarrow \Delta)$ be the union of the antecedents (succedents) in a branch from the endsequent up to $\Gamma \Rightarrow \Delta$.

Let us define the following sets of formulas:

$$\mathcal{F}^1_{\Gamma\Rightarrow\Delta}(w) \equiv \{A \mid w : A \in\downarrow\Gamma\} \cup \{\Box_x A \mid y : \Box_x A, yR_x w \in \Gamma\}$$

$$\mathcal{F}^2_{\Gamma\Rightarrow\Delta}(w) \equiv \{A \mid w : A \in\downarrow\Delta\}$$

and let $w \leq_{\Gamma\Rightarrow\Delta} y$ iff $\mathcal{F}^i_{\Gamma\Rightarrow\Delta}(w) \subseteq \mathcal{F}^i_{\Gamma\Rightarrow\Delta}(y)$ for $i = 1, 2$.

Definition 6.1. *A branch in a proof search up to a sequent $\Gamma \Rightarrow \Delta$ is* **saturated** *if the following conditions are satisfied:*

1. *If w is a label in Γ, Δ, then $w = w$ and $wR_x w$ are in Γ.*
2. *If $wR_x y$ and $yR_x z$ are in Γ, then $wR_x z$ is.*
3. *If $wR_w x$ is in Γ, then $x = w$ is.*
4. *If w, x, y are labels in Γ, Δ, then either $wR_x y$ or $yR_x w$ is in Γ*
5. *There is no w such that $w : \bot$ is in Γ.*
6. *If $w : A\&B$ is in $\downarrow\Gamma$, then $w : A$ and $w : B$ are in $\downarrow\Gamma$.*
7. *If $w : A\&B$ is in $\downarrow\Delta$, then either $w : A$ or $w : B$ is in $\downarrow\Delta$.*
8. *If $w : A \vee B$ is in $\downarrow\Gamma$, then either $w : A$ or $w : B$ is in $\downarrow\Gamma$.*
9. *If $w : A \vee B$ is in $\downarrow\Delta$, then $w : A$ and $w : B$ are in $\downarrow\Delta$.*
10. *If $w : A \supset B$ is in $\downarrow\Gamma$, then either $w : A$ is in $\downarrow\Delta$ or $w : B$ is in $\downarrow\Gamma$.*
11. *If $w : A \supset B$ is in $\downarrow\Delta$, then $w : A$ is in $\downarrow\Gamma$ and $w : B$ is in $\downarrow\Delta$.*
12. *If $w : \Box_x A$ and $wR_x y$ are in Γ, then $y : A$ is in $\downarrow\Gamma$.*
13. *If $w : \Box_x A$ is in $\downarrow\Delta$, then either*

    a. for some y, there is $wR_x y$ in Γ and $y : A$ is in $\downarrow\Delta$, or
    b. for some y such that $y \neq w$, there is $yR_x w$ in Γ and $w \leq_{\Gamma\Rightarrow\Delta} y$.

14. *If $w : A \mathrel{\Box\!\!\rightarrow} B$ is in Γ, then either $z : A$ is in $\downarrow\Delta$ for z in Γ, Δ, or for some y, $y : A, y : \Box_w(A \supset B)$ is in Γ.*
15. *If $w : A \mathrel{\Box\!\!\rightarrow} B$ is in $\downarrow\Delta$, then $y : A$ is in $\downarrow\Gamma$ and either $z : A$ or $z : \Box_w(A \supset B)$ is in $\downarrow\Delta$ for z in Γ, Δ.*

Notice that this definition blocks the proof search in the example above when it produces the formula $t : \Box_x A$ because of clause 13.b (since we then have $t \leq_{\Gamma\Rightarrow\Delta} z$). The finite countermodel is defined as in the proof of Theorem 5.5 starting from the sets $\downarrow\Gamma, \downarrow\Delta$.

Proposition 6.2. *The finite countermodel defined by the saturation procedure is a comparative similarity system.*

For the proof, see the Appendix, on Section 8.5. We further observe that by the subterm property the number of distinct formulas in the sequents of an attempted proof is bounded. Since duplication of the same labelled formulas is not possible by hp-admissibility of contraction, we can prove the following result:

Theorem 6.3. *The system* **G3LC** *allows a terminating proof search.*

Proof. Let F be the set of (unlabelled) subformulas of the endsequent and consider a string of labels $w_0 R_x w_1, w_1 R_x w_2, w_2 R_x w_3, \ldots$ generated by the saturation procedure. For an arbitrary x_j consider the values of the sets $\mathcal{F}^i(x_k)$ for $k < j$ at the step in which x_j was introduced. Clearly $\mathcal{F}^i(x_j) \nsubseteq \mathcal{F}^i(x_k)$ or else x_j would not have been introduced. So each new label corresponds to a new subset of $F \times F$. Since the number of these subsets is finite, also the length of each chain of labels must be finite. QED

7 Conclusion

We presented **G3LC**, a Gentzen-style sequent calculus for David Lewis' logic of counterfactuals **VC**, and proved it sound and complete with respect to Lewis' semantics. In **G3LC**, substitution of labels and left and right weakening and contraction are height-preserving admissible and cut is admissible. Moreover, all the rules are invertible. Finally, we proved a decidability result based on a bounded procedure of root-first proof search that for any given sequent either provides a derivation or a countermodel.

What remains to be spelled out in detail are similar results for the calculus **G3LC4**, i.e. the embedding of **G3LC** in an S4 modal logic, which we presented in Section 2. We expect these further results to be achieved by routine application of the methods used here.

In his book *Countefactuals*, Lewis presents a class **V** of axomatic systems for conditional logics, among which is **VC**. We leave for further work a detailed deductive analysis of the entire class, as well as of conditional logics that are based on alternative versions of Lewis' semantics.

With respect to related work, it is worth mentioning that Lellman and Pattinson (2012) present an unlabelled sequent calculus for Lewis' logic with the binary connective "at least as possible as" as primitive. Since, according to Lewis, *if p were the case, then q would be the case* is definable in terms of *p is at least as possible as q* plus propositional material, their work can be seen as indirectly providing an analysis of Lewis conditional logics. However, their proof-theoretic results fail to carry over straightforwardly when the calculus is extended by the explicit addition of Lewis counterfactual (Lellman and Pattinson 2012, p.13). The work by Olivetti et al. (2007) presents a labelled sequent calculus for Lewis conditional logics and is methodologically close to our approach but rests crucially on the limit assumption. In so far as Lewis' preferred interpretation of the counterfactual conditional notoriously rejects the limit assumption, the strategy followed in the present paper appears to be a more faithful proof-theoretic analysis of Lewis' work. Priest (2008) gives a survey of the closely related topic of tableau systems for conditional logics.

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References

Dyckhoff, R. and S. Negri (2012) Proof analysis in intermediate logics. *Archive for Mathematical Logic*, vol. 51, pp. 71–92, 2012.

Dyckhoff, R. and S. Negri (2013) A cut-free sequent system for Grzegorczyk logic with an application to the Gödel-McKinsey-Tarski embedding. *Journal of Logic and Computation*, DOI: 10.1093/logcom/ext036.

Dyckhoff, R. and S. Negri (2014) Geometrization of first-order theories, ms.

Fitting, M. (2007) Modal proof theory. In P. Blackburn, J. Van Benthem, and F. Wolter (eds) *Handbook of Modal Logic*, pp. 85–138, Elsevier.

Garg, D., V. Genovese, and S. Negri (2012) Countermodels from sequent calculi in multimodal logics. *LICS 2012*, IEEE Computer Society, pp. 315–324.

Gasquet, O. and A. Herzig (1996) From Classical to Normal Modal Logics. In H. Wansing (ed) *Proof Theory of Modal Logic* Applied Logic Series vol. 2, pp. 293-311, Springer.

Giordano, L., V. Gliozzi, N. Olivetti, and C. Schwind (2003) Tableau calculi for preference-based conditional logics. *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Computer Science vol. 2796, pp. 81–101.

Kripke, S. (1963) Semantical analysis of modal logic I. Normal modal propositional calculi. *Zetschrift für mathematische Logik und Grundlagen der Math.*, vol. 9, pp. 67–96.

Lellman, B. and D. Pattinson (2012) Sequent Systems for Lewis' conditional logics. In L. Farinas del Cerro, A. Herzig and J. Mengin (eds.), *Logics in Artificial Intelligence*, pp. 320-332.

Lewis, D. (1973) *Counterfactuals*. Blackwell.

Lewis, D. (1973a) Counterfactuals and comparative possibility. *Journal of Philosophical Logic*, vol, 2, pp. 418–446, 1973.

Negri, S. (2003) Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem. *Archive for Mathematical Logic*, vol. 42, pp. 389–401.

Negri, S. (2005) Proof analysis in modal logic. *Journal of Philosophical Logic*, vol. 34, pp. 507–544.

Negri, S. (2009) Kripke completeness revisited. In G. Primiero and S. Rahman (eds.), *Acts of Knowledge - History, Philosophy and Logic*, pp. 247–282, College Publications.

Negri, S. (2014) Proof analysis beyond geometric theories: from rule systems to systems of rules. *Journal of Logic and Computation*, doi: 10.1093/logcom/exu037.

Negri, S. (2014a) Proofs and countermodels in non-classical logics. *Logica Universalis*, vol. 8, pp. 25-60.

Negri, S. and J. von Plato (1998) Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, vol. 4, pp. 418–435.

Negri, S. and J. von Plato (2001) *Structural Proof Theory*. Cambridge University Press.

Negri, S. and J. von Plato (2011) *Proof Analysis*. Cambridge University Press.

Negri, S. and J. von Plato (2014) Meaning in use. In H. Wansing (ed) *Dag Prawitz on Proofs and Meaning*, Trends in Logic, Springer, in press.

Olivetti, N., G. L. Pozzato and C. Schwind (2007) A Sequent Calculus and a Theorem Prover for Standard Conditional Logics. *ACTM Transactions on Computational Logics (TOCL)*, vol. 8/4, 22/pp. 1–51.

Priest, G. (2008) *An Introduction to Non-Classical Logic*, Cambridge University Press.

8 Appendix

8.1 Proof of Proposition 4.1

Proof. (1) The rule of modus ponens of an axiomatic system is translated as the sequent calculus rule

$$\frac{\Rightarrow x : A \quad \Rightarrow x : A \supset B}{\Rightarrow x : B}$$

The rule is shown admissible in **G3LC** using invertibility of $R\supset$ and admissibility of cut.

(2) Assume $\Rightarrow z : A_1\& \ldots \& A_n \supset B$ for an arbitrary z. By invertibility of $R\supset$ and $L\&$ we obtain $z : A_1, \ldots, z : A_n \Rightarrow z : B$. Then for an arbitrary x we have to derive $\Rightarrow x : ((D \Box\!\!\rightarrow A_1)\& \ldots \&(D \Box\!\!\rightarrow A_n)) \supset (D \Box\!\!\rightarrow B)$. Consider the following proof search:[6]

$$\dfrac{\dfrac{\dfrac{\dfrac{z : D \Rightarrow z : D \qquad \dfrac{y_i : D \Rightarrow y_i : D \qquad \dfrac{\dfrac{y_1R_xw_1, \ldots, y_nR_xw_n, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n), w_1 : D, \ldots, w_n : D \Rightarrow w_1 : B, \ldots, w_n : B}{y_1R_xw_1, \ldots, y_nR_xw_n, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow w_1 : D \supset B, \ldots, w_n : D \supset B}{\scriptstyle R\supset, n\ times}}{y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow y_1 : \Box_x(D \supset B), \ldots, y_n : \Box_x(D \supset B)}{\scriptstyle R\Box_x, n\ times}}{y_1 : D, \ldots, y_n : D, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow x : D \Box\!\!\rightarrow B}{\scriptstyle R\Box\!\!\rightarrow_2, n\ times}}{z : D, x : D \Box\!\!\rightarrow A_1, \ldots, x : D \Box\!\!\rightarrow A_n \Rightarrow x : D \Box\!\!\rightarrow B}{\scriptstyle L\Box\!\!\rightarrow, n\ times}}{x : D \Box\!\!\rightarrow A_1, \ldots, x : D \Box\!\!\rightarrow A_n \Rightarrow x : D \Box\!\!\rightarrow B}{\scriptstyle R\Box\!\!\rightarrow_1}}{x : (D \Box\!\!\rightarrow A_1)\& \ldots \&(D \Box\!\!\rightarrow A_n) \Rightarrow x : D \Box\!\!\rightarrow B}{\scriptstyle L\&, n\ times}}{\Rightarrow x : ((D \Box\!\!\rightarrow A_1)\& \ldots \&(D \Box\!\!\rightarrow A_n)) \supset (D \Box\!\!\rightarrow B)}{\scriptstyle R\supset}$$

Observe that we cannot conclude by applications of $L\Box_x$ to obtain a sequent derivable from the assumption because the formulas in the right topsequent of the above tree have different labels by the freshness conditions. Instead we apply at most n times *SConn* with the following pattern on the right premiss:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ y_1R_xy_3, y_1R_xy_2, \Gamma'' \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\vdots \\ y_3R_xy_1, y_1R_xy_2, \Gamma'' \Rightarrow \Delta\end{matrix}}{y_1R_xy_2, \Gamma' \Rightarrow \Delta} \qquad \dfrac{\begin{matrix}\vdots \\ y_2R_xy_3, y_2R_xy_1, \Gamma''' \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\vdots \\ y_3R_xy_2, y_2R_xy_1, \Gamma''' \Rightarrow \Delta\end{matrix}}{y_2R_xy_1, \Gamma' \Rightarrow \Delta}}{y_1R_xw_1, \ldots, y_nR_xw_n, \Gamma \Rightarrow \Delta}$$

Thus by *Trans* at each topsequent there is a y_h such that $y_1R_xy_h, \ldots, y_nR_xy_h$. Correspondingly, there is a w_h such that (again by n applications of *Trans*), $y_1R_xw_h, \ldots, y_nR_xw_h$. On this w_h we apply n times $L\Box_x$, obtaining the following (the relational atoms are omitted for brevity):

$$\dfrac{\dfrac{w_h : D \Rightarrow w_h : D \quad w_h : A_1, \ldots, w_h : A_n \Rightarrow w_h : B}{w_h : D \supset A_1, \ldots, w_h : D \supset A_n, w_h : D \Rightarrow w_h : B}{\scriptstyle L\supset, n\ times}}{y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n), w_h : D \Rightarrow w_h : B}{\scriptstyle L\Box_x, n\ times}$$

[6] For brevity, some of the repetitions of principal formulas in the premisses as well as the derivable left premisses of $R\Box\!\!\rightarrow_2$ have been omitted. Use of indexes for formulas should be obvious.

The right premiss now derivable from the assumption (with the arbitrary label z chosen to be w_h).

(3) By induction on the structure of Φ. If Φ is $\bot$ or an atomic formula the claim is trivial; if it is a conjunction, disjunction, implication, or an indexed modality, corresponding steps and invertibility of $R\supset$ give the claim. If Φ is a conditional and the interchange is done in the succedent, the conclusion follows from (2). If the interchange is in the antecedent, the claim follows from admissibility of

$$\dfrac{\vdash A \supset\subset B}{A \Box\!\!\to C \vdash B \Box\!\!\to C}$$

First, we observe that from the assumption we have that for arbitrary y, z the sequents $y : A \Rightarrow y : B$ and $z : B \Rightarrow z : A$ are derivable, hence also the sequent $y : \Box_w(A \supset C) \Rightarrow y : \Box_w(B \supset C)$ is. We then have the following derivation

$$\dfrac{\dfrac{z : B \Rightarrow z : A \qquad \dfrac{\dots y : A \Rightarrow y : B \dots \qquad \dots y : \Box_w(A \supset C) \Rightarrow y : \Box_w(B \supset C) \dots}{y : B, y : \Box_w(A \supset C) \Rightarrow w : B \Box\!\!\to C}\,{\scriptstyle R\Box\!\!\to_2}}{z : B, w : A \Box\!\!\to C \Rightarrow w : B \Box\!\!\to C}\,{\scriptstyle L\Box\!\!\to}}{w : A \Box\!\!\to C \Rightarrow w : B \Box\!\!\to C}\,{\scriptstyle R\Box\!\!\to_1}$$

QED

8.2 Proof of Proposition 4.2

Proof. All propositional tautologies are clearly derivable because **G3LC** is an extension of a complete calculus for classical propositional logic. For items 2–6 we have the following derivations:

(2)

$$\dfrac{\dfrac{z : A \Rightarrow z : A \qquad \dfrac{\dfrac{zR_xy, z : A, y : A \Rightarrow y : A}{zR_xy, z : A \Rightarrow y : A \supset A}\,{\scriptstyle R\supset}}{z : A \Rightarrow z : \Box_x(A \supset A)}\,{\scriptstyle R\Box_x}}{z : A \Rightarrow x : A \Box\!\!\to A}\,{\scriptstyle R\Box\!\!\to_2}}{\Rightarrow x : A \Box\!\!\to A}\,{\scriptstyle R\Box\!\!\to_1}$$

(3)

$$\dfrac{\dfrac{\dfrac{z : B \Rightarrow z : B \qquad \dfrac{\dfrac{\dfrac{\dfrac{y : A \Rightarrow y : A}{\Rightarrow y : A, y : \neg A}\,{\scriptstyle R\supset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w : A \Rightarrow w : A}{\Rightarrow w : \neg A, w : A}\,{\scriptstyle R\supset}}{w : \neg A \Rightarrow w : \neg A}\,{\scriptstyle L\supset} \qquad \dfrac{w : A \Rightarrow w : A}{w : \neg A, w : A \Rightarrow}\,{\scriptstyle L\supset}}{wR_xw, w : \neg A, w : \neg A \supset A \Rightarrow}\,{\scriptstyle L\supset}}{wR_xw, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow}\,{\scriptstyle L\Box_x}}{w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow}\,{\scriptstyle Ref}}{zR_xy, y : B, x : \neg A \Box\!\!\to A \Rightarrow y : A}\,{\scriptstyle L\Box\!\!\to}}{zR_xy, x : \neg A \Box\!\!\to A \Rightarrow y : B \supset A}\,{\scriptstyle R\supset}}{x : \neg A \Box\!\!\to A \Rightarrow z : \Box_x(B \supset A)}\,{\scriptstyle R\Box_x}}{z : B, x : \neg A \Box\!\!\to A \Rightarrow x : B \Box\!\!\to A}\,{\scriptstyle R\Box\!\!\to_2}}{x : \neg A \Box\!\!\to A \Rightarrow x : B \Box\!\!\to A}\,{\scriptstyle R\Box\!\!\to_1}}{\Rightarrow x : (\neg A \Box\!\!\to A) \supset (B \Box\!\!\to A)}\,{\scriptstyle R\supset}$$

(4)

$$\dfrac{\dfrac{\vdots}{\Rightarrow x : A \Box\!\!\to \neg B, x : A\&B \Box\!\!\to C \supset\subset A \Box\!\!\to (B \supset C)}\,{\scriptstyle R\&}}{\Rightarrow x : (A \Box\!\!\to \neg B) \lor (A\&B \Box\!\!\to C \supset\subset A \Box\!\!\to (B \supset C))}\,{\scriptstyle R\lor}$$

The proof branches; we continue below with the two premisses

$$\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : ((A\&B) \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} (B \supset C))$$

and

$$\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : (A \mathrel{\Box\!\!\rightarrow} (B \supset C)) \supset ((A\&B) \mathrel{\Box\!\!\rightarrow} C)$$

Proof of the former:[7]

$$
\begin{prooftree}
\AxiomC{$y : A \Rightarrow y : A$}
\AxiomC{$y : B \Rightarrow y : B$}
\RightLabel{$R\&$}
\BinaryInfC{$y : A, y : B \Rightarrow y : A\&B$}
\AxiomC{$t : A \Rightarrow t : A$}
\AxiomC{$t : B \Rightarrow t : B$}
\RightLabel{$R\&$}
\BinaryInfC{$t : A, t : B \Rightarrow t : A\&B$}
\AxiomC{$t : C \Rightarrow t : C$}
\RightLabel{$L\supset$}
\BinaryInfC{$t : A\&B \supset C, t : A, t : B \Rightarrow t : C$}
\RightLabel{$R\supset$}
\UnaryInfC{$t : A\&B \supset C, t : A \Rightarrow t : B \supset C$}
\RightLabel{$R\supset$}
\UnaryInfC{$wR_x t, t : A\&B \supset C \Rightarrow t : A \supset (B \supset C)$}
\RightLabel{$L\Box_x$}
\UnaryInfC{$wR_x t, w : \Box_x(A\&B \supset C) \Rightarrow t : A \supset (B \supset C)$}
\RightLabel{$R\Box_x$}
\UnaryInfC{$w : A, w : B, w : \Box_x(A\&B \supset C) \Rightarrow w : \Box_x(A \supset (B \supset C))$}
\RightLabel{$R\Box\!\!\rightarrow_2$}
\UnaryInfC{$w : A, w : B, w : \Box_x(A\&B \supset C) \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$L\&$}
\UnaryInfC{$w : A\&B, w : \Box_x(A\&B \supset C) \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$L\Box\!\!\rightarrow$}
\BinaryInfC{$y : A, y : B, x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\supset$}
\UnaryInfC{$y : A, x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow y : \neg B, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\supset$}
\UnaryInfC{$zR_x y, x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow y : A \supset \neg B, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\Box_x$}
\UnaryInfC{$x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow z : \Box_x(A \supset \neg B), x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\Box\!\!\rightarrow_2$}
\UnaryInfC{$z : A, x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\Box\!\!\rightarrow_1$}
\UnaryInfC{$x : A\&B \mathrel{\Box\!\!\rightarrow} C \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\RightLabel{$R\supset$}
\UnaryInfC{$\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C \supset A \mathrel{\Box\!\!\rightarrow} (B \supset C)$}
\end{prooftree}
$$

Proof of the latter:[8]

$$
\begin{prooftree}
\AxiomC{$y : A \Rightarrow y : A$}
\AxiomC{$y : B \Rightarrow y : B$}
\RightLabel{$R\&$}
\BinaryInfC{$y : A, y : B \Rightarrow y : A\&B$}
\AxiomC{$z : A \Rightarrow z : A$}
\AxiomC{$z : B \Rightarrow z : B$}
\AxiomC{$z : C \Rightarrow z : C$}
\RightLabel{$L\supset$}
\BinaryInfC{$z : B, z : B \supset C \Rightarrow z : C$}
\RightLabel{$L\supset$}
\BinaryInfC{$z : A, z : B, z : A \supset (B \supset C) \Rightarrow z : C$}
\RightLabel{$L\&$}
\UnaryInfC{$wR_x z, wR_x y, yR_x z, z : A\&B, z : A \supset (B \supset C) \Rightarrow z : C$}
\RightLabel{$L\Box_x$}
\UnaryInfC{$wR_x z, wR_x y, yR_x z, z : A\&B, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : C$}
\RightLabel{$Trans$}
\UnaryInfC{$wR_x y, yR_x z, z : A\&B, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : C$}
\RightLabel{$R\supset$}
\UnaryInfC{$wR_x y, yR_x z, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : A\&B \supset C$}
\RightLabel{$R\Box_x$}
\UnaryInfC{$wR_x y, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : \Box_x(A\&B \supset C)$}
\RightLabel{$R\Box\!\!\rightarrow_2$}
\BinaryInfC{$wR_x y, y : A, y : B, w : \Box_x(A \supset (B \supset C)) \Rightarrow x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\supset$}
\UnaryInfC{$wR_x y, y : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\supset$}
\UnaryInfC{$wR_x y, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : A \supset \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\Box_x$}
\UnaryInfC{$w : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow w : \Box_x(A \supset \neg B), x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\Box\!\!\rightarrow_2$}
\UnaryInfC{$w : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$L\Box\!\!\rightarrow$}
\UnaryInfC{$t : A, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C) \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\Box\!\!\rightarrow_1$}
\UnaryInfC{$x : A \mathrel{\Box\!\!\rightarrow} (B \supset C) \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A\&B \mathrel{\Box\!\!\rightarrow} C$}
\RightLabel{$R\supset$}
\UnaryInfC{$\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} \neg B, x : A \mathrel{\Box\!\!\rightarrow} (B \supset C) \supset A\&B \mathrel{\Box\!\!\rightarrow} C$}
\end{prooftree}
$$

[7] Here an below, to save space, the (derivable) premisses of $R\Box\!\!\rightarrow_2$ and $L\Box\!\!\rightarrow$ that have the same formula in the antecedent and succedent have been omitted.

[8] Observe that this derivation, among others, shows that the left premiss of $R \Box\!\!\rightarrow_2$ is not dispensable.

(5)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{w : A \Rightarrow w : A \quad w : B \Rightarrow w : B}{w : A \supset B, w : A \Rightarrow w : B}\,{\scriptstyle L\supset}
}{wR_x w, w : \Box_x(A \supset B), w : A \Rightarrow w : B}\,{\scriptstyle L\Box_x}
}{x = w, xR_x w, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle Repl}
}{xR_x w, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle LMin}
\quad
\dfrac{
\dfrac{x : A \Rightarrow x : A \quad x : B \Rightarrow x : B}{x : A \supset B, x : A \Rightarrow x : B}\,{\scriptstyle L\supset}
}{wR_x x, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle L\Box_x}
}{w : A, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle SConn}
}{
\dfrac{
\dfrac{x : A \mathrel{\Box\!\!\rightarrow} B, x : A \Rightarrow x : B}{x : A \mathrel{\Box\!\!\rightarrow} B \Rightarrow x : A \supset B}\,{\scriptstyle R\supset}
}{\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} B \supset (A \supset B)}\,{\scriptstyle R\supset}
}\,{\scriptstyle L\Box\!\rightarrow}
$$

(6)

$$
\dfrac{
\dfrac{
x : A \Rightarrow x : A
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{y : A, y : A, y : B \Rightarrow y : B}{y : A, y : B \Rightarrow y : A \supset B}\,{\scriptstyle R\supset}
}{x = y, xR_x y, x : A, x : B \Rightarrow y : A \supset B}\,{\scriptstyle Repl}
}{xR_x y, x : A, x : B \Rightarrow y : A \supset B}\,{\scriptstyle LMin}
}{x : A, x : B \Rightarrow x : \Box_x(A \supset B)}\,{\scriptstyle R\Box_x}
}{x : A, x : B \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} B}\,{\scriptstyle R\Box\!\rightarrow_2}
}{x : A\&B \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} B}\,{\scriptstyle L\&}
$$

$$
\dfrac{x : A\&B \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} B}{\Rightarrow x : (A\&B) \supset (A \mathrel{\Box\!\!\rightarrow} B)}\,{\scriptstyle L\supset}
$$

QED

8.3 Proof of Theorem 5.3 (*Soundness*)

Proof. The proof is by induction on the height of the derivation and follows the methodology employed for modal and non-classical logics in Negri and von Plato (2011), Negri (2009) and Negri (2014a). We use Φ' to denote the set Φ without the principal formula of the inference step considered, which should be everywhere clear from the context.

Assume that $\Gamma \Rightarrow \Delta$ is derivable and that $[\![\cdot]\!]$ makes true all the formulas in Γ.

If $\Gamma \Rightarrow \Delta$ is an initial sequent, it is valid since both Γ and Δ contain some formula $x : A$, and then the claim is obvious. Likewise, if it is the conclusion of $L\bot$, since no interpretation would then make all the formulas in Γ true.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $L\&$, then $[\![x]\!] \Vdash A\&B$, whence $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash B$ by $\mathcal{V}_\&$. By inductive hypothesis, $x : A, x : B, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid. The argument is routine for the remaining propositional rules, and similarly for *Ref* and $Repl_{At}$.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *Trans*, then $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$ and $[\![z]\!] \preceq_{[\![w]\!]} [\![y]\!]$, whence $[\![z]\!] \preceq_{[\![w]\!]} [\![x]\!]$ by *Transitivity* of $\mathcal{S}$. By inductive hypothesis, $xR_w z, xR_w y, yR_w z, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *SConn*, then for any w, x, y, either $[\![x]\!] \preceq_{[\![w]\!]} [\![y]\!]$ or $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, by *Strong connectedness* of $\mathcal{S}$. By inductive hypothesis, both (i) $xR_w y, \Gamma \Rightarrow \Delta$ and (ii) $yR_w x, \Gamma \Rightarrow \Delta$ are valid. So if $[\![x]\!] \preceq_{[\![w]\!]} [\![y]\!]$, then by (i) $[\![\cdot]\!]$ makes true some formula in Δ. Otherwise if $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, then by (ii) $[\![\cdot]\!]$ makes true some formula in Δ. In either case, $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *LMin*, then $[\![x]\!] \preceq_{[\![w]\!]} [\![w]\!]$, whence $[\![x]\!] = [\![w]\!]$ by *Minimality* of $\mathcal{S}$. By inductive hypothesis, $x = w, wR_w x, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box_w$, let z be an arbitrary element of $\mathcal{W}$ such that $z \preceq_{[\![w]\!]} [\![x]\!]$ holds in $\mathcal{S}$. Let $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ except possibly on $[\![y]\!]$, where we set $[\![y]\!]' = z$. Then $[\![\cdot]\!]'$ makes all the formulas in xR_wy, Γ true. By inductive hypothesis, $xR_wy, \Gamma \Rightarrow \Delta', y : A$ is valid, so either some formula in Δ' or $y : A$ is true under $[\![\cdot]\!]'$. If the former holds, then $\Gamma \Rightarrow \Delta$ is valid independently of the choice of $[\![\cdot]\!]'$. If the latter, then $[\![y]\!]' \Vdash A$ for arbitrary $[\![y]\!]'$. Hence $[\![x]\!]' \Vdash \Box_w A$ by $\mathcal{V}_{\Box_w}$. But $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ w.r.t. $[\![x]\!]$, and so $[\![x]\!] \Vdash \Box_w A$. Either case, $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box_w$, then $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$ and $[\![x]\!] \Vdash \Box_w A$, whence $[\![y]\!] \Vdash A$ by $\mathcal{V}_{\Box_w}$. By inductive hypothesis, $xR_wy, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R \Box\!\!\rightarrow_1$, choose $z \notin \Gamma, \Delta$ and let $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ except possibly on $[\![z]\!]$, which we set $[\![z]\!]' = y$ for some arbitrary $y \in \mathcal{W}$ such that $y \Vdash A$. By inductive hypothesis, $z : A, \Gamma \Rightarrow \Delta', w : A \Box\!\!\rightarrow B$ is valid. The case in which there is such a y is obvious. If there is no such y, notice that $[\![\cdot]\!]' \equiv [\![\cdot]\!]$ and $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$ by $\mathcal{V}_{\Box\!\rightarrow}$. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R \Box\!\!\rightarrow_2$, then both $\Gamma \Rightarrow \Delta, x : A$ and $\Gamma \Rightarrow \Delta, x : \Box_w(A \supset B)$ are valid by inductive hypothesis. If $[\![\cdot]\!]$ makes some formula in Δ true, then $\Gamma \Rightarrow \Delta$ is valid. Otherwise, $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash \Box_w(A \supset B)$. So by $\mathcal{V}_{\Box\!\rightarrow}$, $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $L \Box\!\!\rightarrow$, then $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$, whence either for no z, $[\![z]\!] \Vdash A$, or for some x, $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash \Box_w(A \supset B)$, by $\mathcal{V}_{\Box\!\rightarrow}$. Suppose the former. By inductive hypothesis, $\Gamma \Rightarrow \Delta, z : A$ is valid, so $[\![\cdot]\!]$ makes true some formula in Δ, since $[\![z]\!] \Vdash A$ is ruled out. Hence $\Gamma \Rightarrow \Delta$ is valid. Suppose now the latter. By inductive hypothesis, $x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

QED

8.4 Proof of Theorem 5.5

Proof. We define inductively a reduction tree for an arbitrary sequent $\Gamma \Rightarrow \Delta$ in the language of **G3LC** by applying the rules of **G3LC** root first in every possible way. If the construction terminates, we have a proof of the sequent. If it doesn't, then by König's lemma the reduction tree has an infinite branch which is used to define a countermodel to $\Gamma \Rightarrow \Delta$.

1. *Construction of the reduction tree.* Let us take an arbitrary sequent $\Gamma \Rightarrow \Delta$ and take it as the root of the tree (stage 0). At stage n, we distinguish two cases:

Either all topmost sequents are an initial sequent or a conclusion of $L\bot$, in which case the construction terminates.

Otherwise, we continue the construction by applying every possible rule to all topmost sequents in a given order. There are 11 + 5 different stages, 11 for the rules of **G3LC**, 5 for the similarity rules. At stage $n = 11 + 5 + 1$ we repeat stage 1, at stage $n = 11 + 5 + 2$ we repeat stage 2, and so on.

We start for $n = 1$ with $L\&$. Suppose all topmost sequents have the form

$$x_1 : B_1\&C_1, ..., x_m : B_m\&C_m, \Gamma' \Rightarrow \Delta$$

On top of each, we write

$$x_1 : B_1, x_1 : C_1, ..., x_m : B_m, x_m : C_m, \Gamma' \Rightarrow \Delta$$

This step corresponds to applying root first m times rule $L\&$.

For $n = 2$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1\&C_1, ..., x_m : B_m\&C_m, \Delta'$$

We write on top of each 2^m sequents of the form

$$\Gamma \Rightarrow x_1 : D_1, ..., x_m : D_m, \Delta'$$

where D_i is either B_i or C_i and all possible choices are taken. This step corresponds to applying root first $R\&$ for each conjunction in the right-hand side.

For $n = 3$ we consider all sequents of the form

$$x_1 : B_1 \vee C_1, ..., x_m : B_m \vee C_m, \Gamma' \Rightarrow \Delta$$

We write on top of each 2^m sequents of the form

$$x_1 : D_1, ..., x_m : D_m\Gamma' \Rightarrow \Delta$$

This case is analogous to step $n = 2$ and corresponds to the application root first of $L\vee$.

For $n = 4$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \vee C_1, ..., x_m : B_m \vee C_m, \Delta'$$

On top of each, we write

$$\Gamma \Rightarrow x_1 : B_1, x_1 : C_1, ..., x_m : B_m, x_m : C_m, \Delta'$$

This step corresponds to applying root first m times rule $R\vee$.

For $n = 5$ we consider all sequents of the form

$$x_1 : B_1 \supset C_1, ..., x_m : B_m \supset C_m, \Gamma' \Rightarrow \Delta$$

On top of each, we write 2^m sequents of the form

$$x_{i_1} : C_{i_1}, ..., x_{i_k} : C_{i_k}\Gamma' \Rightarrow x_{j_{k+1}} : B_{j_{k+1}}, ..., x_{j_m} : B_{j_m}\Delta$$

where $i_1, ...i_k \in \{1, ..., m\}$ and $j_{k+1}, ..., j_m \in \{1, ..., m\} - \{i_1, ..., i_k\}$. This step correspond to the root first application of $L \supset$.

For $n = 6$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \supset C_1, ..., x_m : B_m \supset C_m, \Delta'$$

On top of each, we write

$$x_1 : B_1, ..., x_m : B_m, \Gamma \Rightarrow C_1, ..., x_m : C_m, \Delta'$$

This step corresponds to applying root first m times rule $R \supset$.

For $n = 7$ we consider all sequents with relational atoms $x_1R_wy_1, ..., x_mR_wy_m$ and formulas $x_1 : \Box_wB_1, ..., x_m : \Box_wB_m$ on the left-hand side. On top of each, we write

$$x_1R_wy_1, ..., x_mR_wy_m, x_1 : \Box_wB_1, ..., x_m : \Box_wB_m, y_1 : B_1, ..., y_m : B_m, \Gamma' \Rightarrow \Delta$$

This step corresponds to applying root first m times rule $L\Box_w$.

For $n = 8$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : \Box_w B_1, ..., x_m : \Box_w B_m, \Delta'$$

Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree. Then on top of each sequent we write

$$x_1 R_w y_1, ..., x_m R_w y_m, \Gamma \Rightarrow y_1 : B_1, ..., y_m : B_m \Delta'$$

This step corresponds to applying root first m times rule $R\Box_w$.

For $n = 9$ we consider all sequents of the form

$$x_1 : B_1 \Box\!\!\rightarrow C_1, ..., x_m : B_m \Box\!\!\rightarrow C_m, \Gamma' \Rightarrow \Delta$$

Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree, and suppose $z_1, ..., z_m$ are any variables already used in the reduction tree, hence disjoint from the y's. Then on top of each sequent, we write 2^m sequents with formulas $y_{i_1} : B_{i_1}, ..., y_{i_k} : B_{i_k}$ together with $y_{i_1} : \Box_x(B_{i_1} \supset C_{i_1}), ..., y_{i_k} : \Box_x(B_{i_k} \supset C_{i_k})$, as well as formulas $x_{j_{k+1}} : B_{j_{k+1}} \Box\!\!\rightarrow C_{j_{k+1}}, ..., x_{j_m} : B_{j_m} \Box\!\!\rightarrow C_{j_m}$, on the left-hand side, and formulas $z_{j_{k+1}} : B_{j_{k+1}}, ..., z_{j_m} : B_{j_m}$ on the right-hand side, where $i_1, ...i_k \in \{1, ..., m\}$ and $j_{k+1}, ..., j_m \in \{1, ..., m\} - \{i_1, ..., i_k\}$. This step correspond to the root first application of $L\Box\!\!\rightarrow$.

For $n = 10$, we consider all sequents with formulas $x_1 : B_1 \Box\!\!\rightarrow C_1, ..., x_m : B_m \Box\!\!\rightarrow C_m$ on the right-hand side. Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree. We write on top of each the sequent

$$y_1 : B_1, ...y_m : B_m, \Gamma \Rightarrow x_1 : B_1 \Box\!\!\rightarrow C_1, ..., x_m : B_m \Box\!\!\rightarrow C_m, \Delta'$$

This step corresponds to applying root first m times rule $R\Box\!\!\rightarrow_1$.

For $n = 11$, we consider again all sequents with formulas $x_1 : B_1 \Box\!\!\rightarrow C_1, ..., x_m : B_m \Box\!\!\rightarrow C_m$ on the right-hand side. Then suppose $z_1, ..., z_m$ are any variables already used in the reduction tree and write 2^m sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \Box\!\!\rightarrow C_1, ..., x_m : B_m \Box\!\!\rightarrow C_m, z_1 : D_1, ..., z_m : D_m, \Delta'$$

on top of each, where D_i is either $z_i : B_i$ or $z_i : \Box_w(B_i \supset C_i)$, and all possible choices are taken. This step corresponds to applying root first m times rule $R\Box\!\!\rightarrow_2$.

Finally, for $n = 11 + j$ we consider the generic case of a frame rule, viz. rules for the relation R_w and identity. Because of the subterm property, these rules need to be instantiated only on terms in the conclusion (notice there are no frame rules with eigenvariables). Thus, e.g., for *Ref*, a step corresponding to the rule consists in adding to the left-hand side all atoms $x = x$ for all x in $\Gamma \Rightarrow \Delta$ and then writing the sequent thus obtained on top of each topmost sequent.

For any n, for sequents that are neither initial nor conclusions of $L\bot$ nor treatable by any one of the above reductions, we copy the sequent above itself. This step is needed to treat uniformly the failure of proof search in the following two cases: the case in which the search goes on for ever because new rules always become applicable, and the case in which a sequent is reached which is not a conclusion of any rule nor an initial sequent.

If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, and the tree read from the leaves to the root, yields a derivation by spelling out in individual rule applications the simultaneous reduction steps.

2. *Construction of the countermodel.* If the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \Rightarrow \Delta_0 \equiv \Gamma \Rightarrow \Delta, \Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_i \Rightarrow \Delta_i, ...$ be one such branch. Consider the set of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$. Consider the system $\mathcal{S}$ whose elements are all the labels appearing in $\mathbf{\Gamma}$ and whose relations are all those expressed by the relations xR_wy in $\mathbf{\Gamma}$. The system $\mathcal{S}$ is transitive, strongly connected and L-minimal by construction. The model is defined as follows: for all atomic formulas P such as $x : P$ is in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$.

We now show inductively on the weight of formulas that A is forced at world x if $x : A$ is in $\mathbf{\Gamma}$ and is not forced at world x if $x : A$ is in $\mathbf{\Delta}$.

If $x : A \equiv x : \bot$, $x : A$ cannot be in $\mathbf{\Gamma}$ since no sequent in $\Gamma_0 \Rightarrow \Delta_0$ has $x : \bot$ on the left-hand side, so it is not forced at any world of the model.

The antecedent atomic case has already been taken care of by definition, and the succedent one follows from the fact that no initial sequent is in the branch.

If $x : A \equiv x : B\&C$ is in $\mathbf{\Gamma}$, there exists i such that $x : B\&C$ appears first in Γ_i and therefore, for some $g \geq 0$, $x : B$ and $x : C$ appear in Γ_{i+g}. By inductive hypothesis, $x \Vdash B$ and $x \Vdash C$, hence $x \Vdash B\&C$.

If $Ax :\equiv x : B\&C$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for $x : B\&C$ applies. This gives a branching, such that one of the branches belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $x : B$ or $x : C$ is in $\mathbf{\Delta}$, and so by inductive hypothesis, either $x \nVdash B$ or $x \nVdash C$. Hence $x \nVdash B\&C$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \vee C$ applies. This gives a branching, such that one of the branches belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $x : B$ or $x : C$ is in $\mathbf{\Gamma}$, and so by inductive hypothesis, either $x \Vdash B$ or $x \Vdash C$. Hence $x \Vdash B \vee C$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Delta}$, there exists i such that $x : B \vee C$ appears first in Δ_i and therefore, for some $g \geq 0$, $x : B$ and $x : C$ appear in Δ_{i+g}. By inductive hypothesis, $x \nVdash B$ and $x \nVdash C$, hence $x \nVdash B \vee C$.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \supset C$ applies. This gives a branching, and one of the premisses belongs to the infinite branch. So either $x : B$ is in $\mathbf{\Delta}$ or $x : C$ is in $\mathbf{\Gamma}$. In the former case, by inductive hypothesis we have $x \nVdash B$ and so $x \Vdash B \supset C$. Otherwise, by inductive hypothesis we have $x \Vdash C$ and so $x \Vdash B \supset C$. Hence $x \Vdash B \supset C$.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Delta}$, there exists i such that $x : B \supset C$ appears first in Δ_i and therefore, for some $g \geq 0$, $x : B$ appears in Γ_{i+g} and $x : C$ in Δ_{i+g}. By inductive hypothesis, $x \Vdash B$ and $x \nVdash C$, hence $x \nVdash B \supset C$.

If $x : A \equiv x : \Box_w B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms xR_wy that occur in $\mathbf{\Gamma}$ (notice that at least there occurs the atom xR_wx). For any such atom, we find an occurrence of $y : B$ in $\mathbf{\Gamma}$ by construction of the reduction tree. So by inductive hypothesis, $y \Vdash B$, hence $x \Vdash \Box_w B$.

If $x : A \equiv x : \Box_w B$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for $x : \Box_w B$ applies. Then for some $g \geq 0$, we find $y : B$ in Δ_{i+g} for some xR_wy in Γ_{i+g}. So by inductive hypothesis, $y \nVdash B$, hence $x \nVdash \Box_w B$.

If $x : A \equiv x : B \Box\!\!\rightarrow C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \Box\!\!\rightarrow C$ applies. This gives a branching, such that one of the premisses belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $z : B$ is in $\mathbf{\Delta}$ for any given z already occurring in the reduction tree, or else we find some $y : B$ and $y : \Box_x(B \supset C)$ in $\mathbf{\Gamma}$. In the latter case, by inductive hypothesis we have $y \Vdash B$ and $y \Vdash \Box_x(B \supset C)$, and so $x \Vdash B \Box\!\!\rightarrow C$. In the former case we have by inductive hypothesis $z \nVdash B$, which does not yet allow us to infer $x \Vdash B \Box\!\!\rightarrow C$. However, $x : B \Box\!\!\rightarrow C$ always gets repeated in the premiss and for each branching determined by an application of $L \Box\!\!\rightarrow$ either the first or the second premiss belongs to the infinite branch. It is enough that the right premiss belongs to the infinite branch once to conclude that $x \Vdash B \Box\!\!\rightarrow C$. In the opposite case, we have that the left premiss always belongs to the infinite branch, and therefore $z \nVdash B$ for each available label. So, by the forcing condition for $\Box\!\!\rightarrow$, we have that $x \Vdash B \Box\!\!\rightarrow C$ because the antecedent is never satisfied.

If $x : A \equiv x : B \Box\!\!\rightarrow C$ is in $\mathbf{\Delta}$, consider the step i in which the first reduction for $x : B \Box\!\!\rightarrow C$ applies. Then $z : B$ appears in Γ_i. So by inductive hypothesis, $z \Vdash B$. This is not sufficient to conclude that $x \nVdash B \Box\!\!\rightarrow C$, but proves that the first disjunct in the semantic clause for $\Box\!\!\rightarrow$ is false. We need to prove that also the second disjunct is false, i.e. that there is no y in the model such that $y \Vdash B$ and $y \Vdash \Box_x(B \supset C)$. By the saturation clause for $R \Box\!\!\rightarrow_2$ we have that for all available labels y, either $y : B$ or $y : \Box_x(B \supset C)$ are in the succedent in the infinite branch. In case the former holds for all available labels, then in particular we would have that $z : B$ would be in the succedent in the branch, with $z : B$ in the antecedent introduced by the clause for $R \Box\!\!\rightarrow_1$. Since the infinite branch cannot contain an initial (or derivable) sequent, this is excluded. So for at least one label the latter case holds and we obtain by inductive hypothesis that $y \nVdash \Box_x(B \supset C)$. Since y was an arbitrary y in the model such that $y \Vdash A$, together with the first part, we have the conclusion, namely $x \nVdash B \Box\!\!\rightarrow C$. QED

8.5 Proof of Proposition 6.2

Proof. We need to show (i) that the saturated branch provides a countermodel, and (ii) that the countermodel thus obtained is a comparative similarity system. For a given $\Gamma \Rightarrow \Delta$, let $\mathcal{W}$ be the set of labels in Γ, the relation $\mathcal{R}_w$ be the reflexive and transitive closure of R_w together with $\leq_{\Gamma\Rightarrow\Delta}$, the definition of interpretation and valuation being as usual. Then:

(i) $\langle \mathcal{W}, \mathcal{R}_w \rangle$ is a countermodel to $\Gamma \Rightarrow \Delta$ iff $x \Vdash A$ for all $x : A$ in Γ, and $x \nVdash A$ for all $x : A$ in Δ. Notice that this is a consequence of the following:

(a) If A is in $\mathcal{F}^1(x)$, then $x \Vdash A$.

(b) If A is in $\mathcal{F}^2(x)$, then $x \nVdash A$.

The two claims are proved by induction on the length of the formula. If it is atomic, they hold by definition of $\Vdash$. If it is a conjunction, disjunction, implication or conditional, they hold by the corresponding saturation clauses and the inductive hypothesis. If $A \equiv \Box_w B$ is in $\mathcal{F}^1(x)$, let $x \equiv x_0 \mathcal{R}_w \ldots \mathcal{R}_w x_n \equiv y$. For $n = 0$, either $x : \Box_w B$ is in $\downarrow\Gamma$, hence in Γ, or for some y, $y : \Box_w B$ and $y R_w x$ are in Γ. In both cases we get the conclusion by step (12) of the saturation procedure. For the inductive step, suppose first $A \equiv \Box_w B$ is in $\mathcal{F}^2(x)$. Then either for some y we have $x R_w y$ in Γ and $y : B$ in $\downarrow\Delta$, or there is a y distinct form x and such that $y R_w x$ is in Γ and $x \leq y$. From the former, we have $B \in \mathcal{F}^2(y)$ and so by inductive hypothesis, $y \nVdash B$, whence $x \nVdash \Box_w B$. From the latter, $\mathcal{F}^2(x) \subseteq \mathcal{F}^2(y)$, and so $\Box_x B \in \mathcal{F}^2(y)$. By inductive hypothesis (y is a smaller label in the R_w-ordering), $y \nVdash \Box_w B$, whence $x \nVdash \Box_w B$.

(ii). $\langle \mathcal{W}, \mathcal{R}_w \rangle$ is a comparative similarity system iff $\mathcal{R}_w$ is transitive, strongly connected and L-minimal. The result follows immediately from points (2), (3) and (4) of the definition of saturation. QED