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PROOF AND ARGUMENTATION IN MATHEMATICS EDUCATION RESEARCH

INTRODUCTION

In the chapter on proof in the previous PME Research Handbook, Mariotti (2006) observed that there had seemed to be "a general consensus on the fact that the development of a sense of proof constitutes an important objective of mathematics education" and also "a general trend towards including the theme of proof in the curriculum" (p. 173). A decade after the publication of the Handbook, Mariotti's observations are equally, if not more, applicable: There is currently a widespread agreement among mathematics educators on the significance of proof in students' learning of mathematics, with a growing number of educational policy documents or curriculum frameworks in different countries calling for an important place for proof in all students' mathematical experiences and as early as the elementary school (see, e.g., the U.S. *Common Core State Standards for School Mathematics* (CCSSI, 2010) and the most recent *National Mathematics Curriculum* in England (Department for Education, 2013)).

It has been suggested that mathematics education research has influenced or even pressured curriculum authors into giving proof a place in the mathematics curricula of different countries (Hoyles, 1997; Mariotti, 2006). There are several arguments for the importance of proof in students' mathematical experiences from the beginning of their education. These arguments have been elaborated in various publications including PME reports (e.g., Stylianides & Stylianides, 2006; Yackel & Hanna, 2003), so we will not repeat them here. Yet, there is a big difference between recommending or accepting the idea that proof should have an important place in school mathematics and making this recommendation a reality for all students.

Although during the past few decades mathematics education research has cast light on many different issues related to proof thus generating useful knowledge with implications for teaching practice, there are still many open debates in the field and important research questions remaining to be addressed. Over the past

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decade in particular there has been an upsurge of research activity related to the teaching and learning of proof, including many articles in all major journals in the field, a number of books or edited volumes (e.g., Stylianou, Blanton, & Knuth, 2010), and an international study conference on proof that was conducted under the auspices of the International Commission on Mathematical Induction (ICMI) (Hanna & de Villiers, 2012).

The PME community is, and continues to be, a main contributor to debates and research advances in the area of proof, with a plethora of relevant reports published in the PME proceedings following the period covered by Mariotti's (2006) review. Our aim in this chapter is to review and reflect on major research advances of the PME community in the area of proof, based primarily on the PME proceedings during the decade 2005-14. More comprehensive reviews of the state of the art in the field as a whole can be found elsewhere (Harel & Sowder, 2007; Stylianides, Stylianides, & Weber, forthcoming).

In what follows, we explain our decision to widen the scope of this review by considering issues related to argumentation and proof rather than just proof, and we discuss the meanings of these two key terms. In the last part of this introductory section, we describe the methodology we followed in the review and how the rest of the chapter is organized.

Argumentation and Proof

The concepts of *argumentation* and *proof* have been discussed in detail by Mariotti (2006, pp. 181-184) who also presented part of the debate about whether the relationship between the two concepts can be more productively viewed as a possible rupture (e.g., Duval, 1989) or as a possible continuity (e.g., Boero, Garuti, & Mariotti, 1996). No matter which position one takes in this debate, with which the PME community has engaged from its early stages as illustrated by the previous two references, the following points stand: (1) argumentation and proof are closely related, and (2) considering both argumentation and proof helps draw attention to a wider range of important processes related to proving than when considering them separately. Indeed, these two points, together with the increased attention that argumentation and proof have received at PME conferences over the years and elsewhere (e.g., Durand-Guerrier, Boero, Douek et al., 2012), have guided our decision to address in this chapter issues related to both argumentation and proof.

There seems to be a fairly shared understanding among researchers about the meaning of *argumentation*, a term which is generally used to describe the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false (e.g., Boero et al., 1996; Duval, 1989; Krummheuer, 1995). Thus argumentation focuses on the epistemic value of a given statement and can embody a link between the process of *ascertaining* (i.e., the process employed by an individual to remove his or her own doubts about the truth or falsity of a statement) and the process of *persuading* (i.e., the process employed by an individual or a group to remove the doubts of others about the truth or falsity of a statement) (Harel & Sowder, 2007). Argumentation is often situated in the context of a broader mathematical activity which has been described using different terms (e.g., proving or reasoning-and-proving) and can involve the following: exploration of examples or particular cases, generation or refinement of conjectures, and production of arguments for these conjectures that may not necessarily qualify as proofs or support the development of proofs (e.g., Buchbinder & Zaslavsky, 2009; Komatsu, 2011; Lockwood, Ellis, Dogan et al., 2012; Morselli, 2006; Stylianides, 2008; Zaslavsky, 2014; Zazkis, Liljedahl, & Chernoff, 2008).

In contrast to the rather consistent meaning attributed to argumentation in the field, the meaning of *proof* has been subject to debate among researchers at PME conferences and elsewhere (e.g., Balacheff, 2002; Reid, 2005; Stylianides, 2007; Weber, 2014). Some of these researchers have reviewed definitions of proof used in different research studies thus illustrating the multiplicity of perspectives in the field, while others proposed specific definitions of proof and discussed their affordances or domains of application in research on proof. The fact that different definitions may be better suited to serve different research purposes implies that it may be neither possible nor desirable for all researchers to adopt a common definition. Yet, it is important that researchers specify their perspective on proof so as to facilitate understanding of their claims or findings and support comparisons between different research reports (Balacheff, 2002; Reid, 2005).

Accepting the importance for such specificity, we describe our perspective on proof, without however suggesting that this is better than alternative perspectives. We begin with Mariotti's (2006) observation that "the crucial point that has emerged from different research contributions [in the field of mathematics education] concerns the need for proof to be acceptable from a *mathematical* point of view but also to make sense for *students*" (p. 198; italics added). Following up on this observation, we define *proof* in the context of a classroom community as a mathematical argument for the truth or falsity of a mathematical statement that meets both of the following criteria, where criterion 1 reflects a mathematical consideration and criterion 2 a student consideration (Stylianides, 2007):

– *Criterion 1*: An argument qualifying as a proof should use true statements, valid modes of reasoning, and appropriate modes of representation, where the terms "true," "valid," and "appropriate" are meant to be understood with reference to what is typically agreed upon nowadays in the field of mathematics, in the context of specific mathematical theories.
– *Criterion 2*: An argument qualifying as a proof should use statements, modes of reasoning, and modes of representation that are accepted, known, or within the conceptual reach of students in a given classroom community.

While not comprehensive, this definition is sufficiently "elastic" to allow description of proof across different levels of education, which is a pressing issue given the current (positive) trend towards making proof part of the mathematics curriculum from elementary school. Also, the definition integrates different perspectives on proof discussed in the literature. These include, for example, the view of proof as a logical deductive chain of reasoning linking premises with conclusions in the context of a mathematical theory (e.g., Healy & Hoyles, 2001; Knuth, 2002; Mariotti, 2000), as well others that highlight the cognitive or social aspects of proof thus viewing proof as an argument that establishes the truth or falsity of a statement for a person or a community (Harel & Sowder, 2007) or as an argument that is accepted by a community at a given time (Balacheff, 1988).

Methodology for the Review and Chapter Organization

As we noted earlier, our aim in this chapter is to review and reflect on major research advances of the PME community in the area of proof, based primarily on the PME proceedings during the decade 2005-14. We used the following two complementary and partly overlapping approaches to identify which reports to include in our review. The term "report" refers in this chapter to any published piece in the proceedings with length more than 1 page (such pieces could be labelled in the proceedings as plenary papers, plenary panels, research reports, research forums, discussion groups, or working sessions).[1]

– *Approach 1*: We included all of the reports with any of the keywords "proof/proving" and "argument/argumentation" in their titles or abstracts, though we did filter out few reports where the use of these terms was incidental (as in the phrase "In this paper we make the *argument* that…").[2]
– *Approach 2*: We included all of the reports that were listed under the domain "Proof, proving and argumentation" in the section typically called "Index of Presentations by Research Domain" and found in volume 1 of the proceedings.[3] (The particular domain under which a report is listed is specified by the authors of the report at the point of submission.)

Approach 1 offered a rather objective way of identifying relevant reports, while Approach 2 gave voice to authors themselves to indicate whether they considered their reports to be primarily about argumentation and proof. Interestingly, a considerable number of reports were identified by one approach but not the other; this emphasizes the complementarity of the two approaches and helps justify our choice to consider in the review the union of their returns.

Over 150 reports qualified for inclusion in the review, with more than 80% of them being categorized into the following three general themes according to the approximate ratio 2:2:1. The bulk of this chapter is a discussion of reports under these three themes.

– *Theme 1*: Research on student conceptions and learning;
– *Theme 2*: Classroom-based research; and
– *Theme 3*: Research on teacher knowledge and development.

The reports that did not fit under any of these themes addressed a fairly large variety of topics that defied broader grouping. Yet, one topic received relatively more attention than others, and so we comment on it briefly. This concerned the place or treatment of concepts related to argumentation and proof in curricular resources, notably mathematics textbooks. The studies on this topic were typically comparative analyses of textbooks in different countries (e.g., Miyakawa, 2012) or of different textbooks in the same country (e.g., Dolev & Even, 2012). Their findings showed a multiplicity of approaches to the place and treatment of argumentation and proof in textbooks not only across but also within countries. In educational contexts where teachers rely rather heavily on textbooks for their everyday planning and teaching, these findings raise a concern about the presumably large variation in the learning opportunities offered to students in different classes, even within the same country, depending on which textbook their teachers follow.[4] A recently published journal special issue on the place and treatment of concepts related to argumentation and proof in mathematics textbooks (Stylianides, 2014) offered a forum for more reports of empirical findings in this area and also for discussion of methodological issues surrounding textbook analyses.

In the sections that follow, we discuss separately the three themes we listed earlier. In each section, we begin with a general description of the theme and any sub-themes within it, we continue with review of the reports belonging to the theme, and we conclude with a reflection on the state of PME research within the particular theme, including possible directions for future research.

THEME 1: RESEARCH ON STUDENT CONCEPTIONS AND LEARNING

General Description of Theme 1

In this section, we review the PME reports that focused primarily on issues of learners' conceptions when engaging in argumentation and proof. Although the set of studies reviewed for this theme included some research on mathematicians' strategies and conceptions, the vast majority of studies focused on students' conceptions and learning in secondary and undergraduate grade levels. Somewhat troubling, though not surprising, was that there were a smaller number of studies focused on middle school students and even fewer studies with elementary school students. Given the well-documented difficulties students face when learning to prove (e.g., Harel & Sowder, 2007), as well as the greater prominence of argumentation and proof in educational policy documents aimed at the elementary and middle school levels over the past decade (e.g., CCSSI, 2010; Department for Education, 2013), relatively little research we reviewed addressed critical needs for understanding these mathematical practices in elementary and middle school levels.

Among the reports we reviewed, there were three related sub-themes:

- The use of examples in argumentation and proving;
- Actor-oriented perspectives on argumentation and proof; and
- Strategies for learning to prove.

In what follows, we review separately reports belonging to each of these three sub-themes. We conclude with a reflection on the state of PME research within Theme 1.

Review of PME Reports Belonging to Theme 1

The use of examples in argumentation and proving. The reports within this sub-theme generally aimed to identify how example use can be a productive, generative part of the argumentation and proving process. This focus is an evolution from the body of existing literature documenting students' over-reliance upon empirical evidence when proving, as well as a general interest in the pedagogical value of examples in mathematics. The research reported here speaks to the practices of example use across a range of participants engaged in argumentation and proof; half of the reports focused on example use by middle (Chrysostomou & Christou, 2013; Ellis, Lockwood, Dogan et al., 2013; Lin & Wu, 2007) and high school students (Buchbinder & Zaslavsky, 2013), whereas the other half examined practices with example use by undergraduates (Morselli, 2006; Watson, Sandefur, Mason et al., 2013*)* as well as by professional mathematicians (Antonini, 2006; Ellis et al., 2013). As such, this collection of reports represents a significant body of literature to inform the field about the nature of students' (and mathematicians') example use and support further inquiry into this common practice of proving.

Several of the reports in this sub-theme focused on identifying types of example use that emerged in students' or mathematicians' argumentation and, in some cases, identifying which types led to desirable outcomes (e.g., a proof). The researchers used qualitative approaches with small numbers of participants to gather rich data about example use. One of the studies with a larger sample was Morselli (2006), who conducted interviews with 47 university students and found that participants' argumentative processes could be classified into four profiles: (1) work exclusively through algebraic manipulation; (2) short explorations with examples and shift to algebraic proof; (3) extended explorations with examples leading to reasoning about the conjecture; and (4) unfocused explorations with examples. She identified, in particular, that participants exhibiting argumentation habits categorized into the fourth profile were less successful than other students. This suggests that exploration with examples can be very productive for proving as long as the exploration is focused and purposeful.

Similarly, Lin and Wu (2007) found that the type of examples students investigate influences successful conjecturing. They posed conjecturing tasks in interviews with sixth grade students. As Figure 1 shows, the researchers provided students with given information instantiated with three examples. One example was considered to be *typical*, meaning that the example represented the typical representation in textbooks. A *conjunctive* example satisfied all of the given conditions, as well as others. The *extreme* example was one that satisfied all of the givens, but also contained some boundary features such as very large or small angle measures. The researchers randomized the order in which the examples were organized for each interview. During the interview, the researchers noted the number of conjectures generated as the participants considered each example, as well as noted each conjecture. A key finding was that conjectures students generated while analyzing extreme conjectures were fewer in number and more likely to be incorrect than if they conjectured from observations of conjunctive or typical examples. Otten, Gilbertson, Males, and Clark (2014) raised questions about the influence of the typical example accompanying claims to be proven in geometry textbooks, which they called a case of *general with particular instantiation.* The findings of Lin and Wu suggest that, as Otten et al. hypothesized, the features of given examples influence the kinds of generalizations that students make. In cases where students are asked to reason from examples to prove a given conjecture, it may be more desirable for a range of examples to be given or for students to generate their own examples so that they can determine which features are variant under the given conditions.

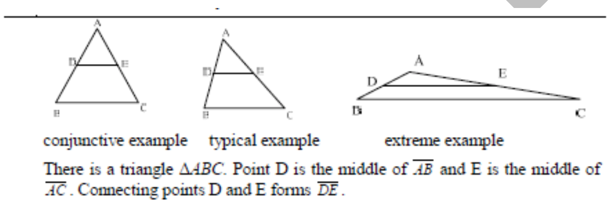

Figure 1. A conjunctive, a typical, and an extreme example (derived from Lin and Wu, 2007, p. 211).

Ellis et al. (2013) compared the example choice and use of middle school students with that of expert provers (mathematicians). Their analysis focused exclusively on the practices of successful provers within each participant group, and they developed categories of example choice and use to characterize example choice and use of successful provers. Their findings suggest that successful provers insightfully navigate a range of examples. Experts (mathematicians) tend to reflect upon the utility of a particular example before choosing it and exhibit more metacognitive awareness of the utility of examples in the proving process.

The work of Buchbinder and Zaslavsky (2013) illustrates that students' notions about the role of empirical evidence in determining the validity of a claim are complex and nuanced. They conducted dyad interviews with seven pairs of ninth and tenth grade students, and provided them arguments to discuss. While students generally recognized the limitations of arguments that over-rely upon empirical evidence, they also tended to believe a statement to be true "unless proven otherwise"; that is, students had particular difficulty accepting a claim to be false when it was initially shown to be true with confirming examples but then they discovered a counterexample to disprove the claim.

Actor-oriented perspectives on argumentation and proof. A relatively large number of reports consisted of what one could call actor-oriented investigations of conceptions and practices of proof and argumentation. Like the theory of actor-oriented transfer perspective (Lobato, 2003), studies with actor-oriented perspectives on argumentation and proof attempt to account for the student's conceptions or approaches without establishing a relationship to expert practice. The importance of attending to students' perspectives is underscored in Knapp and Weber (2006), whose study of advanced calculus students' proving led to reconceptualizing Weber's (2001) construct of strategic knowledge for proof to be strategies, heuristics, and techniques that are used to attain students' goals of proving instead of the Platonic goal of developing a proof by any means.

Fried and Amit (2006), reporting on a sub-study of the large-scale, comparative *Learners' Perspective Study* (Clarke, 2001), investigated eighth graders' perspectives on proof. They argued that students' positioning as mathematical authorities influences their confidence in determining whether their written argument is a proof. They raised an important point from their findings – how we position students as separate from mathematical authorities can exacerbate a belief that there is a definitive notion of "proof." They further challenged mathematics educators to help students "see that their continual debate, defining, and self definition is a normal state of affairs in mathematics" (Fried & Amit, 2006, p. 119).

Another study by Kunimune, Kumakura, Jones, and Fujita (2009) of lower-secondary students (eighth and ninth graders) contributes to the existing body of literature on students' understandings of proof and generality (e.g., Ellis, 2007). Their sample consisted of approximately 400 students' responses to written survey items to assess their conceptions of algebraic proof. The researchers found that students who were consistently successful in producing valid algebraic arguments that met the standard of proof did not necessarily recognize the generality of their proofs. An inability to recognize the generality of a proof may suggest that students do not conceive of proof as a means for establishing truth. Coupled with the work of Fried and Amit (2006) discussed above, these findings also suggest that abilities to produce proof do not need to go hand-in-hand with understanding proof's specific role in the discipline of mathematics. Bieda's (2011) work illustrates similar complexity to students' conceptions of argumentation and proof as shown in the studies of Fried and Amit and Kunimune et al.; Bieda showed that middle grade students' conceptions of what makes a convincing argument for showing the truth of a given statement involves both explanations of why a statement is always true and specific instantiations of the true statement to illustrate the phenomenon to the reader. Additional information regarding the findings of this study can be found in Bieda and Lepak (2014).

Some other studies focused on advanced students' and mathematicians' processes for generating proof. Edwards (2008, 2010) employed embodied cognition perspectives to better understand expert (advanced doctoral students and mathematicians) proof processes. Edwards found her participants evoked a conceptual metaphor of "proof is a journey" as they thought aloud when doing a proof. Wilkerson (2008) also found that embodiments (a term used by the author to describe examples, constructions, and prototypes) emerged in think alouds when mathematicians were interpreting a proof or when they were faced with new content to make sense of. This suggests that examples may play a similar role for novices and possibly relates to the extensive work on students' use of examples in argumentation and proving reviewed earlier in this theme.

Alcock and Weber (2005) illustrated, with two case studies selected from the interviews of 11 undergraduates, that learners might take either a referential or a syntactic approach to attempting proof. In the referential approach, the prover uses particular or generalized instantiations of the statement to guide formal inferences. Those who attempted the proof syntactically tended to stick with manipulating formally stated facts without the use of examples to guide their process. From their analysis of the case studies, Alcock and Weber discovered that students who take a more referential approach have a more difficult time producing formal proof from their intuitions. However, those who approach proof more syntactically tend to generate proofs without a good sense of the meaning of those proofs. Students with more syntactic attempts also tended to describe generalizations regarding when to use particular proof techniques.

Zaskis, Weber, and Mejia-Ramos (2014) investigated the kind of thinking that is needed to move from referential attempts at proof to a successful proof product. They discovered three kinds of action employed by students as they worked on formal proof from informal arguments: syntactifying, re-warranting, and elaborating. From analyzing interviews with 73 undergraduates, they concluded that students who employed all three activities were highly likely to successfully produce a proof. Only in 14% of cases where a student did not use all three activities was a proof produced.

Strategies for learning to prove. The third, and final, subcategory of reports within Theme 1 discussed aspects of students' knowledge that relate with their success in generating proofs, and tasks and tools that can promote students' competencies in generating proofs. Further discussion of tasks and tools for teaching argumentation and proof can be found in Theme 2, which focuses more on classroom-based studies discussing teaching moves and interactions between teachers and students. The studies reviewed in this theme address questions about students' thinking and experiences during the learning process.

One question that emerged in many of the reports in this subcategory could be stated as "What is the knowledge students need to be able to generate a proof?" In the domain of geometry, Hsu (2010) investigated the relationship between students' performance on geometrical calculation items and proof generation questions. Similar in scope to the well-cited Healy and Hoyles (2000) study, Hsu surveyed over 900 eighth- and ninth-grade Taiwanese students and found, unsurprisingly, that ninth graders performed better overall on both types of items. However, of note was that students did better on geometrical calculation items *after* completing a related geometry proof task (where the order of tasks was systematically varied). Order did not matter for performance on the geometry proof task. These findings suggest that doing proof may lead to a deeper understanding of the content.

Ufer, Heinze, and Reiss (2008) reported on a key study of knowledge needed for doing geometry proof, particularly the interactions between key predictors, identified through a review of the literature, and geometry proof performance. They collected survey responses from over 300 students who were enrolled in the highest track within the German secondary school system (Gymnasium). The items on the surveys consisted of proof generation items of various difficulty levels, where an item was assumed to be more difficult if the proof contained additional steps, and questions assessing knowledge of basic facts and problem solving skill. Using a linear regression model, the researchers showed that the three cognitive predictors (declarative basic knowledge, procedural basic knowledge, and problem solving skill) all significantly predicted geometry proof performance. While declarative knowledge was found to be the most significant predictor, problem solving skill was the least. The authors cautioned against an interpretation that there is little relationship between students' problem solving skill and their ability to generate proofs. Indeed, some key studies have yielded valuable information about the process of generating proof by considering proof construction as a problem-solving task (see Weber, 2005). Rather, they highlight that proving involves processes such as associative thinking that may not be a part of problem solving and emphasize that the strength of a student's declarative knowledge likely determines her or his success in producing proof. A later paper by the same authors (Ufer et al., 2009) provided additional information about how the quality of students' geometrical knowledge, particularly the availability of *perceptual chunks*, leads to improved competency in producing proof.

Some other studies discussed the importance of students' declarative knowledge of definitions for better performance in generating proofs. Dickerson and Pitman (2012) showed that knowledge of definitions and ability to use definitions in proof is challenging even for advanced students. Of the five undergraduates interviewed, none were able to make a clear distinction between a mathematical theorem and a mathematical definition and many generated arguments solely from their *concept images* (Tall & Vinner, 1981) rather than the concept definitions. Haj Yahya, Hershkowitz, and Dreyfus (2014) found similarly problematic findings regarding high school students' understanding of geometric concept definitions.

Several studies in this sub-theme discussed aspects of students' thinking and proofs after the implementation of tasks or activities designed to improve students' proof competencies. Lin's (2005) plenary talk on his work studying the effects of refuting and the coloring strategy on students' skill to engage in proof was oft-cited in the PME reports reviewed for this theme. Lin found that the coloring strategy, where students would use colored pens to highlight given diagrams or draw information from the given statements, promoted more complete and valid proofs from his sample of Taiwanese ninth graders. However, he cautioned that they also found the coloring strategy to divert students' attention to extraneous features of the diagram or irrelevant information from the givens. Theme 2 provides further discussion of additional studies on the coloring strategy.

There were a few other studies of note describing particular tasks that promoted meaningful engagement in argumentation and proof. Mamona-Downs (2009) showed positive effects in undergraduates' ability to articulate their reasoning and refine their arguments after reading and interpreting selected work from peers. Brockmann-Behnsen and Rott (2014) investigated the effects of a structured training given to two classes of approximately 30 eighth graders in a German secondary school compared with two similarly-sized classes of their peers who did not receive the training. The training consisted of educating students about the structure of argumentation, such as developing the ability to distinguish claims from evidence, and on problem-solving heuristics, such as working backwards. Using pre-post test design, the treatment students performed significantly better on geometry proof tasks than their counterparts in the control classes. Finally, the work of Cramer (2014) suggests that logical games may foster equitable participation in argumentation, based on her analysis of students' participation in classroom argumentation using Habermas' theory of communicative action.

In addition to investigating the influence of tasks on proof performance, a range of studies focused on learners' interactions with various tools during proving. Several papers discussed the potential of Dynamic Geometry Environments (DGEs) for supporting students' investigation of geometric conjectures and moving from informal argumentation to proof. Notably, Baccaglini-Frank, Antonini, Leung, and Mariotti (2011) refined Leung and Lopez-Real's (2002) notion of *pseudo object* through observations of high school students' work with a DGE, where they defined pseudo object as "a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory" (p. 83). The pseudo object emerged through the actions of construction and dragging, mediating students' reasoning within the DGE and their theoretical knowledge of Euclidean geometry.

Rodriguez and Gutiérrez (2006) studied undergraduate mathematics students' use of DGEs while proving, particularly how these students used DGEs to produce proofs as solutions of geometry proof problems. They sought differences between students' performance when proving without tools and when using DGEs. The authors found that DGEs help students to identify and empirically check conjectures, but it does not provide an advantage over paper-and-pencil when students bridge informal arguments to proof.

Antonini and Martignone (2011) investigated novice and experts' argumentation when explaining the pantograph machine. Designed with pedagogical aims, the pantograph is a machine that consists of "two leads fixed in two plotter points of an articulated system composed by some rigid rods and some pivots" (p. 41, see Figure 2) and will perform geometric transformations. They interviewed three preservice teachers, two university students, and one early career mathematician as they attempted to identify and justify the transformation performed by the machine. The mathematician, unlike the other participants, only referenced features of the physical drawing performed by the machine if they had difficulty identifying the law embodied by the structure of the machine. Similar to DGEs, the learning

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environment afforded by the pantograph exposes the nature of students' declarative knowledge.

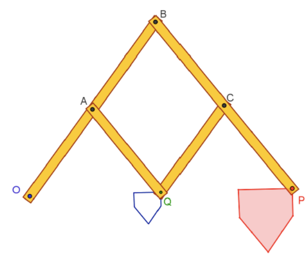

Figure 2: Scheiner's pantograph and its products (derived from Antonini and Martignone, 2011, p. 42).

Reflection on the State of PME Research within Theme 1

Taken together, the reports reviewed for this theme added many important insights to our understanding of students' knowledge, skills, and beliefs for argumentation and proof, especially with regard to students' understanding and use of examples in argumentation and proof. The findings from these studies come from research conducted with a range of methodological approaches. However, our review noted two interesting features of the set of studies that should be considered when considering future work to build upon these studies.

The first feature was the predominance of research on secondary students' argumentation and proof within the content domain of geometry: of the studies we reviewed for this theme, about 40% focused solely on argumentation and proof in secondary school geometry. The second feature was the focus on participants at the upper end of the novice/expert continuum: another about 35% of the studies we reviewed involved undergraduate students, advanced doctoral students, or mathematicians. A minority of reports reviewed addressed issues of learning argumentation and proof in school settings and in contexts other than where argumentation and proof typically appear in the school mathematics curriculum (Euclidean geometry).

Many of the studies presented at PME over the past decade illustrate how learning to prove in upper level mathematics continues to be a persistent challenge for students, but, more importantly, engaging in argumentation is a key activity for developing a foundation of mathematical knowledge that can be applied to learning more mathematics and to doing mathematics in a range of contexts (CCSSI, 2010). Thus, we find it surprising that so few studies reviewed for this theme focus on school students' conceptions of argumentation and proof and in content areas beyond geometry. If generating and critiquing arguments is to be a mathematical habit of mind for learners, then activities that address aspects of students' development in learning to prove should be incorporated throughout the elementary and secondary mathematics curriculum.

THEME 2: CLASSROOM-BASED RESEARCH

General Description of Theme 2

In this section, we review those PME reports that dealt with the role and status of argumentation and proof in the classroom. Some of these reports aimed at understanding the role and status of argumentation and proof in ordinary classrooms (with a focus on students and/or teachers), while others presented and discussed teaching experiments where specific strategies and/or tasks were developed and used in order to improve the teaching and learning of argumentation and proof. Accordingly, we organize our review of PME reports belonging to Theme 2 under the following sub-themes:

- Students' processes of argumentation and proof;
- Teachers' ways of dealing with argumentation and proof in the classroom; and
- Interventions aimed at improving the teaching and learning of argumentation and proof.

The last part of this section will be devoted to a reflection on the state of PME research within Theme 2.

Like in Theme 1, the most represented educational level among the identified reports was secondary school, especially its upper part (grades 8-13). Also, the most prevalent mathematical domain was again geometry, followed by arithmetic and elementary number theory. This may reflect the role that is often attributed to geometry as the "ideal" mathematical domain to deal with issues of argumentation and proof in the classroom, though one may question the originators of this perception (teachers, curricular frameworks and guidelines, researchers, etc.). Only few studies referred to other mathematical domains such as probability.

Review of PME Reports Belonging to Theme 2

Students' processes of argumentation and proof. As outlined in the Introduction, researchers in mathematics education agree on the fact that argumentation and proof are closely related (e.g., Durand-Guerrier et al., 2012). Thus, considering together students' argumentative and proving processes is highly relevant.

A series of PME reports focused primarily on argumentation. Among them, Douek (2006) studied the evolution from everyday to scientific concepts (from a Vygotskyan perspective) in the context of elementary school mathematics. Douek highlighted the key role played by argumentative activities in fostering concept development: "[A] continuous development of argumentative skills allows to nourish the backing of mathematical reasoning on other (and easier to master) kinds of reasoning. As a result, argumentation can effectively move the everyday concepts /scientific concepts under the teacher's guide" (p. 456). This study linked argumentation and concept development, thus showing that argumentation is not only a crucial step towards proof, but has also an educational value in itself. In a similar way, Stoyanova Kennedy (2006) examined conceptual change as it took place through argumentation in a fifth grade class working on the theme of finite and infinite sets as a community of mathematical inquiry. The author presented and discussed the key phases of the activity, starting from the orientation phase, where spontaneous conceptions about finite and infinite sets emerged, to the building phase, where students collaborated to verbalize some solutions, to the conflict phase, where contradictions emerged, to the synthesis phase, where a resolution was found that gave birth to a new conceptual formation.

Another series of PME reports focused on the crucial link between argumentation and proof. These studies relied primarily on the idea of *cognitive unity* (Boero et al., 1996), defined as the continuity between the processes of conjecture production and proof construction. Martinez and Li (2010) focused on the conjecturing process of grade 9-10 students in the domain of arithmetic. They defined *conjecturing* as a complex process "that involves the production of several mathematical statements; from which, one of the conjectures emerges as a conjecture to prove; and, through which a person comes to believe the likely truth-value of the conjecture to prove" (p. 269). The authors noted that, before reaching the conjecture to prove, students explored the problem and tested examples and/or counterexamples. Such a process led them to dismiss or accept conjectures and also to find out mathematical relations that would be employed in the subsequent proving phase. This is well aligned with the hypothesis of cognitive unity. The authors emphasized that the conjecturing phase is a complex and rich process and advocated the diffusion of conjecturing and proving activities in the US, where the curriculum has traditionally centred primarily on proof production and appreciation. Fujita and Jones (2010) investigated the extent to which complex geometrical construction tasks may foster a cognitive unity between conjecturing and proving. Their data derived from classroom experiments carried out in Japanese lower secondary schools. They analyzed excerpts from group discussions using Toulmin's (1958) model for argumentation. They found that, when students were asked to produce conjectures and prove them, cognitive unity was not automatic. Their study also suggested the importance of designing teaching sequences where students, sharing their mathematical arguments with peers, gradually develop their "appreciation of how to use already known facts to proceed with further investigation" (p. 15). This links the study also to the last sub-theme, which concerns intervention studies.

The aforementioned studies all acknowledge the importance of students' interactions during the argumentation and proving processes. Matos and Rodrigues (2011) even more explicitly focused on proving in the classroom as a form of social practice. The authors adopted the *social theory of learning* perspective, according to which mathematics is a situated and social phenomenon. Consequently, the construct of *community of practice* was central to their work.

The authors analyzed excerpts from group interactions in a grade 9 teaching experiment on geometry, focusing on the use of diagrams. Their analysis showed that, although a group may be seen as a community of practice where students share the same concern for the task and develop a shared practice, the members of the group can behave differently in terms of participation depending on their level of mathematical competence. The analysis showed further that the converse can also happen: "all the members of the team increased their ownership of meaning in different degrees depending of the degree of participation" (p. 183). The analysis highlighted also the complexity of the proving process in groupwork and the key role of the teacher in suggesting the use of the diagram as a powerful means for "sharing and increasing the ownership of meaning of proof" (p. 183). The crucial role of the teacher in fostering students' engagement with argumentation and proof has been discussed in many PME reports, some of which we review next.

Teachers' ways of dealing with argumentation and proof in the classroom. Scholars agree that teachers should set up proper actions so as to arouse students' need for proof and proving (e.g., Zaslavsky, Nickerson, Stylianides et al., 2012). Indeed, teachers face many challenges when dealing with proof in the classroom (e.g., Lin, Yang, Lo et al., 2012b): For example, teachers must establish suitable sociomathematical norms, choose or design appropriate tasks and manage them in the proper way so as to foster understanding, and guide the students towards deductive thinking without turning proving into a "ritual" activity. Teachers must also be able to establish a proving culture in the classroom. One key point is the way proof is introduced in the classroom and what the role and purpose of such a treatment is: Furinghetti and Morselli (2011) distinguished between teaching proofs and teaching by proof, with the aim of proof in the second case being to promote understanding.

Several PME reports addressed the role of the teacher in fostering students' engagement with argumentation and proof. Most studies focused on teachers' interactions with students.

Huang (2005) compared Hong Kong and Shanghai lessons on the Pythagorean theorem using video recordings of lessons. The study showed that Hong Kong teachers tended to value visual verification, while Shanghai teachers were keen to present a deductive argument that met the standard of proof. In terms of interacting with students and involving students in the proving process, Shanghai teachers made more efforts to involve students into proof construction. These findings may be interpreted in terms of the influence of the Confucian and British cultural traditions on the educational settings of Shanghai and Hong Kong, respectively. This study highlighted the role of cultural issues in examinations of the teaching of proof, brought to the fore the crucial dialectics between visualization and deductive reasoning, and took into account not only the status of proof in the classroom, but also the way teachers promote students' involvement with proof.

Several reports (e.g., Azmon, Hershkowitz, & Schwarz, 2011; Rigo, Rojano, & Pluvinage, 2008; Schwarz, Hershkowitz, & Azmon, 2006) addressed the role of the teacher when interacting with students, under the theoretical assumptions that social interaction must have a key role in mathematics instruction and that argumentation may foster concept development. Schwarz et al. (2006) identified recurrent patterns of interaction between two teachers and their students dealing with probability concepts in eighth grade. One of the teachers (whom they called teacher A) played a mediating role, while the other (teacher B) called for short and quick answers, with no provision for argumentation and in a sort of Socratic dialogue: "[W]ith teacher A, students feel obligated to support claims by explaining; they are used to crystallize ideas by reaching agreement and negotiating mathematical meanings; with teacher B, students are committed to tune to the teacher's questions and to adopt her explanations as theirs" (p. 71). In a further study by the same group of researchers, Azmon et al. (2011) conducted a quantitative analysis to explore the relationship between teacher-students patterns of interaction in the same two classes and individual students' subsequent argumentative processes. They found that students' explanations in the two classes differed not in terms of correctness, but in terms of richness, with richer explanations offered in the class of teacher A. An interpretation of these findings is that in teacher A's class there were sociomathematical norms concerning students' responsibility for elaboration on their explanations and engagement in knowledge construction. The study highlights the importance of the mediating role of the teacher and suggests the significance of educating teachers so that they can efficiently manage discussions in their classrooms.

The mediating role of the teacher was discussed also by Cusi and Malara (2009), who studied grade 9-10 students' conscious use of algebraic language through teaching experiments on proof in elementary number theory. Drawing on the idea of *cognitive apprenticeship* (Collins et al., 1989), the authors affirmed that the teachers should serve as a "role model," thus fostering students' development of those skills that are crucial for proving. The research was carried out in two steps: in the first, the authors analyzed the teachers' interventions when leading students to prove, highlighting positive and negative behaviours; in the second, the authors drew from their previous analysis a characterization of the theoretical construct of *teacher as a role model*. Some characteristics of the teacher as a role model are: (a) stimulating students' attitude of research and acting as an integral part of the class in the research work; (b) acting as a practical/strategic guide and as a reflective guide in identifying effective practical/strategic models during class activities; (c) maintaining a balance between semantic and syntactic aspects of algebraic language; (d) acting as an "activator" of interpretative processes and anticipating thoughts; (e) acting both as an "activator" of reflective attitudes and as an "activator" of meta-cognitive actions. This is a promising characterization that may also have useful implications for teacher education.

Other studies addressed the role of guide played by the teacher. Ubuz, Dinçer, and Bulbul (2012, 2013, 2014) presented a series of research reports on the structure of argumentation during teacher-students interactions. Their research was conducted in the special context of undergraduate mathematics courses and data analysis was performed using Toulmin's model for argumentation. Their findings suggested that the teacher plays a crucial role, providing guide-backing (approving warrants, backing or intermediate conclusions given by students) and guide-redirecting (proposing examples or suggestions when the students get stuck or do not start the argumentation from a good point).

Interventions aimed at improving the teaching and learning of argumentation and proof. Setting up interventions (i.e. planning task sequences and devising learning strategies) for the teaching and learning of proof is a crucial theme of research in the teaching and learning of argumentation and proof, as evidenced by the recent ICMI Studies 19 "Proof and proving in mathematics education" and 22 "Task design in mathematics education." Lin, Yang, Lee et al. (2012a) discussed principles for task design for conjecturing, proving, and the transition between conjecture and proof. Regarding conjecturing, the authors highlighted the importance of providing students with an opportunity to engage in observation, construction, and reflection. Regarding proving, the authors pointed out the importance of promoting the expression of arguments using different modes of argument representation (verbal arguments, symbolic notations, etc.), asking students to create and share their own proofs and to evaluate proofs produced by the teacher. Finally, regarding the transition from conjecture to proof, the authors suggested that the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317).

Within this general strand of research, a number of PME reports dealt with interventions aimed at promoting students' approach to argumentation and proof. These studies are important, as they bring to the fore a third main element of classroom-based research besides students and teachers, namely tasks. Also, these studies help illustrate the link between theoretical and applied research by examining how theoretical ideas can be turned into proposals for classroom implementation.

An example of the shift from theoretical considerations to classroom implementation is found in a collection of reports including a research forum (Boero, 2006; Boero, Douek, Morselli et al., 2010; Boero & Morselli, 2009; Boero & Planas, 2014) concerning the possible adaptation of Habermas' (1998) construct of *rational behaviour* to study different aspects of proving and other mathematical activities. The construct of rational behaviour deals with the complexity of discursive practices in the intersection of three kinds of rationality: epistemic (relating to the development of knowledge and questions about the validity of judgments), teleological (relating to strategic choices and corresponding actions to achieve a set goal), and communicative (relating to the reflective use of language oriented toward reaching understanding). The work of Boero and his colleagues illustrates how an important theoretical construct from outside mathematics education can be conveniently interpreted and flexibly adapted to offer, in combination with other constructs, a new and promising perspective into the study of discursive practices related to proving. An interesting aspect is the fact that this construct, integrated with other theoretical tools such as Toulmin's model for argumentation, may provide a comprehensive frame that allows: (1) to better analyze students' proving processes and (2) to plan and carry out innovative classroom interventions. Within the integrated model proposed by Boero et al. (2010), two levels of argumentation are outlined: the meta-level, concerning the awareness of the constraints related to the three components of rational behaviour in proving, and the level concerning the proof content. Thus, students' enculturation into the culture of theorems is a long-term process where the teacher must create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological, and communicative requirements of proving.

Another example of theoretical considerations that turn into task design is offered by a series of research reports by Cheng and Lin (2006, 2007, 2008). The authors proposed and tested a learning strategy, called "reading and coloring," aimed at helping students to take into account all the necessary information to develop a proof. The strategy derives from theoretical considerations about the cognitive processes underlying multi-step proof production and is explicitly addressed to low achievers in proving. The authors emphasized that the strategy should be cultivated in line with the following two design principles: the strategy must provide an operative tool to students, and the strategy must be in continuity with the teacher's regular teaching approach. The strategy, tested in geometry grade 9 courses, is found to be efficient in reducing memory workload when organizing several steps into a proof sequence. The strategy is not efficient when colors may cause visual disturbance and for those students who have difficulty in devising intermediate hypothetical conditions. For those students who do not perform hypothetical bridging thinking, Cheng and Lin (2008) proposed a different learning strategy, called "step-by-step unrolled reasoning strategy."

Another strand of research (e.g., Heinze, Reiss, & Groß, 2006; Kuntze, 2008; Miyazaki, Fujita, & Jones, 2014) focused on the development of tasks to foster students' approach to argumentation and proof, with an emphasis on the process of proving and meta-level knowledge about proof. Heinze et al. (2006) proposed worked-out examples as a tool for helping students to learn argumentation and proof. Drawing on Boero's (1999) description of the phases of the proving process and on Schoenfeld's (1983) idea of teaching heuristic methods in problem solving, the authors set up a learning environment based on heuristic worked-out examples. The study was carried out in grade 8. The sample comprised of 243 German students, who were divided into an experimental group (150 students) and a control group (93 students) according to their performance on a pre-test on reasoning and proving and a questionnaire about their interest towards mathematics. The control group received regular instruction on proving, while the experimental group followed a learning path that guided exploration and more reproductive phases. Heuristic worked-out examples were embedded into stories: students could follow the proving process of hypothetical characters, accompanied by meta-level comments and explanations. Moreover, students were involved in self-explanation activities by working with short texts with blanks. In the words of the authors, "heuristic worked-out examples provide scaffolding and might on the other hand encourage students to perform their own mathematical activities" (p. 279). Data analysis showed that the learning path based on heuristic worked-out examples is particularly efficient for low-achievers.

The work by Heinze et al. focused on proving as a process and on the idea of offering students some element of meta-level knowledge about proof. In the same vein, Kuntze (2008) set up a learning environment ("topic study method") where students were asked to write texts on different aspects of the proving process so as to foster their proof-related meta-knowledge. Students, for instance, were asked to evaluate argumentations of hypothetical characters containing mistakes, or to comment on mathematicians' quotations about proof. In the first part of the study, 121 grade 8 students followed the topic study method, while 111 students followed the heuristic examples method (Heinze et al., 2006). The findings showed that the two methods are comparable in efficiency. In the second part of the study, 153 university students were assigned to different groups: 24 received no specific training on proof, 22 solved geometry tasks without proving, 18 worked with the topic study method, and 89 worked with heuristic worked-out examples. Students who worked with the topic study method scored significantly better than students of the first two control groups and comparably with those who followed the worked-out examples method. Kuntze found that the topic study learning environment might improve students' proof-related meta-knowledge. The study also opened up for reflection a possible correlation between meta-knowledge and proof competence.

Miyazaki et al. (2014) addressed the issue of setting up efficient introductory lessons to proof. In order to help students appreciate the structure of a proof, they proposed to combine two pedagogical ideas: flow-chart proofs (showing the "story line" of the proof) and open problems (see Figure 3). Proof construction was an open problem in the sense that students could "construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion" (p. 228).

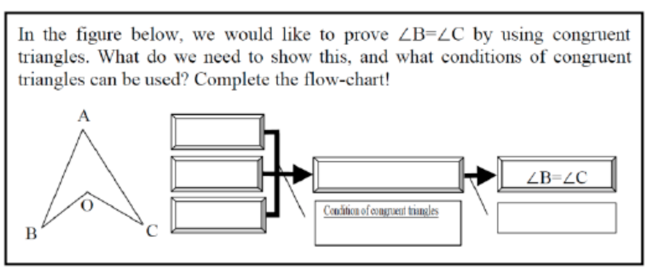

Figure 3. An example of flow-chart proving in an "open problem" situation (derived from Miyazaki et al., 2014, p. 227)

The use of flow-chart proving was tested in grade 8, in a teaching experiment of nine lessons: during the first four lessons students constructed flow-chart proofs in open problem tasks; during two lessons they constructed a proof by reference to a flow chart in a closed problem task; during three lessons they refined proofs by placing them into flow-chart proof format in a closed problem situation. The findings showed that such an approach might foster an understanding of proof. More precisely, the flow chart proof helped students identify necessary conditions and combine them to reach conclusions.

Reflection on the State of PME Research within Theme 2

The review of reports belonging to Theme 2 reveals some common trends and shared consensus among scholars on some key issues concerning the teaching and learning of argumentation and proof. The first issue is the deep interconnection between argumentation and proof and the benefits of addressing argumentative activities. This is also linked to the growing consensus about the importance of social interaction when doing mathematics (e.g., Schwarz, Dreyfus, & Hershkowitz, 2009). Another key issue refers to the importance of providing students with opportunities to appreciate the process of proving, and not only proof as a final product. To this end, proving is conceptualized as a special case of problem solving (e.g., Weber, 2005), thus suggesting the importance of meta-knowledge (Boero et al., 2010; Kuntze, 2008) and heuristics (Heinze et al., 2006). Further research should study the role of heuristics and meta-knowledge and ways of promoting them. It should also aim to address the link between long-term argumentative activities and consequent proving competencies.

Other issues emerging from our review in this section concern task design. The research reports we reviewed proposed and tested tasks or task sequences aimed at improving students' approach to argumentation and proof, with a focus on proving as a process. These reports paid special attention to the theoretical considerations that guided or underpinned task design. The field would benefit from more research that would use theoretical ideas to design practical tools for use in the classroom and in the service or particular learning goals in different areas including argumentation and proof (Stylianides & Stylianides, 2013).

THEME 3: RESEARCH ON TEACHER KNOWLEDGE AND DEVELOPMENT

General Description of Theme 3

The PME reports falling in this theme addressed a range of issues pertaining to teachers' knowledge and development, with a relative balance in focus between preservice and inservice teachers and between the elementary and secondary school levels. The issues addressed could be categorized into the following sub-themes:

– Teachers' knowledge of argumentation and proof (nature and development);

– Teachers' knowledge of teaching argumentation and proof (nature and development); and
– Teachers' beliefs related to argumentation and proof.

In what follows, we review separately reports belonging to each of these three sub-themes. The first sub-theme received by far the most attention in the PME proceedings we reviewed, and this is reflected in the space we have devoted to it. We conclude with a reflection on the state of PME research within Theme 3.

Review of PME Reports Belonging to Theme 3

Teachers' knowledge of argumentation and proof (nature and development). The reports in this sub-theme have predominantly examined the nature of preservice elementary teachers' mathematical knowledge about different aspects of argumentation and proof (e.g., Zazkis & Zazkis, 2013), though there are also examples of studies with inservice teachers including secondary mathematics teachers (e.g., Gabel & Dreyfus, 2013; Tsamir, Tirosh, Dreyfus et al., 2008). Also, there are few studies that specifically aimed to develop teachers' knowledge about argumentation and proof (e.g., Gholamazad, 2007; Reichersdorfer, Vogel, Fischer et al., 2012). Most of the studies were conducted in mathematics courses offered in teacher education or professional development programs, with data collection being directly linked to or forming part of research participants' coursework, supplemented in few cases with individual interviews.

Overall, the contributions made by the reports in this sub-theme fall in one or more of the following three categories: (1) Empirical findings about the nature of teachers' mathematical knowledge about argumentation and proof (What do teachers know?); (2) Empirical findings about the effectiveness of interventions designed to enhance teachers' mathematical knowledge about argumentation and proof (How can teachers' knowledge be developed?); and (3) Theoretical or methodological contributions to research on the nature or development of teachers' mathematical knowledge about argumentation and proof. We present few examples of reports to illustrate these contributions.

The report of Zazkis and Zazkis (2013) is an example of a report making a contribution within categories (1) and (3). The research was conducted in a mathematics course for preservice elementary teachers, with the data comprising the written responses of 24 preservice teachers to a task in an elective course assignment. The task presented a scenario in which two characters (presumably students) had opposing views about the truth or falsity of a mathematical generalization, which was false but this piece of information was not revealed to solvers. The research participants were asked, first, to imagine and write a dialogue in which the two characters attempted to convince each other of their viewpoint, and, second, to comment on their dialogues thus distinguishing between the argumentation attributed to the characters and the argumentation that participants themselves considered appropriate. Only a third of the participants indicated clearly that the generalization was false, with many considering the generalization as "not totally wrong" or "only partly correct." Also, many participants created characters that were not convinced by a single counterexample and found certain counterexamples more convincing than others. While with these dialogues participants demonstrated good understanding of different forms that students' argumentation might take, participants did not clarify in their commentaries whether they themselves considered these forms of argumentation mathematically appropriate, which raises concern about their understanding of the power of a single counterexample to refute a generalization. A similar concern derived from the findings of other PME reports with preservice elementary teachers (e.g., Zeybek & Galindo, 2014).

To examine preservice teachers' mathematical knowledge, Zazkis and Zazkis (2013) used a task that was both *mathematical*, as it asked solvers to comment on their written dialogues thus demonstrating their own mathematical knowledge about argumentation and proof, and *connected to teaching*, as it put solvers in a situation of imagining and articulating different arguments students might offer or consider convincing for the particular generalization. This task is an exemplar of a special category of tasks that Stylianides and Stylianides (2006) called "teaching-related mathematics tasks" and defined as follows: "These are *mathematics* tasks that are *connected to teaching*, and have a dual purpose: (1) to foster [or assess] teacher learning of mathematics that is important for teaching, and (2) to help teachers see how this mathematics relates to the work of teaching" (p. 205).[5] In their report, whose contribution is mainly theoretical and thus illustrative of category (3), Stylianides and Stylianides (2006) argued that teaching-related mathematics tasks might serve as a means to promote or assess inservice or preservice teachers' knowledge of mathematics, including argumentation and proof, by taking seriously the idea that these are adults who are, or are specifically preparing to become, teachers of mathematics. Interestingly, many reports that addressed aspects of teachers' mathematical knowledge about argumentation and proof did not offer a compelling argument about why and how these aspects are, or could be, essential for mathematics teaching.

While the majority of reports making a contribution within category (1) involved (preservice) elementary teachers and identified weaknesses in their mathematical knowledge about argumentation and proof, reports that examined (preservice or inservice) secondary mathematics teachers' knowledge also identified weaknesses (e.g., Gabel & Dreyfus, 2013; Tsamir et al., 2008). For example, in a study with 50 inservice secondary mathematics teachers Tsamir et al. (2008) found the following: While all research participants correctly proved or refuted six given statements using different predicates and quantifiers and also correctly recognized the validity of given symbolically-presented proofs for each of those statements, only about half of them identified as invalid a symbolically-presented argument that was not general.

Few reports made a contribution within category (2), which concerns empirical findings about the effectiveness of interventions to enhance teachers' knowledge. Gholamazad (2007) engaged preservice elementary teachers in writing down dialogues between two imaginary characters: "EXPLORER, the one who tries to prove the proposition, and WHYer, the one who asks all the possible questions related to the process of proof" (p. 266). Analysis of the created dialogues showed that such an activity was efficient in leading preservice teachers to "explain *why* and *how* to do instead of just doing" (p. 271). The study of Reichersdorfer et al. (2012), which involved 119 preservice secondary mathematics teachers, also made a contribution within category (2). Using an experimental research design with pre- and post-tests and random allocation of participants in four intervention groups, the researchers examined the effect on participants' argumentation skills of two collaborative learning settings (one with and another without a collaboration script) and two instructional approaches (one based on heuristic worked examples and another based on authentic problems). Key findings included the following: there was no significant difference between the effects of the two collaborative settings; the instructional approach based on heuristic worked examples was more effective for the development of a certain kind of argumentation skills that the researchers characterized as "low level" (e.g., schematic argumentation skills based on a routine application of simple rules); the instructional approach based on authentic problems was more effective for the development of a different kind of argumentation skills that the researchers characterized as "high level" (e.g., evaluating and proving or refuting conjectures). This research cast some light on the complex network of factors that might determine the effectiveness of an intervention aiming to enhance teachers' (and possibly other individuals') argumentation skills.

Teachers' knowledge of teaching argumentation and proof (nature and development). The reports in this sub-theme collectively examined various aspects of preservice or inservice teachers' knowledge of teaching argumentation and proof, with attention paid to both the nature and the development of that knowledge. Some aspects of knowledge of mathematics teaching that were addressed by the reports related to *knowledge of students*, which we define broadly as knowledge of how students learn or understand argumentation and proof (including knowledge of common student conceptions or misconceptions), while other aspects related to *knowledge of pedagogical practices*, which we also define broadly as knowledge of how to support or assess students' learning or understanding of argumentation and proof.[6]

Overall, the reports in this sub-theme make the point that, while teachers' knowledge of teaching argumentation and proof has weaknesses (some of them having their roots in limitations of teachers' mathematical knowledge about argumentation and proof), improvement of this knowledge is possible. Such an improvement can be purposefully engineered in the context of teacher education or professional development courses, or it can happen more naturalistically in the context of teachers' own professional practice. We present few reports to exemplify aspects of this general point.

Monoyiou, Xistouri, and Philippou (2006) examined the nature of inservice elementary teachers' knowledge of pedagogical practices, with a focus on teachers' assessment of different kinds of student arguments. Specifically, they conducted semi-structured interviews with 16 teachers who were asked to mark on a given scale different kinds of student arguments for three mathematical generalizations. A key finding was that most teachers gave high marks to empirical arguments, which were generally marked at least as highly as valid arguments.

The report of Barkai, Tabach Tirosh, et al. (2009) is an example of a study on the development of teachers' knowledge of students. The research was conducted in a professional development course for inservice secondary mathematics teachers, which aimed to enhance participants' knowledge of mathematics and mathematics teaching related to argumentation and proof. Barkai et al. compared participants' responses before and after the course to the part of a questionnaire that asked them to suggest as many valid and invalid arguments they thought their students would offer for six given statements using different predicates and quantifiers. At the end of the course participants' suggestions of valid and invalid student arguments had increased both in number (by about 50%) and variety. Specifically, participants could offer more valid arguments presented verbally, more invalid arguments based on numerical examples, and more invalid arguments with a repertoire of symbolic lapses. These findings imply that the participants improved their ability to anticipate valid and invalid student arguments expressed with different modes of representation.

Barkai et al. (2009) purposefully engineered the development of teachers' knowledge in a professional development course, which may be viewed as an intervention of long duration. Cirillo (2011) studied in a more naturalistic way the development of a teacher's knowledge of teaching argumentation and proof. Specifically, Cirillo documented the classroom experiences of a beginning secondary mathematics teacher, with strong mathematical background, across his first three years of teaching proof in a geometry class of 15–16-year-olds. With this longitudinal interpretive case study Cirillo cast some light on the challenges faced by beginning teachers in learning to teach proof (even when their mathematical knowledge is not a problem) and on the rather long journey that individual teachers might have to persevere through in order to independently develop their pedagogical practices.

Teachers' beliefs related to argumentation and proof. The reports in this sub-theme have focused predominantly on secondary mathematics teachers (mostly inservice) and have examined teachers' beliefs about the place or purposes of argumentation and proof (or related concepts) in school mathematics (e.g., Chua, Hoyles, & Loh, 2010; Dickerson & Doerr, 2008; Iscimen, 2011), including teachers' views about pedagogical practices related to proof (Dimmel & Herbst, 2014; Miyakawa & Herbst, 2007). We review the findings or broader methodological contributions of some reports in this sub-theme.

The reports of Chua et al. (2010) and Dickerson and Doerr (2008) both examined the beliefs of inservice secondary mathematics teachers. The first was a questionnaire-based study with 29 teachers who took a 9-hour workshop on pattern generalization and were asked to write their thoughts about the purposes of written justification in pattern generalizations, while the second was an interview-based study with 17 teachers concerning their beliefs about the purposes of proof in school mathematics. A key finding of Chua et al. (2010) was that, while almost 60% of the teachers in their study viewed the purpose of a justification to be *explanation*, only one teacher mentioned *conviction*, which is generally recognized to be a core purpose of proof. Chua et al. interpreted this finding with reference to the following distinction between "justification" and "proof" and to the typical form that justification takes in students' work with pattern generalizations:

> A proof is a form of justification given to establish the validity of the rule, but not all justifications are proofs… [In pattern generalization] the explanations provided by learners for justifying how they derive the rule are far less formal than what is expected of in a formal proof. (p. 279)

Dickerson and Doerr's (2008) study focused on proof, and so their findings are not directly comparable to those of Chual et al. A key finding of Dickerson and Doerr was that some teachers believed a major purpose of proof in school mathematics was to develop students' thinking skills and that proofs that deviate from the normative form may undermine this purpose.

Iscimen (2011) examined the development of teachers' beliefs about the place of proof in school mathematics during a geometry course for preservice middle school teachers. Although the course did not focus on proof per se, it did offer plenty of opportunities to participants to engage with proof. Iscimen's findings were based on case studies of six participants who started the course with varying knowledge and beliefs about proof. During the course participants started to appreciate the value of proof and its explanatory power for themselves as teachers. Yet, they questioned the value of proof for their students and their students' ability to engage with proof. Similar disappointing findings concerning preservice secondary mathematics teachers' beliefs about the place of proof in school mathematics were reported by Hallman-Thrasher and Connor (2014), though the participants in their study were teacher candidates with STEM backgrounds and thus not typical of preservice secondary mathematics teachers who are usually mathematics majors.

Dimmel and Herbst (2014) examined inservice secondary mathematics teachers' views about the appropriate level of detail in a proof being scrutinized during a lesson. The researchers used a novel methodological approach to elicit teachers' views that involved use of comics-based, animated representations of lessons in an experimentally controlled way. Findings, derived from application of the methodological approach with a sample of 34 teachers, showed that teachers held different views about the appropriate level of detail in a proof depending on the kind of statements used in a proof. For example, teachers reacted unfavourably to lesson episodes that showed a teacher asking for explicit justification of statements that were tacitly warranted by a diagram, whereas they favoured asking for explicit justification of statements that were tacitly entailed by definitions. In an earlier study that used again representations of lessons to elicit teachers' views about normative practices in instruction, Miyakawa and Herbst (2007) found that secondary mathematics teachers did not always consider that a proof was the best way to convince students about the truth of a theorem. Rather, teachers valued spending time on other kinds of arguments (including empirical) so as to raise students' epistemic value of the theorem. These findings may help explain some of the teacher pedagogical choices and assessments of student work that we reviewed earlier (e.g., Monoyiou et al., 2006): teachers may privilege the kinds of arguments that raise students' epistemic value of the theorem rather than uphold the norms of the discipline. The findings offer further some insight into the possible sources of student misconceptions related to the power of a proof to establish conclusively the truth of a theorem (e.g., Fischbein & Kedem, 1982) and illustrate the tension that may exist between argumentation and proof (e.g., Duval, 1989).

Reflection on the State of PME Research within Theme 3

The focus of PME research on the nature of teachers' mathematical knowledge about argumentation and proof, with an emphasis on limitations of that knowledge and with less attention being paid to ways of developing that knowledge, reflects the state of research on teachers' mathematical knowledge more broadly (e.g., Ponte & Chapman, 2008). It also reflects a general trend in mathematics education research whereby a disproportionally larger number of studies have identified problems of instruction (limitations of teachers' mathematical knowledge being a case in point) than those that have aimed to offer solutions to some of these problems (Stylianides & Stylianides, 2013). More research is thus needed on the development of teachers' mathematical knowledge about argumentation and proof, with the designed interventions taking explicitly into account the idea that effective mathematics teaching requires teachers not only to have good mathematical knowledge but also to be able to use flexibly that knowledge in the course of teaching to support student learning (e.g., Ball, Lubienski, & Mewborn, 2001).

Of course good mathematical knowledge is in itself insufficient for effective teaching (e.g., Kilpatrick et al., 2001), and so a coordinated approach to improving the teaching of argumentation and proof would have to consider also other teacher-related factors, notably, teachers' knowledge of teaching argumentation and proof and teachers' beliefs about the place or purposes of these concepts in school mathematics. Indeed, some PME reports we reviewed earlier and reports published elsewhere have shown, for example, that beginning teachers with good mathematical knowledge still face serious challenges in trying to teach argumentation and proof (e.g., Cirillo, 2011; Stylianides, Stylianides, & Shilling-Traina, 2013) and that teachers tend to have negative beliefs about the appropriateness of proof for their students or their students' ability to engage with proof (e.g., Iscimen, 2011; Knuth, 2002) as well as beliefs that may foster proof-related misconceptions among students (e.g., Miyakawa & Herbst, 2007). Teacher education and professional development programs have a key role to play in preparing or supporting teachers to teach argumentation and proof, though teachers themselves can also view their practice as a context for ongoing inquiry and development (e.g., Ponte & Chapman, 2008) provided that they believe in the importance of argumentation and proof for their students' learning.

CONCLUSION

What this review affords us is a chance to consider the collective approach that is being taken to address the persistent challenge of improving students' experiences with argumentation and proof in school mathematics, and to question how the intellectual resources of the field are being used to address this challenge. In the conclusion of the previous review of PME papers from 1976-2005 concerning proof and proving in mathematics education, Mariotti (2006) cautioned against investigations into the teaching and learning of proof being divorced from the reality of the classroom. It is significant, then, that one of the major themes that has emerged from our review in this chapter of PME reports published over the past decade features classroom-based research. However, while we have more knowledge in the field over the past decade about the nature of argumentation and proof in classrooms, the findings of these and other relevant reports show that the typical school experience of students and the treatment of argumentation and proof in textbooks (e.g., Stylianides, 2014) continue to fall short of what is needed to achieve the intent of educational policy documents or curriculum frameworks (e.g., CCSSI, 2010; Department for Education, 2013).

We note that our review of classroom-based research and research on students' conceptions of argumentation and proof in Themes 1 and 2 focused more on post-elementary school students, while research on teachers' knowledge and development in Theme 3 focused more on elementary teachers (notably pre-service). What might this variation in focus between the three themes imply for the status of proof in school mathematics or for researchers' priorities/assumptions regarding that status? For example, an absence of classroom-based studies in elementary school may be due to the fact that the argumentation activity in most elementary classrooms is sparse. Yet, the same reason could be offered as a justification for the need of more classroom-based research at the elementary school level that would aim to elevate the status of argumentation and proof in elementary classrooms (e.g., Yackel & Hanna, 2003).

Second, while classroom-based research has generally taken a wider view of argumentation and proof, exploring what it might involve to help students appreciate the process of proving and argumentative activity, research on teachers' knowledge and development has focused primarily on teachers' understanding of proof as the final product of an argumentative activity. This is problematic because the teachers' role in teaching and argumentation and proof is multifaceted and not restricted to the judgement of whether different arguments meet the standard of proof (e.g., Herbst, 2006).

Third, both research on students' conceptions and teachers' knowledge of argumentation and proof has placed more emphasis on investigating and understanding the difficulties that students or teachers have with argumentation and proof, paying less attention on the development of instructional interventions to address some of these difficulties. Design-based research might be one tool to respond to this need in the field (Stylianides & Stylianides, 2013); since the heart of design-based research is to iteratively design and investigate classroom-based interventions, the methodology provides an opportunity to engineer situations that answer questions about elementary and middle school students' conceptions of argumentation and proof in the messiness of real classrooms.

Finally, we close by considering what could be new themes in research about argumentation and proof in school mathematics given its increasing prominence in the curriculum frameworks. One such theme should likely focus on equity in students' access to opportunities to engage in argumentation and proof. Given the complex social dynamics at play inside classrooms (e.g., Chazan, 2000), and the difficulties teachers face in managing dialogue about argumentation while also honouring other obligations of their role as teacher (e.g., Stein, Engle, Smith et al., 2008), it is critical that we mobilize intellectual resources in the field to investigate ways of supporting all students to meaningfully participate in argumentation and proof inside classrooms. Another such theme might regard productive ways for assessing students' capacities to not only engage in producing proof, but also for engaging in processes that are "on the road" to proof. Building on research such as the studies reviewed in Theme 1 on students' use of examples or studies published elsewhere (e.g., Zaslavsky, 2014; Zazkis et al., 2008) might support further work into how teachers can identify students' approaches to example use and then act upon their assessments. In addition, as educational standards documents have increasingly featured specific standards regarding knowledge for argumentation (c.f., CCSSI, 2010), a response is needed from the research community about how best to practically assess students' knowledge and understanding of these mathematical practices.

NOTES

1. Our decision to limit this review to reports longer than 1 page implies that we did not consider any short orals or poster presentations.
2. The search function was not available in the 2010 Proceedings and we were thus unable to search for our keywords in the abstracts of that year.
3. We were unable to apply Approach 2 for the 2008 and 2014 Proceedings (which did not include the specific index) and for the 2010 Proceedings (which did not include page numbers or volumes where the relevant reports could be located).
4. Of course, this is not to say that different classes that are taught by the same teacher who uses the same textbook will necessarily receive the same learning experiences (see, e.g., Even, 2008).
5. The notion of "teaching-related mathematics tasks" was further developed and elaborated in Stylianides and Stylianides (2014) under the slightly modified term "pedagogy-related mathematics tasks."
6. Our definitions of these two kinds of teachers' knowledge of mathematics teaching draw on the respective definitions of similar kinds of teacher knowledge discussed in Kilpatrick, Swafford, and Findell (2001, pp. 370-372).

REFERENCES

Alcock, L., & Weber, K. (2005). Referential and syntactic approaches to proof: Case studies from a transition course. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference, 2,* 33-40.

Antonini, S. (2006). Graduate students' processes in generating examples of mathematical objects. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 2,* 57-64.

Antonini, S., & Martignone, F. (2011). Argumentation in exploring mathematical machines: A study on pantographs. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 2,* 41-48.

Azmon, S., Hershkowitz, R., & Schwarz, B. (2011). The impact of teacher-led discussions on students' subsequent argumentative writing. In Ubuz, B. (Ed.), *Proceedings of the 35th PME International Conference*, *2*, 73-80.

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2011). Reasoning by contradiction in dynamic geometry. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 2,* 81-88.

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In I. Wirszup & R. Streit (Eds.), *Proceedings of the Second UCSMP International Conference on Mathematics Education* (pp. 284-297). Reston, VA: National Council of Teachers of Mathematics.

Balacheff, N. (2002). The researcher epistemology: a deadlock for educational research on proof. In F. L. Lin (Ed.), *Proceedings of the 2002 International Conference on Mathematics: understanding proving and proving to understand* (pp. 23-44). Taipei, Taiwan: NSC and NTNU. Pre-publication version retrieved November 25, 2011, from

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). New York, NY: Macmillan.

Barkai, R., Tabach, M., Tirosh, D., Tsamir, P., & Dreyfus, T. (2009). Highschool teachers' knowledge about elementary number theory proofs constructed by students. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 2,* 113-120.

Bieda, K. (2011). Middle grades students' emerging beliefs about argumentation. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 2,* 153-160.

Bieda, K. N., & Lepak, J. (2014). Are you convinced? Middle-grade students' evaluations of mathematical arguments. *School Science and Mathematics*, *114*(4), 166-177.

Boero, P. (1999). Argumentation and mathematical proof: a complex, productive, unavoidable relationship in mathematics and mathematics education. *International newsletter on the teaching and learning of mathematical proof, 7/8*.

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 2*, 185-192.

Boero, P., & Morselli, F. (2009). Towards a comprehensive frame for the use of algebraic language in mathematical modelling and proving. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 2*, 185-192.

Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: a contribution to theoretical perspectives and their classroom implementation. *Proceedings of the 34th PME International Conference*, *1*, 179-209.

Boero, P., Garuti, R., & Lemut, E. (1996). Approaching theorems in grade VIII. In Boero, P. (Ed.), *Theorems in school* (pp. 249-264). Rotterdam: Sense Publisher.

Boero, P., Garuti, R., & Mariotti, M. A. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th PME International Conference, 2,* 121-128.

Boero, P., & Planas, N. (2014). Habermas' construct of rational behavior in mathematics education: New advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterie, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 1*, 205-235.

Brockmann-Behnsen, D., & Rott, B. (2014). Fostering the argumentative competence by means of a structured training. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 2,* 193-200.

Buchbinder, O., & Zaslavsky, O. (2009). A framework for understanding the status of examples in establishing the validity of mathematical statements. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 2,* 225-232.

Buchbinder, O., & Zaslavsky, O. (2013). Inconsistencies in students' understanding of proof and refutation of school mathematics. In. A. Lindemeyer & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference, 2,* 129-137.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. Teachers College Press.

Cheng, Y. H., & Lin, F. L. (2006). Using reading and coloring to enhance incomplete prover's performance in geometry proof. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference*, *2*, 289-296.

Cheng, Y. H., & Lin, F. L. (2007). The effectiveness and limitation of reading and coloring strategy in learning geometry proof. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st PME International Conference*, *2*, 113-120.

Cheng, Y. H., & Lin, F. L. (2008). A study on left behind students for enhancing their competence of geometric argumentation. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the 32nd PME International Conference*, *2*, 305-312.

Chrysostoumou, M., & Christou, C. (2013). Examining 5th grade students' ability to operate on unknowns through their levels of justification. In A. Lindemeyer & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference, 2,* 185-192.

Chua, B. L., Hoyles, C., & Loh, H. C. (2010). Teachers' perceptions of the purposes of mathematical justification. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th PME International Conference, 2,* 273-280.

Cirillo, M. (2011). "I'm like the Sherpa guide": On learning on teach proof in school mathematics. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 2,* 241-248.

Clarke, D. (2001). *Perspectives on practice and meaning in mathematics and science classrooms*. Dordrecht: Kluwer Academic Publishers.

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing and mathematics! In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Cramer, J. C. (2014). Using Habermas to explain why logical games foster argumentation. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 2,* 345-352.

Cusi, A., & Malara, N. A. (2009). The role of the teacher in developing proof activities by means of algebraic language. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference*, *2*, 361-368.

Department for Education. (2013). *Mathematics: Programmes of study: Key Stages 1-2* (National Curriculum in England). Retrieved October 31, 2013, from https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239129/PRIMARY_national_curriculum_-_Mathematics.pdf

Dickerson, D. S., & Doerr, H. M. (2008). Subverting the task" Why some proofs are valued over others in school mathematics. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32th PME International Conference, 2,* 407-414.

Dickerson, D. S., & Pitman, D. J. (2012). Advanced college-level students' categorization and use of mathematical definitions. In T-Y. Tso (Ed.), *Proceedings of the 36th PME International Conference, 2,* 187-193.

Dimmel, J. K., & Herbst, P. G. (2014). What details do geometry teachers expect in students' proofs? A method for experimentally testing possible classroom norms. In C. Nicol, S. Oesterie, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 2,* 393-400.

Dolev, S., & Even, R. (2012). Justifications and explanations in Israeli 7th grade math textbooks. In T. Y. Tso (Ed.), *Proceedings of the 36th PME International Conference, 2,* 203-210.

Douek, N. (2006). Vygotsky's everyday concpets/scientific concepts dialectics in school context: a case study. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th PME International Conference*, *2*, 449-456.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education, New ICMI Study Series 15* (pp. 349-367). Springer Science + Business media B.V.

Duval, R. (1989). Langage et representation dans l'apprentissage d'une demarche deductive. In G. Vergnoud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th PME International Conference, 1,* 228-235.

Edwards, L. D. (2008). Conceptual integration, gesture and mathematics. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd PME International Conference, 2,* 423-430.

Edwards, L. D. (2010). Doctoral students, embodied discourse and proof. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th PME International Conference, 2,* 329-337.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, *38,* 194-229.

Ellis, A. B., Lockwood, E., Dogan, M. F., Williams, C. C., & Knuth, E. (2013). Choosing and using examples: How example activity can support proof insight. In A. Lindemeyer & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference, 2,* 265-273.

Even, R. (2008). Offering mathematics to learners in different classes of the same teacher. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32th PME International Conference, 1,* 51-66.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), *Proceedings of the 6th International Conference of the Psychology of Mathematics Education* (pp. 128–131). Antwerp, Belgium.

Fried, M. N., & Amit, M. (2006). Proof, authority, and agency: Intimations from an 8th grade classroom. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 3,* 113-120.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometrical construction and proving activities: a case of cognitive unity? In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th PME International Conference*, *3*, 9-16.

Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: hows and whys in the teaching of proof. *ZDM – The International Journal on Mathematics Education*, *43*, 587-599.

Gabel, M., & Dreyfus, T. (2013). The flow of a proof – The example of the Euclidean algorithm. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference*, *2*, 321-328.

Gholamazad, S. (2007). Pre-service elementary school teachers' experiences with the process of creating proofs. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st PME International Conference*, *2*, 265-272.

Habermas, J. (1998). *On the pragmatics of communication*. Cambridge, MA: MIT Press.

Haj Yahya, A., Hershkowitz, R., & Dreyfus, T. (2014). Investigating students' geometrical proofs through the lens of students' definitions. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 3,* 217-224.

Hallman-Thrasher, A., & Connor, J. (2014). What and how mathematics should be taught: Views of secondary mathematics teacher candidates with STEM backgrounds. In C. Nicol, S. Oesterie, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 3,* 233-240.

Hanna, G., & de Villiers, M. (Eds.). (2012). *Proof and proving in mathematics education: The 19th ICMI Study*. Springer.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Greenwich, CT: Information Age Publishing.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396-428.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning, 6*, 235-256.

Heinze, A., Reiss, K., & Groß, C. (2006). Learning to prove with heuristic worked-out examples. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th PME International Conference*, *3*, 273-280.

Herbst, P. G. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*, 313–347.

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics, 17*(1), 7-16.

Hsu, H-Y. (2010). Conceptualization of the relations between geometric proof and geometric calculation. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th PME International Conference, 3,* 105-113.

Huang, R. (2005). Verification or proof: Justification of Pythagoras' theorem in Chinese mathematics classrooms. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference*, *3*, 161-168.

Iscimen, F. A. (2011). Preservice middle school teachers' beliefs about the place of proof in school mathematics. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 3,* 57-64.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Knapp, J., & Weber, K. (2006). Reconceiving strategic knowledge in proving from the student's perspective. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 3,* 449-456.

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61-88.

Komatsu, K. (2011). How do students generalize a conjecture through proving? The importance of boundary cases between example and counterexample. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference, 3,* 89-96.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum.

Kunimune, S., Kumakura, H., Jones, K., & Fujita, T. (2009). Lower secondary school students' understanding of algebraic proof. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 3,* 441-448.

Kuntze, S. (2008). Fostering geometrical proof competency by student-centred writing activities. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the 32nd PME International Conference*, *3*, 289-296.

Leung, A., & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: A case of proof by contradiction. *International Journal of Computers for Mathematical Learning, 7,* 145-165.

Lin, F-L. (2005). Modelling students' learning in argumentation and mathematics proof. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference, 1,* 3-18.

Lin, F. L., Yang, K. L., Lee, K. H., Tabach, M., Stylianides, G. (2012a). Principles for task design for conjecturing and proving. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education, New ICMI Study Series 15* (pp. 305-325). Springer Science + Business media B.V.

Lin, F. L., Yang, K. L., Lo, J. J., Tsamir, P., Tirosh, D., & Stylianides, G. (2012b). Teachers' professional learning of teaching proof and proving. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education, New ICMI Study Series 15* (pp. 327-346). Springer Science + Business media B.V.

Lin, M., & Wu, C. (2007). Uses of examples in geometric conjecturing. In J. Woo, K. Park, H. Lew, & D. Sco (Eds.), *Proceedings of the 31st PME International Conference, 3,* 209-216.

Liz, B., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 1,* 126-154.

Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher*, *32*(1), 17-20.

Lockwood, E., Ellis, A. B., Dogan, M. F., Williams, C., & Knuth, E. (2012). A framework for mathematicians' example-related activity when exploring and proving mathematical conjectures. In L. R. Van Zoest, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th PME-NA Conference,* 151–158.

Mamona-Downs, J. (2009). Enhancement of students' argumentation through exposure to others' approaches. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 4,* 89-96.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*, 25-53.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the PME: Past, present and future* (pp. 173-204). Rotterdam, The Netherlands: Sense Publishers.

Martinez, M. V., & Li, W. (2010). The conjecturing process amd the emergence of the conjecture to prove. In M. F. Pinto, & T. F. Kawasaki (Eds.), *Proceedings of the 34th PME International Conference*, *3*, 265-272.

Matos, J. F., & Rodrigues, M. (2011). Proof in classroom social practice. In B. Ubuz (Ed.), *Proceedings of the 35th PME International Conference*, *3*, 177-184.

Miyakawa, T. (2012). Proof in geometry: A comparative analysis of French and Japanese textbooks. In T. Y. Tso (Ed.), *Proceedings of the 36th PME International Conference, 3,* 225-232.

Miyakawa, T., & Herbst, P. (2007). The nature and role of proof when installing theorems: The perspective of geometry teachers. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st PME International Conference, 3,* 281-288.

Miyazaki, M., Fujita, T., & Jones, K. (2014). Functions of open flow-chart proving in introductory lessons in formal proving. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference*, *4*, 226-233.

Monoyiou, A., Xistouri, X., Philippou, G. (2006). Primary students' reasoning in problem solving and teachers' evaluation of their arguments. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th PME International Conference, 4,* 177-184.

Morselli, F. (2006). Use of examples in conjecturing and proving: An exploratory study. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th PME International Conference, 4,* 185-192.

Otten, S., Gilbertson, N. J., Males, L. M., & Clark, D. L. (2014). The mathematical nature of reasoning-and-proving opportunities in geometry textbooks. *Mathematical Thinking and Learning*, *16*(1), 51-79.

Ponte, J. P. & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 223-261). New York: Routledge.

Reichersdorfer, E., Vogel, F., Fischer, F., Kollar, I., Reiss, K., & Ufer, S. (2012). Different collaborative learning settings to foster mathematical argumentation skills. In T. Y. Tso (Ed.), *Proceedings of the 36th PME International Conference, 3,* 345-352.

Reid, D. (2005). The meaning of proof in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4th Conference of the European Society for Research in Mathematics Education* (pp. 458-468). Sant Feliu de Guixols, Spain. Retrieved December 11, 2001, from http://ermeweb.free.fr/CERME4/CERME4_WG4.pdf

Rigo, M., Rojano, T., & Pluvinage, F. (2008). The teacher in the mathematical-argumentation processes within elementary school classroom. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the 32nd PME International Conference*, *4*, 185-192.

Rodriguez, F., & Gutiérrez, A. (2006). Analysis of proofs produced by university mathematics students, and the influence of using Cabri software. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference*, *4*, 433-440.

Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematical problem solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 345-395). New York, NY: Academic Press.

Schwarz, B., Dreyfus, T., & Hershkowitz, R. (Eds.). (2009). *Transformation of knowledge through classroom interaction*. New York: Routledge.

Schwarz, B., Hershkowitz, R., & Azmon, S. (2006). The role of the teacher in turning claims to arguments. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference*, *5*, 65-72.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340.

Stoyanova Kennedy, N. (2006). Conceptual change as dialectical transformation. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference*, *5*, 193-200.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38,* 289–321.

Stylianides, A. J., & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: The case of reasoning and proving. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 5,* 201-208.

Stylianides, A. J., & Stylianides, G. J. (2013). Seeking research-grounded solutions to problems of practice: Classroom-based interventions in mathematics education. *ZDM – The International Journal on Mathematics Education, 45*(3), 333-341.

Stylianides, A. J., & Stylianides, G. J. (2014). Viewing "Mathematics for Teaching" as a form of applied mathematics: Implications for the mathematical preparation of teachers. *Notices of the AMS, 61*(3), 266-276.

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. *For the Learning of Mathematics, 28*(1), 9–16.

Stylianides, G. J. (Ed.). (2014). Special Issue on "Reasoning-and-proving in mathematics textbooks: From the elementary to the university level." *International Journal of Educational Research, 64,* 63-148.

Stylianides, G. J., & Stylianides, A. J. (2006). "Making proof central to pre- high school mathematics is an appropriate instructional goal": Provable, refutable, or undecidable proposition? In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 5,* 209-216.

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education, 11,* 1463-1490.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (forthcoming). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Third Handbook of Research on Mathematics Teaching and Learning*. National Council of Teachers of Mathematics.

Stylianou, D. A., Blanton, M. L., Knuth, E. J. (Eds.). (2010). *Teaching and learning proof across the grades: A K-16 perspective*. Routledge & National Council of Teachers of Mathematics.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Toulmin, S. E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2008). Inservice teachers' judgment of proof in ENT. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd PME International Conference, 4,* 345-352.

Ubuz, B., Dinçer, S., & Bulbul, A. (2012). Argumentation in undergraduate math courses: A study on proof generation. In T. S. Tso (Ed.), *Proceedings of the 36th PME International Conference*, *4*, 163-170.

Ubuz, B., Dinçer, S., & Bulbul, A. (2013). Argumentation in undergraduate math courses: A study on definition construction. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference*, *4*, 313-320.

Ubuz, B., Dinçer, S., & Bulbul, A. (2014). Argumentation in undergraduate math courses: A study on problem solving. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference*, *5*, 305-312.

Ufer, S., Heinze, A., & Reiss, K. (2009). Mental models and the development of geometric proof competency. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME International Conference, 5,* 257-264.

Ufer, S., Heinze, A., & Reiss, K. (2010). Individual predictors of geometrical proof competence. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd PME International Conference, 4,* 401-408.

Watson, A., Sandefur, J., Mason, J., & Stylianides, G. (2013). The use of examples to provide representations in proving. In A. Lindemeyer & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference, 4,* 377-385.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *The Journal of Mathematical Behavior*, *24*(3), 351-360.

Weber, K. (2014). Proof as a cluster concept. In C. Nicol, S. Oesterie, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 5,* 353-360.

Wilkerson, M. (2008). How do mathematicians learn mathematics? In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd PME International Conference, 4,* 409-416.

www.tpp.umassd.edu/proofcolloquium07/reading/Balachef_Taiwan2002.pdf

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 22-44). Reston, VA: National Council of Teachers of Mathematics.

Zaslavsky, O. (2014). Thinking with and through examples. In C. Nicol, S. Oesterie, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 1*, 21-34.

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., Winicki-Landman, G. (2012). The need for proof and proving: Mathematical and pedagogical perspectives. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education, New ICMI Study Series 15* (pp. 215-229). Springer Science + Business media B.V.

Zazkis, D., Weber, K., & Mejia-Ramos, J. P. (2014). Activities that mathematics majors use to bridge the gap between informal arguments and proofs. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 5,* 417-424.

Zazkis, R., & Zazkis, D. (2013). Exploring mathematics via imagined role-playing. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME International Conference, 4,* 433-440.
Zazkis, R., Liljedahl, P., & Chernoff, E. J. (2008). The role of examples in forming and refuting generalizations. *ZDM – The International Journal on Mathematics Education, 40,* 131-141.
Zeybek, Z., & Galindo, E. (2014). Pre-service elementary teachers' misconceptions of proof and counterexamples and their possible influences on their instructional decisions. In C. Nicol, S. Oesterie, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th PME International Conference, 5,* 433-440.

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