COMBINING THE RUNGE APPROXIMATION AND THE WHITNEY EMBEDDING THEOREM IN HYBRID IMAGING

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Abstract. This paper addresses enforcing non-vanishing constraints for solutions to a second order elliptic partial differential equation by appropriate choices of boundary conditions. We show that, in dimension $d \geq 2$, under suitable regularity assumptions, the family of $2d$ solutions such that their Jacobian has maximal rank in the domain is both open and dense. The case of less regular coefficients is also addressed, together with other constraints, which are relevant for applications to recent hybrid imaging modalities. Our approach is based on the combination of the Runge approximation property and the Whitney projection argument [Greene and Wu, Ann. Inst. Fourier (Grenoble), 25(1, vii):215–235, 1975]. The method is very general, and can be used in other settings.

1. Introduction

We consider a general second-order elliptic equation

$$Lu := -\operatorname{div}(a\nabla u + bu) + c \cdot \nabla u + qu = 0 \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded and smooth domain. We assume that L is uniformly elliptic, namely,

$$a(x)\,\xi \cdot \xi \geq \lambda\,|\xi|^2, \qquad \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}^d, \tag{2}$$

for some $\lambda > 0$. The parameters of equation (1) are assumed to satisfy mild regularity assumptions, namely either

$$a \in C^{\ell-1,\alpha}\left(\overline{\Omega}; \mathbb{R}^{d\times d}\right),\ b \in C^{\ell-1,\alpha}(\overline{\Omega}; \mathbb{R}^d),\ c \in W^{\ell-1,\infty}(\Omega; \mathbb{R}^d),\ q \in W^{\ell-1,\infty}(\Omega; \mathbb{R}), \tag{3}$$

with $\ell \geq 1$ and $\alpha \in (0,1)$, or

$$a \in L^\infty\left(\Omega; \mathbb{R}^{d\times d}\right),\ b, c \in L^\infty\left(\Omega; \mathbb{R}^d\right),\ q \in L^\infty\left(\Omega; \mathbb{R}^d\right), \tag{4}$$

which will be referred to as $\ell = 0$. By classical elliptic regularity theory [43, 42, 61], the solutions to (1) belong to $C^{\ell,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R})$ and, provided that the boundary conditions are chosen in the appropriate trace space, such a regularity extends up to the boundary, namely $u \in C^{\ell,\alpha}\left(\overline{\Omega}; \mathbb{R}\right)$. In the case $\ell = 0$, the Hölder exponent α is for example the one given by the De Giorgi–Nash–Moser theorem.

This paper focuses on how to enforce pointwise constraints on the solutions of (1). Our motivation for studying such a question comes from hybrid imaging. Hybrid, or multi-physics, imaging problems are a type of parameter identification problems

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that in many cases involve the reconstruction of the coefficients of a PDE from the knowledge of some internal functional of its solutions [16, 67, 26, 51, 19, 9].

Amongst all these constraints, the most ubiquitous one is the non-vanishing Jacobian problem. It can be reworded as follows: given L as in (1) and a compact set $K \subseteq \Omega$, how can one choose boundary conditions $g_1, \ldots, g_N$ such that

$$\operatorname{rank}(\nabla u_1, \ldots, \nabla u_N) = d \text{ everywhere in } K, \tag{5}$$

where

$$\begin{cases} Lu_i = 0 & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \ldots, N?$$

The difficulty here is that, apart from the fact that a, b, c and q are relatively smooth and coercive, nothing is known about these coefficients, which are the unknowns of the inverse problem.

When $d = 2$, $b = 0, c = 0$ and $q = 0$, that is, L is simply

$$Lu = -\operatorname{div}(a\nabla u),$$

it turns out that the Radó–Kneser–Choquet Theorem can be extended to this setting (without regularity assumptions) [10, 12, 13, 14]. Only two boundary conditions, independent of the matrix valued function a, are required for the constraint to be satisfied globally. This result cannot be extended to higher dimensions [68, 54, 39, 7, 9], even locally: it is not possible to find suitable boundary conditions independently of the (unknown) coefficient. For more general models, when b, c or q are not null, such as the Helmholtz equation, no solace can be found in any dimension, since the Radó–Kneser–Choquet Theorem, whose proof uses the maximum principle, does not apply.

One is therefore drawn to ask whether using a large number of boundary conditions would help. Again, counter-examples to several claims can be derived [9, Corollary 6.18]; nevertheless, it is possible to construct open sets of boundary conditions valid for open sets of parameters for the relevant elliptic operator L. Two main strategies have been used to achieve this goal: complex geometrical optics (CGO) solutions [65, 32, 30, 29, 18, 48, 59, 26, 58, 24, 34, 20, 31, 27] and the Runge approximation property [27, 60, 34, 31]. Both strategies had earlier been used in the context of electrical impedance tomography [49, 64]. Other approaches based on frequency variations [2, 4, 3, 5, 8, 6, 21] or dynamical systems [25] were also developed: these are not discussed here.

CGO solutions are only available for isotropic coefficients a, that is, $a = \gamma I_d$ where γ is a real-valued function. The CGO solution method provides a non-vanishing Jacobian globally inside the domain for a suitable choice of (d complex-valued) boundary conditions. This approach requires high regularity assumptions on the coefficients. On the other hand, the Runge approximation property holds provided that the unique continuation property holds [55], such a property being enjoyed by a much larger class of problems [15]. A drawback is that the argument is local, applied on a covering of the domain by small balls, and so many boundary conditions are needed. Further, while CGO solutions are constructed (depending on the coefficients), the Runge approximation provides an existence result of suitable solutions, but not a constructive method to derive them.

In this work, we combine a Whitney projection argument [66], as described in [45, 44], with the Runge approximation. Conditions of the form (5) are related to the embeddings of manifolds; using Withney's argument to reduce the ambient dimension iteratively, we reduce the number of solutions needed. Not only do we provide an explicit bound on the number of boundary conditions to be considered, but we also obtain that these constitute an open and dense set. For instance, the set of $2d$ boundary conditions such that (5) is satisfied is open and dense in $H^{1/2}(\partial\Omega;\mathbb{R})^{2d}$. Our result applies to more general constraints than (5), so that it is in particular applicable to a variety of imaging problems (see section 2 for details). Our result confirms what has been observed in numerical simulations in the setting of scalar (isotropic) diffusion coefficients, where good reconstructions are obtained for a relatively small set of boundary conditions [59, 60, 25, 34]. After the first version of this manuscript was published, we were made aware of the recent preprint [40], where similar techniques are used for the fractional Calderón problem.

This paper is structured as follows. In section 2, we state our main results and discuss some open problems. Section 3 is devoted to the Runge approximation property. Finally, in section 4 we provide the proof of the main result.

2. Main results

Let $K \subseteq \overline{\Omega}$ be a smooth compact set and

$$\zeta\colon C^{\ell,\alpha}(K) \to C^{0,\alpha}(K)^n$$

be a continuous linear map, with $n \geq 1$. Let $\mathcal{H}(K)$ denote the set of solutions to (1) that are smooth in K, namely

$$\mathcal{H}(K) = \left\{u \in C^{\ell,\alpha}(K) \cap H^1(\Omega) : Lu = 0 \text{ in } \Omega\right\}, \tag{6}$$

equipped with the norm $\|u\|_{\mathcal{H}(K)} = \|u\|_{C^{\ell,\alpha}(K)} + \|u\|_{H^1(\Omega)}$. We are interested in solutions $u_i \in \mathcal{H}(K)$ satisfying the constraint

$$\det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x) \neq 0, \tag{7}$$

in K, locally or globally.

Example 1. Constraints of the form (7) appear in various problems.

- When $n = 1$, $\ell = 0$ and $\zeta(u) = u$, the constraint corresponds to avoiding nodal points, namely $u_1(x) \neq 0$. This is useful whenever a division by u_1 is required.
- When $n = d$, $\ell = 1$, and $\zeta(u) = \nabla u$ (taken as row vector), the constraint imposes a non-vanishing Jacobian. In that case, $(u_1, \dots, u_d)$ defines a local C^2 diffeomorphism. This is the case discussed in the introduction.
- When $n = d+1$, $\ell = 1$, and $\zeta(u) = \begin{bmatrix} u & \nabla u \end{bmatrix}$ (taken as a row vector) the constraint imposes a non-vanishing "augmented" Jacobian. The additional potential u may represent a scaled time derivative, in a time harmonic model. This constraint may also correspond to the non-vanishing Jacobian for $v_i = \frac{u_{i+1}}{u_1}$, $i = 1, \dots, d$.

Such constraints appear in quantitative photoacoustic tomography [22, 34, 32, 29], in quantitative thermoacoustic tomography [30, 20, 3], in acousto-electric tomography (also known as electrical impedance tomography by elastic deformation or ultrasound modulated electrical impedance tomography) [17, 38, 58, 24, 50, 60, 52], in microwave imaging by elastic deformation [18, 2, 11, 65], in current density imaging [47, 56, 62, 27], in dynamic elastography [22, 33, 34] and in other hybrid imaging modalities. We refer to [9] for additional methods and further explanations on some of the models we have mentioned.

We introduce the following notation.

Definition 2. Let $K \subseteq \overline{\Omega}$ be a smooth compact set in $\mathbb{R}^d$ and $\zeta\colon C^{\ell,\alpha}(K) \to C^{0,\alpha}(K)^n$ be a continuous linear map, with $l \geq 0$, $n \geq 1$ and $\alpha \in (0,1)$. The *candidate set* $\mathcal{C}(K)$ is the set of all $x \in K$ for which there exist $u_1, \ldots, u_n \in \mathcal{H}(K)$ so that

$$\det \begin{bmatrix} \zeta\left(u_1\right) \\ \vdots \\ \zeta\left(u_n\right) \end{bmatrix} (x) \neq 0.$$

The *admissible set* $\mathcal{E}(K)$ is the set of all $u = (u_1, \ldots, u_{\left[\frac{d+n}{\alpha}\right]}) \in \mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$ such that

$$\sum_{i_1,\ldots,i_n=1}^{\left[\frac{d+n}{\alpha}\right]} |\det \begin{bmatrix} \zeta\left(u_{i_1}\right) \\ \vdots \\ \zeta\left(u_{i_n}\right) \end{bmatrix} (x)| > 0, \text{ for all } x \in K.$$

Here, $\left[\frac{d+n}{\alpha}\right] = \max\left\{N \in \mathbb{N} : N \leq \frac{d+n}{\alpha}\right\}$ denotes the integer part of $\frac{d+n}{\alpha}$.

Remark 1. In other words, $u \in \mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$ belongs to $\mathcal{E}(K)$ if and only if for every $x \in K$ there exist $i_1, \ldots, i_n \in \{1, \ldots, \left[\frac{d+n}{\alpha}\right]\}$ such that

$$\det \begin{bmatrix} \zeta\left(u_{i_1}\right) \\ \vdots \\ \zeta\left(u_{i_n}\right) \end{bmatrix} (x) \neq 0, \tag{8}$$

namely, if and only if the desired constraint is satisfied everywhere in K for a suitable subset of the solutions $u_1, \ldots, u_{\left[\frac{d+n}{\alpha}\right]}$. The candidate set is the subset of K where satisfying the constraint pointwise is possible at all. If $\mathcal{C}(K) \neq K$, then $\mathcal{E}\left(K\right)$ is empty.

Remark 2. If the coefficients of the PDE are smooth enough, so that $\alpha > \frac{d+n}{d+n+1}$, then the number of solutions is $d+n = \left[\frac{d+n}{\alpha}\right]$. Furthermore, the number of solutions increases as the coefficients become rougher, which reflects the irregular structure of the zero level-sets of (7).

In this general setting, we have the following result.

thm 3. *Take a smooth compact set $K \subseteq \overline{\Omega}$. The admissible set $\mathcal{E}\left(K\right)$ is open in $\mathcal{H}\left(K\right)^{\left[\frac{d+n}{\alpha}\right]}$. If the candidate set satisfies*

$$\mathcal{C}\left(K\right) = K, \tag{9}$$

then $\mathcal{E}\left(K\right)$ is dense in $\mathcal{H}\left(K\right)^{\left[\frac{d+n}{\alpha}\right]}$.

Remark 3. Similar admissibility sets were previously proved to be open in [58]. Since a finite intersection of open and dense sets is open and dense, Theorem 3 immediately extends to the case when finitely many constraints are imposed simultaneously.

Remark 4. As observed in Remark 1, the solutions in the admissible set $\mathcal{E}(K)$ satisfy the constraint (8) in K. It would be interesting to consider a quantitative version of this condition, namely an estimate of the form

$$|\det \begin{bmatrix} \zeta\left(u_{i_1}\right) \\ \vdots \\ \zeta\left(u_{i_n}\right) \end{bmatrix} (x)| \geq C$$

for some constant $C > 0$ given a priori. However, in this case the corresponding admissible set would be neither open nor dense. The study of this scenario would need to make the argument presented below more quantitative.

In section 3 we observe, using the Runge Approximation Property, that assumption (9) is satisfied for a large class of examples, since $\mathcal{C}(\overline{\Omega}) = \overline{\Omega}$.

Our initial focus was on boundary value problems, since such problems are relevant for non-invasive imaging methods, as explained in the introduction. The following corollary is a rewording of our result for boundary value problems.

cor 4. *Take a compact set $K \subseteq \Omega$. Suppose that for every $g \in H^{1/2}(\partial\Omega)$ the problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{10}$$

admits a unique solution $u^g \in H^1(\Omega)$. If (9) holds true, then the set

$$\left\{ \left(g_1, \dots, g_{\left[\frac{d+n}{\alpha}\right]}\right) \in H^{1/2}(\partial\Omega)^{\left[\frac{d+n}{\alpha}\right]} : \left(u^{g_1}, \dots, u^{g_{\left[\frac{d+n}{\alpha}\right]}}\right) \in \mathcal{E}(K) \right\}$$

is open and dense in $H^{1/2}(\partial\Omega)^{\left[\frac{d+n}{\alpha}\right]}$.

Proof. Consider the map

$$\psi\colon H^{1/2}(\partial\Omega)^{\left[\frac{d+n}{\alpha}\right]} \to H^1(\Omega)^{\left[\frac{d+n}{\alpha}\right]}, \qquad \left(g_1, \dots, g_{\left[\frac{d+n}{\alpha}\right]}\right) \mapsto \left(u^{g_1}, \dots, u^{g_{\left[\frac{d+n}{\alpha}\right]}}\right),$$

where u^{g_i} is defined by (10). Because problem (10) is well-posed, we have $\|u^g\|_{H^1(\Omega)} \leq C(L)\|g\|_{H^{1/2}(\partial\Omega)}$ for every $g \in H^{1/2}(\partial\Omega)$. Further, because of our outstanding regularity assumptions on the coefficients (3) we also have $\|u^g\|_{C^{\ell,\alpha}(K)} \leq C(L)\|g\|_{H^{1/2}(\partial\Omega)}$. This shows that the map $\psi\colon H^{1/2}(\partial\Omega)^{\left[\frac{d+n}{\alpha}\right]} \to \mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$ is continuous. Its inverse is given by the trace operator acting component-wise, and is also continuous. In other words, ψ is an isomorphism, and the result immediately follows from Theorem 3, since the set under consideration is $\psi^{-1}(\mathcal{E}(K))$. □

Remark 5. Corollary 4 was stated for simplicity only if K is a proper subset of Ω. When K touches $\partial\Omega$, for instance if $K = \overline{\Omega}$, the same result holds, provided that $H^{1/2}(\partial\Omega)$ is replaced with a suitable trace space consisting of smoother functions, e.g., $H^{\ell+\alpha+\frac{1}{2}(d-1)}(\partial\Omega)$ or $C^{\ell,\alpha}(\partial\Omega)$.

Future perspectives. The regularity assumptions we made are important in all generality, because we use a Unique Continuation Principle argument. For a specific problem, with a given geometry and/or coefficient structure, appropriate extension can often be envisioned (see e.g. [36] for a strategy on how to handle a large class of piecewise regular coefficients). We have limited ourselves to elliptic PDE with real coefficients. Considering the case of complex valued coefficients (which appear in thermo-acoustic tomography [30, 20]) is a natural extension of this work. Maxwell's equation [63, 46, 35, 4, 3] and linear elasticity [37, 57, 53, 23, 28] are not considered here and are natural frameworks where this method could be applied. With the current argument, based on Lemma 7, the number of solutions $\left[\frac{d+n}{\alpha}\right]$ depends monotonically on the regularity of the solutions. It would be interesting to establish whether that is necessary, and if the number of solutions $\left[\frac{d+n}{\alpha}\right]$ is optimal. Finally, note that while a rough description of our result could be that a "random" choice of $\left[\frac{d+n}{\alpha}\right]$ boundary condition suffices, we have not established such a claim. It would be interesting to move from an open and dense set of admissible boundary conditions to a random choice of boundary conditions with high probability (or indeed probability 1). This may allow handling the setup discussed in Remark 4.

3. The Runge approximation property and assumption (9)

For simplicity of exposition, in this section we restrict ourselves to considering only the constraints associated to the maps ζ given in Example 1, namely:

- $n = 1$, $\ell = 0$, $\zeta(u) = u$;
- $n = d$, $\ell = 1$, $\zeta(u) = \nabla u$;
- or $n = d + 1$, $\ell = 1$, $\zeta(u) = \begin{bmatrix} u & \nabla u \end{bmatrix}$.

However, with minor modifications to the argument, many other constraints can be considered, since this approach is very general. The main tool to satisfy (9), namely to show that there always exist global solutions satisfying the desired constraints locally, is the following result: it is sufficient to build suitable solutions of the PDE with constant coefficients, and without lower order terms.

prop 5. *Let L be the elliptic operator defined in (1). In addition to (2), (3) if $\ell = 1$ and (4) if $\ell = 0$, assume that $a(x)$ is a symmetric matrix for every $x \in \overline{\Omega}$ and, if $d \geq 3$, $a \in C^{0,1}\left(\overline{\Omega}; \mathbb{R}^{d\times d}\right)$. Take $x_0 \in \overline{\Omega}$. Let $r > 0$ and $u_1, \ldots, u_n \in C^{\ell,\alpha}(\overline{\Omega}; \mathbb{R})$ be solutions to the constant coefficient problem*

$$-\operatorname{div}\left(a\left(x_0\right)\nabla u_i\right) = 0 \quad \text{in } B(x_0, r), \qquad i = 1, \ldots, n.$$

If

$$\det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x_0) \neq 0, \tag{11}$$

then $x_0 \in \mathcal{C}(\overline{\Omega})$.

Remark 6. This result can in some cases be extended to operators L with piecewise Lipschitz coefficients with possibly countably many pieces, following the strategy given in [36].

Proof. This result, even though not in this exact form, was first derived in [34], and later discussed in [9, Section 7.3] (only in the case $x_0 \in \Omega$). Here we provide only a sketch of the proof in order to highlight the main features; the reader is referred to the references mentioned for the details of the argument.

The proof is split into three steps.

Step 1: approximation of u_i with local solutions v_i to $Lv_i = 0$. Using standard elliptic regularity estimates, it is possible to find $\tilde{r} \in (0, r]$ and $v_i \in H^1(B(x_0, \tilde{r}) \cap \Omega)$ such that $Lv_i = 0$ in $B(x_0, \tilde{r}) \cap \Omega$ and $\|u_i - v_i\|_{C^1\left(\overline{B(x_0,\tilde{r})\cap\Omega}\right)}$ is arbitrarily small (provided that $\tilde{r}$ is chosen small enough). It is worth observing that, even if in [34] the lower order terms are kept in the PDE with constant coefficients, that is not needed [9, Proposition 7.10].

Step 2: approximation of v_i with global solutions w_i to $Lw_i = 0$. Thanks to the regularity assumptions on the coefficients, the elliptic operator L enjoys the unique continuation property [15]. This is equivalent to the Runge approximation property [55], by which it is possible to approximate local solutions with global solutions. Thus, in our setting, there exist $w_i \in H^1(\Omega)$ solutions to $Lw_i = 0$ in Ω such that $\|w_i - v_i\|_{H^1(B(x_0,\tilde{r})\cap\Omega)}$ is arbitrarily small. By elliptic regularity, we can ensure that $\|w_i - v_i\|_{C^1\left(\overline{B(x_0,\tilde{r}/2)\cap\Omega}\right)}$ is arbitrarily small too.

Step 3: $(w_1, \ldots, w_n)$ satisfy the constraint in x_0. Combining the previous steps, we have that $\|u_i - w_i\|_{C^1\left(\overline{B(x_0,\tilde{r}/2)\cap\Omega}\right)}$ is arbitrarily small. For the maps ζ considered above, this immediately implies that $|\zeta(u_i) - \zeta(w_i)|(x_0)$ is arbitrarily small. Thus, by (11), $\tilde{r}$ and w_i may be chosen in such a way that

$$\det \begin{bmatrix} \zeta\,(w_1) \\ \vdots \\ \zeta\,(w_n) \end{bmatrix} (x_0) \neq 0,$$

which shows that $x_0 \in \mathcal{C}(\overline{\Omega})$. □

Let us now verify that for the maps ζ mentioned above, we always have $\mathcal{C}(\overline{\Omega}) = \overline{\Omega}$; in other words, the assumptions of Theorem 3 are satisfied with $K = \overline{\Omega}$.

cor 6. *Let L be the elliptic operator defined in (1). In addition to (2), (3) if $\ell = 1$ and (4) if $\ell = 0$, assume that $a(x)$ is a symmetric matrix for every $x \in \overline{\Omega}$ and, if $d \geq 3$, $a \in C^{0,1}\left(\overline{\Omega}; \mathbb{R}^{d\times d}\right)$. If ζ is one of the maps considered in Example 1, then $\mathcal{C}(K) = K$ for any $K \subseteq \overline{\Omega}$.*

Proof. We consider the three constraints separately:

- $n = 1$, $\ell = 0$, $\zeta\,(u) = u$: set $u_1 = 1$.
- $n = d$, $\ell = 1$, $\zeta\,(u) = \nabla u$: set $u_1 = x_1, \ldots, u_n = x_n$.
- $n = d + 1$, $\ell = 1$, $\zeta\,(u) = \begin{bmatrix} u & \nabla u \end{bmatrix}$: set $u_1 = 1, u_2 = x_1, \ldots, u_{n+1} = x_n$.

Given $x_0 \in \overline{\Omega}$, for any $a(x_0)$, there holds

$$-\operatorname{div}(a(x_0)\nabla u_i) = 0 \quad \text{in } \mathbb{R}^d, \qquad i = 1, \ldots, n,$$

and

$$\det \begin{bmatrix} \zeta\,(u_1) \\ \vdots \\ \zeta\,(u_n) \end{bmatrix} (x_0) \neq 0.$$

The conclusion follows from Proposition 5. □

4. Proof of Theorem 3

We need two lemmata. For $k \geq 2$ and $a \in \mathbb{R}^{k-1}$ let $P_a\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ denote the linear map given by

$$P_a(y) = (y_1 - a_1 y_k, \ldots, y_{k-1} - a_{k-1} y_k)\,.$$

In the following, we shall identify the matrices in $\mathbb{R}^{k\times n}$ with k rows and n columns with the linear maps from $\mathbb{R}^n$ into $\mathbb{R}^k$. We shall denote the Lebesgue measure in $\mathbb{R}^m$ by $|\cdot|_m$.

lem 7. *Take a smooth and compact set $K \subseteq \mathbb{R}^d$ and a positive integer $k > \frac{d+n}{\alpha}$. Let $F\colon K \to \mathbb{R}^{k\times n}$ be of class $C^{0,\alpha}$ and such that $F_x\colon \mathbb{R}^n \to \mathbb{R}^k$ is injective for all $x \in K$. Let $G \subseteq \mathbb{R}^{k-1}$ be the set of those $a \in \mathbb{R}^{k-1}$ for which $P_a \circ F_x\colon \mathbb{R}^n \to \mathbb{R}^{k-1}$ is injective for all $x \in K$, namely*

$$G = \bigcap_{x\in K} \left\{a \in \mathbb{R}^{k-1} : P_a \circ F_x\colon \mathbb{R}^n \to \mathbb{R}^{k-1} \text{ is injective}\right\}.$$

Then $|\mathbb{R}^{k-1} \setminus G|_{k-1} = 0$.

Proof. Note that $\ker P_a = \operatorname{span}\{(a_1, \dots, a_{k-1}, 1)\}$. Thus, since F_x is injective, we have for $x \in K$

$$P_a \circ F_x \text{ is injective} \iff \operatorname{ran} F_x \cap \ker P_a = \{0\} \iff (a_1, \dots, a_{k-1}, 1) \notin \operatorname{ran} F_x.$$

Then, $a \in \mathbb{R}^{k-1} \setminus G$ if and only if there exists $x \in K$ such that $(a_1, \dots, a_{k-1}, 1) \in \operatorname{ran} F_x$, namely $(a_1, \dots, a_{k-1}, 1) \in \cup_{x\in K} \operatorname{ran} F_x$. Therefore, using the projection $\pi\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$, $(b_1, \dots, b_k) \mapsto (b_1, \dots, b_{k-1})$, we can express $\mathbb{R}^{k-1} \setminus G$ as

$$\mathbb{R}^{k-1} \setminus G = \pi(B), \qquad B = \left(\bigcup_{x\in K} \operatorname{ran} F_x\right) \cap \left\{b \in \mathbb{R}^k : b_k = 1\right\}.$$

Hence, it remains to prove that

$$\mathcal{H}^{k-1}(B) = 0,$$

where $\mathcal{H}^{k-1}$ denotes the Hausdorff measure of dimension $k-1$.

Consider the map

$$f\colon K \times \mathbb{R}^n \to \mathbb{R}^k, \qquad (x, v) \mapsto F_x v.$$

By construction, $\operatorname{ran} f = \bigcup_{x\in K} \operatorname{ran} F_x$. Note that $\dim(K \times \mathbb{R}^n) = d + n < \alpha k$, so that $\mathcal{H}^{\alpha k}(K \times \mathbb{R}^n) = 0$. Thus, since f is of class $C^{0,\alpha}$, by Proposition 10 applied to $f_N = f|_{K\times B(0,N)}$ for $N \in \mathbb{N}$, we obtain

$$\mathcal{H}^k\left(\bigcup_{x\in K} \operatorname{ran} F_x\right) = \mathcal{H}^k(\operatorname{ran} f) = \mathcal{H}^k\left(\bigcup_{N\in\mathbb{N}} \operatorname{ran} f_N\right) \le \sum_{N\in\mathbb{N}} \mathcal{H}^k(\operatorname{ran} f_N) = 0.$$

By linearity of $v \mapsto F_x v$, the set $\bigcup_{x\in K} \operatorname{ran} F_x$ is closed under scalar multiplication, and so $\bigcup_{x\in K} \operatorname{ran} F_x \supseteq \mathbb{R}_+ \cdot B$, which implies

$$\mathcal{H}^k(\mathbb{R}_+ \cdot B) = 0.$$

Using the change of variables formula and Tonelli theorem, we deduce $\mathcal{H}^{k-1}(B) = 0$, as desired. □

The following result is in the spirit of Whitney's projection argument, and states that, given k solutions satisfying the necessary constraints, it is possible to reduce the number to $k-1$ by taking suitable linear combinations, provided that k is big enough.

lem 8. *Take a smooth and compact set $K \subseteq \overline{\Omega}$ and a positive integer $k > \frac{d+n}{\alpha}$. Let $u_1, \dots, u_k \in \mathcal{H}(K)$ be such that*

$$\operatorname{rank} \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_k) \end{bmatrix} (x) = n, \qquad x \in K.$$

Let $G \subseteq \mathbb{R}^{k-1}$ be the set of those $a \in \mathbb{R}^{k-1}$ such that

$$\operatorname{rank} \begin{bmatrix} \zeta(u_1 - a_1 u_k) \\ \zeta(u_2 - a_2 u_k) \\ \vdots \\ \zeta(u_{k-1} - a_{k-1} u_k) \end{bmatrix} (x) = n, \qquad x \in K.$$

Then $|\mathbb{R}^{k-1} \setminus G|_{k-1} = 0$.

Proof. Let $F \colon K \to \mathbb{R}^{k \times n}$ be the map of class $C^{0,\alpha}$ defined by

$$F_x = \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_k) \end{bmatrix} (x) \colon \mathbb{R}^n \to \mathbb{R}^k.$$

By assumption, we have that F_x is injective for all $x \in K$. Observe that, by linearity of ζ, we have

$$P_a \circ F_x = \begin{bmatrix} \zeta(u_1)(x) - a_1 \zeta(u_k)(x) \\ \vdots \\ \zeta(u_{k-1})(x) - a_{k-1} \zeta(u_k)(x) \end{bmatrix} = \begin{bmatrix} \zeta(u_1 - a_1 u_k) \\ \vdots \\ \zeta(u_{k-1} - a_{k-1} u_k) \end{bmatrix} (x)$$

and so the conclusion immediately follows by Lemma 7. □

We are now ready to prove Theorem 3.

Proof of Theorem 3.

Step 1: $\mathcal{E}(K)$ is open in $\mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$.

Take $u \in \mathcal{E}(K)$. By definition, we have

$$\sum_{i_1, \dots, i_n = 1}^{\left[\frac{d+n}{\alpha}\right]} \left| \det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x) \right| > 0, \qquad x \in K.$$

Since $\zeta(u_{i_j})$ are $C^{0,\alpha}$ maps in K, and in particular continuous, and K is compact, we have that

$$\sum_{i_1, \dots, i_n = 1}^{\left[\frac{d+n}{\alpha}\right]} \left| \det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x) \right| \geq C, \qquad x \in K,$$

for some constant $C > 0$. Finally, since the map ζ is continuous itself, if $v \in \mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$ is chosen close enough to u we have

$$\sum_{i_1, \dots, i_n = 1}^{\left[\frac{d+n}{\alpha}\right]} \left| \det \begin{bmatrix} \zeta(v_{i_1}) \\ \vdots \\ \zeta(v_{i_n}) \end{bmatrix} (x) \right| \geq \frac{C}{2}, \qquad x \in K,$$

which implies $v \in \mathcal{E}(K)$. This concludes the first step.

Step 2: $\mathcal{E}(K)$ *is dense in* $\mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$.

Take $H = (h_1, \ldots, h_{\left[\frac{d+n}{\alpha}\right]}) \in \mathcal{H}(K)^{\left[\frac{d+n}{\alpha}\right]}$. By assumption, for all $x \in K$ there exist $u_{1,x}, \ldots, u_{n,x} \in \mathcal{H}(K)$ such that

$$|\det \begin{bmatrix} \zeta\left(u_{1,x}\right) \\ \vdots \\ \zeta\left(u_{n,x}\right) \end{bmatrix} (x)| > 0.$$

By continuity of $\zeta\left(u_{i,x}\right)$, there exist neighbourhoods $U^x \ni x$ such that

$$|\det \begin{bmatrix} \zeta\left(u_{1,x}\right) \\ \vdots \\ \zeta\left(u_{n,x}\right) \end{bmatrix} (y)| > 0, \qquad y \in U^x \cap K.$$

Since $K \subseteq \cup_{x \in K} U^x$, by compactness there exist $x_1, \ldots, x_N \in K$ such that $K \subseteq \cup_{j=1}^N U^{x_j}$. Thus, for all $x \in K$ there exists $j \in \{1, \ldots, N\}$ such that

$$|\det \begin{bmatrix} \zeta\left(u_{1,x_j}\right) \\ \vdots \\ \zeta\left(u_{n,x_j}\right) \end{bmatrix} (x)| > 0.$$

Consider all the $M = nN$ corresponding solutions

$$(u_1, \ldots, u_M) = \left(u_{1,x_1}, \ldots, u_{n,x_1}, u_{1,x_2}, \ldots, u_{n,x_2}, \ldots, u_{1,x_N}, \ldots, u_{n,x_N}\right),$$

so that

$$\operatorname{rank} \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_M) \end{bmatrix} (x) = n, \qquad x \in K.$$

In particular, we have

$$\operatorname{rank} \begin{bmatrix} \zeta(h_1) \\ \vdots \\ \zeta(h_{\left[\frac{d+n}{\alpha}\right]}) \\ \zeta(u_1) \\ \vdots \\ \zeta(u_M) \end{bmatrix} (x) = n, \qquad x \in K.$$

By Lemma 8, since $k = \left[\frac{d+n}{\alpha}\right] + M > \left[\frac{d+n}{\alpha}\right]$, we have $k > \frac{d+n}{\alpha}$, and so for almost every $a \in \mathbb{R}^{k-1}$ we have

$$\operatorname{rank} \begin{bmatrix} \zeta(h_1 - a_1 u_M) \\ \vdots \\ \zeta(h_{\left[\frac{d+n}{\alpha}\right]} - a_{\left[\frac{d+n}{\alpha}\right]} u_M) \\ \zeta(u_1 - a_{\left[\frac{d+n}{\alpha}\right]+1} u_M) \\ \vdots \\ \zeta(u_{M-1} - a_{k-1} u_M) \end{bmatrix} (x) = n, \qquad x \in K.$$

Repeating this argument M times (as long as $k > \left[\frac{d+n}{\alpha}\right]$) with very small weights a, we obtain that there exist $\xi_{i,j} \in \mathbb{R}$ ($i = 1, \ldots, \left[\frac{d+n}{\alpha}\right]$, $j = 1, \ldots, M$) which can be chosen arbitrarily small such that

$$\operatorname{rank} \begin{bmatrix} \zeta(h_1 - \xi_{1,j} u_j) \\ \vdots \\ \zeta(h_{\left[\frac{d+n}{\alpha}\right]} - \xi_{\left[\frac{d+n}{\alpha}\right],j} u_j) \end{bmatrix} (x) = n, \qquad x \in K,$$

where we used Einstein summation convention of repeated indices. This implies that

$$(h_1 - \xi_{1,j} u_j, \ldots, h_{\left[\frac{d+n}{\alpha}\right]} - \xi_{\left[\frac{d+n}{\alpha}\right],j} u_j) \in \mathcal{E}(K),$$

which, since the weights $\xi_{i,j}$ are chosen arbitrarily small, concludes the proof. $\square$

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Appendix A. Hausdorff measure and Hölder functions

We recall an elementary property of Hausdorff measures that is used in the proof of Lemma 7. For further details, the reader is referred to [1, section 4.1] or [41, section 2.4.1].

Definition 9. Let $A \subseteq \mathbb{R}^m, 0 \leq s < \infty, 0 < \delta \leq \infty$. We write

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{j=1}^\infty \gamma(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subseteq \bigcup_{j=1}^\infty C_j, \operatorname{diam} C_j \leq \delta \right\},$$

where $\gamma(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$ and Γ is the gamma function. The quantity

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

is the s-dimensional Hausdorff measure of A.

prop 10. *Take $B \subseteq \mathbb{R}^m$, $s \in [0, +\infty)$, $c > 0$ and $\alpha \in (0, 1]$. Let $f \colon B \to \mathbb{R}^k$ be such that*

$$|f(x_1) - f(x_2)| \leq c|x_1 - x_2|^\alpha, \qquad x_1, x_2 \in B.$$

Then

$$\mathcal{H}^s(f(A)) \leq c^s \frac{2^{\alpha s} \gamma(s)}{2^s \gamma(\alpha s)} \mathcal{H}^{\alpha s}(A), \qquad A \subseteq B.$$

In particular, if $\mathcal{H}^{\alpha s}(A) = 0$, then $\mathcal{H}^s(f(A)) = 0$.

Proof. Fix $\delta > 0$ and let $\{C_i\}_{i=1}^\infty \subseteq \mathbb{R}^m$ be such that $\operatorname{diam} C_i \leq \delta$, $A \subseteq \cup_{i=1}^\infty C_i$. Then $\operatorname{diam} f(C_i \cap B) \leq c (\operatorname{diam} C_i)^\alpha \leq c\, \delta^\alpha$ and $f(A) \subseteq \cup_{i=1}^\infty f(C_i \cap B)$. Thus

$$\mathcal{H}^s_{c\delta^\alpha}(f(A)) \leq \sum_{i=1}^\infty \gamma(s) \left(\frac{\operatorname{diam} f(C_i \cap B)}{2} \right)^s \leq c^s \frac{2^{\alpha s} \gamma(s)}{2^s \gamma(\alpha s)} \sum_{i=1}^\infty \gamma(\alpha s) \left(\frac{\operatorname{diam} C_i}{2} \right)^{\alpha s}.$$

Taking infima over all such sets $\{C_i\}_{i=1}^{\infty}$, we find

$$\mathcal{H}^s_{c\delta^\alpha}(f(A)) \le c^s \frac{2^{\alpha s}\gamma(s)}{2^s\gamma(\alpha s)} \mathcal{H}^{\alpha s}_\delta(A).$$

Taking the limit as $\delta \to 0$ the result follows. □

References

[1] Giovanni Alberti. Generalized N-property and Sard theorem for Sobolev maps. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 23(4):477–491, 2012.

[2] Giovanni S. Alberti. On multiple frequency power density measurements. *Inverse Problems*, 29(11):115007, 25, 2013.

[3] Giovanni S. Alberti. Enforcing local non-zero constraints in PDEs and applications to hybrid imaging problems. *Comm. Partial Differential Equations*, 40(10):1855–1883, 2015.

[4] Giovanni S. Alberti. On multiple frequency power density measurements II. The full Maxwell's equations. *J. Differential Equations*, 258(8):2767–2793, 2015.

[5] Giovanni S. Alberti. Absence of Critical Points of Solutions to the Helmholtz Equation in 3D. *Arch. Ration. Mech. Anal.*, 222(2):879–894, 2016.

[6] Giovanni S. Alberti, Habib Ammari, and Kaixi Ruan. Multi-frequency acousto-electromagnetic tomography. In *A panorama of mathematics: pure and applied*, volume 658 of *Contemp. Math.*, pages 67–79. Amer. Math. Soc., Providence, RI, 2016.

[7] Giovanni S. Alberti, Guillaume Bal, and Michele Di Cristo. Critical points for elliptic equations with prescribed boundary conditions. *Arch. Ration. Mech. Anal.*, 226(1):117–141, 2017.

[8] Giovanni S. Alberti and Yves Capdeboscq. On local non-zero constraints in PDE with analytic coefficients. In *Imaging, Multi-scale and High Contrast Partial Differential Equations*, volume 660 of *Contemp. Math.*, pages 89–97. Amer. Math. Soc., Providence, RI, 2016.

[9] Giovanni S. Alberti and Yves Capdeboscq. *Lectures on elliptic methods for hybrid inverse problems*, volume 25 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2018.

[10] G. Alessandrini and R. Magnanini. Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions. *SIAM J. Math. Anal.*, 25(5):1259–1268, 1994.

[11] Giovanni Alessandrini. Global stability for a coupled physics inverse problem. *Inverse Problems*, 30(7):075008, 10, 2014.

[12] Giovanni Alessandrini and Vincenzo Nesi. Univalent σ-harmonic mappings. *Arch. Ration. Mech. Anal.*, 158(2):155–171, 2001.

[13] Giovanni Alessandrini and Vincenzo Nesi. Estimates for the dilatation of σ-harmonic mappings. *Rend. Mat. Appl. (7)*, 35(3-4):215–225, 2014.

[14] Giovanni Alessandrini and Vincenzo Nesi. Quantitative estimates on Jacobians for hybrid inverse problems. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 8(3):25–41, 2015.

[15] Giovanni Alessandrini, Luca Rondi, Edi Rosset, and Sergio Vessella. The stability for the Cauchy problem for elliptic equations. *Inverse Problems*, 25(12):123004, 47, 2009.

[16] H. Ammari. *An introduction to mathematics of emerging biomedical imaging*, volume 62 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2008.

[17] H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink. Electrical impedance tomography by elastic deformation. *SIAM J. Appl. Math.*, 68:1557–1573, 2008.

[18] H. Ammari, Y. Capdeboscq, F. de Gournay, A. Rozanova-Pierrat, and F. Triki. Microwave imaging by elastic deformation. *SIAM J. Appl. Math.*, 71(6):2112–2130, 2011.

[19] H. Ammari, J. Garnier, H. Kang, L. H. Nguyen, and L. Seppecher. Mathematics of super-resolution biomedical imaging. Technical Report 2016-31, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2016.

[20] Habib Ammari, Josselin Garnier, Wenjia Jing, and Loc Hoang Nguyen. Quantitative thermo-acoustic imaging: an exact reconstruction formula. *J. Differential Equations*, 254(3):1375–1395, 2013.

[21] Habib Ammari, Laure Giovangigli, Loc Hoang Nguyen, and Jin-Keun Seo. Admittivity imaging from multi-frequency micro-electrical impedance tomography. *J. Math. Anal. Appl.*, 449(2):1601–1618, 2017.

[22] G. Bal. Explicit Reconstructions in QPAT, QTAT, TE, and MRE. *ArXiv e-prints*, 2012.
[23] G. Bal, C. Bellis, S. Imperiale, and F. Monard. Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements. *Inverse Problems*, 30(12):125004, 22, 2014.
[24] G. Bal, E. Bonnetier, F. Monard, and F. Triki. Inverse diffusion from knowledge of power densities. *Inverse Probl. Imaging*, 7(2):353–375, 2013.
[25] G. Bal and M. Courdurier. Boundary control of elliptic solutions to enforce local constraints. *Journal of Differential Equations*, 255(6):1357 – 1381, 2013.
[26] Guillaume Bal. Hybrid inverse problems and internal functionals. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 325–368. Cambridge Univ. Press, Cambridge, 2013.
[27] Guillaume Bal, Chenxi Guo, and François Monard. Inverse anisotropic conductivity from internal current densities. *Inverse Problems*, 30(2):025001, 21, 2014.
[28] Guillaume Bal, François Monard, and Gunther Uhlmann. Reconstruction of a fully anisotropic elasticity tensor from knowledge of displacement fields. *SIAM J. Appl. Math.*, 75(5):2214–2231, 2015.
[29] Guillaume Bal and Kui Ren. Multi-source quantitative photoacoustic tomography in a diffusive regime. *Inverse Problems*, 27(7):075003, 20, 2011.
[30] Guillaume Bal, Kui Ren, Gunther Uhlmann, and Ting Zhou. Quantitative thermo-acoustics and related problems. *Inverse Problems*, 27(5):055007, 15, 2011.
[31] Guillaume Bal and John C. Schotland. Ultrasound-modulated bioluminescence tomography. *Phys. Rev. E*, 89:031201, Mar 2014.
[32] Guillaume Bal and Gunther Uhlmann. Inverse diffusion theory of photoacoustics. *Inverse Problems*, 26(8):085010, 2010.
[33] Guillaume Bal and Gunther Uhlmann. Reconstructions for some coupled-physics inverse problems. *Appl. Math. Lett.*, 25(7):1030–1033, 2012.
[34] Guillaume Bal and Gunther Uhlmann. Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions. *Comm. Pure Appl. Math.*, 66(10):1629–1652, 2013.
[35] Guillaume Bal and Ting Zhou. Hybrid inverse problems for a system of Maxwell's equations. *Inverse Problems*, 30(5):055013, 17, 2014.
[36] J. M. Ball, Y. Capdeboscq, and B. Tsering-Xiao. On uniqueness for time harmonic anisotropic Maxwell's equations with piecewise regular coefficients. *Math. Models Methods Appl. Sci.*, 22(11):1250036, 11, 2012.
[37] P. E. Barbone, C. E. Rivas, I. Harari, U. Albocher, A. A. Oberai, and Y. Zhang. Adjoint-weighted variational formulation for the direct solution of inverse problems of general linear elasticity with full interior data. *Internat. J. Numer. Methods Engrg.*, 81(13):1713–1736, 2010.
[38] Y. Capdeboscq, J. Fehrenbach, F. de Gournay, and O. Kavian. Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements. *SIAM J. Imaging Sci.*, 2(4):1003–1030, 2009.
[39] Yves Capdeboscq. On a counter-example to quantitative Jacobian bounds. *J. Éc. polytech. Math.*, 2:171–178, 2015.
[40] Mihajlo Cekić, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *arXiv preprint arXiv:1810.04211*, 2018.
[41] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
[42] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.
[43] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[44] R. E. Greene and H. Wu. Embedding of open Riemannian manifolds by harmonic functions. *Ann. Inst. Fourier (Grenoble)*, 25(1, vii):215–235, 1975.
[45] R. E. Greene and H. Wu. Whitney's imbedding theorem by solutions of elliptic equations and geometric consequences. In *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII,*

Part 2, Stanford Univ., Stanford, Calif., 1973), pages 287–296. Amer. Math. Soc., Providence, R. I., 1975.
[46] Chenxi Guo and Guillaume Bal. Reconstruction of complex-valued tensors in the Maxwell system from knowledge of internal magnetic fields. *Inverse Probl. Imaging*, 8(4):1033–1051, 2014.
[47] K.F. Hasanov, A.W. Ma, R.S. Yoon, A.I. Nachman, and M.L. Joy. A new approach to current density impedance imaging. In *Engineering in Medicine and Biology Society, 2004. IEMBS '04. 26th Annual International Conference of the IEEE*, volume 1, pages 1321–1324, Sept 2004.
[48] Ilker Kocyigit. Acousto-electric tomography and CGO solutions with internal data. *Inverse Problems*, 28(12):125004, 2012.
[49] R. V. Kohn and M. S. Vogelius. Determining conductivity by boundary measurements. II. Interior results. *Comm. Pure Appl. Math.*, 38(5):643–667, 1985.
[50] P. Kuchment and D. Steinhauer. Stabilizing inverse problems by internal data. *Inverse Problems*, 28(8):084007, 2012.
[51] Peter Kuchment. Mathematics of hybrid imaging: a brief review. In *The mathematical legacy of Leon Ehrenpreis*, volume 16 of *Springer Proc. Math.*, pages 183–208. Springer, Milan, 2012.
[52] Peter Kuchment and Dustin Steinhauer. Stabilizing inverse problems by internal data. II: non-local internal data and generic linearized uniqueness. *Anal. Math. Phys.*, 5(4):391–425, 2015.
[53] Ru-Yu Lai. Uniqueness and stability of Lamé parameters in elastography. *J. Spectr. Theory*, 4(4):841–877, 2014.
[54] R. S. Laugesen. Injectivity can fail for higher-dimensional harmonic extensions. *Complex Variables Theory Appl.*, 28(4):357–369, 1996.
[55] P. D. Lax. A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations. *Comm. Pure Appl. Math.*, 9:747–766, 1956.
[56] June-Yub Lee. A reconstruction formula and uniqueness of conductivity in MREIT using two internal current distributions. *Inverse Problems*, 20(3):847–858, 2004.
[57] Joyce R. McLaughlin, Ning Zhang, and Armando Manduca. Calculating tissue shear modules and pressure by 2D log-elastographic methods. *Inverse Problems*, 26(8):085007, 25, 2010.
[58] François Monard and Guillaume Bal. Inverse anisotropic diffusion from power density measurements in two dimensions. *Inverse Problems*, 28(8):084001, 20, 2012.
[59] François Monard and Guillaume Bal. Inverse diffusion problems with redundant internal information. *Inverse Probl. Imaging*, 6(2):289–313, 2012.
[60] François Monard and Guillaume Bal. Inverse anisotropic conductivity from power densities in dimension $n \geq 3$. *Comm. Partial Differential Equations*, 38(7):1183–1207, 2013.
[61] Charles B. Morrey, Jr. *Multiple integrals in the calculus of variations*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition.
[62] A. Nachman, A. Tamasan, and A. Timonov. Current density impedance imaging. *Tomography and Inverse Transport Theory. Contemporary Mathematics (G. Bal, D. Finch, P. Kuchment, P. Stefanov, G. Uhlmann, Editors)*, 559:035014, 2011.
[63] J. K. Seo, D. Kim, J. Lee, O. I. Kwon, S. Z. K. Sajib, and E. J. Woo. Electrical tissue property imaging using MRI at dc and Larmor frequency. *Inverse Problems*, 28(8):084002, 26, 2012.
[64] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.
[65] F. Triki. Uniqueness and stability for the inverse medium problem with internal data. *Inverse Problems*, 26(9):095014, 11, 2010.
[66] Hassler Whitney. Analytic coordinate systems and arcs in a manifold. *Ann. of Math. (2)*, 38(4):809–818, 1937.
[67] T. Widlak and O. Scherzer. Hybrid tomography for conductivity imaging. *Inverse Problems*, 28(8):084008, 28, 2012.
[68] J. C. Wood. Lewy's theorem fails in higher dimensions. *Math. Scand.*, 69(2):166 (1992), 1991.

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