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Financial models in continuous time with self-decomposability: application to the pricing of energy derivatives

Supervisors
Piergiacomo SABINO
Emanuela SASSO

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Matteo Gardini

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Aeneadum genetrix, hominum divomque voluptas,
alma Venus, caeli subter labentia signa
quae mare navigerum, quae terras frugiferentis
concelebras, per te quoniam genus omne animantum
concipitur visitque exortum lumina solis:
te, dea, te fugiunt venti, te nubila caeli
adventumque tuum, tibi suavis daedala tellus
summittit flores, tibi rident aequora ponti
placatumque nitet diffuso lumine caelum.

[Lucretius, De Rerum Naturae, Chapter I]

Datta. Dayadhvam. Damyata.
Shantih shantih shantih

[T.S. Elliot, The Waste Land, What the Thunder Said]

Abstract

Based on the concept of self-decomposability we extend some recent multidimensional Lévy models by using multivariate subordination. Our aim is to construct multi-asset market models in which the systemic risk instead of affecting all markets at the same time presents some stochastic delay. In particular we derive new multidimensional versions of the well known Variance Gamma and inverse Gaussian processes. To this end, we extend some known approaches keeping their mathematical tractability, we study the properties of the new processes, we derive closed form expressions for their characteristic functions and, finally, we detail how new and efficient Monte Carlo schemes can be implemented.

As second contribution of the work, we construct a new Lévy process, termed the Variance Gamma++ process, to model the dynamic of assets in illiquid markets. Such a process has the mathematical tractability of the Variance Gamma process and is obtained relying upon the self-decomposability of the gamma law. We give a full characterization of the Variance Gamma++ process in terms of its characteristic triplet, characteristic function and transition probability density. These results are instrumental to apply Fourier-based option pricing and maximum likelihood techniques for the parameter estimation. Furthermore, we provide efficient path simulation algorithms, both forward and backward in time. We also obtain an efficient "integral-free" explicit pricing formula for European options.

Finally, we illustrate the applicability of our models in the context of gas, power and emission markets focusing on their calibration, on the pricing of spread options written on different underlying commodities and on the evaluation of exotic American derivatives, giving an economical interpretation to the obtained results.

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Chapter 1

Introduction

The last fifty years have seen a remarkable develop of financial markets all over the world. Nowadays, many companies are listed on exchange and all financial markets are strictly related each other due to the globalization. Moreover, many strategic fields, such as energy markets have been liberalized in many states leading to the creation of new financial markets together with the rise of new business opportunities and financial risks.

Over the years, in order to properly model those markets and to understand their behaviour, many mathematicians have proposed several different approaches. Undoubtedly, one of the most powerful and successful approach is the one which is based on *continuous time stochastic processes*. The pioneer in this field was Bachelier [8]: in his doctoral thesis he first introduced a mathematical model based on Brownian motion and he used it for stock options valuation. Nevertheless Bachelier discussed his thesis on the 29th March 1900, we have to wait more than seventy years to see the birth of the modern mathematical finance when, in 1973, Fisher Black, Myron Scholes and Robert Merton proposed a "new method to determine the value of derivatives" that worth them the Nobel Prize in Economics in 1995. The dynamic of the asset price S in the Black-Scholes model is given by a Geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{1.1}$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma \in \mathbb{R}^+$ is the volatility and $W = \{W(t); t \geq 0\}$ is a standard Brownian motion. Despite the success of Black-Scholes model, its assumptions are too narrow and, for this reason, in the following years some of them were relaxed: in particular, Merton [91] added jumps to the price dynamic whereas Heston [70] proposed to substitute the constant volatility σ in the Black-Scholes model, with a stochastic mean-reverting process of the CIR type and Bates [15] combined the two approaches together creating a new model with stochastic volatility and jumps.

In addition, many different successful approaches based on the notion of Brownian subordination were proposed, for instance, by Madan and Seneta [88] and Barndorff-Nielsen and Shephard [13], whereas Dupire [51] introduced a so called *local volatility model*. The guests of honour in all these approaches are Lévy processes in continuous

time which, at a first glance, can be thought as a generalization of the Brownian motion. Modelling techniques based on Lévy processes leads to richer models than the classical one proposed by Black-Scholes, retaining both the mathematical and numerical tractability.

All these modelling approaches combined with *martingale theory* and together with numerical algorithms developed for path simulation and pricing purposes, allow to properly evaluate several types of derivatives in different modelling framework, at least in a univariate setting.

Nevertheless, in real world it is common to price derivative contract written on two or more underlying assets and hence it would be desirable to scale the aforementioned models to a multidimensional settings. Unfortunately, as opposed the extension of the Black-Scholes model to a multivariate setting is easy, the generalization of models based on more general Lévy processes is far from the triviality. In the latter case two main issues come up: on one hand using Lévy processes, different dependence structures can be considered: if, in Gaussian processes, all the dependence relations between random variables are modelled by the covariance matrix (see the model proposed by Heath et al. [69]), when we consider Lévy processes the dependence can be modelled in several ways that cannot be caught only by the covariance matrix. On the other hand, all numerical pricing and calibration techniques which are available in the univariate setting must be adapted to the multidimensional framework. In particular Monte Carlo methods might be revisited in order to generate random variable with the introduced dependence structure and those methods based on Fourier techniques and on the resolution of partial derivatives equations must be adjusted and this might be a hard task.

Particular dependence structures can be obtained by using a copulas, namely multivariate cumulative distribution functions for which the marginal probability distribution of each variable is uniform on the interval $[0, 1]$ (see Sklar [119]). As summarized in Cherubini et al. [41], in finance copulas are applied to risk management, portfolio optimization and to derivatives pricing. Moreover, Cont and Tankov [42] shows how multivariate Lévy processes can be fully characterized by Lévy copulas and this gives us a systematic method to construct multidimensional Lévy processes with specified dependence. However, the latter approach leads to models that may be mathematically intractable and somewhat hard to implement. Therefore, even if this approach is very flexible from a mathematical point of view, it results to be hard to apply in practical situations.

A simple multidimensional non Gaussian model with jumps can be obtained by taking a multivariate Brownian motion and time change it with a univariate subordinator as proposed by Eberlein [52] and Prause [102]. This approach is easy to apply and leads to tractable models both from a theoretical and numerical point of view but, unfortunately, it lacks of flexibility and the range of possible dependence and patterns is quite limited. Nevertheless the just mentioned approach can be improved as proposed by Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9]. The common idea of these papers is to model a general

multidimensional stochastic process $\boldsymbol{Y} = \{(Y_1(t), \ldots, Y_n(t)); t \geq 0\}$ in the following way:

$$\begin{aligned} Y_1(t) &= X_1 + a_1 Z(t), \\ &\ldots \\ Y_n(t) &= X_n + a_n Z(t), \end{aligned} \tag{1.2}$$

where $\boldsymbol{X} = \{(X_1(t), \ldots, X_n(t)); t \geq 0\}$ and $Z = \{Z(t); t \geq 0\}$ are independent stochastic processes and $a_j \, j = 1, \ldots, n$ are real numbers. Clearly the resulting process $\boldsymbol{Y}$ has dependent components because of the presence of the common process Z and, moreover, $\boldsymbol{X}$ and Z have a clear economic interpretation: $\boldsymbol{X}$ can be viewed as the *idiosyncratic risk* whilst Z represents the *systemic risk*.

Even if in a process defined as in Equation (1.2) the systemic risk impacts on all processes at the same time, in many situations, it is common to observe that, sometimes, a sudden event happens is one market and it has an effect on the others markets after a *stochastic time delay*: this type of interaction between markets is sometimes referred as *synaptic risk*. Such an empirical feature is not captured in any existing model in literature. Nevertheless, as shown by Cufaro Petroni and Sabino [47], the stochastic delay can be modelled relying upon the probabilistic notion of *self-decomposability*. A random variable X has a self-decomposable law if for all $a \in (0, 1)$ there exist two independent random variable Y (with the same law of X) and Z_a such that:

$$X \stackrel{d}{=} aY + Z_a,$$

and we call Z_a the a-remainder.

The first goal of this thesis is to investigate if stochastic processes of the form (1.2) can be enriched including a synaptic risk component. In particular, we develop new multivariate versions of the well known Variance Gamma and Normal inverse Gaussian processes including stochastic delay. We show how some multidimensional pricing techniques can be used for option pricing and, moreover, we develop several new Monte Carlo schemes to generate their paths. In particular we derive explicitly the law of Z_a when X has a gamma or an inverse Gaussian law and we present new numerical algorithms to efficiently sample from the distribution of Z_a without using an acceptance rejection method. Moreover, we study the mathematical properties of the multidimensional version of the Variance Gamma and Normal inverse Gaussian with stochastic delay we introduce, we derive their characteristic functions at a given time t and the expression of the linear correlation coefficient. Furthermore, we calibrate the derived processes on real data from energy markets and we price several derivative accordingly by using both Monte Carlo and Fourier techniques. The models we present are easy to interpret from an economic point of view and are both theoretically and numerically tractable: for these reasons they can be used for derivatives pricing in industry, especially when the number of underlying assets is not to large.

As we stated before, the range of financial products that can be traded on exchange is huge but, sometimes, the related traded volume is very low: in this case the considered contingent claim is said to be illiquid. For example, the traded volume of derivatives written on forex exchange or interest rate markets is huge whilst other markets, such as electricity markets in many European country, tends to be much less liquid. When a market is not liquid, the number of transaction is small and, since the price is strictly linked to the number of closed trades, it is not rare to observe time intervals in which the price of the asset remains constant. It is extremely important to consider market liquidity when we deal with derivatives pricing: indeed, if a market operator sells an option he must implement an hedging strategy, such as a delta hedging. Of course, if the underlying asset is not liquid, it follows that the hedging policy might result hard to implement or, sometimes, impossible. The option seller has to consider the fact that the market might not be liquid when he establishes the price of the option. For these reason, the second main contribution of this work is the introduction of a new Lévy process, that we call the Variance Gamma ++ ($VG++$) process which aims at modelling market illiquidity. This goal can be achieved starting from the notion of self-decomposability we define a new process by Brownian subordination, where the subordinator is the Lévy process associated to the a-remainder of a gamma law, which is self-decomposable. We fully characterize the $VG++$ process in terms of its characteristic triplet, we derive its transition probability density function, its characteristic function and we introduce a new Monte Carlo scheme for its path simulations. An interesting result is that within this modelling framework a closed pricing formula for vanilla options can be found: in particular the value of an European call option can be expressed as the infinite sum of call options, as it happens in the Merton [91] and Kou [80] models. Furthermore, relying upon Lévy bridges techniques, we show how simulate the $VG++$ process backward in time and we show how this technique can be used to efficiently price American style derivatives. Finally, we apply this model to European electricity markets with different levels of liquidity and we investigate its performance. Relying on the same approach we used to derive multidimensional versions of the Variance Gamma model we show how the $VG++$ model can be extended to a multi-asset market.

In conclusion, the main contributions of this work are the following:

- Derive new multidimensional version of the Variance Gamma and Normal inverse Gaussian process incorporating the stochastic delay.
- Introduce a new Lévy process, named $VG++$ process, that can be successfully used to model those markets with different levels of liquidity.
- Provide all those numerical techniques, such as Monte Carlo schemes, Fourier pricing methods, closed formulas ones need to successfully use these models in practice.

Although we deeply investigated the above modelling framework, some limits and open questions are present. We obtain the $VG++$ process subordinating a Brownian motion with the Lévy process associated to the law of Z_a when X has a gamma law. A similar process can be constructed if we consider the subordinated Brownian motion where X has a inverse Gaussian law, which is self-decomposable too. This could be a very interesting topic to investigate, but it is outside the scope of this thesis.

The mathematical framework we proposed to derive multivariate versions of the Variance Gamma and Normal inverse Gaussian process can be easily applied to any dimension $n \geq 2$. Nevertheless, in numerical applications we mainly focus on bivariate processes. Indeed, as it will be clear from the sequel, the number of parameters we need to model the market dramatically increases when we deal with more that two underlying assets. The main problems arise in the calibration step: the large number of parameters to be fitted makes the calibration hard to perform and, sometimes, unstable. For this reason, the model we present can be successfully applied only when the number of asset is not too large. We focus on a case with three risky assets, but applications with more than three asset are not considered in this work.

As we stated before, we apply the derived models to energy markets, in particular to electricity and gas markets. Clearly the approach we proposed is very general and can be adapted in many market contexts: we think that some other possible real-world applications should be investigated, but we left them for a possible future research.

This thesis is organized as follows. Chapter 2 discusses about infinitely divisible laws, self-decomposability and Lévy processes: all these notions are fundamental to deeply understand the presented subject. In Chapter 3 we introduce the $VG++$ process, we study its mathematical properties and we show how to simulate its trajectories backward and forward in time: moreover we derive a closed formula for European call option pricing. In chapters 4 and 5 multidimensional versions of the Variance Gamma and Normal inverse Gaussian process with stochastic delays are introduced and efficiently simulations of its skeleton discussed. Chapter 6 presents some well known methods for paths simulation and for option pricing, such as Monte Carlo schemes and Fourier based methods. Furthermore, we detail how to simulate the paths of all processes we introduced in previous chapters and we briefly discuss how to price derivatives by solving partial integral differential equations. Chapter 7 shows some possible applications of the aforementioned models to electricity and gas markets. We calibrate the models, we price derivatives and we discuss the obtained results both from a mathematical and economic point of view. Chapter 8 concludes whereas Appendices contain some technical results and useful observations.

Chapter 2

Mathematical background

In this chapter we present the mathematical background one needs to deeply understand the topics we face up in the next chapters. In particular, firstly we briefly show how the notion of infinitely divisible and self-decomposable laws arise as natural answers to different versions of the central limit problem. After that, we show how infinitely divisible laws are connected to a particular class of stochastic processes called Lévy processes and we discuss their properties and characterization. Besides that, we give some examples of such processes and of their related Lévy bridges and, finally, we show how the aforementioned processes can be used in finance to model the price dynamic and to derive arbitrage free market models in continuous time.

2.1 The rise of infinitely divisible laws

The notion of *infinitely divisible distribution* is crucial to study Lévy processes and was introduced for the first time by Finetti [56] in 1929. In many books the link between Lévy processes and infinitely divisible law is highlighted: it turns out that there is a correspondence between Lévy processes and infinitely divisible law (see Cont and Tankov [42, Proposition 3.1]). In the study of infinitely divisible laws and Lévy processes an important role in played by the so called Lévy measures. Many times the definition of Lévy measures is given at the very beginning, but the origin of such an important notion sometimes remains obscure. The goal of this section is to quickly recall the underlying problem that leads to the definition of infinitely divisible laws and that of Lévy measure: we do not presume to be neither rigorous nor exhaustive and we refer the interested reader to Varadhan [122] and Breiman [27] for a complete exposition of the topic.

In this chapter we assume that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and we remember that a random variable $X : \Omega \to E$ is a measurable function from a set of outcomes Ω to a measurable space E. In many applications it is assumed $E = \mathbb{R}$ embedded with $\mathcal{B}(\mathbb{R})$. The notion of *infinitely divisible law* naturally arises when one consider the following problem. Assume we have a sequence $(k_n)_{n\geq 1}$ such that

k_n monotonically increases to infinity $k_n \uparrow \infty$ as $n \to \infty$ and consider the array of independent random variables $(X_{n,j}, 1 \leq j \leq k_n)$ i.e. an object of the following form:

$$\begin{array}{l} X_{1,1},\ \ldots,\ X_{1,k_1} \\ X_{2,1},\ \ldots,\ \ldots,\ X_{2,k_2} \\ \ldots,\ \ddots,\ \ddots,\ \ddots,\ \ldots \\ X_{n,1},\ \ldots,\ \ldots,\ \ldots,\ \ldots,\ X_{n,k_n}. \end{array}$$

In each row the random variables we consider are independent whereas independence through rows is not assumed. We wish to investigate the limit in distribution of the following random variable:

$$S_n = \sum_{j=1}^{k_n} X_{n,j}. \tag{2.1}$$

This problem is the well known *central limit problem (CLP)* and reads as follow: given an array of independent random variables $(X_{n,j}, 1 \leq j \leq k_n)$ find the family of all the limit laws of the consecutive sums (2.1) and the corresponding convergence conditions.

Example 2.1.1. *Assume that $(X_{n,j}, 1 \leq j \leq k_n)$ are independent and identically distributed random variables such that:*

$$\mathbb{P}(X_{n,1} = 1) = p_n = 1 - \mathbb{P}(X_{n,1} = 0).$$

If $\lim_{n\to\infty} np_n = \lambda \in [0, \infty)$ we have that $S_n \xrightarrow{d} S$ as $n \to \infty$ where S has Poisson distribution (see Breiman [27, Theorem 9.4]). We remember that a random variable X is said to be distributed according to a Poisson law if:

$$\mathbb{P}(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n \in \mathbb{N}. \tag{2.2}$$

and we write $X \sim \mathcal{P}(\lambda)$.

As additional condition to study the convergence of S_n, we suppose that all the random variables $(X_{n,j}, 1 \leq j \leq k_n)$ are *uniformly negligible* (*U.N.*), namely that:

$$\forall \epsilon > 0, \qquad \lim_{n\to\infty} \max_{1\leq j\leq k_n} \mathbb{P}(|X_{n,j}| > \epsilon) = 0. \tag{2.3}$$

The *U.N.* condition is the minimum condition one needs to impose in order to study the convergence of (2.1).

In Example 2.1.1 S_n is the sum of independent identically distributed random variables and we have that $S_n \xrightarrow{d} S$. It seems reasonable to expect that S can be seen as "the sum of independent random variables", and this motivates the following definition.

Definition 2.1.1. *(Infinite divisible laws) A random variable X is said to have an infinitely divisible law if for every $k \geq 1$ there exist k independent and identically distributed random variables $X_1^{(k)}, \ldots, X_k^{(k)}$ such that:*

$$X \stackrel{d}{=} X_1^{(k)} + \cdots + X_k^{(k)}.$$

The following proposition shows how to construct a random variable with an infinitely divisible law.

Proposition 2.1.1. *(Breiman [27, Proposition 9.9]) Assume that $(X_{n,j}, 1 \leq j \leq k_n)$ is an array of independent identically distributed random variables, $k_n \uparrow \infty$ and that S_n is defined as in* (2.1)*.*

$$S_n \stackrel{d}{\to} S \iff S \text{ has an infinitely divisible law.}$$

The previous proposition states that if we have convergence in distribution of the sum defined in Equation (2.1) the law of the random variable limit must be infinitely divisible. The converse is also true: if a random variable S has an infinitely divisible law, there exists an array of independent random variables $(X_{n,j}, 1 \leq j \leq k_n)$ such that $S_n \stackrel{d}{\to} S$, as $n \to \infty$. Examples of infinite divisible laws are Dirac, Gaussian, Poisson, gamma and Cauchy distribution.

So far we know that if the sum S_n converges to some limit random variable S, the law of S is infinitely divisible. We now wish to investigate under which conditions S_n converges. Therefore, the next goal is to find some sufficient conditions on the array $(X_{n,j}, 1 \leq j \leq k_n)$ in order to have the convergence in distribution of the following (more general) quantity:

$$\sum_{j=1}^{k_n} X_{n,j} - A_n \stackrel{d}{\to} S, \tag{2.4}$$

where $(A_n)_{n\geq 1}$ is a sequence of real numbers. In order to simplify the analysis of the problem we define the following quantities:

$$a_{n,j} = \mathbb{E}\left[X_{n,j} \mathbb{1}_{\{|X_{n,j}|\leq 1\}}\right], \qquad \tilde{X}_{n,j} = X_{n,j} - a_{n,j},$$
$$\tilde{Y_{n,j}} = \tilde{X}_{n,j}^{[1]}, \qquad B_n = \sum_{j=1}^{k_n} a_{n,j},$$
$$a_n = B_n - A_n,$$

where $X^{[\lambda]}$ for $\lambda > 0$ denotes the *Poisson λ transform* of the random variable X and is defined as:

$$X^{[\lambda]} = \sum_{j=1}^{N} X_j,$$

where $N \sim \mathcal{P}(\lambda)$, $(X_j, 1 \leq j \leq N)$ are N independent copies of X and each of them is independent of N. It can be proven (Varadhan [122, Chapter 3.7]) that:

$$\sum_{j=1}^{k_n} X_{n,j} - A_n \xrightarrow{d} S, \text{ as } n \to \infty \iff \sum_{j=1}^{k_n} \tilde{Y}_{n,j} - a_n \xrightarrow{d} S, \text{ as } n \to \infty.$$

This means that, if we are interested in the convergence in distribution of the right-hand side to a given random variable S it is enough to study the convergence in distribution of the left-hand side to S, where $\tilde{Y}_{n,j}$ and a_n have been defined above. By Lévy continuity theorem (Breiman [27, Theorem 8.28]), in order to analyse the behaviour of $\sum_{j=1}^{k_n} X_{n,j} - A_n$ as $n \to \infty$ it is enough to study the limiting behaviour of the characteristic function of $\sum_{j=1}^{k_n} \tilde{Y}_{n,j} - a_n$ which has the following form[1]:

$$\phi_{\sum_j \tilde{Y}_{n,j} - a_n}(t) = \exp\left\{ \sum_{j=1}^{k_n} \int \left(e^{itx} - 1 \right) d\tilde{\mu}_{n,j}(x) - i a_n t \right\}, \tag{2.5}$$

where $\tilde{\mu}_{n,j}$ is the law associated to the random variable $\tilde{X}_{n,j}$. Despite it seems we have complicated the problem, it turns out that this problem is easier to study than the original one. Looking at Equation (2.5), we can define the following positive measure on $\mathcal{B}(\mathbb{R})$:

$$\nu_n(A) = \sum_{j=1}^{k_n} \tilde{\mu}_{n,j}(A), \quad \forall A \in \mathcal{B}(\mathbb{R}). \tag{2.6}$$

It is worth noting that $\nu_n(\mathbb{R}) = k_n$, and hence ν_n is not a probability measure on $\mathcal{B}(\mathbb{R})$. Moreover, we have that $\lim_{n\to\infty} \nu_n(\mathbb{R}) = \infty$.
A very technical result (Varadhan [122, Section 3.7]) shows that if

$$\sum_{j=1}^{k_n} \tilde{Y}_{n,j} - a_n \xrightarrow{d} S,$$

then:

- $\exists\, 0 < C_0 < \infty$ such that $\sum_{j=1}^{k_n} \mathbb{E}\left[\tilde{X}_{n,j}^2 \mathbb{1}_{|\tilde{X}_{n,j}| \leq 1} \right] \leq C_0, \ \forall n \in \mathbb{N}$.
- $\forall \delta > 0\ \exists\, 0 < C_\delta < \infty$ such that $\sum_{j=1}^{k_n} \mathbb{P}\left[|\tilde{X}_{n,j}| \geq \delta \right] \leq C_\delta, \ \forall n \in \mathbb{N}$.

From these relations it easily follows that:

$$\sum_{j=1}^{k_n} \mathbb{E}\left[\tilde{X}_{n,j}^2 \mathbb{1}_{|\tilde{X}_{n,j}| \leq 1} \right] = \sum_{j=1}^{k_n} \int_{|x| \leq 1} x^2 d\tilde{\mu}_{n,j}(x) = \int_{|x| \leq 1} x^2 d\nu_n(x) < C_0 < \infty, \ \forall n \in \mathbb{N}$$

[1] With a slight abuse of notation we indicate in the same way the characteristic function of a random variable and that of its law.

and

$$\sum_{j=1}^{k_n} \mathbb{P}\left[|\tilde{X}_{n,j}| > \delta\right] = \sum_{j=1}^{k_n} \tilde{\mu}_{n,j}\left(\{x, x > \delta\}\right)$$
$$= \nu_n\left(\{x, x > \delta\}\right) = \int_{|x|>\delta} \nu_n\left(dx\right) < C_\delta < \infty,\ \forall n \in \mathbb{N}.$$

The limiting Lévy measure ν is finite outside the origin and puts a infinite mass at zero but, near the origin, it integrates x^2. A measure with such properties is called Lévy measure.

Definition 2.1.2. *(Lévy measure) A positive measure ν defined on $\mathcal{B}(\mathbb{R})$ is said to be a Lévy measure if:*

- $\forall \delta > 0,\ \exists\ 0 < C_\delta < \infty$ *such that:*
$$\int_{|x|>\delta} \nu\left(dx\right) < C_\delta,$$
- $\exists\ 0 < C_0 < \infty$ *such that:*
$$\int_{|x|\leq 1} x^2 d\nu(x) < C_0.$$

The measures ν_n we introduced in Equation (2.6) are Lévy measures. The characteristic function of $\sum_{j=1}^{k_n} \tilde{Y}_{n,j} - a_n$ can be rewritten in the following way:

$$\phi_{\sum_j \tilde{Y}_{n,j}-a_n}(t) = \exp\left\{\int \left(e^{itx} - 1\right) d\nu_n(x) - ia_n t\right\} \tag{2.7}$$

and we want to investigate its behaviour when $n \to \infty$. Since Lévy measure is integrable outside of the origin whereas it integrates x^2 around zero, it is useful to introduce a *truncation function* $\theta : \mathbb{R} \to \mathbb{R}$ such that:

$$|\theta(x) - x| \leq C|x^3|,$$

for some $0 < C < \infty$ and rewrite Equation (2.7) as:

$$\phi_{\sum_j \tilde{Y}_{n,j}-a_n}(t) = \exp\left\{\int \left(e^{itx} - 1 - it\theta(x)\right) d\nu_n(x) + ib_n t\right\},$$

where $b_n = \int \theta(x) d\nu_n(x) - a_n$.

The following theorem, states that the characteristic function of an infinitely divisible law has a precise form and it depends on the Lévy measure.

Theorem 2.1.2. *(Varadhan [122, Theorem 3.20]) For every admissible Lévy measure ν, real number b and $\sigma^2 > 0$,*

$$\phi_{\nu,b,\sigma^2}(t) = \exp\left\{\int \left(e^{itx} - 1 - it\theta(x)\right) d\nu(x) + ibt - \frac{\sigma^2 t^2}{2}\right\}, \tag{2.8}$$

is the characteristic function of an infinitely divisible law.

Moreover, it can be proven that such a representation is unique, in the sense that if

$$\phi_{\nu_1,b_1,\sigma_1^2}(t) = \phi_{\nu_2,b_2,\sigma_2^2}(t), \quad \forall t \in \mathbb{R},$$

then $\nu_1 = \nu_2$, $\sigma_1^2 = \sigma_2^2$ and $b_1 = b_2$ (see Varadhan [122, Corollary 3.22]). Therefore, we can claim that, the triple (ν, σ^2, b) somehow characterizes the distribution of an infinite divisible random variable (we will see that a similar results holds for Lévy processes we will discuss in Section 2.4).
The second issue we address consists in finding sufficient conditions for the convergence of a series of random variables defined in (2.4). The answer is provided by the following theorem.

Theorem 2.1.3. *(Varadhan [122, Theorem 3.21]) A sequence of random variables X_{ν_n,b_n,σ_n^2} converges to a random variable X if and only if there exists a Lévy measure ν, $\sigma^2 > 0$ and $b \in \mathbb{R}$ such that $X = X_{\nu,b,\sigma^2}$ and*

(a) for all $f \in C_b(\mathbb{R})$[2] for which there exists $\delta > 0$ such that $f(x) = 0$ for all $|x| \leq \delta$, we have:

$$\int f(x) d\nu_n(x) \to \int f(x) d\nu(x), \quad \text{as } n \to \infty.$$

(b) There exists $x_0 > 0$ such that $\nu(\{-x_0\} \cup \{x_0\}) = 0$ and

$$\lim_{n\to\infty} \left\{ \int_{-x_0}^{x_0} x^2 \nu_n(dx) + \sigma_n^2 \right\} = \left\{ \int_{-x_0}^{x_0} x^2 \nu(dx) + \sigma^2 \right\}.$$

(c) $\lim_{n\to\infty} b_n = b$

Given an array of independent random variables $(X_{n,j}, 1 \leq j \leq k_n)$, $k_n \uparrow \infty$ such that the uniform negligibility condition (2.3) holds, the previous results can be summarized as shown in Figure 2.1.

We conclude that, in order to check the convergence in distribution of the quantity $\sum_{j=1}^{k_n} X_{n,j} - A_n$ it is enought to study the simplified problem for $\sum_{j=1}^{k_n} \tilde{Y}_{n,j} - a_n$, construct the triple $(\nu_n, 0, a_n)$ and look for the existence of $(\nu, 0, a)$ such that $S = X_{\nu,0,a}$ and conditions $a), b)$ and $c)$ hold.

An important subclass of infinitely divisible laws consists in the so called *self-decomposable laws* which we are introduced in the next section.

2.2 Self-Decomposable laws

Important cases of the central limit problem arise when we specialize the sequence $(X_{n,j}, 1 \leq j \leq k_n)$. The interested reader can refer to Cufaro Petroni [45] for a concise primer on this topic or to Loeve [82] for a more detailed overview.

[2] Here $C_b(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$.

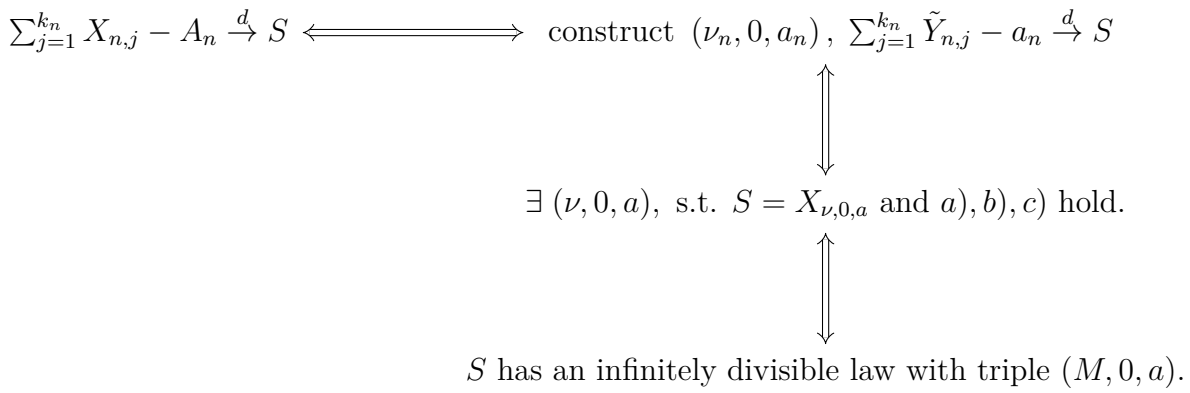

Figure 2.1. Summary of the results of Chapter 2.1.

Assume that there is a sequence $(X_j)_{j\geq 1}$ of independent but in general not identically distributed random variables and two sequences $(a_n)_{n\geq 1}$ with $a_n > 0$ and $(b_n)_{n\geq 1}$ with $b_n \in \mathbb{R}$ such that for every k and n:

$$X_{n,j} = \frac{1}{a_n}\left(X_j - \frac{b_n}{n}\right).$$

Going from row n to row n' we just get a centereing and a rescaling of every X_j. The consecutive sums S_n defined in (2.1) take the form of *normed sums* of independent random variables:

$$S_n = \sum_{j=1}^{k_n} X_{n,j} = \sum_{j=1}^{k_n} \frac{1}{a_n}\left(X_j - \frac{b_n}{n}\right) = \frac{1}{a_n}\left(\sum_{j=1}^{k_n} X_j - b_n\right) = \frac{\tilde{S}_n - b_n}{a_n}, \qquad (2.9)$$

where we defined:

$$\tilde{S}_n = \sum_{j=1}^{k_n} X_j.$$

A second version of the central limit problem reads as follows: *find the family of all the limit laws of the normed sums* (2.9) *and the corresponding convergence conditions* (CLP_2). In particular given a sequence of independent random variables $(X_j)_{j\geq 1}$ find whether there exist sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that X_j/a_n satisfies the *U.N.* condition (2.3) and such that we have convergence in distribution of $(S_n)_{n\geq 1}$ to some random variable S.

The answer to this problem is given in Loeve [82, Section 24]: before proceeding, we give the following important definition.

Definition 2.2.1. *(Self-decomposable laws) The law of a random variable X is said to be self-decomposable if for all $a \in (0,1)$ there exist two independent random variables Y and Z_a such that Y has the same distribution of X and:*

$$X \stackrel{d}{=} aY + Z_a.$$

We call Z_a the a-remainder.

Given a random variable X with law μ we denote by $\phi_X(u)$ for $u \in \mathbb{R}$ its characteristic function defined as:

$$\phi_X(u) = \mathbb{E}\left[e^{iuX}\right] = \int_{\mathbb{R}} e^{iux} d\mu(x).$$

In particular, if the law of a random variable X is self decomposable, we have the following relation:

$$\phi_X(u) = \phi_X(au)\phi_{Z_a}(u).$$

It can be shown that if a law is self-decomposable then it is infinitely divisible (Loeve [82, p.335]) and, moreover, it can be proven that the law of the a-remainder is infinitely divisible (Sato [112, Proposition 15.5]). The Gaussian, Student, inverse Gaussian, gamma, exponential and Laplace laws are examples of self-decomposable law. On the other hand the Poisson law is not self-decomposable but it is infinitely divisible.

The just defined family of laws represents the answer to the second formulation of the central limit problem CLP_2: more precisely self-decomposable laws coincide with the limit distributions of the normed sums defined in Equation (2.9).

As we observed, the law of a self-decomposable random variable is also infinitely divisible, hence its characteristic function admits a representation given by the Lévy-Khinchine formula (2.8). Furthermore, it can be proven that the law of a random variable is self-decomposable if and only if its Lévy measure is absolutely continuous and its density is:

$$d\nu(x) = \frac{k(x)}{|x|} dx,$$

where the function $k(x)$ is non negative, increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ (Sato [112, Theorem 15.10]).

Another version of the central limit problem can be considered if we suppose that the random variables $(X_j)_{j \geq 1}$ are not only independent but also identically distributed. In this case the class of limiting distributions is given by *stable laws*: in particular when X_j has finite variance their normed sums converge to a normal law (see Cufaro Petroni [45] for additional details on this last version of the central limit problem).

As we will see in Section 2.4, infinitely divisible laws are related to Lévy processes. Before moving to deeply investigate Lévy processes we need some technical results about random measures, Poisson processes and jump measures.

2.3 Random measures, Poisson process and jump measures.

In this section we quickly recall all the notions we need to understand Lévy processes. By following Cont and Tankov [42, Chapter 2-3], we do not aim to be neither rigorous nor exhaustive, we just wrap up the main concepts and results one needs to understand the topic of this work.

The notion of *cadlag function*[3] is crucial to introduce Lévy processes.

Definition 2.3.1. *(Cadlag function) A functon $f : [0,T] \mapsto \mathbb{R}^d$ is said to be* cadlag *if it is right-continuous with left limits. For each $t \in [0,T]$ the limits:*

$$f(t-) = \lim_{s \to t^-} f(s), \quad f(t+) = \lim_{s \to t^+} f(s)$$

exist and $f(t) = f(t+)$.

Of course a cadlag funcion can have discontinuities and we call the quantity:

$$\Delta f(t) = f(t) - f(t-)$$

a "jump" of f at t. It is worthing to observe that the set $\{t \in [0,T], f(t) \neq f(t-)\}$ is finite or countable (Fristedt and Gray [58]) and that given an $\epsilon > 0$ the number of discontinuities larger than ϵ should be finite. We can conclude that a cadlag function has a finite number of "large" jumps and a possible infinite but countable number of small jumps. In the context of financial modelling jumps represent sudden market events so the choice of right-continuity is natural. On the other hand, if we want to model a discontinuous process whose values are predictable we should use a *cadlad process*.

In order to proceed we recall the definition of *Radon measure*.

Definition 2.3.2. *(Radon measure) Let $E \subset R^d$. A* Radon measure *on $(E, \mathcal{B}(E))$ is a measure μ such that for every compact measurable set $B \in \mathcal{B}(E)$, $\mu(B) < \infty$.*

For example, the well known Lebesgue measure on $\mathcal{B}(\mathbb{R})$ is a Radon measure. The notion of Radon measure is fundamental to deeply investigate Poisson processes and Poisson measures we are going to introduce in the following section.

2.3.1 Poisson processes and Poisson measures

One of the simplest process we can define is the so called *Poisson process*. A random variable Y is said to follow an exponential distribution with parameter $\lambda > 0$ its probability density function has the form:

$$f(x;\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{\{x \geq 0\}}(x).$$

[3] Cadlag is a french acronym for "continu à droite, limite à gauche" which means "right-continuous with left limits".

An integer valued random variable N is said to follow a Poisson distribution of parameter λ if

$$\mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!},\ \forall n \in \mathbb{N}.$$

The definition of *Poisson process* is then the following.

Definition 2.3.3. *(Poisson process) Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter $\lambda > 0$ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $N = \{N(t); t \geq 0\}$ defined by:*

$$N(t) = \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}}, \tag{2.10}$$

is called a Poisson process *with intensity λ.*

Some properties of the Poisson process are reported in Cont and Tankov [42, Proposition 2.12]. In particular it can be proven that:

- For any $t > 0$, $N(t)$ follows a Poisson distribution with parameter λt:

$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ \forall n \in \mathbb{N}.$$

- The characteristic function of $N(t)$ is given by:

$$\mathbb{E}\left[e^{iuN(t)}\right] = \exp\left\{\lambda t\left(e^{iu} - 1\right)\right\},\ \forall u \in \mathbb{R}.$$

A possible trajectory of the process N is displayed if Figure 2.2 and, as it should be clear from the definition of Poisson process, the jump we observe at stochastic times T_n has size equal to one.

The process we just defined is a counting process: indeed it simply counts the number of events T_n occurred before t: if $T_1, T_2, \ldots$ is the sequence of jumps times of N, then $N(t)$ is the number of jumps between 0 and t:

$$N(t) = \#\{i \geq 1; T_i \in [0, t]\}.$$

Similarly if $s < t$ then:

$$N(t) - N(s) = \#\{i \geq 1; T_i \in (s, t]\}.$$

This counting procedure defines a measure M on $[0, \infty)$: for any measurable set $A \subset \mathbb{R}^+$:

$$M(\omega, A) = \#\{i \geq 1; T_i(\omega) \in A\} \tag{2.11}$$

Then, for $\omega \in \Omega$ fixed, $M(\omega, \cdot)$ is a positive, integer valued measure. On the other hand, observe that if $M(\cdot, A)$, sometimes denoted by $M(A)$, for $A \subset \mathbb{R}^+$ fixed, is a function from $\Omega \to \mathbb{N}$ and hence it is a random variable. The measure $M(\omega, \cdot)$

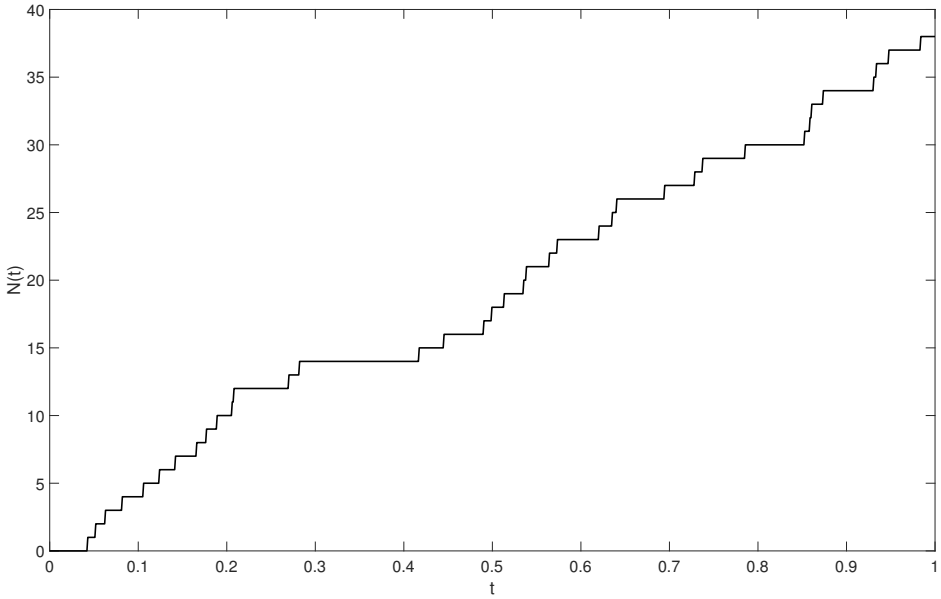

Figure 2.2. A possible path of the Poisson process with parameter $\lambda = 40$, for $t \in [0, T]$ with $T = 1$.

depends on ω and hence it is a *random measure*. The intensity λ of the Poisson process determines the average value of the random measure M, indeed:

$$\mathbb{E}\left[M(A)\right] = \lambda \left|A\right|,$$

where A denotes the Lebesgue measure of A.

M is called the *random jump measure* associated to the Poisson process N. The Poisson process may be expressed in terms of the random measure M in the following way:

$$N(t, \omega) = M\left(\omega, [0, t]\right) = \int_{[0,t]} M\left(\omega, ds\right),$$

where $N(t, \omega)$ denotes the the realization of the random variable $N(t)$ for $\omega \in \Omega$.

The random measure M can also be viewed as the "derivative" of the Poisson process. Each trajectory $t \mapsto N(t, \omega)$ of the Poisson process is an increasing step function. Hence its derivative (in the sense of distributions) is a positive measure. In fact it is the super position of *Dirac masses locate at the random jump times* and, at least formally, we have:

$$\frac{d}{dt} N\left(t, \omega\right) = M(\omega, [0, t]),$$

where:

$$M(\omega, [0, t]) = \sum_{\substack{i \geq 1, \\ T_i(\omega) \leq T}} \delta_{T_i(\omega)},$$

which coincides with (2.11). Roughly speaking we say that the derivative of the trajectory of the process is zero everywhere except at T_i.

Therefore, we have seen that a Poisson process defines a counting measure on measurable sets of $\mathbb{R}^+$, that the Poisson process itself can be defined as an integral with respect its counting measure and, finally, that the counting measure M can be seen as the superposition of Dirac measures located at random times $T_i(\omega)$.

The measure M defined in (2.11) defines a random counting measure on $\mathbb{R}^+$ such that for any measurable set $A \subset \mathbb{R}^+$, $\mathbb{E}[M(A)]$ is given by λ times the Lebesgue measure of A. We can replace $\mathbb{R}^+$ by $E \subset R^d$ and the Lebesgue measure by a Radon measure μ on E and we can define what is called the *Poisson random measure*.

Definition 2.3.4. *(Poisson random measure) Let be $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $E \subset \mathbb{R}$ and μ a given (positive) Radon measure μ on $(E, \mathcal{E})$. A Poisson random measure on E with intensity μ is an integer random measure:*

$$\begin{aligned} M : \Omega \times \mathcal{E} &\to \mathbb{N}, \\ (\omega, A) &\mapsto M(\omega, A), \end{aligned}$$

such that:

1. *For (almost all) $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on E: for any bounded measurable set $A \subset E$, $M(A) < \infty$ is an integer valued random variable.*

2. *For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random vaiable with parameter $\mu(A)$:*

$$\mathbb{P}(M(A) = k) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}, \qquad \forall k \in \mathbb{N}.$$

3. *For disjoint measurable sets $(A_i)_{1 \leq i \leq n} \in \mathcal{E}$, the random variables $(M(A_i))_{1 \leq i \leq n}$ are independent.*

The existence of a Poisson random measure M for any Radon measure μ on $E \subset R^d$ is shown in Cont and Tankov [42, Proposition 2.14]. In that proof a direct construction of M is given and we have that any Poisson random measure on E can be represented as a counting measure associated to a random sequence of points in E: in particular, there exists $(X_n)_{n \geq 1}$ sequence of random variables such that:

$$\forall A \in \mathcal{E}, \qquad M(\omega, A) = \sum_{n \geq 1} \mathbb{1}_A(X_n(\omega)).$$

Thus, as before, M is the sum of Dirac masses located and the random points $(X_n)_{n \geq 1}$:

$$M = \sum_{n \geq 1} \delta_{X_n}.$$

2.3.2 Jump processes from Poisson random measures

So far we have seen how a random measure is somehow induced by a Poisson process. The converse can also be done. Consider, for example, a Poisson random measure M on $E = [0,T] \times \mathbb{R}^d \setminus \{0\}$: as we have shown above it can be described as a counting measure associated to a random configuration of points $(T_n, Y_n) \in E$.

$$M = \sum_{n \geq 1} \delta_{(T_n, Y_n)}.$$

Intuitively, each point (T_n, Y_n) corresponds to an observation made at time T_n and described by a (non zero) random variable $Y_n(\omega) \in \mathbb{R}^d$.
If we introduce the filtration $(\mathcal{F}_t)_{t \geq 0}$, the measure M is said to be a non-anticipating Poisson random measure (or a Poisson random measure adapted to $(\mathcal{F}_t)_{t \geq 0}$) if:

- $(T_n)_{n \geq 1}$ are stopping times, which means that the processes $\mathbb{1}_{T_n \leq t}$ are adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$.
- Y_n is "revealed" at T_n: Y_n is $\mathcal{F}_{T_n}$-measurable.

For each $\omega \in \Omega$ fixed, $M(\omega, \cdot)$ is a measure on $E = [0,T] \times \mathbb{R}^d \setminus \{0\}$ and hence we can define the integral with respect to that measure that we call $M(f)$: following the usual procedure starting from a simple function $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ where $c_i \geq 0$ and $A_i \subset E$ disjoint and measurable sets define:

$$M(f) = \sum_{i=1}^n c_i M(A_i).$$

Clearly $M(f)$ is a random variable, since M is a random measure, and its expected value is given by:

$$\mathbb{E}[M(f)] = \sum_{i=1}^n c_i \mu(A_i) = \mu(f).$$

By proceeding for non negative function and then for general functions f we can define $M(f) \coloneqq M(f_+) - M(f_-)$ and thus $M(f)$ is a random variable with expected value given by:

$$\mathbb{E}[M(f)] = \mu(f) = \int_{[0,T]} \int_{\mathbb{R}^d} f(s,y) \mu(dx \times ds).$$

We can restrict the integal to $[0,t] \times \mathbb{R}^d \setminus \{0\}$ and obtain the non anticipating stochastic process:

$$X(t) = \int_0^t \int_{\mathbb{R}^d} f(s,y) M(ds, dy) = \sum_{\{n, T_n \in [0,t]\}} f(T_n, Y_n).$$

The second sum runs over the events (T_n, Y_n) which have occurred before t. i.e. $T_n \leq t$. $X = \{X(t); t \in [0,T]\}$ is thus a jump process whose jumps happen at the random times T_n and have amplitudes given by $f(T_n, Y_n)$.

2.3.3 Marked point processes

As observed in the previous section, a Poisson random measure on $[0,T]\times\mathbb{R}^d$ can be represented as a counting measure:

$$M(\omega,\cdot)=\sum_{n\geq 1}\delta_{(T_n(\omega),Y_n(\omega))} \tag{2.12}$$

for some random sequence $(T_n(\omega),Y_n(\omega))_{n\geq 1}$ of points in $[0,T]\times\mathbb{R}^d$.
Given a random sequence $(T_n(\omega),Y_n(\omega))\in[0,T]\times E$ where $(T_n)_{n\geq 1}$ is a sequence of random times describing the occurrence of some events $Y_n\in E$, $E\subset\mathbb{R}^d$ observed at time T_n, we can define M by (2.12). M is called an integer valued measure on $[0,T]\times E$ and the random sequence $(T_n(\omega),Y_n(\omega))\in[0,T]\times E$ is called *marked point process*.

Marked point processes do not have the independence properties of Poisson random measures: moreover, $M([0,t]\times A)$ is not a Poisson random variable. For a function $f:[0,T]\times E\to\mathbb{R}^d$ verifying $\int_{[0,T]\times E}|f(t,y)|\,\mu(dt,dy)<\infty$ one can construct the integral of f with respect such measure and hence define the jump process $X=\{X(t);t\geq 0\}$ as in the previous section. Such a process is a non anticipating jump process with cadlag trajectories whose jumps are described by the marked point process M.

Conversely, to each cadlag process $X=\{X(t);t\geq 0\}$ with values in $\mathbb{R}^d$ one can associate a random measure J_X on $[0,T]\times\mathbb{R}^d$ called the jump measure. X has at most a countable number of jumps, i.e. the set $\{t\in[0,T],\Delta X(t)=X(t)-X(t-)\neq 0\}$ is countable. Its elements can be rearranged in a sequence $(T_n)_{n\geq 1}$ which are random jump times of X. At time T_n the process has a discontinuity of size $Y_n=X_{T_n}-X_{T_n-}\in\mathbb{R}^d\setminus\{0\}$, hence $(T_n,Y_n)_{n\geq 1}$ defines a marked point process on $[0,T]\times\mathbb{R}^d$ which contains all information about the jumps of the process X. The associated random measure, denoted by J_X, is called *the jump measure of the process* X:

$$J_X(\omega,\cdot)=\sum_{n\geq 1}\delta_{(T_n(\omega),Y_n(\omega))}=\sum_{t\in[0,T],\,\Delta X(t)\neq 0}\delta_{(t,\Delta X(t))}.$$

In intuitive terms, for any measurable subset $A\subset\mathbb{R}^d$, $J_X([0,t]\times A)$ can be defined as the number of jumps of X occurring between 0 and t whose amplitude belongs to A. The random measure J_X contains all information about the discontinuities of the process X: it tells us when the jumps occur and how big they are. All quantities involving the jumps of X can be computed by integrating various functions against J_X. For example if $f(x,y)=y^2$ then one obtains the sum of the squared of jumps of X:

$$\int_{[0,T]\times\mathbb{R}}y^2J_x(dtdy)=\sum_{t\in[0,T]}(\Delta X(t))^2.$$

Such expression may involve infinite sums and hence we must take care about its convergence.

For example, the jump measure of the Poisson process is given by:

$$J_N = \sum_{n\geq 1} \delta_{(T_n,1)},$$
$$J_N\left([0,t]\times A\right) = \#\left\{i \geq 1, T_i \in [0,T]\right\}.$$

At this point we have all the ingredients we need to introduce Lévy processes, which is the topic of the next section.

2.4 Lévy Processes

Lévy processes are mathematical objects that find several applications in many fields such as mathematical finance, physics and biology and they can be viewed as a generalization of the more famous Brownian motion. In this section we focus on the characterization of Lévy processes, we discuss their main properties and we give some conditions under which a Lévy process is a martingale. This last topic is crucial when want to use Lévy processes in mathematical finance where martingales play a very important role, as we will explain in Section 2.8.1. A deep discussion of the theory of Lévy processes can be found in Sato [112] and Applebaum [6], whereas Cont and Tankov [42] focus on their applications to mathematical finance.

We now recall the definition of Lévy process and we show the connection between Lévy processes and infinitely divisible laws.

Definition 2.4.1. *(Lévy process) A cadlag stochastic process $X = \{X(t); t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X(0) = 0$ is called Lévy process if it possesses the following properties:*

i) Independent increments: for every increasing sequence of times $t_0, \ldots, t_n$, the random variables $X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independent.

ii) Stationary increments: the law of $X(t+h) - X(t)$ does not depend on t.

iii) Stochastic continuity: $\forall \epsilon$, $\lim_{h\to 0} \mathbb{P}\left(|X(t+h) - X(t)| \geq \epsilon\right) = 0$.

If we sample a Lévy process at regular time intervals $0, \Delta, 2\Delta, \ldots$ we obtain a random walk: defining $S_n(\Delta) = X(n\Delta)$ we can write $S_n(\Delta) = \sum_{k=0}^{n-1} Y_k$ where $Y_k = X((k+1)\Delta) - X(k\Delta)$ are independent identically distributed random variables whose distribution is the same of $X(\Delta)$. Choosing $n\Delta = t$, we see that for any $t > 0$ and any $n \geq 1$, $X(t) = S_n(\Delta)$ can be represented as a sum of independent identically distributed random variables whose distribution is that of $X(t/n)$: $X(t)$ can be "divided" into n independent identically distributed parts: then the distribution of $X(t)$ is infinitely divisible. Thus, if X is a Lévy process, for any $t > 0$ the distribution of $X(t)$ is infinitely divisible. This puts a constraint on the possible choices of distributions for $X(t)$: whereas the increments of a discrete-time random

walk can have arbitrary distribution, the distribution of increments of a Lévy process has to be infinitely divisible.

Conversely, given an infinitely divisible distribution F, it is easy to see that for any $n \geq 1$ by chopping it into n independent identically distributed components we can construct a random walk model on a time grid with step size $1/n$ such that the law of the position at time $t = 1$ is given by F. In the limit, this procedure can be used to construct a continuous time Lévy process $X = \{X(t); t \geq 0\}$ such that the law of $X(1)$ is given by F. We have partially proved the following proposition which converse implication is Sato [112, Corollary 11.6].

Proposition 2.4.1. *(Cont and Tankov [42, Proposition 3.1]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process. Then for every $t > 0$, $X(t)$ has an infinite divisible distribution. Converseley, if F is an infinitely divisible distribution then there exists a Lévy process X such that the distribution of $X(1)$ is given by F.*

We recall that the characteristic function for a generic random variable X with values in $\mathbb{R}^d$ is defined as:

$$\phi_X(u) = \mathbb{E}\left[e^{i\langle u, X\rangle}\right],\ u \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. For $s < t$ by writing $X(t+s) = X(s) + (X(t+s) - X(s))$ and using the property of independence of increments, we obtain that $t \mapsto \phi_{X(t)}(u)$ is a multiplicative function:

$$\phi_{X(t+s)}(u) = \phi_{X(t)}(u)\phi_{X(s)}(u),$$

since $X(t+s) - X(s)$ has the same law of $X(t)$. Moreover, the stochastic continuity of $t \mapsto X(t)$ implies in particular that $X(s) \xrightarrow{d} X(t)$ when $s \to t$. Therefore, by Cont and Tankov [42, Proposition 2.6], $\phi_{X(s)}(u) \to \phi_{X(t)}(u)$ when $s \to t$ so $t \mapsto \phi_{X(t)}(u)$ is a continuous function of t. Together with the multiplicative property 2.4 this implies that $t \mapsto \phi_{X(t)}(u)$ is an exponential function. Therefore, we can state the following proposition.

Proposition 2.4.2. *(Cont and Tankov [42, Proposition 3.2]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ called the* characteristic exponent *of X, such that:*

$$\phi_{X(t)}(u) := \mathbb{E}\left[e^{iu \cdot X(t)}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d.$$

Given a random variable $X : \Omega \to \mathbb{R}$ with characteristic function $\phi_X(u)$ the *cumulant generating function* Ψ_X is defined to be the logarithm of the characteristic function. It is clear that if $\psi(u)$ is the cumulant generating function of $X(1)$: $\psi(u) = \Psi_{X(1)}(u)$ and that the cumulant generating function of $X(t)$ varies linearly in t:

$$\Psi_{X(t)}(u) = t\Psi_{X(1)}(u) = t\psi(u).$$

The law of $X(t)$ is therefore determinated by the knowledge of the law of $X(1)$: the only degree of freedom we have in specifying a Lévy processes is to specify the distribution of $X(t)$ for a single time (say, $t = 1$). Note that the converse is not true. Namely if the characteristic function of the process $X = \{X(t); t \geq 0\}$ at time t is associated to an infinitely divisible law does not imply in general that the process X is a Lévy process. Nevertheless, there exists a Lévy process $Y = \{Y(t); t \geq 0\}$ which can be different from X such that its characteristic function at time t coincides with the one of the process X at the same time. An example is shown in Appendix A.

So far we have shown that there is a strong connection between Lévy processes and infinitely divisible law. Since we have show that an infinitely divisible law is fully characterized by a triple of the form (ν, b, σ^2) it seems reasonable that a similar triplet should play an important role in Lévy processes. Indeed, the Lévy measure for Lévy processes can be defined as follow:

Definition 2.4.2. *(Lévy measure) Let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$. The measure ν on $\mathcal{B}\left(\mathbb{R}^d\right)$ defined by:*

$$\nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X(t) \neq 0, \Delta X(t) \in A\right\}\right], \; A \in \mathcal{B}\left(\mathbb{R}^d\right), \tag{2.13}$$

where $\mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-algebra on $\mathbb{R}^d$, is called the Lévy measure of X: $\nu(A)$ is the expected number, per unit of time, of jumps whose size belongs to A.

The definition of Lévy measure for Lévy processes together with the notion of jumps measure we introduced above, allows us to state the following fundamental theorem known as *Lévy-Itô decomposition.*

Theorem 2.4.3. *(Lévy-Itô decomposition Cont and Tankov [42, Proposition 3.7]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ and ν its Lévy measure, given by* (2.13)*.*

- *ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ and verifies:*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \qquad \int_{|x|\geq 1} \nu(dx) < \infty.$$

- *The jump measure of X, denoted by J_X, is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity given by $\nu(dx)\,dt$.*
- *There exist a vector γ and a d-dimensional Brownian motion $W = \{W(t); t \geq 0\}$ with covariance matrix A such that:*

$$X(t) = \gamma t + W(t) + \tilde{X}(t) + \lim_{\epsilon \to 0} X(t)^{\epsilon}, \tag{2.14}$$

where

$$\tilde{X}(t) = \int_{|x|\geq 1, s\in[0,t]} xJ_X\left(dx \times ds\right),$$
$$X^{\epsilon}(t) = \int_0^t \int_{\epsilon<|x|<1} x\left(J_x(dx \times ds) - \nu(dx)ds\right)$$

The terms in in (2.14) *are independent and the convergence of the last term is almost sure and* L^2 *uniformly in finite intervals.*

The theorem states that any Lévy process is a combination of a Brownian motion with drift γ, diffusion A and a possibly infinite sum of independent compound Poisson processes and a positive measure ν that uniquely determine its distribution. The triplet (A, ν, γ) is called *Lévy triplet* or *characteristic triplet* of the process X. The condition $\int_{|y|\geq 1} \nu(dy) < \infty$ means that X has a finite number of jumps with absolute value larger that one. So the sum:

$$\tilde{X}(t) = \sum_{\substack{0\leq s\leq t,\\ |\Delta X(s)|\geq 1}} \Delta X(s),$$

contains almost surely a finite number of terms and $\tilde{X}$ is a compound Poisson process. The sum:

$$X(t)^{\epsilon} = \sum_{\substack{0\leq s\leq t,\\ \epsilon\leq|\Delta X(s)|<1}} \Delta X(s) = \int_{\epsilon\leq|x|<1, s\in[0,t]} xJ_X(dx \times ds),$$

is again a well-defined compound Poisson. However, ν might have a singularity at zero: there can be infinitely many small jumps and their sum does not necessarily converge. This prevents us from taking $\epsilon \to 0$ directly. In order to have convergence we need to center the process and then we can apply a martingale convergence argument to show convergence. Such a results has both theoretical and practical consequence. First of all it tells us that any Lévy process can be approximated with arbitrary precision by a jump-diffusion process, that is by the sum of Brownian motion with drift and a compound Poisson process. Moreover, without additional work we can obtain the expression of the characteristic function of a Lévy process at time t in terms of its characteristic triplet (A, ν, γ).

Theorem 2.4.4. *(Lévy-Khinchin representation Cont and Tankov [42, Theorem 3.1]) Let* $X = \{X(t); t \geq 0\}$ *be a Lévy process on* $\mathbb{R}^d$ *with characteristic triplet* (A, ν, γ)*. Then:*

$$\mathbb{E}\left[e^{iu\cdot X(t)}\right] = e^{t\psi(u)},\ u \in \mathbb{R}^d,$$
$$\text{with } \psi(u) = -\frac{1}{2}u \cdot Au^T + i\gamma \cdot u + \int_{\mathbb{R}^d} \left(e^{iu\cdot x} - 1 - iu \cdot x\mathbb{1}_{|x|\leq 1}\right)\nu(dx).$$

Example 2.4.1. *(Lévy-Khinchin representation of a Poisson process) Let* $N = \{N(t); t \geq 0\}$ *defined as in* (2.10)*. Then its characteristic exponent* $\psi(u)$ *is given by:*

$$\psi(u) = \lambda t \int_{\mathbb{R}} \left(e^{iux} - 1\right) \delta_1(x),$$

where $\delta_1(x)$ *is the Dirac measure concentrated at* $x = 1$*.*

2.4.1 Properties of the Lévy processes

A lot of properties of the paths of a Lévy process can be deduced in terms of its characteristic triplet (A, ν, γ) (the interested reader can refer to Cont and Tankov [42, Chapter 3] for an extensive discussion).

First of all we observe that the Lévy measures can be a finite measures or not. If $\nu(\mathbb{R}) < \infty$ then the expected number of jumps for a unit time interval is finite: such a process is said to be of *finite activity*. Finite activity processes are such that almost all paths have only a finite number of jumps along finite time intervals. Conversely, if $\nu(\mathbb{R}) = \infty$ then the expected number of jumps in a finite time interval is infinite and the process is called of *infinite activity*. The following criterion characterizes Lévy processes with piecewise constant trajectories.

Proposition 2.4.5. *(Cont and Tankov [42, Proposition 3.8]) A Lévy process has piecewise constant trajectories if and only if its characteristic triplet satisfies the following conditions:*

$$A = 0, \qquad \int_{\mathbb{R}^d} \nu(dx) < \infty, \qquad \gamma = \int_{|x| \leq 1} x\nu(dx).$$

or equivalently if its characteristic exponent is of the form:

$$\psi(u) = \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(dx), \text{ with } \nu(\mathbb{R}) < \infty.$$

We observe that the Poisson process satisfies this criterion and indeed its trajectories are piecewise constant.

We recall that for a function $f : [a, b] \mapsto \mathbb{R}^d$ the total variation is defined as:

$$TV(f) \coloneqq \sup_{\Pi} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \right\},$$

where the supremum is taken over all finite partitions Π, $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of the interval $[a, b]$. In particular, in one dimension every increasing or decreasing function is of finite variation and every function of finite variation is a difference of two increasing function. A Lévy process is said to be of finite variation if its trajectories are functions of finite variation with probability one. The following proposition characterizes the finite variation Lévy processes.

Proposition 2.4.6. *(Cont and Tankov [42, Proposition 3.9]) A Lévy process is of finite variation if and only if its characteristic triplet (A, ν, γ) satisfies:*

$$A = 0 \quad \text{and} \quad \int_{|x|\leq 1} |x|\nu(dx) < \infty.$$

Increasing Lévy processes are also called subordinators because they can be used as time changes for other Lévy processes and they are important building-blocks for financial modelling. It can be shown that every Lévy process of finite variation can be written as difference of two subordinators. Subordinators are fully characterized by the following proposition.

Proposition 2.4.7. *(Cont and Tankov [42, Proposition 3.10]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}$. Then the following are equivalent.*

i) $X(t) \geq 0$ a.s. for some $t > 0$.

ii) $X(t) \geq 0$ a.s. for all $t > 0$.

iii) Sample paths of X are almost surely non decreasing: $s < t \implies X(s) \leq X(t)$ a.s.

iv) The characteristic triplet of X satisfies:

$$\begin{aligned} &A = 0, && \nu\left((-\infty, 0]\right) = 0, \\ &\int_0^\infty (x \wedge 1)\, \nu(dx) < \infty, && b \geq 0, \end{aligned}$$

that is X has no diffusion component, only positive jumps and positive drift.

The characteristic triplet (A, ν, γ) of a Lévy process is extremely useful also for practical applications. In particular, the Lévy measure ν allows us to check the existence of n-th moment of $X(t)$. The n-th cumulant of $X(t)$ is defined as:

$$c_n(X(t)) = (-i)^n \frac{\partial^n \Psi_{X(t)}(u)}{\partial u^n}\bigg|_{u=0},$$

where $\Psi_{X(t)}(u),\ u \in \mathbb{R}$ is the cumulant generating function, which has been previously defined as the logarithm of the characteristic function and it can be expressed in terms of the Lévy measure of the process X. Such results are summarized in the following proposition.

Proposition 2.4.8. *(Cont and Tankov [42, Proposition 3.13]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet (A, ν, γ). The n-th absolute moment of $X(t)$, $\mathbb{E}\left[|X(t)|^n\right]$ is finite for some t or, equivalently, for all $t > 0$ if and only if:*

$$\int_{|x|\geq 1} |x|^n \nu\left(dx\right) < \infty.$$

In this case the moments of $X(t)$ can be computed from its characteristic function by differentiation. In particular, the form of cumulants of $X(t)$ is especially simple:

$$\begin{aligned}
\mathbb{E}\left[X(t)\right] &= t\left(\gamma + \int_{|x|\geq 1} x\nu(dx)\right), \\
c_2(X(t)) = Var\left[X(t)\right] &= t\left(A + \int_{-\infty}^{\infty} x^2\nu\left(dx\right)\right) \\
c_n\left(X(t)\right) &= t\int_{-\infty}^{\infty} x^n\nu\left(dx\right), \ for\, n \geq 3.
\end{aligned}$$

The knowledge of theoretical expression for the moments of a Lévy process X at time t is useful from a numerical point of view to check the correctness and the convergence of the algorithms that samples from a given distribution. Indeed from the cumulants $k_n \coloneqq c_n(X)$ of a general random variable X one can obtain the expression of the general moment $\mu_n = \mathbb{E}\left[X^n\right]$, and compare their theoretical value with the numerical one obtained from a generated sample. In particular it can be shown that the n-th moment is an n-th degree polynomial in its first n cumulants. For example we have that the first four moments can be represented in term of the first four cumulants by:

$$\begin{aligned}
\mu_1 &= k_1, \\
\mu_2 &= k_2 + k_1^2, \\
\mu_3 &= k_3 + 3k_2k_1 + k_1^3, \\
\mu_4 &= k_4 + 4k_3k_1 + 3k_2^2 + 6k_2k_1^2 + k_1^4.
\end{aligned}$$

The knowledge of the theoretical moments in terms of the distribution parameters is crucial if one need to apply a parameters estimation technique as the Generalized Method of Moments proposed by Hansen [66].

2.4.2 Lévy processes and martingales

The notion of martingale in mathematical finance is essential. Indeed, as we will show in Section 2.8.1, it is strictly related with the absence of arbitrage in a given market. Several martingales can be constructed from Lévy processes using the independent increments property. The following propositions show how to obtain a martingale starting from a given Lévy process and how to check that a general Lévy process actually satisfies the martingale condition.

Proposition 2.4.9. *(Cont and Tankov [42, Proposition 3.17]) Let $X = \{X(t); t \geq 0\}$ be a real-valued process with independent increments. Then:*

1. $\left\{\frac{e^{iuX(t)}}{\mathbb{E}\left[e^{iuX(t)}\right]}; t \geq 0\right\}$ *is a martingale for all $u \in \mathbb{R}$.*
2. *If for some $u \in \mathbb{R}$, $\mathbb{E}\left[e^{uX(t)}\right] < \infty$ for all $t \geq 0$ then $\left\{\frac{e^{iuX(t)}}{\mathbb{E}\left[e^{iuX(t)}\right]}; t \geq 0\right\}$ is a martingale.*

3. *If* $\mathbb{E}[X(t)] < \infty$ *for all* $t \geq 0$ *then* $M = \{M(t); t \geq 0\}$ *where* $M(t) = X(t) - \mathbb{E}[X(t)]$ *is a martingale (and also a process with independent increments).*

4. *If* $Var[X(t)] < \infty$ *for all* $t \geq 0$ *then* $(M(t))^2 - \mathbb{E}[(M(t))^2]$ *is a martingale, where* M *is the martingale defined above.*

If X *is a Lévy process, for all of the processes of this proposition to be martingales it suffices that the corresponding moments be finite for one value of* t*. (see Sato [112, Theorems 2.17, 25.3]).*

In finance it is important to check if, given a Lévy process, its exponential is a martingale since, in many cases, the price process is modelled as the exponential of a given Lévy process in order to ensure its positivity.

Proposition 2.4.10. *(Cont and Tankov [42, Proposition 3.18]) Let* $X = \{X(t); t \geq 0\}$ *be a Lévy process on* $\mathbb{R}$ *with characteristic triplet* (A, ν, γ)*.*

- X *is a martingale if and only if:*

$$\int_{|x|\geq 1} |x|\nu(dx) < \infty \text{ and } \gamma + \int_{|x|<1} x\nu(dx) = 0.$$

- $\{\exp(X(t)); t \geq 0\}$ *is a martingale if and only if:*

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty \text{ and } \frac{A}{2} + \gamma + \int_{-\infty}^{\infty} \left(e^x - 1 - x\mathbb{1}_{|x|\leq 1}\right)\nu(dx) = 0.$$

2.5 Examples of Lévy processes

In this section we introduce some Lévy processes that will be used in the sequel. They are the Brownian motion, the gamma process, the inverse Guassian process and the Compound Poisson process. Furthermore, the aforementioned Poisson process is a Lévy process. For each of them we retrieve the characteristic triplet and we recall their main properties. As in the previous section we refer to Cont and Tankov [42], Sato [112] and Applebaum [6] for details.

2.5.1 The Brownian motion

The most famous Lévy process is clearly the Brownian motion. For an introduction on Brownian motion with some applications to finance one can refer to Shreve [118] or to Bjork [21]. The Brownian motion can be constructed as a limit of a symmetric random walk and naturally inherits its properties.

Definition 2.5.1. *(Brownian Motion) Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. For each* $\omega \in \Omega$ *suppose there is a continuous function* $W(t)$ *of* $t \geq 0$ *that depends on* ω *and satisfies:*

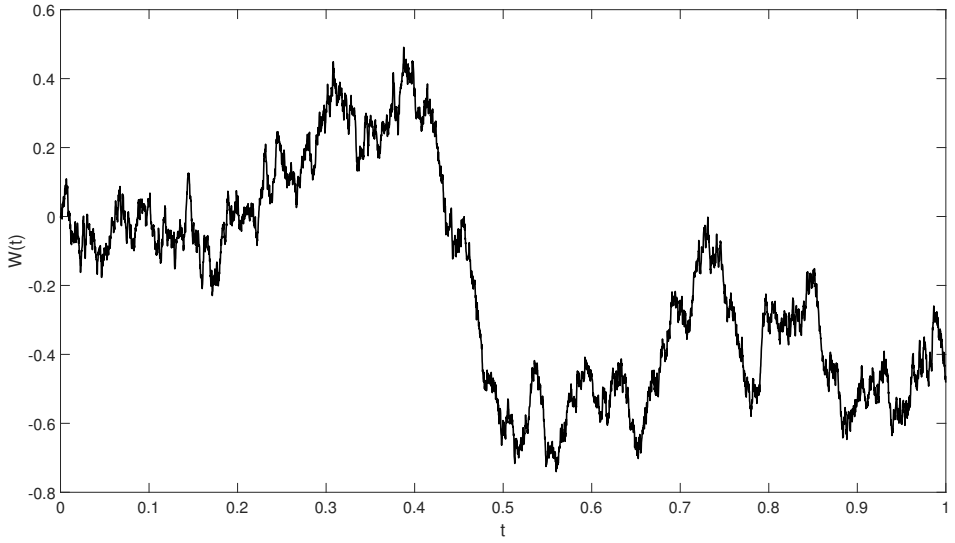

Figure 2.3. A possible path of the standard Brownian motion with parameter $\mu = 0$ and $\sigma = 1$ for $t \in [0, T]$ with $T = 1$.

- $W(0) = 0$, *a.s.*
- *For all* $0 = t_0 < t_1 < \cdots < t_m$ *the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

  are independent.

- *For all* $0 \leq s < t$ *the increment* $\Delta W = W(t) - W(s) \sim \mathcal{N}(0, t - s)$.

The trajectories of the Brownian motion are almost surely continuous but nowhere differentiable. In particular its paths are cadlag and hence the Brownian Motion is a Lévy process. Since the Lévy measure is null if and only if the random variable has a normal distribution, it follows that the characteristic function at time t of a Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion $\sigma^2 \in \mathbb{R}^+$ assumes the simple form:

$$\phi_{W(t)}(u) = \exp\left\{i\mu ut + \frac{\sigma^2 u^2 t}{2}\right\}, \quad u \in \mathbb{R}.$$

A realization of the Brownian motion is shown in Figure 2.3, whereas its characteristic triplet is given by (μ, σ^2, ν) where $\nu(x) = 0$. This is reasonable from an intuitive point of view, because the trajectories of the Brownian motion are continuous and hence no jumps can occurs.

Brownian motion is widely used in finance because it leads to tractable models both from a mathematical and numerical point of view, also in a multivariate setting. The progenitor of these models was proposed by Bachelier [8] who proposed to model the stock price process $S = \{S(t); t \geq 0\}$ as:

$$S(t) = S(0) + \mu t + \sigma W(t),$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The most famous model in finance is due to Black and Scholes [23]: in this case the price dynamics is modelled using a Geometric Brownian motion:

$$S(t) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

Observe that the Brownian motion appears to the exponential and thise guarantees that the trajectories of the process are positive for all $t \geq 0$. We will come back to this modelling technique in Section 2.8.1. On the other hand, a multivariate model for interest rates based on a n-dimensional Brownian motion was developed by Heath et al. [69] and, in recent years, it found applications also in energy markets (see Benth et al. [19]).

2.5.2 The gamma process

The gamma process is a Lévy process that can be constructed starting from the gamma distribution whose law is infinitely divisible.

Definition 2.5.2. *(Gamma distribution) Let $X : \Omega \to \mathbb{R}$ be a random variable. We say that X has a gamma distribution if its probability density function has the following form:*

$$f(x) = \frac{\beta}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0}, \tag{2.15}$$

where $\alpha, \beta \in \mathbb{R}^+$ and Γ is the gamma function at $z > 0$:

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-z} dy.$$

We write $X \sim \Gamma(\alpha, \beta)$.

By using the properties of the conditional expectation it is easy to compute the characteristic function of a random variable X with gamma law and use this results to compute the characteristic function of the gamma process at time t. In particular, given $X \sim \Gamma(\alpha, \beta)$, its characteristic function $\phi(u)$ for $u \in \mathbb{R}$ is given by:

$$\phi(u) = \left(\frac{\beta}{\beta - iu}\right)^\alpha. \tag{2.16}$$

By observing the characteristic function (2.16) it follows that if $G \sim \Gamma(\alpha, \beta)$ then for all $n \in \mathbb{N}$ then

$$X \stackrel{d}{=} Y_1^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_i^{(n)}$ are independent random variables such that $Y_i^{(n)} \sim \Gamma(\alpha/n, \beta)$. It follows that the gamma law is infinitely divisible. Moreover, as proved by Grigelionis [64] the gamma law is also self-decomposable. The gamma process can be defined as follows.

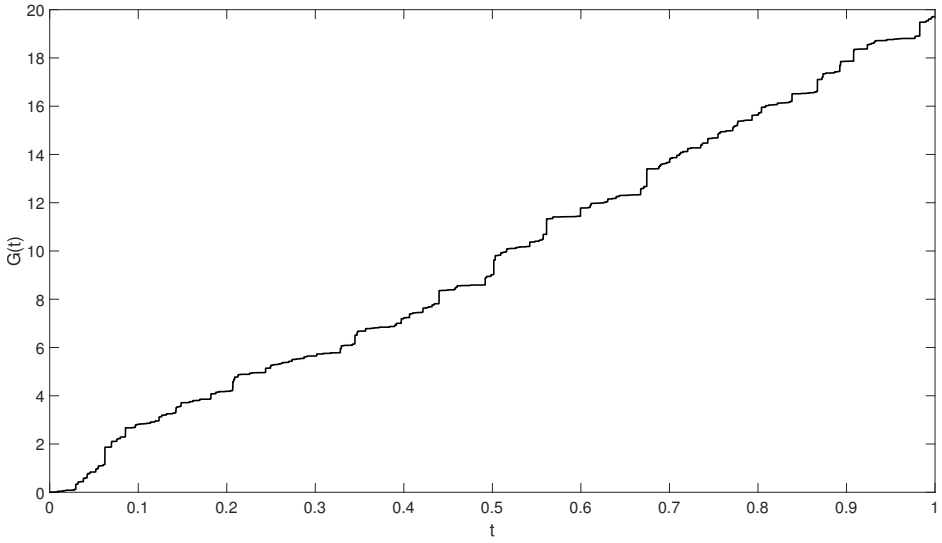

Figure 2.4. A possible path of the Gamma process with parameter $\alpha = 100$ and $\beta = 5$ for $t \in [0, T]$ with $T = 1$.

Definition 2.5.3 (Gamma process)**.** *Let $G = \{G(t); t \geq 0\}$ be a stochastic process. We say that X is a gamma process if for $u < v \leq s < t$ the increments $G(v) - G(u)$ and $G(t) - G(s)$ are independent random variables such that $G(v) - G(u) \sim \Gamma(\alpha(v-u), \beta)$ and $G(t) - G(s) \sim \Gamma(\alpha(t-s), \beta)$.*

A possible path of the gamma process is reported in Figure 2.4: we observe that its trajectories are increasing in time and indeed it turns out that the gamma process is a subordinator. In particular the gamma process is the milestone in the construction Variance Gamma model proposed by Madan and Seneta [88], which finds many applications in financial modelling. The following proposition gives the explicit expression of the characteristic triplet of the gamma process.

Proposition 2.5.1. *For $u \in \mathbb{R}$, The characteristic function of the gamma process at time $t \geq 0$ is given by:*

$$\phi_{G(t)}(u) = e^{t\psi(u)}, \quad u \in \mathbb{R}.$$

where $\psi(u)$ is the characteristic exponent of G and it is given by:

$$\psi(u) = -\frac{1}{2}u^2\sigma^2 + i\gamma u + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1}\right)\nu(dx), \tag{2.17}$$

where:

$$\begin{aligned} \gamma &= \frac{\alpha t}{\beta}(1 - e^{-\beta}), \\ \sigma &= 0, \\ \nu(x) &= \alpha x^{-1} e^{-\beta x}\mathbb{1}_{x>0}. \end{aligned}$$

Since $\sigma^2 = 0$, $\gamma \geq 0$ and $\nu((-\infty, 0]) = 0$ by Proposition 2.4.7 it follows that the gamma process is a subordinator. By using the Lévy-Khinchine expression given in Proposition 2.4.4, it can be shown that its characteristic function at t can be expressed as:

$$\phi_{G(t)}(u) = \exp\left\{t\left(ibu + \int_{\mathbb{R}}\left(e^{iux} - 1\right)\nu\left(dx\right)\right)\right\},$$

where $b = \gamma - \int_{|x|\leq 1}|x|\nu\left(dx\right)$. For the gamma process we have that:

$$b = \gamma - \int_0^1 x\alpha t x^{-1}e^{-\beta x}dx = 0.$$

Using characteristic functions, it is easy to prove the summation an scaling properties fo the gamma law.

- If $X_1 \sim \Gamma\left(\alpha_1, \beta\right)$ and $X_2 \sim \Gamma\left(\alpha_2, \beta\right)$ and assume they are independent random variables, then:
$$X_1 + X_2 \sim \Gamma\left(\alpha_1 + \alpha_2, \beta\right).$$
- If $X \sim \Gamma\left(\alpha, \beta\right)$, then for $c > 0$ we have that:
$$cX \sim \Gamma\left(\alpha, \frac{\beta}{c}\right).$$

In particular, these last two properties can be used to combine together gamma processes obtaining a new gamma process with different parameters. This construction is used in Chapter 4 to build a multidimensional gamma process.

2.5.3 The inverse Gaussian process

Another well known process is the so called inverse Gaussian (IG) process. Its construction is based on the inverse Gaussian law which describes the distribution of the time a Brownian motion with positive drift takes to reach a fixed positive level (see Shreve [118]).

The characterization of the probability density function of an inverse Gaussian law is not unique. For example, Cont and Tankov [42] proposed a parameters setting in (μ, λ), that we denoted by $IG_T\left(\mu, \lambda\right)$ where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. Within this setting the *pdf* of an Inverse Gaussian law is given by:

$$f_Z\left(x; \mu, \lambda\right) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda\left(x-\mu\right)^2}{2\mu^2 x}\right\}\mathbb{1}_{x>0}, \tag{2.18}$$

and its characteristic function is:

$$\phi_Z\left(u\right) = \exp\left\{\frac{\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2iu\mu^2}{\lambda}}\right]\right\}. \tag{2.19}$$

Moreover let be $X \sim IG_T(\mu, \lambda)$ then we have that:

$$\mathbb{E}[X] = \mu, \quad Var[X] = \frac{\mu^3}{\lambda}.$$

The original parameter setting of a inverse Gaussian law proposed by Barndorff-Nielsen [11] is denoted with $IG_B(a, b)$, where a can is the scale parameter and b represents the shape of the distribution. Its probability density function is given by:

$$f_Z(x; a, b) = \frac{a}{\sqrt{2\pi}} \exp(ab)\, x^{-3/2} \exp\left(-\frac{1}{2}\left(a^2 x^{-1} + b^2 x\right)\right) \mathbb{1}_{x>0}, \tag{2.20}$$

and the characteristic function has the following form:

$$\phi_Z(u) = \exp\left\{-a\left(\sqrt{-2iu + b^2} - b\right)\right\}. \tag{2.21}$$

If $X \sim IG_B(a, b)$ then we have that:

$$\mathbb{E}[X] = \frac{a}{b}, \quad Var[X] = \frac{a}{b^3}.$$

Both parametrizations can be adopted and it is possible to switch from one the other by observing that:

$$\mu = \frac{a}{b}, \qquad \lambda = a^2. \tag{2.22}$$

It can be shown that, as the gamma law, the inverse Gaussian law is infinitely divisible and self-decomposable (see Halgreen [65]).

The IG process can be constructed in the same way we construct the gamma process.

Definition 2.5.4 (Inverse Gaussian process)**.** *Let $G = \{G(t); t \geq 0\}$ be a stochastic process. We say that X is a inverse Gaussian process if for $u < v \leq s < t$ the increments $G(v) - G(u)$ and $G(t) - G(s)$ are independent random variables such that $G(v) - G(u) \sim IG_B(a(v-u), b)$ and $G(t) - G(s) \sim IG_B(a(t-s), b)$.*

A possible realization of the inverse Gaussian process is given in Figure 2.5. Again we observe that the process shows increasing paths in time and indeed it turns out that the IG process is a subordinator.

The following proposition fully characterizes the IG process in terms of its characteristic function.

Proposition 2.5.2. *(Barndorff-Nielsen [11]) Let $X = X\{X(t); t \geq 0\}$ be an IG process. Then its Lévy triplet is given by:*

$$\gamma = \frac{a}{b}\Phi\left(\frac{\sqrt{2}}{2}b\right), \quad \sigma^2 = 0$$
$$\nu(x) = \frac{a}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{1}{2}b^2 x\right\} \mathbb{1}_{(0,\infty)}(x),$$

where $\Phi(u) = \frac{\sqrt{\pi}}{2}\operatorname{erf}(u) = \int_0^u e^{-x^2} dx$.

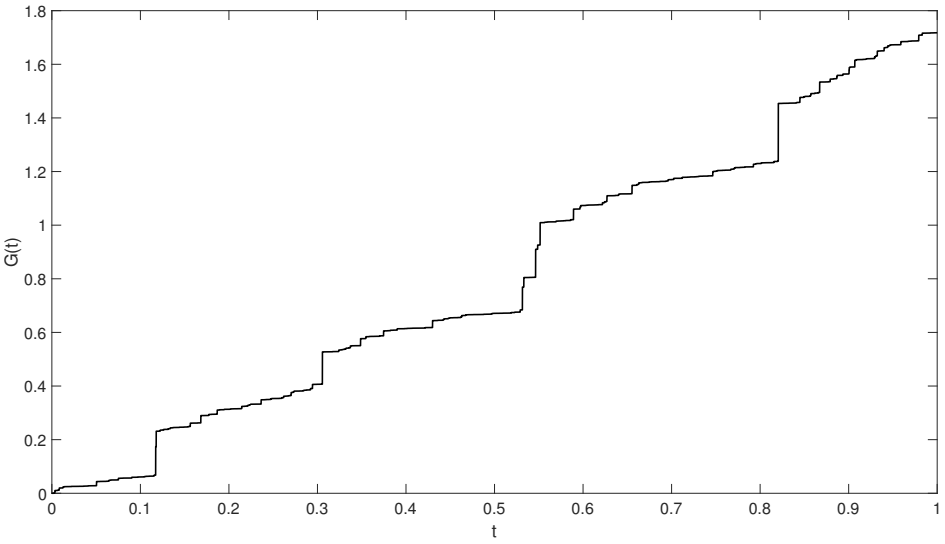

Figure 2.5. A possible path of the inverse Gaussian process with parameter $a = 5$ and $b = 2$ for $t \in [0, T]$ with $T = 1$.

As the gamma law, the inverse Gaussian law enjoys some summation and scaling properties that can be used to combine together independent random variables with inverse Gaussian law leading to a new random variable with the same law.

- Let be $X \sim IG_B(a_1, b)$ and $Y \sim IG_B(a_2, b)$ and let X and Y be independent. Then:
$$cX \sim IG_B\left(ca_1, \frac{b}{c}\right), \quad X + Y \sim IG_B(a_1 + a_2, b).$$
- Let be $X \sim IG_T(\mu_0 w_1, \lambda_0 w_1^2)$ and $Y \sim IG_T(\mu_0 w_2, \lambda_0 w_2^2)$ and let X and Y be independent. Then:
$$cX \sim IG_T\left(c\mu_0 w_1, c\lambda_0 w_1^2\right), \quad X + Y \sim IG_T\left(\mu_0 (w_1 + w_2), \lambda_0 (w_1 + w_2)^2\right),$$
for any $c > 0$.

Such processes will be used in Chapter 5 to build a multivariate version of the inverse Gaussian process.

2.5.4 The Compound Poisson process

Another widely used process in finance is the so called Compound Poisson process. The compound Poisson process is simply a Poisson process defined in Section 2.3.1 which jump size is not equal to one but has a random size according some specified distribution. We follow Cont and Tankov [42, Chapter 2.5.3]. The definition of the Compound Poisson process is the following.

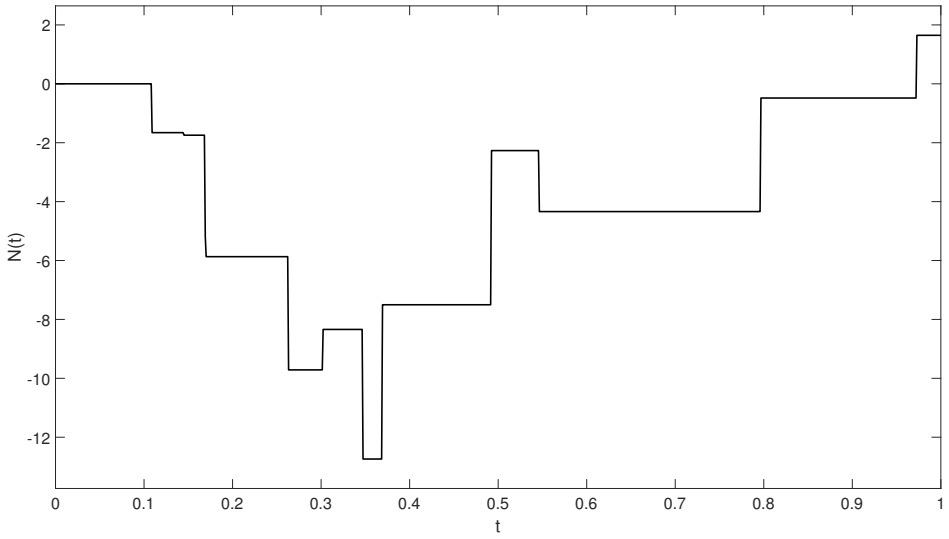

Figure 2.6. A possible path of the Compound Poisson process with parameter $\lambda = 10$ and normal jumps size with mean $\mu_J = 0$ and $\sigma_J = 4$ for $t \in [0, T]$ with $T = 1$.

Definition 2.5.5. *(Compound Poisson process) A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X = \{X(t); t \geq 0\}$ defined as:*

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where jump size Y_i are independent identically distributed random variables with distribution f and $N = \{N(t); t \geq 0\}$ is a Poisson process with intensity λ independent from $(Y_i)_{i \geq 1}$.

A possible realization of the compound Poisson process where jumps are distributed according to a normal distribution is shown in Figure 2.6.

It can be proven that X is a compound Poisson process if and only if it is a Lévy process and its sample paths are piecewise constant functions (see Cont and Tankov [42, Proposition 3.3]). Moreover the characteristic function of a compound Poisson process has the simple form given by the following proposition.

Proposition 2.5.3. *(Cont and Tankov [42, Proposition 3.4]) Let $X = \{X(t); t \geq 0\}$ be a compound Poisson process on $\mathbb{R}$. Its characteristic function has the following representation:*

$$\mathbb{E}\left[e^{iuX(t)}\right] = \exp\left\{t\lambda \int_R \left(e^{iux} - 1\right) f(dx)\right\}, \quad \forall u \in R,$$

where λ denotes the jump intensity and f the jump size distribution.

From this proposition we deduce that the Lévy measure of a compound process is given by $\nu(x) = \lambda t f(x)$ and hence the characteristic triplet of a Compound Poisson

process is the following:

$$\gamma = \int_{|x|\leq 1} x\nu(dx), \qquad \sigma^2 = 0, \qquad \nu(x) = \lambda t f(x).$$

In particular we observe that $\nu(\mathbb{R}) < \infty$ and hence the Compound Poisson process is of finite activity.

2.6 Brownian Subordination

So far we presented some well know Lévy processes which are common in literature. A natural arising question is the following: how can we construct new Lévy processes? To this aime, several techniques are available. As observed before, given a infinitely divisible law it is always possible to build the associated Lévy process (Proposition 2.4.1). Another direct approach is to specify the Lévy triplet, since each Lévy process is fully characterized by its triplet by the Lévy-Itô decomposition. Is it possible to build a Lévy process from known ones? It is easy to prove that a linear combination of Lévy processes is still a Lévy process (Cont and Tankov [42, Theorem 4.1]). Otherwise, given a characteristic triplet (γ, σ^2, ν) it is possible to modify the triplet and, in particular, via *exponential tilting*, the Lévy measure in order to get a new Lévy process (Cont and Tankov [42, Section 4.2.3]). Another common way to obtain a new Lévy process from existing one is by *subordinating a Lévy process*, which simply means to time change a Lévy process with another increasing Lévy process, namely a subordinator. Throughout this work we use this latter technique and hence it is usefull to recall some results. A more technical discussion of this topic and of other possible ways to construct Lévy processes can be found in Cont and Tankov [42, Chapter 4].

If $G = \{G(t); t \geq 0\}$ is a subordinator it means that it satisfies any of the conditions of Proposition 2.4.7. Since $G(t)$ is positive for all $t \geq 0$ we can describe it in terms of its Laplace transform rather that Fourier transform. Recall that if X is a non negative real-valued random variable the its Laplace transform is defined as:

$$L_X(u) = \mathbb{E}\left[e^{-uX}\right] = \int_0^\infty e^{-ux}\mu_X(dx), \ for\, u \geq 0.$$

Let the characteristic triplet of G be $(0, \rho, b)$. Then the Laplace transform of $G(t)$ is given by:

$$\mathbb{E}\left[e^{uG(t)}\right] = e^{tl(u)}, \quad \forall u \geq 0, \text{ where } l(u) = bu + \int_0^\infty (e^{ux} - 1)\,\rho(dx)$$

and we call $l(u)$ the Laplace exponent of G. If we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $X = \{X(t); t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ the subordinated Lévy process $Y = \{Y(t); t \geq 0\}$ for every $\omega \in \Omega$ is defined follows:

$$Y(t, \omega) = X(G(t, \omega), \omega), \ \forall t \geq 0. \tag{2.23}$$

The following theorem ensures that Y is a Lévy process and shows how to compute both its Lévy triplet and its characteristic function.

Theorem 2.6.1. *(Cont and Tankov [42, Theorem 4.2]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process with characteristic exponent $\Psi(u)$ and Lévy triplet (A, ν, γ). Consider a subordinator $G = \{G(t); t \geq 0\}$ with Laplace exponent $l(u)$ and triplet $(0, \rho, b)$. Then the subordinated process $Y = \{Y(t); t \geq 0\}$ defined as in (2.23) is a Lévy process. Its characteristic function is given by:*

$$\mathbb{E}\left[e^{iuY(t)}\right] = e^{tl(\Psi(u))}.$$

Moreover, the Lévy triplet $\left(A^Y, \nu^Y, \gamma^Y\right)$ of Y is given by:

$$\begin{aligned}
A^Y &= bA, \\
\nu^Y(B) &= b\nu(B) + \int_0^\infty p_s^X(B)\rho(ds), \quad \forall B \in \mathcal{B}\left(R^d\right), \\
\gamma^Y &= b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx),
\end{aligned}$$

where p_t^X is the probability distribution of $X(t)$.

From a theoretical point of view, any Lévy processes can be subordinated. Nevertheless, in mathematical finance it is customary to subordinate a Brownian motion with drift and this operation is called *Brownian subordination*. Although the representation via Brownian subordination is a nice property, which makes the model easier to understand and adds tractability, it imposes some important limitations on the form of the Lévy measure. The following theorem fully characterizes Lévy measures of processes that can be represented as subordinated Brownian motion with drift. Recall that a function $f : [a, b] \mapsto \mathbb{R}$ is called *completely monotonic* if all its derivatives exist and if:

$$(-1)^k \frac{d^k f(u)}{du^k} > 0, \quad \forall k \geq 1.$$

Theorem 2.6.2. *(Cont and Tankov [42, Theorem 4.3]) Let ν be a Lévy measure on $\mathbb{R}$ and $\mu \in \mathbb{R}$. There exists a Lévy process $X = \{X(t); t \geq 0\}$ with Lévy measure ν such that $X(t) = \mu Z(t) + W(Z(t))$ for some subordinator $Z = \{Z(t); t \geq 0\}$ and some Brownian motion $W = \{W(t); t \geq 0\}$ independent from Z if and only if the following conditions are satisfied.*

1. *ν is absolutely continuous with density $\nu(x)$.*
2. *$\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$ for all x.*
3. *$\nu(\sqrt{u})e^{-\mu\sqrt{u}}$ is a completely monotonic function on $(0, \infty)$.*

This theorem allows to describe the jump structure of a process that can be represented as time changed Brownian motion with drift. In particular this theorem states that by using Brownian subordination only a particular type of Lévy processes can be obtained: for example the Lévy measures of such subordinate processes (if $\mu = 0$) are always symmetric.

2.6.1 The Variance Gamma process

One of the most common Lévy processes based on Brownian subordination in financial modelling is the Variance Gamma process introduced by Madan and Seneta [88].

Definition 2.6.1. *(Variance Gamma process) Consider the gamma process* $G = \{G(t); t \geq 0\}$ *as in Definition 2.5.3 and consider a Brownian motion* W *with drift* $\theta \in \mathbb{R}$ *and diffusion* $\sigma^2 \in \mathbb{R}^+$*. The process* $X = \{X(t); t \geq 0\}$ *defined as:*

$$X(t) = \theta G(t) + \sigma W(G(t)), \quad t \geq 0, \tag{2.24}$$

is called Variance Gamma *process.*

By using the properties of the conditional expectation the characteristic function of the Variance Gamma model at time t can be easily computed and its form is given by:

$$\phi_{X(t)}(u) = \left(1 - \frac{i}{\beta}\left(u\theta + iu^2\frac{\sigma^2}{2}\right)\right)^{-\alpha t}, \quad u \in \mathbb{R}. \tag{2.25}$$

It is a well known fact that a Lévy process is of finite variation if and only if it can be written as the difference of two subordinators. In particular, it can be verified that the Variance Gamma process is of finite variation.

Proposition 2.6.3. *A Variance Gamma process* X *can be written as the difference of two independent gamma subordinators.*

Proof.

$$\begin{aligned}\phi_{X(t)}(u) &= \left(\frac{1}{1 - \frac{i}{\beta}\left(\theta u + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t} = \left(\frac{1}{1 - \frac{i}{\beta_1}u}\right)^{\alpha t}\left(\frac{1}{1 + \frac{i}{\beta_2}u}\right)^{\alpha t} \\ &= \left(\frac{1}{1 - iu\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) + u^2\frac{1}{\beta_1\beta_2}}\right)^{\alpha t}.\end{aligned}$$

We get:

$$\frac{\theta}{\beta} = \frac{1}{\beta_1} - \frac{1}{\beta_2}, \qquad \frac{1}{\beta_1\beta_2} = \frac{\sigma^2}{2\beta}$$

and solving for β_1 and β_2 and considering only the positive solutions we get:

$$\begin{aligned}\beta_1 &= \frac{\sqrt{\theta^2 + 2\sigma^2\beta} - \theta}{\sigma^2}, \\ \beta_2 &= \frac{\sqrt{\theta^2 + 2\sigma^2\beta} + \theta}{\sigma^2}.\end{aligned}$$

Then we have that the characteristic function of the Variance Gamma process X can be written as:

$$\begin{aligned}\phi_{X(t)}(u) &= \left(\frac{1}{1-\frac{i}{\beta}\left(\theta u + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t} = \left(\frac{1}{1-\frac{i}{\beta_1}u}\right)^{\alpha t}\left(\frac{1}{1+\frac{i}{\beta_2}u}\right)^{\alpha t} \\ &= \phi_{Y(t)}(u)\phi_{Z(t)}(-u)\end{aligned}$$

where $Y(t) \sim \Gamma(\alpha t, \beta_1)$ and $Z(t) \sim \Gamma(\alpha t, \beta_2)$ and $Y(t)$ and $Z(t)$ are independent. Then $X(t)$ is the difference of two independent random variable with gamma law. We can conclude that the Variance Gamma process can be written as difference of two independent gamma processes. ■

By using the previous result we can easily compute the Lévy measure of the Variance Gamma process. By Cont and Tankov [42, Theorem 4.1] and Cont and Tankov [42, Proposition 5.3], if two independent real values Lévy processes Y and Z has Lévy triplet $(\gamma_1, \Sigma_1, \nu_1)$ and $(\gamma_2, \Sigma_2, \nu_2)$ respectively, then the process $X = Y + Z$ is a Lévy process and its Lévy measure ν is given by:

$$\nu(B) = \nu_1(B) + \nu_2(B), \quad \forall B \in \mathcal{B}(\mathbb{R})$$

and moreover its diffusion component is given by:

$$\Sigma = \Sigma_1 + \Sigma_2.$$

It immediately follows that the Lévy measure of the Variance Gamma process X is given by:

$$\nu(x) = \underbrace{\frac{\alpha}{x}e^{-\left(-\frac{\theta}{\sigma^2}+\frac{\sqrt{\theta^2+2\sigma^2\beta}}{\sigma^2}\right)x}\mathbb{1}_{\{x>0\}}}_{\nu_1(x)} + \underbrace{\frac{\alpha}{-x}e^{-\left(\frac{\theta}{\sigma^2}+\frac{\sqrt{\theta^2+2\sigma^2\beta}}{\sigma^2}\right)-x}\mathbb{1}_{\{x<0\}}}_{\nu_2(x)}.$$

If we define

$$\begin{aligned}A &= \frac{\theta}{\sigma^2}, \\ B &= \frac{\sqrt{\theta^2+2\sigma^2\beta}}{\sigma^2},\end{aligned}$$

we have that:

$$\nu(x) = \frac{\alpha}{x}e^{-(B-A)x}\mathbb{1}_{\{x>0\}} + \frac{\alpha}{-x}e^{-(B+A)x}\mathbb{1}_{\{x<0\}} = \frac{\alpha}{|x|}e^{Ax-B|x|}.$$

If $\int_{|x|\leq 1}|x|\nu(x) < \infty$ and $b = 0$ by Proposition 2.4.6 we obtain that the process X is of finite variation and its characteristic function has the following form:

$$\phi_{X(t)}(u) = \exp\left\{t\left(ibu + \int_{\mathbb{R}}\left(e^{iux}-1\right)\nu(dx)\right)\right\},$$

where $b = \gamma - \int_{|x|\leq 1} x\nu(dx)$ (see Cont and Tankov [42, Corollary 3.1]). Since X is the sum of two independent processes Y and Z defined as in the proof of Proposition 2.6.3 with characteristic triplet $(\gamma_1, 0, \nu_1)$ and $(\gamma_2, 0, \nu_2)$ where $\gamma_1 = \int_0^1 x\nu_1(dx)$ and $\gamma_2 = \int_{-1}^0 x\nu_2(dx)$ respectively, and therefore $b_1 = b_2 = 0$, its characteristic function can be written as:

$$\begin{aligned}\phi_{X(t)}(u) &= \phi_{Y(t)}(u)\phi_{Z(t)}(u) \\ &= \exp\left\{t\left(iu(b_1+b_2) + \int_{\mathbb{R}}\left(e^{iux}-1\right)(\nu_1(dx)+\nu_2(dx))\right)\right\} \\ &= \exp\left\{t\left(\int_{\mathbb{R}}\left(e^{iux}-1\right)\nu(dx)\right)\right\}.\end{aligned} \tag{2.26}$$

it follows that $b = 0$ and hence $\gamma = \int_{|x|\leq 1} |x|\nu(x)$.

For the Variance Gamma process the probability density function of the process at time t is known in a closed form. Following Brigo et al. [28], when the probability density function is know it is easy to implement an efficient Maximum Likelihood method to calibrate the Variance Gamma model on historical observations. Indeed, even if at a first glance the expression 2.26 seems scaring, it can be numerically computed in a very efficient way (see Amos [3]).

The probability density function of the Variance Gamma process $X = \{X(t); t \geq 0\}$ at time $t \geq 0$ is given by:

$$f_{X(t)}(x) = K_{\alpha t - \frac{1}{2}}\left(|x|\frac{\sqrt{2\sigma^2\beta+\theta^2}}{\sigma^2}\right)\frac{\exp(\theta x/\sigma^2)}{\sqrt{2\pi\sigma^2}}\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}\left(2\sigma^2\beta+\theta^2\right)^{\frac{1}{4}-\frac{\alpha t}{2}} 2|x|^{\alpha t-\frac{1}{2}}.$$

Finally, a possible realization of the Variance Gamma process is shown in Figure 2.7.

2.6.2 The Normal Inverse Gaussian process

Another widely used Lévy process in mathematical finance base od Brownian subordination is the Normal Inverse Gaussian process (NIG). Here we follow the original costruction of the NIG process proposed by Barndorff-Nielsen [12], Barndorff-Nielsen and Shephard [13] and hence we use the IG_B parametrization, whereas another approach is proposed by Cont and Tankov [42]. As the Variance Gamma process, the Normal Inverse Gaussian process can be constructed by Brownian subordination via an Inverse Gaussian process.

Definition 2.6.2. *(Normal Inverse Gaussian Process) Consider an inverse Gaussian process $G = \{G(t); t \geq 0\}$ such that $G(t) \sim IG_B\left(\delta t, \sqrt{\alpha^2-\beta^2}\right)$ where $\delta > 0$, and $\alpha, \beta \in \mathbb{R}$. Consider a standard Brownian motion W independent of G. The process $X = \{X(t); t \geq 0\}$ defined as:*

$$X(t) = \beta Z(t) + W(Z(t)), \quad t \geq 0,$$

is called Normal Inverse Gaussian process.

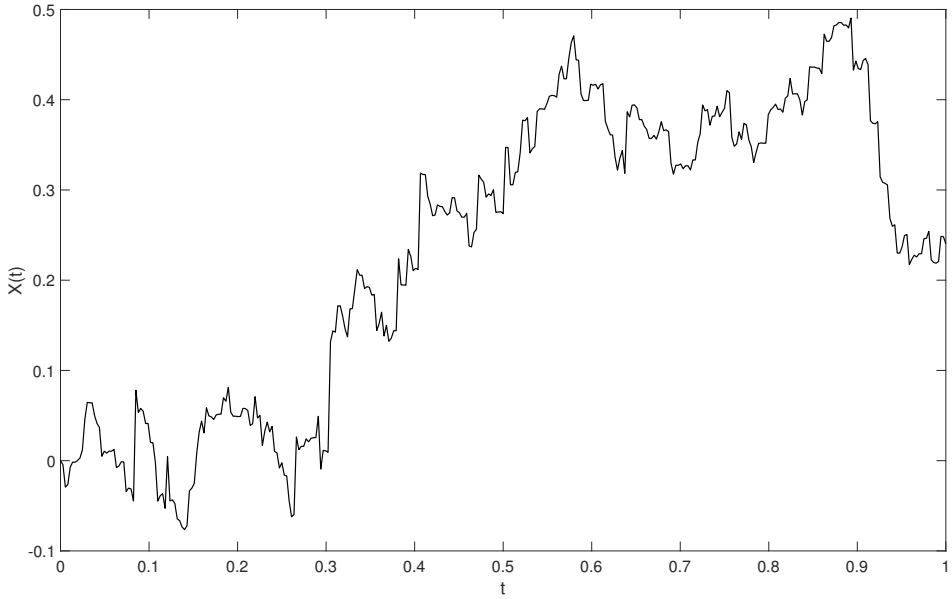

Figure 2.7. A possible path of the Variance Gamma process with parameters $\theta = 0.04$, $\sigma = 0.34$ and $\alpha = \beta = 125$ for $t \in [0, T]$ with $T = 1$.

One can easily show that the characteristic function of the Normal Inverse Gaussian process at time t is given by:

$$\phi_{X(t)}(u) = \mathbb{E}\left[e^{iuX}\right] = \exp\left\{\delta t\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)\right\}.$$

As shown in Barndorff-Nielsen [12], relying upon some results about exponential family distributions presented in Bar-Lev et al. [10], the Lévy triplet (γ, σ^2, ν) is given by:

$$\begin{aligned}
\gamma &= \frac{2\alpha\delta}{\pi}\int_0^1 \sinh(\beta x)\, K_1(\alpha x)\, dx, \\
\sigma^2 &= 0, \\
\nu(x) &= \frac{\alpha\delta}{\pi}\frac{\exp(\beta x)\, K_1(\alpha|x|)}{|x|},
\end{aligned}$$

where

$$K_\lambda = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp\left(-\frac{1}{2}z\left(u + u^{-1}\right)\right) du.$$

Since $\sigma^2 = 0$, we deduce that the Normal Inverse Gaussian process essentially moves by jumps. Unlike the Variance Gamma process, as shown by Cont and Tankov [42], the Normal Inverse Gaussian process is of infinite variation, whereas its probability density function at time $t \geq 0$ is given by:

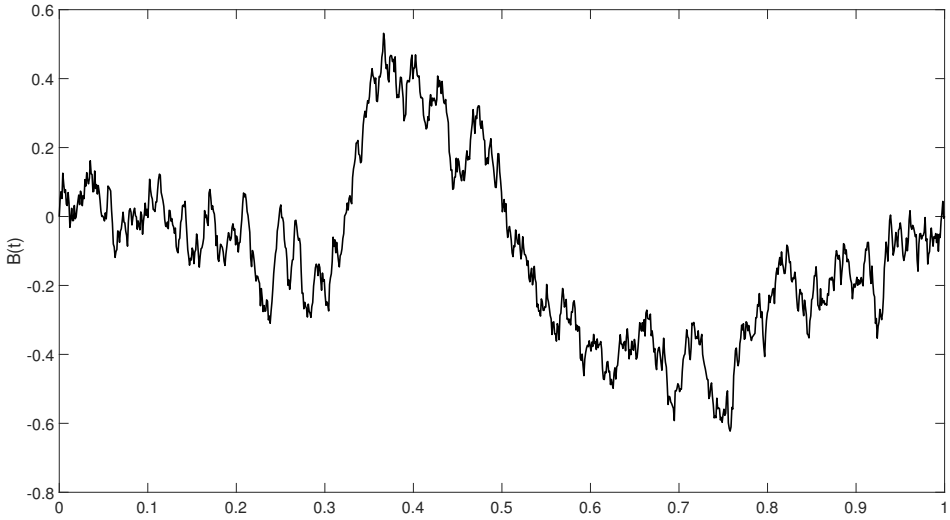

Figure 2.8. A possible path of the standard Brownian Bridge pinned at 0 at time $T = 1$.

$$f_{X(t)}(x) = \frac{\alpha\delta t K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\pi\sqrt{\delta^2 (x-\mu)^2}} e^{\delta t\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)}.$$

2.7 Lévy Bridges

In this section we briefly discuss the concept of Lévy bridges. A bridge is a stochastic process that is pinned to some fixed point at a fixed future time. For example, a Brownian bridge is a continuous-time stochastic process $B = \{B(t); t \in [0, T]\}$ whose probability distribution is the conditional probability distribution of a standard Brownian motion $W = \{W(t); t \in [0, T]\}$ subject to the condition (when standardized) that $W(T) = 0$, so that the process is pinned at the origin at both $t = 0$ and $t = T$. More precisely:

$$B = \{W(t); t \in [0, T], W(T) = 0\}.$$

A possible realization of the standard Brownian Bridge is shown in Figure 2.8 where we have that $W(0) = W(T) = 0$, by construction.

From a practical point of view, Lévy bridges techniques are commonly used in finance to overcome some problems that arise when we consider derivatives which payoff depends on the monitoring that an event occurs. For example suppose that an agent buys an option and get a payoff at maturity T if and only if the underlying asset hit the barrier value $\bar{S}$ in $[0, T]$. A Monte Carlo scheme can be implemented but, if the monitoring of the barrier value is done continuously in time any discretization of the path may lead to some pricing error which can also be surprisingly wide. A Monte Carlo algorithm based on the Lévy bridge leads to a more accurate results overcoming the problem (see Ribeiro and Webber [105] or Acworth et al. [2] for other

applications of the Lévy bridges). The interested reader may refer to Hoyle [71] for an overview on this topic and to Fitzsimmons et al. [57] for a discussion in the more general setting of Markov processes.

We only summarize the main results about discrete and continuous Lévy bridges and we show how to derive the Poisson, the Brownian and the gamma bridges. Following the idea proposed by Hu and Zhou [72] and Sabino [106], these results are instrumental to simulate backward in time the $VG++$ process we will introduce in Chapter 3.

2.7.1 Bridges of Lévy processes with discrete law

Consider the process $M = \{M(t); t \in [0,T]\}$ and assume that the law of $M(t)$ is discrete for some $t > 0$ (and hence for all $t \in [0,T]$). Consider the state space of the process $S = (a_i)_{i\geq 1}$ where $(a_i)_{i\geq 1}$ is an increasing sequence of real numbers. Let $Q_t : \{a_i\} \to [0,1]$ the probability mass function of $M(t)$. Then we have that: $\mathbb{P}(M(t) = a_i) = Q_t(a_i)$. Consider $0 \leq s < t \leq T$ and define by $M_T^{(k)} = \left\{M_T^{(k)}(t); t \in [0,T]\right\}$ the Lévy bridge to the value a_k at time T: this means that the process $M_T^{(k)}$ at time T is almost surely equal to the value a_k. Moreover we require that: $\mathbb{P}(M_T = a_k) = Q_T(a_k) > 0$. By Bayes theorem we obtain that:

$$\begin{aligned}\mathbb{P}\left(M_T^{(k)}(t) = a_j | M_T^{(k)}(s) = a_i\right) &= \mathbb{P}(M(t) = a_j, M(s) = a_i, M(T) = a_k) \\ &= \frac{\mathbb{P}(M(t) = a_j, M(T) = a_k | M(s) = a_i)}{\mathbb{P}(M(T) = a_k | M(s) = a_i)} \\ &= \frac{Q_{t-s}(a_j - a_i)\, Q_{T-t}(a_k - a_j)}{Q_{T-s}(a_k - a_i)}.\end{aligned} \tag{2.27}$$

Equation (2.27) means that the probability that the process M assumes the value a_j at time t, given that $M(s) = a_i$ and $M(T) = a_k$, depends on the its independent increments on $[s,t]$, $[t,T]$ and $[s,T]$.

Observe that Equation (2.27) holds provided that we substitute the probability mass function with the probability density function. In particular, consider the Lévy process $L = \{L(t); t \in [0,T]\}$ and denote by $f_t(x)$ the density of $L(t)$, for $t \geq 0$. Consider $0 \leq s < t \leq T$ and let $L_T^{(z)} = \left\{L_T^{(z)}(t); t \in [0,T]\right\}$ be the Lévy bridge to the value of $z \in \mathbb{R}$ at time T and $L_T^z(s) = x$. We require that $0 < f_T(z) < \infty$ and hence we have that:

$$f_t(y|x,z) = \frac{f_{t-s}(y-x) f_{T-t}(z-y)}{f_{T-s}(z-x)}.$$

2.7.2 Examples

In this section we give some examples of Lévy bridges both in discrete and continuous time focusing on the Poisson, Brownian and gamma bridges. Nevertheless, the

procedure we follow is very general and can be adapted to determine the bridge of any given Lévy process.

The Poisson bridge

Consider the Poisson process $N = \{N(t); t \in [s,T]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $N(t) \sim \mathcal{P}(\lambda t)$. Consider $0 \leq s < t < T$ and the Poisson bridge process $N^{(k)} = \left\{N_T^{(k)}(t); t \in [0,T]\right\}$ where $N_T^{(k)}(t)$ denotes the Poisson process N pinned at a fix value $k \in \mathbb{N}$ at $t = T$, i.e. $N(T) = k$. Moreover assume that $N(s) = i$, for $i \leq k$. Then, by straightforward computations, it follows that for $i \leq j \leq k$:

$$\begin{aligned}\mathbb{P}\left(N_T^{(k)}(t) = j | N_s^{(k)} = i\right) &= \mathbb{P}\left(N(t) = j | N(s) = i, N(T) = k\right) \\ &= \binom{k-i}{j-i}\left(\frac{t-s}{T-s}\right)^{j-i}\left(1 - \frac{t-s}{T-s}\right)^{(k-i)-(j-i)}.\end{aligned}$$

and hence $N_T^{(k)}(t)$ is such that:

$$N_T^{(k)}(t) \sim Bin\left(k-i, \frac{t-s}{T-s}\right),$$

where $Bin(n,p)$ denotes the binomial distribution with parameters $n \in N$ and $p \in [0,1]$. Therefore, assuming that $N(T) = k$ and $N(s) = i$, the value of the process N at time t can be simulated by sampling from a Binomial law with parameters $n = k - i$ and $p = \frac{t-s}{T-s}$.

The Brownian bridge

Consider a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with its natural filtration $\{\mathcal{F}_t; t \geq 0\}$ and consider $0 < u < s < t$. Assume also that $W(u) = x$, $W(t) = y$: we want to recover the law of $W(s)$. As shown in Glasserman [62] we have that:

$$\begin{pmatrix} W(s) \\ W(u) \\ W(t) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & u & s \\ u & u & u \\ s & u & t \end{pmatrix}\right)$$

Assume that $W = \{W(t); t \geq 0\}$ be a Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion $\sigma \in \mathbb{R}^+$. From straightforward computations it follows that, for $0 < u < s < t$, the distribution of $W(s)$, given $W(u) = x$ and $W(t) = y$ is normal with mean and variance given by:

$$\mathbb{E}\left[W(s) | W(u) = x, W(t) = y\right] = \frac{(t-s)\,x + (s-u)\,y}{t-u}, \tag{2.28}$$

$$Var\left[W(s) | W(u) = x, W(t) = y\right] = \sigma^2 \frac{(s-u)\,(t-s)}{(t-u)}. \tag{2.29}$$

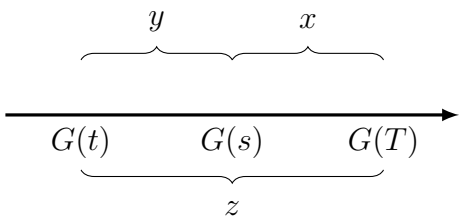

Figure 2.9. A gamma bridge at s with between $G(t)$ and $G(T)$.

We can conclude that, if we want to simulate the Brownian bridge at time $s \in (u, t)$) it is enough to sample from a normal distribution with mean and variance given by Equations (2.28) and (2.29) respectively.

Gamma bridge

In this section we sketch how to construct a gamma bridge following Hoyle [71], Émery and Yor [53]. The idea is the same of the previous section: considering a gamma process G on $[0, T]$, $t < s < T$ and assuming that we know the value of the process at time t and T, the distribution of the process gamma process G at time s is given by the following proposition.

Proposition 2.7.1. *Consider a Gamma process $G = \{G(t); t \geq 0\}$ such that $G(t) \sim \Gamma(\alpha t, \beta)$ and let be $0 \leq t < s < T$. Suppose that the value of the process G at times t and T is given and that $G(t) = g_t$ and $G(T) = g_t$. Then:*

$$\frac{G(s) - G(t)}{G(T) - G(t)} \sim B(\alpha(s-t), \alpha(T-s))$$

where where B denotes the Beta distribution. Therefor we have that:

$$G(s) = g_t + (g_T - g_t)\beta,$$

where $\beta \sim B(\alpha(s-t), \alpha(T-s))$.

Proof. Refer to Figure 2.9. We have that X, Y are independent and that $Z = X + Y$. Observe that:

$$\begin{aligned} X &\sim \Gamma(\alpha(s-t), \beta), \\ Y &\sim \Gamma(\alpha(T-s), \beta), \\ Z &\sim \Gamma(\alpha(T-t), \beta) \end{aligned}$$

and recalling the expression of the probability density function of a gamma law given

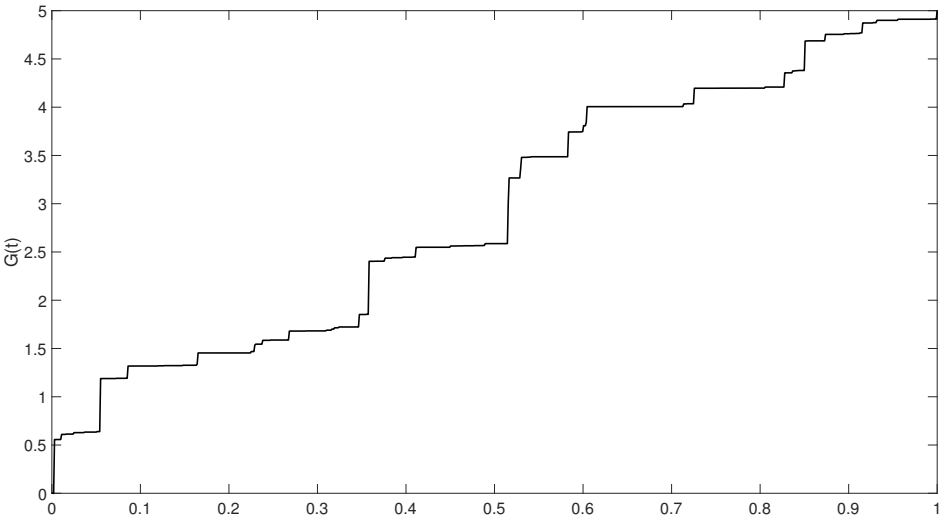

Figure 2.10. A possible path of the gamma Bridge pinned at 5 at time $T = 1$.

by 2.15, we get:

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{f_X(x) f_Y(z-x)}{f_Z(x)} \\ &= \frac{\Gamma\left(\alpha(T-s)+\alpha(s-t)\right)}{\Gamma\left(\alpha(s-t)\right)\Gamma\left(\alpha(T-s)\right)} x^{\alpha(s-t)-1} \frac{(z-x)^{\alpha(T-s)-1}}{z^{\alpha(T-s)+\alpha(s-t)-1}} \\ &= \frac{1}{B\left(\alpha(s-t), \alpha(T-s)\right)} \left(\frac{x}{z}\right)^{\alpha(s-t)-1} \left(1-\frac{x}{z}\right)^{\alpha(T-s)-1} z. \end{aligned}$$

Recalling that

$$z = G(T) - G(t), \quad x = G(s) - G(t),$$

we have that:

$$\frac{G(s)-G(t)}{G(T)-G(t)} \sim B\left(\alpha\left(s-t\right), \alpha\left(T-s\right)\right)$$

and hence:

$$G(s) \stackrel{d}{=} g_t + \left(g_T - g_t\right)\beta.$$

■

A possible realization of the Gamma Bridge with $g_T = 5$ and $g(t) = 0$ is shown in Figure 2.10.

2.8 Basics of Financial modelling and Risk-Neutral Pricing

In this section we recall fundamental concepts behind risk neutral pricing, absence of arbitrage and equivalent martingale measures. We will follow Cont and Tankov [42,

Chapter 8] and the interested reader can refer to Harrison and Kreps [67], Harrison and Pliska [68] and Delbaen and Schachermayer [49] for a more technical exposition.

Consider a market whit d assets whose prices are modelled by a vector stochastic process $S = \{S(t); t \geq 0\}$:

$$S(t) = (S_0(t), S_1(t), \ldots, S_d(t)),$$

which is supposed to be cadlag. $S_0(t)$ is a *numeraire*, a tradable economic entity in terms of whose price the relative prices of all other tradables are expressed and commonly is chooses to be: $S_0(t) = e^{rt}$. We refer to the quantity $\frac{S_i(t)}{S_0(t)}$ as the *discounted stock price*.

A portfolio is a vector:

$$\phi = \left(\phi^1, \ldots, \phi^d\right),$$

describing the amount of each asset held by the investor. The (random) value of such a portfolio at time t is given by:

$$V_\phi(t) = \sum_{k=1}^{d} \phi_k S_k(t).$$

A trading strategy consist of maintaining a dynamic portfolio $\phi = \{\phi(t); t \geq 0\}$ by buying and selling assets at different dates. Let us denote the transaction dates by: $0 = T_0 < T_1 < \cdots < T_n < T_{n+1} = T$. Between two transaction dates T_i and T_{i+1} the portfolio remains unchanged and we will denote its composition by $\phi(t)$. The porfoglio ϕ held at time t can be expressed as:

$$\phi(t) = \phi_0 \mathbb{1}_{t=0} + \sum_{i=0}^{n} \phi_i \mathbb{1}_{(T_i, T_{i+1}]}(t).$$

The transaction dates T_i can be fixed but, more realistically, they are not known in advance and an investor will decide to buy or sell at T_i depending on the information reveleated before T_i. Therefore, the transaction times T_i should be defined as stopping times. Since the portfolio ϕ_i is based on the information available at T_i, ϕ_i is $\mathcal{F}_{T_i}$-measurable.

In finance modelling arbitrages should be avoided. Loosely speaking, an arbitrage is the possibility of having a sure gain with no risks exposition. We recall that a porfolio is said to be self-financing if its value changes because the value of the stocks changes but not because we take or inject money in the portfolio. Now we can define what we mean by *arbitage*.

Definition 2.8.1. *(Arbitrage) An arbitrage is a self-financing strategy ϕ which can lead to a positive terminal gain, without the probability of intermediate loss:*

$$\mathbb{P}\left(\forall t \in [0, T], V_\phi(t) \geq 0\right) = 1, \qquad \mathbb{P}\left(V_\phi(T) > V_\phi(0)\right) \neq 0.$$

The probability measure $\mathbb{P}$ is used only to state which events are possible and which are not. A direct consequence of the absence of arbitrage is the so called *law of one price*: two self-financing strategies with the same terminal payoff must have the same value at all times, otherwise the difference would generate an arbitrage.
Suppose we have a *contingent claim* i.e. a derivative whose future payoff depends on the value of another "underlying" asset, with maturity T. Its value at T, called *payoff*, is a $\mathcal{F}_T$-measurable random variable $H : \Omega \mapsto \mathbb{R}$. A central problem in finance is the valuation problem: how can we attribute a notion of value to each contingent claim H? A procedure which attributes to each contingent claim H a value for each time t is called *pricing rule* and it is denoted by $\Pi_H(t)$. The goal is to define a pricing rule that does not lead to arbitrage opportunities. It can be shown that specifying an arbitrage-free pricing rule on $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ is equivalent to specify a new measure $\mathbb{Q}$ under which the discounted trade assets are martingales. $\mathbb{Q}$ is required to be equivalent to $\mathbb{P}$ in the sense of the following definition.

Definition 2.8.2. *(Equivalent probability measures) Consider a measurable space $(\Omega, \mathcal{F})$. Two measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if:*

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0, \quad \forall A \in \mathcal{F}.$$

The following proposition provides us a powerful tool to compute the price of any contingent claim.

Proposition 2.8.1. *(Cont and Tankov [42, Proposition 9.1]) In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage free pricing rule Π can be represented as:*

$$\Pi_H(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{F}_t\right]. \tag{2.30}$$

where $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$ such that:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_i(T)}{S_0(T)}\middle|\mathcal{F}_t\right] = \frac{S_i(t)}{S_0(t)}.$$

The measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and such that the discounted prices $S_i(t)/S_0(t)$ are martingales is called *equivalent martingale measure* or, sometimes, *rik neutral measure*.
The following theorem exploits the link between arbitrage-free market models and the existence of an equivalent martingale measure (see Harrison and Kreps [67], Harrison and Pliska [68] and Delbaen and Schachermayer [49] for details).

Theorem 2.8.2. *(First Foundamental theorem of asset pricing) The market model defined by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and by the asset prices $S = \{S(t); t \in [0, T]\}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted assets are martingales with respect to $\mathbb{Q}$.*

Another important theorem is related to the financial notion of *market completeness*. Besides the idea of arbitrage, another important concept, originating by Black and Scholes [23], is the notion of *replication strategy* for a contingent claim H i.e. a strategy such that:

$$H = V(0) + \int_0^T \phi(t)dS(t) + \int_0^T \phi_0(t)dS_0(t), \quad \mathbb{P} - a.s.$$

where $V(0)$ is the initial wellness. A market is said to be *complete* if any contingent claim admits a replicating portfolio ϕ. In particular, $\phi = \{\phi(t), t \in [0,T]\}$ should be of the form (2.8). In a complete market there is a only way to define the value of the contingent claim: the value of any contingent claim is given by the initial capital needed to set up a replicating strategy for H. We can state that, in complete markets, contingent claims are redundant instruments since their value can be replicated at any time t by a proper self-financing strategy. The following theorem is sometimes know as *second fundamental theorem of asset pricing* (see Cont and Tankov [42]).

Theorem 2.8.3. *(Second Foundamental Theorem of Asset Pricing) A market defined by the stochastic process $S = \{S(t); t \in [0,T]\}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is complete if and only if there exist a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.*

While most stochastic models used in option pricing are arbitrage-free, only a few of them are complete. Stochastic volatility models, exponential Lévy models, Jump-diffusion models fall into the category of incomplete models. By contrast the Black and Scholes model defines a complete market.

2.8.1 Modelling with exponential Lévy processes

In this section we briefly show how Lévy processes can be used to build financial models in continuous time and how they can be used for derivatives evaluation. The interested reader can refer to Cont and Tankov [42, Chapter 11] for a broad discussion on this topic.

Consider an arbitrage-free market where asset prices are modelled by a stochastic process $S = \{S(t); t \in [0,T]\}$ defined as in (2.8) and where $\mathcal{F}_t$ is the natural filtration generated by the process S. In previous section we have seen that under the pricing rule given by the measure $\mathbb{Q}$ the value $\Pi_H(t)$ of an option with payoff $H(T)$ can be computed by:

$$\Pi_t(H(T)) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H(T)|\mathcal{F}_t\right].$$

Under the hypothesis of the Black and Scholes [23] model, using the Girsanov theorem (see Shreve [118, Chapter 5]) it can be shown that under the risk-neutral measure $\mathbb{Q}$ the asset prices has the following dynamics:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad t \in [0,T], \ S(0) = S_0, \text{ a.s.}, \tag{2.31}$$

where r is the risk-free rate and σ is the diffusion. By using Itô's Lemma we have that the solution of (2.31) is given by:

$$S(t) = S(0) \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right\}.$$

This suggest us that the asset can be modelled by the exponential of a Brownian motion $B = \{B(t); t \in [0, T]\}$ with drift by:

$$S(t) = S(0) \exp\{B(t)\}, \tag{2.32}$$

where $B(t) = rt + \sigma W(t)$ and where $W = \{W(t); t \in [0, T]\}$ denotes a standard Brownian motion.
Following this approach we can replace the Brownian motion with drift by a Lévy process $X = X\{X(t); t \in [0, T]\}$. One way to make this is to make the substitution in (2.32):

$$S(t) = S(0) \exp\{rt + X(t)\}.$$

This model is commonly called *exponential Lévy model*. In order to guarantee that the discounted price process is a martingale, by Proposition 2.4.10 we need to impose some conditions on the Lévy triplet (γ, σ^2, ν) of the chosen process X. In particular we require that:

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty, \tag{2.33}$$

$$\frac{\sigma^2}{2} + \gamma + \int_{-\infty}^{\infty} \left(e^x - 1 - x\mathbb{1}_{|x|\leq 1}\right) \nu(dx) = 0. \tag{2.34}$$

X is then a martingale if $\mathbb{E}\left[e^{X(t)}\right] = 1$ for all $t \in [0, T]$. Hence it is sufficient to impose that the characteristic function of X computed at $u = -i$ is equal to one.

Sometimes it is common to model del stock process S as:

$$S(t) = S(0)e^{rt+\omega t+X(t)},$$

where ω is a parameter that must be chosen in order to guarantee that the discounted stock price is a martingale (see Carr and Madan [37], Luciano and Schoutens [86] and Madan and Seneta [88] among the others). The two modelling techniques are equivalent and in this work we follow the latter approach. In the following section we briefly show how the Lévy exponentiation can be used to model stock price using the Variance Gamma process we introduced in Section 2.6.1.

2.8.2 The Variance Gamma model for asset prices

Considering the Variance Gamma process $X = \{X(t); t \in [0, T]\}$, we model the risky asset process $S = \{S(t); t \in [0, T]\}$ as:

$$S(t) = S(0) \exp\left\{rt + \omega t + X(\theta, \sigma, \nu)\right\}. \tag{2.35}$$

Since the gamma subordinator $G = \{G(t); t \in [0, T]\}$ we used to construct the Variance Gamma model can be interpreted as a *business time*, it is customary to require that in mean the stochastic time runs as fast as the deterministic time t. In more mathematical terms we require that:

$$\mathbb{E}\left[G(t)\right] = t.$$

Since $G(t) \sim \Gamma(\alpha t, \beta)$ then $E\left[G(t)\right] = \alpha t/\beta$ and hence we impose that $\alpha = \beta$. Moreover, we have that $Var\left[G(t)\right] = \alpha t/\beta^2$ and defining $\nu = 1/\alpha = 1/\beta$ we have that $Var\left[G(1)\right] = \nu$ and hence ν is the variance of the subordinator G at time $t = 1$. $\theta \in \mathbb{R}$ and $\sigma > 0$ are respectively the drift and the volatility of the subordinated Brownian motion.
By imposing risk-neutrality condition (2.34) we get that:

$$\omega = \frac{1}{\nu} \log\left(1 - \frac{\sigma^2 \nu}{2} - \theta\nu\right).$$

A nice feature of the Variance Gamma model is that there exists a semi-analytical option pricing formula for European call options. This is important from a numerical point of view because closed formulas are frequently used for market calibration purposes, in order to obtain efficient calibration procedures (see Cont and Tankov [42, Chapter 13]).

For the sake of generality, assume that the gamma process G is such that $G(t) \sim \Gamma(\alpha t, \beta)$ and that the risky asset process S is modelled as in (2.35). Consider now a European call option with maturity T which payoff is given by $(S(T) - K)^+$. We compute the value of the call option with strike K at time $t = 0$, namely $C(0, K)$, using the risk-neutral pricing formula (2.30):

$$C(0, K) = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[(S(T) - K)^+\right] = \int_0^\infty c(g) \frac{\beta^\alpha}{\Gamma(\alpha)} g^{\alpha-1} e^{-\beta g} dg, \tag{2.36}$$

where:

$$c(g) = S(0) e^{\theta \frac{\sigma^2}{2} g + \omega T} \mathcal{N}(d1(g)) - K e^{-rT} \mathcal{N}(d_2(g)), \tag{2.37}$$

and

$$d_2(g) = \frac{\log\left(S(0)/K + rT + \theta g + \omega T\right)}{\sigma\sqrt{g}}, \qquad d_1(g) = d_2(g) + \sigma\sqrt{g},$$

$$\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

This integral in Equation (2.36) can be computed numerically, for example using a quadrature methods. Nevertheless, a more efficient semi-analytical formula can be

deduced by using the confluent hypergeometric function of two variables, introduced by Humbert [73], which is defined by:

$$\Phi(\alpha,\beta,\gamma,x,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}e^{uy}du. \quad (2.38)$$

The integral (2.38) can be computed in a very efficient way and hence its valuation is not a metter of concern. Defining the following quantities:

$$\tilde{a}=\frac{\log(S(0)/K)+(r+\omega)T}{\sigma}\sqrt{A}, \qquad \tilde{b}=\frac{\theta+\sigma^2}{\sigma\sqrt{A}},$$
$$\tilde{c}=\frac{\log(S(0)/K)+(r+\omega)T\sqrt{\beta}}{\sigma}, \qquad \tilde{d}=\frac{\theta}{\sigma\sqrt{\beta}}.$$

we conclude that the price of the Call option at time $t=0$ is given by:

$$C(0,K)=\frac{\beta^\alpha S(0)e^{\omega T}}{A^\alpha}\Psi(\tilde{a},\tilde{b};\alpha)-Ke^{-rT}\Psi(\tilde{c},\tilde{d};\alpha) \quad (2.39)$$

It can be shown that the quantity $\Psi(a,b;\gamma)$ can be expressed in terms of confluent hypergeometric function Φ of two variables (2.38) and in terms of the Bessel Modified function of the second type $K_\nu(z)$ by the following relation:

$$\begin{aligned}\Psi(a,b;\gamma)=&\frac{c^{\gamma+1/2}\exp[sign(a)\,c](1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma}\cdot K_{\gamma+1/2}(c)\\ &\Phi\left(\gamma,1-\gamma,1+\gamma,\frac{1+u}{2},-sign(a)\,c(1+u)\right)\\ &-sign(a)\frac{c^{\gamma+1/2}\exp[sign(a)\,c](1+u)^{\gamma+1}}{\sqrt{2\pi}\Gamma(\gamma)(1+\gamma)}\cdot K_{\gamma-1/2}(c)\\ &\Phi\left(1+\gamma,1-\gamma,2+\gamma,\frac{1+u}{2},-sign(a)\,c(1+u)\right)\\ &+sign(a)\frac{c^{\gamma+1/2}\exp[sign(a)\,c](1+u)^{\gamma}}{\sqrt{2\pi}\Gamma(\gamma)\gamma}\cdot K_{\gamma-1/2}(c)\\ &\Phi\left(\gamma,1-\gamma,1+\gamma,\frac{1+u}{2},-sign(a)\,c(1+u)\right),\end{aligned}$$

where $c=|a|\sqrt{2+b^2}$ and $u=\frac{b}{\sqrt{2+b^2}}$.

After this smattering of theory, in the next sections we introduce new Lévy processes and we apply them to asset pricing modelling.

Chapter 3

The $VG++$ process: a model for illiquid markets

In this chapter we introduce a new Lévy process related to the Variance Gamma process which inherits its mathematical tractability and financial interpretation. It has only an additional parameter which measures the trading activity and therefore the liquidity regime. We call such a new process Variance Gamma++ ($VG++$).

Models based on the Variance Gamma distribution are widely used in finance since the introduction of the Variance Gamma process by Madan and Seneta [88]. Such a process presents many interesting properties: both characteristic function and density are available in a closed form and, moreover, a closed formula for European options is known. Finally, efficient methods for path simulations can be used in order to simulate the process and hence to price exotic contingent claims. All these properties together with the fact that the model overcomes some well known limits of the model proposed by Black and Scholes [23], make it a good candidate for financial markets modeling.

In contrast to the classical Black-Scholes market where real data description is based on the standard Brownian diffusion-type processes, the Variance Gamma assumes that dynamics of the price or of the returns depends on a time-changed Brownian motion where the time-change is given by a gamma process. Such a random time process, called subordinator, can be interpreted as trading activity, in the sense that the price does not evolve in terms of the physical time but instead in terms of the random transactions exchanged in the market.

This interpretation has be explored using different types of subordinator processes, for instance Barndorff-Nielsen [12] takes the Inverse Gaussian process for the the Brownian subordination and also the CGMY model, introduced in Carr et al. [38] which generalizes the Variance Gamma, under some parameter constrains can be seen a time-changed Brownian motion. All these models are pure jumps models with infinite activity that differ from jump-diffusion models (see for instance Merton [91] and Kou [80]) where the jumps are interpreted as sudden news in the market.

However, some real data exhibit characteristic periods of constant values especially

in illiquid markets like some not so mature energy markets. In such cases, adopting the financial interpretation that the subordinating process represents the trading activity, the gamma process (and the other subordinators mentioned above) imply that in any finite time-interval the number of trades cannot be zero because its trajectory is strictly increasing. The Variance Gamma process essentially exhibits an infinite number of jumps in any finite time interval and hence its trajectories can not be constant over time (see Cont and Tankov [42, Lemma 2.1]). Market liquidity is generally strictly related to the amount of registered transactions between counterparts. Therefore, a zero variation of the price over the time period Δt usually appears when no market transactions occur.

The main idea is to replace the gamma process by another process related to it which may be constant in time and keeps the right properties to still behave as a subordinator. The new subordinator is then of finite activity and the probability of having no transactions in a finite period of time will not be null.

To this end we use the well-know self-decomposabilty of the gamma law (see Grigelionis [64]). We recall from the previous chapters that a random variable X is said to have a self-decomposable law if for all $a \in (0,1)$ there exist two independent random variables Y and Z_a such that $X \stackrel{d}{=} Y$ and:

$$X \stackrel{d}{=} aY + Z_a.$$

In the following we will refer to Z_a as the a-remainder of the *sd* law. It turns out that the law of Z_a is infinitely divisible (see Sato [112, Proposition 15.5]) and one can construct the associated Lévy process $Z_a^{++} = \{Z_a^{++}(t); t \geq 0\}$.

Our approach consists in taking the subordinator Z_a^{++}, from the a-remainder of the gamma law to construct the new $VG++$ process $X = \{X(t); t \geq 0\} = \{W(Z_a(t)); t \geq 0\}$ where $W = \{W(t); t \geq 0\}$ is a Brownian motion with drift $\theta \in \mathbb{R}$ and diffusion coefficient $\sigma > 0$. If X represents the log-price process of a risky asset denoting $\Delta X = X(t + \Delta t) - X(t)$ the increment of the process X over the time interval Δt we show that $\mathbb{P}(\Delta X = 0) > 0$ therefore we have non zero probability to have no transactions in the time interval Δt. In particular, we show that the parameter a plays the role of an indicator of the trading activity. apply the $VG++$ process to model power future markets some of which are not very liquid.

Accordingly, we derive the Lévy measure, the transition density, the characteristic function expressions in closed form. However, the new process has finite activity but can also be written as the difference of two independent subordinators and keeps the mathematical tractability of the Variance Gamma process. As a consequence, we obtain a closed formula for the European call option pricing which is an infinite weighted sum of call options under the Variance Gamma model, where the shape parameter of the underlying gamma subordinator is an integer. Such a formula does not require any numerical integration, but can be reduced to matrix multiplications which are faster than numerical integration algorithms.

3.1 Notation and preliminary remarks

In Section 2.5.2 we said that X has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its law has a probability density function of the form (2.15). When $\alpha = n \in \mathbb{N}$ such a law coincides with the Erlang distribution denoted $\mathcal{E}_n(\beta)$, for simplicity we drop $n = 1$ for the exponential distribution. A random variable X is said to have a uniform distribution over $[a, b]$ if its probability density function $f_X(x)$ is of the form:

$$f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x).$$

We write $\mathcal{U}([0,1])$ to denote the uniform distribution in $[0,1]$.

3.1.1 Preliminary remarks: the Z_a^{++} process

Recall that X is said to have a *sd* law if for all $a \in (0,1)$ there exist a *rv* Y with the same law of X and a *rv* Z_a independent of Y such that:

$$X \stackrel{d}{=} aY + Z_a.$$

As observed in the introduction, if we denote by $\phi_X(u)$ the *chf* of X and by $\phi_{Z_a}(u)$ the *chf* of Z_a we have that:

$$\phi_X(u) = \phi_X(au)\,\phi_{Z_a}(u). \tag{3.1}$$

As we stated before, it can be shown that the a-remainder Z_a of a self-decomposable law is infinitely divisible. On the other hand, it is well-known that the gamma law is *sd* (as was shown by Grigelionis [64]) and hence the law of its a-remainder Z_a is also infinitely divisible. Therefore, by Proposition 2.4.1 there exists a Lévy process Z_a^{++} associated to this infinitely divisible law in the sense of the following definition.

Definition 3.1.1. *We say that Z_a has a gamma++ law, and we write $Z_a \sim \Gamma^{++}(a, \alpha, \beta)$, if Z_a is the a-remainder of a $\Gamma(\alpha, \beta)$ distribution.*

We can compute the characteristic function of Z_a by observing that, if $X \sim \Gamma(\alpha, \beta)$ its characteristic function is given by Equation (2.16), which inserted in equation (3.1) gives:

$$\phi_{Z_a}(u) = \frac{\left(\frac{\beta}{\beta - iu}\right)^{\alpha}}{\left(\frac{\beta}{\beta - iua}\right)^{\alpha}} = \left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}, \quad u \in \mathbb{R}.$$

Starting from the characteristic function we can retrieve the moment generating function $M_{Z_a}(u)$ using the relation $M_{Z_a}(u) = \phi_{Z_a}(-iu)$ and hence we can compute all the moments of Z_a. In particular its mean the variance are given by:

$$\mathbb{E}[Z_a] = (1-a)\frac{\alpha}{\beta}, \qquad Var[Z_a] = \left(1 - a^2\right)\frac{\alpha}{\beta^2}.$$

Based on the observations above and the findings of Sabino and Cufaro-Petroni [108], in this section we construct the Lévy process $Z_a^{++} = \{Z_a^{++}(t); t \geq 0\}$ associated to the law of the a-remainder of the gamma law, e.g. $Z_a^{++}(1) \stackrel{d}{=} Z_a$. To this end, we recall the following known results (see Sabino and Cufaro-Petroni [108] for details and proofs).

Definition 3.1.2. *A discrete rv S is said to be Polya distributed, $S \sim \overline{\mathfrak{B}}(\alpha, p)$, with parameters $\alpha > 0$ and $p \in (0,1)$, if its probability mass function has the following form:*

$$\mathbb{P}(\{S = k\}) = \binom{\alpha + k - 1}{k} (1-p)^{\alpha} p^k, \qquad k = 0, 1, \ldots$$

where:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}, \qquad \binom{\alpha}{0} = 1.$$

It is easy to check that if $X \sim \Gamma(\alpha, \beta)$, then

$$Z_a \stackrel{d}{=} \begin{cases} \sum_{i=1}^{S} X_i, & \text{with } S > 0 \\ 0, & \text{when } S = 0 \end{cases} \tag{3.2}$$

when $X_i \sim \mathcal{E}(\beta/a)$ is a sequence of *iid rv*'s and $S \sim \overline{\mathfrak{B}}(\alpha, 1-a)$. In particular $Z_{a|S=s} \sim \Gamma(s, \beta/a)$, when $s > 0$.

Proposition 3.1.1. *The pdf $g_a(x)$ of $Z_a \sim \Gamma^{++}(a, \alpha, \beta)$ is given by:*

$$g_a(x) = a^{\alpha}\delta_0(x) + \sum_{n \geq 1} \binom{\alpha + n - 1}{n} a^{\alpha} (1-a)^n f_{n,\beta/a}(x) \mathbb{1}_{(0,\infty)}(x)\, dx \tag{3.3}$$

where $\delta(x)$ is the Dirac function, $f_{n,\beta/a}(x)$ is pdf of an Erlang law with parameters n and β/a which is given by:

$$f_{n,\beta/a}(x) = \left(\frac{\beta}{a}\right)^n \frac{x^{n-1} e^{-\beta x/a}}{(n-1)!} \mathbb{1}_{[0,\infty)}(x).$$

We remark that the of law of Z_a can be seen as a mixture of Erlang laws with parameter β/a where the mixing distribution is a Polya distribution, plus a degenerate law at $x = 0$.

From Equation (3.2) we can define the process Z_a^{++} as follows:

$$Z_a^{++}(t) \stackrel{d}{=} \begin{cases} \sum_{i=1}^{S(t)} X_i, & \text{when } S(t) > 0, \\ 0, & \text{when } S(t) = 0 \end{cases}, \tag{3.4}$$

where $X_i \sim \mathcal{E}(\beta/a)$ is a sequence of *iid rv*'s and $S = \{S(t); t \geq 0\}$ is a Polya process such that for each $t \geq 0$, $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$. The construction is mathematically

consistent since the Polya distribution is infinitely divisible and therefore the Polya process is a Lévy process. The Lévy-Khintchine representation of the Polya process S is given by:

$$\phi_{S(t)}(u) = \exp\left\{-\alpha t \int_{-\infty}^{\infty} \left(e^{iux} - 1\right) \frac{(1-a)^x}{x} \delta_k(x) dx\right\},$$

where $\delta_k(x)$ is the Dirac function at $k \geq 1$.

We proceed then in the derivation of the characteristic Lévy triplet of the process Z_a^{++}. We rely on the the following proposition proven in Cufaro-Petroni and Sabino [48] that relates the characteristic triplet of a *sd* law with that of its a-remainder.

Proposition 3.1.2. *Consider a sd law with Lévy triplet (γ, σ, ν), where $\sigma > 0$ is the diffusion and ν is the Lévy measure. Then for every $a \in (0,1)$ the law of its a-remainder has Lévy triplet $(\gamma_a, \sigma_a, \nu_a)$:*

$$\begin{aligned}
\gamma_a &= \gamma(1-a) - a \int_{\mathbb{R}} sign(x) \left(\mathbb{1}_{|x| \leq 1/a} - \mathbb{1}_{|x| \leq 1}\right) |x| \, \nu(x) \, dx, \\
\sigma_a &= \sigma\sqrt{1-a^2}, \\
\nu_a(x) &= \nu(x) - \frac{\nu(x/a)}{a}.
\end{aligned}$$

Proposition 3.1.3. *Consider the process Z_a^{++}, then*

(i) The characteristic triplet $(\gamma_a, \sigma_a, \nu_a)$ of Z_a^{++} is given by:

$$\begin{aligned}
\gamma_a &= \left(1 - e^{-\beta}\right) - a\left(1 - e^{-\beta/a}\right), \\
\sigma_a &= 0, \\
\nu_a(x) &= \frac{\alpha}{x}\left(e^{-\beta x} - e^{-\beta x/a}\right) \mathbb{1}_{(0,\infty)}(x).
\end{aligned}$$

(ii) Z_a^{++} has finite variation and, in particular, is a subordinator.

(iii) Z_a^{++} has finite activity and therefore is a compound Poisson process with intensity $\lambda = \alpha \log(1/a)$ and the distribution of the jumps $f(x)$ is given by:

$$f(x) = \int_1^{1/a} \frac{1}{y \log(1/a)} \cdot \beta y e^{-\beta x y} dy = \frac{e^{-\beta x} - e^{-\frac{\beta}{a}x}}{x \log(1/a)}$$

Proof. (i) As a direct consequence of Proposition 3.1.2.

$$\begin{aligned}
\sigma_a &= 0, \\
\nu_a(x) &= \frac{\alpha}{x} e^{-\beta x} \mathbb{1}_{(x,\infty)}(x) - \frac{1}{a}\left(a \cdot \frac{\alpha}{x} e^{-\beta x/a}\right) \mathbb{1}_{(x,\infty)}(x) = \frac{\alpha}{x}\left(e^{-\beta x} - e^{-\beta x/a}\right) \mathbb{1}_{(x,\infty)}(x) \\
\gamma_a &= \frac{\alpha}{\beta}\left(\left(1 - e^{-\beta}\right) - a\left(1 - e^{-\beta/a}\right)\right).
\end{aligned}$$

(ii) By Cont and Tankov [42, Proposition 3.9] a Lévy process with characteristic triplet (A, ν, γ) is of finite variation if and only if:

$$A = 0 \quad \text{and} \quad \int_{|x|\leq 1} |x|\nu\left(dx\right) < \infty.$$

$A = \sigma_a = 0$ and the computation of the integral is straightforward:

$$\begin{aligned}\int_{|x|\leq 1} |x|\nu_a\left(dx\right) &= \int_0^1 \alpha t\left(e^{-\beta x} - e^{-\beta x/a}\right) dx \\ &= \frac{\alpha}{\beta}\left(1 - e^{-\beta} - a\left(1 - e^{-\beta/a}\right)\right) < \infty\end{aligned}$$

By Cont and Tankov [42, Proposition 3.10] since $\sigma_a = 0$, $\nu_a\left((-\infty, 0]\right) = 0$ and $b = \gamma - \int_0^1 x\nu_a\left(x\right)$ it follows that Z_a is a subordinator.

(iii) As a direct consequence of Gradshteyn and Ryzhik [63, 3.434] we have:

$$\nu_a\left(\mathbb{R}\right) = \alpha\int_{-\infty}^{\infty} \frac{e^{-\beta x} - e^{-\beta x/a}}{x} \mathbb{1}_{(0,\infty)}\left(x\right) dx = \alpha \log\left(\frac{1}{a}\right) < \infty,$$

hence, Z_a^{++} has finite activity and is a compound Poisson process such that its Lévy measure can be written as $\nu\left(x\right) = \lambda h\left(x\right)$ where $h(x)$ represents the *pdf* of the jumps and λ is the intensity. Define $\Lambda = \log\left(1/a\right)$, it follows that:

$$\begin{aligned}\nu_a\left(x\right) &= \Lambda\alpha \cdot \frac{1}{\Lambda x}\left(e^{-\beta x} - e^{-\beta/\alpha x}\right) = \Lambda\alpha \cdot \frac{1}{\Lambda x}\int_1^{1/a} \beta x e^{-\beta x y} dy \\ &= \Lambda\alpha \int_1^{1/a} \frac{\beta}{\Lambda} e^{-\beta x y} dy = \Lambda\alpha \int_1^{1/a} \frac{\beta}{\log(1/a)} e^{-\beta x y} dy \\ &= \Lambda\alpha \int_1^{1/a} \frac{\beta y}{y \cdot \log(1/a)} e^{-\beta x y} dy \\ &= \underbrace{\Lambda\alpha}_{\lambda} \cdot \underbrace{\int_1^{1/a} \frac{1}{y\log(1/a)} f_{\mathcal{E}}\left(x|\mu = \beta y\right) dy}_{h(x)},\end{aligned}$$

where $f_{\mathcal{E}}\left(x|\mu\right)$ is the *pdf* of an exponential distribution with parameter $\mu > 0$ and that concludes the proof.

■

We remark that *iii)* in Proposition 3.1.3 states that the distribution of the jump sizes can be seen as a mixture of an exponential law with stochastic rate given by βY where Y is a *rv* whose *pdf* is given by $g_Y(y) = \frac{1}{y\log(1/a)} \mathbb{1}_{[1,1/a]}(y)$. The cumulative distribution function of Y is given by:

$$F_Y\left(x\right) = \frac{1}{\log(1/a)} \int_1^x \frac{1}{y} dy = \frac{\log x}{\log(1/a)}, \quad 1 \leq x \leq 1/a,$$

Moment	T	Algorithm 1	Algorithm 2
$\mathbb{E}(X)$	5.6471	5.6468	5.6530
$Var(X)$	19.9308	19.8998	19.9664
$s(X)$	1.3615	1.3594	1.3561
$k(X)$	5.7083	5.7019	5.7680

Table 3.1. Comparison of theoretical moments (T) of the random variable Z_a^{++} with the numerical ones obtained using both proposed algorithms. We set $a = 0.2$, $\alpha = 2.4$ and $\beta = 0.34$ and we use 10^6 independent realizations.

and it is then easy to verify that

$$Y \stackrel{d}{=} \left(\frac{1}{a}\right)^U, \quad U \sim \mathcal{U}([0,1]),$$

which simplifies the simulation of the skeleton of the process Z_a^{++} as illustrated in Algorithm 1.

Algorithm 1 Simulation of $Z_a(t)$.

1: Simulate $n \sim \mathcal{P}(\alpha t \log(1/a))$
2: Simulate n *iid rv*'s $u_i \sim \mathcal{U}([0,1])$ and set $y_i = \left(\frac{1}{a}\right)^{u_i}$
3: Simulate n *iid rv*'s $J_i \sim \mathcal{E}(\beta\, y_i)$.
4: Set $Z_a^{++}(t) = \sum_{i=0}^{n} J_i$.

Alternatively, as shown in Sabino and Cufaro-Petroni [108] the skeleton of Z_a^{++} can be simulated as a stochastic sum of independent exponentially distributed *rv*'s with parameter β/a where the number of terms is given by $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$ as summarized in Algorithm 2. Therefore, we derived two different equivalent algorithms

Algorithm 2 Simulation of $Z_a(t)$.

1: Simulate $s \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$.
2: Set $Z_a^{++}(t) \sim \mathcal{E}_s(\beta/a)$.

to simulate the process Z_a^{++}. A possible realization of the process is shown in Figure 3.1. In Table 3.2 we summarize the performance of both Algorithms 1 and 2: we can conclude that both procedures are equally efficient in time. In Table 3.1 we compare the numerical mean, variance, skewness and kurtosis of a sample of 10^6 points obtained using both Algorithms 1 and 2 with the theoretical value: we can conclude that both algoritms produces accurate results.

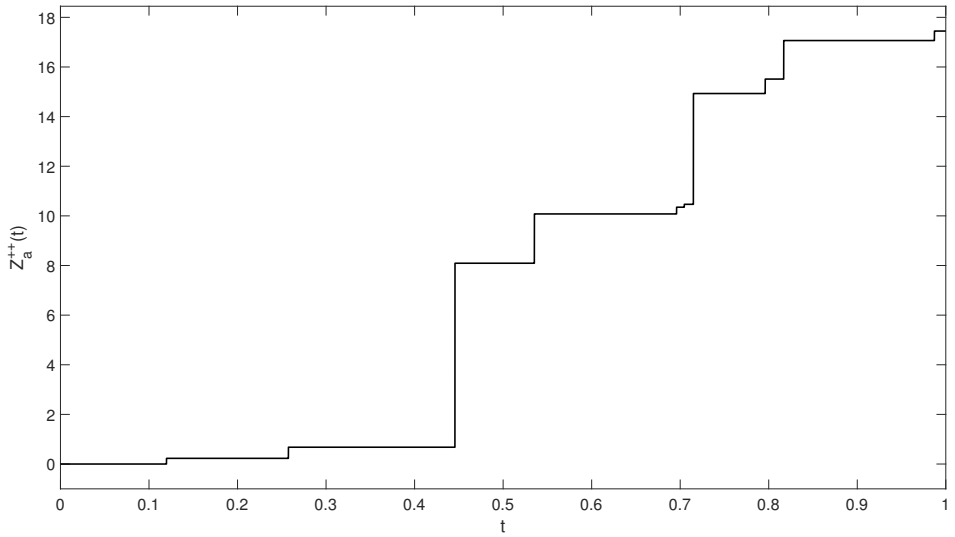

Figure 3.1. A possible path of the Z_a^{++} process with parameter $\alpha = 5.4$, $\beta = 0.34$ and $a = 0.2$ for $t \in [0, T]$ with $T = 1$.

Algorithm	Computational time (s)
Algorithm 1	1.4439
Algorithm 2	1.2048

Table 3.2. Computational time to simulate $N_{sim} = 10^6$ realizations of $Z_a^{++}(t)$. The resulting time is the average of 500 runs of both Algorithms.

3.2 Variance Gamma++ process

In Section 3.1.1 we have shown that Z_a^{++} is a subordinator and hence can be used to time change a Brownian motion following the ideas of Section 2.6.

Definition 3.2.1. *Consider a Brownian motion $W = \{W(t); t \geq 0\}$, with drift $\theta \in \mathbb{R}$, diffusion $\sigma \in \mathbb{R}^+$ independent of Z_a^{++}. We call the process $X = \{X(t); t \geq 0\}$ defined as*

$$X(t) = \theta Z_a^{++}(t) + \sigma W\left(Z_a^{++}(t)\right), \qquad t \geq 0 \tag{3.5}$$

$VG++$ process.

In the following we detail its properties.

Proposition 3.2.1. *For $u \in \mathbb{R}$, the chf of X at time t is given by:*

$$\phi_{X(t)}(u) = \phi_{Z_a^{++}(t)}\left(\theta u + iu^2\frac{\sigma^2}{2}\right) = \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right)a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)}\right)^{\alpha t}. \tag{3.6}$$

Proof. We have that the *chf* of $Z_a^{++}(t)$ is given by:

$$\phi_{Z_a^{++}}(u) = \exp\left\{t\log\left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}\right\}, \tag{3.7}$$

and hence:

$$\begin{aligned}
\mathbb{E}\left[e^{iuX(t)}\right] &= \mathbb{E}\left[e^{iu\left(\theta Z_a^{++}(t) + \sigma W(Z_a(t))\right)}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iu\left(\theta Z_a^{++}(t) + \sigma W(Z_a(t))\right)}\middle| Z_a^{++}(t)\right]\right] \\
&= \mathbb{E}\left[e^{iu\theta Z_a^{++}(t) - \frac{\sigma^2}{2}u^2 Z_a^{++}(t)}\right] = \mathbb{E}\left[e^{i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)Z_a^{++}(t)}\right] = \phi_{Z_a^{++}(t)}\left(u\theta + i\frac{\sigma^2}{2}u^2\right) \\
&= \exp\left\{\log\left(\frac{\beta - i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)a}{\beta - i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t}\right\} = \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right)a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)}\right)^{\alpha t}.
\end{aligned}$$

that concludes the proof. ■

Proposition 3.2.2. *The $VG++$ process can be written as difference of two independent processes $Z_{a_p}^{++} = \left\{Z_{a_p}^{++}(t); t \geq 0\right\}$ and $Z_{a_n}^{++} = \left\{Z_{a_n}^{++}(t); t \geq 0\right\}$ where $Z_{a_p}^{++}(t) \sim \Gamma^{++}(a_p, \alpha t, \beta_p)$ and $Z_{a_n}^{++}(t) \sim \Gamma^{++}(a_n, \alpha t, \beta_n)$.*

Proof. Given the definition of the *chf* of $X(t)$, it results

$$\phi_{X(t)}(u) = \phi_{Z_a^{++}}\left(u\theta + \frac{iu^2\sigma^2}{2}\right) = \frac{\left(\frac{1}{1 - \frac{i}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t}}{\left(\frac{1}{1 - \frac{ia}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t}} = \frac{A}{B}.$$

Consider the term A:

$$A = \left(\frac{1}{1 - \frac{i}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t} = \left(\frac{1}{1 - \frac{iu}{\beta_p}}\right)^{\alpha t}\left(\frac{1}{1 + \frac{iu}{\beta_n}}\right)^{\alpha t},$$

and its denominator

$$1 - iu\frac{\theta}{\beta} - i^2u^2\frac{\sigma^2}{2\beta} = 1 - iu\left(\frac{1}{\beta_p} - \frac{1}{\beta_n}\right) - iu^2\frac{1}{\beta_p\beta_n}.$$

It turns out then:

$$\frac{\theta}{\beta} = \frac{1}{\beta_p} - \frac{1}{\beta_n}, \qquad \frac{1}{\beta_p\beta_n} = \frac{\sigma^2}{2\beta}.$$

By solving the previous system of equations with respect to β_p and β_n and taking only the positive solution we have that:

$$\beta_n = \frac{\sqrt{\theta^2 + 2\sigma^2\beta} + \theta}{\sigma^2}, \qquad \beta_p = \frac{\sqrt{\theta^2 + 2\sigma^2\beta} - \theta}{\sigma^2}.$$

Similarly, the term B can be decomposed as:

$$\tilde{\beta}_n = \frac{\sqrt{\theta^2 + 2\sigma^2\beta/a} + \theta}{\sigma^2}, \qquad \tilde{\beta}_p = \frac{\sqrt{\theta^2 + 2\sigma^2\beta/a} - \theta}{\sigma^2}.$$

It follows that:

$$\phi_{X(t)} = \frac{\left(\frac{1}{1 - iu/\beta_p}\right)^{\alpha t}\left(\frac{1}{1 + iu/\beta_n}\right)^{\alpha t}}{\left(\frac{1}{1 - iu/\tilde{\beta}_p}\right)^{\alpha t}\left(\frac{1}{1 + iu/\tilde{\beta}_n}\right)^{\alpha t}} = \left(\frac{1 - iu\left(\frac{\beta_p}{\tilde{\beta}_p}\right)/\beta_p}{1 - \frac{iu}{\beta_p}}\right)^{\alpha t}\left(\frac{1 + iu\left(\frac{\beta_n}{\tilde{\beta}_n}\right)/\beta_n}{1 + \frac{iu}{\beta_n}}\right)^{\alpha t} \tag{3.8}$$

Because $0 < \beta_p/\tilde{\beta}_p < 1$ we can define $a_p = \beta_p/\tilde{\beta}_p$ and $a_n = \beta_n/\tilde{\beta}_n$ and we obtain:

$$\phi_{X(t)}(u) = \left(\frac{1 - iua_p/\beta_p}{1 - iu/\beta_p}\right)^{\alpha t}\left(\frac{1 + iua_n/\beta_n}{1 + iu/\beta_n}\right)^{\alpha t}$$

which is the *chf* of the difference of two independent *rv*'s $Z^{++}_{a_p}(t) \sim \Gamma^{++}_{a_p}(\alpha t, \beta_p)$ and $Z^{++}_{a_n}(t) \sim \Gamma^{++}_{a_n}(\alpha t, \beta_n)$. Therefore the process X can be expressed as difference of two independent subordinators $Z^{++}_{a_p} = \left\{Z^{++}_{a_p}(t); t \geq 0\right\}$ and $Z^{++}_{a_n} = \left\{Z^{++}_{a_n}(t); t \geq 0\right\}$. ■

As a simple consequence of Proposition 3.2.2 and Proposition 3.1.3 we have that the Lévy measure of the $VG++$ process X is given by:

$$\begin{aligned}\nu(x) = &\left(\alpha x^{-1}e^{-x\beta_p} - \alpha x^{-1}e^{-x\beta_p/a_p}\right)\mathbb{1}_{(0,\infty)}(x) \\ &+ \left(-\alpha x^{-1}e^{x\beta_n} + \alpha x^{-1}e^{x\beta_n/a_n}\right)\mathbb{1}_{(-\infty,0]}(x).\end{aligned} \tag{3.9}$$

The process X is of finite activity and therefore of finite variation.

Since the $VG++$ process is of finite activity and therefore it is a Compound Poisson process. By Cont and Tankov [42, Proposition 3.5] we have that the Lévy measure $\nu(dx)$ of a compound Poisson is of the form $\nu(dx) = \lambda f(dx)$ where λ is the intensity of the Poisson process and $f(dx)$ is the jumps' density. The Lévy measure in Equation 3.9 can be rewritten as:

$$\nu(dx) = \underbrace{\alpha\left[\log(\frac{1}{a_p}) + \log(\frac{1}{a_n})\right]}_{\lambda}$$
$$\underbrace{\frac{dx}{x\left[\log(\frac{1}{a_p}) + \log(\frac{1}{a_n})\right]}\left[\frac{e^{-\beta_p x} - e^{-\beta_p x/a_p}}{x}\mathbb{1}_{(0,\infty)} - \frac{e^{\beta_n x} - e^{\beta_n x/a_n}}{x}\mathbb{1}_{(-\infty,0)}\right]}_{f(dx)}.$$

Therefore, the $VG++$ X process at time t is such that:

$$X(t) \stackrel{d}{=} \sum_{i=1}^{N(t)} Y_i,$$

where $N(t) \sim \mathcal{P}(\alpha t\left[\log(\frac{1}{a_p}) + \log(\frac{1}{a_n})\right])$ and Y_i are *iid* random variables with density given by $f(x)$. The $VG++$ process can be simulated as a compound Poisson process. Sampling from the distribution of $N(t)$ is straightforward whereas realizations of the random variable Y_i with *pdf* $f(x)$ can be obtained by following the approach of point *iii)* of Proposition 3.1.2 and by observing that:

$$\mathbb{P}(Y_i \geq 0) = \frac{\log(1/a_p)}{\log(1/_{a_p}) + \log(1/a_n)}.$$

Algorithm 3 summarize how to simulate realizations of Y_i.

We recall that the cumulant generating function $\psi_Y(u)$ and the cumulants of a *rv* Y with *chf* $\phi_Y(u)$ are defined, respectively, as:

$$\psi_Y(0) = 0, \quad \phi_Y(u) = e^{\psi_Y(u)},$$

$$c_n(X) = \frac{1}{i^n}\frac{\partial^n \psi_X}{\partial u^n}(0).$$

Proposition 3.2.3. *The first four cumulants of the process X at time $t \geq 0$ are*

Algorithm 3 Simulation of Y with *pdf* $f(x)$.

1: Simulate $w \sim \mathcal{U}([0,1])$.
2: Simulate $u \sim \mathcal{U}([0,1])$.
3: Set $p = \frac{\log(1/a_p)}{\log(1/a_p)+\log(1/a_n)}$.
4: **if** $w \leq p$ **then**
5: Set $y = (1/a_p)^u$.
6: Simulate $J \sim \mathcal{E}(\beta_p y)$.
7: Return $Y = J$.
8: **else if** $w > p$ **then**
9: Set $y = (1/a_n)^u$.
10: Simulate $J \sim \mathcal{E}(\beta_n y)$.
11: Return $Y = -J$
12: **end if**

given by:

$$
\begin{aligned}
c_1(X(t)) &= \mathbb{E}[X(t)] = \alpha t \left(\frac{1}{\beta_p} - \frac{1}{\tilde{\beta}_p} - \frac{1}{\beta_n} + \frac{1}{\tilde{\beta}_n} \right), \\
c_2(X(t)) &= Var[X(t)] = \alpha t \left(\frac{1}{\beta_p^2} - \frac{1}{\tilde{\beta}_p^2} + \frac{1}{\beta_n^2} - \frac{1}{\tilde{\beta}_n^2} \right), \\
c_3(X(t)) &= 2\alpha t \left(\frac{1}{\beta_p^3} - \frac{1}{\tilde{\beta}_p^3} - \frac{1}{\beta_n^3} + \frac{1}{\tilde{\beta}_n^3} \right), \\
c_4(X(t)) &= 6\alpha t \left(\frac{1}{\beta_p^4} - \frac{1}{\tilde{\beta}_p^4} + \frac{1}{\beta_n^4} - \frac{1}{\tilde{\beta}_n^4} \right),
\end{aligned}
$$

where β_p, $\tilde{\beta}_p$, β_n, $\tilde{\beta}_n$ are defined in Proposition 3.2.2.

Proof. Using Cont and Tankov [42, Proposition 13.3] and Proposition 3.1.2, it results that if the law of Y is *sd* the n-th cumulant of the a-remainder Z_a is:

$$
c_n(Z_a) = t \int_{-\infty}^{\infty} x^n \nu_a(x)\, dx = (1 - a^n)\, c_n(Y), \tag{3.10}
$$

where $\nu_a(x)$ is the Lévy measure of the process Z_a.
Moreover, it is easy to prove that for two independent *rv*'s X and Y with finite cumulants of order n, taking $U = X - Y$, it holds:

$$
c_n(U) = c_n(X) + (-1)^n c_n(Y). \tag{3.11}
$$

Combining (3.10) and (3.11) and the fact that from Proposition 3.2.2 the $VG++$ process can be written as the difference of two independent subordinators Z_{a_p} and Z_{a_n} it results

$$
c_n(X(t)) = \left(1 - a_p^n\right) c_n(G_1(t)) + (-1)^n \left(1 - a_n^n\right) c_n(G_2(t)),
$$

where $G_1 = \{G_1(t); t \geq 0\}$ and $G_2 = \{G_2(t); t \geq 0\}$ are Gamma processes with parameters (α, β_p) and (α, β_n) respectively. The proof is simply concluded recalling the expression of the cumulants of the gamma laws $\Gamma(\alpha t, \beta_p)$ and $\Gamma(\alpha t, \beta_n)$, respectively:

$$c_n(G_1(t)) = (n-1)!\frac{\alpha t}{\beta_p^n},$$
$$c_n(G_2(t)) = (n-1)!\frac{\alpha t}{\beta_n^n}.$$

■

Proposition 3.2.4. *The pdf of the $VG++$ process $X = \{X(t); t \geq 0\}$ at $t \geq 0$ is given by:*

$$f_{X(t)}(x) = a^{\alpha t}\delta_0(x) + \sum_{k\geq 1}\binom{\alpha t + k - 1}{k} a^{\alpha t}(1-a)^k f^{VG}_{k,\beta/a}(x). \tag{3.12}$$

where $\delta_0(x)$ is the Dirac function and $f^{VG}_{k,\beta/a}(x)$ is the pdf of a Variance Gamma law with parameters $k \in \mathbb{N}$ and β/a which is given by:

$$f^{VG}_{k,\beta/a}(x) = K_{k-\frac{1}{2}}\left(|x|\frac{\sqrt{2\sigma^2\beta/a + \theta^2}}{\sigma^2}\right)\frac{\exp(\theta x/\sigma^2)}{\sqrt{2\pi\sigma^2}}\frac{(\beta/a)^k}{\Gamma(k)}\left(2\sigma^2\beta + \theta^2\right)^{\frac{1}{4}-\frac{k}{2}} 2|x|^{k-\frac{1}{2}}.$$

Proof. From Equation (3.6) we have that:

$$\begin{aligned}\phi_{X(t)}(u) &= \left(\frac{\beta - i(\theta u + iu^2\sigma^2/2)a}{\beta - i(\theta u + iu^2\sigma^2/2)}\right)^{\alpha t} = \left(\frac{a}{1-(1-a)\frac{\beta}{\beta - ia(\theta u + iu^2\sigma^2/2)}}\right)^{\alpha t} \\ &= \sum_{k=0}^{\infty}\binom{\alpha t + k - 1}{k} a^{\alpha t}(1-a)^k\left(\frac{\beta}{\beta - ia(\theta u + iu^2\sigma^2/2)}\right)^k \\ &= a^{\alpha t} + \sum_{k\geq 1}\binom{\alpha t + k - 1}{k} a^{\alpha t}(1-a)^k\left(\frac{\beta}{\beta - ia(\theta u + iu^2\sigma^2/2)}\right)^k.\end{aligned} \tag{3.13}$$

One can notice that $X(t)$ is a mixture of Variance Gamma *rv*'s where the weights are given by a Polya distribution plus a degenerate distribution at $x = 0$. By taking the inverse Fourier transform of (3.13) we get the *pdf* in (3.12).

■

Remark. For $n \in \mathbb{N}$ the modified Bessel function of the second kind $K_{n+\frac{1}{2}}(x)$ can be written in terms of elementary functions (see Abramowitz and Stegun [1, pag. 443]):

$$\sqrt{\frac{\pi}{2x}}K_{n+\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)e^{-x}\sum_{k=0}^{n}\frac{(n+k)!}{k!\Gamma(n-k+1)}(2x)^{-k}.$$

This fact is instrumental to obtain an efficient formula for the pricing of an European call option when the evolution of the market is modelled by a Variance Gamma process with $\frac{t}{\nu} \in \mathbb{N}$ and, as we shall show, by a $VG++$ process as well.

Proposition 3.2.5. *Consider the $VG++$ process X and let S be a Polya process such that $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$. In addition let $(I_k)_{k\geq 1}$ and $(J_k)_{k\geq 1}$ be two independent sequences of iid rv's, with $I_k \sim \mathcal{E}\left(\tilde{\beta}_p\right)$, $J_k \sim \mathcal{E}\left(\tilde{\beta}_n\right)$ where $\tilde{\beta}_n$ and $\tilde{\beta}_p$ are defined in Equation (3.8). Finally take $\delta_k = I_k - J_k$ and define the process $C = \{C(t); t \geq 0\}$ as:*

$$C(t) = \sum_{k=0}^{S(t)} \delta_k, \quad C(t) = 0 \text{ when } S(t) = 0.$$

Then:

$$X(t) \stackrel{d}{=} C(t),\ t > 0.$$

Proof. First we prove that the $VG++$ process at time t can written as a Polya sum of independent *rv*'s. For $u \in \mathbb{R}$, consider the *chf* $\phi_{X(t)}(u)$ at time t of the $VG++$ process given in (3.6) and define $g(u) = i(\theta u + iu^2\sigma^2/2)$. We have:

$$\phi_{X(t)}(u) = \left(\frac{1}{\frac{\beta - g(u)}{\beta - ag(u)}}\right)^{\alpha t} = \left(\frac{a}{\frac{a\beta+\beta-\beta a g(u)}{\beta-ag(u)}}\right)^{\alpha t} = \left(\frac{a}{1-(1-a)\frac{\beta}{\beta-ag(u)}}\right)^{\alpha t}$$
$$\stackrel{a=1-p}{=} \left(\frac{1-p}{1-p\frac{1}{1-\frac{a}{\beta}g(u)}}\right)^{\alpha t} = \left(\frac{1-p}{1-p\varphi(u)}\right)^{\alpha t},$$

where:

$$\varphi(u) = \frac{1}{1-\frac{a}{\beta}g(u)} = \frac{\beta/a}{\beta/a - iu\theta + u^2\sigma^2/2}.$$

Therefore, $X(t)$ can be represented as a Polya sum of independent *rv*'s whose *chf* is given by $\varphi(u)$. We can write:

$$\varphi(u) = \frac{1}{1 - \frac{iua\theta}{\beta} - \frac{i^2u^2\sigma^2}{2}}$$

and the denominator can be decomposed as:

$$1 - \frac{iua\theta}{\beta} - \frac{i^2u^2\sigma^2}{2} = \left(1 - \frac{iu}{\tilde{\beta}_p}\right)\left(1 + \frac{iu}{\tilde{\beta}_n}\right) = 1 - iu\left(\frac{1}{\tilde{\beta}_p} - \frac{1}{\tilde{\beta}_n}\right) - i^2u^2\frac{1}{\tilde{\beta}_p\tilde{\beta}_n}.$$

Taking:

$$\frac{1}{\tilde{\beta}_p} - \frac{1}{\tilde{\beta}_n} = \frac{a\theta}{\beta}, \qquad \frac{1}{\tilde{\beta}_p\tilde{\beta}_n} = \frac{2\beta}{a\sigma^2},$$

solving with respect to $\tilde{\beta}_n$ and $\tilde{\beta}_p$ and considering only positive solutions we have:

$$\tilde{\beta}_p = \frac{\sqrt{\theta^2 + 2\sigma^2 - \beta/a} - \theta^2}{\sigma^2}, \quad \tilde{\beta}_n = \frac{\sqrt{\theta^2 + 2\sigma^2 - \beta/a} + \theta^2}{\sigma^2}.$$

Model name	Z_a^{++} process	$VG++$ process X
Model type	Finite variation Finite activity Subordinator	Finite variation Finite activity
Parameters	$\alpha > 0$ shape, $\beta > 0$ rate and $a \in (0,1)$ *sd*	$\alpha, \beta, a + \theta$ drift and σ diffusion of the Brownian motion
Lévy measure	$\nu_a(x) = \frac{\alpha}{x}\left(e^{-\beta x} - e^{-\beta x/a}\right)\mathbb{1}_{(0,\infty)}(x)$	$\nu(x) = \left(\alpha x^{-1}e^{-x\beta_p} - \alpha x^{-1}e^{-x\beta_p/a_p}\right)\mathbb{1}_{(0,\infty)}(x)$ $+\left(-\alpha x^{-1}e^{x\beta_n} + \alpha x^{-1}e^{x\beta_n/a_n}\right)\mathbb{1}_{(-\infty,0]}(x)$
chf	$\phi_{Z_a(t)}(u) = \left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha t}$	$\phi_{X(t)}(u) = \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right)a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)}\right)^{\alpha t}$
pdf	$g_a(x) = a^{\alpha}\delta_0(x)$ $+\sum_{n\geq 1}\binom{\alpha+n-1}{n}a^{\alpha}(1-a)^n$ $\cdot f_{n,\beta/a}(x)\,\mathbb{1}_{(0,\infty)}(x)\,dx$ where $f_{n,\beta/a}(x)$ is the density of an Erlang distribution.	$f_{X(t)}(x) = a^{\alpha t}\delta_0(x)$ $+\sum_{n\geq 1}\binom{\alpha t+n-1}{n}a^{\alpha t}(1-a)^n$ $\cdot f^{VG}_{n,\beta/a}(x)\,dx$ where $f^{VG}_{n,\beta/a}(x)$ is the density of a Variance Gamma distribution.
Cumulants	$c_1\left(Z_a^{++}(t)\right) = \alpha t\frac{1-a}{\beta}$, $c_2\left(Z_a^{++}(t)\right) = \alpha t\frac{1-a^2}{\beta^2}$, $c_3\left(Z_a^{++}(t)\right) = 2\alpha t\frac{1-a^3}{\beta^3}$, $c_4\left(Z_a^{++}(t)\right) = 6\alpha t\frac{1-a^4}{\beta^4}$.	$c_1(X(t)) = \alpha t\left(\frac{1}{\beta_p} - \frac{1}{\tilde{\beta}_p} - \frac{1}{\beta_n} + \frac{1}{\tilde{\beta}_n}\right)$, $c_2(X(t)) = \alpha t\left(\frac{1}{\beta_p^2} - \frac{1}{\tilde{\beta}_p^2} + \frac{1}{\beta_n^2} - \frac{1}{\tilde{\beta}_n^2}\right)$, $c_3(X(t)) = 2\alpha t\left(\frac{1}{\beta_p^3} - \frac{1}{\tilde{\beta}_p^3} - \frac{1}{\beta_n^3} + \frac{1}{\tilde{\beta}_n^3}\right)$, $c_4(X(t)) = 6\alpha t\left(\frac{1}{\beta_p^4} - \frac{1}{\tilde{\beta}_p^4} + \frac{1}{\beta_n^4} - \frac{1}{\tilde{\beta}_n^4}\right)$. with $\tilde{\beta}_n, \tilde{\beta}_p, \beta_n, \beta_p$ as in Proposition 3.2.2.

Table 3.3. Characterization of Z_a^{++} and of the $VG++$ process.

Finally, $\varphi(u)$ can be written as:

$$\varphi(u) = \frac{1}{1 - \frac{iu}{\tilde{\beta}_p}} \cdot \frac{1}{1 + \frac{iu}{\tilde{\beta}_n}},$$

which is the *chf* of the difference of two independent exponentially distributed *rv*'s with parameters $\tilde{\beta}_p$ and $\tilde{\beta}_n$ respectively.
By computing the *chf* of $C(t)$ it is easy to check that:

$$\phi_{C(t)}(u) = \phi_{X(t)}(u),$$

that means that $X(t) \stackrel{d}{=} C(t)$ which concludes the proof. ■

Finally, Table 3.3 summarizes the properties of the processes Z_a^{++} and $VG++$.

3.2.1 An option pricing formula under $VG++$ model

Following Cont and Tankov [42], we model the evolution of a risky asset by the process $F = \{F(t); t \geq 0\}$ defined as

$$F(t) = F(0)\, e^{rt+\omega t+\theta Z_a^{++}(t)+\sigma W\left(Z_a^{++}(t)\right)} = F(0)\, e^{rt+\omega t+X(t)}, \tag{3.14}$$

where:

$$\omega = \log\left(\frac{\beta - (\theta+\sigma^2/2)}{\beta - a\,(\theta+\sigma^2/2)}\right)^{\alpha},$$

to have non-arbitrage conditions.

The following proposition provides a closed formula for the price of a European call option.

Proposition 3.2.6. *Consider the market model of Equation* (3.14) *where X is a $VG++$ process, the price at time* 0 *of a European call option with strike price K and maturity T is given by:*

$$C(0,K) = C(0)\, a^{\alpha T} + \sum_{n\geq 1}\binom{\alpha T+n-1}{n}(1-a^n)\, a^{\alpha T} C^{VG}_{n,\beta/a}(0,K), \tag{3.15}$$

where

$$C(0) = \max\left(F(0)e^{\omega T} - e^{-rT}K, 0\right)$$

and $C^{VG}_{n,\beta/a}(0,K)$ is the price of a call option with strike K and maturity T under the Variance Gamma model with parameters n and β/a.

Proof. Consider $X(T) = \theta Z_a^{++}(T) + \sigma W(Z_a^{++}(T))$ whose *pdf* $f_{X(T)}(x)$ is given by Equation (3.12). The value of the call option at $t = 0$ is the discounted expected value under the risk-neutral measure:

$$\begin{aligned}
C(0,T) &= e^{-rT}\mathbb{E}\left[(F(T)-K)^+\right] = e^{-rT}\int_{-\infty}^{\infty}\left(F(0)e^{rT+\omega T+x}-K\right)^+ f_{X(T)}(x)dx \\
&= e^{-rT}\Bigg(\int_{-\infty}^{\infty}\left(FS(0)e^{rT+\omega T+x}-K\right)^+ a^{\alpha T}\delta_0(x)\,dx \\
&\int_{-\infty}^{\infty}\left(F(0)e^{rT+\omega Tx}-K\right)^+ \cdot\left(\sum_{n\geq 1}\binom{\alpha T+n-1}{n}a^{\alpha T}(1-a)^n f^{VG}_{n,\beta/a}(x)\right)dx\Bigg) \\
&= \underbrace{a^{\alpha T}\left(F(0)e^{\omega T}-e^{-rT}K\right)^+}_{C(0)} \\
&+ \sum_{n\geq 1}\binom{\alpha T+n-1}{n}a^{\alpha T}(1-a)^k \underbrace{\int_{-\infty}^{\infty}\left(F(0)e^{rT+\omega T+x}-K\right)^+ f_{k,\beta/a}(x)dx}_{C^{VG}_{n,\beta/a}(0,T)}
\end{aligned}$$

where in the last step we used the monotone convergence theorem to interchange the order of the integral and the summation. ■

Shape parameter domain	Computational time (s)
$\mathbb{N}$	$7.61 \cdot 10^{-7}$
$\mathbb{R}$	$3.02 \cdot 10^{-3}$

Table 3.4. Computational time to price a European option if the shape parameter is a real or a natural number.

Remark. The option price in Equation (3.15) can be computed in very efficient way using the results about EPT-distributions discussed in Sexton and Hanzon [116] and summarized in Appendix B.4. Indeed, when the shape parameter $n \in \mathbb{N}$, the computation of $C^{VG}_{n,\beta/a}(0,T)$ is easier than when it is a real number. This fact directly stems from what we observed in Remark 3.2, namely that the Bessel function $K_n(x)$ can be written as a sum of elementary functions when $n \in \mathbb{N}$. The advantage is that one does not need to compute any integral when we evaluate $C^{VG}_{n,\beta/a}(0,T)$ because this term can be simply obtained as matrix multiplications which are faster than numerical integration.

Table 3.4 shows the comparison of the computational times required to price a call option when the shape parameter is either an integer or a positive real number using *MATLAB* on a PC with an Intel Core i5-10210U 2.11 GHz processor. Apparently, the computation taking an integer shape parameter is four times faster.

3.2.2 $VG++$ backward simulation

So far, we have presented algorithms for the simulation of the trajectories of the $VG++$ process forward in time over a given time grid $t_0, t_1, \ldots, t_d$. On the other hand, we are not restricted to generate the random points of the trajectory in sequence, the only strict requirement is to generate points with the correct transition density.

In this section we illustrate how to simulate the $VG++$ process backward in time taking advantage of the notion of *Lévy random bridges* (see Hoyle [71] for details) which are stochastic processes pinned to a fixed point at a fixed future time. Applications of Lévy bridge-based techniques in finance are for instance, the pricing with Monte Carlo (MC) methods of barrier options with continuous monitoring to avoid the bias arising by the use of the Euler discretization scheme, or the combination with Quasi-Monte Carlo methods (see for instance Caflisch et al. [34] and Glasserman [62]).

Lévy bridges naturally lead to the construction of *backward simulations* as described in Pellegrino and Sabino [100], Hu and Zhou [72] and Sabino [106]. In principle, the computational cost of backward and forward strategies is the same, however numerical analysis showed that in most cases the forward construction is the faster solution (see Sabino [106]).

On the other hand, the path generation is only one component of the overall

pricing of derivative contracts with MC simulations. When the pricing of contracts with complex American optionality is based on the Least Squares Monte Carlo (LSMC) approach introduced by Longstaff and Schwartz [83], what matters in the stochastic dynamic programming is the comparison between the intrinsic value and the continuation value at a given time step t. If, for instance, we consider a F-factor market model and we want to price an American option with LSMC, each step of the Bellman backward recurrence requires to know the simulated prices or indices at two consecutive times t and $t+\Delta$, nothing else. To this end, the forward generation requires storing $d \times N \times F$ numbers where d is the number of time steps and N is the number of simulations, whereas the backward solution requires storing a far lower number, $2 \times N \times F$. The forward construction may become computationally unfeasible for contracts with long maturities in contrast, although sometimes slower, the backward construction is more reliable because one could generate a far higher number of trajectories that is often necessary for the computation of the Greek letters.

In order to conceive a backward simulation scheme for the $VG++$ process we start showing how to simulate the process Z_a^{++} backward in time. Indeed, the backward simulation of the $VG++$ will then consists of applying the well-known backward simulation of a Brownian motion on the stochastic grid generated by Z_a^{++}.

Proposition 3.2.7 (Polya Bridge)**.** *Consider a process $S = \{S(t); t \geq 0\}$ such that $S(0) = 0$ a.s. and $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$. For $0 < t \leq T$, define the rv $S_{tT}^{(k)}$, $k \in \mathbb{N}$ with probability mass function:*

$$\mathbb{P}\left(S_{tT}^{(k)} = j\right) := \mathbb{P}\left(S(t) = j \middle| S(T) = k\right).$$

It results:

$$\mathbb{P}\left(S_{tT}^{(k)} = j\right) = \binom{k}{j} \frac{\mathrm{B}\left(\alpha t + j, \alpha\left(T-t\right) + k - j\right)}{\mathrm{B}\left(\alpha t, \alpha\left(T-t\right)\right)}$$

namely, $S_{tT}^{(k)}$ is distributed according to a beta-binomial law $\mathcal{B}\left(\alpha t, \alpha\left(T-t\right), k\right)$ where $\mathrm{B}(\alpha, \beta)$ denotes the Beta function (see Abramowitz and Stegun [1]).

Proof. Knowing that S has independent and stationary increments, the proof is verified as follows:

$$\begin{aligned}
\mathbb{P}\left(S_{tT}^{k} = j\right) &= \frac{P\left(S(t) = j, S(T) = k\right)}{\mathbb{P}\left(S(T) = k\right)} = \frac{\mathbb{P}\left(S(t) = j\right)\mathbb{P}\left(S(T-t) = k-j\right)}{\mathbb{P}\left(S(T) = k\right)} \\
&= \binom{k}{j} \frac{\Gamma\left(\alpha t + j\right)}{\Gamma\left(\alpha t\right)} \frac{\Gamma\left(\alpha(T-t) + k - j\right)}{\Gamma\left(\alpha(T-t)\right)} \frac{\Gamma\left(\alpha T\right)}{\Gamma\left(\alpha T + k\right)} \\
&= \binom{k}{j} \frac{\mathrm{B}\left(\alpha t + j, \alpha(T-t) + k - j\right)}{\mathrm{B}\left(\alpha t, \alpha(T-t)\right)},
\end{aligned}$$

where we used the relations:

$$(\alpha t+j-1)(\alpha t+j-2)\dots\alpha t=\frac{\Gamma(\alpha t+j)}{\Gamma(\alpha t)},\qquad \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\mathrm{B}(x,y).$$

■

Based on Proposition 3.2.2 we can show that the process Z_a^{++} is a gamma process G subordinated by a Polya process S. This simple fact provides us with an easy way to simulate the process Z_a^{++}.

Proposition 3.2.8. *Consider a gamma process $G=\{G(t);t\geq 0\}$, such that $G(t)\sim\Gamma(t,\beta/a)$, $\beta>0$, $a\in(0,1)$, and a Polya process $S=\{S(t);t\geq 0\}$ such that $S(t)\sim\overline{\mathfrak{B}}(\alpha t,1-a)$. Define the process $Y=\{Y(t);t\geq 0\}$ as:*

$$Y(t)=G(S(t)),\qquad t\geq 0.$$

It results:

$$Z_a^{++}(t)\overset{d}{=}Y(t),\qquad t\geq 0,$$

where Z_a^{++} is the Lévy process associated to the a-remainder of a gamma law with parameters α and β, as defined in (3.4).

Proof. We compute the *chf* of $Y(t)$ for $u\in\mathbb{R}$.

$$\begin{aligned}\mathbb{E}\left[e^{iuY(t)}\right]&=\mathbb{E}\left[\mathbb{E}\left[e^{iuG(S(t))}\middle|S(t)\right]\right]=\mathbb{E}\left[\left(\frac{\beta}{\beta-iua}\right)^{S(t)}\right]\\&=\mathbb{E}\left[\left(\frac{\beta/a}{\beta/a-iu}\right)^{S(t)}\right].\end{aligned}\tag{3.16}$$

From Equation (3.2) we have that

$$Z_a^{++}(t)=\sum_{n=0}^{S(t)}E_n,$$

where E_n are *iid rv*'s with exponential law with parameter β/a. The *chf* of $Z_a^{++}(t)$ is given by:

$$\begin{aligned}\mathbb{E}\left[e^{iuZ_a^{++}(t)}\right]&=\mathbb{E}\left[e^{iu\sum_{n=0}^{S(t)}E_n}\right]=\mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{n=0}^{S(t)}E_n}\middle|S(t)\right]\right]\\&=\mathbb{E}\left[\prod_{n=0}^{S(t)}\mathbb{E}\left[e^{iuE_1}\right]\right]=\mathbb{E}\left[\prod_{n=0}^{S(t)}\frac{\beta/a}{\beta/a-iu}\right]=\mathbb{E}\left[\left(\frac{\beta/a}{\beta/a-iu}\right)^{S(t)}\right]\end{aligned}$$

which is the same as Equation (3.16), therefore we can conclude that $Z_a(t)\overset{d}{=}Y(t)$. ■

Proposition 3.2.8 illustrates how to simulate the process Z_a^{++} backward in time: firstly, one simulates Polya process S backward in time, and secondly one simulates the gamma process G backward in time on the stochastic time grid generated by S (see Sabino [106] for the backward simulation of a gamma process).

Assume, indeed, that given $Z_a^{++}(0) = 0$ the value of the process Z_a^{++} at time T is equal to z_T, then $Z_a^{++}(t)$, $t \in (0,T)$ can be simulated by generating the Polya bridge at time t in the first step and the gamma bridge at a random time $S(t) \in (0, S(T))$ in the second step. This procedure is summarized in Algorithm 4.

Algorithm 4 Backward simulation of Z_a.

1: Generate $s_T \sim \overline{\mathfrak{B}}(\alpha T, 1-a)$.
2: Generate $z_T \sim \Gamma(s_T, b/a)$ and set $Z_a^{++}(T) = z_T$.
3: Consider $t \in (0,T)$ and $p \sim Beta(\alpha t, \alpha(T-t))$.
4: Simulate $s_t \sim Bin(s_T, p)$.
5: Simulate $\beta \sim Beta(s_t, s_T - s_t)$.
6: Set $Z_a^{++}(t) = z_T\,\beta$.

In a similar way, the backward simulation of the $VG++$ process can be accomplished implementing the backward simulation of the Brownian motion oven a random grid given by the backward simulation of Z_a^{++} as illustrated in Algorithm 5.

Algorithm 5 Backward simulation of X.

1: Set $X(0) = 0$ and $Z_a^{++}(0) = 0$.
2: Simulate $Z_a^{++}(T)$ and $Z_a^{++}(t)$ using Algorithm 4.
3: Simulate $x_T \sim \mathcal{N}(\theta Z_a^{++}(T), \sigma^2 Z_a^{++}(T))$.
4: Simulate $x_t \sim \mathcal{N}\left(x_T \frac{Z_a^{++}(t)}{Z_a(T)}, \frac{Z_a(t)\left(Z_a^{++}(T) - Z_a(t)\right)}{Z_a^{++}(T)}\sigma^2\right)$.
5: Set $X(t) = x_t$.

Table 3.5 compares the theoretical mean, variance, skewness and kurtosis of X at time $T = 1$ with the ones obtained by numerical forward and backward simulations, where we used the following set of parameters: $\theta = 1.025$, $\sigma = 0.2$, $\alpha = 5$, $\beta = 15$, $a = 0.7$, and 10^6 simulations.

Moment	T	F	B
$\mathbb{E}(X)$	0.10250	0.10234	0.10234
$Var(X)$	0.01591	0.01584	0.01582
$s(X)$	1.73973	1.73637	1.73569
$k(X)$	7.11923	7.12693	7.09786

Table 3.5. Comparison of theoretical moments (T) of the $VG{+}{+}$ process with the numerical ones obtained by forward (F) and backward (B) simulations.

Chapter 4

A Multivariate Variance Gamma process with stochastic delay

In this chapter we use the notion of self-decomposability we introduced in Section 2.2 to build a multivariate version of the Variance Gamma process we introduced in Section 2.6.1. During the last decades a lot of efforts have been done in financial modeling to go beyond the Black and Scholes [23] framework. The Black-Scholes (BS) formula is widely used by practitioners, but its limitations are well-known and over the years several researchers - Merton [91], Madan and Seneta [88] and Heston [70] among others - have proposed more sophisticated models to overcome its shortcomings. On the other hand, their main focus was the single asset modeling framework.

In a multi-asset market one has to take care of the modeling of the dependence structure, which can easily become a challenging task. Mainly, one comes up against three issues:

- How to extend a univariate model to a multivariate setting preserving mathematical tractability?
- How to calibrate this model?
- Which techniques can be used for derivative pricing?

Beyond the Gaussian world, some choices have been proposed to model dependence in the context of Lévy processes. Among others, Cont and Tankov [42], Cherubini et al. [41], Panov and Samarin [95], Panov and Sirotkin [96] have discussed the use of Lévy copulas or of Lévy series representations.

In this chapter, we address the three issues above in the context of multidimensional processes using multivariate subordination. To this end, several approaches are available in the literature: for instance, Barndorff-Nielsen et al. [14] have introduced multivariate subordination and have provided general results and applications. In the same spirit, in a series of papers Semeraro [115], Luciano and Semeraro [87], Ballotta and Bonfiglioli [9], Buchmann et al. [30, 33] have proposed

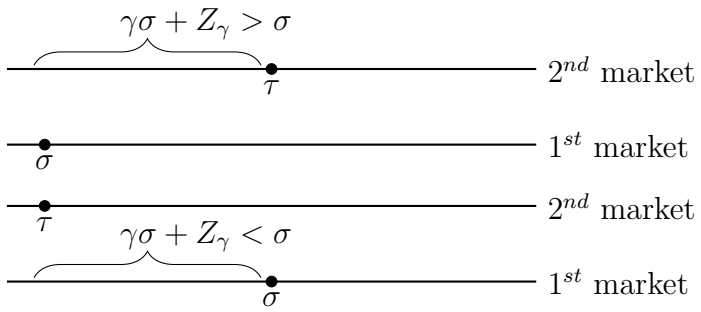

Figure 4.1. Representation of possible realizations of stochastic delay between two events at σ and τ.

models based on subordination to introduce dependence among Lévy processes. The common idea of these papers is to define multivariate processes that are the sum of an independent process and a common one. For example Ballotta and Bonfiglioli [9] define a multivariate process $\mathbf{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ in the following way:

$$\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_n(t))^T = (X_1(t) + a_1 Z(t), \ldots, X_n(t) + a_n Z(t))^T, \quad (4.1)$$

where $Z = \{Z(t); t \geq 0\}$, $X_j = \{X_j(t); t \geq 0\}$, $j = 1, \ldots, n$ are independent Lévy processes. From a financial standpoint, the common process Z can be viewed as a systemic risk, whereas the independent processes X_j can be considered as the idiosyncratic components.

On the other hand, particularly in illiquid markets, it is not so rare to observe that some news or shocks in a certain market do not have a simultaneous impact on the other ones. One rather observes a sort of "delay in the propagation of the information" across markets namely, a "delay in reacting to the given shock". Here we alert the reader observing that the concept of "delay" might be misleading. The reader might be tempted to think that there is one driving market and other markets follow the principal one with a certain stochastic dalay and that any single sudden event has first an impact on the leading market and then all the effects propagate in the other ones. This is not what we want to model. The idea is that there is not a driving market but that all the markets are strictly connected: a general event might appear in one of them and the effect spreads across the other markets with stochastic delays. Maybe the word "asynchronous" instead of "delayed" markets would be better, but it this case we would lose the idea on the back of delay in propagation of the information. In particular, the idea of asynchronous markets is not new: α-subordinators of the kind of the ones presented by Semeraro [115] allow each asset to have its own business time, but the idea of "delay in the propagation of the information" is missing.

In this case we capture this last feature by adding one parameter to the approaches mentioned in Equation (4.1) and at the same time retaining mathematical tractability. Our applications are relative to the energy markets nevertheless this technique can be applied to other contexts for instance, to credit risk. Following the ideas of Cufaro

Petroni and Sabino [46, 47], one can assume that a certain market shock occurs at a random time σ and that has effect on a related market at time τ. Cufaro Petroni and Sabino [47] describe this type of interaction as *synaptic risk* that can also be seen in terms of random delays. With the additional condition that σ and τ follow the same (marginal) distribution with different parameters, a natural way to introduce dependence between these two stochastic times is to set:

$$\tau = \gamma\sigma + Z_\gamma,$$

where $\gamma > 0$, $Z_\gamma \geq 0$ almost surely, and σ and Z_γ are independent. It turns out that the parameter γ is related to the linear correlation between σ and τ and Z_γ plays the role of a delay. Clearly if $\gamma > 1$, we have $\tau > \sigma$ almost surely, and this situation is shown in the top picture of Figure 4.1. On the other hand, if $0 < \gamma < 1$ it might happen that $\tau < \sigma$ and in this case the interpretation is the following: the *sudden event* first occurs in the latter market at time τ and then we observe its effect on the former market at time $\sigma > \tau$. This latter case is illustrated at the bottom picture of Figure 4.1. For example, σ can represent the default event of a bank and τ is the default of another related bank or company after a random time Z_γ.

The modeling above is then strictly related to the mathematical concept of self-decomposability, based on which, following the approach proposed by Sabino and Petroni [109], we extend the multivariate Lévy models presented by Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9] in order to include a "delay in the propagation of the impact" of the systemic risk component. It is worthwhile mentioning that an alternative approach to obtain multivariate Lévy processes can be found in Buchmann et al. [30], Michaelsen and Szimayer [94], Michaelsen [93], Buchmann et al. [33] and Buchmann et al. [32] relying on the notion of *weak-subordination*.

As far as the second, calibration issue is concerned, in Chapter 7 we show how general techniques, such as Non-Linear-Least-Square (NLLS) or Generalized Method of Moments (GMM), can be adopted in our framework, implementing a two-steps method as the one presented by Ballotta and Bonfiglioli [9].

Moreover, we derive the characteristic functions (*chf*'s) of the log-prices process, namely the logarithm of the price process, in closed form therefore in Chapter 7 we can tackle the third and last issue of the derivative pricing based on the Fourier transform methods introduced in Hurd and Zhou [74], Pellegrino [97] and Caldana and Fusai [35]. Finally, standard path generation schemes can be adapted to our models, allowing a numerical pricing via Monte Carlo simulations.

All the results we present in this chapter and the related numerical experiments of Chapter 7 can also be found in Gardini et al. [60].

4.1 Preliminaries - Self-decomposable subordinators

In this section we show how to use the notion of self-decomposability we introduced in Section 2.2 to build what we call self-decomposable subordinators. As mentioned above, two random variables with self-decomposable law $X \overset{d}{=} Y$ related by:

$$X \overset{d}{=} aY + Z_a,$$

can be viewed as two stochastic times delayed one respect the other by the parameter a and the random variable Z_a independent of Y. As we observed in Section 2.8.1 subordinators can be viewed as business time: here the goal is to construct coupled subordinators which runs one delayed with respect the other one, i.e. what we call *Self-decomposable subordinators*.

If H is $\mathbb{P}$-*a.s.* non-negative random variables with *sd* law we can build *sd* subordinators as follows

Definition 4.1.1 (Self-decomposable subordinators)**.** *Let $\tilde{H}_1$ and $\tilde{H}_2$ be* $\mathbb{P}$-a.s. *non-negative rv with sd laws, where $\tilde{Z}_a$ is the a-remainder, and define $H_1 = \{H_1(t); t \geq 0\}$ and $Z_a = \{Z_a(t); t \geq 0\}$ as Lévy processes such that $(H_1(1)) \overset{d}{=} \left(\tilde{H}_1\right)$ and $Z_a(1) \overset{d}{=} \tilde{Z}_a$. A sd subordinator $\boldsymbol{H} = \{\boldsymbol{H}(t); t \geq 0\}$, where $\boldsymbol{H}(t) = (H_1(t), H_2(t))$ is defined as:*

$$H_2(t) = aH_1(t) + Z_a(t). \tag{4.2}$$

Note that the process H_2 defined in (4.2) is a Lévy process because it is a linear combination of two Lévy processes (Cont and Tankov [42, Theorem 4.1]).

The construction proposed by Equation (4.2) has a clear financial interpretation. Stochastic times processes H_1, H_2 "run together" with a stochastic delay, given by the parameter a and by the term $Z_a(t)$, one with respect to the other. In Figure 4.2 different paths of the process $\boldsymbol{H}$ are shown, varying the parameter a: for fixed t the difference between $H_1(t)$ and $H_2(t)$ can be viewed as stochastic delay. Roughly speaking one can observe if $a \to 1$, then processes processes H_1 and H_2 are essentially indistinguishable.

This construction provides us a powerful tool to model those markets in which, whenever an event occurs in one of them, the effect on the other ones is not immediate but it occurs with a certain time delay. Observe that the parameter a is the only parameter we have to add to include this feature in our model and this do not leads to a significant model complication.

The former construction can be extended to the case $n > 2$ as it will showed in the sequel.

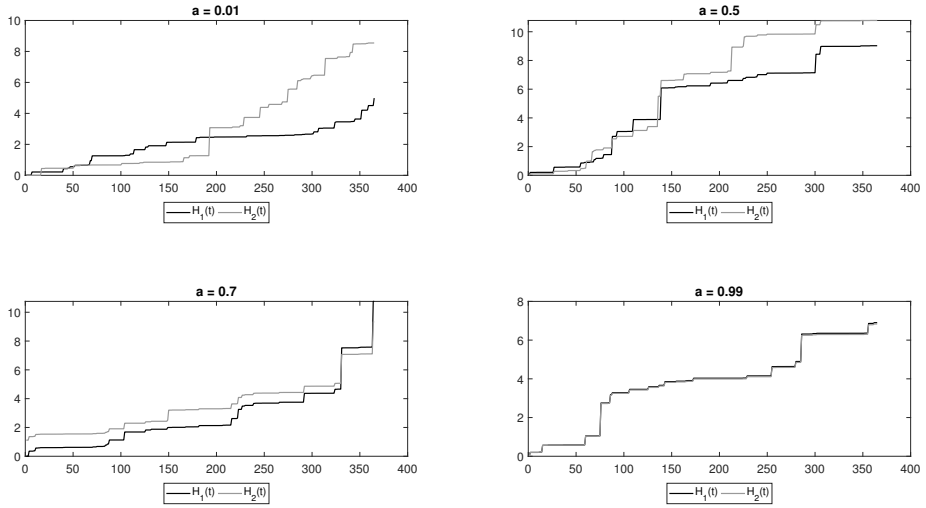

Figure 4.2. Dependent subordinators H_1 and H_2 with different values of a.

4.2 Model extensions with Self-Decomposability

In this section we extend the models presented by Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9] using *sd* subordinators introduced in Section 4.1 and we show how to build a bivariate versione of the Variance Gamma model with stochastic delay.

4.2.1 Semeraro's *sd*-VG model

The first model we extend using *sd* subordinators was proposed by Semeraro [115].

Definition 4.2.1 (*sd*-Semeraro Model)**.** *Let $I_j = \{I_j(t); t \geq 0\}$ $j = 1, 2$ be independent sd subordinators, and H_1, H_2 be sd subordinators defined in Equation* (4.2)*, independent of I_j. Define the subordinator $G_j = \{G_j(t); t \geq 0\}$ as:*

$$G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, 2, \tag{4.3}$$

with $\alpha_j \in \mathbb{R}^+$. Let $\mu_j \in \mathbb{R}$, $\sigma_j \in \mathbb{R}^+$, $W_j = \{W_j(t); t \geq 0\}$ be standard independent BM's and let G_j as is (4.3)*. Define the subordinated* BM *with drift $Y = \{Y_j(t); t \geq 0\}$ as:*

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, 2. \tag{4.4}$$

Observe that the "delay in time effect" appears at the level of subordinators G_j and it is given by the process $\boldsymbol{H}$. Moreover, it is easy to check that, since I_j and H_j are assumed to be *sd* and independent the obtained subordinator G_j is *sd*.

It is worth noting that obtained process $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \geq 0\}$ is Lévy because W_1 and W_2 are independent as observed in Barndorff-Nielsen et al. [14] and in Buchmann et al. [31].

The joint *chf* of the process defined in (4.4) has a nice closed expression.

Proposition 4.2.1 (Characteristic Function)**.** *Let be* $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$*. The joint chf* $\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u})$ *of the process* $\boldsymbol{Y}$ *at time* $t \geq 0$ *defined in* (4.4) *is given by:*

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \phi_{I_1(t)}\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right)\phi_{I_2(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\phi_{Z_a(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\\ & \phi_{H_1(t)}\left(\alpha_1\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right) + a\alpha_2\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\right).\end{aligned} \tag{4.5}$$

Proof. Substituting the expression of $Y_j(t)$, conditioning with respect $G_j(t)$ and since $W_j(t)$ are independent we get:

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y}(t)\rangle}\right] = \mathbb{E}\left[e^{iu_1Y_1(t) + iu_2Y_2(t)}\right]\\ =& \mathbb{E}\left[e^{i\left(u_1\mu_1 + i\frac{\sigma_1^2u_1^2}{2}\right)G_1(t)}e^{i\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)G_2(t)}\right].\end{aligned}$$

Using the definition of $G_j(t)$ we have:

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \mathbb{E}\Bigg[& e^{i\left(u_1\mu_1 + i\frac{\sigma_1^2u_1^2}{2}\right)I_1(t)}e^{i\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)I_2(t)}e^{i\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)\alpha_2 Z_a(t)}\\ & e^{i\left(\left(u_1\mu_1 + i\frac{\sigma_1^2u_1^2}{2}\right)\alpha_1 + \left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)\alpha_2 a\right)H_1(t)}\Bigg]\end{aligned}$$

and, observing that $I_j(t)$, $H_1(t)$ and $Z_a(t)$, are mutually independent the thesis follows. ■

Note. Observe that the derived model is an extension of the one presented by Semeraro [115]. By taking the limit for $a \to 1$ in (4.5) we have that:

$$\begin{aligned}\lim_{a\to 1}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \phi_{I_1(t)}\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right)\phi_{I_2(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\\ & \phi_{H_1(t)}\left(\alpha_1\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right) + \alpha_2\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\right)\end{aligned}$$

and this coincides with the *chf* of the original model.

Starting from the explicit expression of the *chf* one can easily compute the linear correlation coefficient at time t.

The correlation at time $t \geq 0$ $\rho_{Y_1(t),Y_2(t)}$ is given by:

$$\rho_{Y_1(t),Y_2(t)} = \frac{\mu_1\mu_2\alpha_1\alpha_2 aVar\left[H_1\left(t\right)\right]}{\sqrt{Var\left[Y_1\left(t\right)\right]Var\left[Y_2\left(t\right)\right]}}. \tag{4.6}$$

This simply follows from straightforward computations by observing that

$$cov\left(Y_1\left(t\right), Y_2\left(t\right)\right) = \mathbb{E}\left[Y_1\left(t\right)Y_2\left(t\right)\right] - \mathbb{E}\left[Y_1\left(t\right)\right]\mathbb{E}\left[Y_2\left(t\right)\right],$$

substituting the expressions of $Y_j\left(t\right)$ and $G_j\left(t\right)$ and noting that

$$\mathbb{E}\left[H_1\left(t\right)H_2\left(t\right)\right] = aVar\left[H_1\left(t\right)\right].$$

We observe that the value of correlation ρ is lower than the one obtained by Semeraro [115]. This is obvious from an intuitive point of view: in the original model the author modeled the systemic risk component using a common subordinator whilst we use two processes, H_1, H_2. On the other hand, as observed before, if $a \to 1$, then H_1 and H_2 are indistinguishable and we retrieve the value of correlation ρ obtained by Semeraro [115].

2D - *sd* Variance Gamma Semeraro

So far we analyzed the general model without assuming a particular form for the law of any of the processes involved. Since gamma *rv*'s have *sd* law, then they are suitable candidates for our construction. Assuming that $\tilde{H}_1, \tilde{H}_2$ has gamma law (with a specific parameters choice) we extend Semeraro's model for the Variance Gamma process using *sd*-subordinators.
In Section 2.5.2 we introduced the gamma distribution and we have shown that if $X \sim \Gamma\left(\alpha, \beta\right)$, then $cX \sim \Gamma\left(\alpha, \frac{\beta}{c}\right)$ and if $X \sim \Gamma\left(\alpha_1, \beta\right)$ and $Y \sim \Gamma\left(\alpha_2, \beta\right)$ are independent, then $X + Y \sim \Gamma\left(\alpha_1 + \alpha_2, \beta\right)$. Now set in (4.4):

$$I_j \sim \Gamma\left(A_j, \frac{B}{\alpha_j}\right), \quad H_j \sim \Gamma\left(A, B\right), \quad j = 1, 2$$

and noting that $\alpha_j H_j \sim \Gamma\left(A, \frac{B}{\alpha_j}\right)$ we have

$$G_j \sim \Gamma\left(A_j + A, \frac{B}{\alpha_j}\right), \quad j = 1, 2.$$

Remembering that $A_j, A, B, \alpha_j \in \mathbb{R}^+$ we have the following conditions:

$$\frac{1}{A_j + A} = \frac{\alpha_j}{B}, \quad j = 1, 2, \tag{4.7}$$

$$0 < \alpha_j \leq \frac{B}{A}, \quad j = 1, 2. \tag{4.8}$$

Given the condition (4.7) and (4.8) we have that $\mathbb{E}\left[G_j\right] = 1$ and hence $\mathbb{E}\left[G_j\left(t\right)\right] = t$.

Note. If we request condition (4.7), we have that:

$$1 = \alpha_1 \frac{(A_1 + A)}{B} = \alpha_2 \frac{(A_2 + A)}{B}$$

and so the parameter B is somehow redundant and we can assume $B = 1$.

We get the same conclusion observing that, in Equation (4.6), we fit only the variance of $H_1(t)$: for this reason assuming $B = 1$ is not restrictive.

As direct consequences of 4.2.1 and of the computation we did to obtain the linear correlation coefficient of Equation (4.6) we get that the *chf* in 2D - *sd* Variance Gamma case is given by:

$$\begin{aligned} \phi_{H_j(t)}(u) &= \left(1 - i\frac{u}{B}\right)^{-tA}, \quad j = 1, 2, \\ \phi_{I_j(t)}(u) &= \left(1 - \alpha_j i\frac{u}{B}\right)^{-tA_j}, \quad j = 1, 2, \\ \phi_{Z_a(t)}(u) &= \frac{\phi_{H_1(t)}(u)}{\phi_{H_1(t)}(au)} = \left(\frac{B - iu}{B - iau}\right)^{-tA} \end{aligned} \tag{4.9}$$

and so *chf* $\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u})$ in (4.5) can be computed. Furthermore the linear correlation coefficient at time $t \geq 0$ in 2D Variance-Gamma assumes the following simple expression:

$$\rho_{Y_1(t),Y_2(t)} = \frac{\mu_1 \mu_2 \alpha_1 \alpha_2 a A}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1}\sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}.$$

4.2.2 Semeraro-Luciano's *sd*-VG model

The model presented by Luciano and Semeraro [87], which was developed in order to capture those correlation levels in log-returns that the model proposed by Semeraro [115] is not able to get (see Wallmeier and Diethelm [124]), can be extended in a similar way to what we showed in Section 4.2.1.

Definition 4.2.2 (*sd*-Luciano and Semeraro's model)**.** *Let* $I_j = \{I_j(t); t \geq 0\}$, $j = 1, 2$, *be sd subordinators and let* $H_1 = \{H_1(t); t \geq 0\}$ *and* $H_2 = \{H_2(t); t \geq 0\}$ *two sd subordinators independent from* I_j. *Define the process* $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ *as follows:*

$$\boldsymbol{Y}(t) = \begin{pmatrix} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1 \mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\ \mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) + \alpha_2 \mu_2 H_2(t) + \sqrt{\alpha_2}\sigma_2 U(t) \end{pmatrix}, \tag{4.10}$$

where $U(t) = W_2^{\rho}(aH_1(t)) + \tilde{W}(Z_a(t))$, $\boldsymbol{W} = \{(W_1(t), W_2(t)); t \geq 0\}$ *is a standard* BM *with independent components,* $\boldsymbol{W}^{\rho} = \{(W_1^{\rho}(t), W_2^{\rho}(t)); t \geq 0\}$ *is a standard* BM *such that:*

$$\mathbb{E}\left[dW_1^{\rho}(t)\, dW_2^{\rho}(t)\right] = \rho dt$$

and $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ *is a* BM *independent from* $\boldsymbol{W}$ *and* $\boldsymbol{W}^{\rho}$*. Moreover* $\boldsymbol{W}$ *and* $\boldsymbol{W}^{\rho}$ *are independent.*

Unlike the process derived in the previous section, it can be shown that, since the covariance matrix of increments

$$cov\left(\boldsymbol{Y}(t-s), \boldsymbol{Y}(s)\right)$$

is not null $\forall s < t$, then $\boldsymbol{Y}$ is not a Lévy process, even if their its components are. Actually, it can be proven that the process $\boldsymbol{Y}$ is not a Markov process neither. The proof of these claims is reported in Appendix C.

Nevertheless if in Equation 4.10 we consider $U(t) = \sqrt{a}W_2^{\rho}\left(H_1(t)\right) + \tilde{W}\left(Z_a(t)\right)$ the obtained process $\hat{\boldsymbol{Y}}$ is a Lévy process since it is easy to check that the independence of the increments is preserved. Moreover, by following the same computations of the following proposition we get that the characteristic functions of $\hat{\boldsymbol{Y}}$ and $\boldsymbol{Y}$ are the same. This result is not surprising and agrees with Sato [112, Theorem 7.10]. Given the characteristic function of a infinitely divisible law, there exists a Lévy process such that its law at time $t = 1$ coincides with the given law. However, nothing prevents the existence of another process which can be neither Lévy nor Markov such that the characteristic function of the process at time t is associated to an infinitely divisible law. The distribution of $\hat{\boldsymbol{Y}}$ and $\boldsymbol{Y}$ at a general time $t \geq 0$ is the same but the first process is not a Lévy process whilst the second one is. Another example of these facts can be found in Appendix A.
Here too the *chf* of the process $\boldsymbol{Y}$ at time $t \geq 0$ has a nice closed expression.

Proposition 4.2.2 (Characteristic Function)**.** *Let be* $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$*. The joint chf* $\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u})$ *of the process* $\boldsymbol{Y}$ *at time* $t \geq 0$ *defined in* (4.10) *is given by:*

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \phi_{I_1(t)}\left(u_1\mu_1 + \frac{i}{2}\sigma_1^2 u_1^2\right)\phi_{I_2(t)}\left(u_2\mu_2 + \frac{i}{2}\sigma_2^2 u_2^2\right)\\ &\phi_{H_1(t)}\left(\frac{i}{2}u_1^2\alpha_1\sigma_1^2(1-a) + \boldsymbol{u}^T\boldsymbol{\mu} + \frac{i}{2}\boldsymbol{u}^T a\Sigma\boldsymbol{u}\right)\phi_{Z_a(t)}\left(u_2\mu_2\alpha_2 + \frac{i}{2}u_2^2\alpha_2\sigma_2^2\right),\end{aligned}$$

where $\boldsymbol{\mu} = [\alpha_1\mu_1, a\alpha_2\mu_2]$ *and*

$$\Sigma = \begin{bmatrix}\alpha_1\sigma_1^2 & \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho\\ \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho & \alpha_2\sigma_2^2\end{bmatrix}.$$

Proof. Rewrite $\boldsymbol{Y}(t)$ as:

$$\boldsymbol{Y}(t) = \boldsymbol{Y}_{\boldsymbol{I}(t)} + \boldsymbol{Y}_{\boldsymbol{H}(t)},$$

where:

$$\boldsymbol{Y_I}(t) = \begin{pmatrix} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) \\ \mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) \end{pmatrix}$$

and:

$$\boldsymbol{Y_H}(t) = \begin{pmatrix} \alpha_1\mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t)) \\ \alpha_2\mu_2 H_2(t) + \sqrt{\alpha_2}\sigma_2\left(W_2^\rho(aH_1(t)) + \tilde{W}(Z_a(t))\right) \end{pmatrix}.$$

The characteristic function is given by:

$$\begin{aligned} \phi_{\boldsymbol{Y}(t)^\rho}(\boldsymbol{u}) =& \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y}(t)\rangle}\right] = \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y}_I(t) + \boldsymbol{Y}_H(t)\rangle}\right] \\ =& \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y_I}(t)\rangle}\right]\mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y_H}(t)\rangle}\right]. \end{aligned} \tag{4.11}$$

We now compute the two last term separately. Substituting the expression of $\boldsymbol{Y_I}$, conditioning respect $I_j(t)$, $j = 1, 2$ and remebering that $W_1(t)$ and $W_2(t)$ are idependent we have:

$$\begin{aligned} \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y_I}(t)\rangle}\right] =& \mathbb{E}\left[e^{i\left(u_1\mu_1 + \frac{i}{2}u_1^2\sigma_1^2\right)I_1(t)}\right]\mathbb{E}\left[e^{i\left(u_2\mu_2 + \frac{i}{2}u_2^2\sigma_2^2\right)I_2(t)}\right] \\ =& \phi_{I_1(t)}\left(u_1\mu_1 + \frac{i}{2}\sigma_1^2u_1^2\right)\phi_{I_2(t)}\left(u_2\mu_2 + \frac{i}{2}\sigma_2^2u_2^2\right). \end{aligned} \tag{4.12}$$

Following the same approach we can compute the second term, obtaining:

$$\begin{aligned} \mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{Y_H}(t)\rangle}\right] =& \mathbb{E}\Big[\mathbb{E}\Big[e^{iu_1\left(\alpha_1\mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t))\right)} \\ & e^{iu_2\left(\alpha_2\mu_2 aH_1(t) + iu_2\sqrt{\alpha_2}\sigma_2 W_2^\rho(aH_1(t))\right)}|H_1(t)\Big] \\ & \mathbb{E}\left[e^{iu_2\alpha_2\mu_2 Z_a(t) + iu_2\sqrt{\alpha_2}\sigma_2\tilde{W}(Z_a(t))}|Z_a(t)\right]\Big]. \end{aligned}$$

Now we compute the inner expected values separately. The second inner expected value is:

$$\mathbb{E}\left[e^{iu_2\alpha_2\mu_2 Z_a(t) + iu_2\sqrt{\alpha_2}\sigma_2\tilde{W}(Z_a(t))}|Z_a(t)\right] = e^{i\left(u_2\alpha_2 + \frac{i}{2}u_2^2\alpha_2\sigma_2\right)Z_a(t)}.$$

For the second therm we have that, since $H_1(t)$ is known:

$$\begin{aligned} & \mathbb{E}\left[e^{iu_1\alpha_1\mu_1 H_1(t) + iu_1\sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t)) + iu_2\alpha_2\mu_2 aH_1(t) + iu_2\sqrt{\alpha_2}\sigma_2 W_2^\rho(aH_1(t))}|H_1(t)\right] \\ =& e^{iu_1\alpha_1\mu_1 H_1(t) + iu_2\alpha_2\mu_2 aH_1(t)}\mathbb{E}\left[e^{iu_1\sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t)) + iu_2\sqrt{\alpha_2}\sigma_2 W_2^\rho(aH_1(t))}|H_1(t)\right]. \end{aligned}$$

The only unknown terms is the expected value. We have that:

$$\mathbb{E}\left[e^{iu_1\sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t))+iu_2\sqrt{\alpha_2}\sigma_2 W_2^{\rho}(aH_1(t))}|H_1(t)\right] = e^{-\frac{1}{2}u_1^2\alpha_1\sigma_1^2(1-a)H_1(t)}e^{-\frac{1}{2}a\boldsymbol{u}^T a\Sigma\boldsymbol{u}H_1(t)},$$

where

$$\Sigma = \begin{bmatrix} \alpha_1\sigma_1^2 & \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho \\ \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho & \alpha_2\sigma_2^2 \end{bmatrix}$$

and $\boldsymbol{u} = (u_1, u_2)$. Setting $\boldsymbol{\mu} = (\alpha_1\mu_1, a\alpha_2\mu_2)$ we can conclude that:

$$\begin{aligned}\mathbb{E}\left[e^{i\langle\boldsymbol{u},\boldsymbol{Y_H}(t)\rangle}\right] =& \phi_{Z_a(t)}\left(u_2\alpha_2 + \frac{i}{2}u_2^2\alpha_2\sigma_2\right) \\ & \phi_{H_1(t)}\left(\boldsymbol{u}^T\boldsymbol{\mu} + \frac{i}{2}u_1^2\alpha_1\sigma_1^2(1-a) + \frac{i}{2}a\boldsymbol{u}^T a\Sigma\boldsymbol{u}\right).\end{aligned} \tag{4.13}$$

Using (4.12) and (4.13) in (4.11) we have the thesis. ■

Following the technique proposed in the previous section one can show that the correlation at time $t \geq 0$, $\rho_{Y_1(t),Y_2(t)}$ is given by:

$$\rho_{Y_1(t),Y_2(t)} = \frac{a\left(\mu_1\mu_2\alpha_1\alpha_2 Var\left[H_1(t)\right] + \rho\sigma_1\sigma_2\sqrt{\alpha_1\alpha_2}\mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_1(t)\right]Var\left[Y_2(t)\right]}}. \tag{4.14}$$

All considerations about correlation coefficient and *chf* we pointed out in Section 4.2.1 are still valid.

2D - *sd* Variance Gamma Luciano-Semeraro

Here too it's possible to build a 2D-Variance Gamma process by choosing

$$I_j \sim \Gamma\left(A_j, \frac{B}{\alpha_j}\right),\ H_j \sim \Gamma(A, B),\ j = 1, 2.$$

We have that:

$$I_j + \alpha_j H_j \sim \Gamma\left(A_j + A, \frac{B}{\alpha_j}\right),\ j = 1, 2.$$

and, imposing conditions (4.7) and (4.8), we have get $\mathbb{E}[G_j] = 1$ and, consequently, $\mathbb{E}[G_j(t)] = t$ for $j = 1, 2$. Following the same argument of Section 4.2.1, expressions of linear correlation coefficient and the *chf* for the 2D Variance Gamma case can be derived. It is easy to show that the linear correlation coefficient at time $t \geq 0$ in 2D - *sd* Variance Gamma case is given by:

$$\rho_{Y_1(t),Y_2(t)} = \frac{a\left(\mu_1\mu_2\alpha_1\alpha_2 A + \rho A\sigma_1\sigma_2\sqrt{\alpha_1\alpha_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2\alpha_1}\sqrt{\sigma_2^2 + \mu_2^2\alpha_2}}.$$

The *chf* can be obtained by combining Equations 4.9 with Proposition 4.2.2.

4.2.3 Ballotta-Bonfiglioli's *sd*-VG model

The construction technique of dependent Lévy processes proposed by Ballotta and Bonfiglioli [9] is slightly different from what we have seen so far. The dependence between processes is not introduced on subordinators, as in the previous case, but two subordinated *BM* of the same type are added together. Some convolution conditions on parameters guarantee that the resulting process is of the same type of the summed ones. This model, as the previous ones, can be extended using *sd* subordinators.

Definition 4.2.3 (*sd*-Ballotta and Bonfiglioli's model)**.** *Let* $\boldsymbol{H}$ *be a sd subordinator as in* (4.2) *and define the process* $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ *as:*

$$\boldsymbol{Y}(t) = (Y_1(t), Y_2(t)) = (X_1(t) + a_1 R_1(t), X_2(t) + a_2 R_2(t)), \tag{4.15}$$

where:

- $X_j = \{X_j(t); t \geq 0\}$ *is a subordinated* BM *with parameters* $(\beta_j, \gamma_j, \nu_j)$, $j = 1, 2$, *where* $\beta_j \in \mathbb{R}$ *is the drift,* $\gamma_j \in \mathbb{R}^+$ *is the diffusion and* $\nu_j \in \mathbb{R}^+$ *is the variance of the subordinator. Let be* $G_j = \{G_j(t); t \geq 0\}$ *the subordinator of* X_j *and let be* G_1 *and* G_2 *be independent. We define:*

$$X_j(t) = \beta_j G_j(t) + W_j(G_j(t)), \quad j = 1, 2.$$

- *Let* $R_1 = \{R_1(t); t \geq 0\}$ *and* $R_2 = \{R_2(t); t \geq 0\}$ *be given by:*

$$\begin{aligned} R_1(t) &= \beta_{R_1} H_1(t) + \gamma_{R_1} W(H_1(t)), \\ R_2(t) &= \beta_{R_2} H_2(t) + \gamma_{R_2} \left(W(aH_1(t)) + \tilde{W}(Z_a(t)) \right), \end{aligned} \tag{4.16}$$

  where $W = \{W(t); t \geq 0\}$ *and* $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ *are independent* BM*'s and* $\beta_{R_j} \in \mathbb{R}$ *and* $\gamma_{R_j} \in \mathbb{R}^+$.

- *Let be* $a_1, a_2 \in \mathbb{R}$.

The following Lemma will help to derive the *chf* of the process.

Lemma 4.2.3. *Let be* $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$. *The chf of the process defined in* (4.16) *at time* $t \geq 0$ *is given by:*

$$\begin{aligned} \phi_{\boldsymbol{R}(t)}(\boldsymbol{u}) =& \phi_{H_1(t)}\left(u_1\beta_{R_1} + u_2\beta_{R_2}a + \frac{i}{2}\left(u_1^2\gamma_{R_1}^2 + 2u_1u_2\gamma_{R_1}\gamma_{R_2}a + u_2^2 a\gamma_{R_2}^2 \right) \right) \\ & \phi_{Z_a(t)}\left(u_2\beta_{R_2} + \frac{i}{2}u_2^2\gamma_{R_2}^2 \right). \end{aligned}$$

Proof. Replacing the definition of $R_1(t)$ and $R_2(t)$ we get:

$$\begin{aligned}\phi_{\boldsymbol{R}(t)}(\boldsymbol{u}) =& \mathbb{E}\left[e^{iu_1R_1(t)+iu_2R_2(t)}\right]\\ =& \mathbb{E}\left[e^{iu_1\beta_{R_1}H_1(t)+iu_2a\beta_{R_1}H_1(t)+iu_2\beta_{R_2}Z_a(t)}\right.\\ & \left.\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}\left(W(aH_1(t))+\tilde{W}(Z_a(t))\right)}|H_1(t),Z_a(t)\right]\right].\end{aligned}$$

We compute now the inner expected value:

$$\begin{aligned}&\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}\left(W(aH_1(t))+\tilde{W}(Z_a(t))\right)}|H_1(t),Z_a(t)\right]\\ &=\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}W(aH_1(t))}|H_1(t)\right]\mathbb{E}\left[e^{iu_2\gamma_{R_2}\tilde{W}(Z_a(t))}|Z_a(t)\right].\end{aligned}$$

The second computation of the second expected value is immediate:

$$\mathbb{E}\left[e^{iu_2\gamma_{R_2}\tilde{W}(Z_a(t))}|Z_a(t)\right]=e^{-\frac{1}{2}u_2^2\gamma_{R_2}^2Z_a(t)}.$$

For the first term we have:

$$\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}W(aH_1(t))}|H_1(t)\right]=e^{-\frac{1}{2}\left(u_1^2\gamma_{R_1}^2+2u_1u_2\gamma_{R_1}\gamma_{R_2}a+au_2^2\gamma_{R_2}^2\right)H_1(t)}.$$

Observing that $H_1(t)$ and $Z_a(t)$ are independent the thesis follows. ■

The *chf* of the process defined in (4.15) is given by the following Proposition.

Proposition 4.2.4 (Characteristic Function)**.** *Let be $\boldsymbol{u}=(u_1,u_2)\in\mathbb{R}^2$. The chf of the process at time $t\geq 0$ defined in* (4.15) *is given by:*

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \phi_{G_1(t)}\left(\beta_1u_1+\frac{i}{2}u_1^2\gamma_1^2\right)\\ & \phi_{G_2(t)}\left(\beta_2u_2+\frac{i}{2}u_2^2\gamma_2^2\right)\\ & \phi_{\boldsymbol{R}(t)}(\boldsymbol{a}\circ\boldsymbol{u}),\end{aligned}\tag{4.17}$$

where $\boldsymbol{a}=(a_1,a_2)\in\mathbb{R}^2$ and $\circ$ is the Hadamard product[1]*.*

Proof. Replacing the expression of Y_j $j=1,2$ we have that:

$$\mathbb{E}\left[e^{\langle\boldsymbol{u},\boldsymbol{Y}(t)\rangle}\right]=\mathbb{E}\left[e^{iu_1X_1(t)}\right]\mathbb{E}\left[e^{iu_2X_2(t)}\right]\phi_{\boldsymbol{R}(t)}(\boldsymbol{a}\circ\boldsymbol{u}).$$

Observe that, conditioning to $G_j(t)$, we have that:

$$\mathbb{E}\left[e^{iu_jX_j(t)}\right]=\mathbb{E}\left[e^{i\left(u_j\beta_j+\frac{i}{2}u_j^2\gamma_j^2\right)G_j(t)}\right]=\phi_{G_j(t)}\left(u_j\beta_j+\frac{i}{2}u_j^2\gamma_j^2\right).$$

This observation jointly with Lemma 4.2.3 complete the proof. ■

[1]Given two matrices with the same dimension $n\times m$ the Hadarmard product of $A\circ B$ is a $n\times m$ matrix such that $(A\circ B)_{i,j}=A_{i,j}\cdot B_{i,j}$.

Note. As in the previous models it is easy to verify that:

$$\lim_{\substack{a\to 1\\ \beta_{R_1},\beta_{R_2}\to\beta_Z\\ \gamma_{R_1},\gamma_{R_2}\to\gamma_Z}} \phi_{\boldsymbol{Y}(t)}(u_1,u_2) = \phi_{G_1(t)}\left(\beta_1 u_1 + \frac{i}{2}u_1^2\gamma_1^2\right)$$
$$\phi_{G_2(t)}\left(\beta_2 u_2 + \frac{i}{2}u_2^2\gamma_2^2\right)\phi_{Z(t)}\left(\beta_Z(a_1u_1+a_2u_2) + \frac{i}{2}(a_1u_1+a_2u_2)^2\gamma_Z^2\right),$$

which is the *chf* obtained by Ballotta and Bonfiglioli [9]. It is worth noting that the process $\boldsymbol{R}$ is not Lévy, since its increments are not independent: for this reason, neither $\boldsymbol{Y}$ is a Lévy process (see Appendix C for details). Following the same argument we used in Section 4.2.2, we can prove that $\boldsymbol{Y}$ is not even a Markov process.

Even then, the correlation coefficient of the process $\boldsymbol{Y}$ can be obtained.

Proposition 4.2.5. *The correlation coefficient at time $t \geq 0$ of the process $\boldsymbol{Y}$ defined in* (4.15) *is given by:*

$$\rho_{Y_1(t),Y_2(t)} = \frac{a_1a_2a\left(\beta_{R_1}\beta_{R_2}Var\left[H_1(t)\right] + \gamma_{R_1}\gamma_{R_2}\mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_1(t)\right]}\sqrt{Var\left[Y_2(t)\right]}}. \tag{4.18}$$

Proof. Computing the covariance between $Y_1(t)$ and $Y_2(t)$ we have that:

$$cov\left(Y_1(t), Y_2(t)\right) = a_1a_2cov\left(R_1(t), R_2(t)\right). \tag{4.19}$$

But, by direct computations, one can show that:

$$cov\left(R_1(t), R_2(t)\right) = \beta_{R_1}\beta_{R_2}aVar\left[H_1(t)\right] + \gamma_{R_1}\gamma_{R_2}a\mathbb{E}\left[H_1(t)\right]. \tag{4.20}$$

where we used the following property:

$$\mathbb{E}\left[W\left(H_1(t)\right)W\left(aH_1(t)\right)\right] = a\mathbb{E}\left[H_1(t)\right].$$

Using (4.19) and (4.20) we have the thesis. ■

Convolution Conditions

It's easy to show that, if $X_j = \{X_j(t); t \geq 0\}$ and $R_j = \{R_j(t); t \geq 0\}$, $j = 1, 2$, are subordinated *BM*'s with subordinators from the same family, then $Y_j = \{Y_j(t); t \geq 0\}$ is a subordinated process of the same type of X_j and R_j if the following Ballotta and Bonfiglioli [9] style convolution conditions hold:

$$\nu_R \coloneqq \nu_{R_1} = \nu_{R_2} \tag{4.21}$$

and

$$\begin{cases} \alpha_j\mu_j = \nu_R a_j\beta_{R_j} & j = 1,2,\\ \alpha_j\sigma_j^2 = \nu_R a_j^2\gamma_{R_j}^2 & j = 1,2. \end{cases} \tag{4.22}$$

Relation (4.21) holds because $H_j = \{H_j(t); t \geq 0\}$ for $j = 1, 2$ have the same law and so they have the same variance ν_R. It is easy to check that if Equations (4.22) are satisfied, then:

$$\mu_j = \beta_j + a_j \beta_{R_j}, \quad \sigma_j^2 = \gamma_j^2 + a_j^2 \gamma_{R_j}^2, \quad \alpha_j = \nu_j \nu_R / (\nu_j + \nu_R).$$

2D - *sd* Variance Gamma Ballotta-Bonfiglioli

We can construct a 2D - *sd* Variance Gamma using Gamma subordinators as follows.

- Let $H_1(t) \sim \Gamma\left(\frac{t}{\nu_R}, \frac{1}{\nu_R}\right)$ be a Gamma subordinator and set $H_2(t) = aH_1(t) + Z_a(t)$.
- Let $R_j(t)$ be a subordinated *BM* (with drift β_{R_j} and diffusion γ_{R_j}) obtained using the Gamma subordinator $H_j(t) \sim \Gamma\left(\frac{t}{\nu_R}, \frac{1}{\nu_R}\right)$, $j = 1, 2$.
- Let $X_j(t)$ be a subordinated *BM* (with drift β_j and diffusion γ_j) obtained using a Gamma subordinator $G_j(t) \sim \Gamma\left(\frac{t}{\nu_j}, \frac{1}{\nu_j}\right)$, $j = 1, 2$.
- Set $Y_j(t) = X_j(t) + a_j R_j(t)$

We obtain that $Y(t) \sim VG(\mu_j, \sigma_j, \alpha_j)$, $j = 1, 2$, where $\mu_j, \sigma_j, \alpha_j$ respect convolution conditions (4.22).

The joint *chf* $\phi_{\boldsymbol{Y}(t)}(u_1, u_2)$ can be easly derived using (4.17) and remembering the expression of the *chf* of a $\Gamma(\alpha, \beta)$ *rv*:

$$\phi(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha}.$$

Applying Proposition 4.2.5 one can derive the correlation coefficient of the 2D - *sd* Variance Gamma process which has the following expression:

$$\rho_{Y_1(t), Y_2(t)} = \frac{a_1 a_2 a \left(\beta_{R_1} \beta_{R_2} \nu_R + \gamma_{R_1} \gamma_{R_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1} \sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}.$$

Chapter 5

A Multivariate NIG process with stochastic delay

This chapter is the natural sequel of the previous one, but now our main focus is on bivariate *sd* inverse Gaussian (IG) subordinators and on the construction of bivariate dependent NIG processes. As we have done for the Variance Gamma process with self-decomposable subordinations, we derive closed form formulas for the linear correlation and the *chf* of the NIG processes. These results are instrumental for the calibration and the pricing of derivative contracts whose valuation is often accomplished via Monte Carlo simulations. To this end, we provide a novel and efficient algorithm to generate the a-remainder of IG laws and therefore to simulate the skeleton of Z_a, of the *sd* IG subordinators and of the bivariate NIG processes. As already observed among others in Taufer and Leonenko [120], Sabino [107] and Sabino and Petroni [109], the transition law between t and $t + \Delta t$ of a Lévy-driven Ornstein-Uhlenbeck (OU) process $X = \{X(t); t \geq 0\}$ having a certain stationary law coincides with that of the a-remainder of such a law by setting $a = e^{-\lambda \Delta t}$, where λ is the mean-reversion rate of $X(t)$. Hence, the simulation of the a-remainder of a IG law is equivalent to the simulation of the skeleton of a IG-OU process, this last one having been illustrated in Zhang and Zhang [125]. We show that our proposal is more efficient than that of Zhang and Zhang [125], because it does not rely on acceptance-rejection methods and therefore it also applicable to the stochastic volatility models of Barndorff-Nielsen [11] and Andersson [5]. The only small difference it that since Z_a is a Lévy process, its simulation requires the same a at all times t while instead for a OU process, $a = e^{-\lambda \Delta t}$ depends on the time step Δt.

5.1 Preliminaries

The NIG process is constructed via the subordination of a BM with an IG process. As we observed in Section 2.5.3, there are different characterizations of the *pdf* of an IG law: on the one hand, we denote the notation using the parameter-setting (μ, λ), adopted for instance in Cont and Tankov [42], with $IG_T(\mu, \lambda)$: within this

setting $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. On the other hand, we refer to the original notation in Barndorff-Nielsen [11] with $IG_B(a, b)$: in this case $a > 0$ and $b > 0$ describe the scale and the shape of the distribution, respectively. In Section 2.5.3 we gave some details on how to switch from one to the other. In general, the IG_B notation is convenient to analyze sums of IG *rv*'s, whereas IG_T is more convenient to work with expectations and *chf*.

5.1.1 Semeraro's *sd*-NIG model

In this subsection we illustrate the steps required to extend the model proposed by Semeraro [115] in order to cope with stochastic delay relying on the *sd* subordinators defined in (4.2).

Let $I_j = \{I_j(t); t \geq 0\}$, $j = 1, 2$ be independent subordinators, and $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ be the *sd* subordinator defined in (4.2), independent of I_j. Define the subordinator $\boldsymbol{G} = \{(G_1(t), G_2(t)); t \geq 0\}$ as:

$$G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, 2, \tag{5.1}$$

where $\alpha_j \in \mathbb{R}^+$. Let $\mu_j \in \mathbb{R}$, $\sigma_j \in \mathbb{R}^+$ and $\boldsymbol{W} = \{(W_1(t), W_2(t)); t \geq 0\}$ a standard *BM*with independent components also independent of $\boldsymbol{G}$: we define the subordinated *BM* $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \geq 0\}$ as:

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, 2. \tag{5.2}$$

We remark that when a in (4.2) tends to 1 there is no time delay and the synaptic risk coincides with the systematic risk as in the original approach of Semeraro [115].

A bivariate NIG process with IG *sd*-subordinators can be defined starting from (5.1) in the following way. Assume $\alpha_j = \gamma_j^2$ and let $I_j(t)$ and $H_j(t)$ be distributed as follows:

$$\begin{aligned} I_j(t) &\sim IG_T\left(\frac{A_j \gamma_j t}{B}, A_j^2 t^2\right), \\ H_j(t) &\sim IG_T\left(\frac{At}{B}, A^2 t^2\right), \end{aligned} \tag{5.3}$$

and hence we get:

$$G_j(t) \sim IG_T\left(\frac{(A_j + A\gamma_j)\gamma_j t}{B}, (A_j + A\gamma_j)^2 t^2\right).$$

Since $G(t)$ is a stochastic time, it is customary to require that $\mathbb{E}[G_j(t)] = t$: this condition can be easily fulfilled by imposing:

$$A_j + A\gamma_j = \frac{B}{\gamma_j}.$$

Consequently, denoting with k_j the variance of the subordinator $G_j(t)$ at time $t = 1$, we have that:

$$k_j \coloneqq Var\left[G_j(1)\right] = \frac{1}{\left(A_j + A\gamma_j\right)^2} = \frac{\gamma_j^2}{B^2}.$$

As observed in Luciano and Semeraro [87], assuming $B = 1$ is not restrictive: hence we can set $k_j = \gamma_j^2$ and then $k_j = \alpha_j$. After simple calculations, one can find that the expression of the (instantaneous) linear correlation coefficient at time t of the process $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \geq 0\}$ is:

$$\rho_{(Y_1(t),Y_2(t))} = \frac{\mu_1\mu_2\alpha_1\alpha_2 aA}{\sqrt{\sigma_1^2 + \mu_1^2\alpha_1}\sqrt{\sigma_2^2 + \mu_2^2\alpha_2}}. \tag{5.4}$$

Compared to the formula of the linear coefficient in Semeraro [115] the equation above has an additional parameter a that tunes the stochastic delay.
Finally, the *chf* of $\boldsymbol{Y}$ at time t is given by the following proposition.

Proposition 5.1.1. *Denote $\phi(u;\mu,\lambda)$ the chf of a rv distributed according to a $IG_T(\mu,\lambda)$ law then the joint chf at time t of the process $\boldsymbol{Y}$ defined by Equation (5.2), where $H_j(t)$ and $I_j(t)$ are distributed as in (5.3) for $j = 1, 2$, is:*

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =& \phi_{I_1(t)}\left(u_1\mu_1 + i\frac{\sigma_1^2u_1^2}{2}\right)\phi_{I_2(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)\phi_{Z_a(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)\\ &\phi_{H_1(t)}\left(\alpha_1\left(u_1\mu_1 + i\frac{\sigma_1^2u_1^2}{2}\right) + a\alpha_2\left(u_2\mu_2 + i\frac{\sigma_2^2u_2^2}{2}\right)\right),\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}\phi_{H_j(t)}(u) &= \phi\left(u; At, A^2t^2\right), \quad j = 1, 2,\\ \phi_{I_j(t)}(u) &= \phi\left(u; A_jt\gamma_j, A_j^2t^2\right), \quad j = 1, 2,\\ \phi_{Z_a(t)}(u) &= \frac{\phi(u; At, A^2t^2)}{\phi(au; At, A^2t^2)}.\end{aligned} \tag{5.6}$$

Proof. The proof follows the scheme we used to prove the Proposition 4.2.2.

I_j and H_1 are IG processes and hence their *chf*'s at time t can be computed starting from the *chf* expression of an IG *rv*, which is reported in Section 2.5.3, whereas $Z_a(t)$ is the a-remainder of $H_1(t)$ and then its *chf* can be easily computed relying on the fact that:

$$\phi_{Y(t)}(u) = \phi_{Y(t)}(au)\phi_{Z_a(t)}(u),\ u \in \mathbb{R}.$$

The obtained *chf*'s of $H_j(t)$, $I_j(t)$ and $Z_a(t)$ are those of Equations (5.6).

Let $\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) := \mathbb{E}\left[e^{iu_1Y_1(t)+iu_2Y_2(t)}\right]$ be the *chf* of the process $\boldsymbol{Y}$ defined in (5.2) at time t: conditioning on $G_1(t)$ and $G_2(t)$ and recalling that $W_1(t)$ and $W_2(t)$ are independent we get:

$$\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \mathbb{E}\left[e^{i\left(u_1\mu_1+i\frac{\sigma_1^2u_1^2}{2}\right)G_1(t)}e^{i\left(u_2\mu_2+i\frac{\sigma_2^2u_2^2}{2}\right)G_2(t)}\right].$$

Substitute in the previous equation the expression of $G_j(t)$, given by (5.1), for $j=1,2$: by the property of the expected value for the product of independent *rv*'s, since $I_j(t), H_1(t)$ and $Z_a(t)$ are mutually independent processes, we finally get the result of the Equation (5.5). ■

5.1.2 Semeraro-Luciano's *sd*-NIG model

In this subsection we extend the model of Luciano and Semeraro [87] and we build bivariate NIG processes with stochastic delays relying on the *sd* subordinator $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ defined in (4.2). Unlike the previous model, we consider a standard *BM*, $\boldsymbol{W}^\rho = \{(W_1^\rho(t), W_2^\rho(t)); t \geq 0\}$ with correlated margins in order to obtain higher correlations in log-returns.

Let $I_j = \{I_j(t); t \geq 0\})$, $j=1,2$, be independent subordinators and let $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ be a *sd* subordinator independent of I_j. We define the process $\boldsymbol{Y}^\rho = \{(Y_1^\rho(t), Y_2^\rho(t)); t \geq 0\}$ as:

$$\boldsymbol{Y}^\rho(t) = \begin{pmatrix} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1\mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t)) \\ \mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) + \alpha_2\mu_2 H_2(t) + \sqrt{\alpha_2}\sigma_2 U(t) \end{pmatrix}, \quad (5.7)$$

where $U(t) = \left(W_2^\rho(aH_1(t)) + \tilde{W}(Z_a(t))\right)$, $\boldsymbol{W} = \{(W_1(t), W_2(t))\}$ is standard a *BM* with independent components, $\boldsymbol{W}^\rho = \{(W_1^\rho(t), W_2^\rho(t))\}$ is standard a *BM* such that $\mathbb{E}[dW_1^\rho(t)\, dW_2^\rho(t)] = \rho dt$ and $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ is another standard *BM* independent of $\boldsymbol{W}$ and $\boldsymbol{W}^\rho$.

A bivariate version of NIG process with *sd*-subordinators can be obtained letting $H_j(t)$ and $I_j(t)$ for $j=1,2$ be distributed as in the previous section. Moreover, the expression of the *chf* of the process $\boldsymbol{Y}^\rho$ at time t is given by the following proposition.

By retracing the idea we used in the proof of Proposition 5.1.1, and recalling, in addition, that for $\boldsymbol{u} \in \mathbb{R}^2$ the *chf* $\varphi(\boldsymbol{u})$ of a multivariate normal *rv* with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is given by:

$$\varphi(\boldsymbol{u}) = \exp\left(i\boldsymbol{\mu}^T\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\right),$$

we can compute the joint characteristic function $\phi_{\boldsymbol{Y}^{\rho}(t)}(\boldsymbol{u})$ of the multidimensional process $\boldsymbol{Y}^{\rho} = \{(Y_1^{\rho}(t), Y_2^{\rho}(t)); t \geq 0\}$ at time t defined in (5.7), which is given by:

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)^{\rho}}(\boldsymbol{u}) =& \phi_{I_1(t)}\left(u_1\mu_1 + \frac{i}{2}\sigma_1^2 u_1^2\right)\phi_{I_2(t)}\left(u_2\mu_2 + \frac{i}{2}\sigma_2^2 u_2^2\right) \\ & \phi_{H_1(t)}\left(\frac{i}{2}u_1^2\alpha_1\sigma_1^2(1-a) + \boldsymbol{u}^T\boldsymbol{\mu} + \frac{i}{2}\boldsymbol{u}^T a\Sigma\boldsymbol{u}\right)\phi_{Z_a(t)}\left(u_2\mu_2\alpha_2 + \frac{i}{2}u_2^2\alpha_2\sigma_2^2\right).\end{aligned}$$

It is easy to show, by direct computation, that the linear correlation coefficient at time t is given by:

$$\rho_{\boldsymbol{Y}^{\rho}(t)} = \frac{a\left(\mu_1\mu_2\alpha_1\alpha_2 A + \rho A\sigma_1\sigma_2\sqrt{\alpha_1\alpha_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2\alpha_1}\sqrt{\sigma_2^2 + \mu_2^2\alpha_2}}. \tag{5.8}$$

Once again, a can be seen as the parameter that activates stochastic delay.

5.1.3 Ballotta-Bonfiglioli's *sd*-NIG model

The construction of bivariate Lévy processes proposed by Ballotta and Bonfiglioli [9] is slightly different from that of Semeraro [115] and Luciano and Semeraro [87] because the dependence is not introduced at the level of the subordinators but rather directly on the subordinated processes. Nevertheless, we can also extend this approach to include stochastic delay.

The construction of the bivariate process with stochastic delay proceeds as follows. Let $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ be the *sd* subordinator as in (4.2): define subordinated *BM* $\boldsymbol{R} = \{(R_1(t), R_2(t)); t \geq 0\}$, for $j = 1, 2$, with drift $\beta_{R_j} \in \mathbb{R}$ and diffusion $\gamma_{R_j} \in \mathbb{R}^+$, as:

$$\begin{aligned}R_1(t) &= \beta_{R_1}H_1(t) + \gamma_{R_1}W(H_1(t)) \\ R_2(t) &= \beta_{R_2}H_2(t) + \gamma_{R_2}\left(W(aH_1(t)) + \tilde{W}(Z_a(t))\right),\end{aligned} \tag{5.9}$$

where $W = \{W(t); t \geq 0\}$ and $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ are standard independent *BM*'s. Consider $\boldsymbol{W} = \{(W_1(t), W_2(t)); t \geq 0\}$ be a standard *BM* with independent margins, $\boldsymbol{G} = \{(G_1(t), G_2(t); t \geq 0)\}$ a subordinator independent of $\boldsymbol{W}$ and consider $\boldsymbol{X} = \{(X_1(t), X_2(t)); t \geq 0\}$ be a subordinated *BM* with drift $\beta_j \in \mathbb{R}$ and diffusion $\gamma_j \in \mathbb{R}^+$, defined as follows:

$$X_j(t) = \beta_j G_j(t) + \gamma_j W_j(G_j(t)). \tag{5.10}$$

Finally, combining the previous processes, we can define a new bivariate process $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \geq 0\}$ as:

$$\boldsymbol{Y}(t) = (Y_1(t), Y_2(t)) = (X_1(t) + a_1 R_1(t), X_2(t) + a_2 R_2(t)), \tag{5.11}$$

where $a_j \in \mathbb{R}$.

As detailed in Ballotta and Bonfiglioli [9] and Gardini et al. [60], for any chosen distribution for the margin $Y_j(t)$, for example a NIG distribution, it is possible to impose convolution conditions on $X_j(t)$ and $R_j(t)$ such that their linear combination has the same given distribution of $Y_j(t)$. The following proposition shows how to build a bivariate NIG process with stochastic delays and gives the closed form expression for its *chf*.

Proposition 5.1.2. *Consider an IG subordinator* $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ *such that* $H_1(t) \sim IG_T\left(t, \frac{t^2}{\nu_R}\right)$, $H_2(t)$ *defined in Equation* (4.2) *and* $R_j(t)$ *given by* (5.9). *Let then* $\boldsymbol{X}$ *be a subordinated* BM, *defined as in* (5.10), *via an IG process* $\boldsymbol{G}$ *such that* $G_j(t) \sim IG_T\left(t, \frac{t^2}{\nu_j}\right)$, *for* $j = 1, 2$.
Then $Y_j(t)$ *in* (5.11) *are distributed according to a NIG law and the joint chf of the process* $\boldsymbol{Y}$ *at time* t *is*

$$\phi_{\boldsymbol{Y}(t)}(u_1, u_2) = \phi\left(\beta_1 u_1 + \frac{i}{2}u_1^2\gamma_1^2; t, \frac{t^2}{\nu_1}\right)\phi\left(\beta_2 u_2 + \frac{i}{2}u_2^2\gamma_2^2; t, \frac{t^2}{\nu_2}\right)\xi(\boldsymbol{a} \circ \boldsymbol{u}), \tag{5.12}$$

where $\phi(u; \mu, \lambda)$ *is the chf of a* $IG_T(\mu, \lambda)$ *distributed rv,* $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{u} = (u_1, u_2)$ *and* $\circ$ *is the Hadamard product. Finally* $\xi(\boldsymbol{u})$ *is given by:*

$$\begin{aligned}\xi(\boldsymbol{w}) =& \phi\left(w_1\beta_{R_1} + w_2\beta_{R_2}a + \frac{i}{2}\left(w_1^2\gamma_{R_1}^2 + 2w_1w_2\gamma_{R_1}\gamma_{R_2}a + w_2^2a\gamma_{R_2}^2\right); t, \frac{t^2}{\nu_R}\right) \\ & \frac{\phi\left(w_2\beta_{R_2} + \frac{i}{2}w_2^2\gamma_{R_2}^2; t, \frac{t^2}{\nu_R}\right)}{\phi\left(w_2a\beta_{R_2} + \frac{i}{2}a^2w_2^2\gamma_{R_2}^2; t, \frac{t^2}{\nu_R}\right)}.\end{aligned} \tag{5.13}$$

Proof. Relying on properties of the IG distribution in Section 2.5.3 it is easy to check that marginal distributions of the $\boldsymbol{Y}$ process at time t have a NIG law.

Since $X_1(t)$, $X_2(t)$ and $\boldsymbol{R}(t) = (R_1(t), R_2(t))$ are mutually independent we have that

$$\phi_{\boldsymbol{Y}(t)}(u_1, u_2) = \mathbb{E}\left[e^{iu_1X_1(t)}\right]\mathbb{E}\left[e^{iu_2X_2(t)}\right]\mathbb{E}\left[e^{iu_1R_1(t)+iu_2R_2(t)}\right]. \tag{5.14}$$

The computation consists of two steps: firstly we compute $\mathbb{E}\left[e^{iu_1R_1(t)+iu_2R_2(t)}\right]$ which is the *chf* of the joint process $\boldsymbol{R} = \{(R_1(t), R_2(t)); t \geq 0\}$ at time t defined in (5.9). This can be done by conditioning with respect $H_1(t)$ and $Z_a(t)$, relying upon the independence of $W(t)$ and $\tilde{W}(t)$ and recalling the expression of the *chf* of a $IG_T\left(t, \frac{t^2}{\nu_R}\right)$ *rv*, which is given in Section 2.5.3, and the *chf* of its a-remainder,

obtained by applying the usual relation between characteristic functions of self-decomposable law which is given by (2.2). By direct computation we obtain that the *chf* of $\boldsymbol{R}(t)$ has the form shown in Equation (5.13) valuated at $\boldsymbol{w} = \boldsymbol{a} \circ \boldsymbol{u}$.

Secondly, we observe that first two terms of the right hand side of the Equation (5.14) are the *chf*'s of subordinated *BM*'s where subordinators are IG processes and hence their expressions are given by:

$$\mathbb{E}\left[e^{iu_j X_j(t)}\right] = \phi\left(\beta_j u_j + \frac{i}{2}u_j^2\gamma_j^2; t, \frac{t^2}{\nu_j}\right), \tag{5.15}$$

where $\phi(u; \mu, \lambda)$ denotes the *chf* of a *rv* with $IG_T(\mu, \lambda)$ law. Combining Equations (5.13), (5.14) and (5.15) we finally obtain (5.12). ■

The linear correlation coefficient of a bivariate *sd*-NIG process at time t can be directly computed and it is given by:

$$\rho_{\boldsymbol{Y}(t)} = \frac{a_1 a_2 a\left(\beta_{R_1}\beta_{R_2}\nu_R + \gamma_{R_1}\gamma_{R_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2\alpha_1}\sqrt{\sigma_2^2 + \mu_2^2\alpha_2}}. \tag{5.16}$$

As expected, if $a = 1$ we retrieve the original expression of correlation coefficient obtained by Ballotta and Bonfiglioli [9].

5.2 Simulation Algorithm

Simulating the paths of the model dynamics defined in Section 5.1 can be accomplished by simulating BM's on a stochastic time grid generated by the relative IG *sd* subordinators. These subordinators are only marginally IG, in order to get the joint trajectories one has to simulate the skeleton of $Z_a = \{Z_a(t); t \geq 0\}$ in (4.2) and therefore must have a way to obtain random samples distributed according to the law of the a-remainder Z_a of an IG distribution.

The methodology that we propose in this section is based on the close relation between *sd* laws and Lévy-driven OU processes. Following the naming convention in Barndorff-Nielsen and Shephard [13] we say that a Lévy-driven OU process $X = \{X(t); t \geq 0\}$ is a IG-OU process if its stationary law is an IG_B distribution with scale parameter δ and shape parameter γ. Now a well known result (see for instance Cont and Tankov [42] or Sato [112]) is that a given one-dimensional distribution D always is the stationary law of a suitable Lévy-driven OU process if and only if D is *sd*. As shown by Halgreen [65] the IG law is *sd* and can be taken as the stationary distribution of a fully-fledged OU process.

We recall that a Lévy-driven OU process is defined as,

$$X(t) = X(0)\, e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} dL(u), \tag{5.17}$$

where $L = \{L(t); t \geq 0\}$ is a Lévy process and $\lambda > 0$. In addition, as observed in Barndorff-Nielsen and Shephard [13], $X = \{X(t); t \geq 0\}$ is stationary if and only if the *chf* $\phi_X(u)$ of its marginal distribution is of the form

$$\phi_X(u) = \phi_X(ue^{-\lambda t})\chi_a(u,t),$$

where $\chi_a(u,t)$ is the *chf* of the second term of (5.17). On the other hand, due to the definition of *sd*, the last equation means that $\chi_a(u,t)$ is the *chf* of the a-remainder of the stationary law if one sets $a = e^{-\lambda t}$. We can then write

$$X(t) = X(0)\, e^{-\lambda t} + Z_{e^{-\lambda t}}(t). \tag{5.18}$$

Note that the parameter $e^{-\lambda t}$ is now time-dependent and the law of $Z_{e^{-\lambda t}}(t)$ coincides with that of $Z_a(t)$ with $a = e^{-\lambda t}$ only at a given time t, indeed $Z_{e^{-\lambda t}}(t)$ is not a Lévy process. Nevertheless, in practice the simulation of the skeleton of a IG-OU process relies on the generation of a *rv* that is distributed according to the law of the a-remainder of the stationary distribution setting $a = e^{-\lambda t}$.

We improve the results of Zhang and Zhang [125] relative to IG-OU processes and we derive an efficient algorithm to simulate the a-remainder of the $IG_B(\delta, \gamma)$, that is the building block for the generation of the trajectory of the process $Z_a(t)$.

Theorem 5.2.1 (Zhang and Zhang [125])**.** *The rv*

$$Z_a^\Delta = \int_0^\Delta e^{-\lambda(\Delta-u)} dL(u), \quad a = e^{-\lambda\Delta}, \quad \Delta > 0,$$

can be represented as

$$Z_a^\Delta \stackrel{d}{=} W_0^\Delta + \sum_{i=1}^{\tilde{N}^\Delta} W_i^\Delta,$$

where $W_0^\Delta \sim IG_B\left(\delta\left(1 - e^{-\frac{1}{2}\lambda\Delta}\right), \gamma\right)$*,* $\tilde{N}^\Delta$ *is a Poisson-distributed rv with parameter* $\delta\left(1 - e^{-\frac{1}{2}\lambda\Delta}\right)\gamma$ *and* W_i^Δ *are independent rv's with pdf:*

$$f_{W^\Delta}(w) = \frac{\gamma^{-1}}{\sqrt{2\pi}} w^{-\frac{3}{2}} \left(e^{\frac{1}{2}\lambda\Delta} - 1\right)^{-1} \left(e^{-\frac{1}{2}\gamma^2 w} - e^{-\frac{1}{2}\gamma^2 w e^{\lambda\Delta}}\right) \mathbb{1}_{\{w>0\}}(w). \tag{5.19}$$

Assuming for simplicity $\Delta = 1$, we can then rely on Theorem 5.2.1 to conceive the simulation procedure of two correlated IG *rv*'s with linear correlation coefficient a and hence of the *sd* subordinators of (4.2) simply setting $\lambda = -\log a$. We get:

$$Z_a \stackrel{d}{=} W_0 + \sum_{i=1}^{\tilde{N}} W_i,$$

where $W_0 \sim IG\left(\delta\left(1 - a^{\frac{1}{2}}\right), \gamma\right)$ and $\tilde{N} \sim Poisson\left(\delta\left(1 - a^{\frac{1}{2}}\right)\gamma\right)$.

Obtaining random samples according to IG and Poisson laws is relatively easy, whereas the simulation of W_i is non-standard and can be generated using the acceptance-rejection algorithm proposed by Zhang and Zhang [125] observing that:

$$f_W(w) \leq c \cdot \Gamma\left(\frac{1}{2}, \frac{1}{2}\gamma^2\right),$$

where $c = \frac{1}{2}\left(1 + e^{\frac{1}{2}\lambda}\right)$ and $\Gamma(\alpha, \beta)$ denote the law of a gamma *rv* with shape $\alpha > 0$ and rate $\beta > 0$.

Although Zhang and Zhang [125] illustrated a more accurate solution to reduce the expected number of iterations before acceptance c, acceptance-rejection algorithms might be slow and then sometimes inadequate for real time applications. This situation is exacerbated if the software implementation relies on interpreted languages like MATLAB, Python or R. In the following, we detail a simple and more efficient way to draw a random variate from the *pdf* $f_{W^\Delta}(w)$ without relying on acceptance-rejection methods.

Assuming once again $\Delta = 1$ and $\lambda = -\log a$, equation (5.19) becomes:

$$f_W(w) = \frac{\gamma^{-1}}{\sqrt{2\pi}} w^{-\frac{3}{2}} \left(a^{-\frac{1}{2}} - 1\right)^{-1} \left(e^{-\frac{1}{2}\gamma^2 w} - e^{-\frac{1}{2}\gamma^2 \frac{w}{a}}\right) \mathbb{1}_{\{w>0\}}(w).$$

We recall that a *rv* is distributed according to a Gamma law with shape $\alpha > 0$ and rate $\beta > 0$ if its *pdf* is:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the Euler Gamma function. Knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and observing that:

$$\int_1^{\frac{1}{a}} e^{-\frac{\gamma^2}{2} wy} \frac{\gamma^2}{2} w dy = e^{-\frac{\gamma^2}{2} w} - e^{-\frac{\gamma^2}{2}\frac{w}{a}},$$

we can write:

$$\begin{aligned} f_W(w) &= \int_1^{\frac{1}{a}} \frac{y^{-\frac{1}{2}}}{2\left(a^{-\frac{1}{2}} - 1\right)} \cdot \frac{\left(\frac{\gamma^2}{2} y\right)^{\frac{1}{2}} w^{-\frac{1}{2}} e^{-\frac{\gamma^2}{2} yw}}{\Gamma\left(\frac{1}{2}\right)} dy \\ &= \int_1^{\frac{1}{a}} f_Y(y) \cdot f_\Gamma\left(w \middle| \alpha = \frac{1}{2}, \beta = \frac{\gamma^2}{2} y\right) dy. \end{aligned}$$

This means that $f_W(w)$ is a mixture of a Gamma law $\Gamma\left(\alpha = \frac{1}{2}, \beta = \frac{\gamma^2}{2} y\right)$ and a law whose *pdf* and *cdf* are respectively:

$$f_Y(y) = \frac{y^{-\frac{1}{2}}}{2\left(a^{-\frac{1}{2}}-1\right)}\mathbb{1}_{1\leq y\leq \frac{1}{a}}$$
$$F_Y(y) = \frac{y^{\frac{1}{2}}-1}{a^{-\frac{1}{2}}-1}\mathbb{1}_{1\leq y\leq \frac{1}{a}}.$$

The simulation of Z_a and of the *rv* Y distributed according to the law with *cdf* $F_Y(y)$ is straightforward as is summarized in Algorithms 6 and 7, respectively.

Algorithm 6 Simulation of Z_a

1: Simulate $W_0 \sim IG\left(\delta\left(1-\sqrt{a}\right),\gamma\right)$
2: Simulate $\tilde{N} \sim Poisson\left(\delta\left(1-\sqrt{a}\right)\gamma\right)$
3: Simulate $W_i, i = 1\ldots\tilde{N}$ using Algorithm 7
4: Set $Z_a = \sum_{i=0}^{\tilde{N}} W_i$

Algorithm 7 Simulation of $W_i, \tilde{N}$

1: Simulate $U_i \sim U([0,1])$
2: Compute $Y_i = \left(1+\left(a^{-\frac{1}{2}}-1\right)U_i\right)^2$
3: Simulate W_i from a $\Gamma\left(\frac{1}{2}, \frac{1}{2}\gamma^2 Y_i\right)$

In Table 5.1 we compare the theoretical values of the first five moments of Z_a against those obtained by MC simulations using Algorithm 6. We observe that the precision of the algorithms is high for different values of $a \in (0,1)$. In Figure 5.1 we draw the probability density function of two correlated *rv* $X, Y \sim IG_B(\delta,\gamma)$ and their scatter plot for two different values of a.

The proposed algorithm is approximately ten times faster than the one presented by Zhang and Zhang [125], as one can see from results reported in Table 5.2. This time complexity analysis was implemented on a PC having an Intel Core i5-10210U 2.11 GHz processor and all codes are written in *MATLAB*.

The simulation of the a-remainder of an IG law provides the generation of the joint trajectories of the *sd* subordinator $\boldsymbol{H} = \{(H_1(t), H_2(t); t \geq 0)\}$ and therefore those of the models presented in Section 5.1. The application of these MC schemes will be shown in Chapter 7 .

Of course Algorithm 6 is instrumental in simulating trajectories of the IG-OU process in (5.17): a possible realization of the process is shown in Figure 5.2.

In order to check that the improvement of the speed of the algorithm we proposed is not due to any particular choice of the parameters, we compare Algorithm 6 with the ones proposed by Zhang and Zhang [125]. In Figure 5.3 we run the algorithm

$\mathbb{E}[Z_a^n]$	T	N
$\mathbb{E}[Z_a^1]$	3.00	3.00
$\mathbb{E}[Z_a^2]$	10.47	10.48
$\mathbb{E}[Z_a^3]$	42.17	42.26
$\mathbb{E}[Z_a^4]$	194.72	195.49
$\mathbb{E}[Z_a^5]$	1021.84	1029.41

(a) $a = 0.1$

$\mathbb{E}[Z_a^n]$	T	N
$\mathbb{E}[Z_a^1]$	1.67	1.67
$\mathbb{E}[Z_a^2]$	3.89	3.89
$\mathbb{E}[Z_a^3]$	11.91	11.90
$\mathbb{E}[Z_a^4]$	45.58	45.46
$\mathbb{E}[Z_a^5]$	209.90	208.97

(b) $a = 0.5$

$\mathbb{E}[Z_a^n]$	T	N
$\mathbb{E}[Z_a^1]$	1.00	1.00
$\mathbb{E}[Z_a^2]$	1.76	1.76
$\mathbb{E}[Z_a^3]$	4.56	4.59
$\mathbb{E}[Z_a^4]$	15.77	15.89
$\mathbb{E}[Z_a^5]$	67.94	68.66

(c) $a = 0.7$

$\mathbb{E}[Z_a^n]$	T	N
$\mathbb{E}[Z_a^1]$	0.33	0.33
$\mathbb{E}[Z_a^2]$	0.39	0.40
$\mathbb{E}[Z_a^3]$	0.85	0.86
$\mathbb{E}[Z_a^4]$	2.66	2.68
$\mathbb{E}[Z_a^5]$	10.71	10.72

(d) $a = 0.9$

Table 5.1. Moments comparison using $N_{sim} = 10^6$ for $\delta = 5$ and $\gamma = 1.5$. T stands for the values of the theoretical n-th moment, whereas N stands for the MC-based estimations.

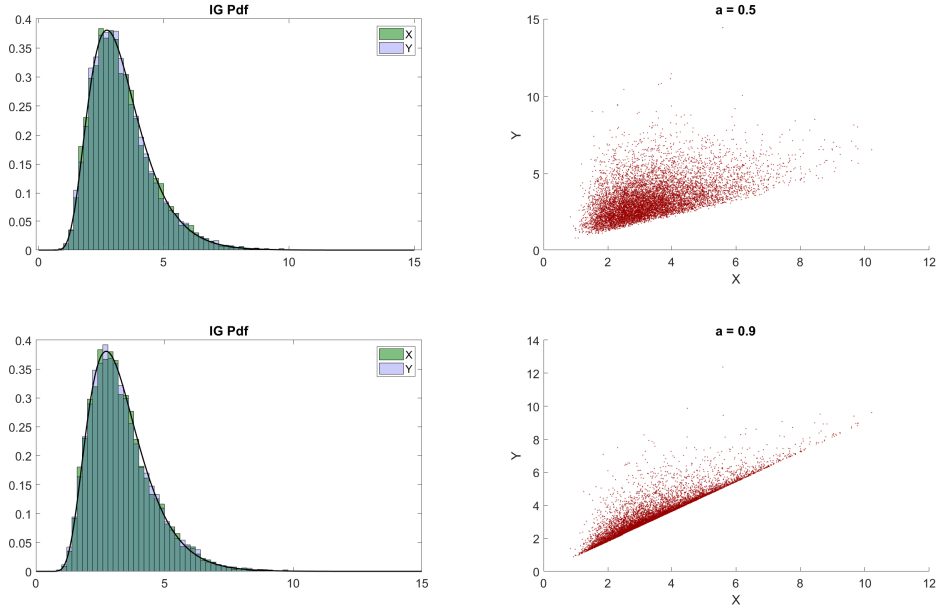

Figure 5.1. Correlated rv X and Y for $\delta = 5$ and $\gamma = 1.5$ and their scatter plots for $a = 0.5$ and $a = 0.9$.

Algorithm	N_{sim}	10^3	10^4	10^5	10^6
Algorithm 6	Time (s)	$2.2 \cdot 10^{-3}$	$2.2 \cdot 10^{-2}$	$1.7 \cdot 10^{-1}$	$2.1 \cdot 10^{0}$
Zhang and Zhang [125]	Time (s)	$1.8 \cdot 10^{-2}$	$2.0 \cdot 10^{-1}$	$1.7 \cdot 10^{0}$	$1.8 \cdot 10^{1}$

Table 5.2. Average computational time of one hundred runs of Algorithm 6, varying the number of simulations, compared with the computational time of the original one proposed by Zhang and Zhang [125].

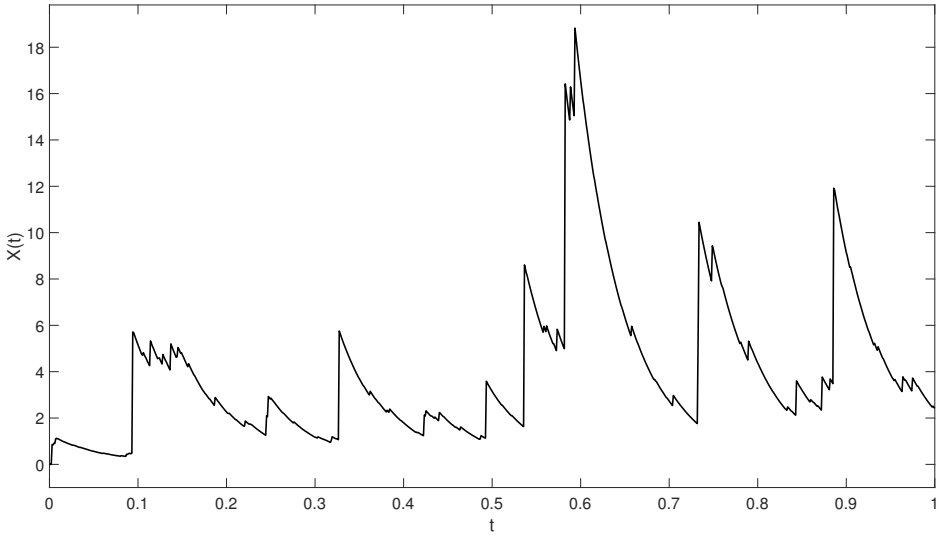

Figure 5.2. A possible realization of the IG-OU process with parameters $\delta = 2$, $\gamma = 0.5$, $\lambda = 0.02$ over the interval $[0, T]$, with $T = 1$.

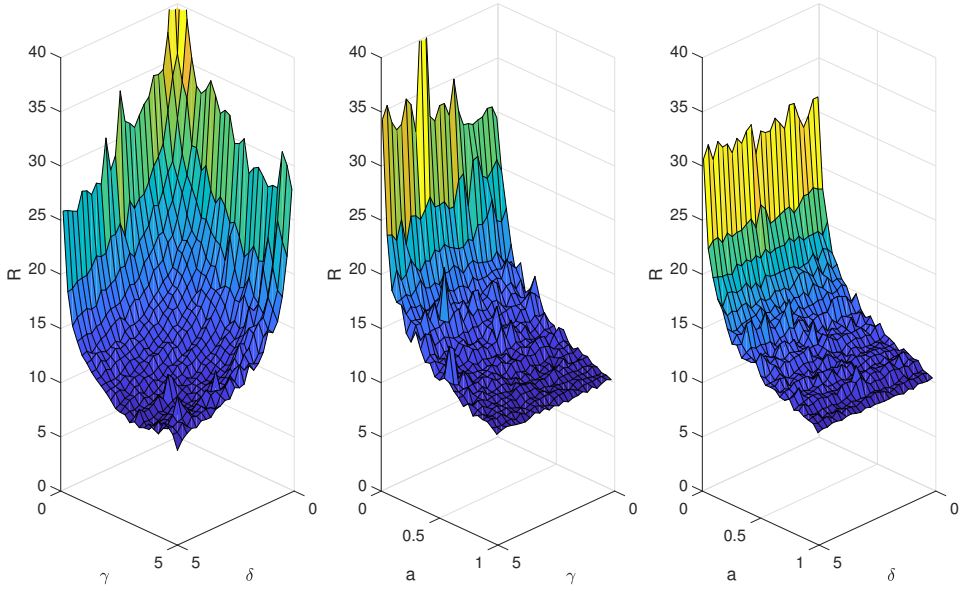

Figure 5.3. Value of R for different choices of the parameters δ, γ and a.

several times fixing one of the three parameters involved, namely δ, γ and a and varying the other ones. We compute the computational time ratio R as:

$$R = \frac{\text{Time Algorithm Zhang}}{\text{Time Algorithm 6}}.$$

We observe that the algorithm we proposed is considerably faster than the original one by Zhang and Zhang [125] for different reasonable choices of the parameters $(\delta, \gamma, \lambda)$.

Finally, we remark that our path-generation procedure is also applicable to the stochastic volatility models based on IG-OU processes proposed by Barndorff-Nielsen [11] and extended by Andersson [5].

In the following section we details how to simulate the process $\boldsymbol{H}$ and the corresponding subordinated Brownian motion.

Chapter 6

Numerical Methods for option pricing

In this chapter we collect all the numerical methods we need to sample from different distributions, to simulate the skeleton of the processes we presented in previous chapters and to price derivatives. All these techniques are widley used in literature and the interested reader can refer to Glasserman [62] for an exhaustive discussion about Monte Carlo methods in finance and to Seydel [117] for an overview on standard techniques such as finite difference, finite elements and Monte Carlo schemes for option pricing when the price dynamics is modeled using the Brownian motion. On the other hand, Cont and Tankov [42] deals with numerical techniques in the more general setting of Lévy processes whereas Devroye [50] provides a bunch of algorithms to sample random numbers according to different kind of distribution.

6.1 Simulation of stochastic processes

In this section we show how we can simulate the paths of a given stochastic process $X = \{X(t); t \geq 0\}$. Roughly speaking, in order to generate possible realizations of a given stochastic process one needs to be able to simulate pseudo-random numbers according to a given distribution.

Almost all computers are able to simulate random variables with uniform distribution on $[0, 1]$ and, starting from an independent identically distributed random samples of uniform random variables on $[0, 1]$, it is possible to generate from a wide range of laws. For instance we can sample independent realizations of $X \sim \mathcal{N}(0, 1)$ by using the Box-Muller algorithm (see Box and Muller [26] and Devroye [50]).

If one need to sample normally distributed correlated random variable X, Y with correlation ρ one can consider $X, Z \sim \mathcal{N}(0, 1)$ with X independent of Z and define:

$$Y = \rho X + \sqrt{1 - \rho^2} Z,$$

obtaining $Y \sim \mathcal{N}(0, 1)$ and $corr(X, Y) = \rho$. Observe that this technique is based on the Cholesky decomposition (Quarteroni et al. [104]).

On the other hand, using the scaling properties of the Gamma distribution we discussed in Section 2.5.2, it is easy to show that to draw random samples of $X \sim \Gamma(c, \lambda)$ it is enough being able to simulate from $X \sim \Gamma(a, 1)$. According to Devroye [50] the cases $a \leq 1$ and $a > 1$ should be considered separately. The algorithm for $a \leq 1$ (which is the most often case in applications) can be found in Cont and Tankov [42, Algorithm 6.8], whereas we can sample from a $\Gamma(a, 1)$ with $a > 1$ using Cont and Tankov [42, Algorithm 6.9]. To sample from an inverse Gaussian distribution $IG_T(\mu, \lambda)$ the algorithm proposed by Michael, Schucany and Hass can be adopted (see Cont and Tankov [42, Algorithm 6.9]).

Sample trajectories of the compound Poisson process can be obtained combining together random sampling from the Poisson and from the uniform law (see Cont and Tankov [42, Chapter 6]), whereas random samples from Poisson distributions can be obtained by using Devroye [50, Lemma 3.1]. The derived algorithm is substantially based on the generation of random independent samples from an exponential law: this last task can be easily achieved by adopting the inversion method presented in Devroye [50, Chapter 2].

6.1.1 Subordinate Brownian motion simulation

All the models we have presented in previous chapters are based on Brownian subordination. The simulation of a subordinated Brownian motion $X = \{W(S(t)); t \in [0, T]\}$ on a fixed time grid $0 = t_0 < t_1 < \cdots < t_n = T$ where W is a Brownian motion with drift μ and diffusion σ and S is a subordinator is done in two steps. First we simulate the path of the subordinator S on the deterministic grid obtaining a "stochastic grid" $0 = S_{t_0} < S_{t_1}, \ldots, < S_{t_n}$ and hence we simulate the Brownian motion on that grid. This procedure is summarized in Algorithm 8 proposed by Cont and Tankov [42].

Algorithm 8 Simulation of a subordinated Brownian motion

1: Simulate the increments of the subordinator: $\Delta S_i = S_{t_i} - S_{t_{i-1}}$, where $S_0 = 0$.
2: Simulate n independent standard random variables $N_1, \ldots, N_n$. Set $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \mu \Delta S_i$.
3: The discretized trajectory is given by $X(t_i) = \sum_{k=1}^{i} \Delta X_k$.

When the increments ΔS_i are distributed according to a gamma law we obtain the Variance Gamma processes of Madan and Seneta [88], whereas when they are distributed according an inverse Gaussian law we get the Normal Inverse Gaussian process of Barndorff-Nielsen [11]. The aforementioned Algorithm can be adapted to simulate the skeleton of the stochastic processes we introduced in previous chapters.

6.1.2 Simulation of self-decomposable subordinators

In order to simulate the paths of the bivariate versions of the *sd* Variance Gamma and that of the *sd* NIG processes we must simulate the process $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$

which is defined according to Definition 4.1.1 as:

$$H_2(t) = aH_1(t) + Z_a(t),$$

where $a \in (0,1)$ and $H_1(t) \sim \Gamma(\alpha t, \beta)$ or $H_1(t) \sim IG_B(\delta t, \gamma)$. We also remember that the increment of the process H_1 over the time interval Δt denoted by ΔH_1 is such that $\Delta H_1 \sim \Gamma(\alpha \Delta t, \beta)$ and $\Delta H_1 \sim IG_B(\delta \Delta t, \gamma)$ respectively. As mention above, sampling algorithms from a gamma or an inverse Gaussian law are known and the simulation of the a-remainder $Z_a(t)$ can be achieved by using Algorithm 2 or Algorithm 6 if $H_1(t)$ is distributed according to a gamma or an inverse Gaussian law respectively. This procedure is reported in Algorithm 9.

Algorithm 9 Simulation of $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \in [0,T]\}$

Simulation of the process $\boldsymbol{H}$ on evenly spaced time grid $0 = t_0 < t_1 < \ldots, < t_n = T$ with step Δt.

1: Simulate the increments of the subordinator H_1: $\Delta H_{1,i} = H_{1,t_i} - H_{1,t_{i-1}}$, where $H_{1,t_0} = 0$.
2: Simulate the increments of the process Z_a: $\Delta Z_{a,i} = Z_{t_i} - Z_{t_{i-1}}$ where $Z_{t_0} = 0$.
3: Set $\Delta H_{2,i} = a\Delta H_{1,i} + \Delta Z_{a,i}$
4: The discretized trajectories are given by:

$$H_1(t_i) = \sum_{k=1}^{i} \Delta H_{1,k}, \qquad H_2(t_i) = \sum_{k=1}^{i} \Delta H_{2,k}.$$

6.1.3 Self-decomposable processes simulation

In this section we use the results of the previous section and that ones we derived in Chapters 3, 4 and 5 to simulate the paths of the $VG++$ process and that of the bivariate Variance Gamma and NIG process with stochastic delay.

The simulation algorithm of the $VG++$ process X is the simplest one to achieve. A realization X can be obtained by using Algorithm 8 where the increment ΔS_i is that of the subordinating process Z_a^{++} whose increments can be obtained by Algorithm 1 or 2.

On the other hand, the simulation of the skeleton of the multivariate version of the Variance Gamma and Normal Inverse Gaussian processes with stochastic delay is trickier. First of all observe that, in order to properly simulate those processes we need to simulate objects of the form:

$$\boldsymbol{X} = \{(W(t), W(at)); t \geq 0\},$$
$$\boldsymbol{Y} = \{(W_1^\rho(t), W_2^\rho(at)); t \geq 0\},$$

where the deterministic time t can be replaced by a general subordinator $H = \{H(t); t \geq 0\}$. The main problem with the simulation is neither process $\boldsymbol{X}$ nor $\boldsymbol{Y}$ are

Lévy processes and hence we cannot use the property of the independent increments to simulate the path of the process.

The procedure to simulate the skeleton of the process $\boldsymbol{X}$ is reported in Algorithm 10 whereas an equivalent numerical routine which relies upon the Brownian bridge technique is shown in Algorithm 11.

Algorithm 10 Simulation $\boldsymbol{X} = \{(W(t), W(at)), t \in [0, T]\}$

Simulation of $\boldsymbol{X}$ on an evenly spaced time grid $\Pi_1 = 0 = t_0 < t_1 < \cdots < t_n = T$ with step Δt.

1: Simulate $2n$ *iid* random variables ΔW_i, $i = 1, \ldots, 2n$.
2: Create a second grid $\Pi_2 = a\Pi_1 = at_0 < at_1 < \cdots < at_n$ and merge it with Π_1 obtaining Π.
3: Sort $\Pi = s_0 < s_1 < \cdots < s_{2n}$.
4: Set $W(s_i) = \sum_{k=1}^{i} \Delta W_i$.
5: For all $t_i \in \Pi_j$ find $s_k \in \Pi$ such that $t_i = s_k$, for $j = 1, 2$.
6: Set $X_j(t_i) = W(s_k)$, for $j = 1, 2$.

Algorithm 11 Simulation $\boldsymbol{X} = \{(W(t), W(at)); t \in [0, T]\}$ (alternative Algorithm)

Simulation of $\boldsymbol{X}$ on an evenly spaced time grid $\Pi_1 = 0 = t_0 < t_1 < \cdots < t_n = T$ with step Δt.

1: Simulate n *iid* random variables ΔW_i, $i = 1, \ldots, n$.
2: Create a second grid $\Pi_2 = a\Pi_1 = at_0 < at_1 < \cdots < at_n$.
3: Set $X_1(t_i) = \sum_{k=1}^{i} \Delta W_i$.
4: For all $s_k \in \Pi_2$ find the time interval $[t_j, t_{j+1}]$ in Π_1 such that $s_k \in [t_j, t_{j+1}]$.
5: Construct a Brownian bridge $W(s_i)$ at s_i, pinned at $X_1(t_j)$ and $X_1(t_{j+1})$ between t_j and t_{j+1}.
6: Set $X_2(t_i) = W(s_k)$.

Algorithms 12, 13 and 14 show how to simulate processes defined, respectively, in Section 4.2.1, Section 4.2.2 and Section 4.2.3, i.e. when we use self-decomposable subordinators H_1 and H_2 which are distributed according to a gamma law simulated by using Algorithm 9. The same algorithms can be adapted to simulate the trajectories of the 2D *sd* NIG process with by switching the gamma with the inverse Gaussian distribution and the a-remainder of a gamma law with the ones of the inverse Gaussian law.

In the following section we show how the aforementioned methods can be used to price derivatives using the Monte Carlo method, relying upon the risk neutral valuation formula we mentioned in Section 2.8.

Algorithm 12 Simulation of 2D Semeraros's *sd* Variance Gamma process

Simulation of a bivariate Variance Gamma $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \in [0,T]\}$ in equispaced time grid $0 = t_0 < t_1 < \cdots < t_n = T$ with step Δt.

1: Simulate n independent gamma variables $\Delta I_{j,i} \sim \Gamma\left(\Delta t \left(\frac{1}{\alpha_j} - A\right), \frac{1}{\alpha_j}\right)$ for $j = 1, 2$, and $i = 1, \ldots, n$.
2: Simulate n independent gamma variables $\Delta H_{1,i} \sim \Gamma(\Delta t \cdot A, 1)$.
3: Generate n independent $\Delta Z_{a,i}$ variables as a-remainder of a Gamma distribution $\Gamma(\Delta t \cdot A, 1)$ using Algorithm 1.
4: Set $\Delta H_{2,i} = a\Delta H_{1,i} + \Delta Z_{a,i}$
5: Set $\Delta G_{j,i} = \Delta I_{j,i} + \alpha_j \Delta H_{j,i}$ for $j = 1, 2$.
6: For $j = 1, 2$ simulate n *iid* $\mathcal{N}(0,1)$ random variables $N_{j,1}, \ldots, N_{j,n}$.
Set $\Delta Y_{j,i} = \sigma_j N_{j,i} \sqrt{\Delta G_{j,i}} + \mu_j \Delta G_{j,i}$ for all i.
7: The discretized trajectories are $Y_j(t_i) = \sum_{k=1}^{i} \Delta Y_{j,k}$ for $j = 1, 2$.

6.2 Monte Carlo methods for option pricing

Monte Carlo methods were first used by Enrico Fermi in 1930 to study neutron diffusion. In 1940 Stanislaw Ulam developed the modern formulation of the Markov Chain Monte Carlo method and John Von Neumann understood its importance. See Metropolis [92] for a brief history on the origin of Monte Carlo method. Nowadays Monte Carlo methods are very popular is many fields of science such as computational biology, computer graphics, applied statistics, artificial intelligence and they find applications even to law. In finance they are used for the evaluation of investment in projects, insurance and risk analysis. Moreover, the Monte Carlo method is a must for derivatives valuations, mainly when one deals with exotic contingent claims for which closed formulas are not available and other methods based on the resolution of the partial differential equation (PDE) or on the Fourier transform are hard to apply.

In this section we briefly sketch how the Monte Carlo scheme can be used to value contingent claims, without claiming to exhaust all possible aspects. We refer to Seydel [117] and Glasserman [62] for a detailed analysis of the Monte Carlo methods in Finance.

The martingale approach to option pricing we discussed in Section 2.8 shows that the price of a given contingent claim H at time $t = 0$ can be computed by the formula (2.30):

$$\Pi_H(0) = e^{-rT} \mathbb{E}^{\mathbb{Q}} [H(T) | \mathcal{F}_0] .$$

Therefore, in order to price derivatives it is sufficient to evaluate the expected value of the random variable $H(T)$ under the risk-neutral measure $\mathbb{Q}$: namely the evaluation an integral is required. To this aim the Monte Carlo scheme can be adopted. If we want to price an European call option which payoff at maturity T is given by $H(T) = \max(S(T) - K, 0)$ then the valuation procedure is extremely simple and it

Algorithm 13 Simulation of 2D Semeraro-Luciano's *sd* Variance Gamma process

Simulation of a bivariate Variance Gamma $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \in [0,T]\}$ on evenly spaced time grid $0 = t_0 < t_1 < \cdots < t_n = T$ with step Δt.

1: Simulate n independent gamma variables $\Delta I_{j,i} \sim \Gamma\left(\Delta t\left(\frac{1}{\alpha_j} - A\right), \frac{1}{\alpha_j}\right)$ for $j = 1, 2$ and $i = 1, \ldots, n$.
2: Simulate n independent gamma variables $\Delta H_{1,i} \sim \Gamma\left(\Delta t \cdot A, 1\right)$ for $i = 1, \ldots, n$.
3: Generate n independent $\Delta Z_{a,i}$ variables as a-remineder of a gamma distribution $\Gamma\left(\Delta t \cdot A, 1\right)$ for $i = 1, \ldots, n$ using Algorithm 1.
4: Set $\Delta H_{2,i} = a\Delta H_{1,i} + \Delta Z_{a,i}$
5: Simulate n *iid* $\mathcal{N}(0,1)$ random variables $N_{j,1}, \ldots, N_{j,n}$ for $j = 1, 2$.
6: Set $\Delta Y_{I_{j,i}} = \sigma_1 N_{j,i}\sqrt{\Delta I_{j,i}} + \mu_j \Delta I_{j,i}$.
7: Set $H_1(t_i) = \sum_{k=1}^{i} \Delta H_{1,k}$ and $H_2(t_i) = \sum_{k=1}^{i} \Delta H_{2,k}$.
8: Consider the partitions $\Pi_1 = H_1(t_0) < H_1(t_1) < \cdots < H_1(t_n)$ and $\Pi_2 = a\Pi_1$. Merge Π_1 and Π_2 together obtaining the partition $\Pi = h_0 < h_1 < \cdots < h_{2n}$.
9: Simulate correlation Brownian motions W_1^ρ and W_2^ρ on partition Π.
10: For each $H_1(t_i) \in \Pi_1$ find $h_k \in \Pi$ such that $H_1(t_i) = h_k$ and for each $H_2(t_i) \in \Pi_2$ find $s_k \in \Pi$ such that $H_2(t_i) = s_k$.
11: Set:

$$W_1^\rho(H_1(t_i)) = W_1^\rho(h_k),$$
$$W_2^\rho(aH_1(t_i)) = W_2^\rho(s_k).$$

12: Simulate a subordinated Brownian motion where the subordinator is Z_a using Algorithm 8. $\tilde{W}(Z_a(t_i))$ represents the value of the subordinated Brownian motion at time t_i.
13: Set $Y_{H_1}(t_i) = \alpha_1\mu_1 H_1(t_i) + \sqrt{\alpha_1}\sigma_1 W_1^\rho(H_1(t_i))$.
14: Set $Y_{H_2}(t_i) = \alpha_2\mu_2 H_2(t_i) + \sqrt{\alpha_2}\sigma_2\left(W_2^\rho(aH_1(t_i)) + \tilde{W}(Z_a(t_i))\right)$.
15: Set $Y_{I_j}(t_i) = \sum_{k=1}^{i} \Delta Y_{I_j,k}$ for $j = 1, 2$.
16: Set $Y_j(t_i) = Y_{I_j}(t_i) + Y_{H_j}(t_i)$ for $j = 1, 2$.

Algorithm 14 Simulation of 2D Ballotta-Bonfiglioli's *sd* Variance Gamma process

Simulation of a bivariate Variance Gamma $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \in [0,T]\}$ on evenly spaced time grid $0 = t_0 < t_1 < \cdots < t_n = T$ with step Δt.

1: Simulate a Variance Gamma process $X_j(t_i)$ with parameters $(\beta_j, \gamma_j, \nu_j)$ $j = 1, 2$ using Algorithm 8.
2: Simulate n independent gamma variables $\Delta H_{1,i} \sim \Gamma\left(\frac{\Delta t}{\nu_R}, \frac{1}{\nu_R}\right)$ for $i = 1, \ldots, n$.
3: Generate n independent $\Delta Z_{a,i}$ variables as a-remainder of a Gamma distribution $\Gamma\left(\frac{\Delta t}{\nu_R}, \frac{1}{\nu_R}\right)$ for $i = 1, \ldots, n$.
4: Set $\Delta H_{2,i} = a\Delta H_{1,i} + \Delta Z_{a,i}$
5: Set $H_1(t_i) = \sum_{k=1}^{i} \Delta H_{1,k}$ and $H_2(t_i) = \sum_{k=1}^{i} \Delta H_{2,k}$.
6: Simulate a subordinated Brownian motion where the subordinator is Z_a using Algorithm 8. $\tilde{W}(Z_a(t_i))$ represents the value of the subordinated Brownian motion at time t_i.
7: Simulate the non Lévy process $\{(W(H_1(t)), W(aH_1(t))); t \geq 0\}$ using Algorithm 10.
8: Set $R_1(t_i) = \beta_{R_1} H_1(t_i) + \gamma_{R_1} W(H_1(t_i))$.
9: Set $R_2(t_i) = \beta_{R_2} H_2(t_i) + \gamma_{R_2} \left(W(aH_1(t_i)) + \tilde{W}(Z_a(t_i)\right)$.
10: Set $Y_j(t_i) = X_j(t_i) + a_j R_j(t_i)$ for $j = 1, 2$.

is summarized in Algorithm 15. In particular it suffices to simulate a large number of trajectories of the underlying price process at maturity T under the risk neutral measure $\mathbb{Q}$, compute the payoff $H(T)$ for all the realizations, take the mean and discount it by the factor e^{-rT}.

It is worth noting that Monte Carlo algorithm (whose order of convergence is $O\left(N^{-1/2}\right)$ independently of the dimension of the problem where N is the number of simulations) might underperform methods based on the Fourier transform and PDE in terms of speed and order of convergence. Nevertheless, Monte Carlo algorithms are flexible, easy to implement and allow to solve derivatives pricing problems which cannot be solved using a PDE or Fourier approach. To this end, several methods for option pricing based on the Monte Carlo technique have been developed over the years. An interested read can refer to Seydel [117], Cont and Tankov [42] for a general discussion and to Longstaff and Schwartz [83], Broadie and Glasserman [29], Fu et al. [59], Tseng and Barz [121] and Boogert and de Jong [25] for American style options pricing. Finally, applications of the Monte Carlo scheme for the evaluation of barrier options can be found in Karatzas and Shreve [76] and Caflisch et al. [34].

6.3 Fourier Methods for option pricing

Contrary to the classical Black-Scholes model, in many exponential-Lévy models there are no explicit formulas for European call option pricing, because the density of the Lévy process is not known in closed form. Nevertheless, the characteristic

Algorithm 15 Monte Carlo simulation for European options

1: Fix a number of simulation N large.
2: Simulate the underlying process $S = \{S(t), t \in [0,T]\}$ over the grid $0 = t_0 < t_1 < \cdots < t_n = T$ N times and get the final results $(S_k(T))_{k=1}^N$.
3: Valuate the payoff function H at T and obtain:

$$(H(S_k(T)))_{k=1}^N = (\max(S_k(T) - K, 0))_{k=1}^N$$

4: Given the value of S at $t = 0$, an unbiased estimate of the risk-neutral expectation $\mathbb{E}^{\mathbb{Q}}[H]$ is given by:

$$\hat{\mathbb{E}}^{\mathbb{Q}}[H] = \frac{1}{N}\sum_{k=1}^{N} H(S_k(T))$$

5: The value $H(0)$ of the European option at time $t = 0$ is given by:

$$H(0) = e^{-rT}\hat{\mathbb{E}}^{\mathbb{Q}}[H],$$

function of Lévy models can be expressed in terms of elementary functions and this has led to the developed of numerical methods based on the Fourier Transform. An overview of the most used Fourier-based methods can be found in Schmelzle [113], whereas a MATLAB implementation of many of Fourier methods can be found in Kienitz and Wetterau [77]. The most famous pricing algorithms based on the Fourier approach have been proposed by Carr and Madan [37], Attari [7], Lewis [81], Lord et al. [84] and Fang and Oosterlee [55]. All these methods present strengths and weaknesses and they have been compared in Schmelzle [113] and Kienitz and Wetterau [77]. Even if they are mainly used for European options evaluations some of them have been extended in order to be applicable also for American and exotic option pricing (see for example Fang [54]).

In particular, the aforementioned methods can be efficiently applied when we deal with financial derivatives written on a single underlying risky asset. If we scale to a multi-asset market these methods become more complicated to use and hard to be implemented. Nevertheless, aiming at pricing a spread option Hu and Zhou [72] proposed an approach based on a double inversion of the Fourier transform whereas Caldana and Fusai [35] derived an approximated formula for Spread option pricing which requires only one Fourier inversion, leading to an easier numerical implementation. Another algorithm for spread option pricing based on cosine expansions have been proposed by Pellegrino and Sabino [99], whereas Pellegrino and Sabino [98] have shown how a moment matching technique for its valuation can be adopted.

In the next section we briefly focus only on two methods we used in our work, namely the method proposed by Carr and Madan [37] and Caldana and Fusai [35].

Moreover, for the sake of completeness, all the analysis we performed by using the Carr-Madan method have been conducted by using the Lewis [81] procedure, obtaining similar results.

6.3.1 One dimensional pricing with FFT

The first pricing algorithm based on the Fast Fourier Method introduced by Cooley and Tukey [44] was proposed by Carr and Madan [37] and proceeds as follows. Suppose we want to price an European call option $C_T(k)$ with strike price K, where $k = \log K$ is the log-strike and T is the maturity. Let the risk-neutral density of the log-price s_T in T denoted by q_T (note that this density could be unknown in closed form), whereas $\tilde{q}_T(s)$ denotes the density of the prices S at T. Let the characteristic function Φ of this density be:

$$\Phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T(s)\, ds, \quad u \in \mathbb{R}. \tag{6.1}$$

Remember that $\Phi_T(u)$ is assumed to have an analytically closed form, which is true for many Lévy processes. The initial call value $C_T(k)$ can be related to risk-neutral density $q_T(s)$ by:

$$\begin{aligned} C_T(k) &= \int_0^{\infty} e^{-rT} (S_T - K)^+ \tilde{q}_T(S)\, dS \\ &= \int_k^{\infty} e^{-rT} \left(e^s - e^k\right) q_T(s)\, ds. \end{aligned} \tag{6.2}$$

Observe that we can not compute the Fourier Transform of $C_T(k)$ because $\lim_{k\to-\infty} C_T(k)$ is constant and so $C_T(k)$ is not integrable. Nevertheless, we can choose $\alpha > 0$ and define the "modified call option value" as $c_T(k) = e^{\alpha k} C_T(k)$: $c_T(k) \to 0$ for $k \to -\infty$ and so we can compute the Fourier transform in k.

$$\Psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk \tag{6.3}$$

Observe that one can compute $C_T(k)$ by inverting (6.3), obtaining:

$$\begin{aligned} C_T(k) &= e^{-\alpha k} c_T(k) \\ &= e^{-\alpha k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \Psi_T(v)\, dv \\ &= e^{-\alpha k} \frac{1}{\pi} \int_0^{\infty} e^{-ivk} \Psi_T(v)\, dv. \end{aligned} \tag{6.4}$$

Therefore, if we are able to compute the last integral in Equation 6.4 we obtain the option value. At this point, all we need is an expression for $\Psi_T(v)$. In particular we have that:

$$\begin{aligned}\Psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk \\ &\overset{(6.2)}{=} \int_{-\infty}^{\infty} e^{ivk} \int_k^{\infty} e^{\alpha k} e^{-rT} \left(e^s - e^k\right) q_T(s)\, ds dk \\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left(\int_{-\infty}^{s} \left(e^{s+\alpha k} - e^{(1+\alpha)k}\right) e^{ivk} dk \right) ds,\end{aligned}$$

where in the last equality we have switched the order of integration. The inner integral can be split up in two parts which are computed separately. In particular we get:

$$\begin{aligned}\int_{-\infty}^{s} e^{s+\alpha k+ivk} dk &= \int_{-\infty}^{s} e^{s+(\alpha+iv)k} dk = \frac{e^{(i+\alpha+iv)s}}{\alpha+iv}, \\ \int_{-\infty}^{s} e^{(\alpha+1+iv)k} dk &= \frac{e^{(1+\alpha+iv)s}}{1+\alpha+iv},\end{aligned}$$

and hence we obtain that:

$$\begin{aligned}\Psi_T(v) &= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left[\frac{e^{(i+\alpha+iv)s}}{\alpha+iv} - \frac{e^{(1+\alpha+iv)s}}{1+\alpha+iv} \right] ds \\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s)\, e^{s(1+\alpha+iv)} \frac{1+\alpha+iv-\alpha-iv}{(\alpha+iv)(1+\alpha+iv)} ds \\ &= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} \int_{-\infty}^{\infty} q_T(s)\, e^{s(1+\alpha+iv)} ds.\end{aligned}$$

Looking at the last integral of the previous equation we note that:

$$\int_{-\infty}^{\infty} q_T(s)\, e^{s(1+\alpha+iv)} = \int_{-\infty}^{\infty} q_T(s)\, e^{is(-\alpha i+v-i)} \overset{(6.1)}{=} \Phi_T(v-(\alpha+1)i),$$

which is the characteristic function of a Lévy processes which is assumed to be known in closed form. To this end we have that:

$$\Psi_T(v) = \frac{e^{-rT}\Phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}. \tag{6.5}$$

By substituting (6.5) in Equation (6.4) we obtain

$$\begin{aligned}C_T(k) &= e^{-\alpha k}\frac{1}{\pi}\int_0^{\infty} e^{-ivk}\Psi_T(v)\, dv \\ &\overset{(6.4)}{=} e^{-\alpha k}\frac{1}{\pi}\int_0^{\infty} e^{-ivk} \frac{e^{-rT}\Phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} dv\end{aligned} \tag{6.6}$$

Solving this integral we get the required call Option price.
We finally remark that if $\alpha = 0$ then the denominator vanishes if $v = 0$ this is the reason why we need the a $\alpha > 0$. The choice of α is discussed in Carr and Madan [37].

6.3.2 FFT algorithm

The FFT algorithm introduced by Cooley and Tukey [44] is a very efficient way to compute the sum:

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \quad k = 1, \ldots, N, \tag{6.7}$$

where N is typically a power of 2, even if this hypothesis can be relaxed.
By taking Equation (6.4) and applying the trapezoid rule for quadratures we get:

$$C_T(k) = e^{-\alpha k}\frac{1}{\pi}\int_0^{\infty} e^{-ivk}\Psi_T(v)\,dv \simeq e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-iv_j k}\Psi_T(v_j)\,\eta, \tag{6.8}$$

where $v_j = \eta(j-1)$. Observe that we truncated the upper limit of the integral, using N points with a step size of η. The upper limit of integration is then $a = N\eta$. The FFT algorithm can be used to efficiently evaluate integrals of the form (6.6) and hence to price European call options.
More precisely, the FFT algorithm returns N values of k so we obtain N option values, spanning from a minimum to a maximum value of the log-strike price, with a step size of λ. In particular, the values of k are given by:

$$k_u = -b + \lambda(u-1), \quad u = 1, \ldots, N. \tag{6.9}$$

Therefore, log-strikes k runs from $-b$ to b where:

$$b = \frac{1}{2}N\lambda.$$

Observe that we discretize in two dimensions: in the first grid N points discretize the integral in Equation (6.4), whilst in the second one N points discretize the space of the log-strike price. Of course the meshes might have different step sizes, namely η for the integral and λ for the log-strike, but number of points is equal to N for both of them.
If we substitute (6.9) in Equation (6.8) we obtain:

$$\begin{aligned} C_T(k_u) &\simeq e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-iv_j k_u}\Psi_T(v_j)\,\eta \\ &\overset{(6.9)}{=} e^{-\alpha k_u}\frac{1}{\pi}\sum_{j=1}^{N} e^{-iv_j[-b+\lambda(u-1)]}\Psi_T(v_j)\,\eta \\ &= e^{-\alpha k_u}\frac{1}{\pi}\sum_{j=1}^{N} e^{-iv_j\lambda(u-1)}e^{ibv_j}\Psi_T(v_j)\,\eta, \quad u = 1, \ldots, N. \end{aligned}$$

Remembering that $v_j = \eta\,(j-1)$ and substituting v_j in the first exponent of the previous equation we get:

$$C_T\,(k_u) \simeq e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(u-1)} e^{ibv_j}\Psi_T\,(v_j)\,\eta, \quad u = 1,\ldots,N.$$

In order to apply the Fast Fourier Transform we must request that $\lambda\eta = \frac{2\pi}{N}$ and hence we obtain:

$$C_T\,(k_u) \simeq e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j}\Psi_T\,(v_j)\,\eta, \quad u = 1,\ldots,N.$$

Using Simpson's rule weightings we get:

$$C_T\,(k_u) \simeq e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j}\Psi_T\,(v_j)\,\frac{\eta}{3}\left[3+(-1)^j-\delta_{j-1}\right], \quad u = 1,\ldots,N.$$

where δ_n is the Kronecker delta function which is equal to one if $n = 0$ and zero otherwise. If we compare the last equation with Equation (6.7):

$$C_T\,(k_u) = e^{-\alpha k}\frac{1}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j}\Psi_T\,(v_j)\,\frac{\eta}{3}\left[3+(-1)^j-\delta_{j-1}\right], \quad u = 1,\ldots,N.$$

$$w\,(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x\,(j), \quad k = 1,\ldots,N.$$

we recognize that the value of the call option $C_T(k_u)$ is obtained by computing the FFT of the quantity:

$$x\,(j) = e^{ibv_j}\Psi_T\,(v_j)\,\frac{\eta}{3}\left[3+(-1)^j-\delta_{j-1}\right].$$

6.3.3 An approximated formula for spread options

In this section we sketch the method proposed by Caldana and Fusai [35] to price a spread option under the assumption that the log-price dynamics is modelled by a stochastic process which characteristic function is known in closed form.

Based on a result given by Bjerksund and Stensland [20] the authors generalize the well known Margrabe [89] formula for spread option pricing. Assume that a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and that $\mathbb{Q}$ is a risk-neutral measure equivalent to $\mathbb{P}$. Let $S_1\,\{S_1\,(t)\,; t \geq 0\}$ and $S_2 = \{S_2\,(t)\,; t \geq 0\}$ be two stock processes: the price of an European Spread-Option with maturity T is given by:

$$C_K(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T) - K)^+\right],$$

If we define the event A:

$$A = \left\{\omega \in \Omega : \frac{S_1(T)}{S_2^{\alpha}(T)} > \frac{e^k}{\mathbb{E}\left(S_2^{\alpha}(T)\right)}\right\},$$

the value:

$$C_K^{k,\alpha}(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T) - K)\,\mathbb{1}_A\right],$$

can be computed in a semi-explicit way and, moreover, $C_K^{\alpha,k}(0)$ is a very good approximation for $C_K(0)$ for suitable choices of α and k.
Let $\boldsymbol{u} = (u_1, u_2)^T \in \mathbb{R}^2$ and $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$, where $\boldsymbol{Y}(t) = (\log S_1(t), \log S_2(t))^T$ and consider the joint characteristic function:

$$\Phi_T(\boldsymbol{u}) = \Phi_T(u_1, u_2) = \mathbb{E}^{\mathbb{Q}}\left[e^{iu_1 \log S_1(T) + iu_2 S_2(T)}\right] = \mathbb{E}\left[e^{i\langle \boldsymbol{u}^T, \boldsymbol{Y}(T)\rangle}\right],$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product. The following proposition shows how to compute the price of spread option.

Proposition 6.3.1. *(Caldana and Fusai [35, Proposition 1]) The approximate spread option value $C_K^{k,\alpha}(0)$ is given in term of a Fourier inversion formula as:*

$$C_K^{k,\alpha}(0) = \left(\frac{e^{-\delta k - rT}}{\pi}\int_0^{+\infty} e^{-i\gamma k}\Psi_T(\gamma;\delta,\alpha)\,d\gamma\right)^+, \tag{6.10}$$

where δ is the dumping factor and

$$\begin{aligned}\Psi_T(\gamma;\delta,\alpha) = \frac{e^{i(\gamma - i\delta)\log(\Phi_T(0,i\alpha))}}{i(\gamma - i\delta)}\big[&\Phi_T((\gamma - i\delta) - i, -\alpha(\gamma - i\delta)) - \\ &\Phi_T(\gamma - i\delta, -\alpha(\gamma - i\delta) - i) - K\Phi_T(\gamma - i\delta, -\alpha(\gamma - i\delta))\big],\end{aligned}$$

and

$$\begin{aligned}\alpha &= \frac{F_2(0,T)}{F_2(0,T) + K},\\ k &= \log(F_2(0,T) + K).\end{aligned}$$

The quantity $F_2(0,T)$ appearing in the previous formulas can be computed in the following way:

$$F_2(0,T) = \mathbb{E}[S_2(T)] = \Phi_T(0,-i).$$

Therefore, we can conclude that the price of a spread option where the joint characteristic function is known can be computed by solving (6.10). Observe that this

algorithm only provides an approximated value of the spread option: nevertheless, such an approximation is acceptable (see Caldana and Fusai [35]). Moreover, the just presented algorithm is simpler to implement that the one proposed for example by Hurd and Zhou [74] since it request a single Fourier Transform inversion.

6.4 A note on PDE methods for option pricing

In the Black and Scholes [23] model, where the dynamic of the underlying asset S under the risk neutral measure $\mathbb{Q}$ is given by:

$$dS(t) = rS(t)dt + \sigma(t,S)S(t)dW(t),$$

the value $C(t,S(t))$ of an European call option can be computed by solving the following parabolic PDE:

$$\frac{\partial C(t,S)}{\partial t} + rS\frac{\partial C(t,S)}{\partial S} + \frac{\sigma^2(t,S)S^2}{2}\frac{\partial^2 C(t,S)}{\partial S^2} - rC(t,S) = 0, \tag{6.11}$$

with the boundary condition $C(T,S) = (S(T)-K)^+$. Such a PDE can be solved in many ways: following Salsa [111], by using a change of variables it can be reduced tot he heat equation and hence it can be analytically solved. Alternatively, it can be numerically solved using finite differences or finite elements as discussed in Seydel [117].

Of course, as we have shown in Section 2.8 the value of the same European call option can be computing by taking the discounted expectation of the payoff under the risk-neutral measure $\mathbb{Q}$:

$$C(t,S(t)) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)^+\right].$$

The connection between the partial differential equation approach and the martingale one is given by the Feynman–Kac formula (see Shreve [118, Theorem 6.4.1] and Kac [75]).

A similar result holds when the risk-neutral dynamics is given by an exponential Lévy model or a jump-diffusion model (see Cont and Tankov [42, Chapter 12]). The value of a European call option is given by $C(t,S(t))$ where $C(t,S(t))$ solves a *second-order partial intego-differential equation* (PIDE):

$$\begin{aligned}\frac{\partial C(t,S)}{\partial t} + rS\frac{\partial C(t,S)}{\partial S} + \frac{\sigma^2(t,S)S^2}{2}\frac{\partial^2 C(t,S)}{\partial S^2} - rC(t,S) \\ \int_{\mathbb{R}}\nu(dy)\left[C(t,Se^y) - C(t,S) - S\left(e^y-1\right)\frac{\partial C(t,S)}{\partial S}\right] = 0,\end{aligned} \tag{6.12}$$

with the usual boundary condition $C(T,S) = (S(T)-K)^+$. The new element is the integral term in Equation 6.12, due to the presence of jumps. Its presence leads to new theoretical and numerical issues making the PIDE harder to solve than the usual PDE.

Remark. PIDE of the type of (6.12) have been studied by Garroni and Menaldi [61] and Bensoussan and Lions [17] whereas an overview of numerical methods (such as multinomial trees, finite difference schemes and Galerkin methods) for its resolution can be found in Cont and Tankov [42, Chapter 12]. Financial examples can be found in the book Prigent [103], whereas an approach combining the Fast Fourier Transform method and the operator splitting technique was proposed by Andersen and Andreasen [4]. Mayer and Van Der Hoek [90] discussed the method of lines for American options valuation, whilst Cont and Voltchkova [43] studied the application of finite difference methods to finite and infinite activity Lévy models.

A PIDE of the form of (6.12) can be written for the $VG++$ model discussed in Chapter 3. If we impose the boundary condition for an European call option and we solve the PIDE we obtain the price of the option which coincides with the one we obtain by the formula of Proposition 3.2.6. Similarly to what has been done for spread options in the standard Black-Scholes model, a PIDE can be written and solved to determine the price of spread options within the models we presented in Chapters 4 and 5. Their study and numerical resolution requires a detailed analysis and this goes beyond the scope of this work. Nevertheless, this is a very interesting topic and it might be the subject of a future inquire.

In the next chapter we use the process we introduced in Chapters, 3, 4 and 5 to model real financial markets: in particular, we deal with calibration issue and we use the algorithms we have presented in this chapter to properly evaluate different contingent claims.

Chapter 7

Numerical results and financial applications to energy markets

In this chapter we use the processes we introduced so far and the relative numerical techniques to model energy markets and to price different types of derivatives. In particular we focus on European power and gas future markets and we highlight how the stochastic processes we discussed so far are able to catch some empirical facts that can be observed in this market context.

7.1 Why Lévy processes?

Since the down of mathematical finance Brownian motion has been a milestone in modeling market prices dynamic. In the Black and Scholes [23] model the risky asset process is modeled by using a geometric Brownian motion and this intuition leads to several advantages: closed formulas for the valuation of many derivative contracts are available and efficient algorithms for path simulations can be easily implemented. Moreover, closed expression for greeks computation can be obtained.

Despite these advantages, the limits of Black-Scholes model are well known and led in recent years to explore more general techniques aiming at describing market dynamics in a more realistic way. Among the others proposed solutions, Lévy models represents one of the most widely used and flexible tools in mathematical finance both for risk-management and pricing purposes.

Everyone who deals with quantitative finance knows the main drawbacks of the Black-Scholes approach: price trajectories are assumed to be continuous in time, volatility is constant, log-returns are assumed to be normally distributed and hence the probability of extrem events is small. All these limits are overcome by using Lévy models which lead to a better description of market dynamics: see Cont and Tankov [42, Chapter 1] for the details. As we observed in previous chapters, Lévy processes are harder to handle that the classical Brownian motion both from a mathematical and a numerical point of view. Nevertheless, if they can be successfully applied and calibrated one can obtain a very good description of many *stylized-facts* observed in

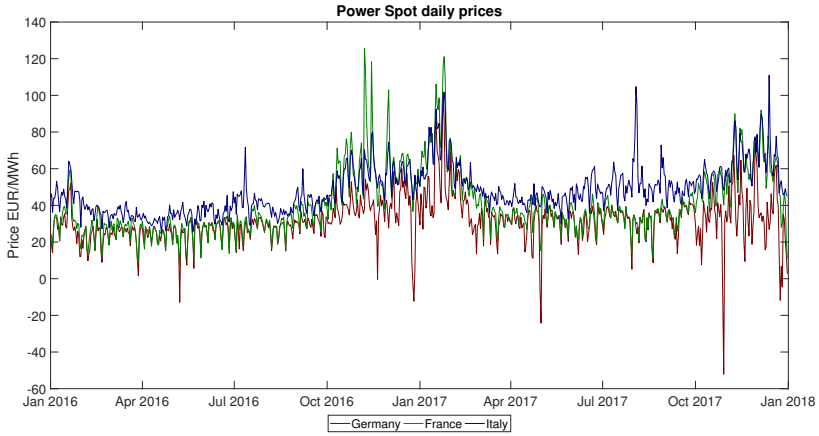

Figure 7.1. Daily spot electricity prices of different European countries.

the market.

7.2 Energy markets: some stylized facts

As many other financial markets, the European energy market is mainly divided in two parts: the spot and the future market. The *spot market* is a public financial market in which energy commodities, such as natural gas, electricity, emissions and so on, are traded for immediate delivery. Loosely speaking, each European country owns its local spot electricity market within which energy producer and consumers agree to buy or sell electricity at some "equilibrium price". The mechanism behind the electricity energy market is complex and it is not the subject of this work[1]. Although each country is characterized by its own spot electricity market, each of them is not independent from the others. Power transmission from one country to the other are frequent: for example the electricity produced in Germany can be exported to France, and then to Italy leading to *de facto* interconnected markets. Because of this dependence, it is customary to observe that a risen in the price level of a given country, Germany for example, leads to a general rise of the electricity prices in the whole Europe. For this reason the European spot electricity markets show the same macro-behavior. For example, in October/November 2016 many problems related to the some nuclear power plants in France let to a rise of electricity prices not only in France but also in other countries such as Germany and Italy, as it can be observed in Figure 7.1. Looking at this last Figure, it turns out that spot prices in electricity markets are very peculiar: they clearly exhibit a mean-reverting behavior, a seasonality component is present and spikes are frequent.

Over the years many approaches have been proposed to model these markets in a univariate setting. In particular, Cartea and Figueroa [39] derive a mean-reverting

[1]A quick overview can be found here: `https://www.epexspot.com/en/basicspowermarket`

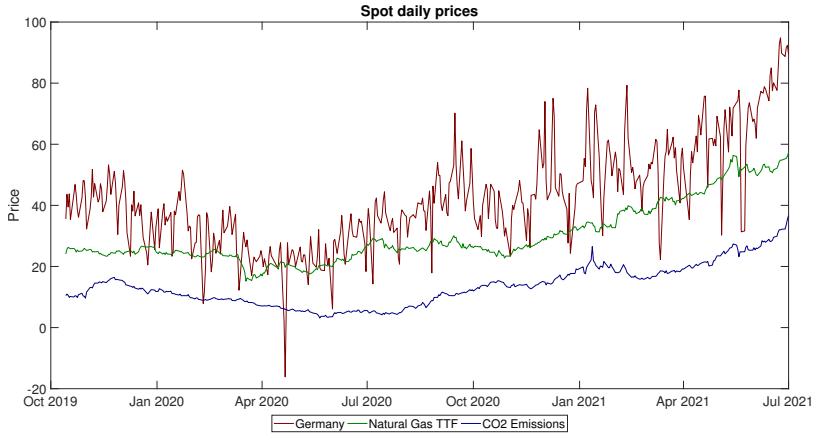

Figure 7.2. Daily spot prices of German electricity, Natural Gas (TTF) and CO_2 emissions.

model with jumps and a deterministic seasonality component, whereas Saifert and Uhrig-Homburg [110] compare different modeling approaches. As in many other markets, electricity derivatives such as call options, virtual power plants (VPP), swing options and storages are traded and hence need to be priced: these issues are tackled in Lucia and Schwartz [85], Cartea and Villaplana [40], Vehvilainen [123] and Kluge [79], whereas specific algorithms for the pricing of VPP and storages was derived by Tseng and Barz [121] and Boogert and de Jong [25] respectively. When we scale to a multi-commodities market these modeling techniques become harder to apply in practice and literature is not as richer as in the one dimensional framework. Petroni and Sabino [101] showed how some standard models such as the ones proposed by Black and Scholes [23], Schwartz and Smith [114] and Cartea and Figueroa [39] can be extended to a multivariate context by adding dependent jumps which are modeled using self-decomposability, whereas Kiesel and Kusterman [78] proposed a structural model to properly catch the dependence between electricity spot markets in the spirit of what is proposed by Carmona and Coulon [36]. Similar dependencies arise also if we consider other commodities which are directly related with the production of electricity, such as the natural gas and CO_2 emissions, as can be observed in Figure 7.2 where spot quotations of German electricity, natural gas and CO_2 emissions prices are reported.

Spot markets discussed so far contrast with futures markets, in which delivery is due at a later date. Usually, when we deal with commodities such as electricity ad gas the delivery of the commodity does not refer to a precise time in the future but to predetermined time interval. For example, if two counterparts A and B stipulate a *future calendar contract on 2022* with future price of F this mean that they agree to exchange energy for the whole year 2022 at the fixed price F. Of course, shorter delivery periods such as quarters, months, weeks, weekend and even days are possible. As in the spot markets, all countries own their related future market but the amount of trades is different for each market. Empirically, it can be observed that mature

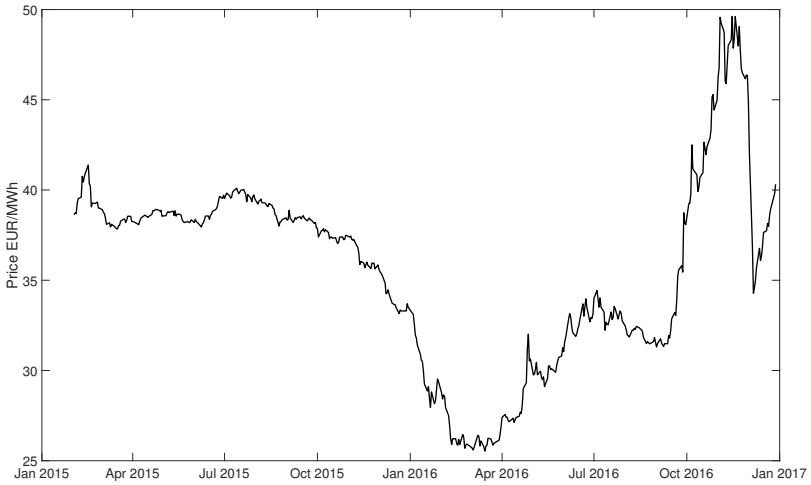

Figure 7.3. Power French Calendar 2017 future price.

markets like German or Nordpool power markets are very liquid, whereas the markets of the Mediterranean area including country such as Italy, Greece and Spain tend to be less liquid than the aforementioned ones. We will come back on this point in Section 7.6 when we will apply the $VG++$ process to model illiquid markets.

The future energy market modelling reveals to be a challenging task, because one has to deal with complex dependence structures and price behavior, which remind what has been observed in the spot markets. A widely recognized approach for their modeling is proposed by Benth and Saltyte-Benth [18] which show how the framework introduced by Heath et al. [69] to model interest rates dynamic can be easily adapted to power energy future market. Another possible approach would be to model the underlying future dynamic using the multivariate version of the model proposed by Black [22]. Nevertheless, energy future markets present a series of empirical facts that cannot be modeled by using the standard Gaussian framework, based on Brownian motions. As can be observed in Figure 7.3, sometimes jumps in the price process occur and hence the Brownian motion seems not to be a good candidate for financial modeling because of the continuity of its trajectories.

In statistical terms the presence of jumps in the price leads to the so called *heavy tails effect* in the empirical distribution of returns: the tail of the distribution decays slowly at infinity and very large moves have a significant probability of occurring. This well-known fact leads to a poor representation of the distribution of the log-returns by a normal distribution, as can be observed if we look at the QQ-plot in Figure 7.4.

Finally we recall that one of the assumptions of the models proposed by Black and Scholes [23] and Black [22] is that the volatility of the asset is constant over time. However, if we compute the rolling volatility over the previous thirty days, we observe that this assumption is not satisfied (see Figure 7.5): volatility is not constant at all over time, but shows a mean-reverting behavior. All these observations, drive

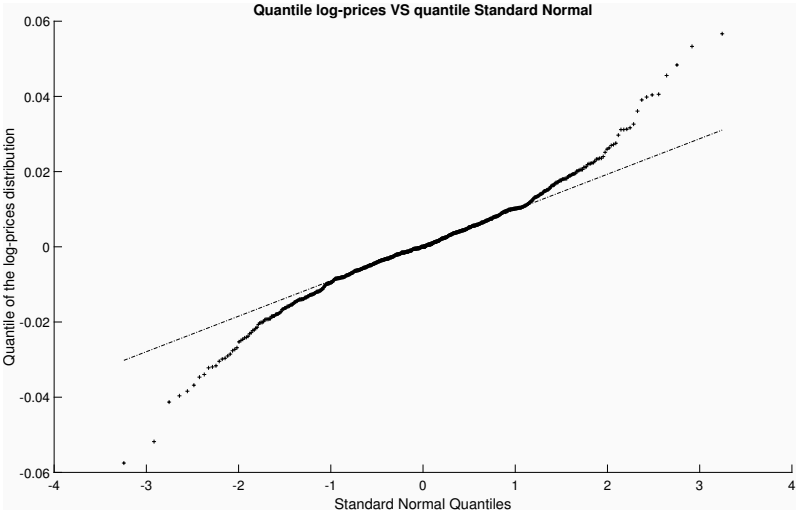

Figure 7.4. QQ-plot of the log-returns of the 2020 France forward calendar versus the standard normal.

us to abandon the idea of modeling energy markets using the well know Gaussian approach and to focus our attention on Lévy processes which overcome the limits of the Gaussian framework introducing, on the other hand, some mathematical and numerical issues.

As we stated in Chapter 2.1 a Lévy process has independent increments. This is an important assumption that we need to check before start modeling financial assets by using Lévy processes. As suggested by Brigo et al. [28], an easy method to check the independence of the increment is to compute the so called *autocorrelation function of lag k* (ACF): given a process $X = \{X(t); t \geq 0\}$, roughly speaking, ACF measures the correlation between $X(t+k)$ and $X(t)$ for $k \in \mathbb{N}$. Its definition is given by:

$$ACF(k) = \frac{1}{(n-k)\hat{v}} \sum_{i=1}^{n-k} (x_i - \hat{m})\,(x_{i+k} - \hat{m})\,, \quad k = 1, 2, \ldots$$

where $\hat{m}$ and $\hat{v}$ are the sample mean and variance of the series we want to model and $x = (x_1, \ldots, x_n)$ is a vector of realizations. The autocorrelation function of log-returns of power France future calendar is displayed in Figure 7.6. As we can observe, no evident dependence between the increments of log-returns is present and hence a Lévy process might be a suitable choice to model the price dynamic.

As stated before, European spot energy markets exhibit a strong dependence and this feature can be observed in energy future markets too (see Figure 7.7 and Figure 7.8). As in the spot markets, the dependence is stronger if we look only at power future prices and weaker if we include other commodities such as natural gas and CO_2 emissions, but in any case it must be considered within the modeling framework.

Finally, looking at the options market on futures, we observe that the so called

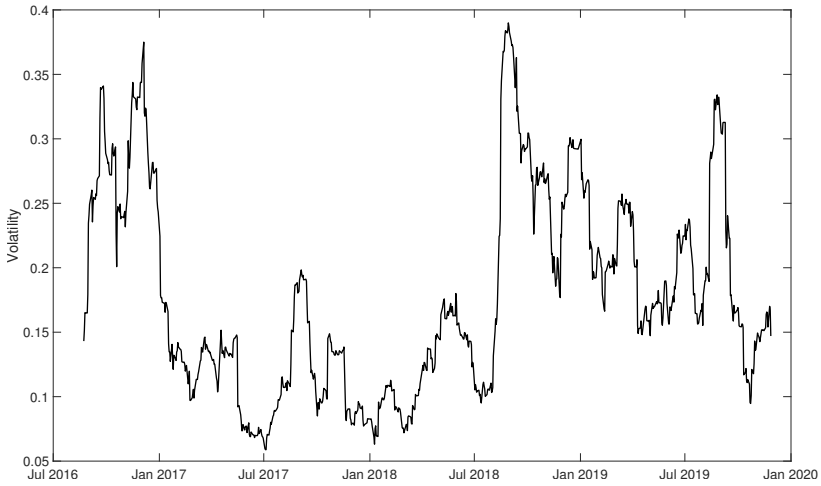

Figure 7.5. Rolling 30-days volatility of the forward calendar France with delivery 2020.

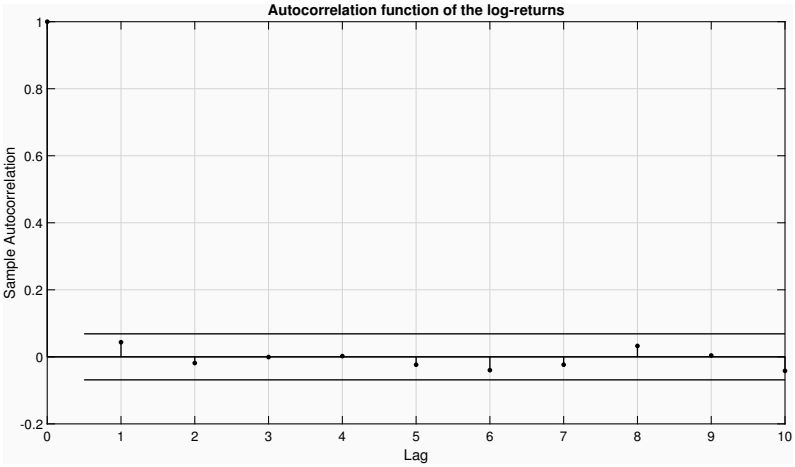

Figure 7.6. Autocorrelation functions of the log-returns of the forward calendar France with delivery 2020.

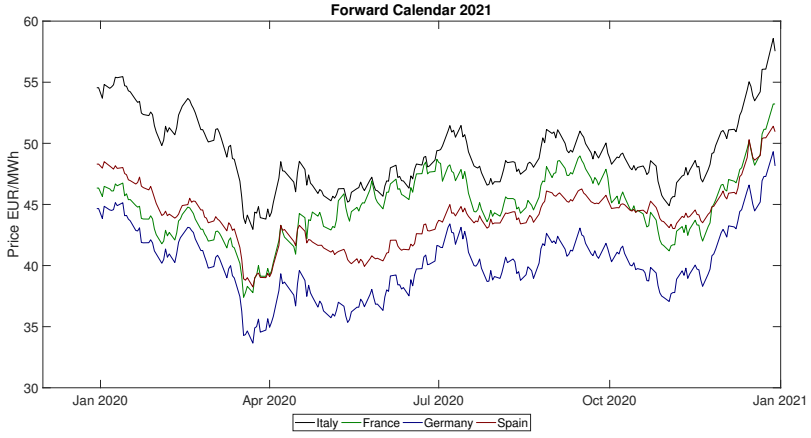

Figure 7.7. Power future market prices.

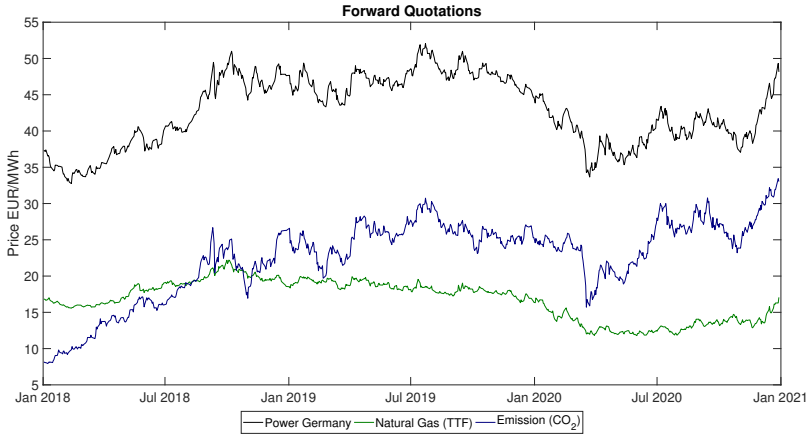

Figure 7.8. Future market prices of German power, natural gas (TTF) and CO_2 emissions with delivery date 2021.

implicit volatility is not constant varying the strike price K but a sort of *volatility smile* effect is present, as it is shown in Figure 7.9. A large part of quantitative finance in the previous years was focused on deriving models that can overcome this problem: among the others we recall Madan and Seneta [88], Barndorff-Nielsen [12], Heston [70], Dupire [51] and Bates [16]. Since many univariate Lévy models can produce a volatility smile effect, we deduce that their multivariate versions might be a good candidate to model future energy market.

In the next sections we focus on of forward markets modeling. The goal is to answer to two main questions:

- Are 2D-*sd* Variance Gamma and Normal Inverse Gaussian models suitable for power forward market modeling purposes?
- Since power forward markets are not always liquid, is the $VG++$ process we introduced a good candidate to model illiquid markets?

We will focus on 2D-*sd* Variance Gamma and Normal Inverse Gaussian processes, we calibrate both models on different data-set, we price derivatives using Monte Carlo and Fourier techniques we presented in Chaper 6 and we give an economic interpretation to the obtained results. Moreover, we briefly discuss how those models can be extended to model a market with more that two commodities: to this aim we provide an example of calibration in a market with three commodities.

Furthermore, we compare the numerical techniques we derived in Chapter 3 to price vanilla call options under the $VG++$ model and hence we show how the $VG++$ process can be successfully applied to properly catch market's illiquidity.

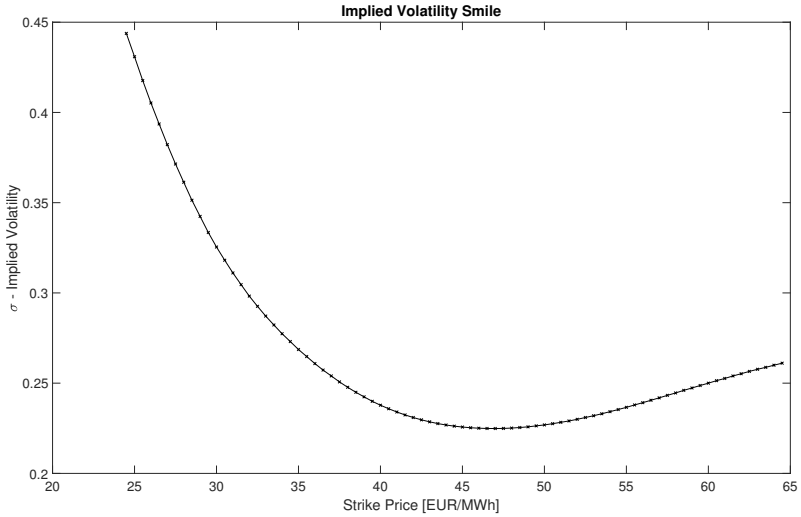

Figure 7.9. Volatility smile for European call options written on German power future quotations.

7.3 2D-*sd* Variance Gamma Process

In Chapter 4 we derived the theoretical modeling framework and we showed how to build correlated Lévy processes using *sd* gamma subordinators. In this section we show concrete applications of the models presented in Section 4.2 to energy markets. Similarly to what we have shown in Section 2.8.1 we model the dynamics of energy forward contracts with exponential processes, whose components are Lévy processes, based on dynamics of the type of $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ derived in Section 4.2[2]. The evolution of the forward price $F_j(t),\ j = 1, 2$ at time t is defined as

$$F_j(t) = F_j(0)\, e^{\omega_j t + Y_j(t)}, \tag{7.1}$$

non-arbitrage conditions can be obtained setting

$$\omega_j = -\varphi_j(-i), \tag{7.2}$$

where $\varphi_j(u)$ denotes the characteristic exponent of the process Y_j.

We adopt the same two-steps calibration procedure of Luciano and Semeraro [87]; to this end, it is worthwhile noticing that the marginal distributions do not depend on the parameters required to model the dependence structure. The vector of the marginal parameters Θ^* is obtained solving the following optimization problem

$$\Theta^* = \arg\min_{\Theta} \sum_{i=1}^{n} \left(C_i^{\Theta}(K, T) - C_i \right)^2, \tag{7.3}$$

[2]Note that even if $\boldsymbol{Y}$ may not be a Lévy process, its components are Lévy process. For this reason, even if $\boldsymbol{F} = \{(F_1(t), F_2(t)); t \geq 0\}$ is not an exponential Lévy process, the dynamics of the forward price $F_j(t),\ j = 1, 2$ is an exponential Lévy process.

where $C_i, i = 1, \ldots, n$ are the values of n quoted vanilla products and $C_i^{\Theta}(K,T), i = 1, \ldots, n$ are the relative model prices. Once we have fitted Θ^* we have to calibrate the remaining parameters for the dependence structure. Generally derivatives written on multiple underlying assets are not very liquid and market quotes are rarely available, therefore the vector $\boldsymbol{\eta}^*$ is estimated fitting the correlation matrix on historical data. The expression of the theoretical correlation matrix has been derived in Section 4.2.

For the first step we have combined the NLLS approach with the FFT method proposed by Carr and Madan [37] (the version proposed by Lewis [81] returns similar results), whereas for the second one we have used the plain NLLS method for the minimization of the distance between the theoretical and the observed correlation coefficient.

As far as the path generation of the skeleton of the 2D-*sd* Variance Gamma processes is concerned, the only non-standard step is the simulation of the process $Z_a = \{Z_a(t); t \geq 0\}$. On the other we recall that the a-remainder Z_a of a gamma distribution $\Gamma(\alpha, \lambda)$ can be exactly simulated knowing that

$$Z_a = \sum_{j=1}^{S} X_j,$$

where

$$S \sim \mathfrak{B}(\alpha, 1-a) \quad X_j \sim \mathfrak{E}(\lambda/a) \quad X_0 = 0 \quad \mathbb{P}-a.s.$$

Here $\mathfrak{B}(\alpha, p)$ denotes a Polya distribution with parameters α and p and $\mathfrak{E}(\lambda)$ denotes an exponential distribution with rate parameter λ.

Since we have derived the *chf*'s of the log-process in closed form, in alternative to the MC schemes we can adopt Fourier methods. Different Fourier based techniques are available for the option pricing in a multivariate setting (see for example Hurd and Zhou [74], Pellegrino [97] and Caldana and Fusai [35]). In this section we use the method proposed by Caldana and Fusai [35] we presented in Section 6.3.3, which gives a good approximation for spread-options prices and has the advantage to require only one Fourier inversion.

The remaining part of the section is split into two parts: in the first one we apply our models to the German and French power forward markets and in the second part we focus on the German power forward market and to the TTF natural gas forward market. In the first case we have selected markets that are strongly positively correlated due to the configuration of European electricity network, whereas in the second case, the correlation between markets is still positive, because natural gas can be used to produce electricity, but it is not as high as in the first one. This gives us the opportunity to test our models for different levels of correlations.
Moreover, as shown in Figure 7.10, due to the particular structure of European electricity grid, power markets are very interconnected and usually react "in the same way at the same time", whereas the reaction of the power market to a shock

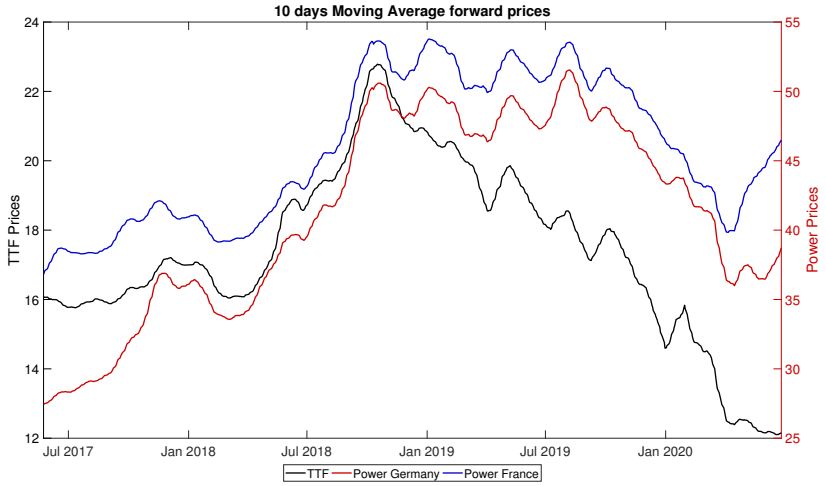

Figure 7.10. German, French and natural gas TTF forward market.

in the natural gas market (and vice versa) is more likely to occur with a stochastic delay. Based on these observations, we expect that the value of the parameter a will be very close to one in the first example and will have a lower value in the second one.

For the sake of concision we use the following notation:

- (*SSD*): *sd*-VG model presented in Section 4.2.1.
- (*LSSD*): *sd*-VG model presented in Section 4.2.2.
- (*BBSD*): *sd*-VG model presented in Section 4.2.3.

In our experiments we price spread options on future prices, denoted $F_i(t), i = 1, 2$, whose payoff is given by:

$$\Phi_T = (F_1(T) - F_2(T) - K)^+ .$$

It is customary to reserve the name *cross-border* or *spark-spread* option if the futures are relative to power or gas markets, respectively. In all our experiments we use the MC technique with $N_{sim} = 10^6$ simulations and the Fourier-based method proposed by Caldana and Fusai [35].

7.3.1 Application to the German and French power markets

In order to calibrate our models we need the quoted prices of the derivatives contracts written on each of the two forward products and their joint historical time series. The data-set[3] we have relied upon is composed as follows:

[3]Data Source: www.eex.com.

- Forward quotations from 25 April 2017 to 12 November 2018 of calendar 2019 power forward. A forward calendar 2019 contract is a contract between two counterparts to buy or sell a specific volume of energy in MWh at fixed price for all the hours of 2019. calendar power forward in German and France are stated respectively with DEBY and F7BY.

- European call options on power forward 2019 quotations for both countries with settlement date 12 November 2018. We used strikes in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract, i.e. we exclude deep ITM and OTM options.

- We assume a risk-free rate $r = 0.015$.

- The estimated historical correlation between markets is $\rho_{mkt} = 0.94$.

From Table 7.17 we see that all models provide the same set of marginal parameters. In the lower box of Figure 7.13 we report the percentage error ϵ_i defined as:

$$\epsilon_i = \frac{C_i^{\Theta}(K,T) - C_i}{C_i}.$$

We can observe that this error is very small and our models are able to replicate market prices for different values of the strike price K.

If we look at the fitted correlation the situation is slightly different. The *SSD* model presented in Section 4.2.1 fits a correlation that is roughly zero, therefore it is not recommendable for *cross-border* option pricing because it overestimates the derivative price as shown in the upper picture in Figure 7.13. In contrast, the correlation estimated selecting the *LSSD* is very close to the one observed in the market, as we can see from Table 7.12; for this reason the *LSSD* model is appropriate to price *cross-border* options. Moreover, the *BBSD* model derived in Section 4.2.3 provides an even better fit of market correlation and therefore we conclude that the *BBSD* model is the best one for the valuation of *cross-border* options. An additional comparison among the models is illustrated in the upper part of Figure 7.13: the option prices returned by the *BBSD* model are the lowest ones due to the highest value of fitted correlation.

Finally, we remark that, as we expected, the fitted value for the *sd* parameters a is very close to one for all the three settings. This fact does not come as a surprise because the German and French forward markets are so strictly interconnected that whenever an event occurs in a market it has an immediate impact on the other one. As already mentioned, if $a \to 1$ we obtain the original models of Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9]. For this reason, for *cross-border* options, there is not an essential difference between original models and those presented in Section 4.2.

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.40	0.61	0.31	0.32	0.02	0.02
LSSD	0.40	0.61	0.31	0.32	0.02	0.02
BBSD	0.40	0.61	0.31	0.32	0.02	0.02

Table 7.1. Fitted marginal parameters the German and French power markets.

Parameter	Value
A	41.89
B	1.00
a	0.99
ρ_{mod}	0.05

Table 7.2. *SSD*

Parameter	Value
A	42.31
B	1.00
ρ	1.00
a	0.99
ρ_{mod}	0.92

Table 7.3. *LSSD*

Parameter	Value	Parameter	Value
β_1	-0.00	β_{R_2}	0.85
β_2	0.09	γ_{R_1}	0.50
γ_1	0.00	γ_{R_2}	0.47
γ_2	0.10	ν_R	0.02
ν_1	1.01	a	0.99
ν_2	0.14	ρ_{mod}	0.94
β_{R_1}	0.62		

Table 7.4. *BBSD*

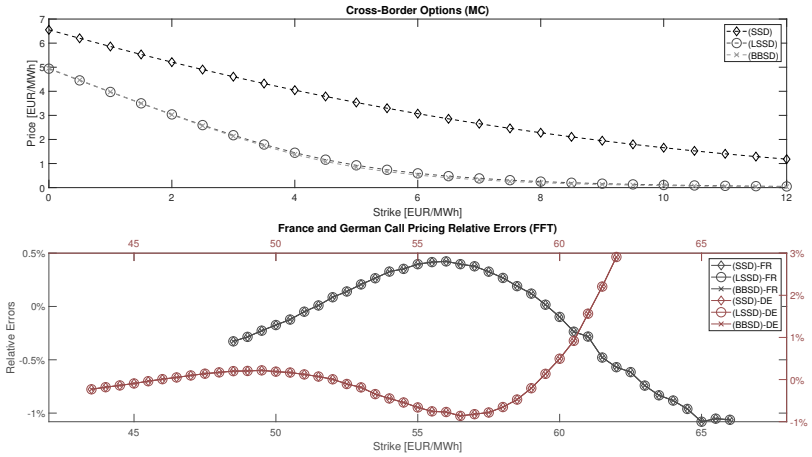

Figure 7.11. Percentage errors and cross-border option prices.

7.3.2 Application to the German power and the TTF gas markets

In this section we consider the German power forward market and the TTF gas forward market. These two markets are not as positively correlated as two purely power markets.
The data-set[4] we relied upon consists of the following assumptions:

- Forward quotations from 1 July 2019 to 09 September 2019 relative to January 2020 for the power forward in Germany and the gas TTF forward.
- Call options relative to January 2020 for both power and gas with settlement

[4]Data Source: www.eex.com and www.theice.com

date 9 September 2019. As done before, we use strikes prices K in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract, i.e. we exclude deep ITM and OTM options.

- We assume a risk-free rate $r = 0.015$.
- The estimated historical correlation between log-returns is $\rho_{mkt} = 0.54$.

In the picture at the bottom of Figure 7.12 we can see that all the models provide a good fit of the quoted market options because the error ϵ is very small. In Figure 7.12 the picture at the top shows that the *SSD* model overprices the *spark-spread* option due to the fact that the estimated correlation is close to zero. In contrast, both models *LSSD* and *BBSD* are able to capture the market correlation and return a lower spread option price. From Table 7.9 we observe that the *sd* parameter a is not as close to one as it was in the previous example.

This result can be explained by the fact that approximately 25% of electricity in Germany is produced using natural gas, hence a certain downward or upward change of the natural gas price will not immediately affect the power prices. Moreover, in contrast to electricity, natural gas can be stored and players of gas markets can subscribe swing contracts to protect against natural gas price movements. Of course, if the natural gas price shock persists for some time it will impact electricity prices as well.

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.46	0.24	0.43	0.33	0.08	0.05
LSSD	0.46	0.24	0.43	0.33	0.08	0.05
BBSD	0.46	0.24	0.43	0.33	0.08	0.05

Table 7.5. Fitted marginal parameters for the power and gas forward markets.

Parameter	Value
A	12.36
B	1.00
a	0.99
ρ_{mod}	0.04

Table 7.6. *SSD*

Parameter	Value
A	9.89
B	1.00
ρ	0.89
a	0.90
ρ_{mod}	0.57

Table 7.7. *LSSD*

Parameter	Value	Parameter	Value
β_1	0.13	β_{R_2}	0.29
β_2	0.12	γ_{R_1}	0.47
γ_1	0.23	γ_{R_2}	0.29
γ_2	0.23	ν_R	0.11
ν_1	0.28	a	0.90
ν_2	0.12	ρ_{mod}	0.54
β_{R_1}	0.47		

Table 7.8. *BBSD*

Model	a
SSD	0.99
LSSD	0.90
BBSD	0.90

Table 7.9. Values for the a parameter of the three models.

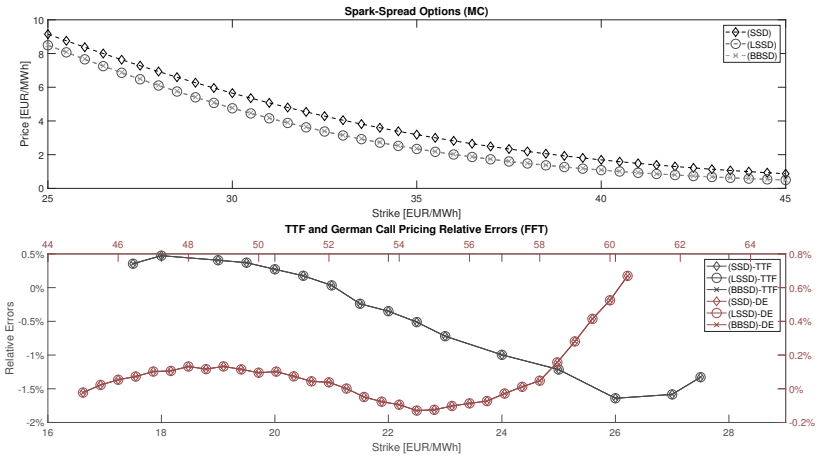

Figure 7.12. Percentage errors and spark-spread option prices.

7.4 2D-*sd* Normal Inverse Gaussian process

In this section we use the 2D-*sd* Normal Inverse Gaussian processes we illustrated in Chapter 5 to model power and gas forward markets. The modeling framework, the calibration procedure and the pricing techniques are the same we adopted in Section 7.3. Moreover, we numerically check that the Monte Carlo and the Fourier methods proposed by Caldana and Fusai [35] provide the same results.

As before, for the sake of concision we introduce the following notation:

- (*SSD*): *sd*-NIG model presented in Section 5.1.1.
- (*LSSD*): *sd*-NIG model presented in Section 5.1.2.
- (*BBSD*): *sd*-NIG model presented in Section 5.1.3.

7.4.1 Application to German and French power markets

In order to calibrate the proposed *sd*-NIG models we consider vanilla contracts written on German and French power forward and the historical quotation of both products. The data-set[5] is:

- Forward quotations from 25 April 2017 to 12 November 2018 of Calendar 2019 power forward. Calendar power forward in Germany and France are stated respectively with DEBY and F7BY.
- European call Options on power forward 2019 quotations for both countries with settlement date 12 November 2018. We used strikes in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract.
- We assume a risk-free rate $r = 0.015$.
- The historical correlation observed between markets is $\rho_{mkt} = 0.94$.

We denote (Θ_1, Θ_2) parameters related to the French and German power forward markets respectively. We define the error ϵ_i as

$$\epsilon_i = \frac{C_i^{\Theta}(K,T) - C_i}{C_i},$$

where $C_i^{\Theta}(K,T)$ is the value of the i-th Call option obtained by the model and C_i is its market price: the picture at the bottom of Figure 7.13 shows that all models provide a good fit for quoted market options because ϵ is negligible. In Figure 7.13 the picture at the top shows that the *SSD* model overprices *cross-border* options: this is because the fitted model correlation is low, as shown by the value ρ_{mod} in Table 7.11, so one should avoid using this model for pricing. For *LSSD* model

[5]Data Source: www.eex.com.

the situation is better, but it is not really able to capture the prevailing market correlation. Fortunately, the *BBSD* model can replicate the market correlation and then can be used to price *cross-border* options. Fitted common parameters are shown in Table 7.10, whereas the dependence parameters for *SSD*, *LSSD* and *BBSD* models are shown in Tables 7.11, 7.12, 7.13, respectively. The value of a, is shown in Table 7.14. We observe that the parameter a is very close to one, as expected. As previously observed in Section 7.3, this result has a very natural economic interpretation: the European electricity network is strongly connected and a certain price signal in either the German or French market is propagated without stochastic delay.

Finally, in Table 7.15 we compare values of *cross-border* options priced using both the FFT method proposed by Caldana and Fusai [35] and the MC scheme that we derived relying upon the results of Chapter 5 and 6. Option prices provided by both algorithms are very close and this allows us to use indistinctly the FFT or the MC method for spread option pricing. Nevertheless, if the valuation of an exotic derivative is required, the Monte Carlo scheme becomes mandatory, since other methods such as the Fourier transform or the partial differential approach become inapplicable.

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.64	0.40	0.31	0.32	0.02	0.03
LSSD	0.64	0.40	0.31	0.32	0.02	0.03
BBSD	0.64	0.40	0.31	0.32	0.02	0.03

Table 7.10. Fitted marginal parameters for German and French power markets.

Parameter	Value
A	40.15
B	1.00
a	0.99
ρ_{mod}	0.05

Table 7.11. *SSD*

Parameter	Value
A	40.15
B	1.00
ρ	0.99
a	0.99
ρ_{mod}	0.88

Table 7.12. *LSSD*

Parameter	Value	Parameter	Value
β_1	-0.001	β_{R_2}	0.800
β_2	0.013	γ_{R_1}	0.448
γ_1	0.002	γ_{R_2}	0.50
γ_2	0.103	ν_R	0.025
ν_1	1.007	a	0.99
ν_2	0.091	ρ_{mod}	0.94
β_{R_1}	0.554		

Table 7.13. *BBSD*

Model	a
SSD	0.99
LSSD	0.99
BBSD	0.99

Table 7.14. Values for the a parameter of the three models.

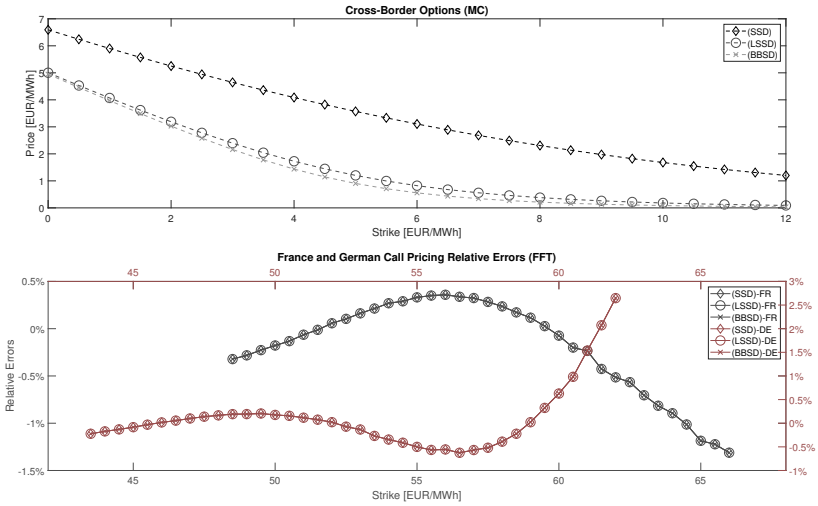

Figure 7.13. Percentage errors and cross-border option prices.

K	*SSD*			*LSSD*			*BBSD*		
-	FFT	MC	Δ	FFT	MC	Δ	FFT	MC	Δ
0.0	6.61	6.59	(0.02)	5.01	5.00	(0.01)	4.95	4.95	(0.00)
1.0	5.92	5.90	(0.02)	4.08	4.07	(0.01)	3.98	3.97	(0.01)
2.0	5.27	5.25	(0.02)	3.20	3.19	(0.01)	3.03	3.03	(0.00)
3.0	4.67	4.65	(0.02)	2.41	2.40	(0.01)	2.16	2.16	(0.00)
4.0	4.11	4.09	(0.02)	1.74	1.73	(0.01)	1.44	1.44	(0.00)
5.0	3.59	3.57	(0.02)	1.22	1.20	(0.02)	0.91	0.90	(0.01)
6.0	3.13	3.11	(0.02)	0.83	0.82	(0.01)	0.56	0.56	(0.00)
7.0	2.71	2.68	(0.03)	0.57	0.56	(0.01)	0.34	0.34	(0.00)
8.0	2.33	2.31	(0.02)	0.39	0.38	(0.01)	0.21	0.21	(0.00)
9.0	2.00	1.97	(0.03)	0.27	0.26	(0.01)	0.14	0.13	(0.01)
10.0	1.70	1.68	(0.02)	0.19	0.18	(0.01)	0.09	0.09	(0.00)
11.0	1.44	1.42	(0.02)	0.13	0.13	(0.00)	0.06	0.06	(0.00)
12.0	1.22	1.20	(0.02)	0.09	0.09	(0.00)	0.04	0.04	(0.00)

Table 7.15. Cross-border option prices comparison between the three *sd*-NIG models using the FFT and Monte Carlo methods. Δ is the difference between the computed prices.

Comparison between *sd*-VG and *sd*-NIG model - Power Markets

In this section we compare the prices of a cross-border option that we evaluate by using both the *sd*-VG and *sd*-NIG processes. Of course, the dataset we used to calibrate the two models is the same. The cross-border option prices we obtained using the two models are reported in Table 7.16. In this case we adopted the Monte Carlo technique but, in view of what we observed in the previous section, we get similar conclusions if we adopt the Fourier approach.

It is worth observing that the obtained prices are very close if we use both the *sd*-VG and the *sd*-NIG process, provided we focus only on *SSD* and *BBSD* models. This is due to the fact that both the *sd*-VG and *sd*-NIG model catch the same level of linear correlation at time T. The situation is a little bit different if we look at *LSSD* model: as observed in Table 7.3, *sd*-VG model provides a correlation $\rho_{mod} = 0.92$, which is very closed to the market one, whereas *sd*-NIG model provides a lower level of correlation between log-returns $\rho_{mod} = 0.88$ of Table 7.12 and this leads to higher cross-border option prices if we use the *sd*-NIG model. A similar analysis applied to the results of Table 7.4 and Table 7.13 lead us concluding that both versions of the *BBSD* model are appropriate for cross-border option pricing. Moreover, we point out that a small change in the correlation might cause a big variation in the spread option price.

K	*SSD*			*LSSD*			*BBSD*		
-	*sd*-VG	*sd*-NIG	Δ	*sd*-VG	*sd*-NIG	Δ	*sd*-VG	*sd*-NIG	Δ
0.0	6.55	6.59	(-0.04)	4.94	5.00	(-0.07)	4.96	4.95	(0.01)
1.0	5.86	5.90	(-0.04)	3.97	4.07	(-0.10)	3.98	3.97	(0.01)
2.0	5.21	5.25	(-0.04)	3.04	3.19	(-0.15)	3.03	3.03	(-0.00)
3.0	4.61	4.65	(-0.04)	2.18	2.40	(-0.22)	2.13	2.16	(-0.03)
4.0	4.05	4.09	(-0.04)	1.45	1.73	(-0.28)	1.38	1.44	(-0.06)
5.0	3.54	3.57	(-0.04)	0.93	1.20	(-0.28)	0.85	0.90	(-0.06)
6.0	3.07	3.11	(-0.03)	0.59	0.82	(-0.23)	0.52	0.56	(-0.04)
7.0	2.65	2.68	(-0.03)	0.38	0.56	(-0.18)	0.32	0.34	(-0.02)
8.0	2.28	2.31	(-0.03)	0.25	0.38	(-0.13)	0.20	0.21	(-0.01)
9.0	1.95	1.97	(-0.03)	0.16	0.26	(-0.10)	0.13	0.13	(-0.00)
10.0	1.66	1.68	(-0.02)	0.11	0.18	(-0.07)	0.08	0.09	(-0.00)
11.0	1.40	1.42	(-0.02)	0.07	0.13	(-0.05)	0.05	0.06	(-0.00)
12.0	1.18	1.20	(-0.02)	0.05	0.09	(-0.04)	0.04	0.04	(-0.00)

Table 7.16. Cross-border option prices comparison between the three *sd*-VG and *sd*-NIG models obtained using Monte Carlo simulations.

7.4.2 Application to German Power market and NCG Gas Market

In this section we present numerical results obtained applying our models to German power forward market (DE) and to natural gas forward market (NCG). These two markets are positively correlated, but the log-return correlation is lower than the one between power futures. The data-set[6] we relied upon is the following:

- Forward quotations from 1 July 2019 to 09 September 2019 relative to the Month January 2020 for the Power Forward in Germany and the Gas NCG Forward.
- Call Options on power forward NCG with settlement date 9 September 2019. As done before, we use strike prices K in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract.
- A risk-free rate $r = 0.015$.
- The historical correlation between log-returns is $\rho_{mkt} = 0.54$.

In the picture at the bottom of Figure 7.14 we observe that all models provide a good fitting of quoted market options because the relative error ϵ_i is small. The picture at the top of Figure 7.14 shows that the *SSD* model overprices the Spark-Spread option due to the fact that fitted model correlation is close to zero, as shown by the value ρ_{mod} in Table 7.18. In contrast, *LSSD* and *BBSD* models provide a lower price and catch the right level of market correlation as shown in Tables 7.19,7.20. We can conclude that both *LSSD* and *BBSD* models can be used to price *Spark-Spread* options. Table 7.17 shows fitted common parameters whereas dependence parameters for *SSD*, *LSSD* and *BBSD* models are shown in Tables 7.18, 7.19, 7.20: the value of a, the *sd* parameter which aims to model the *stochastic delay*, is shown in Table 7.21. The value is still close to one but it is smaller than that estimated for the power forward markets. From the expressions of the linear correlation coefficient reported in equations (5.4), (5.8) and (5.16) it is easy to see that a change in the value of a has an impact on the value of the correlation coefficient and it is a fact that even a small change in correlation has a high impact on the spread option price. On the other hand, unlike electricity, natural gas can be stored and therefore the impact on the power market can be moderated and delayed, for example, using storage contracts or other types of OTC derivatives. If the gas price suddenly rises then it is not rare to observe that the power price is not immediately effected.

[6]Data Source: www.eex.com and www.theice.com

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.37	0.20	0.44	0.33	0.09	0.07
LSSD	0.37	0.20	0.44	0.33	0.09	0.07
BBSD	0.37	0.20	0.44	0.33	0.09	0.07

Table 7.17. Fitted marginal parameters for German and TTF future markets.

Parameter	Value
A	11.27
B	1.00
a	0.99
ρ_{mod}	0.03

Table 7.18. *SSD*

Parameter	Value
A	8.79
B	1.00
ρ	0.87
a	0.90
ρ_{mod}	0.54

Table 7.19. *LSSD*

Parameter	Value	Parameter	Value
β_1	0.11	β_{R_2}	0.23
β_2	0.09	γ_{R_1}	0.56
γ_1	0.24	γ_{R_2}	0.50
γ_2	0.22	ν_R	0.13
ν_1	0.28	a	0.89
ν_2	0.15	ρ_{mod}	0.54
β_{R_1}	0.38		

Table 7.20. *BBSD*

Model	a
SSD	0.99
LSSD	0.90
BBSD	0.89

Table 7.21. Values for the a parameter of three models.

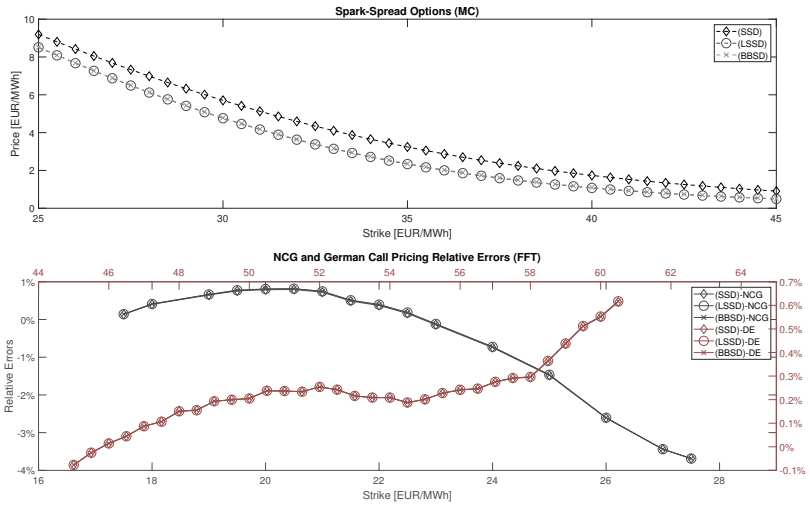

Figure 7.14. Percentage errors and spark-spread option prices.

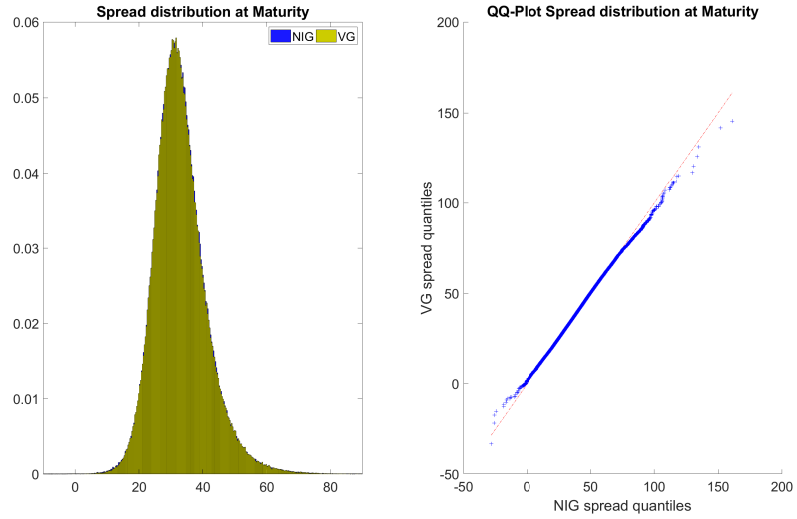

Figure 7.15. Spark spread distribution at maturity obtained using *sd*-VG and *sd*-NIG models. On the left the *pdf* of spark spread distributions, on the right its QQ-plot.

Comparison between *sd*-VG and *sd*-NIG model - Power and Gas Markets

At this point a natural question arises: does the *sd*-NIG model and the *sd*-VG model presented in Chapter 4 provide different prices for spark spread options? Of course, the data-set we used for numerical tests is the same as the one we used in Section 7.3. In Figure 7.15 we compare the spark spread distributions obtained using both *sd*-VG and *sd*-NIG models. Both the histogram and the QQ-plot show that the spark spread distributions obtained using the *sd*-NIG or *sd*-VG model at maturity T are very similar. Since the value of the considered spread options depends only on the spark spread distribution at T we can conclude that both models lead to the same derivative prices in the same way we have shown in Section 7.4.1. The results in Table 7.22 confirm this fact. Note that, as in the power case, the *SSD* model is not recommended to price a spread option since it is not able to catch the right level of log-returns linear correlation we observe in the market.

K	*SSD*			*LSSD*			*BBSD*		
-	*sd*-VG	*sd*-NIG	Δ	*sd*-VG	*sd*-NIG	Δ	*sd*-VG	*sd*-NIG	Δ
25.0	9.15	9.19	(-0.04)	8.49	8.51	(-0.02)	8.51	8.52	(-0.00)
26.0	8.37	8.42	(-0.05)	7.66	7.67	(-0.02)	7.68	7.68	(-0.01)
27.0	7.63	7.68	(-0.05)	6.86	6.87	(-0.02)	6.88	6.89	(-0.01)
28.0	6.93	6.99	(-0.05)	6.10	6.12	(-0.01)	6.13	6.13	(-0.01)
29.0	6.27	6.33	(-0.06)	5.40	5.41	(-0.01)	5.43	5.43	(-0.01)
30.0	5.65	5.71	(-0.06)	4.75	4.76	(-0.01)	4.78	4.78	(-0.01)
31.0	5.07	5.13	(-0.06)	4.16	4.17	(-0.01)	4.18	4.19	(-0.00)
32.0	4.54	4.59	(-0.06)	3.62	3.63	(-0.01)	3.65	3.65	(-0.00)
33.0	4.05	4.10	(-0.06)	3.14	3.14	(-0.00)	3.17	3.17	(-0.00)
34.0	3.60	3.65	(-0.06)	2.71	2.71	(0.00)	2.74	2.74	(-0.00)
35.0	3.19	3.24	(-0.05)	2.34	2.34	(0.00)	2.36	2.36	(-0.00)
36.0	2.82	2.87	(-0.05)	2.01	2.01	(0.01)	2.03	2.03	(0.00)
37.0	2.49	2.54	(-0.05)	1.73	1.72	(0.01)	1.75	1.74	(0.00)
38.0	2.19	2.24	(-0.05)	1.48	1.47	(0.01)	1.50	1.50	(0.00)
39.0	1.93	1.97	(-0.05)	1.27	1.26	(0.01)	1.28	1.28	(0.00)
40.0	1.69	1.74	(-0.05)	1.08	1.07	(0.01)	1.10	1.10	(0.00)
41.0	1.48	1.53	(-0.04)	0.93	0.92	(0.01)	0.94	0.94	(-0.00)
42.0	1.30	1.34	(-0.04)	0.79	0.78	(0.01)	0.80	0.81	(-0.00)
43.0	1.14	1.18	(-0.04)	0.68	0.67	(0.01)	0.69	0.69	(-0.00)
44.0	0.99	1.03	(-0.04)	0.58	0.57	(0.01)	0.59	0.59	(-0.01)
45.0	0.87	0.91	(-0.04)	0.50	0.49	(0.00)	0.50	0.51	(-0.01)

Table 7.22. Spark spread option prices comparison between the three *sd*-VG and *sd*-NIG models obtained using Monte Carlo simulations.

7.5 Modeling, calibration and pricing for $n \geq 3$

So far in this work, for the sake of simplicity, we have considered only the bivariate case but a n-dimensional extension of the presented models is quite immediate. In this section we outline how to extend our approach when the number of commodities is larger than two and we provide a numerical experiment for $n = 3$.
To ease the notation we denote by $\boldsymbol{X} = \{(X_1(t), X_2(t), \ldots, , X_n(t)) \, ; t \geq 0\}$ a general n-dimensional process and by $\boldsymbol{X}(t)$ its value at a fixed time t.

From a mathematical point of view, the procedure consists in defining the process $\boldsymbol{H}$ as follows:

$$\begin{aligned} &H_1(t), \\ &H_2(t) = a_1 H_1(t) + Z_{a_1}(t), \\ &\ldots \\ &H_n(t) = a_{n-1} H_1(t) + Z_{a_{n-1}}(t), \end{aligned} \tag{7.4}$$

where $a_1, \ldots, a_{n-1} \in (0, 1)$ and $Z_{a_1}(t) \ldots, Z_{a_n}(t)$ and $H_1(t)$ are independent: therefore, we need $n - 1$ parameters a_j, $j = 1, \ldots, n-1$. Based on the definition above, the extension of the models presented in Chapters 4 and 5 is straightforward and it is briefly discussed in the following subsections.

7.5.1 n-dimensional *SSD*

Consider a n-dimensional subordinator $\boldsymbol{I}$ with independent components, $\alpha_j > 0$, $j = 1, \ldots, n$ and $\boldsymbol{H}$ defined in Equation (7.4). The multivariate subordinator $\boldsymbol{G}$ is defined as:

$$G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, \ldots, n.$$

Taking a n-dimensional Brownian Motion $\boldsymbol{W}$ with independent components we then define the n-dimensional process $\boldsymbol{Y}$ as:

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W(G_j(t)), \quad j = 1, \ldots, n,$$

where $\mu_j \in \mathbb{R}$ and $\sigma_j > 0$ for $j = 1, \ldots, n$. $\boldsymbol{I}$, $\boldsymbol{H}$ and $\boldsymbol{W}$ are assumed to be mutually independent.

7.5.2 n-dimensional *LSSD*

As far as the extension to a n-dimensional *sd*-Semeraro-Luciano model is concerned, we consider a multivariate subordinator $\boldsymbol{I}$ with independent components and the process $\boldsymbol{H}$ defined in Equation (7.4). Then, we define $\boldsymbol{Y}$ as:

$$\boldsymbol{Y}(t) = \begin{pmatrix} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1\mu_1 H_1(t) + \\ \qquad \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\ \dots \\ \mu_j I_j(t) + \sigma_j W_j(I_j(t)) + \alpha_j\mu_j H_j(t) \\ \qquad +\sqrt{\alpha_j}\sigma_j\left(W_j^{\rho}(a_{j-1}H_1(t)) + \tilde{W}_1\left(Z_{a_{j-1}}(t)\right)\right) \\ \dots \\ \mu_n I_n(t) + \sigma_n W_n(I_n(t)) + \alpha_n\mu_n H_n(t) \\ \qquad +\sqrt{\alpha_n}\sigma_n\left(W_n^{\rho}(a_{n-1}H_1(t)) + \tilde{W}_{n-1}\left(Z_{a_{n-1}}(t)\right)\right) \end{pmatrix},$$

where $a_j \in (0,1)$, $\mu_j \in \mathbb{R}$, $\sigma_j > 0$ and $\alpha_j > 0$ for $j = 1, \dots, n$. $\boldsymbol{W}$ and $\tilde{\boldsymbol{W}}$ are Brownian motions with independent components, $\boldsymbol{W}^{\rho}$ is a Brownian motion with correlated components. $\boldsymbol{I}$, $\boldsymbol{H}$, $\boldsymbol{W}, \boldsymbol{W}^{\rho}$ and $\tilde{\boldsymbol{W}}$ are assumed to be mutually independent.

7.5.3 n-dimensional *BBSD*

Following the same route, we can extend the $BBSD$ to the case in which the market present $n \geq 3$ risky assets. Consider the process $\boldsymbol{H}$ defined in Equation (7.4) and take $\boldsymbol{R}$ defined as:

$$\boldsymbol{R}(t) = \begin{pmatrix} R_1(t) = \beta_{R_1} H_1(t) + \gamma_{R_1} W(H_1(t)) \\ \dots \\ R_j(t) = \beta_{R_j} H_2(t) + \gamma_{R_j} W(a_j H_1(t)) + \tilde{W}_j\left(Z_{a_j}(t)\right) \\ \dots \\ R_n(t) = \beta_{R_n} H_n(t) + \gamma_{R_n} W(a_n H_1(t)) + \tilde{W}_n\left(Z_{a_{n-1}}(t)\right) \end{pmatrix},$$

where $a_j \in (0,1)$, $\beta_{R_j} \in \mathbb{R}$ and $\gamma_{R_j} > 0$ for $j = 1, \dots, n$. In addition, take two independent processes, a one-dimensional Brownian motion W and an independent multivariate Brownian motion $\tilde{\boldsymbol{W}}$ with independent components. Consider now a n-dimensional subordinator $\boldsymbol{G}$ with independent components, a standard Brownian motion $\boldsymbol{W}$ with independent components, and independent of $\boldsymbol{G}$, and define the process $\boldsymbol{X}$ the subordinated n-dimensional Brownian motion with drift $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n_+$, by:

$$X_j(t) = \beta_j G_j(t) + \sigma_j W_j(G_j(t)) \quad j = 1, \dots, n.$$

Finally, we define the general n-variate process $\boldsymbol{Y}$ as follows:

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\alpha} \circ \boldsymbol{R}, \quad Y_j(t) = X_j(t) + \alpha_j R_j(t), \quad j = 1, \dots, n,$$

where $\circ$ denotes the Hadamard's product and $\boldsymbol{\alpha} \in \mathbb{R}^n$.

7.5.4 Calibration

The two-steps calibration procedure adopted in Section 7.3 can be easily adapted for $n \geq 3$; of course, the fit of the marginal distributions is independent of the number of underlying assets or commodities.

On the other hand, the estimation of correlation matrix can be accomplished knowing the theoretical expression of correlation between $Y_j(t)$ and $Y_i(t)$ for all $i, j = 1, \ldots, n$. By direct computation or by deriving the *chf* function of the process $\boldsymbol{Y}$ at time t, it can be shown that the correlation coefficient between $Y_j(t)$ and $Y_i(t)$, $i, j \in 1, \ldots n$ for the $BBSD$ model is given by:

$$\rho_{Y_i(t),Y_j(t)} = \frac{\alpha_i \alpha_j \left(\beta_{R_i}\beta_{R_j} a_{i-1}a_{j-1} Var\left[H_1(t)\right] + \gamma_{R_i}\gamma_{R_j} \min(a_{i-1}, a_{j-1}) \mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_i(t)\right]}\sqrt{Var\left[Y_j(t)\right]}},$$

with the convention $a_0 = 1$.

Given the theoretical expression of the correlation matrix, one can use the NLLS method or the GMM to find the set of parameters that minimize the distance between the theoretical and the historical correlation matrices. This approach could be applied to an arbitrary number of assets n, nevertheless, it is important to investigate how stable is this methodology and how good is the fit when the dimension of the problem increases. In the next section we illustrate this calibration procedure for $n = 3$, limiting our analysis to the *sd*-VG $BBSD$ model for sake of brevity.

7.5.5 A numerical application with $n = 3$

In this section we illustrate a financial application with three commodities using the *sd*-VG $BBSD$ model, we omit the application of the other two models for short. We consider future prices of the power Germany $F_1(t)$, the natural gas TTF $F_2(t)$ and the CO_2 emissions $F_3(t)$. We use the same data set of the previous section with the addition of European call options on CO_2 and the relative forward quotations.
Following the usual two-steps calibration, we first fit the margins on vanilla quoted options, we re-compute the European call options prices and quantify the error against the corresponding market data. In Table 7.23 we report the mean absolute percentage errors (MAPE) defined as:

$$MAPE = \frac{100}{n} \sum_{i=1}^{n} \left| \frac{C_i - C_i^{\Theta}(T, K)}{C_i} \right|$$

where n is the number of European call option, C_i is the market prices and $C_i^{\Theta}(T, K)$ is the model price. As mentioned, adding new commodities does not impact the robustness of the calibration of the parameters of the marginal distribution and in fact the MAPE is very low. In the second step, we fit the dependence parameters: we observe that the fitted correlation matrix $\tilde{\boldsymbol{\rho}}$ is very similar to the historical one, $\boldsymbol{\rho}$,

as reported in Table 7.24; of course the correlation between power and gas coincides with that estimated in the $n = 2$ case. We can conclude that the fit is also reliable when we deal with $n = 3$ commodities and the parameters can be used to price spread options with a third leg. Derivatives contracts written on these commodities are frequent in the energy sector and it is customary to reserve them the name of *clean-spark-spread* options whose payoff is

$$\Phi_T = \left(F_1(T) - F_2(T) - F_3(T) - K\right)^+,$$

we omit this calculation for short.

Commodity	Power Germany	Natural Gas	CO_2
MAPE	0.08%	0.41%	0.27%

Table 7.23. MAPE of re-computed option prices against the corresponding market data.

$$\boldsymbol{\rho} = \begin{bmatrix} 1.00 & 0.54 & 0.59 \\ 0.54 & 1.00 & 0.68 \\ 0.59 & 0.68 & 1.00 \end{bmatrix} \qquad \tilde{\boldsymbol{\rho}} = \begin{bmatrix} 1.00 & 0.54 & 0.59 \\ 0.54 & 1.00 & 0.67 \\ 0.59 & 0.67 & 1.00 \end{bmatrix}$$

Historical correlation matrix. Fitted correlation matrix.

Table 7.24. Comparison of the historical correlation matrix against the fitted one.

However, because the number of parameters rapidly increases as n increases, both the quality and the robustness of the fit of the correlation will be affected if n is large. This fact has been already observed and detailed in Ballotta and Bonfiglioli [9] and Luciano and Semeraro [87] relatively to the original models. For this reasons, we are inclined to conclude that all presented models should be applied only when the number of assets is relatively small in order to avoid biased results in the option pricing.

7.6 $VG++$ model for illiquidity markets

In this section we show concrete applications of the $VG++$ model we presented in Chapter 3 to energy markets. First, we price European call options using three different approaches: the closed formula of Proposition 3.2.6, Monte Carlo (MC) simulations, and the FFT method of Carr and Madan [37]. This is mandatory to be sure that the algorithms we derived are stable and adapted to pricing purposes.

Secondly, we calibrate the $VG++$ model on historical data focusing on power future market quotations adopting the *Maximum Likelihood Estimator* (*MLE*) approach. Finally, we fit the model on quoted vanilla contracts using the standard Non-Linear-Least-Squares (*NLLS*) technique and then we price non standard derivatives with backward simulations.

7.6.1 Option pricing methods

In this subsection we compare the following three different methods for vanilla options pricing:

- The closed formula derived in Section 3.2.1.
- The MC method relying upon the Algorithm 2 to simulate the process Z_a^{++}.
- The FFT method of Carr and Madan [37] based on the *chf* of the $VG++$ process given by Proposition 3.2.1.

In this first analysis we select the set of parameters reported in Table 7.25, nevertheless, we carried out tests with different parameter sets getting similar results which we do not report here for sake of brevity. We use the MC technique with 10^6 simulations and we impose $\beta = (1-a)\,\alpha$ in order to have $\mathbb{E}\left[Z_a(t)\right] = t$. As far as the computation with the closed formula (3.15) is concerned, we fix a cut-off rule for the computation of the infinite sum, namely we truncate the sum as soon as its $(n+1)$-th term contributes less than 0.01% to the sum up n. Finally, we model the risky asset process $F = \{F(t); t \geq 0\}$ as in Equation (3.14).

F_0	r	σ	θ	a	α
100	0.01	0.2	-0.1436	0.5	10

Table 7.25. Set of parameters for the numerical experiment.

In Figure 7.16 we graphically compare the difference (error in the figures) of the FFT and MC methods with respect to the closed formula of the European call option varying the strike price K and the maturity T. The size of the error of the FFT algorithm is approximately 10^{-3} and is smaller than that of the MC scheme which is around 10^{-2}. Indeed, due to its accuracy and efficiency, the FFT method is preferable for standard contracts, whereas the MC approach is more appropriate for the pricing of more exotic derivatives.

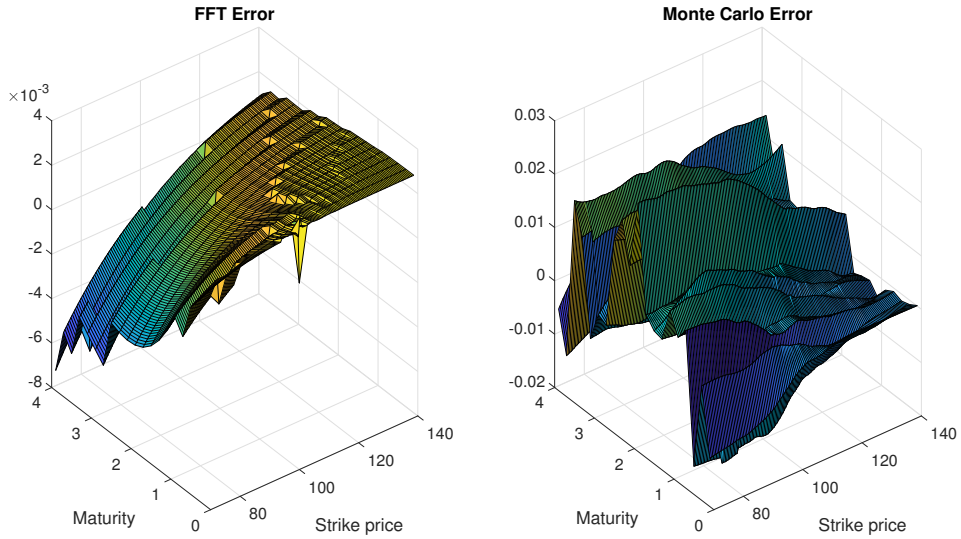

Figure 7.16. Fourier and Monte Carlo methods error for different values of the maturity T and of the strike price K.

7.6.2 Calibration

In this subsection we show how to calibrate the $VG++$ model on real market observations and find the set of unknown parameters $\Theta = (\theta, \sigma, \alpha, a)$[7]. The data-set we rely upon is the following:

- Market quotations from 23 August 2017 to 12 November 2019 of the German, Italian and Spanish power future Calendar 2020.
- Call options written on the German, Italian and Spanish power future Calendar 2020 with settlement date 19 November 2019 and expiration date on 13 December 2019.
- The risk-free rate is assumed to be $r = 0.015$.

We perform the historical calibration with a *MLE* relying on the closed form of the transition density of the $VG++$ process given by Proposition 3.2.4 and then numerically maximize the log-likelihood $\log \mathcal{L}(\Theta)$ with respect to Θ.

On the other hand, one could also adopt the *Generalized Method of Moments* (GMM) and minimize "a distance" between theoretical moments and their empirical analog, with respect to Θ. Therefore, the *GMM* method can be easily applied, by using Proposition 3.2.3 recalling that the first cumulant is the mean, the second one is the variance and that skewness $s(X)$ and kurtosis $k(X)$ can be derived from higher order cumulants as follows:

$$s(X) = \frac{c_3(X)}{c_2(X)^{3/2}}, \qquad k(X) = \frac{c_4(X)}{c_2(X)^2}.$$

[7]Note that parameter β does not appear because we imposed $b = (1-a)\alpha$ such that $\mathbb{E}[Z(t)] = t$.

The historical calibration is generally suitable for risk-management purposes, while instead the calibration on option quotes must be considered in order to properly price derivative contracts (see Cont and Tankov [42]).

If the market quotes n products[8] $\{C_i\}_{i=1}^n$, the goal is then to find the set of parameters Θ^* which minimizes the usual quantity:

$$\Theta^* = \arg\min_{\Theta} \sum_{i=1}^{n} \left(C_i - C_i^{\Theta}(K,T) \right)^2,$$

where $C_i(\Theta)$ is the price obtained by using the $VG++$ model. The optimization problem consists in a numerical Non-Linear-Least-Squared ($NLLS$) problem. In Table 7.26, Table 7.27 and Table 7.28 we report the parameters obtained per each country with the historical calibration (MLE) and with the calibration of option quotes ($NLLS$)[9], whereas in Figure 7.18 we draw the cumulative distribution functions of the $VG++$ process at maturity T[10].

European power future markets are not always liquid and, in some cases, prices tend to remain constant over time. As is shown in Figure 7.17 the power future calendar 2020 is not very liquid, especially when the delivery is far out but its liquidity increases as the delivery approaches. For these reasons, power future markets offers a natural setting to test our model. Indeed, the value of the parameters a and α can be interpreted as the *liquidity activity* of the market. Taking the change $\Delta X = X(t) - X(t-1)$ of the log-price over the time interval Δt, from Equation (3.12) we observe that the probability that the increment equals zero over the time interval Δt is strictly larger than zero and, more precisely, it is given by

$$\mathbb{P}(\Delta X = 0) = a^{\alpha \Delta t},$$

since the density of the $VG++$ process has an atom in zero. This is the main financial difference from the standard VG process which does imply that non-zero trading activity takes place in every time interval. Nevertheless, our model inherits the mathematical tractability of the standard VG process which is in any case recovered when a tends to zero.

In financial markets the liquidity is strictly related to the amount of registered transactions: if the number of trades is high, the prices fluctuate faster than when a small number of contracts is exchanged. In the extreme case where no products are traded the price remains constant over time, once again this feature cannot be captured by Brownian subordination where the subordinator has infinite activity. Therefore, illiquid markets are characterized by high values of the probability $\mathbb{P}(\Delta X = 0)$. We remark once again that since the transition density of the Variance Gamma process is atom-less, such a process always presents a non zero increment over the time period Δt and hence their paths cannot be constant over time.

[8]Usually, European Call or Put options are quoted and liquid for many markets whereas more

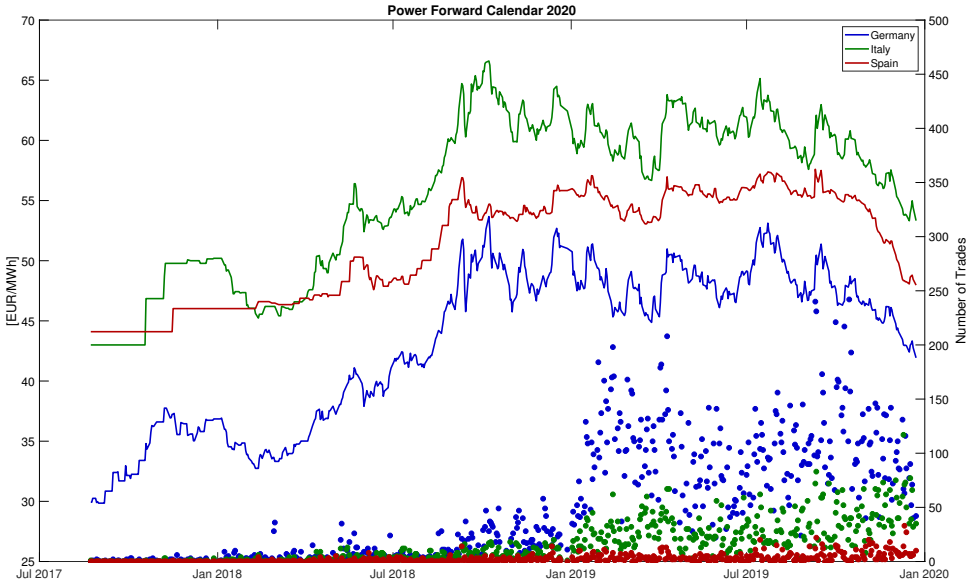

Figure 7.17. Prices of the German, Italian and Spanish power forward Calendar 2020 and respective number of trades.

The results reported in Table 7.26, Table 7.27 and Table 7.28 are coherent with some empirical facts observed in power markets: first of all, future products are more liquid than the corresponding options: this is clear if we compare the values of $\mathbb{P}(\Delta X = 0)$ obtained calibrating the model on historical forward quotations (MLE) with the ones we get when we calibrate it on European option prices ($NLLS$). Moreover, as a matter of fact, the German power future market is more liquid than the Italian and Spanish ones, as it can be observed in Figure 7.17: the number of trades in German future power markets is significantly higher than the one we observe in the other markets. This empirical evidence is coherent with the value of $\mathbb{P}(\Delta X = 0)$ we estimate for the three markets: such a probability is smaller in the German power market than in the other ones. Finally, the Spanish power future market is the most illiquid one, as it can be deduced observing the number of trades in Figure 7.17: consequently, the values of $\mathbb{P}(\Delta X = 0)$ in Table 7.28 are significantly higher than the ones reported in Table 7.26 and Table 7.27.

7.6.3 Pricing of exotic derivatives

Once that the $VG++$ model is calibrated on quoted derivatives, it is possible to price illiquid contingent claims in a consistent way. For illustrative purposes we price American put options written on the Italian power future calendar with the Least-Square Monte Carlo introduced by Longstaff and Schwartz [83] combined with

complex derivatives are traded over the counter (OTC).

[9]For brevity we focus on the MLE method and do not use the GMM.

[10]Note that the density has a non-zero mass at point $x = 0$.

Method	σ	θ	a	α	$\mathbb{P}(\Delta X = 0)$
MLE	0.16	0.18	0.46	1255.7	0.02
NLLS	0.20	0.0.39	0.54	650.71	0.21

Table 7.26. Set of parameters Θ for the Italian power future market.

Method	σ	θ	a	α	$\mathbb{P}(\Delta X = 0)$
MLE	0.24	0.02	0.27	872.83	0.01
NLLS	0.28	0.91	0.52	1044.43	0.06

Table 7.27. Set of parameters Θ for the German power future market.

Method	σ	θ	a	α	$\mathbb{P}(\Delta X = 0)$
MLE	0.09	0.05	0.38	6430.06	0.08
NLLS	0.13	0.83	0.49	616.35	0.18

Table 7.28. Set of parameters Θ for the Spanish power future market.

the backward simulations described in Section 3.2.2 and for completeness, with the sequential (forward) simulation approach. The results are reported in Figure 7.19, where we fix the strike price $K = 56$ and the maturity $T = 0.26$ years and we set different values of the process F at time $t = 0$. As observed, for example, in Seydel [117], the value of the American put options is never lower than the payoff and, as expected, the sequential simulation and the backward simulation return indistinguishable results. This result is not surprising, since the interpretation of the index set $I = \{t \geq 0\}$ of the stochastic process X as time is just a convention: the mathematical object $X = \{X(t); t \in I\}$ is well defined even if the index set I has not an order relation. A simple question then arises: is there any advantage in using backward simulations instead of the standard forward approach? Backward simulations are not necessarily faster than forward simulations as observed in Sabino [106]: nevertheless, the backward recursion of the stochastic optimization at each time step t_j requires the path simulations at time t_j and t_{j+1} only, which is perfectly consistent with backward approach in contrast, with the forward strategy one has to store the entire set of paths. For example, using the standard forward simulation approach to price an American contract with maturity one year, daily early exercise and 10^6 simulations, $2.52 \cdot 10^8$ values need to be stored instead of $2 \cdot 10^6$ values which are necessary with the backward simulations strategy. This gives a remarkable computational advantage especially if the contract has a large maturity or if one deals with the pricing of more complex derivatives such as gas storages (Boogert and de Jong [24]) or virtual power plants (Tseng and Barz [121]), for which additional discretization grids are needed.

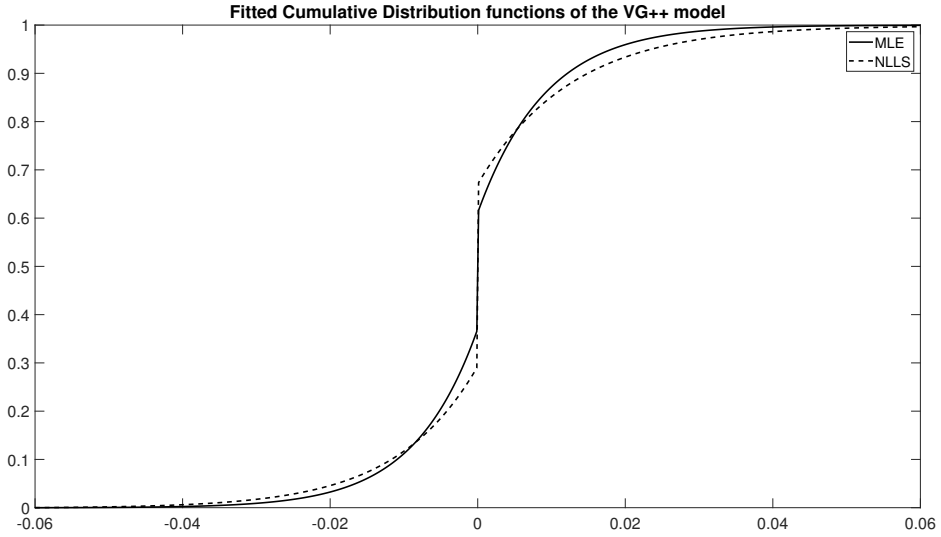

Figure 7.18. Fitted cumulative distribution functions of the $VG++$ process obtained at maturity T using *MLE* and *NLLS* methods on Italian power forward quotations.

In order to point out differences between the Variance Gamma and the $VG++$ processes we apply them to the same market framework: to this aim, we consider the pricing of Lookback call options with MC simulations. We stress out once again that the transition density of the $VG++$ process has an atom at zero and then the changing ΔX in the log-price over the time interval Δt can be zero with strictly positive probability: this is equivalent to say that no trades have been exchanged over that time interval. On the other hand, in the Variance Gamma model a zero trading activity is not possible over any finite time interval Δt. This difference between the two models has an impact on derivative valuation. Indeed, from a financial perspective, whenever an agent sells derivatives, a hedging strategy has to be implemented. If the underlying asset is not liquid, such a hedging strategy, a delta-hedging for example, might be expensive and hard to implement.

Indeed, if an option seller decides to adopt the delta-hedging strategy it may happen that the underlying asset is not available therefore, the strategy can not be implemented at all. On the other hand, if the underlying asset is exchanged but the bid-ask spread is extremely wide, the hedging strategy results to be highly expansive. For these reasons, the price of options in illiquid markets should be higher than that of the same contingent claim traded in a liquid market: the price of the contingent claim must take into account the cost of the "impracticable" hedging strategy.

In Figure 7.20 we show the price of Lookback call options on the maximum in the Spanish future market, which is the most illiquid one between the markets we analyzed. It is worth noting that the value of the option computed with the Variance Gamma model is lower than the one we obtain using the $VG++$ model. As stated before, unlike the Variance Gamma model, the $VG++$ considers the possibility that

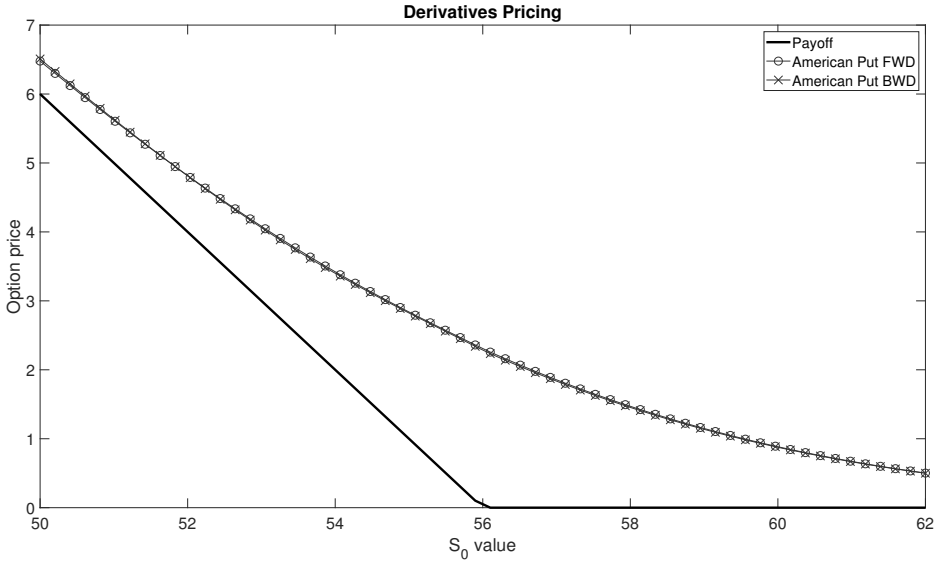

Figure 7.19. Price of the American put option with different values of starting point $F(0)$ using Least-Square Monte Carlo with forward and backward simulations.

the market becomes illiquid leading to possible difficulties in the implementation of an adequate hedging strategy. Accordingly, when the market is illiquid, in order to mitigate his risk exposure, the only thing that the option seller can do is to increase the option value. We finally observe that the price differences in Figure 7.20 might not seem remarkable: indeed, even if the Spanish future market has the 8% of probability of not being liquid on a given day, such a level of liquidity guarantees to the option seller to secure himself against derivative price fluctuations.

We conclude that, when we consider illiquid markets, the $VG++$ model is a better choice because it allows the option seller to include in the option price a sort of "cost of market illiquidity", which somehow mitigates the risk of not having a proper hedging strategy.

7.6.4 A Multivariate framework for the $VG++$ process

At this point, it should be clear that one of the most challenging tasks in financial modelling is the extension of continuous time Lévy models from a univariate to a multivariate framework. As we stated befeor, in a Gaussian setting, as the one proposed by Black and Scholes [23] or Heath et al. [69], the extension is easy since the whole dependence structure is caught by the covariance matrix. Multi-asset versions of commonly used Lévy models have been proposed by Buchmann et al. [30, 32, 33], Michaelsen and Szimayer [94] and Michaelsen [93], Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9] among the others.

As observed by Sabino and Cufaro-Petroni [108], the scaling and summation properties of the Gamma laws also hold for their a-remainder's, namely:

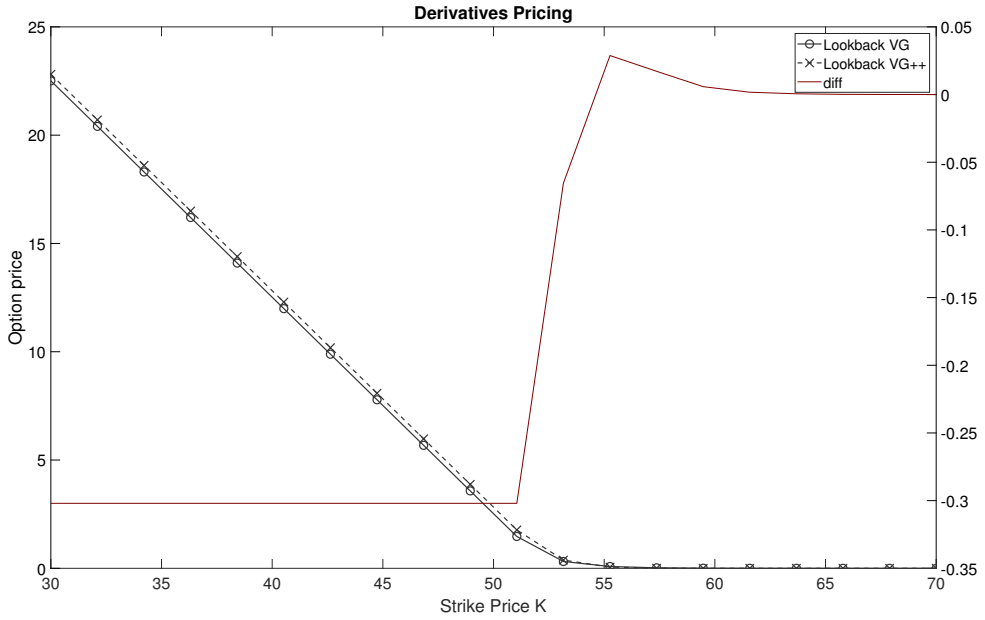

Figure 7.20. Price of Lookback call option over the maximum in the Spanish power future market. The prices are computed using the Variance Gamma model and the $VG++$ model, calibrated on the same data-set of vanilla options.

- If $Z_a \sim \Gamma^{++}(a, \alpha, \beta)$ for every $c > 0$ it results:

$$cZ_a \sim \Gamma^{++}\left(a, \alpha, \frac{\beta}{c}\right). \tag{7.5}$$

- If $Z_{a,i} \sim \Gamma^{++}(a, \alpha_i, \beta)$, $i = 1, \ldots, n$ and are independent then:

$$\sum_{i=1}^{n} Z_{a,i} \sim \Gamma^{++}\left(a, \sum_{i=1}^{n} \alpha_i, \beta\right). \tag{7.6}$$

For this reason, the same construction proposed by Semeraro [115], Luciano and Semeraro [87] and Ballotta and Bonfiglioli [9] can be used to build a multivariate subordinator $\boldsymbol{H} = \{(H_1(t), \ldots, H_n(t)); t \geq 0\}$ whose marginal distributions have a Γ^{++} law with suitable parameters. The construction is the following: consider independent $X_i = \{X_i(t); t \geq 0\}$ for $i = 1, \ldots, n$ with $X_i(t) \sim \Gamma^{++}\left(a, \alpha_i t, \frac{\beta}{c_i}\right)$ and consider Z_a^{++} defined in Section 3.1.1. We define the process $\boldsymbol{H}$ as:

$$H_i(t) = X_i(t) + c_i Z_a^{++}(t), \qquad i = 1, \ldots, n.$$

where $c_i > 0$ for all $i = 1, \ldots, n$. Using properties (7.5) and (7.6) it is easy to check that $H_i(t) \sim \Gamma^{++}\left(a, (\alpha_i + \alpha)t, \frac{\beta}{c_i}\right)$. All the components of the process $\boldsymbol{H}$ are dependent, because of the presence of the common process Z_a^{++}. $\boldsymbol{H}$ is a multivariate subordinator and it can be used to derive multidimensional versions of the $VG++$ process: this topic will be the subject of future investigations.

Chapter 8

Conclusions and further inquiries

In this work we have shown how the probabilistic notion of self-decomposability can be used in finance both to build multivariate versions of the well known Variance Gamma and Normal inverse Gaussian processes with *stochastic delay* and to construct the process we called $VG++$ which is particularly useful to model illiquid markets.

All the models we have obtained are easy to handle both from a mathematical and numerical point of view: we deeply investigated their mathematical properties and we pointed out which processes are Markovian or Lévy and which are not. In particular, the characteristic functions and the linear correlation coefficients of the multidimensional Variance Gamma and Normal Inverse Gaussian process with stochastic delay have been derived in a closed form. The knowledge of the expression of the characteristic function allowed us to adopt numerical algorithms based on the Fourier transform for spead option pricing, whereas the closed form expression of the linear correlation coefficient at time t is crucial to calibrate the dependence structure of the multidimensional process on real market data.

In addition, for the $VG++$ process we have derived a closed form expression for the price of an European call option. In particular, we have shown how the price of an European call option can be obtained as an infinite sum of call options under the Variance Gamma model where the shape parameter of the underlying subordinating gamma process is an integer. The evaluation of this formula can be easily done numerically and its computation is extremely fast, since it requires only matrices multiplications and no integral computations.

As we observed, in real financial applications Monte Carlo algorithms are frequently used, since closed formula for complex contingent claims evaluation are usually not available: for these reason, a remarkable part of the thesis consist in the derivation of efficient numerical schemes for the path simulations of the processes we have introduced. Such results can be used to evaluate derivative contracts following the Monte Carlo approach.

In particular, another significant contribution of this work is the derivation of an efficient algorithm to sample from the distribution of the so called a-remainder Z_a when the law of the random variable is inverse Gaussian. Such an algorithm is faster

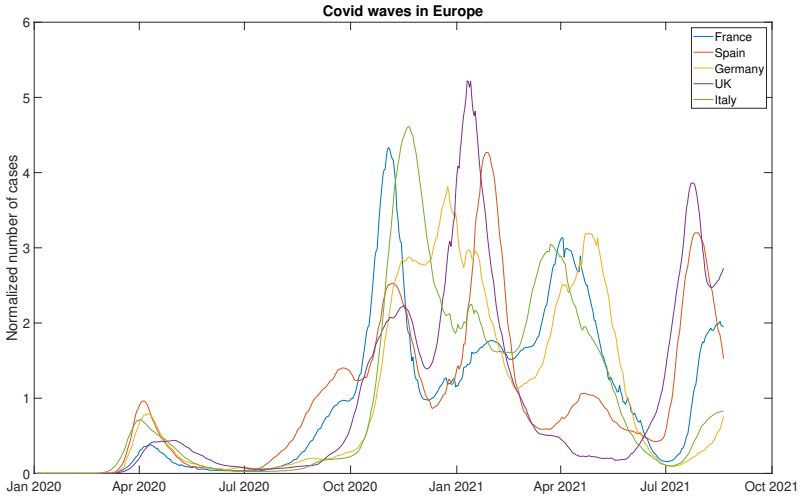

Figure 8.1. Number of Covid-19 cases divided by the mean of the whole considered period.

than the one are presented in literature since it does not relies upon an acceptance rejection algorithm and it can also be used to simulate so called Ornstein-Uhlenbeck backward driven Lévy process with inverse Gaussian stationary distribution.

Another new contribution of the thesis is the backward in time $VG++$ process simulation, which is accomplished by using Lévy bridges techniques: this result reveals to be particularly useful if one needs to prices complex American derivatives: adopting the backward simulation approach a large amount of RAM can be saved, especially when the maturity is far.

In the last part of this work, we have applied all the derived models to energy markets comparing the results and giving them an economic interpretation. In particular, we have calibrated the models on real financial data and we used several numerical techniques to price different type of contracts. Unfortunately, it turns out that the multidimensional version of the Variance Gamma and of the Normal Inverse Gaussian processes are easy to calibrate only if the number of the considered underlying asset is small: indeed, if the number of the considered risky asset grows, the number of the parameters rapidly increases and hence the fitting of the dependence structure may be difficult to achieve. Nevertheless, it is hard to find derivatives contracts written on more than three underlying assets and, for this reason, our models can be used is many practical situations.

In addition, we point out that the modelling framework we introduced by using the so called *stochatic delay* is very general and can be used beyond financial applications to model all those physical systems which present a delay in time one with respect to the other ones. A possible application of this approach could be the modelling of the waves of an pandemic disease across different areas of the world as it has been observed, for example, for the Covid-19 pandemic. As we can observe in Figure 8.1, when infection cases increase in one country and, after a stochastic time, they start increasing also in other countries. We think that the application of

the proposed approach to build an epidemiological mathematical model based on stochastic differential equations might be investigated.

Focusing on financial applications, even if we applied our processes to model energy markets, it is worth observing that such a framework can also be used, for example, in equity derivatives, with an arbitrary number of stocks, or in credit risk to model a chain of defaults caused by a market shock that propagates with a stochastic delay. Moreover, a multidimensional application of the Variance Gamma++ process could be the object of a future inquire. Finally, a topic deserving further investigation is the possibility to apply the procedure adopted to construct the Variance Gamma++ process to the inverse Gaussian law, whose law is a self-decomposable as well, and accordingly study its mathematical properties and potential financial applications.

Appendices

Appendix A

A non Lévy process

In this section we show that if the characteristic function $\phi_{X(t)}(u),\ u \in \mathbb{R}$ of a given process X at time t is associated to an infinitely divisible law it is not guaranteed that X is a Lévy process. In other words, the fact that $\phi_{X(t)}(u)$ is associated to an infinitely divisible distribution is only a necessary condition to state that X is a Lévy process. On the other hand, the law of a Lévy process at time t is infinitely divisible. Moreover, if given an infinitely divisible law there exists a Lévy process associated to the given law. These results are summarized in Sato [112, Theorem 7.10].

For example, consider the following processes:

$$\begin{aligned}\boldsymbol{X} &= \{(W(t), W(at)); t \geq 0\}, \\ \boldsymbol{Z} &= \left\{(\sqrt{1-a}W_1(t) + \sqrt{a}W_2(t), \sqrt{a}W_2(t)); t \geq 0\right\}\end{aligned}$$

where $a \in (0,1)$ and $W_1 = \{W_1(t); t \geq 0\}$ and $W_2 = \{W_2(t); t \geq 0\}$ are independent Brownian motions. The increments of the process $\boldsymbol{X}$ are not independent and hence it is not a Lévy process. If we take $M = \begin{bmatrix}\sqrt{1-a} & \sqrt{a} \\ 0 & \sqrt{a}\end{bmatrix}$ and $W = [W_1, W_2]$ we can write $Z = MW^T$ and hence by Cont and Tankov [42, Theorem 4.1] it is a Lévy process. If we compute the characteristic function at time t of both processes X and Z we get that:

$$\phi_{X(t)}(\boldsymbol{u}) = \phi_{Z(t)}(\boldsymbol{u}) = e^{-\frac{t}{2}\boldsymbol{u}\Sigma\boldsymbol{u}^T}, \qquad \forall t \geq 0,$$

where $\boldsymbol{u} = [u_1, u_2]$ and $\Sigma = \begin{bmatrix}1 & a \\ a & a\end{bmatrix}$. We have proven the following Proposition.

Proposition A.0.1. *Let $W = \{W(t); t \geq 0\}$, $W_1 = \{W_1(t); t \geq 0\}$ and $W_2 = \{W_2(t); t \geq 0\}$ be mutually independent standard Brownian motions and $a \in (0,1)$. Consider the two processes $\boldsymbol{X} = \{\boldsymbol{X}(t); t \geq 0\}$ and $\boldsymbol{Z} = \{\boldsymbol{Z}(t); t \geq 0\}$ defined as follows:*

$$\boldsymbol{X}(t) = (W(t), W(at)), \tag{A.1}$$

$$\boldsymbol{Z}(t) = \left(\sqrt{1-a}W_1(t) + \sqrt{a}W_2(t), \sqrt{a}W_2(t)\right). \tag{A.2}$$

Then $\boldsymbol{X}(t) \stackrel{d}{=} \boldsymbol{Z}(t)$ *for all* $t \geq 0$.

Proof. Let be $t \geq 0$ fixed and $\boldsymbol{u} \in \mathbb{R}^2$. Compute the characteristic function of $\boldsymbol{X}$ and $\boldsymbol{Z}$ at time t.

$$\begin{aligned}
\phi_{\boldsymbol{X}(t)}(\boldsymbol{u}) &= \mathbb{E}\left[e^{iu_1W(t)+iu_2W(at)}\right] = \mathbb{E}\left[e^{iu_1(W(t)-W(at)+W(at))+iu_2W(at)}\right] \\
&= \mathbb{E}\left[e^{iu_1(W(t)-W(at))+i(u_1+u_2)W(at)}\right] \\
&= \mathbb{E}\left[e^{iu_1(W(t)-W(at))}\right]\mathbb{E}\left[e^{i(u_1+u_2)W(at)}\right] \\
&= e^{-\frac{1}{2}u_1^2(t-at)}e^{-\frac{1}{2}(u_1+u_2)^2at} \\
&= \exp\left\{-\frac{1}{2}\left(1-a\right)u_1^2t\right\}\exp\left\{-\frac{1}{2}(u_1+u_2)^2at\right\}.
\end{aligned}$$

$$\begin{aligned}
\phi_{\boldsymbol{Z}(t)}\left(\boldsymbol{u}\right) &= \mathbb{E}\left[e^{i\sqrt{1-a}u_1W_1(t)+iu_1\sqrt{a}W_2(t)+iu_2\sqrt{a}W_2(t)}\right] \\
&= \mathbb{E}\left[e^{i\sqrt{1-a}u_1W_1(t)}\right]\mathbb{E}\left[e^{i(u_1+u_2)\sqrt{a}W_2(t)}\right] \\
&= \exp\left\{-\frac{1}{2}\left(1-a\right)u_1^2t\right\}\exp\left\{-\frac{1}{2}\left(u_1+u_2\right)^2at\right\}
\end{aligned}$$

By Lévy continuity theorem we can conclude that since the characteristic functions coincide for all $t \geq 0$ the two random variables have the same distribution. ■

Hence we can conclude that the condition that requires that the law of a process $\boldsymbol{X}$ at time t is infinitely divisible is only a necessary but not a sufficient condition to guarantees that $\boldsymbol{X}$ is a Lévy process.

Clearly the two considered processes are different and hence their paths are not the same as it can be observed in Figure A.1. Indeed, process $\boldsymbol{X}$ shows two components that seems to be delayed one with respect the other one, whereas the trajectory of process $\boldsymbol{Z}$ seems not to show any particular delay. Proposition A.0.1 shows how to construct the Lévy process $\boldsymbol{Z}$.

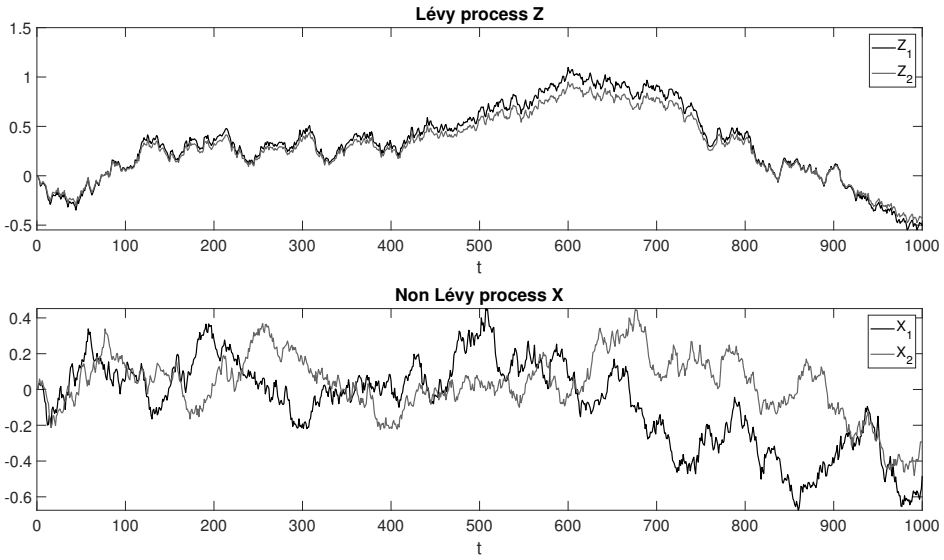

Figure A.1. Possible paths of the processes $\boldsymbol{X}$ and $\boldsymbol{Z}$ defined as by (A.1) and (A.2).

Appendix B

Variance Erlang distribution: derivation and option pricing

In this Appendix we report some results about Exponential Polynomial Trigonometric (EPT) distributions we used in the article. For a complete discussion about this topic refer to Sexton and Hanzon [116].

B.1 2-EPT distributions

The class of EPT functions $f : [0, \infty) \to \mathbb{R}$ is given by:

$$f(x) = \Re\left(\sum_{k=1}^{K} p_k(x)\, e^{\mu_k x}\right)$$

where $\Re(z)$ denotes the real part of a complex number $z \in \mathbb{C}$, $p_k(x)$ is polynomial with complex coefficients for each $k = 1, \ldots, K$ and $\mu_k \in \mathbb{C}$ for $k = 1, 2, \ldots, K$. And EPT function defined on the positive real line can be represented in the following form:

$$f(x) = \boldsymbol{c} e^{\boldsymbol{A}x} \boldsymbol{b}, \qquad x \geq 0$$

where $\boldsymbol{A}$ is a $n \times n$ matrix, $\boldsymbol{c}$ is $1 \times n$ vector and $\boldsymbol{b}$ is a $n \times 1$ vector. We consider probability density functions which can be written as two separate EPT functions:

$$f(x) = \begin{cases} \boldsymbol{c}_N e^{\boldsymbol{A}_N x} \boldsymbol{b}_N, & x \geq 0 \\ \boldsymbol{c}_P e^{\boldsymbol{A}_P x} \boldsymbol{b}_P, & x > 0. \end{cases}$$

B.2 Variance Gamma as 2-EPT function

Variance Gamma density can be views as an 2-EPT function. The density of a random variable X with Variance Gamma law is given by:

$$f_X\left(x;C,G,M\right)=\frac{\left(GM\right)^C}{\sqrt{\pi}\Gamma\left(C\right)}\exp\left(\frac{\left(G-M\right)x}{2}\right)\left(\frac{\left|x\right|}{G+M}\right)^{C-\frac{1}{2}}K_{C-\frac{1}{2}}\left(\frac{\left(G+M\right)\left|x\right|}{2}\right)$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind and $C,G,M\in\mathbb{R}^+$. For $u\in\mathbb{R}$, its characteristic function is given by the following formula:

$$\phi_X\left(u\right)=\left(\frac{GM}{GM+\left(M-G\right)iu+u^2}\right)^C$$

If $C\in\mathbb{N}$, using Abramowitz and Stegun [1, pag. 443] which is:

$$\sqrt{\frac{\pi}{2x}}K_{n+\frac{1}{2}}\left(x\right)=\left(\frac{\pi}{2x}\right)e^{-x}\sum_{k=0}^{n}\left(n+\frac{1}{2},k\right)\left(2z\right)^{-k}$$

where

$$\left(n+\frac{1}{2},k\right)=\frac{\left(n+k\right)!}{k!\Gamma\left(n-k+1\right)},$$

and defining $z\coloneqq\frac{(G+M)|x|}{2}$ we have that:

$$\begin{aligned}
f_X(x)&=\frac{\left(GM\right)^C}{\sqrt{\pi}\Gamma\left(c\right)}\exp\left(\frac{\left(G-M\right)x}{2}\right)\left(\frac{\left|x\right|}{G+M}\right)^{C-\frac{1}{2}}K_{C-\frac{1}{2}}\left(\frac{\left(G+M\right)\left|x\right|}{2}\right)\\
&=\frac{\left(GM\right)^C}{\sqrt{\pi}\Gamma\left(c\right)}\exp\left(\frac{\left(G-M\right)x}{2}\right)\left(\frac{\left|x\right|}{G+M}\right)^{C-\frac{1}{2}}\sqrt{\frac{2z}{\pi}}\sqrt{\frac{\pi}{z}}K_{C-\frac{1}{2}}\left(z\right)\\
&=\frac{\left(GM\right)^C}{\sqrt{\pi}\Gamma\left(c\right)}\exp\left(\frac{\left(G-M\right)x}{2}\right)\left(\frac{\left|x\right|}{G+M}\right)^{C-\frac{1}{2}}\sqrt{\frac{2z}{\pi}}\frac{\pi}{z}e^{-z}\sum_{k=0}^{C-1}\left(C-1-\frac{1}{2},k\right)\left(2z\right)^{-k}\\
&=\frac{\left(GM\right)^C}{\left(C-1\right)!}\exp\left(\frac{\left(G-M\right)x}{2}-\frac{\left(G+M\right)\left|x\right|}{2}\right)\sum_{k=0}^{C-1}\left(C-1-\frac{1}{2},k\right)\left|x\right|^{C-1}\left(G+M\right)^{-k-C}\\
&=\exp\left(\frac{\left(G-M\right)x}{2}-\frac{\left(G+M\right)\left|x\right|}{2}\right)\underbrace{\frac{\left(GM\right)^C}{\left(C-1\right)!}\sum_{k=0}^{C-1}\frac{\left(C-1+k\right)!\left(G+M\right)^{-C-k}\left|x\right|^{C-1-k}}{\left(C-1-k\right)!k!}}_{p(x)}.
\end{aligned}$$

We can split the density around the origin, obtaining:

$$f_X\left(x\right)=\begin{cases}\exp\left(Gx\right)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!(C-1-s)!} & x\leq 0\\ \exp\left(-Mx\right)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!(C-1-s)!} & x>0.\end{cases}\tag{B.1}$$

Observe that the polynomial part of (B.1) are identical for all x and this implies that $\boldsymbol{c}_N=\boldsymbol{c}_P$ and $\boldsymbol{b}_N=\boldsymbol{b}_P$. Set:

$$\begin{aligned}
&\boldsymbol{c}=\left(c_0,\ldots,c_{S-1}\right), && \boldsymbol{c}\in\mathbb{R}^{1\times C}\\
&c_s=\frac{\left(MG\right)^C}{\left(C-1\right)!}\frac{\left(2\left(C-1\right)-s\right)!\left(G+M\right)^{-2C+1+s}}{\left(C-1-s\right)!}, && s\in\left(0,\ldots,C-1\right).
\end{aligned}$$

Similarly $\boldsymbol{b} = (1, 0, \ldots, 0)^T$ is a $C \times 1$ column vector whereas $\boldsymbol{a}$ is given by:

$$\boldsymbol{a} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and finally we get that $p(x) = \boldsymbol{c}e^{-\boldsymbol{a}x}\boldsymbol{b}$. This leads us to write:

$$f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{G\boldsymbol{x}}e^{-\boldsymbol{a}x}\boldsymbol{b} & x \leq 0 \\ \boldsymbol{c}e^{-M\boldsymbol{x}}e^{\boldsymbol{a}x}\boldsymbol{b} & x > 0. \end{cases}$$

$$f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{(G\boldsymbol{I}-\boldsymbol{a})x}\boldsymbol{b} & x \leq 0 \\ \boldsymbol{c}e^{(-M\boldsymbol{I}+\boldsymbol{a})x}\boldsymbol{b} & x > 0. \end{cases}$$

Define $\boldsymbol{A}_N = G\boldsymbol{I} - \boldsymbol{a}$ and $\boldsymbol{A}_P = -M\boldsymbol{I} + \boldsymbol{a}$, therefore:

$$f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{\boldsymbol{A}_N x}\boldsymbol{b} & x \leq 0 \\ \boldsymbol{c}e^{\boldsymbol{A}_P x}\boldsymbol{b} & x > 0. \end{cases}$$

B.3 The price process

We model the risky underlying asset F as:

$$F(t) = F(0)e^{rT+\omega T+X(T)}, \quad F(0) = F_0$$

where $T \geq 0$, r is the risk-free rate and ω is such that the discounted prices are martingales. In order to work under the risk-neutral measure $\mathbb{Q}$ we must require that:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\omega T X(T)}\right] = 1$$

and this leads to:

$$\omega = C \log\left(\left(1 - \frac{1}{M}\right)\left(1 + \frac{1}{G}\right)\right).$$

If we add the constrain $CT \in \mathbb{N}$, we observe that ω is defined only if $M > 1$. Moreover, if $CT \in \mathbb{N}$ a closed formula for a Call option with maturity T can be derived (In the original article you have $\tau = T - t$, which is the time to maturity, instead of T: here we considered $t = 0$ and hence τ and T coincides).

B.4 A closed formula for Call option pricing

Consider a Call option with strike price K and maturity T. The value of the underlying asset at $t = 0$ is $F(0)$ and we consider a constant risk free rate $r \geq 0$. Define:

$$d = \log\left(\frac{F(0)}{K}\right) + (r + \omega)\, T.$$

The price of the Call option $C(0, K)$, where $X(T)$ has a infinitely divisible distribution with 2-EPT density distribution with realizations $(\boldsymbol{A}_N, \boldsymbol{b}_N, \boldsymbol{c}_N, \boldsymbol{A}_P, \boldsymbol{b}_P, \boldsymbol{c}_P)$, is given by:

- If $d > 0$:

$$\begin{aligned} C(0,K) = F(0)e^{\omega T}\left(\boldsymbol{c}_N\left(\boldsymbol{A}_N + \boldsymbol{I}\right)^{-1}\right)\boldsymbol{b}_P - \boldsymbol{c}_N\left(\boldsymbol{A}_N\ + \boldsymbol{I}\right)^{-1} e^{-(\boldsymbol{A}_N+\boldsymbol{I})d}\boldsymbol{b}_N \\ - \boldsymbol{c}_P\left(\boldsymbol{A}_p + \boldsymbol{I}\right)^{-1}\boldsymbol{b}_P - Ke^{-rT}\left(1 - \boldsymbol{c}_N\boldsymbol{A}_N^{-1}e^{-\boldsymbol{A}_N d}\boldsymbol{b}_N\right) \end{aligned}$$

- If $d \leq 0$:

$$C(0,K) = -F(0)e^{\omega T}\boldsymbol{c}_P\left(\boldsymbol{A}_P + \boldsymbol{I}\right)^{-1} e^{-(\boldsymbol{A}_p+\boldsymbol{I})d}\boldsymbol{b}_p + Ke^{-rT}\boldsymbol{c}_P\boldsymbol{A}_P^{-1}e^{-\boldsymbol{A}_p d}\boldsymbol{b}_P.$$

In contrast to many option pricing formulas available in finance, observe that no integrals appear: the computation of $C\left(0, K\right)$ requires only linear algebra techniques which are usually faster than numerical integration procedures.

B.5 From C, G, M to $\alpha, \beta, \sigma, \theta$

Usually in literature, the parametrization of the Variance Gamma process is given in term of α, β, σ and θ, whereas in the previous section the 2-EPT version of the Variance Gamma as function of C, G and M. Since these equivalent parametrization may be a source of confusion, in this section we show how to easily switch from one to the other. For the sake of completeness, we recall how the Variance Gamma process is defined.

Definition B.5.1. *Consider the gamma process $G = \{G(t); t \geq 0\}$ such that $G(t) \sim \Gamma\left(\alpha t, \beta\right)$ and consider a Brownian motion W with drift $\theta \in \mathbb{R}$ and diffusion $\sigma \in \mathbb{R}^+$. The process $X = \{X(t); t \geq 0\}$ defined as:*

$$X(t) = \theta G(t) + \sigma W(G(t)) \quad t \geq 0, \tag{B.2}$$

is called Variance Gamma process.
Its characteristic function is given by:

$$\phi_{X(t)}(u) = \left(1 - \frac{i}{\beta}\left(u\theta + iu^2\frac{\sigma^2}{2}\right)\right)^{-\alpha t}. \tag{B.3}$$

Observe that Equation (B.3), can be rewritten as:

$$\phi_{X(t)} = \left(1 - \frac{1}{\beta}\left(u\theta + i\frac{\sigma^2}{2}u^2\right)\right)^{-\alpha T} = \left(\frac{2\frac{\beta}{\sigma^2}}{2\frac{\beta}{\sigma^2} - iu\frac{2\theta}{\sigma^2} + u^2}\right)^{\alpha T},$$

that has to be compared to:

$$\phi_{X(t)}(u) = \left(\frac{GM}{GM + (M - G)\, iu + u^2}\right)^{C},$$

and hence,

$$GM = 2\frac{\beta}{\sigma^2},$$
$$M - G = -2\frac{\theta}{\sigma^2}.$$

Finally we obtain:

$$G = \frac{1}{\sigma^2}\left(\theta + \sqrt{\theta^2 + \beta\sigma^2}\right),$$
$$M = \frac{\sqrt{\theta^2 + \beta\sigma^2}}{\sigma^2} - \frac{\theta}{\sigma^2}.$$

Appendix C

Markovian and non Markovian processes

In this Appendix we clarify better we clarify better which processes introduced in Chapter 4 are Lévy and which are not. Moreover we show that some of the process we introduced are not even Markov processes.

In the proof of Proposition 4.2.2, we write $\boldsymbol{Y}(t)$ as:

$$\boldsymbol{Y}(t) = \boldsymbol{Y_I}(t) + \boldsymbol{Y_H}(t),$$

where:

$$\boldsymbol{Y_I}(t) = \begin{pmatrix} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) \\ \mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) \end{pmatrix}$$

and:

$$\boldsymbol{Y_H}(t) = \begin{pmatrix} \alpha_1\mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\ \alpha_2\mu_2 H_2(t) + \sqrt{\alpha_2}\sigma_2 \left(W_2^{\rho}(aH_1(t)) + \tilde{W}(Z_a(t))\right) \end{pmatrix}.$$

The first term $\boldsymbol{Y_I}(t)$ is a Lévy process because is based on a linear combination of independent Lévy processes, on the other hand $\boldsymbol{Y_H}(t)$ can be split into two terms: $\boldsymbol{Y_H}(t) = \boldsymbol{Y_H^1}(t) + \boldsymbol{Y_H^2}(t)$.

$$\boldsymbol{Y_H^1}(t) = \begin{pmatrix} \alpha_1\mu_1 H_1(t) \\ \alpha_2\mu_2 H_2(t) + \tilde{W}(Z_a(t)) \end{pmatrix}.$$

$$\boldsymbol{Y_H^2}(t) = \begin{pmatrix} \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\ \sqrt{\alpha_2}\sigma_2 W_2^{\rho}(aH_1(t)) \end{pmatrix}.$$

In the following, we take for simplicity $\alpha_1 = \alpha_2 = 1$ and $\sigma_1 = \sigma_2 = 1$. $\boldsymbol{Y_H^1}(t)$ is once again a Lévy process because it is a linear combination of independent Lévy processes, in contrast, we show that $\boldsymbol{Y_H^2}(t)$ is neither a Lévy process nor a Markov process. Take B_1 and B_2, two independent Brownian motions and $\mathbf{B} = \{W_1^{\rho}(t) = B_1(t), W_1^{\rho}(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t), t \geq 0\}$, then

$$\boldsymbol{Y_H^2}(t) = \begin{pmatrix} B_1(H_1(t)) \\ \rho\, B_1(a\, H_1(t)) + \sqrt{1-\rho^2}\, B_2(a\, H_1(t)) \end{pmatrix}.$$

We follow the same route as in the first point-to-point reply to the referee, where we proved that the increment of the process are correlated, once again we take the cross term. For $s < t$, $0 < a < 1$, we have

$$\begin{gathered}\mathbb{E}\left[(B_1(a\,H_1(t)) - B_1\left(a\,H_1(s)\right)\,B_1\left(H_1(s)\right)\right] = \\ \mathbb{E}\left[B_1(a\,H_1(t))\,B_1\left(H_1(s)\right)\right] - \mathbb{E}\left[(B_1(a\,H_1(t))\,B_1\left(H_1(s)\right)\right] = \\ \mathbb{E}\left[\min\left(B_1(a\,H_1(t)), B_1\left(H_1(s)\right)\right)\right] - a\,s \neq 0\end{gathered} \tag{C.1}$$

therefore the process $\boldsymbol{Y}^2_{\boldsymbol{H}}$ is not Lévy and hence $\mathbf{Y}$ neither. In order to check if the process $\boldsymbol{Y}^2_{\boldsymbol{H}}(t)$ is a Markov process, it suffices to focus on the process $\mathbf{T} = \{T_1(t) = B_1(H_1(t)), T_2(t) = \rho\,B_1(a\,H_1(t))\}$. Denote with $\mathbb{F} = \{\mathcal{F}(t), t \geq 0\}$ the natural filtration of $\mathbf{T}$, then if $\mathbf{T}$ a Markov process, for $s < t$ and for any Borel function g, it must hold

$$\mathbb{E}\left[g(T_1(t), T_2(t)|\mathcal{F}(s)\right] = \mathbb{E}\left[g(T_1(t), T_2(t)|T(s)\right]. \tag{C.2}$$

However, that cannot be true because $aH_1(t)$ can be smaller than $H(s)$, therefore the process cannot be independent of the past before s. We also detail the apparent contradiction that $\phi_{\boldsymbol{Y}^2_{\boldsymbol{H}}(t)}(u) = \phi_{\boldsymbol{Y}^2_{\boldsymbol{H}}(1)}(u)^t$ although $\boldsymbol{Y}^2_{\boldsymbol{H}}$ is not a Lévy process, once again, it suffices to focus on the process $\mathbf{T}$. The *chf* of $\mathbf{T}(t)$ is

$$\begin{aligned}\phi_{\mathbf{T}(t)}(u) &= \mathbb{E}\left[e^{i\,u_1\,B_1(H_1(t))+i\,u_2\,\rho\,B_1(a\,H_1(t))}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{i\,u_1(B_1(H_1(t))-B_1(a\,H_1(t)))+i\,u_1\,B_1(a\,H_1(t))+i\,u_2\,\rho\,B_1(a\,H_1(t))}|H_1(t)\right]\right] \\ &= \mathbb{E}\left[e^{-\frac{H_1(t)}{2}\left(u_1^2(1-a)+a(u_1+\rho\,u_2)^2\right)}\right] \\ &= \mathbb{E}\left[e^{-\frac{H_1(t)}{2}\left(u_1^2+a\,\rho^2\,u_2^2+2\,a\rho\,u_1\,u_2\right)}\right]\end{aligned} \tag{C.3}$$

and apparently $\phi_{\mathbf{T}(t)}(u) = \phi_{\mathbf{T}(1)}(u)^t$. However, according to Sato [112, Theorem 7.10], such a condition is only a necessary condition and does not imply that $\mathbf{T}$ is a Lévy process but rather that there exist a Lévy $\tilde{\mathbf{T}}$ whose marginal *chf* at time t coincides with (C.4). Indeed, consider $\tilde{\mathbf{T}} = \{\tilde{T}_1(t) = Z_1(H_1(t)), \tilde{T}_2(t) = \rho\sqrt{a}\,Z_2(H_1(t))\}$ where $Z_1(t)$, $Z_2(t)$ are two correlated Brownian motions with correlation $\sqrt{a}$. Now take $Z_1(t) = \tilde{B}_1(t)$ and $Z_2(t) = \sqrt{a}\tilde{B}_1(t) + \sqrt{a(1-a)}\tilde{B}_2(t)$, where $\tilde{B}_1(t)$ and $\tilde{B}_2(t)$ are two independent Brownian motions. Then $\tilde{\mathbf{T}}$ is a Lévy process whose *chf* at time t is

$$\begin{aligned}\phi_{\tilde{\mathbf{T}}(t)}(u) &= \mathbb{E}\left[e^{i\,u_1\,\tilde{B}_1(H_1(t))+i\,u_2\,\rho\,a\tilde{B}_1(H_1(t))+i\,u_2\rho\sqrt{a(1-a)}H_1(t)}\right] \\ &= \mathbb{E}\left[e^{-\frac{H_1(t)}{2}\left(u_1^2+a^2\,\rho^2\,u_2^2+2\,a\rho\,u_1u_2+u_2^2\rho^2\,a-a^2\,\rho^2\,u_2^2\right)}\right] \\ &= \mathbb{E}\left[e^{-\frac{H_1(t)}{2}\left(u_1^2+a\,\rho^2\,u_2^2+2\,a\rho\,u_1\,u_2\right)}\right]\end{aligned} \tag{C.4}$$

which coincides with that of $\mathbf{T}(t)$, therefore the two random vectors $\mathbf{T}(t)$ and $\tilde{\mathbf{T}}(t)$ share the same law although the two processes are different. Summarizing, we can

conclude that the process $\mathbf{Y}$ is not a Markov process and therefore not a Lévy process. In addition, in contrast to the original models where the idiosyncratic component affects all the marginal processes at the same time, our models encompass what in Cufaro Petroni and Sabino [46, 47] is defined as synaptic risk that can be also seen in terms of random delays or interaction between the dependent (self-decomposable) subordinators.

Following the same ideas we used above, it can be proven that the process $\boldsymbol{Y}$ we defined in Section 4.2.3 is neither a Markov nor a Lévy process even if its marginal are Lévy processes.

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