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About determinantal Cremona maps

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*"Car le futur n'est jamais complètement écrit,
il reste un espace de liberté à conquérir, mais aussi
à préserver."*

Jean Lassègue, *Lettres*

Ringraziamenti

Ogni tesi comporta la sua parte di fallimenti, dubbi, solitudine e vorrei ringraziare ora le persone che mi hanno permesso di superare tutto questo buio inerente all'esercizio.

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Parigi, il 24 novembre 2021.

Sintesi

Studiamo trasformazioni di Cremona determinantale, cioè trasformazioni birazionale il cui ideale di base è l'ideale dei minori massimali di una matrice Φ, via la risoluzioni dei sistemi di polinomi definiti da Φ. Usando geometria convessa, questo approccio porta in particolare a descrivere i gradi proiettivi di alcuni trasformazioni di Cremona determinantale raccolte.

Parole chiavi: Geometria algebrica, Commutative algebra, Algebra commutativa, Teoria delle singolarità, ipersuperfici homaloidale, Algebra simmetrica e algebra di Rees, Syzygies, Risoluzione, Gradi proiettivi d'una trasformazione razionale, Trasformazioni determinantale, Trasformazioni di Cremona, polinomi ed ipersuperfici homaloidale, trasformazioni di Cremona pfaffiane, volumi misti dei politopi, teorema di Bernstein sugli sistemi di polinomi, trasformazioni di Cremona determinantale raccolte.

Abstract

We study determinantal Cremona maps, i.e. birational maps whose base ideal is the maximal minors ideal of a given matrix Φ, via the resolution of the polynomials systems defined by Φ. Using convex geometry, this approach leads in particular to describe the projective degrees of some glued determinantal maps.

Keywords: Algebraic Geometry, Commutative algebra, Singularity theory, Homaloidal hypersurfaces, Symmetric and Rees algebra, Syzygies, Resolutions, projective degrees of a rational map, determinantal maps, Cremona maps, homaloidal polynomials and homaloidal hypersurfaces, pfaffian Cremona maps, Mixed volumes of polytopes, Bernstein theorem on sparse polynomial systems, glued determinantal Cremona map.

Contents

Introduction

This work is about *determinantal Cremona maps* $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ where $n \in \mathbb{N}^*$. Recall that a *rational map* $f : X \dashrightarrow Y$ between two algebraic varieties X and Y, endowed with their Zariski topology, is a representative of an equivalence class for the equivalence relation that identifies the morphisms coinciding on a dense open subset of X. Given an integer $n \geqslant 1$, when X and Y are both the n-dimensional projective space $\mathbb{P}^n_{\mathrm{k}}$ over a field k, a rational map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ is then represented by $n+1$ homogeneous polynomials $f_0, \ldots, f_n \in \mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$ of the same degree and without common factor, a fact that we sum up in the notation $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$.

In this framework, *determinantal maps* $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ are maps whose defining polynomials $f_0, \ldots, f_n$ are the n-minors of a given $(n+1) \times n$ matrix $\Phi_f \in \mathrm{R}^{(n+1)\times n}$ where all the entries of a given column are homogeneous of the same degree. The matrix Φ_f is then called the *Hilbert-Burch* matrix of f, see below for justifications about this designation.

On the other hand, a map $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ and another map $g = (g_0 : \ldots : g_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ which is *dominant* i.e. whose image is dense on the target space, can be composed, the composition $f \circ g$ being defined by the $n+1$ polynomials $f_0(g_0, \ldots, g_n), \ldots, f_n(g_0, \ldots, g_n)$ given by substituting $g_0, \ldots, g_n$ to the variables $x_0, \ldots, x_n$ in $f_0, \ldots, f_n$. A *Cremona maps* is a map which has an inverse $g : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ for the composition law, that is to say $f \circ g$ and $g \circ f$ are equal, as rational maps, to the identity morphism of $\mathbb{P}^n_{\mathrm{k}}$. Another way to get familiar with Cremona maps is to consider the polynomial systems they define: let $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be a dominant map and assume that the base field k is algebraically closed. Then, given any general point $\mathbf{y} = (y_0 : \ldots : y_n) \in \mathbb{P}^n_{\mathrm{k}}$ i.e. $\mathbf{y}$ is such that its components $y_0, \ldots, y_n \in \mathrm{k}$ are random coefficients, the number of distinct solutions $\mathbf{x} = (x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathrm{k}}$ of the system

$$\begin{cases} f_0(x_0, \ldots, x_n) & = y_0 \\ \vdots & \vdots \\ f_n(x_0, \ldots, x_n) & = y_n \end{cases}$$

does not depend on $\mathbf{y}$. This latter quantity, denoted $d_0(f)$ in the following, is called the *topological degree* of f. A Cremona map being by definition an isomorphism between two open subsets of $\mathbb{P}^n_{\mathrm{k}}$, f is a Cremona map if and only if $d_0(f) = 1$. Keeping this latter criterion in mind, a naive strategy to determine if f is a Cremona map is to compute explicitly the preimages $\mathbf{x} \in f^{-1}(\{\mathbf{y}\})$ of a general point $\mathbf{y} \in \mathbb{P}^n_{\mathrm{k}}$.

To look through this strategy, we develop it now in a structuring example for this work: the one given by the so-called *standard Cremona map* of $\mathbb{P}^2_{\mathrm{k}}$.

Example 1. Let $\tau_2 = (x_1x_2 : x_0x_2 : x_0x_1) : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ over an algebraically closed ground field k. Its open subset of definition is the set $\mathbb{P}^2_{\mathrm{k}} \setminus \{(1:0:0),(0:1:0),(0:0:1)\}$. Moreover the components x_1x_2, x_0x_2, x_0x_1 of τ_2 do not share a common component and are the 2-(signed) minors of the matrix

$$\Phi_{\tau_2} = \begin{pmatrix} x_0 & 0 \\ -x_1 & x_1 \\ 0 & -x_2 \end{pmatrix}$$

so τ_2 is a determinantal map. A general point $\mathbf{y} = (y_0 : y_1 : y_2) \in \mathbb{P}^2_{\mathrm{k}}$ is the intersection of two lines, say $L_a = \mathbb{V}(a_0y_0 + a_1y_1 + a_2y_2)$ and $L_b = \mathbb{V}(b_0y_0 + b_1y_1 + b_2y_2)$ and let us compute the preimages $\mathbf{x} \in \tau_2^{-1}(\mathbf{y})$ of $\mathbf{y}$ as follows. Consider the four 3-minors of the matrix

$$\begin{pmatrix} x_0 & 0 & a_0 & b_0 \\ -x_1 & x_1 & a_1 & b_1 \\ 0 & -x_2 & a_2 & b_2 \end{pmatrix}$$

defined by concatenating the matrix of Φ_2 and the 3×2-matrix with entries in k defined by the coefficients of L_a and L_b. Among these four 3-minors, consider in particular the point $\mathbf{x} = (x_0 : x_1 : x_2)$ defined as zero locus of the ones containing the coefficients of L_a and L_b, that is $\mathbf{x}$ satisfies:

$$x_0 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + x_1 \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} = 0$$
$$x_1 \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} + x_2 \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} = 0.$$

The so-called *liaison theory*, see below for more about liaison theory, applied on codimension 2 determinantal ideal asserts that this latter zero locus point defines in general the only solutions of the polynomial system

$$\begin{cases} y_0 &= x_1x_2 \\ y_1 &= x_0x_2 \\ y_2 &= x_0x_1. \end{cases}$$

Since it is only one point, namely

$$\mathbf{x} = \left(\begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} : - \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \right),$$

one has $d_0(\tau_2) = 1$ so τ_2 is a Cremona map.

The previous example illustrates a specificity when computing the topological degree of a determinantal map and one of our starting motivations was to understand the different tools, for instance provided by liaison theory, in order to describe determinantal Cremona maps. Moreover, given a determinantal Cremona map $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$, we are not only interested in its topological degree $d_0(f)$ but also in all its *projective degrees* $d_0(f), \ldots, d_n(f)$ which are the quantities such that for any $i \in \{0, \ldots, n\}$,

$$d_i(f) = \# f^{-1}(H^{n-i}_{\mathbf{y}}) \cap H^i_{\mathbf{x}}$$

is the cardinal of the intersection of the preimage $f^{-1}(H^{n-i}_{\mathbf{y}})$ of a general codimension $n-i$ linear space $H^{n-i}_{\mathbf{y}}$ in the target space of f with a general codimension i linear space $H^i_{\mathbf{x}}$ in the source space of f. One motivation was to understand the distribution of these quantities when considering determinantal Cremona maps.

Let us now describe in more details the influences which motivated this work.

State of the art about determinantal maps

Maps with palindromic projective degrees

Our starting points are the articles [GSP06] and [DH17] describing, among other results, a family of *cubo-cubic* determinantal Cremona maps and a family of *quarto-quartic* determinantal Cremona maps in the projective 3-dimensional space $\mathbb{P}^3_{\Bbbk}$ where a *cubo-cubic* map $f : \mathbb{P}^3_{\Bbbk} \dashrightarrow \mathbb{P}^3_{\Bbbk}$ is a map such that

$$\big(d_0(f), d_1(f), d_2(f), d_3(f)\big) = (1, 3, 3, 1)$$

and a *quarto-quartic* map $f : \mathbb{P}^3_{\Bbbk} \dashrightarrow \mathbb{P}^3_{\Bbbk}$ is a map such that

$$\big(d_0(f), d_1(f), d_2(f), d_3(f)\big) = (1, 4, 4, 1).$$

In the first case, the cubo-cubic determinantal maps are defined by the 3-minors of a 4×3-matrix with linear entries without algebraic relations between them. This latter condition is in particular fulfilled when the linear entries are *general* i.e. that the coefficients of each entries are taken randomly in $\Bbbk$ (say $\Bbbk$ is an algebraically closed field) and in which case the associated map is called a *general determinantal Cremona map* [GSP06]. The determinantal quarto-quartic maps described in [DH17] are particular instances of determinantal maps $\mathbb{P}^3_{\Bbbk} \dashrightarrow \mathbb{P}^3_{\Bbbk}$ whose defining 4×3 Hilbert-Burch matrix is *almost linear* that is, in this case, that two of its columns are filled with homogeneous linear polynomials and the remaining column is filled with homogeneous polynomials of degree 2. If those latter entries were all general polynomials, the associated determinantal map would not be a Cremona map and in particular not a quarto-quartic map, see Example 1.1.2 for more explanations. However, if all the degree one entries of one column and all the degree 2 of Φ_f belongs to the ideal of a line in $\mathbb{P}^3_{\Bbbk}$, say (x_0, x_1), and all the entries of Φ_f are general under these conditions, one obtains a quarto-quartic Cremona map.

One of the motivations for this present work was to generalize the previous constructions of determinantal Cremona maps and to clarify the description of their projective degrees. To complete this picture about projective degrees, let us emphasize that all the developments we will present in this manuscript fit in line with a long standing question that one can find in [Dol11, 7.1.3]: given $n \geqslant 1$, let $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ be a Cremona map. Then its projective degrees $d_0(f) = 1, d_1(f), \ldots, d_{n-1}(f), d_n(f) = 1$ are subject to some constraints defined by the so-called *Cremona's inequalities* and *Hodge-type inequalities* [Dol11, end of 7.1.3], namely for $0 \leqslant i, j \leqslant n$:

$$\begin{cases} d_{n-i-j}(f) \leqslant d_{n-i}(f) d_{n-j}(f) \\ d_{i+j}(f) \leqslant d_i(f) d_j(f) \\ d_{n-i+1}(f) d_{n-i-1}(f) \leqslant d_{n-i}(f)^2. \end{cases} \tag{E}$$

Problem 2. *Given $d_0 = 1, d_1, \ldots, d_{n-1}, d_n = 1$, n integers verifying (E), does there exist a Cremona map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ such that $d_0(f) = d_0, \ldots, d_n(f) = d_n$?*

If $n = 3$, the answer to Problem 2 is yes by a result in [Pan13] but it is however not known in more generality. Even if we won't bring much to this question, apart from explicit sequence $d_0, d_1, \ldots, d_{n-1}, d_n$ that are realized as the projective degrees of explicit Cremona maps, it is a background problem we had in mind when looking at determinantal Cremona maps.

In relation with the quarto-quartic maps, we relied also on [KPU11] where the three authors described, inter alia, the equations of the *graph* of a determinantal map $f : \mathbb{P}^1_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ whose $(n+1) \times n$-Hilbert-Burch matrix Φ_f is almost linear (that is $n-1$ columns of Φ_f are filled with linear homogeneous polynomials and the remaining columns is filled with homogeneous polynomials of degree $d \geqslant 2$). Recall that the graph Γ_f of a map $f : X \dashrightarrow Y$ between two varieties X and Y is the closure $\overline{\{(x, f(x)) \in X \times Y, x \in U}$ in the Zariski topology of $X \times Y$ of the set of couple $(x, f(x)) \in U \times Y$ for an open subset $U \subset X$ such that the restriction f_U of a representative of f to U is a morphism from U to Y. Since the projective degrees of f can be defined via Γ_f, the description of the equations of Γ_f provide precious information when determining conditions under which f is a Cremona map or not.

Liaison and residuality theory applied to determinantal maps

The article [KPU11] is one of the many works released over the past fifty years applying the ideas of liaison theory and its generalization, residuality theory, to the study of rational maps. Let us refer to [PS74] and [MDP90] (resp. [Hun83]) for introductions and results about liaison theory (resp. residuality) and only mention that the computation we explained in Example 1 to compute the topological degree $d_0(\tau_2)$ was nothing but an illustration of those ideas continuously developed over the past years. We refer now to few of those works that particularly influenced us.

Given a rational map $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ $(n \geqslant 1)$, a presentation matrix $\Phi_f \in \mathrm{R}^{(n+1)\times m}$ $(\mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$, $m \geqslant n)$ of the *base ideal*

$$\mathrm{I}_f = (f_0, \ldots, f_n) \subset \mathrm{R}$$

provides a first approximation of the equations of Γ_f, see [RS01], [BCJ09], [BCRD20] and [CR21] for developments and applications in broader contexts. In restriction to Cremona maps, the article [HS12] was, as far as we know, the first work to consider specifically (plane) determinantal Cremona maps by studying numerical properties of their base locus. The starting idea is that the graph Γ_f is the proj $\mathrm{Proj}\, \mathcal{R}(\mathrm{I}_f)$ of the Rees algebra $\mathcal{R}(\mathrm{I}_f) = \bigoplus_{k \geqslant 0} \mathrm{I}_f^k t^k \subset \mathrm{R}[t]$ of I_f. The surjection

$\mathrm{Sym}(\mathrm{I}_f) \twoheadrightarrow \mathcal{R}(\mathrm{I}_f)$ defined by the natural maps $\mathrm{I}_f^{\otimes k} \twoheadrightarrow \mathrm{I}_f^k$ provides thus an embedding $\Gamma_f = \mathrm{Proj}\,\mathcal{R}(\mathrm{I}_f) \subset \mathrm{Proj}\,\mathrm{Sym}(\mathrm{I}_f) = \mathbb{P}(\mathrm{I}_f)$ of Γ_f in the *residual scheme* $\mathbb{P}(\mathrm{I}_f) = \mathrm{Proj}\,\mathrm{Sym}(\mathrm{I}_f)$ of I_f where the equations of $\mathbb{P}(\mathrm{I}_f)$ in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ are the entries of the line matrix

$$\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} \Phi_f$$

where $y_0, \dots, y_n$ are the variables of the second factors of $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$. The ideal I_f is said to be of *linear type* if $\mathbb{P}(\mathrm{I}_f) = \Gamma_f$ and this latter property is characterized by the codimension of the ideals of the t-minors $I_t(\Phi_f)$ of Φ_f for $t \in \{1, \dots, \min(n, m)\}$ [RS01, (3),beginning of 2.1]. Provided that the ideal I_f is of linear type, see [RS01] or [SUV94], the jacobian dual criterion characterizes the fact that f is a Cremona map and, assuming that f is determinantal, a refinement of the latter criterion provides a very effective way to check if f is a Cremona map or not, see [DHS12].

Focusing on the graph Γ_f of a determinantal Cremona map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ (whose base ideal I_f is of linear type or not), [HS17, Th.3.7] provides a bound on the degree of the generators of I_f if Γ_f is Cohen-Macaulay (assuming k infinite), a structuring conditions on the generators and the syzygies of the ideal of Γ_f in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$, see [BH93] for a definition of Cohen-Macaulay schemes. Regarding specifically the equations of the graph, our starting point, in addition to [KPU11], is [BCRD20] where the three authors described, inter alia, the equations of a determinantal Cremona map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ defined by an almost linear Hilbert-Burch matrix Φ_f characterized in the same way as the Hilbert-Burch matrix defining quarto-quartic maps of $\mathbb{P}^3_{\mathrm{k}}$ in [DH17]. The result in [BCRD20] take its roots in [CHW08] and, more generally, in Jouanolou's duality describing the equations of the graph Γ_f of a determinantal map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ via inertia forms, see [Jou97] and [BCJ09] for more explanations and results about Jouanolou's duality.

Parallel to those initial results about (determinantal Cremona) maps, let us also quote a last influencial result about *monomial Cremona maps* i.e. maps whose base ideal is monomial. In [DS16], the two authors describe some key properties of monomial Cremona maps. Let us however already clear up that determinantal Cremona maps do not behave as one could expect from the monomial case. For instance, [CS12a] or the more recent [DS16], established that the inverse of a monomial Cremona map is a monomial map and one can naively wonder if the inverse of a determinantal Cremona map is also a determinantal map.

Problem 3. *Is the inverse of a determinantal Cremona map determinantal?*

The answer to this latter question is no as for instance:

Example 4. Let $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ be the map defined by the 3-minors of the matrix

$$\Phi_f = \begin{pmatrix} 0 & x_0 & x_0^2 \\ x_0 & 0 & x_0x_2 - x_2^2 + x_1x_3 \\ x_2 & x_1 & -x_2^2 + x_1x_3 \\ x_3 & x_2 & x_0x_1 + x_0x_2 \end{pmatrix}.$$

Then applying for instance the following Macaulay2 code using the package Cremona[Sta17]:

```
loadPackage "Cremona"
k = QQ
R = k[x_0..x_3]

Phif = matrix{
    { 0 ,x_0,         x_0^2          },
    {x_0, 0 , x_0*x_2-x_2^2+x_1*x_3 },
    {x_2,x_1,     -x_2^2+x_1*x_3    },
    {x_3,x_2,    x_0*x_1+x_0*x_2    }
        };
f = toMap(minors(3,Phi)); projectiveDegrees f
```

one has that f is a determinantal Cremona map. Moreover, adding the following command

```
g = inverseMap f; Ig = ideal(g(x_0),g(x_1),g(x_2),g(x_3));
syz gens Ig
```

one can observe that the minimal syzygy matrix of the base ideal $\mathrm{I}_{f^{-1}}$ of f^{-1} is a 4×4 matrix which, as we will explain below, shows that f^{-1} is not a determinantal map.

Given these initial developments and results, let us now sum up our contributions about determinantal Cremona maps that we will develop in the body of this manuscript.

Contents of the manuscript

Part I is dedicated to define formally the projective degrees $d_0(f), \dots, d_n(f)$ of a rational map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ and to illustrate in this context their estimations via the residual scheme $\mathbb{P}(\mathrm{I}_f) = \operatorname{Proj}\operatorname{Sym}(\mathrm{I}_f)$ where I_f is the base ideal of f, see Chapter 2 and Chapter 3.

The knowledge of the bidegrees of the generators of the ideal I_{Γ_f} of the graph Γ_f of f and the bidegree of all the successive syzygies of I_{Γ_f} (i.e. the knowledge of a bigraded free resolution of I_{Γ_f}) provides a universal definition of the projective degrees of f which coincides with the previous definition of projective degrees assuming that the ground field is algebraically closed, see Chapter 1. It emphasizes the importance of estimating I_{Γ_f}.

As a first application of this point of view, we describe in Section 1.2, a very simple construction of a *glued determinantal map* $[g|g'] : \mathbb{P}^{m+m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+m'}_{\mathrm{k}}$ starting with two initial determinantal maps $g : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ and $g' : \mathbb{P}^{m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m'}_{\mathrm{k}}$. The Hilbert-Burch matrix $\Phi_{[g|g']}$ is the concatenation of the Hilbert-Burch matrices Φ_g and $\Phi_{g'}$ of, respectively, g and g' and our starting result, which assumes that the graph of $[g|g']$ has the expected bigraded free resolution, reads:

Proposition 5. *Let $m, m' \geqslant 1$, $\mathrm{R}_m = \mathrm{k}[x_0, \dots, x_m]$, $\mathrm{R}_{m'} = \mathrm{k}[x_m, \dots, x_{m+m'}]$ and $\mathrm{R} = \mathrm{k}[x_0, \dots, x_{m+m'}]$ and let $g : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ (resp. $g' : \mathbb{P}^{m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m'}_{\mathrm{k}}$) be a determinantal map such that I_g (resp. $\mathrm{I}_{g'}$) is the m-minors ideal of a matrix $\Phi_g \in \mathrm{R}_m^{(m+1)\times m}$ (resp. the m'-minors ideal of $\Phi_{g'} \in \mathrm{R}_{m'}^{(m'+1)\times m'}$).*

Put

$$\Phi_{[g|g']} = \begin{matrix} m+1\left\{ \vphantom{\begin{matrix} a\\ b\\ c \end{matrix}} \right. \\ m'\left\{ \vphantom{\begin{matrix} a \end{matrix}} \right. \end{matrix} \overbrace{\left(\begin{array}{c:c} \Phi_g & 0_{m\times m'} \\ \hdashline 0_{m'\times m} & \Phi_{g'} \end{array} \right)}^{m \quad m'} \begin{matrix} \left.\vphantom{\begin{matrix} a\\ b \end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix} a\\ b \end{matrix}}\right\} m'+1 \end{matrix} \in \mathrm{R}^{(m+m'+1)\times(m+m')}$$

and assume that the ideal of $(m+m')$-minors of $\Phi_{[g|g']}$ is the base ideal of a map $[g|g'] : \mathbb{P}_{\mathrm{k}}^{m+m'} \dashrightarrow \mathbb{P}_{\mathrm{k}}^{m+m'}$.

Suppose that the tensor product $\mathbb{F}_g \otimes \mathbb{F}_{g'}$ of a free resolution $\mathbb{F}_g$ of the graph of g and a free resolution $\mathbb{F}_{g'}$ of the graph of g' provides a free resolution of the graph of $[g|g']$. Then, given any $k \in \{0, \ldots, m+m'\}$

$$d_k([g|g']) = \sum_{p=0}^{k} d_p(g) d_{k-p}(g') \tag{1}$$

with the convention that $d_p(g) = 0$ (resp. $d_p(g') = 0$) if $p > m$ (resp. $p > m'$).

As we will develop it in Proposition 1.2.3, Lemma 1.2.1 shed also some lights on the distribution of the projective degrees of the standard Cremona map

$$\tau_n = (x_1 \cdots x_n : x_0 x_2 \cdots x_n : \ldots : x_0 \cdots x_{n-1}) : \mathbb{P}_{\mathrm{k}}^n \dashrightarrow \mathbb{P}_{\mathrm{k}}^n$$

which can be interpreted as the gluing of n standard Cremona maps $\tau_1^{(1)} = (x_1 : x_0) : \mathbb{P}_{\mathrm{k}}^1 \dashrightarrow \mathbb{P}_{\mathrm{k}}^1$, $\tau_1^{(2)} = (x_2 : x_1) : \mathbb{P}_{\mathrm{k}}^1 \dashrightarrow \mathbb{P}_{\mathrm{k}}^1$, …, $\tau_1^{(n)} = (x_n : x_{n-1}) : \mathbb{P}_{\mathrm{k}}^1 \dashrightarrow \mathbb{P}_{\mathrm{k}}^1$. Indeed, I_{τ_n} is the n-minors ideal of the $(n+1)\times n$ matrix

$$\Phi_{\tau_n} = \begin{pmatrix} x_0 & 0 & \ldots & \ldots & 0 \\ -x_1 & x_1 & \ddots & & \vdots \\ 0 & -x_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & x_{n-1} \\ 0 & \ldots & \ldots & 0 & -x_n \end{pmatrix}$$

which is the concatenation of the presenting matrices $\Phi_{\tau_1^{(1)}} = \begin{pmatrix} x_0 \\ -x_1 \end{pmatrix}, \ldots, \Phi_{\tau_1^{(n)}} = \begin{pmatrix} x_{n-1} \\ -x_n \end{pmatrix}$ of, respectively, $\mathrm{I}_{\tau_1^{(1)}}, \ldots, \mathrm{I}_{\tau_1^{(n)}}$ completed with zeros entries. Even if our results are far from being complete regarding the description of the graph of glued maps, it gives however new perspectives about maps with palindromic projective degrees and among which cubo-cubic and quarto-quartic determinantal maps, see Conjecture 1.2.4. It motivates also our study of glued maps via indirect approach in Part II.

Leaving our hopes to fully determine bigraded free resolution of the ideal of the graph of determinantal Cremona maps in Chapter 1, we focus specifically in

Chapter 2 and Chapter 3 on the generators of the ideal of the graph of a determinantal map starting from the equations of those of the residual scheme associated to the base ideal. Let us point out another time that by the jacobian dual criterion, see [SUV94], one can determine if a determinantal map is a Cremona map via its Hilbert-Burch matrix. It explains why we focus on the equations of the residual scheme.

Chapter 2 falls under this approach in the context of rational maps that are defined by the polar of projective hypersurface. Recall that given a hypersurface $\mathsf{H} = \mathbb{V}(\mathsf{h}) \subset \mathbb{P}^n_{\mathrm{k}}$ for a given homogeneous polynomial $\mathsf{h} \in \mathrm{k}[x_0, \dots, x_n]$, the polar f_{h} of H is the map

$$f_{\mathsf{h}} = (\mathsf{h}_0 : \dots : \mathsf{h}_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$$

where, given $i \in \{0, \dots, n\}$, $\mathsf{h}_i = \frac{\partial \mathsf{h}}{\partial x_i}$ is the i-th partial derivative of h. The hypersurface H and the polynomial h are called *homaloidal* if f_{h} is a Cremona map. Over $\mathbb{C}$, a result of I.V. Dolgachev states that the complex homaloidal polynomials in three variables, i.e. the homaloidal complex curves are of degree at most three. Answering [DHS12, Question 3.7], we describe homaloidal polynomials in three variables of arbitrarily large degree in positive characteristic, namely:

Theorem 2.0.4. *Let* k *be a field of characteristic* p *and let* $n \in \mathbb{N}_{>0}$ *be a multiple of* p*. Then the near-pencil arrangement of* $n+1$ *lines is homaloidal.*

From the point of view of determinantal maps, the relevance of such a result comes from its proof. It consists in studying the reduction modulo p of the Hilbert-Burch matrix of the base ideal of f_{h} which are determinantal in all the cases we will consider, see the developments in Chapter 2. In addition to this result, we also provide the classification of homaloidal line arrangements at the end of Chapter 2.

In line with [KPU11] and [DH17], Chapter 3 is dedicated to the equations of the graph Γ_f of a Cremona map $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ whose 4×3 Hilbert-Burch matrix Φ_f is almost linear. This work was lead under the supervision of B.Ulrich and C.Polini at Purdue University in winter 2020. Following [KPU11], the approach is to express Γ_f as a divisor on the complete intersection scheme, denoted $\mathbb{X}_f$ only in this introduction, defined by the $n-1$ equations of bi-degree $(1,1)$ (meaning the equations of degree 1 in the variables $x_0, \dots, x_3$ and 1 in the variables $y_0, \dots, y_3$) among the entries of the line matrix

$$\begin{pmatrix} y_0 & \dots & y_3 \end{pmatrix} \Phi_f.$$

When the divisor class group $\mathrm{Cl}(\mathbb{X}_f)$ is cyclic, a condition that we will explicit in the development, the cycle $[\Gamma_f] \in \mathrm{Cl}(\mathbb{X}_f)$ defined by Γ_f is then the symbolic power of the generator of $\mathrm{Cl}(\mathbb{X}_f)$, a property that provides an effective insight on all the equations of Γ_f. Via the normal form of the 4×2 matrices with linear entries in $\mathrm{R} = \mathrm{k}[x_0, \dots, x_3]$, a classification of the possible divisor class, cyclic or not, can be obtained, see Subsection 3.1.2. The strength of such an approach is that it gives a clear insight about all the possible ideals of graphs of Cremona maps $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ with an almost linear Hilbert-Burch matrix and whose associated scheme $\mathbb{X}_f$ has a cyclic divisor class group. More precisely:

Proposition 3.0.1. *Let* $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ *be a dominant determinantal map such that:*

(i) the Hilbert-Burch matrix Φ_f of f reads

$$\Phi_f = \begin{pmatrix} x_0 & x_3 & \phi_{03} \\ x_1 & 0 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_2 & \phi_{33} \end{pmatrix}$$

with $\phi_{i3} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \dots, 3\}$. Then the ideal of Γ_f is minimally generated by one element in the following bi-degree:

$$(d, 1), (d-1, 2)(d-2, 3), \dots, (1, d)$$

and two extra other generators in bidegree $(1, 1)$.

(ii) The Hilbert-Burch matrix Φ_f of f reads

$$\Phi_f = \begin{pmatrix} x_0 & x_2 & \phi_{03} \\ x_1 & x_3 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_1 & \phi_{33} \end{pmatrix}$$

with $\phi_{i3} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \dots, 3\}$. Then the ideal of Γ_f is minimally generated by the elements in the following bi-degree if d is even:

$$(d, 1), 2(d-1, 2), 3(d-2, 3), \dots, \frac{d}{2}(\frac{d}{2}+1, \frac{d}{2}), \frac{d}{2}(\frac{d}{2}, \frac{d}{2}+1), \dots, 2(2, d-1), (d, 1)$$

and two extra other generators in bidegree $(1, 1)$ (here $m(d_1, d_2)$ means that the component of degree (d_1, d_2) of the ideal of Γ_f is minimally generated by m elements). If d is odd, Γ_f is minimally generated by the elements in the following bi-degree:

$$(d, 1), 2(d-1, 2), 3(d-2, 3), \dots, \frac{d+1}{2}(\frac{d+1}{2}, \frac{d+1}{2}), \dots, 2(2, d-1), (d, 1)$$

and two extra other generators in bidegree $(1, 1)$.

These two possible behaviors in the case of $\mathbb{P}^3_{\Bbbk}$ have to be compared with its analogue in $\mathbb{P}^2_{\Bbbk}$, see [BCRD20, Th.5.12], in which case only one possible behavior is possible.

Part II is dedicated to studying specifically the projective degrees of determinantal Cremona maps using tools from convex geometry applied to the resolution of polynomial systems. When interested in the topological degree $d_0(f)$ of a determinantal map $f = (f_0 : \dots : f_n) : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$, remark that two polynomial systems can be considered. An initial one, in the end the main object of this work, is defined by f itself: given $\mathbf{y} = (y_0 : \dots : y_n) \in \mathbb{P}^n_{\Bbbk}$ in the target space of f, one wants to find the pre-images $\mathbf{x} \in \mathbb{P}^n_{\Bbbk}$ of $\mathbf{y}$, i.e. wants to solve the polynomial system

$$f(\mathbf{x}) = \mathbf{y} \Leftrightarrow \begin{cases} y_0 & = f_0(x_0, \dots : x_n) \\ & \vdots \\ y_n & = f_n(x_0, \dots : x_n) \end{cases}$$

Expressing $\mathbf{y}$ as the intersection of n hyperplans in the target space of f and assuming that $\mathbf{y}$ is general enough, Bézout theorem then asserts that $d_0(f) = \#f^{-1}(\{\mathbf{y}\}) \leqslant d^n$ where $d = d_{n-1}(f)$ is the common degree of the polynomials f_i generating the base ideal $\mathrm{I}_f = (f_0, \ldots, f_n)$ of f. An intermediate polynomial system is moreover defined by a presentation matrix Φ_f of I_f. Indeed, by definition of Φ_f, one has $(f_0\ \ldots\ f_n)\Phi_f = 0$ so the polynomial system

$$(y_0\ \ldots\ y_n)\Phi_f = 0 \tag{A}$$

whose number of equations is the number of columns of Φ_f, contains by definition $f^{-1}(\{y\})$. Now consider the situation where f is a determinantal map of $(n+1)\times n$ Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}$ $(n+1)\times n$ and such that the residual scheme $\mathbb{P}(\mathrm{I}_f)$ of I_f is a complete intersection in its embedding in $\mathbb{P}^n_k k \times \mathbb{P}^n_{\mathrm{k}}$ defined by Φ_f, a special situation in which case we will say that f is *Koszul-determinantal* in reference to the fact that the ideal of $\mathbb{P}(\mathrm{I}_f)$ is generated by a regular sequence. Thus the system (A) is 0-dimensional in $\mathbb{P}^n_{\mathrm{k}}$ and it's number of solutions is described by Bézout theorem. Our observation is that in case the entries of Φ_f are sparse polynomials that all verify the same algebraic constraints on their coefficients, one can refine the bound on the number of solutions of (A) by using Bernstein's theorem on sparse polynomials. It then gives a combinatorial translation, via the computation of *mixed volumes* associated to the polynomial entries of Φ_f, to the problem of detecting determinantal Cremona maps. Our motivation was here to understand the degrees of Koszul-determinantal maps defined by an almost linear Hilbert-Burch matrix starting from the polytopes defined by Φ_f, thus providing an alternative approach to the ones in Part I. It provides in particular a common view on cubo-cubic and quarto-quartics maps that one can interpret as specific instances of glued maps. Our result in this direction being:

Proposition 5.3.3. *Let $d \geqslant 2$ and let $\Phi_{[g|g']} = (\phi_{ij})_{\substack{0\leqslant i\leqslant 2+n\\ 1\leqslant j\leqslant 2+n}}$ be such that:*

- *all the entries ϕ_{i1} of the 1-st column of $\Phi_{[g|g']}$ are general linear combinations of x_0 and x_1,*
- *all the entries ϕ_{i1} of the 2-nd column of $\Phi_{[g|g']}$ are general linear combinations of the generators of the ideal*

$$(x_0, x_1)^{d-1}\cdot(x_0, x_1, x_2) = (x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, \ldots, x_1^{d-1}x_2, x_1^d),$$

- *for all $l \in \{3, \ldots, 2+n\}$, all the entries ϕ_{il} of the l-th column of $\Phi_{[g|g']}$ are general linear combinations of $x_2, \ldots, x_{2+n}$.*

Then the glued map $[g|g'] : \mathbb{P}^{2+n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{2+n}_{\mathrm{k}}$ whose base ideal $\mathrm{I}_{[g|g']}$ is the $(m+n)$-minors ideal $\Phi_{[g|g']}$ is a determinantal Cremona map and moreover:

$$\forall k \in \{0, \ldots, 2+n\},\ d_k([g|g']) = \binom{n}{n-k} + (d+1)\binom{n}{n-k+1} + \binom{n}{n-k+2}$$

with the convention that $\binom{j}{i} = 0$ if $i < 0$ or $i > j$. In particular, the projective degrees of $[g|g']$ is a palindromic sequence.

Let us precise that, even if we did not find any trace of the previous construction in the litterature, other processes of building Cremona maps starting from an initial one in a smaller space have already been described for instance in [CS12b]. Our work can also be put in line with another aspect of [CS12b] or [DS16] and the Newton complementary dual construction presented therein : the composition of two maps. Indeed, under restrictions, the composition of two determinantal Cremona maps is a determinantal Cremona map, see Proposition B.1.1 in the annex.

Chapter 4 provides the basic material about convex geometry and mainly follows [CLO05]. The main observation, labeled Proposition 5.1.6, is that given a Koszul-determinantal map $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ of Hilbert-Burch matrix Φ_f, and provided that the scheme $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ is contained in the axis $\overset{n}{\underset{i=0}{\cup}} \mathbb{V}(x_i)$, the projective degrees $d_0(f), \dots, d_n(f)$ of f are computed by the mixed volumes associated to the polytopes defined by Φ_f, using notations and assumptions of Proposition 5.3.3, results of convex geometry, in particular [ST10, Lemma 6] (see Lemma 4.2.5 below), provide then formulas to compute the mixed volumes of the polytopes defined by the glued Hilbert-Burch matrix $\Phi_{[g|g']}$, emphasizing that the projective degrees are controlled by the same Kunneth-like formulas as in Proposition 5.

In addition to apply convex geometry to the definition and study of glued map, we use the tools provided by the mixed volumes to study determinantal plane Cremona maps with *non* almost linear Hilbert-Burch matrix, see Section 5.2 which leads for instance to the following result:

Example 5.2.6. The determinantal map $f : \mathbb{P}^2_{\Bbbk} \dashrightarrow \mathbb{P}^2_{\Bbbk}$ of Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} 0 & x_0(x_0x_2 - x_0x_1)^2 \\ x_0x_2 - x_0x_1 & x_1(x_1x_2 - x_0x_2)^2 \\ x_1x_2 - x_0x_2 & 0 \end{pmatrix}$$

is a Cremona map ($d(f) = (1, 7, 1)$).

See Proposition 5.2.3 and Proposition 5.2.5 for our extended results.

In Chapter 6, we focus on determinantal Cremona map $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ which are not Koszul-determinantal or, equivalently, such that the associate residual scheme $\mathbb{P}(\mathrm{I}_f)$ is not a complete intersection. We provides examples of such maps and we explain how convex geometry still sheds some lights on their projective degrees.

The annex is dedicated to the study of *pfaffian Cremona maps* i.e. maps $\mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ ($n \geqslant 4$ odd) whose base ideal is defined by the n-pfaffians of a $(n+1) \times (n+1)$ skew symmetric matrix Φ_f whose non zero entries are polynomials of the same degree. With the definitions we use in this manuscript, such maps are not determinantal but, by [BE77], the free resolution of their base ideal is characterized in an analogue way as in the case of determinantal base ideal, provided some natural technical conditions. It seemed thus relevant to add the analysis of the projective degrees of some of the pfaffian maps. In particular, we will explain how residuality and especially results in [KU92] provides a description of the projective degrees of pfaffian Cremona maps defined by an $(n+1) \times (n+1)$ skew symmetric matrix whose entries are general linear polynomial which, as far as we know, was not written in the existing literature. For instance:

Proposition A.0.1. *The projective degrees* $(d_0(p), d_1(p), d_2(p), d_3(p), d_4(p))$ *of a pfaffian map* $\mathbb{P}^4_{\mathrm{k}} \dashrightarrow \mathbb{P}^4_{\mathrm{k}}$ *defined by the* 4*-pfaffians of a* 5×5*-matrix* Φ_p *whose subdiagonal is filled with general linear homogeneous polynomials reads*

$$(1, 3, 2^2, 2, 1).$$

We end this manuscript by summarizing some of the questions that arose while we studied determinantal maps and are still open to us, see Appendix B.

Part I

Computation of the projective degrees of a determinantal map

Chapter 1

Resolution of the graph of a map

1.1 Background about bigraded free resolutions

In this section, we introduce the basic notions and material about the projective degrees of a map, especially in the determinantal case. As previously explained in [MS05, Chapter 8], the theory of multigraded Hilbert series is the best context for defining and studying the multidegree of subschemes in multiprojective spaces. We adapt here previous expositions as in [MS05] or the more recent one in [CR21] to the following bi-graded context: given a map $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ with base ideal $\mathrm{I}_f = (f_0, \ldots, f_n) \subset \mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$ the *symmetric algebra* $\mathrm{Sym}(\mathrm{I}_f)$ of I_f is $\mathbb{Z}^2$-graded, as we will explain. The point of this section is to clarify the contribution of the torsion of the symmetric algebra to the computation of the projective degrees of f provided f is determinantal.

For the rest of the section, we let $n \geqslant 2$ be an integer, k be any field (unless further assumptions are specified), $\mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$ and $\mathrm{S} = \mathrm{R}[y_0, \ldots, y_n]$ be the $\mathbb{Z}^2$-graded polynomial ring with the standard graduation $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$ for all $i \in \{0, \ldots, n\}$. Given a bi-graded algebra B, we let BiProj(B) be the set of bi-graded prime ideals of B. Thus, a subscheme $\mathbb{X}$ of $\mathrm{BiProj}(\mathrm{S}) \simeq \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ is defined by a bi-graded ideal $\mathcal{A} \subset \mathrm{S}$ and one associates to $\mathbb{X} = \mathrm{BiProj}(\mathrm{S}/\mathcal{A})$ its bi-variate Hilbert series

$$\mathrm{H}_{\mathrm{S}/\mathcal{A}}(T_0, T_1) = \sum_{(n_0,n_1)\in\mathbb{Z}^2} (\mathrm{length}(\mathrm{S}/\mathcal{A})_{(n_0,n_1)} T_0^{n_0} T_1^{n_1}.$$

Then, writing $\mathrm{H}_{\mathrm{S}/\mathcal{A}}(T_0, T_1) = \frac{\mathrm{Num}_{\mathrm{S}/\mathcal{A}}(T_0,T_1)}{(1-T_0)^n(1-T_1)^n}$ and assuming that the scheme $\mathbb{X} = \mathrm{BiProj}(\mathrm{S}/\mathcal{A})$ has codimension c, the coefficients of the homogeneous component of $\mathrm{Num}_{\mathrm{S}/\mathcal{A}}(1 - T_0, 1 - T_1)$ of total degree c define the multidegree of $\mathbb{X}$.

Definition 1.1.1 (multidegree of a subscheme in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$)**.** Let $\mathcal{A}$ be a bi-graded ideal of S and assume that $\mathbb{X} = \mathrm{BiProj}(\mathrm{S}/\mathcal{A})$ has codimension c in $\mathrm{BiProj}(\mathrm{S}) \simeq \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$.

Given $k \in \{0, \ldots, c\}$, define $\deg^{c-k,k}_{\mathbb{P}} \mathbb{X}$ as the coefficient of the monomial of bi-degree $(c - k, k)$ of $\mathrm{Num}_{\mathrm{S}/\mathcal{A}}(1 - T_0, 1 - T_1)$.

The *multidegree* $\deg_{\mathbb{P}} \mathbb{X}$ of $\mathbb{X}$ is the $c + 1$-uple

$$\deg_{\mathbb{P}} \mathbb{X} = \big(\deg^{c,0}_{\mathbb{P}} \mathbb{X}, \ldots, \deg^{0,c}_{\mathbb{P}} \mathbb{X}\big).$$

Example 1.1.2 (multidegree of complete intersection in $\mathbb{P}^3_{\mathrm{k}} \times \mathbb{P}^3_{\mathrm{k}}$)**.** Put $n = 3$, fix three positive integer $d_1, d_2, d_3 \geqslant 1$ and consider a matrix $\Phi = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}}$ such that $\deg(\phi_{ij}) = (d_j, 0)$ for all $i \in \{0, \dots, 3\}, j \in \{1, \dots, 3\}$ (i.e. the entries of Φ do not depend on the y-variables).

Let $\mathfrak{J}$ be the ideal generated by the entries of the line matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix} \Phi$$

and assume that $\mathbb{X} = \mathrm{BiProj}(\mathrm{S}/\mathfrak{J})$ has codimension 3 as it is for instance the case if given $j \in \{1, 2, 3\}$ and $i \in \{0, \dots, 3\}$, ϕ_{ij} is a general polynomial of degree d_j. The subscheme $\mathbb{X}$ being thus a complete intersection, the Koszul complex on the entries of $\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix} \Phi$ provides a bi-graded free resolution of $\mathrm{S}/\mathfrak{J}$ [Eis95, Chapter 16]:

$$0 \to \mathrm{S}(-\sum_{k=1}^{3} d_k, -3) \to \bigoplus_{k<k'} \mathrm{S}(-d_k - d_{k'}, -2) \to \bigoplus_{k=1}^{3} \mathrm{S}(-d_k, -1) \to \mathrm{S}$$

from which we can compute that

$$\mathrm{Num}_{\mathrm{S}/\mathfrak{J}}(T_0, T_1) = 1 - \sum_{k=1}^{3} T_0^{d_k} T_1 + \sum_{k<k'} T_0^{d_k + d_{k'}} T_1^2 - T_0^{d_1+d_2+d_3} T_1^3.$$

So, focusing on the component of total degree 3 of $\mathrm{Num}_{\mathrm{S}/\mathfrak{J}}(1 - T_0, 1 - T_1)$, one has that:

$$\begin{aligned}
\deg_{\mathbb{P}}^{3,0} \mathbb{X} &= \sigma_{3,3}(d_1, d_2, d_3) = d_1 d_2 d_3, \\
\deg_{\mathbb{P}}^{2,1} \mathbb{X} &= \sigma_{2,3}(d_1, d_2, d_3) = \sum_{k<k'} (d_k + d_{k'}), \\
\deg_{\mathbb{P}}^{1,2} \mathbb{X} &= \sigma_{1,3}(d_1, d_2, d_3) = \sum_{k=1}^{3} d_k, \\
\deg_{\mathbb{P}}^{0,3} \mathbb{X} &= \sigma_{0,3}(d_1, d_2, d_3) = 1
\end{aligned}$$

where for $k \in \{0, \dots, 3\}$, $\sigma_{k,3}(u_1, u_2, u_3) = \sum_{\{i_1,\dots,i_k\} \subset \{1,2,3\}} u_{i_1} \dots u_{i_k}$ is the k-th elementary symmetric polynomial in 3 variables.

Remark 1.1.3. As it is explained for instance in [Har92, Proposition 7.16] or [CR21, Theorem 4.7], the multidegree $\deg_{\mathbb{P}} \mathbb{X} = \left(\deg_{\mathbb{P}}^{c,0} \mathbb{X}, \dots, \deg_{\mathbb{P}}^{0,c} \mathbb{X}\right)$ of a c-dimensional subscheme $\mathbb{X} \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ has the following geometric interpretation. Since the Chow ring $A(\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}})$ (see [EH16, Chapter 1]) of $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ is isomorphic to the ring $\mathbb{Z}[\xi_{\mathbf{x}}, \xi_{\mathbf{y}}]/(\xi_{\mathbf{x}}^{n+1}, \xi_{\mathbf{y}}^{n+1})$ [EH16, Theorem 2.10] where $\xi_{\mathbf{x}}$ (resp. $\xi_{\mathbf{y}}$) is the pull-back of the hyperplane class of $\mathbb{P}^n_{\mathrm{k}}$ via the first (resp. second) projection map $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}} \to \mathbb{P}^n_{\mathrm{k}}$, the class $[\mathbb{X}]$ of $\mathbb{X}$ in $A^c(\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}})$ satisfies the relation

$$[\mathbb{X}] = \sum_{k=c-n}^{n} (\deg_{\mathbb{P}}^{c-k,k} \mathbb{X}) \xi_{\mathbf{x}}^{n-k} \xi_{\mathbf{y}}^{n-c+k}$$

Hence, assuming the base field k is algebraically closed and letting $k \in \{0, \ldots, c\}$, we have:

$$\deg_{\mathbb{P}}^{c-k,k} \mathbb{X} = \operatorname{length}(H_{\mathbf{x}}^k \cap \mathbb{X} \cap H_{\mathbf{y}}^{c-k})$$

where $H_{\mathbf{x}}^k = \mathbb{V}(l_{1,0}, \ldots, l_{k,0})$ is the zero locus of k general linear forms $l_{1,0}, \ldots, l_{k,0}$ in the x-variables (that is $\deg(l_{i,0}) = (1,0)$ for all $i \in \{0, \ldots, k\}$) and $H_{\mathbf{y}}^k = \mathbb{V}(l_{0,1}, \ldots, l_{0,c-k})$ is the zero locus of $c-k$ general linear forms $l_{0,1}, \ldots, l_{0,c-k}$ in the y-variables.

Let us now define the projective degrees of a rational map.

Definition 1.1.4 (projective degrees of a map)**.** Let $f : \mathbb{P}_{\mathrm{k}}^n \dashrightarrow \mathbb{P}_{\mathrm{k}}^n$, regular on a dense open subset $U \subset \mathbb{P}_{\mathrm{k}}^n$, and let $\Gamma_f = \overline{\{(x, f(x)), x \in U\}} \subset \mathbb{P}_{\mathrm{k}}^n \times \mathbb{P}_{\mathrm{k}}^n$ be the graph of f. It is of pure dimension n so, for $k \in \{0, \ldots, n\}$, define the *k-th projective degree of f*, written $d_k(f)$, by

$$d_k(f) := \deg_{\mathbb{P}}^{n-k,k} \Gamma_f$$

.

Following Remark 1.1.3, observe that $d_0(f) = 1$ if and only if f is an isomorphism between two dense open subsets of $\mathbb{P}_{\mathrm{k}}^n$ in which case f is called *birational* or, equivalently in this case, is a Cremona map.

Remark 1.1.5. Following [Tru01, After Th.4.6], let us point out that the k-th projective degree $d_k(f)$ of a map $f = (f_0 : \ldots : f_n)$ can be equivalently defined as the multiplicity of the ring $\mathrm{R}/(a_1, \ldots, a_k : \mathrm{I}_f^\infty)$ where $a_1, \ldots, a_k$ are general linear combinations of $f_0, \ldots, f_n$ and $(a_1, \ldots, a_k : \mathrm{I}_f^\infty)$ stands for the saturation of the ideal $(a_1, \ldots, a_k)$ by $\mathrm{I}_f = (f_0, \ldots, f_n)$. In this context, the quantities $d_k(f)$ are usually called the *mixed multiplicities of* I_f, see also below for more explanations about this algebraic definition of the projective degrees.

Algebraically, the graph Γ_f of a map $f = (f_0 : \ldots : f_n) : \mathbb{P}_{\mathrm{k}}^n \dashrightarrow \mathbb{P}_{\mathrm{k}}^n$ is the Proj of the Rees algebra $\mathcal{R}(\mathrm{I}_f) = \bigoplus_{k \geqslant 0} \mathrm{I}_f^k t^k$ of the base ideal I_f of f and the embedding of Γ_f in $\mathbb{P}_{\mathrm{k}}^n \dashrightarrow \mathbb{P}_{\mathrm{k}}^n$ is defined by the surjection $\mathrm{S} \twoheadrightarrow \mathcal{R}(\mathrm{I}_f)$ sending the variables y_i to $f_i t$. In practice, the kernel J of the latter surjection may be difficult to compute and an approximation of J is more accessible by considering the symmetric algebra of I_f, this is a classical approach that we now summarize briefly in our context of rational maps, see [Vas05] for an introduction in a broader context and pointers to references about this procedure. The presentation matrix Φ_f of I_f, by definition verifying the following short exact sequence:

$$\mathrm{R}^m \xrightarrow{\phi} \mathrm{R}^{n+1} \xrightarrow{(f_0 \ \ldots \ f_n)} \mathrm{I}_f \rightarrow 0,$$

defines an embedding of $\mathbb{P}(\mathrm{I}_f) = \operatorname{Proj}\big(\operatorname{Sym}(\mathrm{I}_f)\big)$ in $\mathbb{P}_{\mathrm{k}}^n \dashrightarrow \mathbb{P}_{\mathrm{k}}^n$ whose ideal J_1 is generated by the m entries of the line matrix $(y_0 \ \ldots \ y_n)\phi$ (here we consider that ϕ is a matrix both in R and $\mathrm{S} = \mathrm{R}[y_0, \ldots, y_n]$). Now, since the natural surjection $\operatorname{Sym}(\mathrm{I}_f) \twoheadrightarrow \mathcal{R}(\mathrm{I}_f)$ factorizing the maps $\mathrm{I}_f^{\otimes k} \twoheadrightarrow \mathrm{I}_f^k$ defines an embedding of Γ_f in $\mathbb{P}(\mathrm{I}_f)$, one has that $\mathrm{J}_1 \subset \mathrm{J}$, i.e. J_1 provides some equations of Γ_f and the ideal I_f is of linear type precisely when $\mathrm{J}_1 = \mathrm{J}$.

At the level of the projective degrees, assuming that $\mathbb{P}(\mathrm{I}_f)$ has codimension n, the inclusion $\mathrm{J}_1 \subset \mathrm{J}$ provides moreover the following estimations:

$$\forall k \in \{0, \ldots, n\}, \qquad d_k(f) \leqslant \deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f). \tag{1.1.1}$$

Let us now present a first special feature of determinantal rational maps after presenting basic facts about them.

Definition 1.1.6 (Hilbert-Burch matrix of a (Koszul)-determinantal map)**.** Given that the base ideal $\mathrm{I}_f = (f_0 : \ldots : f_n) \subset \mathrm{R}$ of f verifies $\operatorname{codim}\big(\mathbb{V}(\mathrm{I}_f)\big) \geqslant 2$ and that I_f is the n-minors ideal of a $(n+1)\times n$-matrix, Hilbert-Burch theorem [Eis95, Theorem 20.15] states that I_f is the n-minors ideal of its presentation matrix Φ_f, i.e. the presentation of I_f reads:

$$0 \;\to \mathrm{R}^n \xrightarrow{\Phi_f} \mathrm{R}^{n+1} \xrightarrow{(f_0\, \cdots\, f_n)} \mathrm{I}_f \;\to\; 0.$$

It motives why the matrix Φ_f is called the *Hilbert-Burch matrix* of f in the following.

Actually, when keeping track of the graduation in the previous short exact sequence

$$0 \;\to \bigoplus_{k=1}^{n} \mathrm{R}(-d_k) \xrightarrow{\Phi_f} \mathrm{R}^{n+1} \xrightarrow{(f_0\, \cdots\, f_n)} \mathrm{I}_f(d) \;\to\; 0$$

where $d = \sum_{k=1}^{n} d_k$, we will call the positive integer $d_1, \ldots, d_n \geqslant 1$ the *syzygetic degrees of* f.

At this point, observe also from the equations $(y_0\ \ldots\ y_n)\Phi_f$ defining the embedding of $\mathbb{P}(\mathrm{I}_f)$ in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ that the conditions:

$$\forall k \in \{1, \ldots, n-1\},\ \operatorname{codim} \mathbb{V}\big(\mathrm{I}_k(\Phi_f)\big) \geq n+1-k \tag{1.1.2}$$

are the required conditions in order that $\operatorname{codim}\big(\mathbb{P}(\mathrm{I}_f)\big)$ has codimension n in which case $\mathbb{P}(\mathrm{I}_f)$ is a complete intersection and a minimal free resolution of its coordinate ring is the Koszul complex on the entries of $(y_0\ \ldots\ y_n)\Phi_f$. Thus from now on, a *Koszul-determinantal map* $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ is a map such that the following two conditions are verified:

- a minimal presentation of $\mathrm{I}_f = (f_0, \ldots, f_n)$ reads:

$$0 \;\to \mathrm{R}^n \xrightarrow{\Phi_f} \mathrm{R}^{n+1} \xrightarrow{(f_0\, \cdots\, f_n)} \mathrm{I}_f \;\to\; 0.$$

- the conditions (1.1.2) are satisfied.

Remark 1.1.7. As it is stated for instance in [RS01, Subsection 2.1], I_f is of linear type if

$$\forall k \in \{1, \ldots, n\},\ \operatorname{codim} \mathbb{V}\big(\mathrm{I}_k(\Phi_f)\big) \geq n+2-k$$

so studying Koszul-determinantal maps is a first step toward understanding determinantal Cremona maps not necessarily of linear type.

Remark also that the notion of determinantal and Koszul-determinantal map coincide in the plane case $n = 2$.

Proposition 1.1.8. *Let $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be a Koszul-determinantal map of syzygetic degree $d_1, \ldots, d_n \geqslant 1$, then for all $k \in \{0, \ldots, n\}$,*

$$d_k(f) \leqslant \sigma_{n-k,n}(d_1, \ldots, d_n) = \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} d_{i_1} \ldots d_{i_k} \tag{1.1.3}$$

where $\sigma_{k,n}(u_1, \ldots, u_n)$ is the k-th symmetric polynomial in n variables.

Proof. Letting Φ_f be the $(n+1)\times n$ Hilbert-Burch matrix of f, the ideal J_1 of $\mathbb{P}(\mathrm{I}_f)$ in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ is generated by the n entries $\phi_1, \ldots, \phi_n$ of the line matrix $(y_0 \ \ldots \ y_n)\phi$. Since conditions (1.1.2) are moreover satisfied, $\mathbb{P}(\mathrm{I}_f)$ has codimension n in $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ so $\mathbb{P}(\mathrm{I}_f)$ is a complete intersection and the Koszul complex on $\phi_1, \ldots, \phi_n \in \mathrm{S}$

$$0 \to \mathrm{S}(-\sum_{k=1}^{n} d_k, -n) \to \ldots \to \bigwedge^2\Big(\bigoplus_{k=1}^{n} \mathrm{S}(-d_k, -1)\Big) \to \bigoplus_{k=1}^{n} \mathrm{S}(-d_k, -1) \to \mathrm{S}$$

provides a minimal bi-graded free resolution of the coordinate ring of $\mathbb{P}(\mathrm{I}_f)$. As it was done in Example 1.1.2, we compute the numerator $\mathrm{Num}_{\mathrm{S}/\mathrm{J}_1}(T_0, T_1)$ of the Hilbert series of S/J_1 from which we extract the component of total degree n of $\mathrm{Num}_{\mathrm{S}/\mathrm{J}_1}(1 - T_0, 1 - T_1)$:

$$\forall k \in \{0, \ldots, n\},$$
$$\deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f) = s_{n-k,n}(d_1, \ldots, d_n) = \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} d_{i_1} \ldots d_{i_k}.$$

The conclusion of the proposition follows then from (1.1.1). □

We point out that the conclusion of Proposition 1.1.8 is classical and had been established in more general context of maps $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^{n'}_{\mathrm{k}}$ with $n \leqslant n'$, see for instance [CR21, Theorem 5.7]. It explains however why, when interested in determinantal Cremona maps, one has also to consider base ideal not of linear type or else, if the base ideal is of linear type, the entries of the presentation matrix can only contain linear polynomials (case $d_1 = \ldots = d_n = 1$ of the previous proposition). In other words, one has to consider the kernel of the surjection $\mathrm{Sym}(\mathrm{I}_f) \twoheadrightarrow \mathcal{R}(\mathrm{I}_f)$ in order to understand the level of approximations of the inequalities (1.1.3). This kernel is the R-torsion of $\mathrm{Sym}(\mathrm{I}_f)$ [Mic64] described by the Fitting ideals of I_f, i.e. by the ideals $\mathrm{I}_k(\Phi_f)$ of k-minors of the presentation matrix Φ_f of I_f for $k \in \{1, \ldots, n-1\}$, see [Vas05, Prop. 1.1 and below] and [BCJ09]. In our context of determinantal base ideal I_f, these ideals of k-minors of Φ_f have an expected codimension [RS01, Subsection 2.1]:

$$\forall k \in \{1, \ldots, n\}, \ \mathrm{codim}\, \mathbb{V}\big(\mathrm{I}_k(\Phi_f)\big) \geq n + 2 - k,$$

conditions ensuring that I_f is of linear type (see also Remark 1.1.7). Accordingly, let us now describe for each $k \in \{1, \ldots, n\}$, the contribution of the possible defect $\mathrm{codim}\, \mathbb{V}\big(\mathrm{I}_k(\Phi_f)\big) = n + 1 - k$ on the projective degrees of f.

Proposition 1.1.9. *Let $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be a Koszul-determinantal rational map of Hilbert-Burch matrix Φ_f and let*

$$j_0 := \min\{j \in \{1, \ldots, n\}, \ \mathrm{codim}\, \mathbb{V}\big(I_j(\Phi_f)\big) \geqslant n + 2 - j\}.$$

Then:

$$\forall k \in \{j_0-1,\dots,n\},\ d_k(f) = \deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f).$$

Moreover if $j_0 > 1$*,* $d_{j_0-2}(f) < \deg_{\mathbb{P}}^{n-j_0+2,j_0-2} \mathbb{P}(\mathrm{I}_f)$*.*

Proof. First, since the computation of the projective degrees of a map depends on the free resolution of the coordinate ring of its graph Γ_f and since the multidegree are invariant when taking the algebraic closure $\overline{\mathrm{k}}$ of k, we can assume that k is algebraically closed. Thus, to show Proposition 1.1.9, we consider the geometric interpretation of the projective degrees of f as described in Remark 1.1.3.

Now remark that $\mathbb{P}(\mathrm{I}_f)$ decomposes is set-theoretically the union $\Gamma_f \cup \overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ where $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ is the closure of $\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f$ in the Zariski topology. This decomposition is a consequence of the fact that $\mathbb{P}(\mathrm{I}_f)$ is a complete intersection and that Γ_f is residual to the torsion in $\mathbb{P}(\mathrm{I}_f)$. In addition, $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ is supported on the zero locus $\operatorname{Proj}\big(\mathrm{S}/\,\mathrm{I}_{j_0-1}(\Phi_f)\big) \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ defined by $\mathrm{I}_{j_0-1}(\Phi_f)$ (as an ideal in $\mathrm{S} = \mathrm{R}[y_0,\dots,y_n]$).

But by assumption, since $\operatorname{codim}\operatorname{Proj}\big(\mathrm{S}/\,\mathrm{I}_{j_0-1}(\Phi_f)\big) \geqslant n-j_0+2$, for any $k \in \{j_0-1,\dots,n\}$, the intersection $H^{n-k}_{\mathbf{x}} \cap \mathbb{V}\big(\mathrm{I}_{j_0-1}(\Phi_f)\big) \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ and consequently the intersection $H^{n-k}_{\mathbf{x}} \cap \overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ are empty for any general linear space $H^{n-k}_{\mathbf{x}}$ of codimension $n-k$ in the x-variables (i.e. $H^{n-k}_{\mathbf{x}} = \mathbb{V}(l_{1,0},\dots,l_{cnk,0}) \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ is the zero locus of $n-k$ general linear forms $l_{1,0},\dots,l_{c-k,0}$ in the x-variables). Hence, for all $k \in \{j_0-1,\dots,n\}$, the intersection $H^{n-k}_{\mathbf{x}} \cap \Gamma_f$ and $H^{n-k}_{\mathbf{x}} \cap \mathbb{P}(\mathrm{I}_f)$ coincide so $d_k(f) = \deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f)$.

Actually, since $\operatorname{codim}\operatorname{Proj}\big(\mathrm{S}/\,\mathrm{I}_{j_0}(\Phi_f)\big) = n-j_0+1$, the intersection $H^{j_0-2}_{\mathbf{x}} \cap \operatorname{Proj}\big(\mathrm{S}/\,\mathrm{I}_{j_0}(\Phi_f)\big)$ and the intersection $H^{n-k}_{\mathbf{x}} \cap \overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ are not empty for any general linear space $H^{n-k}_{\mathbf{x}}$ of codimension $n-j_0$ in the x-variables. The intersection $H^{j_0-2}_{\mathbf{x}} \cap \Gamma_f$ is thus strictly included in $H^{j_0-2}_{\mathbf{x}} \cap \mathbb{P}(\mathrm{I}_f)$ so

$$d_{j_0-2}(f) < \deg_{\mathbb{P}}^{n-j_0+2,j_0-2} \mathbb{P}(\mathrm{I}_f)$$

which shows the last assertion in case $j_0 > 1$. □

From the geometric intuition, assuming $j_0 > 2$, one could believe that for all $k \in \{0,\dots,j_0-2\}$, $d_k(f) < \deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f)$ but it may happen that the torsion has an actual impact only on one term, as illustrated by the following example:

Example 1.1.10. Let $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3$ be the rational map whose Hilbert-Burch matrix is

$$\Phi_f = \begin{pmatrix} x_0 & 0 & x_0x_1 \\ x_1 & x_0 & x_0^2 \\ 0 & x_1 & x_0x_3 \\ x_2 & x_3 & x_1x_2 \end{pmatrix}.$$

Then $\operatorname{codim}\mathbb{V}\big(I_2(\Phi_f)\big) = 2 < 3$ and $\operatorname{codim}\mathbb{V}\big(I_3(\Phi_f)\big) = 4$. Hence, by Proposition 1.1.9, $d_1(f) < \deg_{\mathbb{P}}^{2,1} \mathbb{P}(\mathrm{I}_f) = 5$, however in this example one has also $d_0(f) = \deg_{\mathbb{P}}^{3,0} \mathbb{P}(\mathrm{I}_f) = 2$, the complete multidegree being

$$d(f) = (2,4,4,1),$$
$$\deg_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = (2,5,4,1)$$

as it can be computed from a computer system as Macaulay2.

In fact, assuming that the coordinate ring of the defect $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_g}$ is Cohen-Macaulay [Eis95, Chapter 18], one can compute directly the bi-graded free resolution of the graph of a map f starting from the bi-graded free resolution of $\mathbb{P}(\mathrm{I}_f)$ and the bi-graded free resolution of $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_g}$. Thus, assuming that the coordinate ring of the torsion is Cohen-Macaulay, one can compute algebraically the projective degrees of f without geometric interpretation. This latter remark relies on [Eis95, Exercise 21.23] and we give it an illustration now.

Example 1.1.11. Let $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ be the determinantal map whose Hilbert-Burch matrix reads:

$$\Phi_f = \begin{pmatrix} x_0 & 0 & x_0^2 \\ x_1 & 0 & x_1^2 \\ 0 & x_2 & x_0x_1 - x_1x_2 \\ 0 & x_3 & x_0x_2 \end{pmatrix}.$$

From the fact that $\mathrm{I}_2(\Phi_f) = (x_0, x_1)$, one can read from the matrix Φ_f that the ideal I_t of the torsion $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}$ is equal to $(x_0, x_1, x_2y_2 + x_3y_3)$ and hence it is arithmetically Cohen-Macaulay (as it is *perfect* i.e. the length of its resolution is equal to its codimension [BH93, Definition 1.4.15]). From the minimal bi-graded free resolution of $\mathbb{P}(\mathrm{I}_f)$

$$0 \to \mathrm{S}(-4,-3) \to \begin{matrix} \mathrm{S}(-2,-2) \\ \oplus \\ \mathrm{S}(-3,-2)^2 \end{matrix} \to \begin{matrix} \mathrm{S}(-1,-1)^2 \\ \oplus \\ \mathrm{S}(-2,-1) \end{matrix} \to \mathrm{S}$$

and the minimal bi-graded free resolution of I_t

$$0 \to \mathrm{S}(-3,-1) \to \begin{matrix} \mathrm{S}(-2,0) \\ \oplus \\ \mathrm{S}(-2,-1)^2 \end{matrix} \to \begin{matrix} \mathrm{S}(-1,0)^2 \\ \oplus \\ \mathrm{S}(-1,-1) \end{matrix} \to \mathrm{S}$$

one can construct the following free resolution of Γ_f as a mapping cone of the previous two, see [Eis95, Exercise 21.23]

$$0 \to \mathrm{S}(-3,-3)^2 \to \begin{matrix} \mathrm{S}(-3,-2) \\ \oplus \\ \mathrm{S}(-2,-2)^3 \\ \oplus \\ \mathrm{S}(-2,-3) \end{matrix} \to \begin{matrix} \mathrm{S}(-2,-1) \\ \oplus \\ \mathrm{S}(-1,-1)^2 \\ \oplus \\ \mathrm{S}(-1,-2) \end{matrix} \to \mathrm{S}$$

from which one has that $d(f) = (1,4,4,1)$ is palindromic (for more about Cremona maps whose graph has a Cohen-Macaulay coordinate ring, see for instance [HS17]).

1.2 A first application: the projective degrees of some glued maps

Notation 6. When recalling generalities about rational maps, we consider a positive integer $n \in \mathbb{N}^*$, $\mathsf{R} = \mathrm{k}[\mathsf{x}_0, \ldots, \mathsf{x}_0]$, $\mathbb{P}^n_{\mathrm{k}} = \mathrm{Proj}(\mathsf{R})$ and a rational map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$.

When considering glued maps, we let $m, m' \geqslant 1$, $\mathrm{R}_m = \mathrm{k}[x_0, \ldots, x_m]$, $\mathrm{R}_{m'} = \mathrm{k}[x_m, \ldots, x_{m+m'}]$, $\mathrm{R} = \mathrm{k}[x_0, \ldots, x_{m+m'}]$ and we always consider that R_m and $\mathrm{R}_{m'}$ are embedded in R. Let also $g : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ (resp. $g' : \mathbb{P}^{m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m'}_{\mathrm{k}}$), where $\mathbb{P}^m_{\mathrm{k}} = \mathrm{Proj}(\mathrm{R}_m)$ (resp. $\mathbb{P}^{m'}_{\mathrm{k}} = \mathrm{Proj}(\mathrm{R}_{m'})$). The associated glued map is denoted $[g|g'] : \mathbb{P}^{m+m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+m'}_{\mathrm{k}}$ where $\mathbb{P}^{m+m'}_{\mathrm{k}} = \mathrm{Proj}(\mathrm{R}_{m+m'})$.

Now let us clarify in which embedding we consider the graphs Γ_g and $\Gamma_{g'}$ of, respectively, a map $g : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ and $g' : \mathbb{P}^{m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m'}_{\mathrm{k}}$ see Notation 6. Denote by $\mathrm{S}_m = \mathrm{R}_m[y_0, \ldots, y_m]$, $\mathrm{S}_{m'} = \mathrm{R}_{m'}[y_m, \ldots, y_{m+m'}]$, $\mathrm{S} = \mathrm{R}[y_0, \ldots, y_{m+m'}]$ and $\mathrm{I}_{\Gamma_g} \subset \mathrm{S}$ (resp. $\mathrm{I}_{\Gamma_{g'}} \subset \mathrm{S}$) be the ideal of the graph $\Gamma_g \subset \mathbb{P}^{m+m'}_{\mathrm{k}} \times \mathbb{P}^{m+m'}_{\mathrm{k}}$ of g (resp. $\Gamma_{g'} \subset \mathbb{P}^{m+m'}_{\mathrm{k}} \times \mathbb{P}^{m+m'}_{\mathrm{k}}$ of g') where $\Gamma_g \subset \mathbb{P}^{m+m'}_{\mathrm{k}} \times \mathbb{P}^{m+m'}_{\mathrm{k}} = \mathrm{BiProj}(\mathrm{S})$. Geometrically, Γ_g (resp. $\Gamma_{g'}$) is thus the cone of vertex $\Gamma_g \cap \mathbb{P}^m_{\mathrm{k}} \times \mathbb{P}^m_{\mathrm{k}}$ (resp. $\Gamma_{g'} \cap \mathbb{P}^{m'}_{\mathrm{k}} \times \mathbb{P}^{m'}_{\mathrm{k}}$) where $\mathbb{P}^m_{\mathrm{k}} \times \mathbb{P}^m_{\mathrm{k}} = \mathrm{BiProj}(\mathrm{S}_m)$ (resp. $\mathbb{P}^{m'}_{\mathrm{k}} \times \mathbb{P}^{m'}_{\mathrm{k}} = \mathrm{BiProj}(\mathrm{S}_{m'})$). Proposition 5 follows from the following results about $\Gamma_g \cap \Gamma_{g'}$.

Lemma 1.2.1. *Assume that the tensor product $\mathbb{F}_g \otimes \mathbb{F}_{g'}$, of a bigraded free resolution $\mathbb{F}_g$ of Γ_g with a bigraded free resolution $\mathbb{F}_{g'}$ of $\Gamma_{g'}$, is a free resolution of $\Gamma_g \cap \Gamma_{g'}$. Then for any $k \in \{0, \ldots, m+m'\}$:*

$$d_k([g|g']) = \sum_{p=0}^{k} d_p(g) d_{k-p}(g').$$

Proof. By assumptions, the numerator $\mathrm{Num}_{\mathrm{S}/\mathrm{I}_{\Gamma_g \cap \Gamma_{g'}}}(T_0, T_1)$ of the Hilbert series of $\mathrm{S}/\mathrm{I}_{\Gamma_g \cap \Gamma_{g'}}$ is the product $\mathrm{Num}_g(T_0, T_1)\,\mathrm{Num}_{g'}(T_0, T_1)$ of the numerators of the Hilbert series of respectively S/I_g and $\mathrm{S}/\mathrm{I}_{g'}$.

Focusing on the homogeneous component of total degree $m+m'$ of the polynomial $\mathrm{Num}_{[g|g']}(1-T_0, 1-T_1)$, we thus have:

$$\begin{aligned}
\big(\mathrm{Num}_{\mathrm{S}/\mathrm{I}_{\Gamma_g \cap \Gamma_{g'}}}(1-T_0,&1-T_1)\big)_{m+m'} \\
&= \big(\mathrm{Num}_g(1-T_0, 1-T_1)\,\mathrm{Num}_{g'}(1-T_0, 1-T_1)\big)_{m+m'} \\
&= \big(\mathrm{Num}_g(1-T_0, 1-T_1)\big)_m \big(\mathrm{Num}_{g'}(1-T_0, 1-T_1)\big)_{m'}
\end{aligned}$$

the last equality holds because $\big(\mathrm{Num}_g(1-T_0, 1-T_1)\big)_m$ (resp. $\big(\mathrm{Num}_{g'}(1-T_0, 1-T_1)\big)_{m'}$) is also the homogeneous component of smallest total degree of $\mathrm{Num}_g(1-T_0, 1-T_1)$ (resp. $\mathrm{Num}'_g(1-T_0, 1-T_1)$) as Γ_g, being the cone over the graph of g, is irreducible of codimension m (resp. $\Gamma_{g'}$ is irreducible of codimension m'). ☐

Given these preliminaries and before proving Proposition 5, let us briefly underline why we restrict to inputted determinantal maps g and g' when building a glued map $[g|g']$, see Definition 1.1.6 for their definition. If the gluing is as in Lemma 1.2.1, the $(m+m')$-minors ideal of the concatenated matrix $\Phi_{[g|g']}$ always defines a map $\mathbb{P}^{m+m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+m'}_{\mathrm{k}}$ and, if $\mathrm{codim}\,\mathbb{V}\big(\mathrm{I}_{m+m'}(\Phi_{[g|g']})\big) = 2$, the base ideal $\mathrm{I}_{[g|g']}$ of $[g|g']$ is equal to $\mathrm{I}_{m+m'}(\Phi_{[g|g']})$. Let us mention here that a glued map $[g|g']$ can also be defined when g and g' are not necessarily determinantal however in this case $[g|g']$ is not necessarily a map from $\mathbb{P}^{m+m'}_{\mathrm{k}}$ to $\mathbb{P}^{m+m'}_{\mathrm{k}}$.

When f is determinantal of Hilbert-Burch matrix $\Phi_f \in \mathsf{R}^{(n+1)\times n}$, the kernel of $\mathrm{Sym}(\mathrm{I}_f) \twoheadrightarrow \mathcal{R}(\mathrm{I}_f)$ is the R-torsion of $\mathrm{Sym}(\mathrm{I}_f)$ [Mic64] and are described by the *Fitting ideals* of I_f, i.e. by the ideals $\mathrm{I}_k(\Phi_f)$ for $k \in \{1, \dots, n-1\}$ of k-minors ideals of the presentation matrix Φ_f of I_f, see [Vas05, Prop. 1.1 and below] and [BCJ09]. Hence, the irreducible components of $\mathbb{P}(\mathrm{I}_f)$ are the graph Γ_f of f and eventual additional pieces lying above closed strict subschemes of the source space $\mathbb{P}^n_{\mathrm{k}}$ of f.

With all these facts, let us now show Proposition 5.

Proof of Proposition 5. Assume that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_{m+m'}(\Phi_{[g|g']})\big) = 2$ so the base ideal $\mathrm{I}_{[g|g']}$ of the glued map $[g|g'] : \mathbb{P}^{m+m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+m'}_{\mathrm{k}}$ is the $(m+m')$-minors ideal of the matrix $\Phi_{[g|g']}$. It is clear that $\Gamma_{[g|g']}$ is included in $\Gamma_g \cap \Gamma_{g'}$. Indeed, by definition, $\Gamma_{[g|g']} \subset \mathbb{P}(\mathrm{I}_{[g|g']}) = \mathbb{P}(\mathrm{I}_g) \cap \mathbb{P}(\mathrm{I}_{g'})$ so $\Gamma_{[g|g']} \subset \mathbb{P}(\mathrm{I}_g)$ (resp. $\Gamma_{[g|g']} \subset \mathbb{P}(\mathrm{I}_{g'})$) and since $\Gamma_{[g|g']}$ cannot be included in a component of $\mathbb{P}(\mathrm{I}_g)$ (resp. $\mathbb{P}(\mathrm{I}_{g'})$) lying above a closed strict subscheme of $\mathbb{P}^{m+m'}_{\mathrm{k}}$, it is necessarily included in Γ_g (resp. $\Gamma_{g'}$).

Now, since, by assumption, $\mathbb{F}_g \otimes \mathbb{F}_{g'}$ is a resolution of $\Gamma_{[g|g']}$ and $\Gamma_{[g|g']} \subset \Gamma_g \cap \Gamma_{g'}$, one has $\Gamma_{[g|g']} = \Gamma_g \cap \Gamma_{g'}$.

The conclusion of Proposition 5 follows then from Lemma 1.2.1. □

Note that, as sets, $\Gamma_{[g|g']}$ and $\Gamma_g \cap \Gamma_{g'}$ are not always equal as illustrated by the following example.

Example 1.2.2. Let $\Phi_g = \begin{pmatrix} x_2 & 0 \\ x_1 & x_0x_2 \\ 0 & x_1^2 \end{pmatrix}$ and $\Phi_{g'} = \begin{pmatrix} x_3 & 0 \\ x_4 & x_2x_3 \\ 0 & x_4^2 \end{pmatrix}$ and $\Phi_{[g|g']}$ be the concatenate matrix of Φ_g and $\Phi_{g'}$ as in Proposition 5. Then, as it can be checked by direct computation on a computer algebra system such as Macaulay2, one has that $\operatorname{codim} \mathbb{V}\big(\Phi_{[g|g']}\big) \geqslant 2$ and that $d([g|g']) = (1, 6, 10, 6, 1)$, $d(g) = d(g') = (1, 3, 1)$ and

$$d_2([g|g']) = 10 < 11 = d_0(g)d_2(g') + d_1(g)d_1(g') + d_2(g)d_0(g').$$

In this example $\mathbb{F}_g \otimes \mathbb{F}_{g'}$ provides a bigraded free resolution of $\Gamma_g \cap \Gamma_{g'}$ and $\Gamma_{[g|g']} \subsetneq \Gamma_g \cap \Gamma_{g'}$.

Let us also mention that there exist situations where the ideal of $\Gamma_g \cap \Gamma_{g'}$ decomposes as the intersection of the ideal of $\Gamma_{[g|g']}$ and the ideal of embedded components, so that $\Gamma_{[g|g']}$ and $\Gamma_g \cap \Gamma_{g'}$ are not scheme-theoretically equal but have the same projective degrees.

Even if it might seem a bit artificial (since the results about the standard Cremona maps are well known, in particular its projective degrees, and computable by other means, see [GSP06]), we illustrate an application of Proposition 5 in the example of the standard Cremona maps $\tau_{m+m'} = [\tau_m|\tau_{m'}]$.

Proposition 1.2.3. *Following Notation 6, let*

$$\tau_m = (x_1 \cdots x_m : \dots : x_0 \cdots x_{m-1}) : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$$
$$\tau_{m'} = (x_{m+1} \cdots x_{m+m'} : \dots : x_m \cdots x_{m+m'-1}) : \mathbb{P}^{m'}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m'}_{\mathrm{k}}$$

with associated Hilbert-Burch matrices $\Phi_{\tau_m} \in \mathrm{R}_m^{(m+1)\times m}$ *and* $\Phi_{\tau_{m'}} \in \mathrm{R}_{m'}^{(m'+1)\times m'}$. *Let also* $\Phi_{[\tau_m|\tau_{m'}]} \in \mathrm{R}^{(m+m'+1)\times(m+m')}$ *be as in Proposition 5. Then:*

(1) $\operatorname{codim} \mathbb{V}(\mathrm{I}_{m+m'}\left(\Phi_{[\tau_m|\tau_{m'}]})\right) = 2$,

(2) *the ideal of* $\mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]})$ *is minimally resolved by* $\mathbb{F}_{\tau_m} \otimes \mathbb{F}_{\tau_{m'}}$ *which is the Koszul complex on the entries of the line matrix*

$$\begin{pmatrix} y_0 & \dots & y_{m+m'} \end{pmatrix} \Phi_{[\tau_m|\tau_{m'}]}.$$

(3) $\mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]}) = \Gamma_{[\tau_m|\tau_{m'}]}$ *and, consequently, for any* $k \in \{0, \dots, m+m'\}$

$$d_k(\tau_{m+m'}) = \sum_{p=0}^{k} d_p(\tau_m) d_{k-p}(\tau_{m'}) = \binom{m+m'}{k}. \tag{1.2.1}$$

Proof. Let $m, m' \geqslant 1$ and

$$\Phi_{\tau_{m+m'}} = \begin{pmatrix} x_0 & 0 & \dots & \dots & 0 \\ -x_1 & x_1 & \ddots & & \vdots \\ 0 & -x_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & x_{m+m'-1} \\ 0 & \dots & \dots & 0 & -x_{m+m'}. \end{pmatrix} \tag{1.2.2}$$

Since

$$\mathrm{I}_{m+m'}(\Phi_{[\tau_m|\tau_{m'}]}) = (x_1 \cdots x_{m+m'}, \dots, x_0 \cdots x_{m+m'-1}),$$

one has that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_{m+m'}(\Phi_{[\tau_m|\tau_{m'}]})\big) = 2$ (since the generators of the ideal $\mathrm{I}_{m+m'}(\Phi_{[\tau_m|\tau_{m'}]})$ do not share a common factor) and thus Item (1) is verified.

The proof of Item (2) and Item (3) relies on an induction on m and m'. The inductive step consists in first showing that the tensor product $\mathbb{F}_{\tau_m} \otimes \mathbb{F}_{\tau_{m'}}$ of a free resolution $\mathbb{F}_{\tau_m}$ of $\mathbb{P}(\mathrm{I}_{\tau_m})$ and a free resolution $\mathbb{F}_{\tau_{m'}}$ of $\mathbb{P}(\mathrm{I}_{\tau_{m'}})$ provides a free resolution of $\mathbb{P}(\mathrm{I}_{\tau_m}) \cap \mathbb{P}(\mathrm{I}_{\tau_{m'}}) = \mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]})$. The second step consists in showing that $\mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]})$ is irreducible or, in other words, that $\mathrm{I}_{\mathbb{P}(\mathrm{I}_{\tau_m})} + \mathrm{I}_{\mathbb{P}(\mathrm{I}_{\tau_{m'}})}$ is prime. This latter property insures that

$$\Gamma_{\tau_m} \cap \Gamma_{\tau_{m'}} = \mathbb{P}(\mathrm{I}_{\tau_m}) \cap \mathbb{P}(\mathrm{I}_{\tau_{m'}}) = \mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]}) = \Gamma_{[\tau_m|\tau_{m'}]}$$

and Equation (1.2.1) follows then from a direct application of Proposition 5.

- Initial case: Item (2) and Item (3) are verified in the case $m = 1$, this follows from the fact that $\mathbb{P}(\mathrm{I}_{\tau_1}) = \mathbb{V}(x_0y_0 - x_1y_1)$ whose minimal free resolution is the Koszul complex on the single irreducible polynomial $x_0y_0 - x_1y_1 \in \mathrm{k}[x_0, x_1, y_0, y_1]$.

- Inductive step: let $m, m' \geqslant 1$ and assume that Item (2) and Item (3) hold for τ_m and $\tau_{m'}$. In particular the ideal $\mathcal{I}_m$ of $\mathbb{P}(\mathrm{I}_{\tau_m}) = \Gamma_{\tau_m}$ (resp. the ideal $\mathcal{I}_{m'}$ of $\mathbb{P}(\mathrm{I}_{\tau_{m'}}) = \Gamma_{\tau_{m'}}$) is minimally resolved by the Koszul complex $\mathbb{F}_{\tau_m}$ on the entries of the line matrix $\begin{pmatrix} y_0 & \dots & y_m \end{pmatrix} \Phi_{\tau_m}$ (resp. by the Koszul complex $\mathbb{F}_{\tau_{m'}}$ on the entries of the line matrix $\begin{pmatrix} y_m & \dots & y_{m'} \end{pmatrix} \Phi_{\tau_{m'}}$). We show Item (2) and Item (3) via classical methods involving Gröbner bases and we refer to [AL94] for the definitions associated to this argument. Remark that $\mathbb{F}_{\tau_m} \otimes \mathbb{F}_{\tau_{m+m'}}$ is the Koszul complex on the entries of $\begin{pmatrix} y_0 & \dots & y_{m+m'} \end{pmatrix} \Phi_{[\tau_m|\tau_{m'}]}$ and one has that $\mathbb{F}_{\tau_m} \otimes \mathbb{F}_{\tau_{m'}}$ resolves the ideal of $\mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]})$ if and only if

$$\forall i \geqslant 1, \mathbb{T}^i\big(\mathrm{S}/\mathcal{I}_m, \mathrm{S}/\mathcal{I}_{m'}\big) = 0,$$

S being the polynomial ring $\mathrm{k}[x_0, \dots, x_{m+m'}, y_0, \dots, y_{m+m'}]$. Actually, by $\mathbb{T}$ rigidity, the latter conditions are verified if and only if

$$\mathbb{T}^1\big(\mathrm{S}/\mathcal{I}_m, \mathrm{S}/\mathcal{I}_{m'}\big) = \mathcal{I}_m \cap \mathcal{I}_{m'}/\mathcal{I}_m \cdot \mathcal{I}_{m'} = 0.$$

To show this last condition, i.e. to show that

$$\mathcal{I}_m \cap \mathcal{I}_{m'} = \mathcal{I}_m \cdot \mathcal{I}_{m'},$$

we compute $\mathcal{I}_m \cap \mathcal{I}_{m'}$ by eliminating t in the ideal $t\mathcal{I}_m + (1-t)\mathcal{I}_{m'} \subset \mathrm{S}[t]$ see [AL94, Prop. 2.3.5] for this standard use of Gröbner bases. Using the graded reverse lexicographic order, the variable t being bigger than the x and y variables, and denoting

$$\forall i \in \{0, \dots, m-1\},\ g_i = t(x_i y_i - x_{i+1} y_{i+1}),$$
$$\forall j \in \{m, \dots, m+m'\},\ g_j = (1-t)(x_j y_j - x_{j+1} y_{j+1}).$$

and $\mathcal{G} = \{g_0, \dots, g_{m+m'}\}$, a direct computation shows that for any $i \in \{0, \dots, m-1\}$ and $j \in \{m, \dots, m+m'\}$ one has that $S(g_i, g_j)$ reduces to $(x_i y_i - x_{i+1} y_{i+1})(x_j y_j - x_{j+1} y_{j+1}) \in \mathcal{I}_m \cdot \mathcal{I}_{m'}$ modulo $g_0, \dots, g_{m+m'}$ (where $S(g_i, g_j)$ is the S-polynomial associated to g_i and g_j). Since $S(g_i, g_{i'})$ reduces to 0 modulo $g_0, \dots, g_{m+m'}$ if i, i' are both elements of $\{0, \dots, m-1\}$ or both elements $\{m, \dots, m+m'\}$ and since $S(g_i, S(g_{i'}, g_j))$ reduces to 0 modulo $g_0, \dots, g_{m+m'}, S(g_0, g_m), \dots, S(g_{m-1}, g_{m+m'-1})$, one has then that

$$(t\mathcal{I}_m + (1-t)\mathcal{I}_{m'}) \cap \mathrm{S} \subset \mathcal{I}_m \cdot \mathcal{I}_{m'}$$

which concludes the first step and shows Item (2).

We show that the ideal $\mathcal{I}_{m+m'}$ of $\mathbb{P}(\mathrm{I}_{[\tau_m|\tau_{m'}]})$ (which is generated by the entries $\begin{pmatrix} y_0 & \dots & y_{m+m'} \end{pmatrix} \Phi_{[\tau_m|\tau_{m'}]}$) is prime again by computing Gröbner basis. More precisely, a direct computation shows that the set $\mathcal{H} = \{h_0 = x_0 y_0 - x_1 y_1, \dots, h_{m+m'} = x_{m+m'-1} y_{m+m'-1} - x_{m+m'} y_{m+m'}\}$ is a Gröbner basis of $\mathcal{I}_{m+m'}$ (all S-polynomials in the h_i reduces to 0 modulo $h_0, \dots, h_{m+m'-1}$). We then show that $\mathcal{I}_{m+m'}$ is prime by applying the primality test [AL94, Algoritm 4.4.1] since, given $i \in \{0 \dots, m+m'-1\}$, $h_i x_i y_i - x_{i+1} y_{i+1}$ is irreducible in $\mathrm{k}'[x_i]$ where k' is the quotient field of the ring $\mathrm{k}[x_{i+1}, \dots, x_n, y_0, \dots, y_n]/(h_{i+1}, \dots, h_{m+m'-1})$.

The last equality (1.2.1) follows then from applying classical formulas between binomial numbers. ☐

Let us emphasize again that all the previous results are well known and could be summed up by the fact that the base ideal of the standard Cremona maps is of linear type (see for instance [RS01, Subsection 2.1]). However the scheme of our proof of Proposition 1.2.3, could virtually be applied to more general situations. Even if it is at the moment out of our reach, let us present the kind of situations we have in mind and that we verified experimentally in all the examples we considered.

Conjecture 1.2.4. *Following Notation 6 about glued maps, assume moreover that* k *is algebraically closed, and let*

$$\Phi_{[g|g']} = (\Phi_{ij})_{\substack{0\leqslant i\leqslant m+m'+1\\ 1\leqslant j\leqslant m+m'}} = \begin{array}{c} m+1\left\{\right. \\ m'\left\{\right. \end{array} \overbrace{\left(\begin{array}{c|c} \Phi_g & 0_{m\times m'} \\ \hline 0_{m'\times m} & \Phi_{g'} \end{array}\right)}^{m \quad m'} \begin{array}{l} \left.\right\} m \\ \left.\right\} m'+1 \end{array} \in \mathrm{R}^{(m+m'+1)\times(m+m')}$$

be the matrix defined by the following data:

- *for any* $j \in \{1,\dots,m\}$*, let* $k_j \geqslant 2$, $\lambda_{1,j},\dots,\lambda_{k_j,j} \in \mathrm{R}_m = \mathrm{k}[x_0,\dots,x_m] \subset \mathrm{R}$ *and:*
  - *for* $i \in \{1,\dots,m+1\}$*, let* $\Phi_{ij} \in |\lambda_{1,j},\dots,\lambda_{k_j,j}|$ *be a general linear combination of* $\lambda_{1,j},\dots,\lambda_{k_j,j}$*,*
  - *for* $i \in \{m+2,\dots,m+m'+1\}$, $\Phi_{ij} = 0$.
- *for any* $j \in \{m+1,\dots,m+m'\}$*, let* $k_j \geqslant 2$ *and* $\lambda_{1,j},\dots,\lambda_{k_j,j} \in \mathrm{R}_{m'} = \mathrm{k}[x_m,\dots,x_{m+m'}] \subset \mathrm{R}$ *and:*
  - *for* $i \in \{1,\dots,m\}$, $\Phi_{ij} = 0$,
  - *for* $i \in \{m+1,\dots,m+m'+1\}$*, let* $\Phi_{ij} \in |\lambda_{1,j},\dots,\lambda_{k_j,j}|$ *be a general linear combination of* $\lambda_{1,j},\dots,\lambda_{k_j,j}$.

Let also $\Phi_g = (\Phi_{ij})_{\substack{0\leqslant i\leqslant m+1\\ 1\leqslant j\leqslant m}} \in \mathrm{R}_m^{(m+1)\times m}$ *and* $\Phi_{g'} = (\Phi_{ij})_{\substack{m+1\leqslant i\leqslant m+m'+1\\ m+1\leqslant j\leqslant m+m'}} \in \mathrm{R}_n^{(m'+1)\times m'}$ *and* $[g|g'] : \mathbb{P}^{m+m'} \dashrightarrow \mathbb{P}^{m+m'}$, $g : \mathbb{P}^m \dashrightarrow \mathbb{P}^m$ *and* $g' : \mathbb{P}^{m'} \dashrightarrow \mathbb{P}^{m'}$ *be the determinantal maps defined by* $\Phi_{[g|g']}$, Φ_g *and* $\Phi_{g'}$ *(where* $\mathbb{P}^m = \mathrm{Proj}(\mathrm{R}_m) \subset \mathrm{Proj}(\mathrm{R}) = \mathbb{P}^{m+m'}$ *and* $\mathbb{P}^{m'} = \mathrm{Proj}(\mathrm{R}_{m'}) \subset \mathrm{Proj}(\mathrm{R}) = \mathbb{P}^{m+m'}$*).*

Then for all $i \in \{0,\dots,m+m'\}$*:*

$$d_k([g|g']) = \sum_{p=0}^{k} d_p(g)d_{k-p}(g'). \tag{1.2.3}$$

Chapter 2

Construction of homaloidal plane curves via syzygies

Introduction

Given a homogeneous polynomial $\mathsf{h} \in \mathrm{k}[x_0, \dots, x_m]$ over a field k, the *polar map* $f_{\mathsf{h}} : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ of h is the rational map defined by the polynomials $\frac{\partial \mathsf{h}}{\partial x_0}, \dots, \frac{\partial \mathsf{h}}{\partial x_m}$. The polynomial h and the associated hypersurface $\mathsf{H} = \mathbb{V}(\mathsf{h})$ of $\mathbb{P}^n_{\mathrm{k}}$ are called *homaloidal* if the polynomials $\frac{\partial \mathsf{h}}{\partial x_0}, \dots, \frac{\partial \mathsf{h}}{\partial x_m}$ do not share a common factor and f_{h} is birational.

It was established by I.V. Dolgachev [Dol00, Theorem 4] that if $\mathrm{k} = \mathbb{C}$, the homaloidal polynomials in three variables are either of degree 2, defining a smooth conic in the projective plane, or of degree 3, defining either a union of three lines in general position or a union of a smooth conic with one of its tangents. These polynomials remain homaloidal when the base field k has characteristic greater than 2. This leads to the following question which is a generalization of [DHS12, Question 3.7].

Problem 2.0.1. *Over a field* k *of positive characteristic, are there other homaloidal polynomials than the ones in Dolgachev's classification?*

In [BC18, Proposition 4.6], a first example of a homaloidal polynomial of degree 5 over a field of characteristic 3 was produced, answering both Problem 2.0.1 and [DHS12, Question 3.7]. Very recently, the following example of a homaloidal curve of degree 5 in characteristic 3 was also described.

Example 2.0.2. In characteristic 3, the polynomial $\mathsf{h} = x_0(x_1^2 + x_0x_2)(2x_1^2 + x_0x_2)$, whose zero locus is the union of two conics intersecting with multiplicity two in two distinct points with the tangent at one intersection point, is homaloidal.

The negative answer to Problem 2.0.1 leads to the following question.

Problem 2.0.3. *Does there exist homaloidal polynomials in three variables of arbitrary large degree over fields of arbitrarily large characteristic?*

We answer this question in this chapter, see Theorem 2.2.4 for our expanded result. Following the designation in [Hir83] we say that a union of n distinct lines through a given point z_0 with another line not passing through z_0 is a *near-pencil* arrangement of $n+1$ lines. In addition, given a reduced projective curve $\mathsf{H} = \mathbb{V}(\mathsf{h})$ which is the zero locus $\mathbb{V}(\mathsf{h})$ in $\mathbb{P}^2_{\mathrm{k}}$ of a homogeneous polynomial h, we say that H is *homaloidal* if h is homaloidal.

Theorem 2.0.4. *Let* k *be a field of characteristic* p *and let* $n \in \mathbb{N}_{>0}$ *be a multiple of* p*. Then the near-pencil arrangement of* $n+1$ *lines is homaloidal.*

This result provides an answer to Problem 2.0.3. For instance, let $n \in \mathbb{N}_{>0}$ be such that $n \equiv 0 \mod 5$ and let k be a field of characteristic 5. Then the near-pencil arrangement in Theorem 2.0.4 has degree $n+1$ and is homaloidal. Moreover, given a prime number p, a field k of characteristic p and a positive integer m, Theorem 2.0.4 gives a homaloidal curve of degree $mp+1$, so homaloidal curves (and homaloidal polynomials) exist in arbitrarily large degree and in any prime characteristic.

Remark that the base ideal $\mathrm{I}_{f_\mathsf{h}} = (\mathsf{h}_0, \ldots, \mathsf{h}_n)$ of the polar map f_h defines the singular locus of the curve $\mathsf{H} = \mathbb{V}(\mathsf{h}) \subset \mathbb{P}^2_{\mathrm{k}}$. In this direction, the proof given by I.V.Dolgachev about the classification of homaloidal complex polynomials relies on the *Jung-Milnor's formula* over $\mathbb{C}$ relating several invariants of singularities [Dol00, Lemma 3]. In contrast, our proof of Theorem 2.0.4 relies on the study of the torsion of the symmetric algebra of the base ideal of f_h, an approach that fits in line with previous works such as [RS01], [DHS12], and [BC18]. We emphasize that most of the polynomials we consider in this note define *free curves* i.e. curves such that the base ideal of their polar map is determinantal [Dim17a, Def 2.1]). We specially focus on this case since, when the base ideal has a linear syzygy, free curves are the curves whose singular schemes have maximal length [Dim17a, Cor 1.2].

Contents of the chapter

In the first section, we recall briefly the relations between the symmetric algebra of the base ideal of a plane rational map f and the graph of f.

The second section constitutes the heart of our work. As a central idea, one can study the reduction modulo p of the presentation matrix of the base ideal in order to predict a drop of the *topological degree* of the polar map, see Subsection 2.2.1. The next step is then to evaluate this drop. We carry on this evaluation by describing the generic fiber of the residual scheme and thus describing the generic fiber of the graph itself. This implies in particular that the polynomials h_n are homaloidal (Lemma 2.2.3). We end this section by providing another example of a polynomial of degree 5 which is homaloidal in characteristic 3. Its zero locus in $\mathbb{P}^2_k$ is the union of the unicuspidal ramphoïd quartic and the tangent cone at its cusp (Example 2.2.6).

In the third section, we focus on line arrangements and we show that, over an algebraically closed field of characteristic $p > 0$, the only homaloidal line arrangements are the ones defining unions of three general lines or near-pencil of $lp+1$ lines for any $l \geqslant 1$, see Proposition 2.3.1. This classification follows from the description of the singularities defined by line arrangements.

The explicit computations given in this paper were made using basic functions of the software systems Polymake and Macaulay2 with the Cremona package [Sta17] associated.

2.1 Graph and residual scheme

In this section, all the fields are assumed to be algebraically closed and denoted by the same letter k.

Let $f = (f_0 : f_1 : f_2) : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ be a rational map with base ideal $\mathrm{I}_f = (f_0, f_1, f_2)$ in the coordinate ring $\mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$ of $\mathbb{P}^2_{\mathrm{k}}$. Recall that the epimorphism $\mathrm{Sym}(\mathrm{I}_f) \twoheadrightarrow \mathcal{R}(\mathrm{I})$ between the symmetric algebra $\mathrm{Sym}(\mathrm{I}_f)$ of I_f and the Rees algebra $\mathcal{R}(\mathrm{I}_f) := \oplus_{i\geq 0} \mathrm{I}_f^i\, t^i \subset \mathrm{R}[t]$ of I_f defines the embedding of the graph $\Gamma_f = \mathrm{Proj}\, \mathcal{R}(\mathrm{I}_f)$ of f in the residual scheme $\mathbb{P}(\mathrm{I}) = \mathrm{Proj}\,\big(\mathrm{Sym}(\mathrm{I})\big) \subset \mathbb{P}^2_{\mathrm{k}} \times \mathbb{P}^2_{\mathrm{k}}$. Moreover the ideal of $\mathbb{P}(\mathrm{I}_f)$ is generated by the entries of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix} \Phi_f$ where $\mathrm{S} = \mathrm{R}[y_0, y_1, y_2]$ stands for the coordinate ring of $\mathbb{P}^2_{\mathrm{k}} \times \mathbb{P}^2_{\mathrm{k}}$ and Φ_f stands for a presentation matrix of $\mathrm{I}_f \subset \mathrm{R}$, see Chapter 1 for more details.

Example 2.1.1. Let $f = (f_0 : f_1 : f_2) : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ be a determinantal map whose 3×2-Hilbert-Burch matrix Φ_f is such that all its entries in the first column are homogeneous of degree a and all its entries in the second column are homogeneous of degree b. Hence a free resolution of I_f reads:

$$0 \longrightarrow \mathrm{R}^2 \xrightarrow{\Phi_f} \mathrm{R}^3 \xrightarrow{(f_0\ f_1\ f_2)} \mathrm{I}_f \longrightarrow 0.$$

and $\mathbb{P}(\mathrm{I}_f)$ is the intersection of two divisors of $\mathbb{P}^2_{\mathrm{k}} \times \mathbb{P}^2_{\mathrm{k}}$ of bidegree $(a, 1)$ and $(b, 1)$ respectively, see Definition 1.1.6 for more details. If I_f is of linear type, the projective degrees of f are then given by Proposition 1.1.8 and Proposition 1.1.9:

$$\begin{cases} d_0(f) = \deg^{2,0}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = ab \\ d_1(f) = \deg^{1,1}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = a + b \\ d_2(f) = \deg^{0,2}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = 1. \end{cases} \tag{2.1.1}$$

The complement subscheme $\mathbb{T} = \overline{\mathbb{P}(\mathrm{I})\backslash\Gamma}$ of Γ in $\mathbb{P}(\mathrm{I}_f)$ is moreover supported on

$$\bigcup_{\mathbf{x}\in\mathbb{V}\big(\mathrm{Fitt}_2(\mathrm{I}_f)\big)} \{\mathbf{x}\} \times \mathbb{P}^2_{\mathrm{k}} \tag{2.1.2}$$

where $\mathrm{Fitt}_2(\mathrm{I}_f)$ is the second Fitting ideal of I [Eis95, Corollary-Definition 20.4], by definition generated by the entries of Φ_f, see [BC18, Corollary 1.4] for a reference.

Example 2.1.2. If I_f is not of linear type in Example 2.1.1, we still have that

$$\begin{cases} \deg^{2,0}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = ab \\ \deg^{1,1}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = a + b \\ \deg^{0,2}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = 1. \end{cases}$$

because $\mathbb{P}(\mathrm{I})$ is still a complete intersection. However, there is an extra part $\mathbb{T} = \overline{\mathbb{P}(\mathrm{I})\backslash\Gamma}$ in $\mathbb{P}(\mathrm{I})$ with support as in (2.1.2) such that

$$\begin{cases} \deg^{2,0}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = \deg^{2,0}_{\mathbb{P}} \Gamma_f + \deg^{2,0}_{\mathbb{P}} \mathbb{T} = d_0(f) + \deg^{2,0}_{\mathbb{P}} \mathbb{T} \\ \deg^{1,1}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = \deg^{1,1}_{\mathbb{P}} \Gamma_f + \deg^{1,1}_{\mathbb{P}} \mathbb{T} = d_1(f) + \deg^{1,1}_{\mathbb{P}} \mathbb{T} \\ \deg^{0,2}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_f) = \deg^{0,2}_{\mathbb{P}} \Gamma_f + \deg^{0,2}_{\mathbb{P}} \mathbb{T} = d_2(f) + \deg^{0,2}_{\mathbb{P}} \mathbb{T}. \end{cases}$$

So the topological degree $d_0(f)$ is strictly smaller than $\deg_{\mathbb{P}}^{2,0}\mathbb{P}(\mathrm{I}_f)$ because $\deg_{\mathbb{P}}^{2,0}\mathbb{T}$ is greater or equal to length $\big(\mathbb{V}\big(\mathrm{Fitt}_2(\mathrm{I})\big)$ which is non zero. The quantity $d_0(f) = \deg_{\mathbb{P}}^{2,0}\mathbb{P}(\mathrm{I}_f) - \deg_{\mathbb{P}}^{2,0}\mathbb{T}$ depends moreover on the scheme structure of $\mathbb{T}$ and is the object of Subsection 2.2.2.

2.2 Contribution of the torsion

In this section, following the situation described in Example 2.1.2, we illustrate first on an example how to estimate the drop of the topological degree in positive characteristic compared to characteristic 0. We analyze then in greater generality how this modification impacts the computation of the topological degree of Φ.

2.2.1 Reduction of the presentation matrix modulo p

In what follows, for a homogeneous ideal $\mathcal{I}$ of $\mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$ and an integer t, we denote by $\mathcal{I}_t$ the homogeneous piece of $\mathcal{I}$ of degree t. Let $n \in \mathbb{N}_{>1}$ and let H_n be the union of n distinct lines through a point $z_0 \in \mathbb{P}^2_{\mathrm{k}}$ with any other line not passing through z_0. We can reduce to the situation where $\mathbb{V}(x_2)$ is the latter line and two lines among the n^{th} firsts are $\mathbb{V}(x_0)$ and $\mathbb{V}(x_1)$ so that $z_0 = (0:0:1)$. We can consequently assume without loss of generality that an equation of H_n reads $\mathsf{h} = x_0x_1l_2\cdots l_{n-1}x_2$ where, for all $i \in \{2,\dots,n-1\}$, l_i belongs to $(x_0,x_1)_1$ (set $\mathsf{h} = x_0x_1x_2$ if $n = 2$). The ideal $\mathrm{I}_{f_\mathsf{h}}$ of partial derivatives of h is then equals to

$$\mathrm{I}_{f_\mathsf{h}} = (x_1l_2\cdots l_{n-1}x_2 + x_0x_1\frac{\partial}{\partial x_0}(l_2\cdots l_{n-1})x_2,$$
$$x_0l_2\cdots l_{n-1}x_2 + x_0x_1\frac{\partial}{\partial x_1}(l_2\cdots l_{n-1})x_2, x_0x_1l_2\cdots l_{n-1}).$$

Lemma 2.2.1. *A minimal presentation matrix of* $\mathrm{I}_{f_\mathsf{h}}$ *reads*

$$\Phi_{f_\mathsf{h}} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_1l_2\cdots l_{n-1} \\ -nx_2 & -l_2\cdots l_{n-1}x_2 - x_1\frac{\partial}{\partial x_1}(l_2\cdots l_{n-1})x_2 \end{pmatrix}$$

Proof. The ideal $\mathrm{I}_{f_\mathsf{h}}$ has depth 2 for otherwise the first two generators of $\mathrm{I}_{f_\mathsf{h}}$ would be divisible by either x_0, x_1 or l_i for $i \in \{2,\dots,n-1\}$ which is excluded by the assumption that all the lines in H_n are distinct. A direct computation shows that

$$\begin{cases} x_1x_2l_2\dots l_{n-1} + x_0x_1x_2\frac{\partial}{\partial x_0}(l_2\dots l_{n-1}) = \Phi_1 \\ x_0x_2l_2\dots l_{n-1} + x_0x_1x_2\frac{\partial}{\partial x_1}(l_2\dots l_{n-1}) = \Phi_2 \\ x_0x_1l_2\dots l_{n-1} = \Phi_3 \end{cases}$$

where, given $j \in \{1,2,3\}$, Φ_j stands for $(-1)^j$ times the minor obtained from Φ_{f_h} by leaving out the j^{th} row (in order to check these equalities, remark that, for any $i \in \{2,\dots,n-1\}$, $l_i = x_0\frac{\partial l_i}{x_0} + x_1\frac{\partial l_i}{x_1}$). Hence $\mathrm{I}_{f_\mathsf{h}}$ is a determinantal ideal given by the 2-minors of Φ_{f_h}. Since it has the expected depth, the Hilbert-Burch theorem asserts that a free resolution of $\mathrm{I}_{f_\mathsf{h}}$ reads:

$$0 \longrightarrow \mathrm{R}^2 \xrightarrow{\Phi_{f_\mathsf{h}}} \mathrm{R}^3 \longrightarrow \mathrm{I}_{f_\mathsf{h}} \longrightarrow 0.$$

Moreover, since it does not have constant entries, $\Phi_{f_{\mathsf{h}}}$ is a minimal presentation matrix of $\mathrm{I}_{f_{\mathsf{h}}}$. □

Proposition 2.2.2. *Let $n \in \mathbb{N}_{>1}$ and k be an algebraically closed field such that $p = \mathrm{char}(\mathrm{k})$ does not divide n. Then $\mathrm{I}_{f_{\mathsf{h}}}$ is of linear type and the polar map f_{h} of h has multidegree $(n-1, n, 1)$.*

Proof. Since $\Phi_{f_{\mathsf{h}}}$ has one column of linear entries and one column of entries of degree n, the naive multidegree is $(n-1, n, 1)$, see Example 2.1.1. Moreover $\mathrm{Fitt}_2(\mathrm{I}_{f_{\mathsf{h}}}) = (x_0, x_1, x_2)$ is not supported on any point of $\mathbb{P}^2_{\mathrm{k}}$ so, by (2.1.2), I is of linear type. Hence $\Gamma = \mathbb{P}(\mathrm{I}_{f_{\mathsf{h}}})$ coincide, so

$$\begin{cases} d_0(f_{\mathsf{h}}) = \deg^{2,0}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_{f_{\mathsf{h}}}) \\ d_1(f_{\mathsf{h}}) = \deg^{1,1}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_{f_{\mathsf{h}}}) \\ d_2(f_{\mathsf{h}}) = \deg^{0,2}_{\mathbb{P}} \mathbb{P}(\mathrm{I}_{f_{\mathsf{h}}}). \end{cases}$$

□

We consider now the case where char(k) divides n in a more general situation.

2.2.2 Weight of the torsion

In all the section, we put $n \in \mathbb{N}_{>1}$, k be any field unless otherwise specified, $\mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$ and we consider the ideal I generated by the 2-minors of the matrix

$$\Phi = \begin{pmatrix} \lambda_0 & p_0 \\ \lambda_1 & p_1 \\ \lambda_2 & p_2 \end{pmatrix}$$

with entries in $\mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$ such that for all $j \in \{0, 1, 2\}$, λ_j belongs to $(x_0, x_1)_1$ and p_j belongs to $(x_0, x_1)^{n-2}_{n-1}$, the homogeneous piece of degree $n-1$ of the ideal $(x_0, x_1)^{n-2}$. We assume moreover that I has height 2 and that there exists $j \in \{0, 1, 2\}$ such that $p_j \in (x_0, x_1)^{n-2}_{n-1} \backslash (x_0, x_1)^{n-1}_{n-1}$.

In $\mathrm{S} = \mathrm{R}[y_0, y_1, y_2]$, consider now the ideal

$$\mathrm{I}_{\mathbb{P}(\mathrm{I})} = (\lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2, y_0 p_0 + y_1 p_1 + y_2 p_2)$$

of the embedding of $\mathbb{P}(\mathrm{I})$ in $\mathbb{P}^2_{\mathbf{x}} \times \mathbb{P}^2_{\mathbf{y}}$ generated by the entries of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix} \Phi$. Following the computation in Example 2.1.1, $\mathbb{P}(\mathrm{I})$ being a complete intersection of two divisors of bidegree $(1,1)$ and $(n-1, 1)$, one has

$$\big(d_0(\mathbb{P}(\mathrm{I})), d_1(\mathbb{P}(\mathrm{I})), (d_2(\mathbb{P}(\mathrm{I}))\big) = (n-1, n, 1).$$

Moreover since all the entries of Φ are in the ideal (x_0, x_1), the radical $\sqrt{\mathrm{Fitt}_2(\mathrm{I})}$ of the ideal $\mathrm{Fitt}_2(\mathrm{I})$ of entries of Φ is contained in (x_0, x_1). Actually, since I has height 2 and the polynomials in the first column of Φ are linear in x_0 and x_1 one has $\mathrm{Fitt}_2(\mathrm{I}) = (x_0, x_1)$. Hence, as previously stated in (2.1.2), $\mathbb{P}(\mathrm{I}) = \mathbb{V}(\mathrm{I}_{\mathbb{P}(\mathrm{I})}) \subset \mathbb{P}^2_{\mathbf{x}} \times \mathbb{P}^2_{\mathbf{y}}$ is the union of a torsion part $\mathbb{T}$ supported on $\mathbb{V}(x_0, x_1) = \{(0:0:1)\} \times \mathbb{P}^2_{\mathrm{k}}$, and of the graph $\Gamma_f = \overline{\mathbb{P}(\mathrm{I}) \backslash \mathbb{V}(x_0, x_1)}$ of the map f whose base ideal is the 2-minors ideal I of Φ. The next result is a consequence of [BCRD20, Theorem 5.14] but we will give a self-contained proof.

Lemma 2.2.3. *Under the previous conditions on* Φ*, the torsion component* $\mathbb{T}$ *of* $\mathbb{P}(\mathrm{I})$ *has multidegree* $(n-2,0,0)$ *and the graph* Γ_f *of* f *has multidegree* $(1,n,1)$.

Proof. We analyze separately each element of the multidegree

$$\begin{cases} \deg_{\mathbb{P}}^{2,0} \Gamma_f = d_0(f) \\ \deg_{\mathbb{P}}^{1,1} \Gamma_f = d_1(f) \\ \deg_{\mathbb{P}}^{0,2} \Gamma_f = d_2(f). \end{cases}$$

- **Case** $i = 0$**.** Take a general point $\mathbf{y} \in \mathbb{P}^2_{\mathrm{k}}$ as the intersection of two general lines H_1, H_2 of $\mathbb{P}^2_{\mathrm{k}}$ and consider the intersection

  $$\mathbb{P}(\mathrm{I})_{\mathbf{y}} = \mathbb{P}(\mathrm{I}) \cap p_2^{-1}(H_1) \cap p_2^{-1}(H_2) \subset \mathbb{P}^2_{\mathbf{y}}.$$

  Under our assumptions, this intersection is a complete intersection of a line and a curve of degree $n-1$ in $\mathbb{P}^2_{\mathbf{y}}$. Moreover, since $\lambda_j \in (x_0, x_1)_1$ for all $j \in \{0,1,2\}$ and since there exists $j \in \{0,1,2\}$ such that $p_j \in (x_0,x_1)^{n-2}_{n-1} \backslash (x_0,x_1)^{n-1}_{n-1}$, this complete intersection decomposes as the union of the point $\mathbb{V}(x_0,x_1)_{\mathbf{y}} \in \mathbb{P}^2_{\mathbf{y}}$ with multiplicity $n-2$ and of another point with multiplicity 1. By the generality assumption on $\mathbf{y}$, we can assume that the subscheme $\mathbb{P}(\mathrm{I}) \cap \mathbb{P}^2_{\mathbf{y}}$ is defined by the ideal $\big(x_1, x_0^{n-2}(x_0+\alpha)\big)$ for some $\alpha \in \mathrm{k}\backslash\{0\}$. Since $\Gamma = \overline{\mathbb{P}(\mathrm{I}) \backslash \mathbb{V}(x_0,x_1)}$ is defined by the saturation $[\mathrm{I}_{\mathbb{P}(\mathrm{I})} : (x_0 : x_1)^\infty]$ of the ideal $\mathrm{I}_{\mathbb{P}(\mathrm{I})}$ of $\mathbb{P}(\mathrm{I})$ by the ideal $(x_0,x_1) = \mathrm{Fitt}_2(\mathrm{I})$, see (2.1.2), the only points of $\mathbb{P}^2_k$ over which the fiber of $\mathbb{P}(\mathrm{I})$ contributes to $d_0(\Gamma)$ are those different from $\mathbb{V}(x_0,x_1)$. Thus the other point is the only element in $\Gamma \cap p_2^{-1}(H_1) \cap p_2^{-1}(H_2)$. Hence

  $$\deg_{\mathbb{P}}^{2,0} \Gamma_f = \mathrm{length}\left(\Gamma_f \cap p_2^{-1}(H_1) \cap p_2^{-1}(H_2)\right) = 1$$

  and

  $$\deg_{\mathbb{P}}^{2,0} \mathbb{T} = \deg_{\mathbb{P}}^{2,0} \mathbb{P}(\mathrm{I}) - \deg_{\mathbb{P}}^{2,0} \Gamma_f = n-2.$$

- **Case** $i = 1$**.** Since I has height 2, the linear system defined by the 2-minors of M does not have fixed components so $\deg_{\mathbb{P}}^{1,1} \Gamma_f = \deg_{\mathbb{P}}^{1,1} \mathbb{P}(\mathrm{I}) = n$ and thus $\deg_{\mathbb{P}}^{1,1} \mathbb{T} = 0$.

- **Case** $i = 2$**.** The torsion component $\mathbb{T}$ being supported over $\mathbb{V}(x_0,x_1)$, the intersection $\mathbb{T} \cap p_1^{-1}(H_1) \cap p_1^{-1}(H_2)$ of $\mathbb{T}$ with inverse images of general lines in $\mathbb{P}^2_{\mathrm{k}}$ is empty so $\deg_{\mathbb{P}}^{0,2} \mathbb{T} = 0$ and $\deg_{\mathbb{P}}^{0,2} \Gamma_f = \deg_{\mathbb{P}}^{0,2} \mathbb{P}(\mathrm{I}) - \deg_{\mathbb{P}}^{0,2} \mathbb{T} = 1$.

To sum up, $\mathbb{T}$ has multidegree $(n-2,0,0)$ and Γ_f has multidegree

$$(n-1,n,1) - (n-2,0,0) = (1,n,1).$$

□

We have the following extension of Theorem 2.0.4.

Theorem 2.2.4. (1) *Let* $n \in \mathbb{N}_{>1}$ *and assume that* $p = \mathrm{char}\,\mathrm{k}$ *divides* n*, then the near-pencil arrangements of* $n+1$ *lines is homaloidal.*

(2) *Let* $n \in \mathbb{N}_{>1}$ *and assume that* $p = \operatorname{char} \mathrm{k}$ *divides* $n(n-1)-1$*, then the curve* $\mathsf{G}_n = \mathbb{V}\big(x_0x_1(x_1^{n-1}+x_0^{n-2}x_2)\big)$ *is homaloidal.*

Proof. (1) By Lemma 2.2.1, a presentation matrix of the ideal I of partial derivatives of h verifies the conditions of Lemma 2.2.3. Hence f_{h} is birational and since the associated linear system has no fixed component, the polynomial h and $\mathsf{H}_n = \mathbb{V}(\mathsf{h})$ are homaloidal.

(2) Let $n \in \mathbb{N}_{>1}$. The ideal

$$\mathrm{I}_{f_{\mathsf{g}}} = (x_1^n + (n-1)x_0^{n-2}x_1x_2, nx_0x_1^{n-1} + x_0^{n-1}x_2, x_0^{n-1}x_1)$$

of partial derivatives of $\mathsf{g} = x_0x_1(x_1^{n-1}+x_0^{n-2}x_2)$ has presentation matrix

$$\Phi = \begin{pmatrix} nx_0 & 0 \\ -x_1 & x_0^{n-2}x_1 \\ -(n(n-1)-1)x_2 & -nx_1^{n-1} - x_0^{n-2}x_2 \end{pmatrix},$$

Indeed, $\mathrm{I}_{f_{\mathsf{g}}}$ has height 2 for otherwise x_0 or x_1 would divide $x_1^n + (n-1)x_0^{n-2}x_1x_2$ and $nx_0x_1^{n-1} + x_0^{n-1}x_2$ which is not the case. Moreover

$$\begin{cases} x_1^n + (n-1)x_0^{n-2}x_1x_2 = M_1 \\ nx_0x_1^{n-1} + x_0^{n-1}x_2 = M_2 \\ x_0^{n-1}x_1 = M_3, \end{cases}$$

where given $j \in \{1,2,3\}$, Φ_j is equal to $(-1)^j$ times the minor obtained from Φ by leaving out the j^{th} row. Hence $\mathrm{I}_{f_{\mathsf{g}}}$ is a determinantal ideal and, by application of Hilbert-Burch theorem, $\Phi_{f_{\mathsf{g}}}$ is a minimal presentation matrix of $\mathrm{I}_{f_{\mathsf{g}}}$. Now, if p divides $n(n-1)-1$, the matrix M verifies the conditions of Lemma 2.2.3. Thus, in this case, f_{g} is birational so g and $\mathsf{G}_n = \mathbb{V}(\mathsf{g})$ are homaloidal.

□

Remark 2.2.5. The method of reduction modulo p we just described also applies to Example 2.0.2 and to the quintic $Q_5 = \mathbb{V}\big(x_0(x_1^2+x_0x_2)(x_1^2+x_0x_2+x_0^2)\big)$ described in [BC18].

2.2.3 Limits and perspectives

The fact that the presentation matrix of the ideal of partial derivatives reduces well modulo p does not always occur, as illustrated by the following example.

Example 2.2.6. Let $\mathsf{h} = x_2(x_1^4 - 2x_0x_1^2x_2 + x_0^2x_2^2 - x_1x_2^3) \in \mathrm{k}[x_0,x_1,x_2]$. Its zero locus in $\mathbb{P}^2_{\mathrm{k}}$ is the union of the unicuspidal ramphoïd quartic with the tangent cone at its cusp, see [Moe08]. Over a field k of characteristic 0, a computation with Macaulay2 shows that a presentation matrix of the ideal I of partial derivatives of h reads:

$$\begin{pmatrix} 15x_1^2 + 3x_0x_2 & 72x_0x_1 + 15x_2^2 \\ 8x_1x_2 & 2x_1^2 + 30x_0x_2 \\ -2x_2^2 & -8x_1x_2 \end{pmatrix}.$$

We can a priori not expect to apply Lemma 2.2.3 after reduction modulo p. However, after reducing modulo 3, a presentation matrix of the reduction of I modulo 3 reads

$$\begin{pmatrix} 0 & x_1^3 - x_0x_1x_2 - x_2^3 \\ x_1 & x_0x_2^2 \\ -x_2 & -x_1x_2^2 \end{pmatrix}.$$

This implies that the polar map of h is birational by Lemma 2.2.3 (here, remark that the torsion is supported on $\mathbb{V}(x_1, x_2)$ and that the maximal power of x_0 is 1 is the second column). By application of Hilbert-Burch theorem, we also have that the induced linear system does not have fix components so h is actually homaloidal.

Remark 2.2.7. As pointed out by Example 2.2.6 and Item (2) of Theorem 2.2.4, the classification of homaloidal plane curves in any characteristic seems to be a challenging problem, especially by only looking to the reduction modulo p of the syzygies of the jacobian ideal. One can however restrict first to the classification of homaloidal line arrangements and this is the object of next section.

2.3 Classification of homaloidal line arrangements in positive characteristic

As a guideline for the section, let us state first our result about the classification of homaloidal line arrangements.

Proposition 2.3.1. *Given an algebraically closed field* k *of characteristic* $p > 0$*, the only homaloidal line arrangements are:*

(i) the union of three general lines,

(ii) the near-pencils of $n + 1$ *lines where* p *divides* n*.*

Our proof of Proposition 2.3.1 mainly relies on the observation that, as far as the topological degree of the polar map of an arrangement is concerned, the only quantity to consider is the numbers of lines defining the singularities of the arrangement. More precisely, a singularity z of a line arrangement $\mathcal{A} = \mathbb{V}(\mathsf{h})$ being the intersection of $r \geqslant 2$ lines of $\mathcal{A}$, the numerical contribution of z in the computation of $d_0(f_\mathsf{h})$ only depends on whether the characteristic p divides r or not, see Lemma 2.3.2 for our complete result. Given this fact, the proof of Proposition 2.3.1 aims to characterize combinatorially near-pencils of $p + 1$ lines among all arrangements of $p+1$ lines and this combinatorial characterization follows from [dBE48, Th.1].

In the following, given an integer $d \geqslant 4$, we let $\mathsf{h} = l_1 \cdots l_d$ be the product of d homogeneous linear polynomials $l_1, \ldots, l_d \in \mathrm{k}[x_0, x_1, x_2]$ and $\mathcal{A} = \mathbb{V}(\mathsf{h})$ be the line arrangement defined by h. Moreover, using the designation in [Hir83], a point z in the singular locus of $\mathcal{A}$ which is the intersection point of r lines is called a *r-fold point*.

The field k being algebraically closed, the topological degree $d_0(f_\mathsf{h})$ of f_h is the degree of the fiber of a generic point of $\mathbb{P}^2$, that is:

$$d_0(f_\mathsf{h}) = \deg \mathbb{V}(\mathrm{I}^g : \mathrm{I}^\infty)$$

where $\mathrm{I} = (\frac{\partial \mathsf{h}}{\partial x_0}, \frac{\partial \mathsf{h}}{\partial x_1}, \frac{\partial \mathsf{h}}{\partial x_2})$ is the jacobian ideal of h, $\mathrm{I}^g = (a\frac{\partial \mathsf{h}}{\partial x_0} + b\frac{\partial \mathsf{h}}{\partial x_1} + c\frac{\partial \mathsf{h}}{\partial x_2}, \alpha\frac{\partial \mathsf{h}}{\partial x_0} + \beta\frac{\partial \mathsf{h}}{\partial x_1} + \gamma\frac{\partial \mathsf{h}}{\partial x_2})$ is the ideal defined by two generic linear combinations of $\frac{\partial \mathsf{h}}{\partial x_0}, \frac{\partial \mathsf{h}}{\partial x_1}, \frac{\partial \mathsf{h}}{\partial x_2}$ and $\mathrm{I}^g : \mathrm{I}^\infty$ stands for the saturation ideal of I^g by I, see [Dol11, 7.1.3] for this computation of the topological degree. Since $\mathbb{V}(\mathrm{I}^g : \mathrm{I}^\infty)$ is set-theoretically equal to $\mathbb{V}(\mathrm{I}^g)\backslash\mathbb{V}(\mathrm{I})$, one has thus:

$$d_0(f_{\mathsf{h}}) = (d-1)^2 - \sum_{z \in \mathbb{V}(\mathrm{I})} m_z \tag{2.3.1}$$

where m_z is the multiplicity of z in the scheme $\mathbb{V}(\mathrm{I}^g)$ (note that this latter expression of $d_0(f_{\mathsf{h}})$ is true for any reduced plane curve $\mathbb{V}(\mathsf{h})$ and not only for line arrangements). Over the field of complex numbers $\mathbb{C}$, by [Dim17b, 4.2], given an r-fold point $z \in \mathbb{V}(\mathrm{I})$ one has

$$m_z = \mu_{\mathsf{h},z} = (r-1)^2$$

where $\mu_{\mathsf{h},z}$ stands for the local Milnor number of $\mathcal{A}$ at z, see [Dim17b, Definition 2.17] for the definition of Milnor numbers. Over a field of positive characteristic, the relation between $d_0(f_{\mathsf{h}})$ and Milnor numbers of the singularities of $\mathcal{A}$ is much blurred, in particular because Milnor number is not an invariant under contact equivalence anymore (see [HRS19] for the definition of contact equivalence and more precision about the definition of Milnor number in positive characteristic). In other words, over a field k of positive characteristic, Equation (2.3.1) is still valid by definition but the numbers m_z cannot be interpreted as the Milnors numbers of the singularities defined by h (even if we won't need it, let us however precise that the numbers m_z appeared to be related to the *Milnor number of a hypersurface* $\mu(\mathcal{O}_{\mathsf{h}})$, a contact equivalent invariant defined in [HRS19, end of section 3]. We also point out that typical behaviors of singularities in positive characteristic prevent the classification of homaloidal polynomials via Dolgachev's approach in [Dol00, Lemma 3] over $\mathbb{C}$, see [MHW01] and [Ngu16] for instances of such behaviors when reducing modulo p).

Lemma 2.3.2. *Let $p = \mathrm{char}(\mathrm{k})$ and z an r-fold point of a line arrangement $\mathcal{A} = \mathbb{V}(\mathsf{h})$, $\mathsf{h} = l_1 \cdots l_d$. Denote by m_z the multiplicity of z in $\mathbb{V}(\mathrm{I}^g)$ as in Equation (2.3.1):*

(1) *if p divides r, then $m_z = (r-1)^2 + (r-2)$,*

(2) *if p does not divide r, then $m_z = (r-1)^2$.*

Proof. To describe m_z, we first explain why it is enough to make the computation in the case that $\mathcal{A}$ is a near-pencil of $r+1$ lines such that z is the intersection point of r lines. Once we have our local model for z, we use Theorem 2.2.4 to compute m_z.

Let $\mathcal{A} = \mathbb{V}(\mathsf{h})$, $\mathsf{h} = l_1 \cdots l_d$, write $z = (0:0:1)$ by choosing coordinates of $\mathbb{P}^2_{\mathrm{k}}$ and label the linear polynomials $l_1, \ldots, l_d$ defining $\mathcal{A}$ such that $l_1, \ldots, l_r \in (x_0, x_1)$ and $l_{r+1}, \ldots, l_d \in (x_0, x_1)^c$, $(x_0, x_1)^c$ being the complementary of the ideal (x_0, x_1).

Now, by applying the elementary rules of derivations, one has that a generic linear combination $a\frac{\partial \mathsf{h}}{\partial x_0}+b\frac{\partial \mathsf{h}}{\partial x_1}+c\frac{\partial \mathsf{h}}{\partial x_2}$ of $\frac{\partial \mathsf{h}}{\partial x_0}, \frac{\partial \mathsf{h}}{\partial x_1}, \frac{\partial \mathsf{h}}{\partial x_2}$ reads:

$$a\frac{\partial \mathsf{h}}{\partial x_0}+b\frac{\partial \mathsf{h}}{\partial x_1}+c\frac{\partial \mathsf{h}}{\partial x_2} = (\sum_{j=1}^{r}(l_1\cdots l_{j-1}(a\frac{\partial l_j}{\partial x_0}+b\frac{\partial l_j}{\partial x_1})l_{j+1}\cdots l_r)(l_{r+1}\cdots l_d)$$
$$+(l_1\cdots l_r)(\sum_{j=r+1}^{d}(l_{r+1}\cdots l_{j-1}(a\frac{\partial l_j}{\partial x_0}+b\frac{\partial l_j}{\partial x_1}+c\frac{\partial l_j}{\partial x_2})l_{j+1}\cdots l_d)$$

In the localization $\mathrm{k}[x_0,x_1,x_2]_{(x_0,x_1)}$ of $\mathrm{k}[x_0,x_1,x_2]$ at the prime (x_0,x_1), remark that $(\sum_{j=r+1}^{d}(l_{r+1}\cdots l_{j-1}(a\frac{\partial l_j}{\partial x_0}+b\frac{\partial l_j}{\partial x_1}+c\frac{\partial l_j}{\partial x_2})l_{j+1}\cdots l_d)$ is a unit since $(a:b:c)\in\mathbb{P}^2_{\mathrm{k}}$ is generic.

Hence denoting $\overline{v}\in\mathrm{k}[x_0,x_1,x_2]_{(x_0,x_1)}$ the localization of $v\in\mathrm{k}[x_0,x_1,x_2]$ at the prime (x_0,x_1), one has:

$$\overline{a\frac{\partial \mathsf{h}}{\partial x_0}+b\frac{\partial \mathsf{h}}{\partial x_1}+c\frac{\partial \mathsf{h}}{\partial x_2}} = u\times(\overline{(\sum_{j=1}^{r}(l_1\cdots l_{j-1}(a'\frac{\partial l_j}{\partial x_0}+b'\frac{\partial l_j}{\partial x_1})l_{j+1}\cdots l_r)l_{r+1}}$$
$$\overline{+c'(l_1\cdots l_r)})$$

where u is a unit of $\mathrm{k}[x_0,x_1,x_2]_{(x_0,x_1)}$. In other words, $(\mathrm{I}^g)_{(x_0,x_1)}$ is equal to $(a'\frac{\partial \mathsf{h}'}{\partial x_0}+b'\frac{\partial \mathsf{h}'}{\partial x_1}+c'\frac{\partial \mathsf{h}'}{\partial x_2},\alpha'\frac{\partial \mathsf{h}'}{\partial x_0}+\beta'\frac{\partial \mathsf{h}'}{\partial x_1}+\gamma'\frac{\partial \mathsf{h}'}{\partial x_2})_{(x_0,x_1)}$ where

$$\mathsf{h}'=l'_1\cdots l'_r l'_{r+1}$$

such that $l'_1\cdots l'_r\in(x_0,x_1)$ are distinct lines passing by $z=(0:0:1)$, $l'_{r+1}\in(x_0,x_1)^c$ and $(a':b':c')$, $(\alpha':\beta':\gamma')$ are generic in $\mathbb{P}^2_{\mathrm{k}}$.

Hence, to compute the multiplicity m_z of the component supported at the r-fold point $z=(0:0:1)$ of $\mathbb{V}(\mathrm{I}^g)$, it is enough to consider that z is the r-fold point of a near-pencil of $r+1$ lines.

We treat now the case $r=2$. The near-pencil $\mathcal{A}'=\mathbb{V}(l_1l_2l_3)$ has three singular point z,z_2,z_3 and is always homaloidal, thus:

$$d_0(f_{l_1l_2l_3})=4-m_z-m_2-m_3=1$$

and since $m_z,m_2,m_3\geqslant 1$, one has thus $m_z=m_2=m_2=1$. This ends the proof of Lemma 2.3.2 in the case $r=2$ whether $p=2$ or not.

Consider now the case $r>2$. Put $\mathcal{A}'=\mathbb{V}(\mathsf{h}')$ for the near-pencil of $r+1$ lines:

(1) By Theorem 2.2.4, if $p|r$, then

$$d_0(f_{\mathsf{h}'})=r^2-m_z-\sum_{i=1}^{r}m_{z_i}=1$$

where $z_1,\dots,z_n$ are the r singularities defined by the intersection of the r lines of the pencil and the other extra line. These r singularities are all 2-fold points so, from the case $r=2$, $m_{z_i}=1$ for all $i=1,\dots,r$ and $m_z=r^2-r-1=(r-1)^2+(r-2)$.

(2) If $p \nmid r$, then $d_0(f_{\mathsf{h}'}) = r-1$ by Proposition 2.2.2 (case $n = r$ of Proposition 2.2.2). Hence

$$d_0(f_{\mathsf{h}'}) = r^2 - m_z - \sum_{i=1}^{r} m_{z_i} = r - 1$$

where $z_1, \ldots, z_n$ are the r singularities defined by the intersection of the r lines of the pencil and the other extra line. As in the previous case $m_{z_i} = 1$ for all $i = 1, \ldots, r$ so $m_z = (r-1)^2$.

□

Proof of Proposition 2.3.1. As it is stated at the beginning of [Hir83], given any line arrangement $\mathcal{A} = \mathbb{V}(\mathsf{h})$ of d lines, one has the combinatorial identity:

$$\frac{d(d-1)}{2} = \sum_{r=2}^{d} t_r \frac{r(r-1)}{2}$$

where t_r is the number of r-fold point defined by $\mathcal{A}$. This identity can be re-write as:

$$(d-1)^2 - \sum_{r=2}^{d} t_r (r-1)^2 - \sum_{r=2}^{d} t_r (r-2) = 1 + (\sum_{r=2}^{d} t_r - d) \qquad (2.3.2)$$

Hence, assuming that the characteristic p of k divides all $r \geqslant 3$ such that $t_r \neq 0$, one has by Lemma 2.3.2:

$$d_0(f_{\mathsf{h}}) = 1 + (\sum_{r=2}^{d} t_r - d).$$

Remark that a near-pencils of d lines verifies $\sum_{r=2}^{d} t_r = d$ so, if p divides all $r \geqslant 3$ such that $t_r \neq 0$, showing Proposition 2.3.1 aims to show that the identity $\sum_{r=2}^{d} t_r = d$ characterizes near-pencils of d lines among all arrangements of d lines and this fact is established in [dBE48, Theorem 1] (see also [Beu95, Theorem 5.1] for a more recent treatment).

In case p does not divide at least one $r \geqslant 3$ such that $t_r \neq 0$, then by Item (2) of Lemma 2.3.2, Equation (2.3.2) implies that $d_0(f_{\mathsf{h}}) > 1$ so h is not homaloidal. □

Chapter 3

Equations of the graph of determinantal Cremona maps defined by an almost linear Hilbert-Burch matrix

Introduction

In this chapter, we focus on determinantal maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose Hilbert-Burch matrix Φ_f is *almost linear* i.e. Φ_f a $(n+1) \times n$ matrix such that all but one columns are filled by homogeneous linear polynomials, the remaining column being filled by homogeneous polynomials of degree $d \geqslant 2$.

As we already mentioned in the introduction, the equations of the graph of a determinantal plane map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by an almost linear Hilbert-Burch matrix can be computed via a determinantal procedure starting from the equations of the Proj $\mathbb{P}(\mathrm{I}_f) = \mathrm{Proj}\big(\mathrm{Sym}(\mathrm{I}_f)\big)$ of the base ideal I_f of f, provided that the presentation matrix Φ_f of I_f has a prescribed form, see [CHW08] and [BCRD20].

We present now another approach describing the equations of the graph of almost linear determinantal maps. Recall that we always consider $\mathbb{P}(\mathrm{I}_f)$ embedded in $\mathbb{P}^n \times \mathbb{P}^n$ by the mean of Φ_f in which case the equations of $\mathbb{P}(\mathrm{I}_f) \subset \mathbb{P}^n \times \mathbb{P}^n$ are the entries in the line matrix $(y_0 \ \dots \ y_n)\Phi_f$, $y_0, \dots, y_n$ being the variables of the target space $\mathbb{P}^n$ of f. Denoting Φ'_f the $(n+1) \times (n-1)$ sub-matrix of Φ_f defined by the columns of linear entries of Φ_f, the idea consists in expressing the graph Γ_f of f as a divisor on the complete intersection BiProj(A) where $\mathrm{A} = \mathrm{R}[y_0, \dots, y_n]/(y_0 \ \dots \ y_n)\Phi'_f$ and $(y_0 \ \dots \ y_n)\Phi'_f$ is the ideal generated by the entries in the line matrix $(y_0 \ \dots \ y_n)\Phi'_f$. In good cases, that is when the *divisor class group* Div(A) of BiProj(A), which is the group of all divisors of BiProj(A) identified via rational equivalence, is cyclic, Γ_f as a divisor is thus a multiple of the generator of Div(A). As we will explain, it completely describes the equations of Γ_f.

This approach was developed in [KPU09] and in [KPU11] where were in particular described the graph of maps $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$. In this present work, this method is adapted in order to have an insight on the equations of the graph of almost linear determinantal spatial maps $f : \mathbb{P}^3 \to \mathbb{P}^3$. As we will see, it is particularly well suited because one can classify the linear part $\Phi'_f \in \mathrm{R}^{4\times 2}$ of the matrix $\Phi_f \in \mathrm{R}^{4\times 3}$ via normal forms of pencil of matrix. From a detailed analysis of all the cases, it leads then to the classification of all associated divisor class groups. However, contrary to the case of determinantal plane maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ or $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ defined by an almost linear Hilbert-Burch matrix Φ_f, we will see that one has also to deal with non cyclic divisor class groups. In this latter case, a complementary information from the non linear column of Φ_f has to be added in order to describe the equations of the graph of f and we will give several examples of this situation.

Let us already precise that, even in restriction to cyclic divisor class groups, a main surprise was to describe two different classes of ideals of graphs of determinantal Cremona maps defined by an almost linear Hilbert-Burch matrix. A result we summarize as follow.

Proposition 3.0.1. *Let $d \geqslant 2$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a dominant determinantal map such that:*

(i) the Hilbert-Burch matrix Φ_f of f reads

$$\Phi_f = \begin{pmatrix} x_0 & x_3 & \phi_{03} \\ x_1 & 0 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_2 & \phi_{33} \end{pmatrix}$$

with $\phi_{i3} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \dots, 3\}$. Then the ideal of Γ_f is minimally generated by one element in the following bi-degree:

$$(d, 1), (d-1, 2)(d-2, 3), \dots, (1, d)$$

and two extra other generators in bidegree $(1, 1)$.

(ii) the Hilbert-Burch matrix Φ_f of f reads

$$\Phi_f = \begin{pmatrix} x_0 & x_2 & \phi_{03} \\ x_1 & x_3 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_1 & \phi_{33} \end{pmatrix}$$

with $\phi_{i3} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \dots, 3\}$. Then the ideal of Γ_f is minimally generated by the elements in the following bi-degree if d is even:

$$(d, 1), 2(d-1, 2), 3(d-2, 3), \dots, \frac{d}{2}(\frac{d}{2}+1, \frac{d}{2}), \frac{d}{2}(\frac{d}{2}, \frac{d}{2}+1), \dots, 2(2, d-1), (d, 1)$$

and two extra other generators in bidegree $(1, 1)$ (here $m(d_1, d_2)$ means that the component of degree (d_1, d_2) of the ideal of Γ_f is minimally generated by m elements). If d is odd, Γ_f is minimally generated by the elements in the following bi-degree:

$$(d, 1), 2(d-1, 2), 3(d-2, 3), \dots, \frac{d+1}{2}(\frac{d+1}{2}, \frac{d+1}{2}), \frac{d}{2}+1), \dots, 2(2, d-1), (d, 1)$$

and two extra other generators in bidegree $(1, 1)$.

This result has to be compared to the description of the graph of determinantal plane Cremona map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose Hilbert-Burch matrix is almost linear:

Proposition. *[BCRD20, Th.5.12] Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be an almost linear determinantal map whose Hilbert-Burch matrix Φ_f is almost linear and reads*

$$\Phi_f = \begin{pmatrix} x_0 & \phi_{02} \\ x_1 & \phi_{12} \\ 0 & \phi_{22} \end{pmatrix}$$

with $\phi_{i2} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \ldots, 2\}$ and such that it exists $i \in \{0, 1, 2\}$ such that $\phi_{i2} \notin (x_0, x_1)^d$.

Then the ideal of Γ_f is minimally generated by one element in the following bi-degree:

$$(d, 1), (d-1, 2)(d-2, 3), \ldots, (1, d)$$

and one extra other generators in bidegree $(1, 1)$.

In another words, from the planar case, one might have only expected Item (i) in Proposition 3.0.1 with only one generator by bi-degree. As we will explain, Item (i) can also be described from a determinantal procedure as in [BCRD20]. Moreover, by applying the jacobian dual criterion [SUV94] or [RS01, Proposition 2.1], one has that the maps considered in Proposition 3.0.1 are Cremona maps.

Contents of the chapter

In Section 3.1, we recall briefly the main notions we need about the divisor class group of a scheme. We recall also Kronecker's results about the normal forms of pencils of matrix and we explain how they lead to a classification of the linear submatrices Φ'_f of the almost linear determinantal maps. In the end, a description of the divisor class group can be done in all the cases and not only in the cyclic cases. However, for a more concise presentation, we only focus on the matrix Φ'_f associated to cyclic divisor class group in Section 3.2 resulting in the proof of Proposition 3.0.1. In the last section, we briefly set foot on the non cyclic cases in order to describe a last non expected ideal of an almost linear determinantal Cremona map, see Example 3.3.1.

Let me mention again my gratitude to Claudia Polini and Bernd Ulrich with who was carried out this work at the end of winter 2020. Let me also mention that I chose to present here only a part of the all the results and conclusion obtained during this collaboration, especially regarding the non cyclic cases.

3.1 Divisor class groups and their description

3.1.1 Generators of divisor class groups

We follow here the presentation given in [BH93, p315]. Let A be a Noetherian normal domain. Recall that a *fractionary ideal* I of A is a submodule of the field of fraction Frac(A) of A for which there exists a non zero elements $a \in$ Frac(A) such that $aI \subset$ A (see [HS06, Def. 2.4.4] for this definition of fractionary ideal) and that a fractionay ideal is *divisorial* if it is a reflexive A-module (one has that $\mathfrak{p} \in \mathrm{Spec}(\mathrm{A})$ is divisorial if and only if $\mathrm{codim}(\mathfrak{p}) = 1$).

The *divisor class group* $\mathrm{Cl}(\mathrm{A})$ of A is the set of isomorphism classes of fractionary ideal I, denoted $[I]$, with law group:

$$\forall [I], [J] \in \mathrm{Cl}(\mathrm{A}),\ [I] + [J] = [(IJ)^*]$$

where $-^*$ denotes the A-dual $\mathrm{Hom}(-, \mathrm{A})$.

One has $[I] = 0$ if and only if I is principal (note that the right to left implication can be directly deducted from the law group) so in particular, A is factorial if and only if $\mathrm{Cl}(\mathrm{A}) = \{0\}$ using the property that any codimension 1 ideal of a factorial domain is principal[Eis95, Corollary 10.6].

For the computation we will carry out, a central tool is the following result:

Theorem 3.1.1 (Nagata's theorem)**.** *If $S \subset \mathrm{A}$ is a multiplicative closed subset of* A*, then assignments $[I] \to [I\mathrm{A}S^{-1}]$ maps $\mathrm{Cl}(\mathrm{A})$ surjectively onto $\mathrm{Cl}(\mathrm{A}S^{-1})$; the kernel of this map is generated by classes $[\mathfrak{p}]$ of the divisorial prime ideals $\mathfrak{p}$ with $\mathfrak{p} \cap S \neq \emptyset$.*

Notice that, assuming $\mathrm{Cl}(\mathrm{A}) =< g >= \{ng,\ n \in \mathbb{Z}\}$ is cyclic generated by an element $g \in \mathrm{Cl}(\mathrm{A})$, the definition of the law group implies to not consider the embedded components of the ordinary power of g, that is:

$$\forall n \in \mathbb{Z},\ \underbrace{ng}_{n\ times\ g} = g^{(n)}$$

where $g^{(n)}$ stands for the n-th symbolic power of g. Recall that the n-th symbolic power of prime ideal P in a Noetherian ring is the P-primary component of P^n and the n-th symbolic power of an ideal I is the intersection of the n-th symbolic powers of the minimal primes associated to I (see [Eis95, 3.9] and [DdSH+18] for more about symbolic powers of ideals). As we are going to explain in Section 3.2, the two main behaviors we describe in Proposition 3.0.1 are direct consequences of the facts the generator g of the class group verifies $g^{(2)} = g^2$ or $g^{(2)} \neq g^2$.

3.1.2 Classification of rational normal scrolls via normal forms of matrices' pencils

For the rest of the section, we focus on 4×2 matrices $\phi' \in \mathrm{R}^{4\times 2}$ with homogeneous linear entries ($\mathrm{R} = \mathrm{k}[x_0, x_1, x_2, x_3]$).

Given two extra indeterminates T_0, T_1, remark that we can write

$$\begin{pmatrix} T_0 & T_1 \end{pmatrix} (\phi'^t) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} M$$

where $M \in \mathrm{k}[T_0, T_1]^{4\times 4}$ is a 4×4-matrix with homogeneous linear entries in T_0, T_1. In other words

$$M = y_0 M' + y_1 M''$$

where $M_1, M_2 \in \mathrm{k}^{4\times 4}$ is a pencil of matrix that has a *Kronecker canonical form* (see [KPU11] and [BTW16, Th. 4.2]), that is it exists common changes of basis of

k^4 such that the linear map defined by M' and M'' have matrix M_1 and M_2 which are block-diagonal with blocks of the following forms:

$$T_0\operatorname{Id} + T_1 J,\ T_0 N + T_1 \operatorname{Id}\ T_0 K + T_1 L,\ T_0 K^t + T_1 N^t$$

where for $k \in \{0, \dots, 4\}$,

$$\operatorname{Id} = \underbrace{\left.\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 1\end{pmatrix}\right\} k}_{k}$$

$$N = \underbrace{\left.\begin{pmatrix}0 & \dots & \dots & \dots & 0\\ 1 & \ddots & & & \vdots\\ 0 & \ddots & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 1 & 0\end{pmatrix}\right\} k}_{k} \qquad J = \underbrace{\left.\begin{pmatrix}\lambda & 1 & 0 & \dots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \dots & \dots & 0 & \lambda\end{pmatrix}\right\} k}_{k}$$

$$K = \underbrace{\left.\begin{pmatrix}1 & 0 & \dots & \dots & 0\\ 0 & \ddots & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 1 & 0\end{pmatrix}\right\} k-1}_{k} \qquad L = \underbrace{\left.\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & \dots & \dots & 0 & 1\end{pmatrix}\right\} k-1}_{k}$$

Regarding a classification of the linear submatrices $\Phi'_f \in R^{4\times 2}$ in the almost linear Hilbert-Burch matrix $\Phi_f \in R^{4\times 3}$, these canonical forms are a good way to distinguish between the associated determinantal maps f since the linear changes of basis involved do not modify the properties of the associated maps.

Let us present the classification of matrices Φ_f with respect to the codimension of the subscheme $\mathbb{V}\big(\mathrm{I}_1(\Phi'_f)\big)$ defined by the entries of Φ'_f. We set aside the case $\operatorname{codim}(\mathbb{V}\big(\mathrm{I}_1(\Phi'_f)\big) = 2$ since all the associated maps f are not dominant so in particular not Cremona. Indeed, if only two variables, say x_0, x_1 appear in Φ'_f, there is at least one generator of degree 0 in the $\mathbf{x}$-variables generating the ideal of the graph Γ_f of f by [KPU11] so f cannot be dominant.

We describe now the cases corresponding to $\operatorname{codim}\mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big) = 3$. Via the left action (resp. right action) of $\mathrm{Gl}_4(k)$ (resp. $\mathrm{Gl}_2(k)$) on Φ'_f, only three variables appears in Φ'_f. Hence one can write $M = \begin{pmatrix}\mu\\ 0\,0\,0\,0\end{pmatrix}$ where μ is a 3×4 matrix of the canonical form in the following table where we write on the right the associated matrix Φ'_f via the relation $(x_0\ x_1\ x_2\ x_3)M = (T_0\ T_1)\Phi_f^t$. When possible we do another change of basis in order to have x_0 (resp. x_1) as the first (resp. second) entry of the first column of Φ_f.

	μ	Φ'_f
3.a	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$
3.b	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & 0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ 0 & x_0 \\ 0 & x_1 \end{pmatrix}$
3.c	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & 0 & T_0+\lambda T_1 & T_1 \\ 0 & 0 & 0 & T_0+\lambda T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & \lambda x_1 \\ x_2 & x_1+\lambda x_2 \end{pmatrix}$ $\lambda \in \mathrm{k}$
3.d	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & 0 & \alpha T_0+\beta T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ \alpha x_2 & \beta x_2 \end{pmatrix}$ $\alpha, \beta \in \mathrm{k}$
3.e	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & 0 & \alpha T_0+\beta T_1 & 0 \\ 0 & 0 & 0 & \delta T_0+\epsilon T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ \alpha x_1 & \beta x_1 \\ \delta x_2 & \epsilon x_2 \end{pmatrix}$ $\alpha, \beta, \delta, \epsilon \in \mathrm{k}$

Observe that the transpose of the previous matrices N provides canonical form such that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big) = 4$. We begin by these matrices the following classification of cases such that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big) = 4$.

	M	Φ'_f
4.a	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ 0 & T_1 & T_0 & 0 \\ 0 & 0 & T_1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \\ 0 & 0 \end{pmatrix}$
4.b	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & T_1 & T_0 & 0 \\ 0 & 0 & T_1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \\ 0 & x_1 \\ 0 & 0 \end{pmatrix}$
4.c	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & T_0+\lambda T_1 & 0 & 0 \\ 0 & T_1 & T_0+\lambda T_1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3+\lambda x_2 \\ x_3 & \lambda x_3 \\ 0 & 0 \end{pmatrix}$ $\lambda \in \mathrm{k}$
4.d	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & \alpha T_0+\beta T_1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ \alpha x_3 & \beta x_3 \\ 0 & 0 \end{pmatrix}$ $\alpha, \beta \in \mathrm{k}$
4.e	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & \alpha T_0+\beta T_1 & 0 & 0 \\ 0 & 0 & \delta T_0+\epsilon T_1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_1 \\ \alpha x_2 & \beta x_2 \\ \delta x_3 & \delta x_3 \\ 0 & 0 \end{pmatrix}$ $\alpha, \beta, \delta, \epsilon \in \mathrm{k}$

	M	Φ'_f
4.f	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & \alpha T_0+\beta T_1 & 0 \\ 0 & 0 & 0 & \delta T_0+\epsilon T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ \alpha x_2 & \beta x_2 \\ \delta x_3 & \epsilon x_3 \end{pmatrix}$ $\alpha,\beta,\delta,\epsilon\in \mathrm{k}$
4.g	$\begin{pmatrix} T_0+\lambda T_1 & T_1 & 0 & 0 \\ 0 & T_0+\lambda T_1 & 0 & 0 \\ 0 & 0 & \alpha T_0+\beta T_1 & 0 \\ 0 & 0 & 0 & \delta T_0+\epsilon T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0+\lambda x_1 \\ \alpha x_2 & \beta x_2 \\ \delta x_3 & \epsilon x_3 \end{pmatrix}$ $\alpha,\beta,\delta,\epsilon,\lambda\in \mathrm{k}$
4.h	$\begin{pmatrix} T_1 & 0 & 0 & 0 \\ T_0 & T_1 & 0 & 0 \\ 0 & 0 & \alpha T_0+\beta T_1 & 0 \\ 0 & 0 & 0 & \delta T_0+\epsilon T_1 \end{pmatrix}$	$\begin{pmatrix} x_1 & x_0 \\ 0 & x_1 \\ \alpha x_2 & \beta x_2 \\ \delta x_3 & \epsilon x_3 \end{pmatrix}$ $\alpha,\beta,\delta,\epsilon\in \mathrm{k}$
4.i	$\begin{pmatrix} T_0+\lambda T_1 & T_1 & 0 & 0 \\ 0 & T_0+\lambda T_1 & 0 & 0 \\ 0 & 0 & T_0+\nu T_1 & 0 \\ 0 & 0 & 0 & T_0+\nu T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0+\lambda x_1 \\ x_2 & \nu x_2 \\ x_3 & \nu x_3 \end{pmatrix}$ $\lambda,\nu\in \mathrm{k}$
4.j	$\begin{pmatrix} T_0+\lambda T_1 & T_1 & 0 & 0 \\ 0 & T_0+\lambda T_1 & 0 & 0 \\ 0 & 0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0+\lambda x_1 \\ x_3 & x_2 \\ 0 & x_3 \end{pmatrix}$ $\lambda\in \mathrm{k}$
4.k	$\begin{pmatrix} T_1 & 0 & 0 & 0 \\ T_0 & T_1 & 0 & 0 \\ 0 & 0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \\ 0 & x_0 \\ 0 & x_1 \end{pmatrix}$
4.l	$\begin{pmatrix} T_0+\lambda T_1 & T_1 & 0 & 0 \\ T_0 & T_0+\lambda T_1 & T_1 & 0 \\ 0 & 0 & T_0+\lambda T_1 & 0 \\ 0 & 0 & 0 & \alpha T_0+\beta T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0+\lambda x_1 \\ x_2 & x_1+\lambda x_2 \\ \alpha x_3 & \beta x_3 \end{pmatrix}$ $\lambda,\alpha,\beta\in \mathrm{k}$
4.m	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & T_1 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_3 \\ x_1 & 0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$
4.n	$\begin{pmatrix} T_1 & 0 & 0 & 0 \\ T_0 & T_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & 0 & \alpha T_0+\beta T_1 \end{pmatrix}$	$\begin{pmatrix} x_1 & x_0 \\ x_2 & x_1 \\ 0 & x_2 \\ \alpha x_3 & \beta x_3 \end{pmatrix}$ $\alpha,\beta\in \mathrm{k}$
4.o	$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & 0 & \alpha T_0+\beta T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & x_1 \\ x_2 & 0 \\ 0 & x_2 \\ \alpha x_3 & \beta x_3 \end{pmatrix}$ $\alpha,\beta\in \mathrm{k}$
4.p	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & 0 & T_0 & 0 \\ 0 & 0 & T_1 & T_0 \\ 0 & 0 & 0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$
4.q	$\begin{pmatrix} T_0 & T_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \\ 0 & 0 & 0 & T_0 \end{pmatrix}$	$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ x_3 & x_3 \end{pmatrix}$
4.r	$\begin{pmatrix} T_1 & 0 & 0 & 0 \\ T_0 & T_1 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \\ 0 & 0 & T_0 & T_1 \end{pmatrix}$	$\begin{pmatrix} x_1 & x_0 \\ x_2 & x_1 \\ x_3 & x_2 \\ 0 & x_3 \end{pmatrix}$

From this classification, an analysis depending on the parameters has to be lead to compute the divisor class group Div(A) of the complete intersection of coordinate ring $\mathrm{A} = \mathrm{S}/(y_0 \ \ldots ; y_3)\Phi'_f$. Focusing on the cyclic divisor class group we only treat the cases written in blue in the previous tabular.

Remark 3.1.2. Let us point out here that all the algebras $\mathrm{A} = \mathrm{S}/(y_0 \ \ldots ; y_3)\Phi'_f$ considered in the following are normal so that we can apply Nagata's theorem (Theorem 3.1.1) to describe their divisor class group. Indeed, their are Cohen-Macaulay so checking their normality is equivalent to checking that their are regular in codimension at least 2. This last condition is verified since the 2-minors ideal of the jacobian matrix of $(y_0 \ \ldots ; y_3)\Phi'_f$ has codimension greater or equal to $4 \geqslant 2$. Indeed, the jacobian matrix of $(y_0 \ \ldots ; y_3)\Phi'_f$ decomposes into two submatrices of size 2×4 each one involving separate set of variables (namely one depends only on $\mathbf{x}$ and the the other one depends only on the $\mathbf{y}$) and the 2-minors ideals of these two 2×4-matrices have codimension 2 by construction (this can also be verified via a computer in each case).

3.2 Almost linear determinantal spatial maps associated to cyclic divisor class groups

With respect to the previous table of canonical forms M and associated matrices Φ_f, let us consider the following cases:

(i) Case 3.a: $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$

(ii) Case 4.a: $\Phi'_f = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \\ 0 & 0 \end{pmatrix}$

(iii) $\lambda = \beta = 0$ and another change of variables in Case 4.g: $\Phi'_f = \begin{pmatrix} x_0 & x_3 \\ x_1 & 0 \\ 0 & x_0 \\ 0 & x_2 \end{pmatrix}$

(iv) $\lambda = 0, \alpha \neq 0$ and another change of variables in Case 4.g: $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_2 \\ 0 & x_3 \end{pmatrix}$

(v) $\alpha = \beta = \delta = \epsilon = 1$ in Case 4.h: $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_2 \\ x_3 & x_3 \end{pmatrix}$

(vi) Case 4.m with another change of variables: $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$

(vii) $\alpha = 0, \beta = 1$ and another change of variables in Case 4.n: $\Phi'_f = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_0 \\ 0 & x_1 \\ 0 & x_3 \end{pmatrix}$

Let us denote by $\mathrm{A} = \mathrm{S}/(y_0 \ \dots \ y_3)\Phi'_f$ ($\mathrm{S} = \mathrm{R}[y_0, \dots, y_3]$) the complete intersection whose ideal is generate by the entries in the line matrix $(y_0 \ \dots \ y_3)\Phi'_f$.

Proposition 3.2.1. *Let Φ'_f be one of the preceding seven matrices from Item (i) to Item (vii). Then the divisor class group $\mathrm{Cl}(\mathrm{A})$ of A is cyclic generated by a prime ideal P of codimension 1 verifying:*

$$\forall n \in \mathbb{N}^*, \ P^{(n)} = P^n.$$

Our proof or the fact that symbolic powers are equal to the ordinary powers in Proposition 3.2.1 relies on a result about the Cohen-Macaulayness of the Rees algebra of ideals having small *analytic deviation* in complete intersection algebra [HS92, Cor. 2.21]. Recall that the *analytic deviation* $\mathrm{ad}(I)$ of an ideal I is the quantity:

$$\mathrm{ad}(I) := \ell(I) - \mathrm{codim}(I)$$

where $\ell(I)$ is the *analytic spread* of I (that is the Krull dimension of the special fiber ring of I, let us refer here to [HS06, Def. 5.1.5] and [HS06, Chapter 8] for alternative definition of the analytic spread). Let us also remind [HS92, Cor. 2.21] for self-completeness.

Lemma 3.2.2. *[HS92, Cor. 2.21] Let A be a Cohen-Macaulay local ring and P be a prime ideal of A. Assume that $\mathrm{ad}(P) = 1$ and A/P is nonsingular in codimension 1. Then the following are equivalent.*

(i) $\mathrm{depth}(\mathrm{A}/P) \geqslant \dim(\mathrm{A}/P) - 1$

(ii) the Rees algebra of P is Cohen-Macaulay.

Using the implication Item (i) $\Rightarrow$ Item (ii) showing that the Rees algebra of P is Cohen-Macaulay, we use then the equivalence:

$$P^{(n)} = P^n \Leftrightarrow \ell(P_Q) < \mathrm{codim}\, Q.$$

as stated in [HS92, p.386].

As pointed out by one referee of this work, let us also mention that another argument applies in order to show that the Rees algebra of P is Cohen-Macaulay: indeed, P is Cohen-Macaulay of deviation 2 so it is *strongly Cohen-Macaulay* (i.e. the Koszul homology modules of P are all Cohen-Macaulay) of deviation at most 2 by [AH80, Supplement] which is enough to provide that $\mathcal{R}(P)$ is Cohen-Macaulay by [Hun83, Th.4.2].

Proof of Proposition 3.2.1. We give the complete proof Proposition 3.2.1 in the case Item (iii):

$$\Phi'_f = \begin{pmatrix} x_0 & x_3 \\ x_1 & 0 \\ 0 & x_0 \\ 0 & x_2 \end{pmatrix},$$

the other cases follow by applying the exact same arguments. Let $L_1 := y_0x_0+y_1x_1$ and $L_2 := y_0x_3 + y_2x_0 + y_3x_2$.

Apply first Nagata's theorem (Theorem 3.1.1) by localizing A at the prime (y_0). Hence:

$$0 \ \rightarrow \{\mathfrak{p} \text{ prime of A}, \ \mathfrak{p} \cap (y_0) = \{0\}\} \rightarrow \mathrm{Cl(A)} \rightarrow \mathrm{Cl}(\mathrm{A}_{y_0}) \rightarrow \ 0.$$

In A_{y_0}, one has that: $x_0 = -\frac{y_1x_1}{y_0}$ using L_1 and $x_3 = -\frac{-\frac{y_1y_2x_1}{y_0}+y_3x_2}{y_0}$ using L_2. Hence $\mathrm{A}_{y_0} = \mathrm{k}(y_0)[x_1,x_2,y_1,y_2,y_3]$ is factorial (being a polynomial ring) so $\mathrm{Cl}(\mathrm{A}_{y_0}) = 0$ implying that $\mathrm{Cl(A)} = \{\mathfrak{p}, \ \mathfrak{p} \cap (y_0) = \{0\}\}$.

Computing then the primary decomposition of $(L_1, L_2)/y_0$ (for instance via a software system), one has then that Cl(A) is generated by $P = (y_0, x_1)$ and $P' = (y_0, y_1)$. Moreover

$$PP' = (y_0^2, y_0y_1, x_1y_0, x_1y_1) = y_0(y_0, y_1, x_1, x_0)$$

is principal so $P' = P^{-1}$ in Cl(A) and Cl(A) is thus cyclic generated by $P = (y_0, x_1)$.

We now explain why for any $n \in \mathbb{N}$, $P^{(n)} = P^n$. In our case, since $2 \geqslant \ell(P) \geqslant \mathrm{codim}(I) = 1$ and $\ell(I)$ cannot be equal to 1 or else P would be principal whereas it is not, $\ell(P) = 2$ and

$$\mathrm{ad}(P) = 1.$$

Moreover, as it can be checked for instance from a software system, the ideal of 2-minors of the jacobian matrix of Spec(A) has codimension 3 so A is regular in codimension 2. It is the condition we need to apply Lemma 3.2.2. Since $\mathrm{depth}(\mathrm{A}/P) \geqslant \dim(\mathrm{A}/P) - 1$ (as it can also be directly computed with a software system), the Rees algebra of P is Cohen-Macaulay. The sufficient condition

$$\ell(P_Q) < \mathrm{codim}\, Q \tag{3.2.1}$$

for the symbolic powers of P to be equal to the ordinary powers of P has only to be checked at the primes describing the primary decomposition of the non regular locus of A. But since the ideal of 2-minors of the jacobian matrix of Spec(A) has codimension 3, the condition in (3.2.1) is automatically verified hence:

$$\forall n \in \mathbb{N}^*, \ P^{(n)} = P^n.$$

In the other cases, the same arguments apply mutatis mutandis:

Item (i) $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$, $P = (x_0, y_2^2 - y_1y_3)$ and $P^{-1} = (x_0, x_1, x_2)$ (inverse x_0 to apply Nagata's theorem).

Item (ii) $\Phi'_f = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \\ 0 & 0 \end{pmatrix}$, $P = (y_0, x_2^2 - x_1x_3)$ and $P^{-1} = (y_0, y_1, y_2)$ (inverse y_0).

Item (iv) $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_2 \\ 0 & x_3 \end{pmatrix}$, $P = (x_0, y_1)$ (inverse y_1).

Item (v) $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_2 \\ x_3 & x_3 \end{pmatrix}$, $P = (x_0, y_1)$ (inverse y_1).

Item (vi) $\Phi'_f = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$, $P = (y_3, y_1^2 - y_0y_2)$ and $P^{-1} = (x_0, x_1, y_3)$ (inverse y_3).

Item (vii) $\Phi'_f = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_0 \\ 0 & x_1 \\ 0 & x_3 \end{pmatrix}$, $P = (x_1, y_0)$ (inverse y_0).

□

We describe now a case where symbolic powers differ from the ordinary powers.

Proposition 3.2.3. *Let* $\Phi'_f = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \\ 0 & x_0 \\ 0 & x_1 \end{pmatrix}$ *and* $\mathrm{A} = \mathrm{S}/(y_0 \ \dots \ y_3)\Phi'_f$. *Then* $\mathrm{Cl}(\mathrm{A})$ *is cyclic generated by* $P = (y_0, x_1)$. *In addition:*

$$\mathcal{R}_S(P) = \mathrm{A}[Pt, ft^2]$$

where $\mathcal{R}_S(P)$ *stands for the symbolic Rees algebra* $\oplus_{i \geqslant 0} P^{(i)}t^i$ *of* P *and* $f = y_0x_0 - y_2x_1$.

In particular, $\begin{cases} P^{(2)} & = (P^2, f) \\ P^{(2n)} & = (P^2, f)^n \ \forall n \geqslant 1 \\ P^{(2n+1)} & = P(P^2, f)^n \ \forall n \geqslant 1 \end{cases}$

Proof. Following the same path as in the proof of Proposition 3.2.1, remark that an application of Nagata's theorem (Theorem 3.1.1) by localizing at the prime (y_0) shows that $\mathrm{Cl}(\mathrm{A})$ is generated by $P = (y_0, x_1)$.

However, contrary to the cases in Proposition 3.2.1, one has that the ideal of 2-minors of the jacobian matrix of $\mathrm{Spec}(\mathrm{A})$ has codimension 2. Actually, via a direct computation of the primary decomposition, one has that the only codimension 2 minimal prime associated to the 2-minors ideal of the jacobian matrix is $Q = (x_0, x_1, y_0, y_1)$. Hence the defect $\mathcal{R}_S(P)/\mathcal{R}(P)$ is only supported at Q since away from Q, the same arguments as in the proof of Proposition 3.2.1 applies.

Let us compute now a presentation of $\mathrm{A}[Pt, ft^2]$ via the presentation of the Rees algebra $\mathcal{R}(P)$ of P. Since a presentation of P reads $\Phi_P = \begin{pmatrix} x_0 & x_1 \\ y_1 & -y_0 \end{pmatrix}$, one has

$$\mathcal{R}(P) = \mathrm{Sym}(P) = \mathrm{A}[S, T]/(l_1, l_2)$$

where $l_1 = Sx_0 + Ty_1, l_2 = x_1U - Ty_0$ using that the latter ring is a domain (one can show that the ideal (l_1, l_2) is prime via a direct computation).

Furthermore $P^2 : f = Q$ so, via a computation of the syzygies of $fQ + P^2$, define in $\big(\mathrm{A}[S, T]/(l_1, l_2)\big)[U]$:

- $l_3 = x_0U - x_2ST - y_3T^2$
- $l_4 = x_1U - x_3ST + y_2T^2$
- $l_5 = y_0U - x_3S^2 + y_2ST$
- $l_6 = y_1U + x_2S^2 + y_3ST$.

By computation, one has that (l_3, l_4, l_5, l_6) is included in the kernel of the map

$$\begin{aligned} \big(\mathrm{A}[S, T]/(l_1, l_2)\big)[U] &\to \mathrm{A}[Pt, ft^2] \\ U &\longmapsto ft^2. \end{aligned}$$

This latter map is actually surjective because, via a computation by computer, the ideal $E = (l_1, l_2, l_3, l_4, l_5, l_6)$ is prime in $\mathrm{A}[S, T, U]$ (considering here that l_3, l_4, l_5, l_6 are in $\mathrm{A}[S, T, U]$). Since, via another computation by computer, $\mathrm{codim}_{A[S,T,U]/E}\, Q = 2 > 1$, one has that $\mathrm{A}[Pt, ft^2] = \mathcal{R}_S(P) = \mathrm{A}[S, T, U]/E$. □

Proof of Proposition 3.0.1. Given $d \geqslant 2$, consider now a matrix

$$\Phi_f = \begin{pmatrix} x_0 & x_3 & \phi_{03} \\ x_1 & 0 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_2 & \phi_{33} \end{pmatrix}$$

with $\phi_{i3} \in (x_0, x_1)^{d-1}$ for all $i \in \{0, \dots, 3\}$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the determinantal map whose Hilbert-Burch matrix is Φ_f.

Writing $\mathrm{A} = \mathrm{S}/(y_0x_0 + y_1x_1, y_0x_3 + y_2x_0 + y_3x_2)$, since $\mathrm{Cl}(\mathrm{A})$ is cyclic generated by $P = (x_1, y_0)$, the class $[\Gamma_f]$ of the graph Γ_f of f reads

$$[\Gamma_f] = \frac{H}{L} P^{(\nu)}$$

for given $\nu \in \mathbb{Z}$ and $H, L \in \mathrm{A}$ (this expression a priori in $\mathrm{Frac}(\mathrm{A})$ actually makes sense in A by definition of $\mathrm{Cl}(\mathrm{A})$).

Using that $P^{(\nu)} = P^{\nu} = (y_0^{\nu}, y_0^{\nu-1}x_1, \ldots, x_1^{\nu})$ by Proposition 3.2.1, one has that $\frac{H}{L}$ shifts the bidegree of the generators of $P^{(\nu)}$. Actually, since $L_3 = y_0\phi_{03} + y_1\phi_{13} + y_2\phi_{23} + y_3\phi_{33}$ is a generator of the Rees algebra (as the generator of the component of bidegree $(*,1)$ of the Rees algebra) we know that the bidegree is $(d,1)$ has to appear as the bidegree of a generator of $P^{(\nu)}$ and since there is no generator of bidegree $(0,*)$ in the graph of f, because f is dominant, one has necessarily $L_3 = \frac{H}{L}x_1^{\nu}$ so

$$\frac{H}{L} = \frac{L_3}{x_1^{\nu}}.$$

Hence $\nu = d-1$ and

$$[\Gamma_f] = \frac{L_3}{x_1^{d-1}}P^{(d-1)}.$$

Now, by chasing the bidegree of the generators in $\frac{L_3}{x_1^{d-1}}P^{(d-1)}$, one obtains the result about the bidegree of the generators the graph of f.

The argument is the same in the case $\Phi_f = \begin{pmatrix} x_0 & x_2 & \phi_{03} \\ x_1 & x_3 & \phi_{13} \\ 0 & x_0 & \phi_{23} \\ 0 & x_1 & \phi_{33} \end{pmatrix}$ with $\phi_{i3} \in (x_0, x_1)^{d-1}$ of degree d for all $i \in \{0, \ldots, 3\}$. The only difference comes from the expression of

$$P^{(d-1)} = (y_0^{d-1}, y_0^{d-2}x_1, y_0^{d-3}f, y_0^{d-4}x_1 f, \ldots, y_0 x_1^{d-2}, f x_1^{d-3}, x_1^{d-1})$$

where $f = y_0x_0 - y_2x_1$ by Proposition 3.2.3 which explains the number of each minimal generators in each bidegrees. □

Remark 3.2.4. Let $\Phi_f = \begin{pmatrix} x_0 & x_3 & 0 \\ x_1 & 0 & x_0^3 \\ 0 & x_0 & x_0^2x_2 \\ 0 & x_2 & x_1^2x_3 \end{pmatrix}$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ the associated Cremona map. By Proposition 3.0.1, we know that the graph Γ_f of f is minimally generated by one generator in the following bidegrees:

$$(3,1), (2,2), (1,3)$$

and two of bidegree $(1,1)$. Actually the generator of bidegree $(3,1)$ is $L_3 = y_1x_0^3 + y_2x_0^2x_2 + y_3x_1^2x_3$ (up to scalar) and the two generators of bidegree $(1,1)$ are $L_1 = y_0x_0 + y_1x_1$ and $L_2 = y_0x_3 + y_2x_0 + y_3x_2$. Remark that one can compute the other generators of the ideal of the graph of f using the equality:

$$[\Gamma_f] = \frac{L_3}{x_1^2}P^{(d-1)} = \frac{L_3}{x_1^2}(y_0^2, y_0x_1, x_1^2)$$

where $P = (y_0, x_1)$ is the generator of the divisor class group Cl(A) of A $=$ S$/(L_1, L_2)$.

For instance in A:

$$\begin{aligned} \frac{L_3}{x_1^2}y_0x_1 &= \frac{(y_1x_0^3 + y_2x_0^2x_2 + y_3x_1^2x_3)y_0y_1}{x_1^2} \\ &= \frac{-y_1^2x_0^2x_1^2 - y_1y_2x_0x_1^2x_2 + y_0y_3x_1^3x_3}{x_1^2} = -y_1^2x_0^2 - y_1y_2x_0x_2 + y_0y_3x_1x_3 \end{aligned}$$

which provides a minimal generator of bidegree $(2,2)$ of Γ_f.

Remark 3.2.5. Let $\Phi_f = \begin{pmatrix} x_0 & x_2 & 0 \\ x_1 & x_3 & x_0^4 \\ 0 & x_0 & x_0^3x_2 \\ 0 & x_1 & x_1^3x_3 \end{pmatrix}$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ the associated Cremona map. One can still use the Sylvester form to compute recursively some minimal generators of the graph of the graph of f. For instance:

$$\begin{vmatrix} x_0^3y_1 + x_0^2x_2y_2 & x_1^2x_3y_3 \\ y_0 & y_1 \end{vmatrix} = x_0^3y_1^2 + x_0^2x_2y_1y_2 - x_1^2x_3y_0y_3$$
$$\begin{vmatrix} x_0^2y_1^2 + x_0x_2y_1y_2 & -x_1x_3y_0y_3 \\ y_0 & y_1 \end{vmatrix} = x_0^2y_1^3 + x_0x_2y_1^2y_2 + x_1x_3y_0^2y_3$$
$$\begin{vmatrix} x_0^3y_1^2 + x_2y_1^2y_2 & x_3y_0^2y_3 \\ y_0 & y_1 \end{vmatrix} = x_0y_1^4 + x_2y_1^3y_2 - x_3y_0^3y_3$$

although this computation only provides a part of the equations since it lacks one other generator in bidegree $(3,2)$, for instance:

$$x_0^2x_2y_1^2 + x_0x_2^2y_1y_2 - x_1x_3^3y_0y_3 - x_0^3y_1 + (-x_0^2x_2 + x_1^2x_3)y_1y_3$$
$$x_0x_2y_1^3 + x_2^2y_1^2y_2 + x_3^2y_0^2y_3 - x_0^2y_0^2y_3 - x_1x_3y_0y_2y_3 - x_0x_2y_1y_2y_3$$

3.3 The ideal of the Rees algebra in the non cyclic case: some examples

In the cases where the divisor class group is non cyclic (from 3.b to 3.e, from 4.b to 4.f, 4.i, 4.j, 4.l, from 4.o to 4.r in the table page 48), the description of the ideal of the Rees algebra is not so direct because one has to take into account the last column of the almost linear matrix Φ_f which has non linear entries. In this section, we only illustrate the impact of this last column in an example.

Example 3.3.1. Let $\Phi_f = \begin{pmatrix} x_0 & x_2 & x_3x_0^3 + x_2x_1^3 \\ x_1 & x_0 & x_0^4 + x_1^4 \\ 0 & x_1 & x_0^3x_2 \\ 0 & 0 & x_1^3x_3 \end{pmatrix}$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the associated Cremona map. Via a computation using a computer software, one can see that $d(f) = (1,6,6,1)$ and the ideal of the graph of f is minimally generated by one element in the following bidegree:

$$(4,1),\ (2,2),\ (1,4),\ (1,1).$$

Actually in this case, denoting $L_1 = y_0x_0 + y_1x_1, L_2 = y_0x_2 + y_1x_0 + y_2x_1$ and $\mathrm{A} = \mathrm{S}/(L_1, L_2)$ one has that $\mathrm{Cl(A)}$ is generated by $P_1 = (x_1, y_0, y_1), P_2 = (x_1, x_0, x_2), P_3 = (x_1, x_0^2, y_0^2)$ and $P_1 + P_2 + P_3 = 0_{\mathrm{Cl(A)}}$ so that $\mathrm{Cl(A)}$ is difficult to compute (the class $[\Gamma_f]$ of Γ_f is then a combination of two of these generators).

Hence, contrary to cyclic cases, changing the last column of entries of degree 4 may change the list of bidegrees of the generators of the ideal of the Rees algebra of the associated map. For instance, let now $\Phi_f = \begin{pmatrix} x_0 & x_2 & x_1^3 x_3 \\ x_1 & x_0 & x_0^3 x_1 \\ 0 & x_1 & x_0^4 + x_1^3 x_3 \\ 0 & 0 & x_1^4 + x_0^3 x_1 \end{pmatrix}$ and let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the associated Cremona map (it has bidegree $(1, 5, 6, 1)$). The ideal of the graph of f is minimally generated by one element in the following bidegree:

$$(4, 1),\ (2, 2),\ (1, 3),\ (1, 1).$$

Part II

Detecting determinantal Cremona maps via convex geometry

Chapter 4

Bernstein theorem's bound on the number of solutions of a zero dimensional polynomial system

From now on, we let $\mathrm{k} = \mathbb{C}$ be the field of complex number and let $n \in \mathbb{N}^*$. As explained in the introduction, given a determinantal map $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$, we study the polynomial systems defined by the residual scheme $\mathbb{P}(\mathrm{I}_f)$ of I_f via convex geometry. Section 4.1 is dedicated to briefly present the basic material about convex geometry and mainly follow [CLO05, Chapter 7]. This approach precises also the notion of genericity, a polynomial being generic with respect to a given polytope. We additionally use Bernstein theorem to compute intersection multiplicity in order to translate the detection of plane determinantal Cremona maps as an interpolation problem in Section 5.2.

4.1 Preliminaries in convex geometry

Following [CLO05, 7. Section 4], a set $C \subset \mathbb{R}^n$ is *convex* if it contains any segments between two points in C and the *convex hull* $\mathrm{Conv}(S)$ of a subset $S \subset \mathbb{R}^n$ is the smallest convex set containing S. A *polytope* is the convex hull $\mathrm{Conv}(A)$ of a finite set $A \subset \mathbb{R}^n$ and the polytopes which are the convex hull of points with integer coordinates are called *lattice polytopes*.

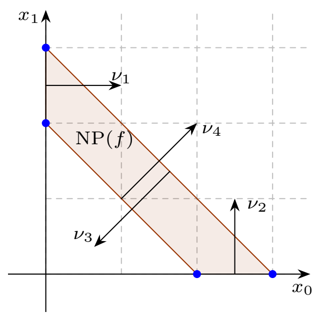
Definition 4.1.1. Given $\phi = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in \mathrm{k}[x_1, \ldots, x_n]$ where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, the *Newton polytope* of ϕ, denoted $\mathrm{NP}(\phi)$, is the lattice polytope $\mathrm{NP}(\phi) = \mathrm{Conv}\{\alpha \in \mathbb{N}^n,\ c_\alpha \neq 0\}$.

A polytope $P \subset \mathbb{R}^n$ has an n-dimensional volume $\mathrm{Vol}_n(P)$.

Example 4.1.2. Let $\phi = x_0^3 + x_0^2 x_1 + x_0 x_1^2 + x_1^3 + x_0^2 + x_0 x_1 + x_1^2$, then $\mathrm{NP}(\phi) = \mathrm{Conv}\{(3,0),(0,3),(2,0),(0,2)\}$ and $\mathrm{Vol}_2\left(\mathrm{NP}(f)\right) = 2 + \frac{1}{2} = \frac{5}{2}$.

Polytopes $P \subset \mathbb{R}^n$ have special subsets defined as follows. Given a vector $\nu \in \mathbb{R}^n \backslash \{0\}$, an affine hyperplane is defined by an equation $m \cdot \nu = -a$ and denoting $a_P(\nu) = -\min_{m \in P}\{m \cdot \nu\}$, the hyperplane of equation $m \cdot \nu = -a_P(\nu)$ is called a *supporting hyperplane* of P in which case ν is called an *inward pointing normal*. Actually, $P_\nu := P \cap \{m \in \mathbb{R}^n,\ m \cdot \nu = -a_P(\nu)\} \neq \emptyset$ [CLO05, 7.1, Ex.13] and P lies in the half space $\{m \in \mathbb{R}^n,\ m \cdot \nu \geqslant -a_P(\nu)\}$, P_ν is then called the *face* of P. The dimension of a polytope Q being the dimension of the affine space it generates, *vertices* (resp. *edges*, *facets*) of P are faces of P of dimension 0 (resp. 1, $\dim(P) - 1$). A facet lies on a unique supporting hyperplane and hence has a unique inward pointing normal up to a positive multiple so, provided P is a lattice polytope, an inward normal ν_F of a facet F of P can be re-scaled so that ν_F has integer coordinates which are moreover relatively prime. It follows that F has a unique such *primitive inward normals* $\nu_F \in \mathbb{Z}^n$.

Example 4.1.3. The polytope $\mathrm{NP}(\phi)$ and its primitive inward normals in Example 4.1.2.



Let us mention that the volume of an n-dimensional lattice polytope can be computed using its facets, namely, putting $P = \bigcap_F \{m \in \mathbb{R}^n,\ m \cdot \nu_f \geqslant -a_f\}$ where the intersection is taken over all the facet F of P whose supporting hyperplane with primitive inward normal ν_f is written $m \cdot \nu_f = -a_f$, one has:

$$\mathrm{Vol}_n(P) = \frac{1}{n} \sum_F a_F \,\mathrm{Vol}'_{n-1}(F) \tag{4.1.1}$$

where the sum is taken over all facets in P and where $\mathrm{Vol}'_{n-1}(F) = \frac{\mathrm{Vol}_{n-1}(F)}{\|\nu_F\|}$ is the *normalized volume* of F and $\|\nu_F\|$ is the euclidean length of ν_f, see [CLO05, 7.Proposition 4.6]

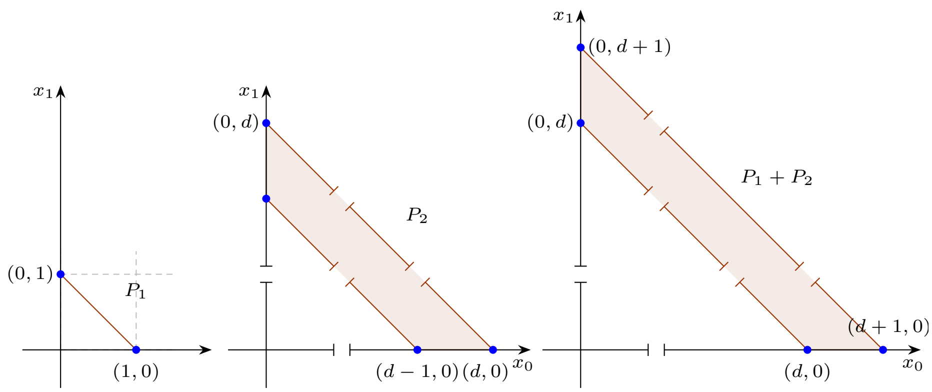
4.2 Mixed volumes and their computations

Given two polytopes $P, Q \subset \mathbb{R}^n$ and a real number $\lambda \geqslant 0$, the *Minkowski sum* of P and Q, denoted $P + Q$ is the set

$$P + Q := \{p + q,\ p \in P, q \in Q\}$$

where $p+q$ denotes the usual vector sum in $\mathbb{R}^n$ and the λP stands for the polytope $\{\lambda p,\ p \in P\}$ where λp is the usual scalar multiplication in $\mathbb{R}^n$.

Example 4.2.1. Given $d \geqslant 1$, let $P_1 = \mathrm{Conv}\{(1,0),(0,1)\}$ and $P_2 = \mathrm{Conv}\{(d,0),$ $(d-1,0),(0,d-1),(0,d)\}$. Then $P_1 + P_2 = \mathrm{Conv}\{(d+1,0),(d,0),(0,d),(0,d+1)\}$ The polytope $\mathrm{NP}(f)$ and its primitive inward normals in Example 4.1.2.



Now given any collection $P_1,\dots,P_r \subset \mathbb{R}^n$ and r non negative scalar $\lambda_1,\dots,\lambda_r \in \mathbb{R}$, then $\mathrm{Vol}_n(\lambda_1 P_1 + \dots \lambda_r P_R)$ is a homogeneous polynomial of degree n in the λ_i [CLO05, 7. Prop.4.9].

Definition 4.2.2 (mixed volume of a collection of polytopes)**.** The n-dimensional *mixed volume* $\mathrm{MV}_n(P_1,\dots,P_n)$ of given polytopes $P_1,\dots,P_n$ is the coefficient of the monomial $\lambda_1,\dots,\lambda_n$ in $\mathrm{Vol}_n(P_1,\dots,P_n)$.

For this present work and all the actual computations of mixed volumes we present, all the material we need is contained in [CLO05, 7. Th.4.12] so let us briefly re-state this toolbox:

Theorem 4.2.3. *[CLO05, 7. Th.4.12]*

(i) *The mixed volume* $\mathrm{MV}_n(P_1,\dots,P_n)$ *is invariant if the* P_i *are replaced by their images under a volume-preserving transformation of* $\mathbb{R}^n$ *(for example, a translation).*

(ii) $\mathrm{MV}_n(P_1,\dots,P_n)$ *is symmetric and linear in each variable (multilinearity of the mixed volume).*

(iii) $\mathrm{MV}_n(P_1,\dots,P_n) \geqslant 0$.

$\mathrm{MV}_n(P_1,\dots,P_n) = 0$ *if one of the* P_i *has dimension zero (i.e. if* P_i *consists of a single point).*

$\mathrm{MV}_n(P_1,\dots,P_n) > 0$ *if every* P_i *has dimension* n.

(iv) $\mathrm{MV}_n(P_1,\dots,P_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} \mathrm{Vol}_n(\sum_{i\in I} P_i)$ *where* $\sum_{i\in I} P_i$ *is the Minkowski sum of polytope.*

Example 4.2.4. Let P_1 and P_2 the polytopes of Example 4.2.1. Then:

$$\begin{aligned}\mathrm{MV}_2(P_1, P_2) &= \mathrm{Vol}_2(P_1 + P_2) - \mathrm{Vol}_2(P_1) - \mathrm{Vol}_2(P_2)\\ &= \frac{1}{2}\big(-d(d+1) + (d+1)(d+2)\big) - \frac{1}{2}\big(-(d-1)d + d(d+1)\big)\\ &= 1\end{aligned}$$

Let us quote a last tool for computing mixed volumes.

Lemma 4.2.5. *[ST10, Lemma 6] Let $n, m \in \mathbb{N}^*$ and let $P_1, \dots, P_m$ be polytopes in $\mathbb{R}^{m+n}$ and $P_{m+1}, \dots, P_{m+n}$ be polytopes in $\mathbb{R}^m \times \{0_{\mathbb{R}^n}\} \subset \mathbb{R}^{m+n}$. Then:*

$$\mathrm{MV}_{m+n}(P_1, \dots, P_{m+n}) = \mathrm{MV}_m(P_1, \dots, P_m)\, \mathrm{MV}_n(\pi_n(P_{m+1}), \dots, \pi_n(P_{m+n}))$$

where $\pi_n : \mathbb{R}^{m+n} \to \mathbb{R}^n$ stands for the projection on the last n coordinates.

See [ST10, Lemma 6] for the proof of Lemma 4.2.5. Let us now give an illustration of this result:

Example 4.2.6. Let $d \geqslant 1$ and let $P_1 = \mathrm{Conv}\{(1,0,0), (0,1,0)\}$, $P_2 = \mathrm{Conv}\{(d,0,0), (d-1,0,0), (0,d-1,0), (0,d,0)\}$ and $P_3 = \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$.

Then:

$$\mathrm{MV}_3(P_1, P_2, P_3) = \mathrm{MV}_2(P_1, P_2)\, \mathrm{MV}_1\left(\pi_1(P_3)\right) = 1 \times 1 = 1.$$

4.3 Number of solutions of a generic polynomial system and Bernstein theorem

Following [CLO05, 7 Section 5], let us first define the genericity of a polynomial with respect to a polytope.

Definition 4.3.1 (genericity with respect to a polytope)**.** Given finite set $A \subset \mathbb{Z}^n$, put $L(A) := \{\sum_{\alpha \in A_i} c_\alpha x^\alpha \in \mathrm{k}[x_1, \dots, x_n]\}$ and remark that each $L(A)$ can be considered as an affine space $\mathrm{k}^{\# A_i}$ with the coordinate c_α as coordinates.

A polynomial $\sum_{\alpha \in A_i} c_\alpha x^\alpha$ is said to be *generic with respect to* $L(A)$ if its coefficients are generic in $L(A)$.

Given $k \geqslant 1$ and finite sets $A_1, \dots, A_k \in \mathbb{Z}^n$, a property is said to *hold generically* for polynomials $(\phi_1, \dots, \phi_n) \in L(A_1) \times \dots \times L(A_k)$ is there is a non zero polynomial in the coefficients of the ϕ_i such that the property holds for all $\phi_1, \dots, \phi_n$ for which the polynomial is non vanishing, in particular if every $\phi_1, \dots, \phi_n$ is generic with respect to its own polytope $L(A_1), \dots, L(A_k)$.

Theorem 4.3.2 (Bernstein theorem)**.** *Given n polynomials in n variables $\phi_1, \dots, \phi_n \in \mathrm{k}[x_1, \dots, x_n]$ with finitely many common zeroes in $(\mathrm{k}^*)^n$, let $P_i = \mathrm{NP}(\phi_i)$. Then the number of common zeroes in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $\mathrm{MV}_n(P_1, \dots, P_n)$. Moreover if each ϕ_i is generic with respect to P_i, the number of common zero solutions is exactly $\mathrm{MV}_n(P_1, \dots, P_n)$.*

Let us refer to [CLO05, Proof of 7. Th. 5.4] and the references therein for highlights about Bernstein theorem.

Remark 4.3.3. Let $d_1, d_2, d_3 \in \mathbb{N}^*$ and consider j polynomials ϕ_1, ϕ_2, ϕ_3 such that for any $j \in \{1, 2, 3\}$, ϕ_j is generic with respect to the polytope

$$P_j = \mathrm{Conv}\{(0,0,0), (d_j, 0, 0), (0, d_j, 0), (0, 0, d_j)\} = d_j S_3$$

where $S_3 = \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ is the unit simplex of $\mathbb{R}^3$.

In this case, using the multilinearity of the mixed volume Theorem 4.2.3. Item (ii), one has:

$$\mathrm{MV}_3(P_1, P_2, P_3) = d_1 d_2 d_3 \, \mathrm{MV}_3(S_3, S_3, S_3) = d_1 d_2 d_3.$$

Thus, the system defined by ϕ_1, ϕ_2, ϕ_3 has $d_1 d_2 d_3$ distinct common zeros all lying in $(\mathrm{k}^*)^3$.

Consider now the Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 3 \\ 1 \leqslant j \leqslant 3}} \in \mathrm{R}^{4\times 3}$ ($\mathrm{R} = \mathrm{k}[x_0, \ldots, x_3]$) of a determinantal map $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ where for any $j \in \{1, 2, 3\}$, every entries ϕ_{ij} of the j-th column is generic with respect to P_j (that is to say, the de-homogenization of ϕ_{ij} with respect to a variable is generic with respect to P_j). When interested in the topological degree $d_0(f)$, remark that considering a general point $\mathbf{y} \in \mathbb{P}^3_{\mathrm{k}}$ in the target space of f is equivalent to take three polynomials generic with respect to the unit simplex S_3 (that is three polynomials whose dehomogenization with respect to a variable is generic with respect to S_3). Moreover, denoting $\phi_1^{(\mathbf{y})}, \phi_2^{(\mathbf{y})}, \phi_3^{(\mathbf{y})}$ the entries of the line matrix $(y_0 \ \ldots \ y_3)\Phi_f \subset \mathrm{S} = \mathrm{R}[y_0, \ldots, y_3]$, remark that for any $j \in \{1, 2, 3\}$, $\phi_j^{(\mathbf{y})}$ is then generic with respect to the polytope $P_j^{(6)} = P_j \times \{0_{\mathbb{R}^3}\} + \{0_{\mathbb{R}^3}\} \times S_3 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and that

$$e := \mathrm{MV}_6(P_1^{(6)}, P_2^{(6)}, P_3^{(6)}, \{0_{\mathbb{R}^3}\} \times S_3, \{0_{\mathbb{R}^3}\} \times S_3, \{0_{\mathbb{R}^3}\} \times S_3)$$

provides a bound to the number of solution $\mathbf{x} \in \mathbb{P}^3_{\mathrm{k}}$ with non zero coordinates to the system $f(\mathbf{x}) = \mathbf{y}$. In this case, using the multilinearity of the mixed volume Theorem 4.2.3. Item (ii), projection formula Lemma 4.2.5 and the vanishing condition Theorem 4.2.3. Item (iii), one has that

$$e = \mathrm{MV}_3(P_1, P_2, P_3) \, \mathrm{MV}_3(S_3, S_3, S_3) = d_1 d_2 d_3$$

which is precisely the topological degree $d_0(f)$ of f.

Chapter 5

Applications: projective degrees vs mixed volumes of determinantal maps

Introduction

The mixed volumes of the polytopes defined by the entries of Φ_f provide an original perspective on the computation of all the sequence of projective degrees

$$d(f) = \big(d_0(f), d_1(f) \dots, d_n(f)\big)$$

of f. Since Bernstein theorem states that the mixed volumes associated to a polynomial system (E) compute the solutions of (E) with non zero coordinates, see [CLO05, 7. Th. 5.4] or Theorem 4.3.2 below, the starting observation of this work is the following

Proposition 5.1.6. *Let $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ be a Koszul-determinantal map and denote by $\Phi_f = (\phi_{ij})_{\substack{0 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}}$ the presenting matrix of the base ideal I_f of f. Then:*

$$\forall k \in \{0, \dots, n\},\ d_k(f) = \mathrm{MV}_{2n}(\underbrace{S^{\mathbf{x}}_n, \dots, S^{\mathbf{x}}_n}_{k}, P^{\mathbf{y}}_1, \dots, P^{\mathbf{y}}_n, \underbrace{S^{\mathbf{y}}_n, \dots, S^{\mathbf{y}}_n}_{n-k})$$

$$\Leftrightarrow \overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f} \subset \mathbb{V}(\prod_{i=0}^{n} x_i) \subset \mathbb{P}^n_{\Bbbk} \times \mathbb{P}^n_{\Bbbk}$$

where for $l \in \{1, \dots, n\}$, $P^{\mathbf{y}}_l \subset \mathbb{R}^n \times \mathbb{R}^n$ is the Newton polytope of the l-th entry of the matrix $(y_0\ \dots\ y_n)\Phi_f$, $S^{\mathbf{x}}_n = S_n \times \{0_{\mathbb{R}^n}\} \subset \mathbb{R}^n \times \mathbb{R}^n$, $S^{\mathbf{y}}_n = \{0_{\mathbb{R}^n}\} \times S_n \subset \mathbb{R}^n \times \mathbb{R}^n$ and S_n is the unit simplex of $\mathbb{R}^n$ (see below for our convention describing the Newton polytope in $\mathbb{R}^n \times \mathbb{R}^n$ of a bi-homogeneous polynomial in $\mathbf{x}$ and $\mathbf{y}$-variables).

Moreover, using tools of combinatorial convex geometry, and in particular the projection formula Lemma 4.2.5 ([ST10, Lemma 6]) decomposing some mixed volumes as the product of mixed volumes in smaller dimension, we describe a *glued determinantal Cremona maps* $[g|g'] : \mathbb{P}^{m+n}_{\Bbbk} \dashrightarrow \mathbb{P}^{m+n}_{\Bbbk}$ starting from two determinantal Cremona maps $g : \mathbb{P}^m_{\Bbbk} \dashrightarrow \mathbb{P}^m_{\Bbbk}$ and $g' : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ of smaller projective spaces. In this framework, we are able to describe the projective degrees of some determinantal maps whose Hilbert-Burch matrix is almost linear:

Proposition 5.3.3. *Let $d \geqslant 2$ and let $\Phi_{[g|g']} = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 2+n \\ 1 \leqslant j \leqslant 2+n}}$ be such that:*

- *all the entries ϕ_{i1} of the 1-st column of $\Phi_{[g|g']}$ are general linear combinations of x_0 and x_1,*
- *all the entries ϕ_{i1} of the 2-nd column of $\Phi_{[g|g']}$ are general linear combinations of the generators of the ideal*

$$(x_0, x_1)^{d-1} \cdot (x_0, x_1, x_2) = (x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, \ldots, x_1^{d-1}x_2, x_1^d),$$

- *for all $l \in \{3, \ldots, 2+n\}$, all the entries ϕ_{il} of the l-th column of $\Phi_{[g|g']}$ are general linear combinations of $x_2, \ldots, x_{2+n}$.*

Then the glued map $[g|g'] : \mathbb{P}^{2+n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{2+n}_{\mathrm{k}}$ whose base ideal $\mathrm{I}_{[g|g']}$ is the $(m+n)$-minors ideal $\Phi_{[g|g']}$ is a determinantal Cremona map and moreover:

$$\forall k \in \{0, \ldots, 2+n\},\; d_k([g|g']n) = \binom{n}{n-k} + (d+1)\binom{n}{n-k+1} + \binom{n}{n-k+2}$$

with the convention that $\binom{j}{i} = 0$ if $i < 0$ or $i > j$. In particular, the sequence of projective degrees of $[g|g']$ is palindromic.

Contents of the chapter

In Section 5.1, we show how the computation of mixed volumes can be used to study the linear type case of determinantal maps. In this ground situation, we only show known result about the projective degrees of such maps and we shed some lights on their distribution, see for instance Example 5.1.2.

In Example 5.1.2, we focus on non linear type cases of plane determinantal Cremona map and we show how their projective degrees can still be estimated and, in some specific situations, computed by the mixed volumes associated to their Hilbert-Burch matrix, see for instance Proposition 5.2.3 and Proposition 5.2.5.

Section 5.3 is dedicated to the definition of some glued map defined via polytopes. The gluing is different as the one explained in Section 1.2 and we explain how the properties of the mixed volumes allow to describe the distribution of the projective degrees of some of such maps, see Proposition 5.3.3.

5.1 Projective degrees of Koszul-determinantal maps defined by sparse polynomials

Following Remark 4.3.3, let us first precise our notation. Given a homogeneous polynomial $\phi \in \mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$, the Newton polytope $\mathrm{NP}(\phi) \subset \mathbb{R}^n$ associated to ϕ is the Newton polytope of the de-homogenization of ϕ with respect to a variable x_i (omitted when irrelevant) and, reciprocally, given a polytope $P \subset \mathbb{R}^n$, we denote $\phi \in P$ for a homogeneous polynomial $\phi \in \mathrm{k}[x_0, \ldots, x_n]$ whose de-homogenization with respect to the variable x_i is in P. Furthermore, we consider more general Newton polytopes $P \subset \mathbb{R}^n \times \mathbb{R}^n$ corresponding to bi-homogeneous polynomials $\phi \in \mathrm{S} = \mathrm{R}[y_0, \ldots, y_n]$, that is the bi-de-homogenization of ϕ with respect to one fixed variable x_i and one fixed variable y_j $(i, j \in \{0, \ldots, n\})$ is in P.

Let us also denote by $S_n = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}),(1,\underbrace{0,\dots,0}_{n-1}),\dots,(\underbrace{0,\dots,0}_{n-1},1)\} \subset \mathbb{R}^n$ the unit simplex of $\mathbb{R}^n$ and let $S_n^{\mathbf{x}} := S_n \times \{0_{\mathbb{R}^n}\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and $S_n^{\mathbf{y}} := \{0_{\mathbb{R}^n}\} \times S_n \subset \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 5.1.1. *Let $f : \mathbb{P}^n_{\mathrm{k}} \to \mathbb{P}^n_{\mathrm{k}}$ be a determinantal map of Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}$ of syzygetic degree $(d_1,\dots,d_n)$. Then for all $k \in \{0,\dots,n\}$:*

$$d_k(f) \leqslant \mathrm{MV}_{2n}(\underbrace{S_n^{\mathbf{x}},\dots,S_n^{\mathbf{x}}}_{k}, P_1^{\mathbf{y}},\dots,P_n^{\mathbf{y}},\underbrace{S_n^{\mathbf{y}},\dots,S_n^{\mathbf{y}}}_{n-k}) \leqslant \sigma_{n-k,n}(d_1,\dots,d_n)$$

where for $l \in \{1,\dots,n\}$, $P_l^{\mathbf{y}} \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ is the Newton polytope of the l-th entry of the matrix $(y_0 \ \dots \ y_n)\Phi_f$ and $\sigma_{k,n}$ is the k-th symmetric polynomial in n-variables.

Proof. As explained in Remark 1.1.3, for all $k \in \{0,\dots,n\}$, $\deg_{\mathbb{P}}^{n-k,k} \mathbb{P}(\mathrm{I}_f)$ is the number of common zero solutions in $\subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ of the polynomials

$$l_{1,0},\dots,l_{k,0},\phi_1,\dots,\phi_n,l_{0,1},\dots,l_{0,n-k} \tag{5.1.1}$$

where $l_{1,0},\dots,l_{k,0}$ (resp. $l_{0,1},\dots,l_{0,n-k}$) are generic with respect to $S_n^{\mathbf{x}}$ (resp. generic with respect to $S_n^{\mathbf{y}}$) and $\phi_1,\dots,\phi_n \in \mathrm{S} = \mathrm{R}[y_0,\dots,y_n]$ are the entries of the matrix $(y_0 \ \dots \ y_n)\Phi_f$ (the associated polynomial system being 0-dimensional by definition of a determinantal map, see Definition 1.1.6).

In this setting, by Theorem 4.3.2 in the generic case, the quantity

$$\mathrm{MV}_{2n}(\underbrace{S_n^{\mathbf{x}},\dots,S_n^{\mathbf{x}}}_{k}, P_1^{\mathbf{y}},\dots,P_n^{\mathbf{y}},\underbrace{S_n^{\mathbf{y}},\dots,S_n^{\mathbf{y}}}_{n-k})$$

is the number of solutions of (5.1.1) whose coordinates are all non zero which shows the right hand side inequality of Proposition 5.1.1 since the number of common zero of (5.1.1) is equal to $\sigma_{n-k,n}(d_1,\dots,d_n)$.

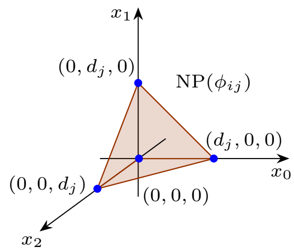
The set of common zeros of (5.1.1) contains moreover the set

$$H_{\mathbf{x}}^k \cap \Gamma_f \cap H_{\mathbf{y}}^{n-k} \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$$

where $H_{\mathbf{x}}^k = \mathbb{V}(l_{1,0},\dots,l_{k,0})$ is the zero locus of $l_{1,0},\dots,l_{k,0}$, Γ_f is the graph of f and $H_{\mathbf{y}}^{n-k} = \mathbb{V}(l_{0,1},\dots,l_{0,n-k})$. However, by the genericity assumptions, remark that the points of $H_{\mathbf{x}}^k \cap \Gamma_f \cap H_{\mathbf{y}}^{n-k}$ have all non zero coordinates which shows the left hand side inequality of Proposition 5.1.1. □

As illustrated by the following example, the inequality of the right hand side of the estimations in Proposition 5.1.1 is an equality in "naive" generic cases. This is nothing but expected since the bound provided by the mixed volume in Bernstein theorem say no more than the bound of Bézout theorem in those "naive" generic cases, see [CLO05, 7. Section 5. Ex.2].

Example 5.1.2. Put momentarily $n = 3$, let $d_1, d_2, d_3 \in \mathbb{N}^*$ and $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}}$ ($\mathrm{R} = \mathrm{k}[x_0,\dots,x_3]$) be the Hilbert-Burch matrix of a determinantal map f where for $j \in \{1,2,3\}$ each entry ϕ_{ij} of the j-th entry of Φ_f is generic with respect to the polytope $d_j S_3$.



Denoting $P_j^{(6)} := d_j S_3 \times \{0_{\mathbb{R}^3}\} + \{0_{\mathbb{R}^3}\} \times S_3 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and using the multilinearity of the mixed volume (Theorem 4.2.3. Item (ii)), projection formula (Lemma 4.2.5) and the vanishing condition (Theorem 4.2.3. Item (iii)), one has:

(a) $\mathrm{MV}_6(P_1^{(6)}, P_2^{(6)}, P_3^{(6)}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}) = d_1 d_2 d_2 = \sigma_{3,3}(d_1, d_2, d_3)$

(b) $\mathrm{MV}_6(S_3^{\mathbf{x}}, P_1^{(6)}, P_2^{(6)}, P_3^{(6)}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}) =$

$$\begin{aligned}
&\mathrm{MV}_6(S_3^{\mathbf{x}}, d_2 S_3 \times \{0_{\mathbb{R}^3}\}, d_2 S_3 \times \{0_{\mathbb{R}^3}\}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}) \\
&+ \mathrm{MV}_6(S_3^{\mathbf{x}}, d_1 S_3 \times \{0_{\mathbb{R}^3}\}, d_3 S_3 \times \{0_{\mathbb{R}^3}\}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}) \\
&+ \mathrm{MV}_6(S_3^{\mathbf{x}}, d_1 S_3 \times \{0_{\mathbb{R}^3}\}, d_2 S_3 \times \{0_{\mathbb{R}^3}\}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}) \\
=& d_2 d_3 \,\mathrm{MV}_3(S_3, S_3, S_3) + d_1 d_3 \,\mathrm{MV}_3(S_3, S_3, S_3) + d_1 d_2 \,\mathrm{MV}_3(S_3, S_3, S_3) \\
=& d_2 d_3 + d_1 d_3 + d_1 d_3 = d_1 + d_2 + d_3 = \sigma_{2,3}(d_1, d_2, d_3)
\end{aligned}$$

(c) $\mathrm{MV}_6(S_3^{\mathbf{x}}, S_3^{\mathbf{x}}, P_1^{(6)}, P_2^{(6)}, P_3^{(6)}, S_3^{\mathbf{y}}) = d_1 + d_2 + d_3 = \sigma_{1,3}(d_1, d_2, d_3)$

(d) $\mathrm{MV}_6(S_3^{\mathbf{x}}, S_3^{\mathbf{x}}, S_3^{\mathbf{x}}, P_1^{(6)}, P_2^{(6)}, P_3^{(6)}) = \mathrm{MV}_6(S_3^{\mathbf{x}}, S_3^{\mathbf{x}}, S_3^{\mathbf{x}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}}, S_3^{\mathbf{y}})$
$= 1 = \sigma_{0,3}(d_1, d_2, d_3)$.

Let us now isolate two technical facts that rely on Lemma 4.2.5:

Lemma 5.1.3. *Let $P_1, \dots, P_n \subset \mathbb{R}^n$ be n polytopes and for $j \in \{1, \dots, n\}$ put $P_j^{\mathbf{y}} := P_j \times \{0_{\mathbb{R}^n}\} + \{0_{\mathbb{R}^n}\} \times S_n^{\mathbf{y}} \subset \mathbb{R}^n \times \mathbb{R}^n$.*
Then
$$\mathrm{MV}_{2n}(P_1^{\mathbf{y}}, \dots, P_n^{\mathbf{y}}, \underbrace{S_n^{\mathbf{y}}, \dots, S_n^{\mathbf{y}}}_{n}) = \mathrm{MV}_n(P_1, \dots, P_n).$$

Proof. It suffices to apply the multilinearity of the mixed volume (Theorem 4.2.3. Item (ii)), the projection formula (Lemma 4.2.5) and the vanishing condition (Theorem 4.2.3. Item (iii)) on the left member of the equality to obtain:

$$\begin{aligned}
\mathrm{MV}_{2n}(P_1^{\mathbf{y}}, \dots, P_n^{\mathbf{y}}, \underbrace{S_n^{\mathbf{y}}, \dots, S_n^{\mathbf{y}}}_{n}) &= \mathrm{MV}_n(P_1, \dots, P_n)\,\mathrm{MV}_n(\underbrace{S_n, \dots, S_n}_{n}) \\
&= \mathrm{MV}_n(P_1, \dots, P_n).
\end{aligned}$$

□

Lemma 5.1.4. *In addition to $n \in \mathbb{N}^*$, let $m \in \mathbb{N}^*$ and:*

- $P_1, \dots, P_m \subset \mathbb{R}^m \times \{0_{\mathbb{R}^n}\} \subset \mathbb{R}^m \times \mathbb{R}^n$,

- $P_{m+1}, \dots, P_{m+n} \subset \{0_{\mathbb{R}^m}\} \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n$

be $n+m$ polytopes and for $j \in \{1, \dots, m+n\}$ put:

$$P_j^{\mathbf{y}} := P_j \times \{0_{\mathbb{R}^{m+n}}\} + \{0_{\mathbb{R}^{m+n}}\} \times S_{m+n}^{\mathbf{y}} \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}.$$

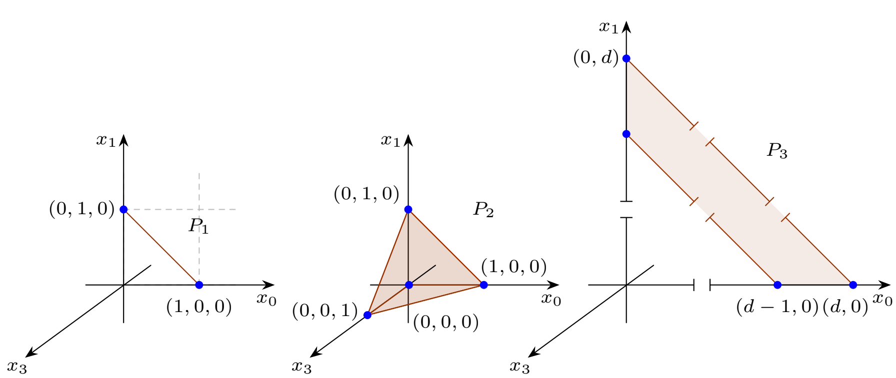
Then for all $k \in \{0, \dots, m+n\}$,

$$\mathrm{MV}_{2(m+n)}(\underbrace{S_{m+n}^{\mathbf{x}}, \dots, S_{m+n}^{\mathbf{x}}}_{m+n-k}, P_1^{\mathbf{y}}, \dots, P_{n+m}^{\mathbf{y}}, \underbrace{S_{m+n}^{\mathbf{y}}, \dots, S_{m+n}^{\mathbf{y}}}_{k})$$
$$=$$
$$\sum_{p=0}^{k}\Big[\Big(\sum_{\substack{\{l_1,\dots,l_p\}\subset\\ \{1,\dots,m\}}} \mathrm{MV}_m(\underbrace{S_m, \dots, S_m}_{m-p}, P_{l_1}, \dots, P_{l_p})\Big)\times$$
$$\Big(\sum_{\substack{\{l_1,\dots,l_{k-p}\}\subset\\ \{m+1,\dots,m+n\}}} \mathrm{MV}_n(\underbrace{S_n, \dots, S_n}_{n+p-k}, P_{l_1}, \dots, P_{l_{k-p}})\Big)\Big]$$

Proof. The formula follows from by decomposing first $S_{m+n}^{\mathbf{x}} \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ (resp. $S_{m+n}^{\mathbf{y}}$) as the sum $S_m^{\mathbf{x}} \times \{0_{\mathbb{R}^n}\} \times \{0_{\mathbb{R}^{m+n}}\} + \{0_{\mathbb{R}^m}\} \times S_n^{\mathbf{x}} \times \{0_{\mathbb{R}^{m+n}}\} \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ (resp. as the sum $\{0_{\mathbb{R}^{m+n}}\} \times S_m^{\mathbf{y}} \times \{0_{\mathbb{R}^n}\} + \{0_{\mathbb{R}^{m+n}}\} \times \{0_{\mathbb{R}^m}\} \times S_n^{\mathbf{y}} \times \{0_{\mathbb{R}^{m+n}}\}$) and then applying the multilinearity of the mixed volume (Theorem 4.2.3. Item (ii)), the projection formula (Lemma 4.2.5), the vanishing condition (Theorem 4.2.3. Item (iii)) and eventually Lemma 5.1.3. ☐

We now describe a situation where the right hand side inequality in Proposition 5.1.1 is strict.

Example 5.1.5. Let $d \geqslant 1$ and let $f : \mathbb{P}_{\mathrm{k}}^3 \dashrightarrow \mathbb{P}_{\mathrm{k}}^3$ be a determinantal map of Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}} \in \mathrm{R}^{4\times 3}$ ($\mathrm{R} = \mathrm{k}[x_0, \dots, x_3]$) such that for all $i \in \{0, \dots, 3\}$, ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(1,0,0),(0,1,0)\}$, ϕ_{i2} is generic with respect to $P_2 = \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$ and ϕ_{i3} is generic with respect to $P_3 = \mathrm{Conv}\{(d,0,0),(d-1,0,0),(0,d-1,0),(0,d,0)\}$ (let us moreover precise that we only consider here the de-homogenization of ϕ_{ij} with respect to x_2).



Using Lemma 5.1.3 one has

$$\begin{aligned}\mathrm{MV}_6(P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}}) &= \mathrm{MV}_3(P_1,P_2,P_3)\\ &= \mathrm{MV}_2\big(\pi_2(P_1),\pi_2(P_3)\big)\,\mathrm{MV}_1\big(\pi_1(P_2)\big)=1\end{aligned}$$

where the first equality follow from applying the projection formula Lemma 4.2.5 since $P_1,P_3\subset\mathbb{R}^2\subset\mathbb{R}^2\times\mathbb{R}$ ($\pi_2:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ and $\pi_1:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ being the first and second projection) and the second equality follows from Example 4.2.6.

Hence, in this example

$$d_0(f)\leqslant \mathrm{MV}_6(P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})=1<\sigma_{3,3}(1,1,d)=d.$$

Additionally one can say even more about the estimation of $d_0(f)$ in this example. Indeed, remark that under our genericity assumption on the entries of Φ_f, one has that $\operatorname{codim}\mathbb{V}\big(\mathrm{I}_2(\Phi_f)\big)=2$ and $\operatorname{codim}\mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big)\geqslant 4$ so the only defect preventing I_f of being be of linear type comes from $\mathrm{I}_2(\Phi_f)$. Actually, in this example, one has even that all the points $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}\cap H^3_{\mathbf{y}}\subset\mathbb{P}^3_{\mathrm{k}}\times\mathbb{P}^3_{\mathrm{k}}$ for $H^3_{\mathbf{y}}=\mathbb{V}(l_{01},l_{02},l_{03})$ (l_{01},l_{02},l_{03} generic with respect to $S_3^{\mathbf{y}}$) lie on the 4-space $\mathbb{V}(x_0,x_1)\subset\mathbb{P}^3_{\mathrm{k}}\times\mathbb{P}^3_{\mathrm{k}}$ so $\mathbb{P}(\mathrm{I}_f)$ and Γ_f coincide away of the coordinate axis $\mathbb{V}(x_0x_1x_2x_3)$. Since $\mathrm{MV}_6(P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})$ is the number of solutions of $\mathbb{P}(\mathrm{I}_f)\cap H^3_{\mathbf{y}}$ away from $\mathbb{V}(x_0x_1x_2x_3)$, it means that the number of solutions of $\Gamma_f\cap H^3_{\mathbf{y}}$ away from $\mathbb{V}(x_0x_1x_2x_3)$ is also equal to $\mathrm{MV}_6(P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})$.

Hence, in this example $d_0(f)=\mathrm{MV}_6(P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})=1$ and f is a determinantal Cremona map.

Actually, the same argument shows that:

$$d_1(f)=\mathrm{MV}_6(S_3^{\mathbf{x}},P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})=d+2<\sigma_{2,3}(1,1,d)=2d+1. \tag{5.1.2}$$

Remark that the actual computation of $\mathrm{MV}_6(S_3^{\mathbf{x}},P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}})$ shed light on the defect in the right hand side inequality in (5.1.2). Namely, using the multilinearity of the mixed volume (Theorem 4.2.3. Item (ii)), the projection formula (Lemma 4.2.5) and the vanishing condition (Theorem 4.2.3. Item (iii)), one has:

$$\begin{aligned}\mathrm{MV}_6(S_3^{\mathbf{x}},P_1^{\mathbf{y}},P_2^{\mathbf{y}},P_3^{\mathbf{y}},S_3^{\mathbf{y}},S_3^{\mathbf{y}}) &= \mathrm{MV}_3(P_2,P_3,S_3)+\mathrm{MV}_3(P_1,P_3,S_3)\\ &\qquad+\mathrm{MV}_3(P_1,P_2,S_3)\\ &= d+1+1\end{aligned}$$

and the defect is due to $\mathrm{MV}_3(P_1,P_3,S_3)$ since here it is equal to 1 whereas it is equal to d for instance if $P_1=\operatorname{Conv}\{(0,0,0),(1,0,0),(0,1,0)\}$ (in which case I_f is of linear type).

In the end, $d(f)=(1,d+2,d+2,1)$ is palindromic.

The discussion in the end of Example 5.1.5 prove more generally:

Proposition 5.1.6. *Let $f:\mathbb{P}^n_{\mathrm{k}}\dashrightarrow\mathbb{P}^n_{\mathrm{k}}$ be a Koszul-determinantal map of Hilbert-Burch matrix $\Phi_f=(\phi_{ij})_{\substack{0\leqslant i\leqslant n\\1\leqslant j\leqslant n}}$. Then:*

$$d_k(f)=\mathrm{MV}_{2n}(\underbrace{S_n^{\mathbf{x}},\ldots,S_n^{\mathbf{x}}}_{k},P_1^{\mathbf{y}},\ldots,P_n^{\mathbf{y}},\underbrace{S_n^{\mathbf{y}},\ldots,S_n^{\mathbf{y}}}_{n-k})$$

$$\Leftrightarrow \overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f}\subset\mathbb{V}(\prod_{i=0}^{n}x_i)\subset\mathbb{P}^n_{\mathrm{k}}\times\mathbb{P}^n_{\mathrm{k}}$$

where for $l \in \{1, \dots, n\}$, $P_l^{\mathbf{y}} \subset \mathbb{R}^n \times \mathbb{R}^n$ is the Newton polytope of the l-th entry of the matrix $(y_0 \ \dots \ y_n)\Phi_f$.

Hence, to know the actual term $d_k(f)$ of the projective degrees of a determinantal map f, the computation of the associated mixed volume has to be completed by a preliminary control on the support of the successive ideal of minors of Φ_f. We illustrate such a control in the next subsection.

5.2 Plane Cremona maps as solutions of an interpolation problem

Following the discussion at the end of the previous section, we describe now plane determinantal Cremona maps using the theory on sparse polynomials. We focus on plane determinantal maps since, given such a map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ of syzygetic degree $(d_1, d_2) \in \mathbb{N}^2$ and Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 2\\ 1\leqslant j\leqslant 2}} \in \mathrm{R}^{3\times 2}$ ($\mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$), there is nothing to control but the topological degree $d_0(f)$ of f which only depend on the single ideal $\mathrm{I}_1(\Phi_f)$ of 1-minors of Φ_f (by definition of a determinantal map, $d_1(f) = d_1 + d_2$ and $\operatorname{codim} \mathbb{V}(\mathrm{I}_f) = 2$, see Definition 1.1.6). Remark that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big) \geqslant 2$ so $\mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big)$ is set-theoretically an intersection of points $p_1, \dots, p_l$ (if non empty) and one has moreover that for any $k \in \{1, \dots, l\}$, $\{p_k\} \times \mathbb{P}^2_{\mathrm{k}} \subset \mathbb{P}(\mathrm{I}_f)$ as set.

Recall also Proposition 5.1.1 and Lemma 5.1.3 that if for any $j \in \{1, 2\}$, all the entries ϕ_{ij} of the j-th column of Φ_f are generic with respect to a given polytope $P_j \subset \mathbb{R}^2$, one has then:

$$d_0(f) \leqslant \mathrm{MV}_2(P_1, P_2).$$

Moreover if the zero locus $\mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big)$ of the 1-minors ideal $\mathrm{I}_1(\Phi_f)$ of Φ_f is included in $\mathbb{V}(x_0x_1x_2)$, then the latter inequality is is actually an equality, see Proposition 5.1.6. A starting example for this situation being the map f where P_1 and P_2 are the polytope of Example 4.2.1 (almost linear determinantal case). In this example, $\mathbb{V}\big(\mathrm{I}_1(\Phi_f)\big) = (0:0:1) \in \mathbb{P}^2_{\mathrm{k}}$ so this construction answer the case $d_1 = 1$ and $d_2 = d$ with one point $p_1 = (0:0:1)$ of multiplicity $d-1$ in the interpolation problem we are now going to define. We refer to [EH16, Chapter 1] for the definition of multiplicity of a component in a intersection of two schemes.

Definition 5.2.1 (Interpolation for plane determinantal Cremona maps)**.** Let two integer $d_1, d_2 \in \mathbb{N}^*$ and a matrix $\Phi = (\phi_{ij})_{\substack{0\leqslant i\leqslant 2\\ 1\leqslant i\leqslant 2}} \in \mathbb{R}^{3\times 2}$ such that for any $i \in \{0, 1, 2\}$ and $j \in \{1, 2\}$, ϕ_{ij} is homogeneous of degree d_j and such that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_2(\Phi)\big) = 2$.

Does it exist $l > 0$ points $p_1, \dots, p_l \in \mathbb{P}^2_{\mathrm{k}}$ and a l-uple $m = (m_1, \dots, m_l) \in (\mathbb{N}^*)^n$ verifying

(1) $\sum_{k=1}^{l} m_i = d_1 d_2 - 1$

(2) $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big) = \{p_1, \dots, p_l\}$

(3) for all $k \in \{1, \ldots, l\}$, $(\{p_k\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}} \in \mathbb{P}^2_{\mathrm{k}} \times \mathbb{P}^2_{\mathrm{k}}$ has multiplicity m_j in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$?

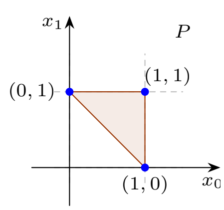
If such a matrix Φ exists, the determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ whose Hilbert-Burch matrix is Φ is a Cremona map and we say that such a map answer the *interpolation problem* with $[d_1, d_2, (p_1, m_1), \ldots, (p_m, m_l)]$.

Remark 5.2.2. Let $c, c' \in \mathrm{R} = \mathrm{k}[x_0, x_1, x_2]$ be two generic cubic polynomials with respect to the polytope $\mathrm{Conv}\{(0,0), (3,0), (0,3)\}$ and denote by $p_1, \ldots, p_9 \in \mathbb{P}^2_{\mathrm{k}}$ the nine distinct point of the intersection $\mathbb{V}(c) \cap \mathbb{V}(c')$. Choosing eight points among $p_1, \ldots, p_9$, say $p_1, \ldots, p_8$, the interpolation problem with $[3, 3, (p_1, 1), \ldots, (p_m, 1)]$ have a negative answer. Indeed if $\Phi = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2}}$ is composed by cubic polynomials all vanishing at $p_1, \ldots, p_8$ then $\{p_1, \ldots, p_8\} \subset \mathbb{V}\big(\mathrm{I}_1(\Phi)\big)$ and, by Chasles' theorem [EGH96, Theorem CB3], $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big) = \{p_1, \ldots, p_9\}$ cannot be just equal $\{p_1, \ldots, p_8$. It emphasize that sometimes the interpolation problem we defined in Definition 5.2.1 can depend on the given configuration of points $p_1, \ldots, p_l$.

Let us now present positive answer to some interpolation problem.

Proposition 5.2.3. *The interpolation problem* $[2, 2, (p_1, 1), (p_2, 1), (p_3, 1)]$ *has a positive answer.*

Proof. For every $i \in \{0, 1, 2\}$ and $j \in \{1, 2\}$, let $\phi_{ij} \in \mathrm{R}$ be generic with respect to $P = \mathrm{Conv}\{(0,1), (1,0), (1,1)\}$. Then $\Phi = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2}}$ is a solution to $[2, 2, (p_1, 1), (p_2, 1), (p_3, 1)]$ where $p_1 = (1:0:0)$, $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$.



Indeed, each entry ϕ_{ij} of Φ is a generic linear combination $a_{ij}x_1x_2 + b_{ij}x_0x_2 + c_{ij}x_0x_1$ for coefficients $a_{ij}, b_{ij}, c_{ij} \in \mathrm{k}$ so $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big) = \{p_1, p_2, p_3\}$. Moreover, by the symmetry of the construction, each $(\{p_k\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}} \subset \mathbb{P}^2_{\mathrm{k}} \times \mathbb{P}^2_{\mathrm{k}}$ has the same multiplicity a in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$ which provide the equation:

$$d_0(f) + 3a = 4.$$

Since by Proposition 5.1.6, $d_0(f) = \mathrm{MV}_2(P, P) = 1$, one has furthermore

$$a = 1.$$

□

Example 5.2.4. The determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ whose Hilbert-Burch matrix is $\Phi_f = \begin{pmatrix} x_1x_2 & x_0x_2 \\ x_0x_2 & x_0x_1 \\ x_0x_1 & x_1x_2 \end{pmatrix}$ is a Cremona map ($d(f) = (1, 4, 1)$).

Proposition 5.2.5. *Given $d \geqslant 1$, the interpolation problem*

$$[2, 2d + 1, (p_1, d + 1), (p_2, d), (p_3, d), (p_4, d)]$$

has a solution.

Proof. Choose two polynomials $q_1, q_2 \in \mathrm{R}$ generic with respect to the polytope $P_1 = \mathrm{Conv}\{(0, 1), (1, 0), (1, 1)\}$. These two polynomials vanish along foor common points $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$, $p_3 = (0 : 0 : 1)$ and p_4, say $p_4 = (\alpha : \beta : \gamma)$ with $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$.

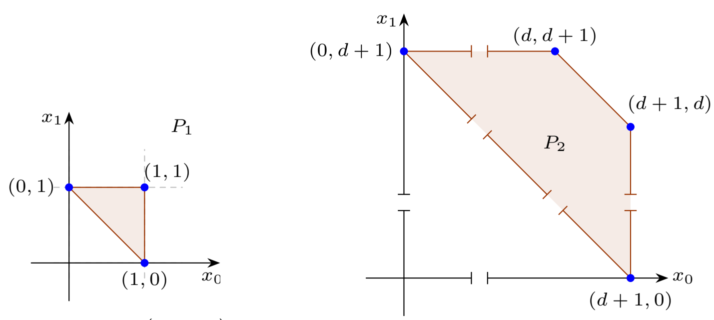
Now for all $i \in \{0, 1, 2\}$, let

$$\phi_{i1} = a_{i,1}q_1 + b_{i,1}q_2$$

be a generic linear combination of q_1 and q_2 and let

$$\phi_{i2} = (a_{i,2}x_0 + b_{i,2}x_1) \prod_{k=1}^{d} (a_{i,k,2}q_1 + b_{i,k,2}q_2)$$

be the product of a generic linear combination $(a_{i,2}x_0 + b_{i,2}x_1)$ of x_0 and x_1 with the product of d generic linear combinations of q_1 and q_2.



By construction, $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big)$ is set theoretically equal to $\{p_1, p_2, p_3, p_4\}$ and by symmetry, $(\{p_2\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_3\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_4\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have the same multiplicity a in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$. Moreover, the multiplicity of $(\{p_1\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}} = (\{(1 : 0 : 0)\} \times$

$\mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ has moreover multiplicity $a+1$ in the previous intersection since there is just one additional general line passing through $(\{p_1\} \times \mathbb{P}^2_{\mathrm{k}})$ in $\mathbb{P}(\mathrm{I}_f)$ compared to $(\{p_k\} \times \mathbb{P}^2_{\mathrm{k}})$ for $k \in \{2,3,4\}$. All this fact together provide the equation:

$$4a + 1 + d_0(f) = 2(2d+1) = 4d + 2$$

where f is the determinantal map with Hilbert-Burch matrix Φ.

Since moreover $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$, one has the additional equation

$$a + d_0(f) = \mathrm{MV}_2(P_1, P_2)$$

where $P_2 = \mathrm{Conv}\{(d+1,0),(d+1,d),(d,d+1),(0,d+1)\}$. Since after computation $\mathrm{MV}_2(P_1,P_2) = d+1$, we have $a = d$ which concludes the proof. ☐

Example 5.2.6. The determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ of Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} 0 & x_0(x_0x_2 - x_0x_1)^2 \\ x_0x_2 - x_0x_1 & x_1(x_1x_2 - x_0x_2)^2 \\ x_1x_2 - x_0x_2 & 0 \end{pmatrix}$$

is a Cremona map $(d(f) = (1,7,1))$.

Proposition 5.2.7. *Given $d \geqslant 2$, the interpolation problem*

$$[2, 2d, (p_1, d), (p_2, d), (p_3, d), (p_4, d-1)]$$

has a solution.

Proof. Choose two polynomials $q_1, q_2 \in \mathrm{R}$ generic with respect to the polytope $P_1 = \mathrm{Conv}\{(0,1),(1,0),(1,1)\}$. These two polynomials vanish along four common points $p_1 = (1:0:0)$, $p_2 = (0:1:0)$, $p_3 = (0:0:1)$ and p_4, say $p_4 = (\alpha:\beta:\gamma)$ with $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$.

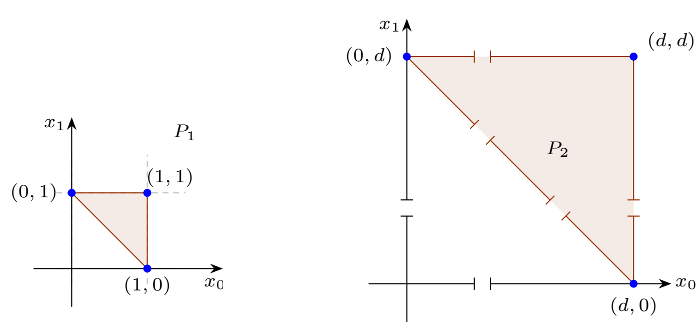
Now for all $i \in \{0,1,2\}$, let

$$\phi_{i1} = a_{i,1}q_1 + b_{i,1}q_2$$

be a generic linear combination of q_1 and q_2 and let

$$\phi_{i2} = q_{i2}\prod_{k=1}^{d-1}(a_{i,k,2}q_1 + b_{i,k,2}q_2)$$

be the product of a polynomial q_{i2} generic with respect to $\mathrm{Conv}\{(0,1),(1,0),(1,1)\}$ with the product of $d-1$ generic linear combinations of q_1 and q_2.



By construction, $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big)$ is set theoretically equal to $\{p_1, p_2, p_3, p_4\}$ and by symmetry, $(\{p_1\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_2\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_3\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have the same multiplicity a in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$. Moreover, the multiplicity of $(\{p_4\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}} = (\{(1:0:0)\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ has moreover multiplicity $a-1$ in the previous intersection since, by the genericity assumptions, there is just one general line less passing through $(\{p_4\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ in $\mathbb{P}(\mathrm{I}_f)$ compared to $(\{p_k\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ for $k \in \{1,2,3\}$. All this fact together describe the system:

$$\begin{cases} 3a + (a-1) + d_0(f) = 2 \times 2d = 4d \\ (a-1) + d_0(f) = \mathrm{MV}_2(P_1, P_2) \end{cases}$$

where $P_2 = dP_1$ and f is the determinantal map with Hilbert-Burch matrix Φ. Since $\mathrm{MV}_2(P_1, P_2) = d\,\mathrm{MV}_2(P_1, P_1) = d$ after computation, we thus have $a = d$ which concludes the proof. □

Example 5.2.8. The determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ of Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} 0 & x_0x_2(x_0x_2 - x_0x_1)^2 \\ x_0x_2 - x_0x_1 & x_1x_2(x_1x_2 - x_0x_2)^2 \\ x_1x_2 - x_0x_2 & x_0x_1(x_1x_2 - x_0x_1)^2 \end{pmatrix}$$

is a Cremona map ($d(f) = (1, 8, 1)$).

Let us finish this section by presenting two sporadic constructions.

Proposition 5.2.9. *Given $d \geqslant 2$, the interpolation problem*

$$[3, 5, (p_1, 6), (p_2, 2), (p_3, 2), (p_4, 2), (p_5, 1), (p_6, 1)]$$

has a solution.

Proof. Choose two polynomials $c_1, c_2 \in \mathrm{R}$ whose de-homogenization with respect to x_2 is generic with respect to $P_1 = \mathrm{Conv}\{(2,0),(2,1),(1,2),(0,2)\}$. Assume moreover that c_1 and c_2 are generic under the condition that they vanish at the point $(1:1:1)$ that is, the coefficients are generic with respect to a hyperplane of $L(\{(2,0),(2,1),(1,2),(0,2)\})$ in the notation of Definition 4.3.1.

By construction, $\mathbb{V}(c_1)$ and $\mathbb{V}(c_2)$ intersect set theoretically at 6 points $p_1 = (0:0:1), p_2 = (1:0:0), p_3 = (0:1:0), p_4 = (1:1:1), p_5 = (\alpha_5 : \beta_5 : \gamma_5), p_6 = (\alpha_6 : \beta_6 : \gamma_6)$ where $\alpha_5 \neq 0, \beta_5 \neq 0, \gamma_5 \neq 0, \alpha_6 \neq 0, \beta_6 \neq 0, \gamma_6 \neq 0$. In addition p_1 has multiplicity 4 in this intersection when p_2, p_3, p_4, p_5, p_6 have multiplicity 1.

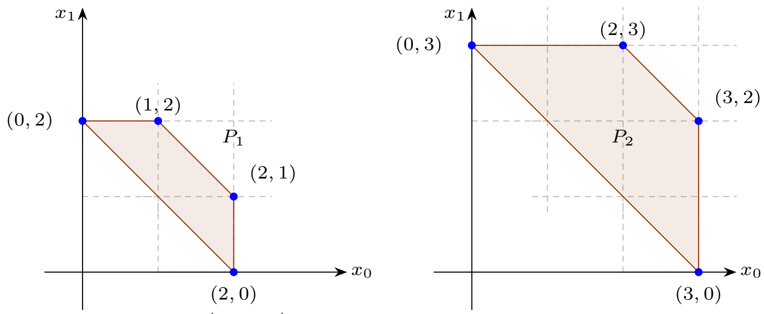
Now for all $i \in \{0, 1, 2\}$, let

$$\phi_{i1} = a_{i,1}c_1 + b_{i,1}c_2$$

be a generic linear combination of c_1 and c_2 and let

$$\phi_{i2} = q_{i2}(a_{i,2}c_1 + b_{i,2}c_2)$$

be the product of a general linear combination $q_{i2} = l_{i2}(x_1x_2 - x_0x_2) + l'_{12}(x_0x_2 - x_0x_1)$ of $x_1x_2 - x_0x_2$ and $x_0x_2 - x_0x_1$ with the product of a generic linear combination of q_1 and q_2.



By construction, $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big)$ is set theoretically equal to $\{p_1, \ldots, p_6\}$ and $(\{p_2\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_3\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_4\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have multiplicity 2 and $(\{p_5\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_6\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have multiplicity 1 in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$. Denoting a the multiplicity of $(\{p_1\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ in $\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$, one has the system:

$$\begin{cases} a + 3 \times 2 + 2 \times 1 + d_0(f) = 3 \times 5 = 15 \\ 4 + d_0(f) = \mathrm{MV}_2(P_1, P_2) \end{cases}$$

where P_2 is the polytope $P_1 + \mathrm{Conv}\{(0,1), (1,1), (1,0)\}$. Since $\mathrm{MV}_2(P_1, P_2) = 5$ after computation, one eventually finds $a = 6$ as expected. □

Example 5.2.10. The determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ of Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} 0 & (x_0^2x_2 - x_0x_1^2)(x_0x_2 - x_0x_1) \\ x_0^2x_2 - x_0x_1^2 & (x_0^2x_1 - x_1^2x_2)(x_1x_0 - x_1x_2) \\ x_0^2x_1 - x_1^2x_2 & 0 \end{pmatrix}$$

is a Cremona map ($d(f) = (1, 8, 1)$).

Proposition 5.2.11. *Given $d \geqslant 2$, the interpolation problem*

$$[4, 6, (p_1, 6), (p_2, 6), (p_3, 6), (p_4, 2), (p_5, 1), (p_6, 1), (p_7, 1)]$$

has a solution.

Proof. Let $c_1 = x_1^2x_2^2 - x_0^2x_2^2, c_2 = x_0^2x_2^2 - x_0^2x_1^2, q_1 = x_1x_2 - x_0x_2, q_2 = x_0x_2 - x_0x_1$ and for all $i \in \{0, 1, 2\}$, let

$$\phi_{i1} = a_{i,1}c_1 + b_{i,1}c_2$$

be a generic linear combination of c_1 and c_2 and let

$$\phi_{i2} = (a_{i,2}c_1 + b_{i,2}c_2)(a'_{i,2}q_1 + b'_{i,2}q_2)$$

be the product of a general linear combination of c_1 and c_2 with the product of a generic linear combination of q_1 and q_2.

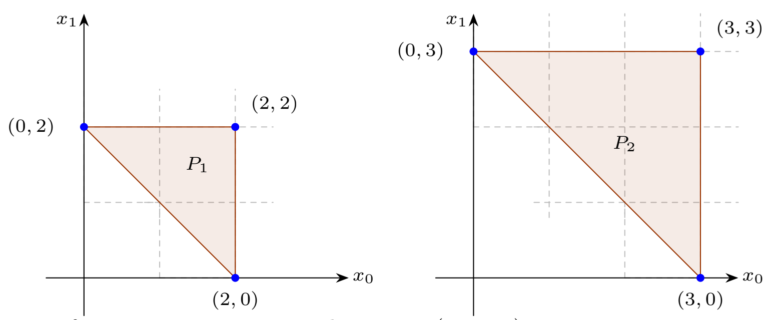
By construction, $\mathbb{V}\big(\mathrm{I}_1(\Phi)\big) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ where $p_1 = (1:0:0), p_2 = (0:1:0), p_3 = (0:0:1), p_4 = (1:1:1), p_5 = (1:-1:1), p_6 = (1:1:-1), p_7 = (1:-1:-1)$. In addition $(\{p_4\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have multiplicity 2 and $(\{p_5\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_6\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ $(\{p_6\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have multiplicity 1 in the intersection

$$\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$$

where $H^2_{\mathbf{y}} = \mathbb{V}(l_{0,1}, l_{0,2})$ is the zero locus of 2 polynomials $l_{0,1}, l_{0,k}$ generic with respect to $S^{\mathbf{y}}_2$. The points $(\{p_1\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$, $(\{p_2\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ $(\{p_3\} \times \mathbb{P}^2_{\mathrm{k}}) \cap H^2_{\mathbf{y}}$ have moreover the same multiplicity a in $\mathbb{P}\big(\mathrm{I}_2(\Phi)\big) \cap H^2_{\mathbf{y}}$ that satisfies the following conditions:

$$\begin{cases} 3a + 5 + d_0(f) = 3 \times 5 = 24 \\ 5 + d_0(f) = \mathrm{MV}_2(P_1, P_2) \end{cases}$$

where f is the determinantal map whose Hilbert-Burch matrix is Φ and where $P_1 = 2 \cdot \mathrm{Conv}\{(1,0), (0,1), (1,1)\}$ and $P_2 = 3 \cdot \mathrm{Conv}\{(1,0), (0,1), (1,1)\}$



Since after computation, one has $\mathrm{MV}_2(P_1, P_2) = 6$, $a = 6$ as expected. □

Example 5.2.12. The determinantal map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ of Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} 0 & (x_0^2x_2^2 - x_0^2x_1^2)(x_0x_2 - x_0x_1) \\ x_0^2x_2^2 - x_0^2x_1^2 & (x_1^2x_2^2 - x_0^2x_2^2)(x_1x_2 - x_0x_2) \\ x_1^2x_2^2 - x_0^2x_2^2 & 0 \end{pmatrix}$$

is a Cremona map $(d(f) = (1, 10, 1))$.

Let us end the descriptions of examples since, as we will see in the next section, there exists procedures which provides new examples of determinantal Cremona maps starting from old ones.

5.3 Glued determinantal Cremona maps: a polytopal approach

We now focus on the construction of a *glued determinantal map* $[g|g'] : \mathbb{P}^{m+n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+n}_{\mathrm{k}}$, where $n, m \in \mathbb{N}^*$, starting from two determinantal Cremona maps $g : \mathbb{P}^m_{\mathrm{k}} \dashrightarrow \mathbb{P}^m_{\mathrm{k}}$ and $g' : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$. Actually, if last subsection fits into the previous studies such as [DS16] where the two authors described the composition with the standard

Cremona map via a polytopal construction, the Newton complementary dual, the results we are going to present fits into the the previous study in [CS12b, Proposition 2.4] where the two authors presented the construction of a Cremona map $\tilde{g} : \mathbb{P}^{n+1}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{n+1}_{\mathrm{k}}$ starting from a Cremona map $g : \mathbb{P}^{n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{n}_{\mathrm{k}}$ (see also [Dol11, Prop. 7.2.8] for a similar procedure). Let us emphasize that this latter process is qualitative in the sense that it applies to any Cremona map $g : \mathbb{P}^{n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{n}_{\mathrm{k}}$ whereas the process we are going to present involves generic arguments due to the polytopal approach it is based on (hence being closer to a quantitative process).

The goal behind this glued construction is to approach combinatorially the projective degrees of the almost linear determinantal map in order to eventually extend in greater dimension previous works such as [DH17].

Proposition-Definition 5.3.1 (glued map)**.** Given any $j \in \{1, \dots, m+n\}$, let $l_j \in \mathbb{N}^*$ and let:

- $\psi_1^{(j)}, \dots, \psi_{l_j}^{(j)} \in \mathrm{R}_m = \mathrm{k}[x_0, \dots, x_m]$ if $j \in \{1, \dots, m\}$
- $\psi_1^{(j)}, \dots, \psi_{l_j}^{(j)} \in \mathrm{R}_n = \mathrm{k}[x_m, \dots, x_{m+n}]$ if $j \in \{m+1, \dots, m+n\}$.

Let also $\Phi_g = (\phi_{ij}^{(g)})_{\substack{0 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant m}} \in \mathrm{R}_m^{(m+1)\times m}$, $\Phi_{g'} = (\phi_{ij}^{(g)})_{\substack{0 \leqslant i \leqslant n \\ m+1 \leqslant j \leqslant m+n}} \in \mathrm{R}_n^{(n+1)\times n}$ and $\Phi_{[g|g']} = (\phi_{ij})_{\substack{0 \leqslant i \leqslant m+n \\ 1 \leqslant j \leqslant m+n}} \in \mathrm{R}_{m+n}^{(m+n+1)\times(m+n)}$ where $\mathrm{R}_{m+n} = \mathrm{k}[x_0, \dots, x_{m+n}]$ be such that:

- for any $j \in \{1, \dots, m\}$, each entry $\phi_{ij}^{(g)} = \sum_{k=1}^{l_j} \lambda_k^{(ij)} \psi_k^{(j)}$ of the j-th column of Φ_g is a general linear combination of $\psi_1^{(j)}, \dots, \psi_{l_j}^{(j)}$.
- for any $j \in \{m+1, \dots, m+n\}$, each entry $\phi_{ij}^{(g')} = \sum_{k=1}^{l_j} \lambda_k^{(ij)} \psi_k^{(j)}$ of the j-th column of $\Phi_{g'}$ is a general linear combination of $\psi_1^{(j)}, \dots, \psi_{l_j}^{(j)}$.
- for any $j \in \{1, \dots, m+n\}$, each entry $\phi_{ij} = \sum_{k=1}^{l_j} \lambda_k^{(ij)} \psi_k^{(j)}$ of the j-th column of $\Phi_{[g|g']}$ is a general linear combination of $\psi_1^{(j)}, \dots, \psi_{l_j}^{(j)}$.

Assume that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_m(\Phi_g)\big) = \operatorname{codim} \mathbb{V}\big(\mathrm{I}_n(\Phi_{g'})\big) = 2$, then necessarily one has that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_{m+n}(\Phi_{[g|g']})\big) = 2$ and we define the *glued map* $[g|g'] : \mathbb{P}^{m+n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m+n}_{\mathrm{k}}$ as the map whose base locus $\mathrm{I}_{[g|g']}$ is the $(m+n)$-minors ideal of $\Phi_{[g|g']}$.

In the following, under the notation of Proposition-Definition 5.3.1 and under the assumption that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_m(\Phi_g)\big) = \operatorname{codim} \mathbb{V}\big(\mathrm{I}_n(\Phi_{g'})\big) = 2$, we let $g : \mathbb{P}^{m}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{m}_{\mathrm{k}}$ (resp. $g' : \mathbb{P}^{n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{n}_{\mathrm{k}}$) be the map whose base locus I_g is equal to $\mathrm{I}_m(\Phi_g)$ (resp. $\mathrm{I}_{g'} = \mathrm{I}_n(\Phi_{g'})$).

Proof. Assume that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_m(\Phi_g)\big) = \operatorname{codim} \mathbb{V}\big(\mathrm{I}_n(\Phi_{g'})\big) = 2$ and suppose by contradiction that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_{m+n}(\Phi_{[g|g']})\big) < 2$. Then there is a common factor to each $m+n$ minors of $\Phi_{[g|g']}$, so, after operation on columns, one column of $\Phi_{[g|g']}$ have all its entries sharing a common factor. But it is impossible under the genericity assumption on the entries of $\Phi_{[g|g']}$. $\square$

We emphasize however that the glued map $[g|g']$ of two Koszul-determinantal maps g and g', though determinantal by the previous Proposition-Definition 5.3.1, may not be Koszul-determinantal, as illustrated by the following example.

Example 5.3.2. Put $m = n = 2$ and let $l_1 = l_2 = l_3 = l_4 = 2$ with:

- $\psi_1^{(1)} = \psi_1^{(2)} = x_1$, $\psi_2^{(1)} = \psi_2^{(2)} = x_2$
- $\psi_1^{(3)} = \psi_1^{(4)} = x_2$, $\psi_2^{(3)} = \psi_2^{(4)} = x_3$.

Then, the maps $g : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ and $g' : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ are Koszul-determinantal but $[g|g'] : \mathbb{P}^4_{\mathrm{k}} \dashrightarrow \mathbb{P}^4_{\mathrm{k}}$ is such that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_1(\Phi_{[g|g']})\big) = 3 < 4$ $(\mathrm{I}_1(\Phi_{[g|g']}) = (x_1, x_2, x_3))$.

Let us now describe the projective degrees of a glued determinantal map in the almost linear setting.

Proposition 5.3.3. *Let $d \geqslant 2$, put $l_1 = 2, l_2 = 3d, l_3 = \ldots = l_{2+n} = 1$ and*

- $\psi_1^{(1)} = x_0$, $\psi_2^{(1)} = x_1$
- $\forall k \in \{1, \ldots, 3d\}$, $\psi_k^{(2)}$ *is the k-th generator of the product*
$$(x_0, x_1)^{d-1} \cdot (x_0, x_1, x_2) = (x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, \ldots, x_1^{d-1}x_2, x_1^d).$$
- $\forall l \in \{3, \ldots, m+3\}, k \in \{2, \ldots, m+3\}$, $\psi_k^{(l)} = x_l$

Then the glued map $[g|g'] : \mathbb{P}^{2+n}_{\mathrm{k}} \dashrightarrow \mathbb{P}^{2+n}_{\mathrm{k}}$ whose base ideal $\mathrm{I}_{[g|g']}$ is the $(m+n)$-minors ideal $\Phi_{[g|g']}$ is a determinantal Cremona map and moreover:

$$\forall k \in \{0, \ldots, 2+n\},\ d_k([g|\tau_n]) = \binom{n}{k-2} + (d+1)\binom{n}{k-1} + \binom{n}{k}$$

with the convention that $\binom{j}{i} = 0$ if $i < 0$ or $i > j$.

Remark that, following [BCRD20, Remark 5.13], any almost linear determinantal Cremona map Φ of $\mathbb{P}^2_{\mathrm{k}}$, can be assumed to verified $\mathrm{I}_1(\Phi_g) = (x_0, x_1)$ via a linear change of variables.

Proof. First, under our generic assumptions, $\operatorname{codim} \mathrm{I}_{1+n}(\Phi) = 2$ and for all $k \in \{1, \ldots, n\}$, $\operatorname{codim} \mathrm{I}_k(\Phi) \geqslant 3+n-k$, hence $\Phi_{f|g}$ is Koszul-determinantal. Moreover $\overline{\mathbb{P}(\mathrm{I}_{[g|g']}) \backslash \Gamma_{[g|g']}} \subset \mathbb{V}(\prod_{i=0}^{n} x_i) \subset \mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ hence, by Proposition 5.1.6, we can use the mixed volumes of the polytopes defined by the polynomials $\psi_i^{(j)}$ to compute the projective degrees of $[g|g']$.

By applying Lemma 5.1.4, given any $k \in \{0, \dots, 2+n\}$, one has the formula:

$$\mathrm{MV}_{2(2+n)}(\underbrace{S^{\mathbf{x}}_{2+n}, \dots, S^{\mathbf{x}}_{2+n}}_{2+n-k}, P^{\mathbf{y}}_1, \dots, P^{\mathbf{y}}_{2+m}, \underbrace{S^{\mathbf{y}}_{2+n}, \dots, S^{\mathbf{y}}_{2+n}}_{k})$$

$$=$$

$$\sum_{p=0}^{k} \Big[\Big(\sum_{\substack{\{l_1,\dots,l_p\}\subset \\ \{1,\dots,2\}}} \mathrm{MV}_2(\underbrace{S_2, \dots, S_2}_{2-p}, P_{l_1}, \dots, P_{l_p})\Big) \times$$

$$\Big(\sum_{\substack{\{l_1,\dots,l_{k-p}\}\subset \\ \{3,\dots,2+n\}}} \mathrm{MV}_n(\underbrace{S_n, \dots, S_n}_{n+p-k}, P_{l_1}, \dots, P_{l_{k-p}})\Big)\Big]$$

$$\Updownarrow$$

$$d_k(f|g) = \sum_{p=0}^{k} \big[d_p(f) d_{k-p}(g) \big]$$

with the convention that $d_p(g) = 0$ if $p > 2$ and $d_{k-p}(g') = 0$ if $p > k - n$.

The result of Proposition 5.3.3 follows from the fact that:

- $d(g) = (d_0(g), d_1(g), d_2(g)) = (1, d+1, 1)$
- $d(g') = (1, \binom{n}{1}, \dots, \binom{n}{n-1}, \binom{n}{n})$ as g' is general determinantal map, see [GSP06, Theorem 2]

□

Let us emphasize that this description of a glued determinantal map lead to divide determinantal maps between *elementary ones*, those which cannot be expressed as the glued determinantal map of two initial ones from smaller spaces, and non elementary ones. In this perspective, the general determinantal maps $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ should be considered as non elementary, being the gluing of n Cremona maps $f_1, \dots, f_n : \mathbb{P}^1_{\mathrm{k}} \dashrightarrow \mathbb{P}^1_{\mathrm{k}}$. The almost linear determinantal Cremona maps $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ should also be thought as non elementary as being the gluing of an almost linear determinantal plane Cremona maps $f_1 : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ and $n-1$ Cremona maps $f_2, \dots, f_{n-1} : \mathbb{P}^1_{\mathrm{k}} \dashrightarrow \mathbb{P}^1_{\mathrm{k}}$.

Let us finish by giving the example of an elementary determinantal map $\mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$.

Example 5.3.4. Let $L = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \in \mathrm{R}_3^{2\times 3}$, ($\mathrm{R}_3 = \mathrm{k}[x_0, x_1, x_2, x_3]$) and denote by ψ_1, ψ_2, ψ_3 the generator of the 2-minors ideal of L (the equations of the twisted cubic).

Let $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3 \\ 1\leqslant j\leqslant 3}} \in \mathrm{R}_3^{4\times 3}$ be the matrix such that:

- the entries ϕ_{i1} (resp. ϕ_{i2}) of the first (resp. second) column of Φ_f are general linear combinations of ψ_1, ψ_2, ψ_3.
- the entries ϕ_{i3} of the third column of Φ_f are general linear combinations of x_0, x_1, x_2, x_3.

Then the determinantal map $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ whose Hilbert-Burch matrix is Φ_f is a Cremona map and $d(f) = (1, 5, 5, 1)$.

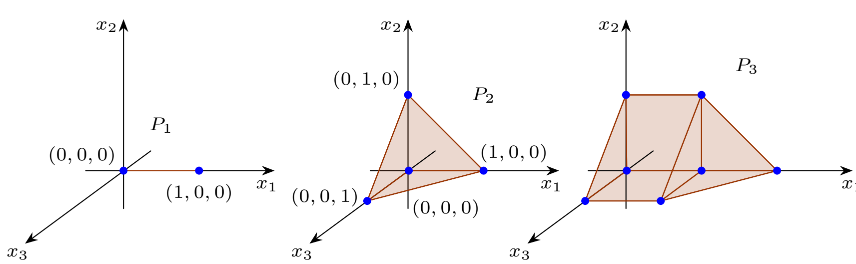
Chapter 6

Determinantal but not Koszul determinantal maps: about the excess intersection case

Introduction

In this section, we want to highlight another generalization of the quarto-quartic construction in [DH17]. By using the notation and convention of Chapter 4, let us present this generalization via polytopes. The quarto-quartic map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ of projective degree $(1, 4, 4, 1)$ described in [DH17] is a determinantal map of Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 3 \\ 1 \leqslant j \leqslant 3}} \in \mathrm{R}^{4\times 3}$ $(\mathrm{R} = \mathrm{k}[x_0, \dots, x_3])$ whose first and second column are filled with polynomials of degree 1 and the third column is filled with polynomial of degree 2. Following Proposition 5.3.3, let us interpret this construction as follows, our convention here is that the considered polytopes in $\mathbb{R}^3$ represent polynomials of R deshomogenized with respect to x_0): given any $i \in \{0, \dots, 3\}$:

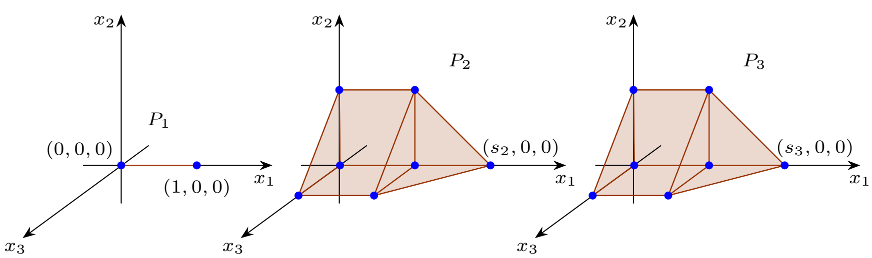
- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(0, 0, 0), (1, 0, 0)\}$
- ϕ_{i2} is generic with respect to $P_2 = S_3 = \mathrm{Conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
- ϕ_{i3} is generic with respect to $P_2 = P_1 + S_3 = \mathrm{Conv}\{(0, 0, 0), (2, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1)\}$



And we saw in the previous section how we can generalized this construction by considering $n \geqslant 4$ polytopes: $P_1 = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}), (1, \underbrace{0,\dots,0}_{n-1}),\ P_2 = \dots = P_{n-1} = S_n$ where S_n is the unit simplex of $\mathbb{R}^n$ and $P_n = dP_1 + S_n$. In this latter case, the associated maps are Koszul-determinantal and associated mixed volumes are the actual projective degrees of the almost linearly presented maps, see Proposition 5.3.3.

Taking another direction, let d_2 and d_3 be two integer and consider a Hilbert-Burch matrix $\Phi_f = (\phi_{ij}) \in \mathrm{R}^{4\times 3}$ ($\mathrm{R} = \mathrm{k}[x_0,\dots,x_3]$) such that given any $i \in \{0,\dots,3\}$:

- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(0,0,0),(1,0,0)\}$
- ϕ_{i2} is generic with respect to $P_2 = (s_2 - 1)P_1 + S_3$ where S_3 is the unit simplex of $\mathbb{R}^3$,
- ϕ_{i3} is generic with respect to $P_2 = (s_3 - 1)P_1 + S_3$.



In higher dimension $n \geqslant 4$, one can generalized this latter construction as follows: let $s_2,\dots,s_n \geqslant 1$ and a matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \in \mathrm{R}^{(n+1)\times n}$ such that for any $i \in \{0,\dots,n\}$:

- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}), (1, \underbrace{0,\dots,0}_{n-1})$
- and for all $j \in \{2,\dots,n\}$, ϕ_{ij} is generic with respect to $P_j = (s_j - 1)P_1 + S_n$ where S_n is the unit simplex of $\mathbb{R}^n$.

A main difference now is that for every any entry Φ_{ij} of Φ_f is in the ideal (x_0, x_1) and, consequently, the associated maps defined by the n-minors of Φ_f are not Koszul-determinantal (for instance $\operatorname{codim} \mathrm{I}_1(\Phi_f) = 2 < n$. However, via an experimental approach, it seems that the mixed volumes associated to the polytopes $P_1,\dots,P_n$ still describe the projective degrees of those maps. In this direction, our goal in this section is to provide strong evidences for the following conjecture:

Conjecture 6.0.1. *Let $n \geqslant 3$, $s_2,\dots,s_n \geqslant 1$, $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \in \mathrm{R}^{(n+1)\times n}$ such that for any $i \in \{0,\dots,n\}$:*

- ϕ_{i1} *is generic with respect to* $P_1 = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}), (1, \underbrace{0,\dots,0}_{n-1})$

- *and for all* $j \in \{2,\dots,n\}$, ϕ_{ij} *is generic with respect to* $P_j = (s_j - 1)P_1 + S_n$ *where* S_n *is the unit simplex of* $\mathbb{R}^n$.

Then the projective degrees of the map f *defined by the* n*-minors of* Φ_f *reads:*

$$\Big(1, \binom{1}{n} + \sum_{i_1 \in \{2,\dots,n\}} (s_{i_1} - 1), \binom{2}{n} + \sum_{\substack{\{i_1, i_2\} \\ \cap \\ \{2,\dots,n\}}} \sum_{j=1}^{2} (s_{i_j} - 1),$$

$$\binom{3}{n} + \sum_{\substack{\{i_1, i_2, i_3\} \\ \cap \\ \{2,\dots,n\}}} \sum_{j=1}^{3} (s_{i_j} - 1), \dots, \binom{n-1}{n} + \sum_{\substack{\{i_1,\dots,i_{n-1}\} \\ \cap \\ \{2,\dots,n\}}} \sum_{j=1}^{n-1} (s_{i_j} - 1), 1\Big)$$

and are in particular palindromic.

Example 6.0.2. Let us illustrate the previous formula in some examples:

- $n = 3, s_2 = 2, s_3 = 3$ and $d(f) = (1, 6, 6, 1)$,
- $n = 4, s_2 = 2, s_3 = 3, s_4 = 4$ and $d(f) = (1, 10, 18, 10, 1)$,
- $n = 5, s_2 = 2, s_3 = 3, s_4 = 4, s_5 = 5$ and $d(f) = (1, 15, 40, 40, 15, 1)$.

As we will explain, this conjecture is supported by the computation of the mixed volumes associated to $P_1, \dots, P_n$ (see Proposition 6.1.1) and some experiments. The only result we will be able to actually show is that, given a map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ as in Conjecture 6.0.1, then $d_{n-1}(f) = 1 + \sum_{i=2}^{n} s_i$ (i.e. f is a determinantal map) and $d_0(f) = 1$ (i.e. f is a Cremona map), see Proposition 6.1.2.

Following [EH16, Chapter 13], about excess intersection descriptions, we will also discuss more precisely the (primary) decomposition of the intersection of three surfaces in $\mathbb{P}^3$ along a line which is our starting case.

One could imagine wilder excess intersection situations (see for instance [EH16, 13.6]) so let us refer to the situation in Conjecture 6.0.1 as a *linear excess intersection* problem emphasizing that the support of the excess intersection is a (codimension 2) linear space. In the third subsection, we will present other linear excess intersection situations which also give rise to Cremona maps.

6.1 Estimation/computation of the projective degrees in the linear excess intersection problem

In this section, we let $n \geqslant 2$, $s_2, \dots, s_n \geqslant 1$ and, using the notations and results about convex geometry in Chapter 4, let:

- $P_1 = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}), (1, \underbrace{0,\dots,0}_{n-1})$ (with respect to any deshomogeneization, say x_0),
- and for all $j \in \{2, \dots, n\}$, $P_j = (s_j - 1)P_1 + S_n$ where S_n is the unit simplex of $\mathbb{R}^n$

In order to support Conjecture 6.0.1, let us first compute the mixed volumes associated to $P_1, \dots, P_n$.

Proposition 6.1.1. *Let $k \in \{1, \dots, n\}$, then given any $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$:*

$$\mathrm{MV}_n(\underbrace{P_{i_1}, \dots, P_{i_k}}_{k}, \underbrace{S_n, \dots, S_n}_{n-k}) = \begin{cases} 1 & \text{if } 1 \in \{i_1, \dots, i_k\} \\ 1 + \sum_{j=1}^{k} (s_{i_j} - 1) & \text{if } 1 \notin \{i_1, \dots, i_k\} \end{cases}.$$

Consequently:

$$\sum_{\substack{\{i_1,\dots,i_k\} \\ \cap \\ \{1,\dots,n\}}} \mathrm{MV}_n(P_{i_1}, \dots, P_{i_k}, S_n, \dots, S_n) = \binom{n}{k} + \sum_{\substack{\{i_1,\dots,i_k\} \\ \cap \\ \{2,\dots,n\}}} \sum_{j=1}^{k} (s_{i_j} - 1) \quad (6.1.1)$$

Moreover this latter quantity is also equal to:

$$\sum_{\substack{\{i_1,\dots,i_{n-k}\} \\ \cap \\ \{1,\dots,n\}}} \mathrm{MV}_n(P_{i_1}, \dots, P_{i_{n-k}}, S_n, \dots, S_n) = \binom{n}{n-k} + \sum_{\substack{\{i_1,\dots,i_{n-k}\} \\ \cap \\ \{2,\dots,n\}}} \sum_{j=1}^{n-k} (s_{i_j} - 1)$$

Proof. First note that Lemma 4.2.5 provides that all mixed volumes involving more than two P_1 terms vanish. Moreover, we denote in the following $\pi_1 : \mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1} \to \mathbb{R}^1$ (resp. $\pi_{n-1} : \mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$) the first (resp. second) projection.

Now let $k \in \{1, \dots, n\}$ and $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$:

- assume that $1 \in \{i_1, \dots, i_k\}$, say $i_1 = 1$. Then using the multilinearity of the mixed volumes Theorem 4.2.3 Item (ii) and the projection formula Lemma 4.2.5:

$$\begin{aligned}\mathrm{MV}_n(P_{i_1}, \dots, P_{i_n}, \underbrace{S_n \dots, S_n}_{n-k}) &= \mathrm{MV}_n(P_1, \underbrace{S_n \dots, S_n}_{n-1}) \\ &= \mathrm{MV}_1\left(\pi_1(P_1)\right) \mathrm{MV}_{n-1}\left(\underbrace{\pi_{n-1}(S_n), \dots, \pi_{n-1}(S_n)}_{n-1}\right) \\ &= \mathrm{MV}_1(S_1)\, \mathrm{MV}_{n-1}(\underbrace{S_{n-1}, \dots, S_{n-1}}_{n-1}) = 1\end{aligned}$$

- Now assume that $1 \notin \{i_1, \dots, i_k\}$, then again by the multilinearity of the mixed volume and the projection formula, one has:

$$\begin{aligned}
&\mathrm{MV}_n(P_{i_1}, \dots, i_k, \underbrace{S_n, \dots, S_n}_{n-k}) \\
&= \mathrm{MV}_n(\underbrace{S_n, \dots, S_n}_{n}) + \sum_{j}^{k} (s_{i_j} - 1)\, \mathrm{MV}_n(\underbrace{S_n, \dots, S_n}_{j-1}, P_1, \underbrace{S_n, \dots, S_n}_{n-j-1}) \\
&= 1 + \sum_{j=1}^{k} (s_{i_j} - 1).
\end{aligned}$$

  Consequently, (6.1.1) follows from the identification $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and the following development:

$$\begin{aligned}
&\sum_{\substack{\{i_1, \dots, i_k\} \\ \cap \\ \{1, \dots, n\}}} \mathrm{MV}_n(P_{i_1}, \dots, P_{i_k}, S_n, \dots, S_n) \\
&\qquad = \sum_{\substack{\{i_2, \dots, i_k\} \\ \cap \\ \{2, \dots, n\}}} \mathrm{MV}_n(P_1, P_{i_2}, \dots, P_{i_k}, S_n, \dots, S_n) \\
&\qquad\quad + \sum_{\substack{\{i_1, \dots, i_k\} \\ \cap \\ \{2, \dots, n\}}} \mathrm{MV}_n(P_{i_1}, \dots, P_{i_k}, S_n, \dots, S_n) \\
&\qquad = \binom{n-1}{k-1} + \sum_{\substack{\{i_1, \dots, i_k\} \\ \cap \\ \{2, \dots, n\}}} \Big(1 + \sum_{j=1}^{k} (s_{i_j} - 1)\Big) \\
&\qquad = \binom{n-1}{k-1} + \binom{n-1}{k} \sum_{\substack{\{i_1, \dots, i_k\} \\ \cap \\ \{2, \dots, n\}}} \sum_{j=1}^{k} (s_{i_j} - 1)
\end{aligned}$$

  The equality

$$\binom{n}{k} + \sum_{\substack{\{i_1, \dots, i_k\} \\ \cap \\ \{2, \dots, n\}}} \sum_{j=1}^{k} (s_{i_j} - 1) = \binom{n}{n-k} + \sum_{\substack{\{i_1, \dots, i_{n-k}\} \\ \cap \\ \{2, \dots, n\}}} \sum_{j=1}^{n-k} (s_{i_j} - 1)$$

  follows from a reorganization of the terms in the sums (and the identification $\binom{n}{k} = \binom{n}{n-k}$).

□

Among the previous numbers, let us now present those for which we can prove that they are the actual projective degrees of a map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose base ideal is the n-minors ideal of a matrix $\Phi_f = (\phi_{ij} \in \mathrm{R}^{(n+1)\times n}$ such that for all $j \in \{1, \dots, n\}$ and $i \in \{0, \dots, n\}$, ϕ_{ij} is generic with respect to P_j.

Proposition 6.1.2. *Let $f = (f_0 : \ldots : f_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the map defined by the n-minors of the matrix Φ_f as in Conjecture 6.0.1. Then:*

(a) $d_{n-1}(f) = 1 + \sum_{j=2}^{n} s_j$ *(i.e. f is determinantal or, in other words, its base ideal I_f is the n-minors ideal of the matrix Φ_f),*

(b) $d_0(f) = 1$ *(i.e. f is a Cremona map).*

Proof. (a) Since the polynomials in Φ_f are generic with respect to the polytopes $P_1, \ldots, P_n$, we can suppose that $x_0^{s_j}$ and $x_1^{s_j}$ appear in each entries of the j-th column of Φ_f for any $j \in \{1, \ldots, n\}$. Hence the monomials $x_0^{1+\sum s_j}$ and $x_1^{1+\sum s_j}$ appear in each n-minor $f_0, \ldots, f_n)$ of Φ_f so, since there are no common component to $f_0, \ldots, f_n$, $\operatorname{codim}\left(\mathrm{I}_n(\Phi_f)\right) = 2$ and f is determinantal.

(b) Let $\mathbf{y} = (y_0 : \ldots : y_n) \in \mathbb{P}^n$ be a general point in the target space of f and consider the product

$$\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} y_0 & \ldots & y_n \end{pmatrix} \Phi_f$$

whose associated zero locus $\mathbb{V}(\phi_1, \ldots, \phi_n)$ in the source space of f contains the actual fiber $f^{-1}(\mathbf{y})$ of $\mathbf{y}$. Via the genericity conditions on the entries of Φ_f and the fact that $\mathbf{y}$ is a general point of $\mathbb{P}^n$, we can assume that:

$$\phi_1 = \lambda_0 x_0 + \lambda_1 x_1 \text{ for } \lambda_1 \neq 0.$$

Consequently, when considering $\mathbb{V}(\phi_1, \ldots, \phi_n) \subset \mathbb{P}^n$, we can set $x_1 = -\frac{\lambda_0}{\lambda_1} x_0$ so that all the other equations $\phi_2, \ldots, \phi_n$ factor as:

$$\phi_2 = x_0 L_2, \phi_3 = x_0^2 L_3, \ldots, \phi_n = x_0^{n-1} L_n,$$

where $L_2, L_3, \ldots, L_n$ are general linear equations in $x_0, x_1, \ldots, x_n$. Hence $\mathbb{V}(\phi_1, \ldots, \phi_n)$ decomposes set-theoretically as an extra point $\mathbf{x} \in \mathbb{P}^n$ away from $\mathbb{V}(x_0, x_1) \subset \mathbb{P}^n$ and the component $\mathbb{V}(x_0, x_1)$ (note that this set-theoretic decomposition is different from the scheme-theoretic one as we will explain in the next subsection). From the genericity assumptions on Φ_f, $\mathbf{x}$ is thus the only point in the fiber $f^{-1}(\mathbf{y})$ hence $d_0(f) = 1$.

□

Remark 6.1.3. In addition to computer experiments and the analogy with the Koszul-determinantal case, let us underline why for any $k \in \{0, \ldots, n\}$ the numbers $\binom{n}{k} + \sum_{\substack{\{i_1, \ldots, i_k\} \\ \cap \\ \{2, \ldots, n\}}} \sum_{j=1}^{k} (s_{i_j} - 1)$ is the k-th projective degree of a determinantal map f whose Hilbert-Burch matrix Φ_f as in Conjecture 6.0.1. In general, given n polytopes in $\mathbb{R}^n$, $Q_1, \ldots, Q_n$, the mixed volume $\mathrm{MV}_n(Q_1, \ldots, Q_n)$ compute the number of solutions with non-zero coordinated (i.e. out of $\mathbb{V}(\prod x_i) = \cup \mathbb{V}(x_i)$) of a generic polynomial system with respect to $Q_1, \ldots, Q_n$, provided this latter polynomial system is zero dimensional. Hence a priori we do not have any interpretation of the mixed volumes in the case of polytopes $Q_1 = P_1, \ldots, Q_n = P_n$ as in Conjecture 6.0.1. However, remark that in this latter case, the excess intersection is located in the codimension 2 linear space $\mathbb{V}(x_0, x_1)$ which is precisely contained in $\cup\mathbb{V}(x_i)$. Consequently, if one would know that given any polynomial system generic with respect to polytopes $Q_1, \ldots, Q_n$, not necessarily zero dimensional but with a finite number δ of solutions with non zero coordinates verify $\mathrm{MV}_n(Q_1, \ldots, Q_n) = \delta$, then Conjecture 6.0.1 could be promoted to a result.

6.2 Residual intersection of three surfaces intersecting along a line in $\mathbb{P}^3$

A remark about *3264 and all that*

Since it appears to be a classical situation, let us focus on the topological degree $d_0(f)$ of a determinantal map f defined by a Hilbert-Burch matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}} \in \mathrm{R}^{4\times 3}$ ($\mathrm{R} = \mathrm{k}[x_0, \ldots, x_3]$) such that for any $i \in \{0, \ldots, 3\}$:

- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(0, 0, 0), (1, 0, 0)\}$,
- ϕ_{i2} is generic with respect to $P_2 = (s_2 - 1)P_1 + S_1$ $(s_2 \geqslant 1)$,
- ϕ_{i3} is generic with respect to $P_3 = (s_3 - 1)P_1 + S_1$, $(s_3 \geqslant 1)$.

As we already explained in the proof of Proposition 6.1.2 Item (b), $d_0(f)$ is the number of points (necessarily of multiplicity 1) away from the line $L = \mathbb{V}(x_0, x_1)$ in the scheme $\mathbb{V}\big(\begin{pmatrix} y_0 & \ldots & y_3\end{pmatrix}\Phi_f\big) \subset \mathbb{P}^3$ where $(y_0 : \ldots : y_3) \in \mathbb{P}^3$ is a general point. Via the genericity assumptions on the entries of Φ_f, remark that this computation aims to describe the points away from L in the intersection $S_1 \cup S_2 \cup S_3 \in \mathbb{P}^3$ where, given $i \in \{1, 2, 3\}$, $S_i = \mathbb{V}(\phi_i)$ for a generic polynomial with respect to P_i. Following the notation in [EH16], write $S_1 \cup S_2 \cup S_3 = L \cap \Gamma$ where Γ is a zero-dimensional subscheme in $\mathbb{P}^3$ (via the genericity assumptions on S_1, S_2 and S_3, one can assume that L is the only strictly positive dimensional subscheme in the intersection $S_1 \cup S_2 \cup S_3$). Actually, [EH16, Chapter 13] establishes an estimation of Γ, provided Γ and L are disjoints, namely:

$$\deg(\Gamma) = 1 \times s_2 \times s_3 - (1 + s_2 + s_3) + 2,$$

see the beginning of [EH16, Chapter 13] for the proof of this formula and, more generally, for the theory about an adapted Bzout's theorem to the excess intersection cases.

Our remark is that in our set up defining a Cremona map, L and the residual scheme $\Gamma = \mathbb{V}\big((\phi_1, \phi_2, \phi_3) : (x_0, x_1)\big)$ are not disjoint since $\deg\Gamma = 1 \times s_2 \times s_3 - (1 + s_2 + s_3) + 2$ whereas $d_0(f) = \deg\mathbb{V}\big((\phi_1, \phi_2, \phi_3) : (x_0, x_1)^{+\infty}\big) = 1$ (where $(\phi_1, \phi_2, \phi_3) : (x_0, x_1)^{\infty}$ stands for the saturation of the ideal (ϕ_1, ϕ_2, ϕ_3) by the ideal (x_0, x_1)). In other words, in our case Γ decomposes as $(1\times s_2\times s_3-(1+s_2+s_3)+2)-1$ points (counted with an eventual multiplicity) on the line L and an extra point (of multiplicity 1 away from L so that a primary decomposition of $\Gamma \cup L$ is:

$$(\phi_1, \phi_2, \phi_3) = (x_0, x_1) \cap \mathrm{I}(\Gamma') \cap (\phi_1, \phi_2, \phi_3) : (x_0, x_1)^{\infty}$$

where $\mathrm{I}(\Gamma')$ is the ideal of the points of Γ on L (and $\deg\mathbb{V}\big(\mathrm{I}(\Gamma')\big) = (1 \times s_2 \times s_3 - (1 + s_2 + s_3) + 2) - 1)$.

In more generality, if $S_1, \dots, S_n \subset \mathbb{P}^n$ ($n \geqslant 2$) are hypersurfaces of respective degree $s_1, \dots, s_n$ such that he only strictly positive dimensional component of $S_1 \cap \dots \cap S_n$ is a codimension 2 linear space L then we decompose the latter intersection as follows:

$$S_1 \cap \dots \cap S_n = L \cup \Gamma$$

and, using again [EH16, Chapter 13], we have compute the degree of Γ:

$$\deg(\Gamma) = \prod_{i=1}^{n} s_i + \sum_{i=2}^{n-2} (-1)^{i-1} \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = n-i}} \prod s_I + (-1)^{n-1}(n-1). \tag{6.2.1}$$

This expression follows from the description of the Chow ring

$$C(\tilde{\mathbb{P}^n}) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^n - \alpha\beta^{n-1})$$

of the blow-up $\tilde{\mathbb{P}^n}$ of $\mathbb{P}^n$ along a codimension 2 linear space ([EH16, Chapter 9]) and the computation in $C(\tilde{\mathbb{P}^n})$ of the product $\prod_{i=1}^{n}(s_i\beta - \epsilon)$ where $\epsilon = \beta - \alpha$ is the class of the exceptional divisor.

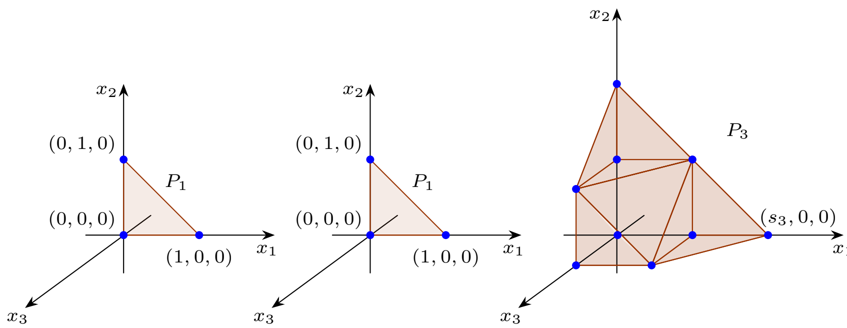
In our case, say $S_1 = \mathbb{V}(\phi_1)$ with ϕ_1 generic with respect to $P_1 = \mathrm{Conv}\{(\underbrace{0,\dots,0}_{n}),$ $(1, \underbrace{0,\dots,0}_{n-1})\}$ and for $i \in \{2, \dots, n\}$, $S_i = \mathbb{V}(\phi_i)$ with ϕ_i generic with respect to $P_i = (s_i - 1)P_1 + S_n$, the zero-dimensional scheme Γ of the scheme-theoretic intersection $S_1 \cap \dots \cap S_n = L \cup \Gamma$ decomposes as 1 point (of multiplicity 1) away from L and ((6.2.1)-1) points (with eventual multiplicity) over L.

6.3 More linear intersection problems

In this last section, we want to present other classes of determinantal Cremona maps related to other linear excess intersection problem. In Section 6.2, we focus on codimension 2 linear excess intersection problem (in $\mathbb{P}^3$ and more generally in $\mathbb{P}^n$), mainly because it is related to classical excess intersection situation such as the one presented in [EH16]. But what about a codimension 3 linear excess intersection in $\mathbb{P}^4$?

Instead of dealing with not particularly enlightening formulas, let us present the problem in $\mathbb{P}^3$ where there is no excess intersection but where we have a visual representation of polytopes. Hence let $s_3 \geqslant 1$ and consider a matrix $\Phi_f = (\phi_{ij})_{\substack{0 \leqslant i \leqslant 3 \\ 1 \leqslant j \leqslant 3}} \in \mathrm{R}^{4\times 3}$ such that:

- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0)\}$,
- ϕ_{i2} is generic with respect to P_1,
- ϕ_{i3} is generic with respect to $P_3 = (s_3 - 1)P_1 + S_3$, where S_3 is the unit simplex of $\mathbb{R}^3$.



Proposition 6.3.1. *Let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the map defined by the 3-minors of the matrix Φ_f defined just above. Then:*

$$d(f) = (1, 2s_3 + 1, s_3 + 1, 1).$$

In particular f is determinantal of Hilbert-Burch matrix Φ_f and whose projective degrees are not palindromic.

Proof. Via the generic conditions on the entries of Φ_f, the only defect of the Fitting ideals of Φ_f comes from the ideal (x_0, x_1, x_2) which has codimension 3. Hence $\operatorname{codim}\left(\mathrm{I}_1(\Phi_f)\right) = 3$ and $\operatorname{codim}\left(I_2(\Phi_f)\right), \operatorname{codim}\left(I_3(\Phi_f)\right) \geqslant 2$. Thus, f is Koszul-determinantal and, since the $\overline{\mathbb{P}(\mathrm{I}_f)\backslash\Gamma_f} = \mathbb{V}(x_0, x_1, x_2) \subset \cup\mathbb{V}(x_i)$, we can compute the projective degree of f via the mixed volumes associated to P_1 and P_3, see Proposition 5.1.6. Namely:

- $d_0(f) = \mathrm{MV}_3(P_1, P_1, (s_3 - 1)P_1 + S_3) = MV_3(P_1, P_1, S_3) = 1,$
-
$$\begin{aligned} d_1(f) =& \mathrm{MV}_3(S_3, P_1, (s_3 - 1)P_1 + S_3) + \mathrm{MV}_3(P_1, S_3, (s_3 - 1)P_1 + S_3) \\ &+ MV_3(P_1, P_1, S_3) \\ =& (s_3 - 1) + 1 + (s_3 - 1) + 1 + 1 = 2s_3 + 1, \end{aligned}$$
- $d_2(f) = s_3 + 1 + 1 = s_3 + 2,$
- $d_3(f) = 1.$

□

In order to compute the topological degree $d_0(f)$ of f, remark that we could also follow the proof of Proposition 6.1.2. Indeed, denote ϕ_1 and ϕ_2 two polynomials generic with respect to P_1 and ϕ_3 a polynomial generic with respect to P_3. Then, via the generic assumptions, the system $\begin{cases} \phi_1 = 0 \\ \phi_1 = 0 \end{cases}$ can be reduced to the system $\begin{cases} x_1 = \lambda x_0 \\ x_2 = \mu x_0 \end{cases}$ for some λ and μ. When substituted in ϕ_3, the latter identities provides the equality $\phi_3 = x_0^{s_3-1}l$ where l is a linear polynomial in x_0 and x_3 which shows, following the proof of Proposition 6.1.2, that $d_0(f) = 1$.

This latter argument applies in the following situation in $\mathbb{P}^4$: let $s_3, s_4 \geqslant 1$ and $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 4\\ 1\leqslant j\leqslant 4}} \in \mathrm{R}^{5\times 4}$ ($\mathrm{R} = \mathrm{k}[x_0, \ldots, x_4]$) such that:

- ϕ_{i1} is generic with respect to $P_1 = \mathrm{Conv}\{(0,0,0,0),(1,0,0,0),(0,1,0,0)\}$,
- ϕ_{i2} is generic with respect to P_1,
- ϕ_{i3} is generic with respect to $P_3 = (s_3-1)P_1 + S_4$, where S_3 is the unit simplex of $\mathbb{R}^4$.
- ϕ_{i4} is generic with respect to $P_4 = (s_4-1)P_1 + S_4$.

Via the computation of the the mixed volumes associated with P_1, P_3, P_4, one can conjecture the following (also supported by experimental computations):

Conjecture 6.3.2. *Let $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 4\\ 1\leqslant j\leqslant 4}} \in \mathrm{R}^{5\times 4}$ be as just above and $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ the map defined by the 4-minors of Φ_f. Then:*

$$d(f) = (1, 2(s_3+s_4), 3s_3+3s_4+(s_3-1)(s_4-1), 1+2s_3+2s_4+s_3s_4, s_3+s_4+2, 1)$$

As in Section 6.1, Conjecture 6.3.2 would be true if we know that Bernstein's theorem could be applied in the situation of a polynomial system with a finite number of solutions with all non-zero coordinates. However following the proof of Proposition 6.1.2, we are able to show:

Proposition 6.3.3. *Let $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 4\\ 1\leqslant j\leqslant 4}} \in \mathrm{R}^{5\times 4}$ and $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ as in Conjecture 6.3.2, then:*

(a) $d_3(f) = s_3 + s_4 + 2$ *(i.e. f is determinantal),*

(b) $d_2(f) = 1 + 2s_3 + 2s_4 + s_3s_4$

(c) $d_0(f) = 1$ *(i.e. f is a Cremona map).*

Proof. Let us only focus on $d_2(f)$ since we already explained how to handle $d_3(f)$ and $d_0(f)$ in the proof of Proposition 6.1.2. Via the generic assumptions on the entries of Φ_f, remark that $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_4(\Phi_f))\big) \geqslant 2$ and $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_3(\Phi_f))\big) \geqslant 3$ so that $d_2(f) = \delta_2(f)$ where $\delta_2(f)$ is the second projective degree of $\mathbb{P}(\mathrm{I}_f) \subset \mathbb{P}^n \times \mathbb{P}^n$ and is equal to the value $\sigma_{2,4}(1,1,s_3,s_4) = 1+2s_3+2s_4+s_3s_4$ of the second symmetric polynomial in four variables evaluated at $1,1,s_3,s_4$, see Proposition 1.1.9. □

Remark 6.3.4. One can even increase the considered classes of Cremona maps. Indeed let a codimension k linear space, say $\mathbb{V}(x_0, \ldots, x_{k-1})$, in $\mathbb{P}^n$, $s_k, \ldots, s_n \leqslant 1$ and a matrix $\Phi_f = (\phi_{ij})_{\substack{0\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}} \in \mathrm{R}^{(n+1)\times n}$ ($\mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$) such that for any $i \in \{0, \ldots, n\}$:

- $\phi_{i1}, \ldots, \phi_{i,k-1}$ is generic with respect to

$$P_1 = \mathrm{Conv}\{(\underbrace{0,\ldots,0}_{n}),(1,\underbrace{0,\ldots,0}_{n-1}),\ldots,(\underbrace{0,\ldots,0}_{k-2},1,\underbrace{0,\ldots,0}_{n-k+1})\}$$

(for a deshomogenization with respect to x_0),

- $for all j \in \{k, \ldots, n\}$, ϕ_{ij} is generic with respect to $P_2 = (s_j - 1)P_1 + S_n$ where S_n is the unit simplex of $\mathbb{R}^n$.

By the same consideration as before, one can show that the maps f defined by the n-minors of Φ_f are determinantal Cremona maps and one can infer their projective degrees.

Appendix

Appendix A

Characteristic classes of pfaffian Cremona maps

Introduction

In this section, we focus on *pfaffian Cremona maps* that is Cremona maps $q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $n \geqslant 4$ even, whose base ideal I_q is generated by the n-pfaffians of a $(n+1) \times (n+1)$ skew-symmetric matrix Φ_q. If the associated ring $\mathrm{R}/\,\mathrm{I}_q$ is arithmetically Gorenstein of codimension 3, which will always be the case in this work, its free resolution is understood in an analogue way that for determinantal rings, see [BH93, 3.4], so pfaffian Cremona maps are a natural next case to study after considering determinantal Cremona maps. A natural case that has indeed been already considered in the literature and let us gather now the result relevant for our work that we could found:

- in [RS01, Example 2.5, *Pfaffians of an odd size skew symmetric matrix*], the two author establish the main properties (for the scope of our work), a pfaffian map $q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ ($n \geqslant 4$ even), given by the n-pfaffians of a $(n+1) \times (n+1)$ skew symmetric matrix Φ_q whose subdiagonal entries are general linear polynomials in $\mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$ (the quoted result is actually more general but let us restrict to this situation), namely:

  · $d_0(q) = 1$,

  · $d_1(q) = n - 1$,

  · $d_{n-1}(q) = \frac{n}{2}$.

  The base locus of the inverse q^{-1} is moreover a codimension 2 arithmetically Buchsbaum variety of degree $\frac{(n+1)(n-2)}{2}$ characterized by its minimal free resolution:

  $$0 \to \mathrm{R}(-n-1) \to \mathrm{R}(-n)^{n+1} \to \mathrm{R}(-n+1)^{n+1} \to \mathrm{R} \to \mathrm{R}/\,\mathrm{I}_{q^{-1}} \to 0.$$

- Even if we will not use it, let us quote [KPU17] describing in particular the equations of the graph of the base ideal of a pfaffian Cremona whose subdiagonal entries of its associated skew-symmetric matrix Φ_f are linear polynomials. Actually, looking retrospectively, the strategy developed to show [KPU17, Prop. 4.2] and compute the topological degree of a pfaffian rational map $q : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{d-1}$ is the one we will outline to compute the other degrees.

Complementary to the previous results, a reason for this work was to provide the systematic computation of all the projective degrees of the *general pfaffian Cremona maps* i.e. maps whose base ideal is the n-pfaffian ideal of a $(n+1)\times(n+1)$ matrix Φ_q whose subdiagonal entries are general linear polynomial, our model being the article [GSP06] performing this description for general determinantal Cremona maps (i.e. maps whose base ideal is the n-minors ideal of a $(n+1)\times n$ matrix filled with general linear polynomials). We will show in particular:

Proposition A.0.1. *(a) the projective degrees* $(d_0(q), d_1(q), d_2(q), d_3(q), d_4(q))$ *of a general pfaffian map of* $\mathbb{P}^4$ *reads*

$$(1, 3, 2^2, 2, 1),$$

(b) the projective degrees $(d_0(q), \ldots, d_6(q))$ *of a general pfaffian map of* $\mathbb{P}^6$ *reads*

$$(1, 5, 11, 13, 3^2, 3, 1),$$

As it was pointed out by Profesor Daniele Faenzi in his report of this manuscript, the study of the Chern classes of the cotangent sheaf $\Omega_{\mathbb{P}^n_{\mathrm{k}}}$ of $\mathbb{P}^n_{\mathrm{k}}$ provides formulae for the projective of general pfaffian maps of $\mathbb{P}^n_{\mathrm{k}}$, see Proposition A.1.8.

Contents of the section

To show Proposition A.0.1, we will rely on [KU92, Theorem 10.5] describing a free resolution of a residual ideal $J = (a : \mathrm{I}_q)$ associated to the (Gorenstein of codimension 3) base ideal I_q of q with respect to an ideal $a = (a_1, \ldots, a_f)$ generated by $3 \leqslant f \leqslant n$ linear combinations of the pfaffians of Φ_q for $f = 3, \ldots, n$ and $n = 4$ or $n = 6$ (to fit with the notation in [KU92], q will stand for a map, g will be the integer $n+1$ and f will be an integer between 2 and n). Since Φ_q is filled with linear polynomials, $(a : \mathrm{I}_q) = (a : \mathrm{I}_q^\infty)$ so the degree of J is the $(n-f)$-th projective of q. In the end, the only difficulty here is to identify the maps in the free resolution of J described in [KU92, Section 2 and 4] and to describe from it a graded free resolution of J. Let us point out that one could a priori compute the projective degrees of a general pfaffian Cremona maps in higher dimension $n \geqslant 8$ with this approach. However, as it was explained to us by Profesor Daniele Faenzi, we will describe those projective degrees via the study of the cotangent sheaf $\Omega_{\mathbb{P}^n_{\mathrm{k}}}$ of $\mathbb{P}^n_{\mathrm{k}}$, see Proposition A.1.8.

Let us also underline that our approach via residuality is different, and in some sense more naïve, than the one developed in [GSP06]. In this latter article, the authors used the identification between the family of base locus of determinantal maps with an open and connected subset of the Hilbert scheme of arithmetically Cohen-Macaulay subscheme of codimension 2 of $\mathbb{P}^n$ whose dimension and other properties were described in [Ell75]. Via a deformation argument, the two authors showed then that the projective degrees of the general determinantal Cremonan maps are the same as those of the standard Cremona map which can be computed via standard arguments of toric geometry. We believe that such a strategy is applicable for general pfaffian Cremona maps, especially because the analogue of Ellinsgrud's result was obtain in [KMR98]. It would then remain to find a "good" monomial pfaffian Cremona map to actually compute the projective degree of any general pfaffian Cremona map (but then, one would probably be anyway in difficulty to describe the general formula for those numbers).

In a second time, in Appendix A.2, we focus on excess intersection numbers that can be attached to the singular locus of the base locus of the standard Cremona map of $\mathbb{P}^n$ and *standard pfaffian Cremona maps* of $\mathbb{P}^n$. We will show how to numerically estimate these numbers, see the output of Algorithm A.2.1.

A.1 Projective degrees of the general pfaffian Cremona maps

In all this section, let $n \geqslant 4$ be an even integer. Before going into the details of the proof of Proposition A.0.1, let us shortly re-explain its main arguments. Under the assumption that $q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a general pfaffian Cremona map, its base ideal I_q is of linear type, see Proposition A.1.1. This initial remark should be contained in works such as [Mor96] and [KPU17] but we will give it a proof via a deformation argument in the spirits of [GSP06]. Since I_q is of linear type, the saturation $J = (a : \mathrm{I}_q^\infty)$ of an ideal a generated by general linear combinations of the generators of I_q by I_q is the same as the colon ideal $(a : \mathrm{I}_q)$, see Corollary A.1.2. Since [KU92] provides (graded) free resolutions of these latter colon ideals, one can then deduce the projective degrees of the associated maps.

Proposition A.1.1. *Let $\Phi_q \in \mathrm{R}^{(n+1)\times(n+1)}$ ($\mathrm{R} = \mathrm{k}[x_0, \ldots, x_n]$) be a skew-symmetric matrix whose subdiagonal entries are general linear polynomials and let I_q be the ideal of n-pfaffians of Φ_f.*

Then I_q is of linear type, i.e. the Rees algebra of I_q is equal to the symmetric algebra of I_q.

We prove Proposition A.1.1 by showing that a particular monomial Gorenstein ideal of codimension 3 is of linear type and that we can extend this result using [KMR98].

Proof. We apply [RS01, Crit. (3)], namely, I_f is of linear type if and only if:

$$\forall t \geqslant 1,\ \operatorname{codim} \mathrm{I}_t(\Phi_q) \geqslant n - t + 2. \tag{Crit.(3)}$$

Actually, we only show Equation (Crit.(3)) when I_q is the n-pfaffian ideal of a particular skew-symmetric matrix

$$\Phi_q = \begin{pmatrix} 0 & -x_1 & 0 & \dots & 0 & -x_0 \\ x_1 & 0 & \ddots & \ddots & & 0 \\ 0 & x_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 0 & -x_n \\ x_0 & 0 & \dots & 0 & x_n & 0 \end{pmatrix}.$$

In this case, the ideal $\mathcal{J}$ of $\mathbb{P}(\mathrm{I}_q)$ in $\mathbb{P}^n \times \mathbb{P}^n$ reads

$$\mathcal{J} = (y_1x_1 + y_nx_0, -y_0x_1 + y_2x_2, \dots, -y_{n-1}x_n + y_{n+1}x_{n+1}, -y_0x_0 - y_nx_{n+1}) \tag{A.1.1}$$

and one can apply [AL94, Prop. 2.3.5] to show that $\mathcal{J}$ is prime. Here let us not write down the details of the computations by only precising that this step requires the computation of a Groebner basis of $\mathcal{J}$ via Burchberger's algorithm, see [AL94]. A Gröbner basis of $\mathcal{J}$ contains actually more elements than the generators appearing in (A.1.1) but the formulas for the new elements of the basis can be given explicitly.

Hence the Rees algebra and the symmetric algebra of I_q coincide and (Crit.(3)) is verified in this case. But by [KMR98, Th.2.6], since all arithmetically Gorenstein subscheme of codimension 3 of $\mathbb{P}^n$ associated to a skew-symmetric matrix whose subdiagonal is filled with linear polynomials belongs to the same open subset of the associated Hilbert scheme, one has:

$$\forall t \geqslant 1,\ \operatorname{codim} \mathrm{I}_t(\Phi_q) \geqslant \operatorname{codim} \mathrm{I}_t(\Phi_q) \geqslant n - t + 2$$

for any skew-symmetric matrix of size $(n+1) \times (n+1)$ whose subdiagonal is filled with general linear polynomials. Hence (Crit.(3)) is verified also for those latter maps whose base ideal is thus of linear type. □

Under the hypothesis of Proposition A.1.1, the ideals of minors of successive size $t \geqslant 1$ of the presentation matrix Φ_q of I_q have the expected dimension so one has:

Corollary A.1.2. *Under the hypothesis of Proposition A.1.1, let $q = (q_0 : \dots : q_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the map defined by the n-pfaffians of Φ_q, then:*

$$\forall f \in \{0, \dots, n\},\ d_{n-f}(q) = \deg J_{(f,g)}$$

where $J_{(f,g)} = (a_{(f)} : \mathrm{I}_q)$ and $a_{(f)}$ is generated by f general linear combinations of $q_0, \dots, q_n$.

The main point in our strategy to show Proposition A.0.1 is that, given $f \in \{2, \dots, n\}$, a (minimal) graded free resolution of the ideal $J_{(f,g)}$ (so in particular their codimension and their degree) can be extracted from [KU92, Th.10.5] and we outline how now. A priori the complexes $\mathfrak{D}^0$ ([KU92, Figure 4.4]) are only free resolution of the ideals $J_{(f,g)}$. Hence to obtain a graded resolution of $J_{(f,g)}$ from $\mathfrak{D}^0$, one has to chase the graduation of the maps involved in $\mathfrak{D}^0$ defined in [KU92, Definition 2.15] (or to recall that the involved maps between free modules are restriction of differentials of a Eagon-Northcott complex associated to Φ_q). For instances, the module

$$L_{ij} \simeq (\bigoplus_{v=0}^{f} L_{i+1}^{j-f+v}(\mathrm{R}^g \otimes \wedge^v \mathrm{R}^g)^*$$

has to be written

$$L_{ij} \simeq \bigoplus_{v=0}^{f} \mathrm{R}\big(-(j-f+v)+1\big)^{rk_v} \otimes \wedge^v \mathrm{R}^f$$

where the rank rk_v are described in [KU92, Formula (2.39)].

Let us now enter in the details of the computation of the free resolution of each degree. We will divide the proof of Proposition A.0.1 in several lemma each one focusing on one projective degree at a time, the simplest case being:

Lemma A.1.3. *Under the notations and hypothesis of Proposition A.0.1, if $n = 4$ or $n = 6$, $d_0(q) = 1$.*

Proof.

- Case $n = f = 4, g = 5$. Following [KU92, Th.10.5], a free resolution of $J_{(4,5)}$ reads:

$$\mathcal{D}^0: \qquad 0 \to L_{2,9} \to L_{1,8} \to L_{0,7} \to Q_0 \to K_{0,0} \to 0$$

where the involved free modules are described in [KU92, Prop.2.41] and [KU92, Definition.2.15]. Hence a graded resolution of $J_{(4,5)}$ reads:

$$0 \to \binom{7}{7}\binom{6}{4}\binom{4}{0}\mathrm{R}(-4) \to \begin{matrix}\binom{6}{5}\binom{4}{3}\binom{4}{0}\mathrm{R}(-3)\\ \oplus \\ \binom{6}{6}\binom{5}{4}\binom{4}{1}\mathrm{R}(-4)\end{matrix}$$

$$\to \begin{matrix}\binom{5}{3}\binom{2}{2}\binom{4}{0}\mathrm{R}(-2)\\ \oplus \\ \binom{5}{4}\binom{3}{3}\binom{4}{1}\mathrm{R}(-3)\\ \oplus \\ \binom{5}{5}\binom{4}{4}\binom{4}{2}\mathrm{R}(-4)\end{matrix} \longrightarrow \begin{matrix}\binom{4}{3}\mathrm{R}(-1)\\ \oplus \\ \binom{4}{1}\mathrm{R}(-2)\end{matrix} \longrightarrow \mathrm{R} \to 0$$

from which we can extract the following minimal free resolution of $J_{(4,5)}$:

$$0 \to \binom{4}{4}\mathrm{R}(-4) \to \binom{4}{3}\mathrm{R}(-3) \to \binom{4}{2}\mathrm{R}(-2) \to \binom{4}{1}\mathrm{R}(-1) \to \mathrm{R} \to 0.$$

Since the latter resolution is nothing but the Koszul complex on four independent linear polynomials one has

$$\text{codim } J_{(4,5)} = 4, \ \deg J_{(4,5)} = 1$$

(which can also be obtained by computing the Hilbert polynomial of $J_{(4,5)}$ from its graded resolution).

- Case $n = f = 6, g = 7$. A free resolution of $J_{(6,7)}$ reads:

$$\mathcal{D}^0: \qquad 0 \to L_{4,13} \to L_{3,12} \to L_{2,11} \to L_{1,10} \to L_{0,9}$$

$$\longrightarrow Q_0 \to K_{0,0} \to 0$$

where the involved free modules are described in [KU92, Prop.2.41] and [KU92, Definition.2.15]. Hence a graded resolution of $J_{(6,7)}$ reads:

$$0 \rightarrow \binom{11}{11}\binom{10}{6}\binom{6}{0}\mathrm{R}(-6) \rightarrow \begin{array}{c}\binom{10}{9}\binom{8}{5}\binom{6}{0}\mathrm{R}(-5)\\ \oplus \\ \binom{10}{10}\binom{9}{6}\binom{6}{1}\mathrm{R}(-6)\end{array} \rightarrow \dots$$

$$\dots \longrightarrow \begin{array}{c}\binom{7}{3}\binom{2}{2}\binom{6}{0}\mathrm{R}(-2)\\ \oplus \\ \binom{7}{4}\binom{3}{3}\binom{6}{1}\mathrm{R}(-3)\\ \oplus \\ \binom{7}{5}\binom{4}{4}\binom{6}{2}\mathrm{R}(-4)\\ \oplus \\ \binom{7}{6}\binom{5}{5}\binom{6}{3}\mathrm{R}(-5)\\ \oplus \\ \binom{7}{7}\binom{6}{6}\binom{6}{4}\mathrm{R}(-6)\end{array} \longrightarrow \begin{array}{c}\binom{6}{5}\mathrm{R}(-1)\\ \oplus \\ \binom{6}{3}\mathrm{R}(-2)\\ \oplus \\ \binom{6}{1}\mathrm{R}(-3)\end{array} \longrightarrow \mathrm{R} \rightarrow 0$$

from which one can extract the following minimal graded free resolution of $J_{(6,7)}$:

$$0 \rightarrow \binom{6}{6}\mathrm{R}(-6) \rightarrow \binom{6}{5}\mathrm{R}(-5) \rightarrow \dots \rightarrow \binom{6}{2}\mathrm{R}(-2) \rightarrow \binom{6}{1}\mathrm{R}(-1) \rightarrow \mathrm{R} \rightarrow 0.$$

Hence

$$\operatorname{codim} J_{(6,7)} = 6, \ \deg J_{(6,7)} = 1.$$

□

Lemma A.1.4. *Under the notations and hypothesis of Proposition A.0.1,*

- *if* $n = 4$, $d_1(q) = 3$,
- *if* $n = 6$, $d_1(q) = 5$.

Proof.
Case $n = 4, f = 3, g = 5$. By [KU92, Th.10.5], a free resolution of $J_{(3,5)}$ reads:

$$\mathcal{D}^0 : \qquad 0 \rightarrow L_{1,8} \rightarrow L_{0,7} \rightarrow Q_0 \rightarrow K_{0,0} \rightarrow 0$$

that can be translate in the following graded free resolution of $J_{(3,5)}$:

$$0 \rightarrow \binom{6}{6}\binom{5}{4}\binom{3}{0}\mathrm{R}(-4) \rightarrow \begin{array}{c}\binom{5}{4}\binom{3}{3}\binom{3}{0}\mathrm{R}(-3)\\ \oplus \\ \binom{5}{5}\binom{4}{4}\binom{3}{1}\mathrm{R}(-4)\end{array} \rightarrow \begin{array}{c}\binom{3}{3}\mathrm{R}(-1)\\ \oplus \\ \binom{3}{1}\mathrm{R}(-2)\end{array} \rightarrow \mathrm{R} \rightarrow 0$$

from which one extract the following minimal free resolution of $J_{(3,5)}$:

$$0 \rightarrow 2\mathrm{R}(-4) \rightarrow 5\mathrm{R}(-3) \rightarrow \begin{array}{c}\mathrm{R}(-1)\\ \oplus \\ 3\mathrm{R}(-2)\end{array} \rightarrow \mathrm{R} \rightarrow 0$$

so, via the computation of the Hilbert polynomial of $J_{(3,5)}$, one has

$$\operatorname{codim} J_{(3,5)} = 3, \ \deg J_{(3,5)} = 3.$$

Case $n = 6, f = 5, g = 7$. By [KU92, Th.10.5], a free resolution of $J_{(5,7)}$ reads:

$$\mathcal{D}^0 : \qquad 0 \to L_{3,12} \to L_{2,11} \to L_{1,10} \to L_{0,9} \to Q_0 \to K_{0,0} \to 0$$

so that, after simplification, a minimal graded free resolution of $J_{(5,7)}$ reads:

$$0 \to 4R(-6) \to 19\mathrm{R}(-5) \to 35\mathrm{R}(-4) \to 30\mathrm{R}(-3) \to \begin{array}{c}\mathrm{R}(-1)\\ \oplus \\ 10\mathrm{R}(-2)\end{array} \to \mathrm{R} \to 0$$

and so one has

$$\operatorname{codim} J_{(5,7)} = 5, \ \deg J_{(5,7)} = 5.$$

□

Lemma A.1.5. *Under the notations and hypothesis of Proposition A.0.1, in the case $n = 6$, $d_2(f) = 11$ and $d_3(f) = 13$.*

Proof.

- Case $n = 6, f = 4, g = 7$. By [KU92, Th.10.5], a free resolution of $J_{(4,7)}$ reads:

  $$\mathcal{D}^0 : \qquad 0 \to L_{2,11} \to L_{1,10} \to L_{0,9} \to Q_0 \to K_{0,0} \to 0$$

  which can be translated, after simplification, into the following minimal graded free resolution:

  $$0 \to 6\mathrm{R}(-5) \to 20\mathrm{R}(-4) \to 21\mathrm{R}(-3) \to \begin{array}{c}4\mathrm{R}(-2)\\ \oplus \\ 4\mathrm{R}(-3)\end{array} \to \mathrm{R} \to 0$$

  so

  $$\operatorname{codim} J_{(4,7)} = 4, \ \deg J_{(4,7)} = 11.$$

- Case $n = 6, f = 3, g = 7$. By [KU92, Th.10.5], a free resolution of $J_{(3,7)}$ reads:

  $$\mathcal{D}^0 : \qquad 0 \to L_{1,10} \to L_{0,9} \to Q_0 \to K_{0,0} \to 0$$

  which can be translated, after simplification, into the following minimal graded free resolution:

  $$0 \to 4\mathrm{R}(-6) \to 7\mathrm{R}(-5) \to \begin{array}{c}\mathrm{R}(-2)\\ \oplus \\ 4\mathrm{R}(-3)\end{array} \to \mathrm{R} \to 0$$

  so

  $$\operatorname{codim} J_{(3,7)} = 3, \ \deg J_{(3,7)} = 13.$$

□

Now let us treat apart the computation of $d_2(q)$, if $n=4$ and of $d_4(q)$, if $n=6$. Indeed, in those cases, we cannot apply [KU92, Th.10.5] because in both cases, we are considering the cases $f = 2 < 3 = \operatorname{codim} \mathrm{I}_q$ which is out of the scope of [KU92, Th.10.5] (let us point out however that in practice the free complex $\mathcal{D}^0$ associated to $J_{(2,5)}$ or $J_{(2,7)}$ actually resolve those ideals).

Lemma A.1.6. *Under the notations and hypothesis of Proposition A.0.1,*

- *if* $n=4$, $d_2(q) = 2^2$,
- *if* $n=6$, $d_4(q) = 3^2$.

Proof. In both cases $n=4$ or $n=6$, let a_1 and a_2 be two general linear combinations of the generators of I_f. But $\operatorname{codim} \mathrm{I}_f = 3$ and $\operatorname{codim}(a_1,a_2) = 2$, one has that $(a_1,a_2) : \mathrm{I}_f = (a_1,a_2)$ so that:

- if $n=4$, $d_2(q) = 2 \times 2 = 2^2$,
- if $n=6$, $d_4(q) = 3 \times 3 = 3^2$.

□

Now, following [GSP06, Proposition 5], one can easily compute the Segre class $s\big(\mathbb{V}(\mathrm{I}_q)\big) = \sum_{k=3}^{n} s_k[H^k]$ of $\mathbb{V}(\mathrm{I}_q)$ in the Chow group of $\mathbb{P}^n$ and where $[H^k]$ is the class of a codimension i linear space of $\mathbb{P}^n$ from the projective degree of a Cremona map q:

Corollary A.1.7. *Let* I_q *be the base ideal of a general pfaffian map* q *of* $\mathbb{P}^n$*, then:*

- *if* $n=4$, $s\big(\mathbb{V}(\mathrm{I}_q)\big) = -25H^4 + 5H^3$,
- *if* $n=6$, $s\big(\mathbb{V}(\mathrm{I}_q)\big) = -1666H^6 + 448H^5 - 98H^4 + 14H^3$.

As explained to us by Profesor Daniele Faenzi in his report of this manuscript, given $n \geqslant 4$ an even integer, the projective degrees $d_0(f), \dots, d_n(f)$ of a general pfaffian map $f : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ are described by the Chern classes $c_0(\Omega_{\mathbb{P}^n}(2)), \dots, c_n(\Omega_{\mathbb{P}^n}(2))$ of a twisted copy of the cotangent sheaf $\Omega_{\mathbb{P}^n}$ of $\mathbb{P}^n$. Indeed, the graph $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$ of f is the zero locus of a global section of the bundle $E = p_1^*(\Omega_{\mathbb{P}^n}(2)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^n})$ vanishing in the expected dimension and where p_1 (resp. p_2) is the first (resp. second) projection of $\mathbb{P}^n \times \mathbb{P}^n$. For each $i \in \{0, \dots, n\}$, looking at the degree of the intersection $p_1^*(H_1^i) \cap \Gamma_f \cap p_2^*(H_2^{n-i})$ where H_1^i is a general linear space of $\mathbb{P}^n$ of codimension i and H_2^{n-i} is a general linear space of $\mathbb{P}^n$ of codimension $n-i$, aims to compute the number $\int_{\mathbb{P}^n \times \mathbb{P}^n} [p_1^*(H_1^i)] \cdot \Gamma_f \cdot [p_2^*(H_2^{n-i})]$ which is equal to the degree of $c_i\big(\Omega_{\mathbb{P}^n}(2)\big)$ (where $[X]$ stands for the class defined by X in the Chow group $A(\mathbb{P}^n_{\Bbbk} \times \mathbb{P}^n_{\Bbbk})$ of $\mathbb{P}^n \times \mathbb{P}^n$, see [EH16, Chapter 1] for the associated notations and background). By standard computations with Chern classes, see [EH16, Ex. 5.14 and 5.5.1], one obtains thus:

Proposition A.1.8. *Let $n \geqslant 4$ be an even integer and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a general pfaffian map, then for any $i \in \{0, \ldots, n\}$:*

$$d_i(f) = \sum_{j=0}^{i} (-1)^{i-j} 2^j \binom{n-i+j}{j} \binom{n+1}{i-j}.$$

Looking back, the Chern classes of $\Omega_{\mathbb{P}^n}$ provide thus a clear understanding of the projective degrees of general pfaffian maps. We have however let our initial approach via residuality in the case $n = 4$ and $n = 6$ since it gave an application of classical results about pfaffian ideals.

Proof of Proposition A.1.8. In the following, we identify a Chern class in the Chow group $A(\mathbb{P}^n) \simeq \mathbb{Z}$ of $\mathbb{P}^n$ with its degree (it is a classical identification since $A(\mathbb{P}^n) \simeq \mathbb{Z}$).

First, for all $l \in \{0, \ldots, n\}$, $c_l(\Omega_{\mathbb{P}^n}) = (-1)^l \binom{n+1}{l}$. Indeed, one computes the Chern polynomial $c_t(\Omega_{\mathbb{P}^n}) = c_0 + c_1 t + \ldots + c_n t^n$ from the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

where $\mathcal{T}_{\mathbb{P}^n}$ is the tangent sheaf of $\mathbb{P}^n$ and the fact that $c_t(\Omega_{\mathbb{P}^n}) = c_{-t}(\mathcal{T}_{\mathbb{P}^n})$, see [EH16, Ex. 5.14].

Given now $i \in \{0, \ldots, n\}$,

$$\begin{aligned} c_i\big(\Omega_{\mathbb{P}^n}(2)\big) &= c_i\big(\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(2)\big) \\ &= \sum_{j=0}^{i} \binom{n-i+j}{j} c_1\big(\mathcal{O}_{\mathbb{P}^n}(2)\big)^j c_{i-j}(\Omega_{\mathbb{P}^n}) \end{aligned}$$

by the formula in [EH16, Proposition 5.17] which concludes the proof. □

Example A.1.9. If $n = 8, 10$ or 12, the projective degrees of a general pfaffian Cremona map $q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ read:

- if $n = 8$, $(1, 7, 22, 40, 46, 34, 4^2, 4, 1)$,
- if $n = 10$, $(1, 9, 37, 91, 148, 166, 130, 70, 5^2, 5, 1)$,
- if $n = 12$, $(1, 11, 56, 174, 367, 553, 610, 496, 295, 125, 6^2, 6, 1)$.

After treating these simple cases of general pfaffian Cremona maps, the question remains to handle/detect other pfaffian Cremona maps whose associated skew-symmetric matrix is filled with polynomials of degree greater than 1. Let us point out that this problem seems more difficult as in the determinantal case because, as far as we know, we cannot use Bernstein's theorem as we did for Koszul-determinantal maps. Nevertheless, analogous to the determinantal case, "non general" pfaffian Cremona maps do exist as illustrated by the following example arising by composing two general pfaffian Cremona maps.

Example A.1.10. By a computation with Macaulay2, one has that the map $q : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ defined by the 4-pfaffians of the matrix

$$\Phi_q = \begin{pmatrix} 0 & -x_1x_3 & -x_0x_2 & -x_1x_4 & x_0x_3 \\ x_1x_3 & 0 & -x_1x_4 & -x_0x_3 & -x_2x_4 \\ x_0x_2 & x_1x_4 & 0 & -x_2x_4 & -x_1x_3 \\ x_1x_4 & x_0x_3 & x_2x_4 & 0 & -x_0x_2 \\ x_0x_3 & x_2x_4 & x_1x_3 & x_0x_2 & 0 \end{pmatrix}$$

is a pfaffian map and has projective degrees:

$$d(q) = (1, 9, 16, 4, 1).$$

The map q is actually the composition $q_1 \circ q_2$ of two pfaffian maps q_1 and q_2 whose defining matrices are filled with linear polynomial, namely:

$$\Phi_{q_2} = \begin{pmatrix} 0 & -x_1 & 0 & 0 & -x_0 \\ x_1 & 0 & -x_2 & 0 & 0 \\ 0 & x_2 & 0 & -x_3 & 0 \\ 0 & 0 & x_3 & 0 & -x_4 \\ x_0 & 0 & 0 & x_4 & 0 \end{pmatrix}$$

and

$$\Phi_{q_1} = \begin{pmatrix} 0 & -x_0 & -x_1 & -x_2 & -x_3 \\ x_0 & 0 & -x_2 & -x_3 & -x_4 \\ x_1 & x_2 & 0 & -x_4 & -x_0 \\ x_2 & x_3 & x_4 & 0 & -x_1 \\ x_3 & x_4 & x_0 & x_1 & 0 \end{pmatrix}$$

emphasizing why $d(q) = (1^2, 3^2, 4^2, 2^2, 1^2)$ as it is expected to be, see Proposition B.1.1.

Let us finish by writing down two of our remaining questions concerning pfaffian Cremona maps:

- Does there exist $q : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ a pfaffian Cremona map of matrix Φ_q such that

  $$? \begin{cases} \forall t \in \{2, 3, 4\}, \ \operatorname{codim} \mathrm{I}_t(\Phi_q) \geqslant 6 - t \\ \operatorname{codim} \mathrm{I}_1(\Phi_q) = 4 \end{cases}$$

  Such a pfaffian Cremona map $q : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ would be of algebraic degree $d_3(q)$ greater than 2 and would not be the composition of two pfaffian maps.

- Given two general pfaffian Cremona maps $q_1, q_2 : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $q_1 \circ q_2^{-1}$ has palindromic projective degrees. What does characterize its base locus?

A.2 An excess intersection question for standard Cremona maps

Let us focus now on a problem related to the singularities of the base locus of standard Cremona maps. To this end, let us explain it in the case of the standard Cremona map of $\mathbb{P}^3$

$$\tau_3 = (x_1x_2x_3 : x_0x_2x_3 : \ldots : x_0x_1x_2) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

The base locus $B_{\tau_3} = \mathbb{V}(x_1x_2x_3, \ldots, x_0x_1x_2)$ of τ_3 is the union of 6 lines $L_1 = \mathbb{V}(x_0, x_1)$, $\ldots, L_6) = \mathbb{V}(x_2, x_3)$ whose singular locus is the unions of the 4 points $p_1 = (1 : 0 : 0 : 0), \ldots, p_4 = (0 : 0 : 0 : 1)$.

Now to compute the topological degree of τ_3, one has to take three general linear combinations a_1, a_2, a_3 of $x_1x_2x_3, ldots, x_0x_1x_2$ and compute the degree of $(a_1, a_2, a_3 : \mathrm{I}_{\tau_3})$ which is equal to one. Now perturb the ideal $a = (a_1, a_2, a_3)$ by considering an ideal $a^t = a + t(b_1, b_2, b_3)$ for b_1, b_2, b_3 three general cubic polynomials and a parameter $t \in \mathbb{R}$. By Bézout's theorem, the zero locus $\mathbb{V}(a^t)$ is a union of 27 distinct points. Consider now the limit $\mathbb{V}(a^t)$ when $t \to 0$. Since $(a : \mathrm{I}_{\tau_3})$ is a degree 1 ideal, 26 of the points in the perturbation $\mathbb{V}(a^t)$ will migrate to B_{τ_3} (the remaining one migrating to $\mathbb{V}(a : \mathrm{I}_{\tau_3})$). The question is then to know how much of these 26 points will migrate to the 4 points $p_1, \ldots, p_4$ and how much will migrate to the 6 lines $L_1, \ldots, L_6$ (excluding $p_1, \ldots, p_4$), see Figure A.1 and Figure A.2.

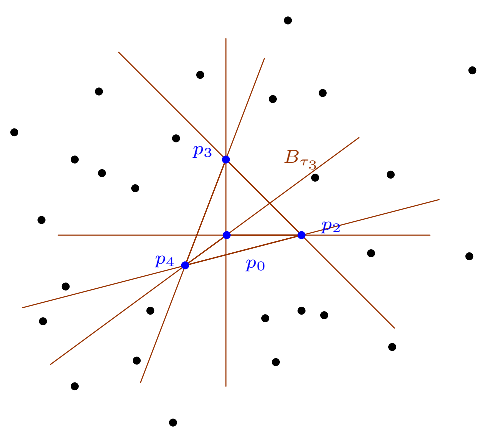


Figure A.1: The 27 points of the perturb locus $\mathbb{V}(a^t)$

Via the internal symmetries of B_{τ_3} none of the four points p_1, p_2, p_3, p_4 can be distinguished between them and none of the lines $L_1, \ldots, L_6$ can be distinguished between them. Hence we can translate this "excess intersection problem" in the

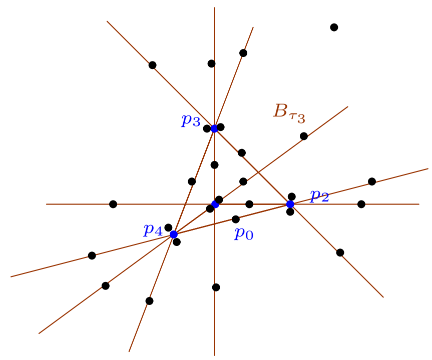


Figure A.2: The limit as $t \to 0$

following numerical constraint: find the quantity β of points migrating to p_1 and the quantity α of points migrating to $L_1 \backslash \{p_1, p_2\}$. Actually, in the case of $\mathbb{P}^3$, since

$$26 = 6\alpha + 4\beta$$

we do not have many choices, namely:

(a) either $(\alpha, \beta) = (1, 5)$,

(b) or $(\alpha, \beta) = (3, 2)$.

Even if we don't know a theoretical answer to this question, one can approach it experimentally by using the package "NumericalAlgebraicGeometry" of Macaulay2, see [Ley11], via the following code:

Algorithm A.2.1.

```
loadPackage "NumericalAlgebraicGeometry"

k = CC
R = k[x_1..x_3]

I = (gens ideal( x_1*x_2*x_3 , x_2*x_3 , x_1*x_3, x_1*x_2) ) *
             random( R^{4:0},R^{3:0} )
time J =  solveSystem ( toList apply(0..2,i-> ( I )_(0,i)) );
J_0 ----the first component of J is point not on the base locus
                                             of the map
for i from 1 to (length J-1) do (
```

```
print toList apply(0..2,j->round  abs (coordinates J_i)_j);
);
```

The output of the last command should be, up to permutation:

```
{1, 0, 0},{0, 1, 0},,{0, 0, 0},{0, 0, 0},{1, 0, 0},{0, 1, 0},
{0, 1, 0},{0, 0, 1},{0, 0, 1},{1, 0, 0},{0, 0, 1}
```

where one has to remember that we are in the affine chart $\{x_0 = 1\}$ of $\mathbb{P}^3$ so one has to read:

```
{1, 0, 0, 1},{0, 1, 0, 1},{0, 0, 0, 1},{0, 0, 0, 1},{1, 0, 0, 1},
{0, 1, 0, 1},{0, 1, 0, 1},{0, 0, 1, 1},{0, 0, 1, 1},{1, 0, 0, 1},
{0, 0, 1, 1}
```

so that two of the solution are located on (or at least very closed to) $p_1 = (0 : 0 : 0 : 1)$, three of the solutions are located on $L_1 = \mathbb{V}(x_0, x_1)$, three are on $L_2 = \mathbb{V}(x_0, x_2)$ and three are on $L_3 = \mathbb{V}(x_1, x_2)$ which are all numbers that we could expect from Item (b).

Let us now comment on this code and why it should not be so surprising that it provides an answer to our excess intersection question. The package "NumericalAlgebraicGeometry" relies *homotopy continuation methods*, see [DE05, Chapter 8 of A.J.Sommese, J.Verschelde, C.W.Wampler] for an introduction about these methods. It computes the solution by approximating an affine polynomial system by a "simpler" one and then by tracking the solutions of the simpler system to the solutions of the initial system. In others words, the command `solveSystem` does precisely what we want to answer our problem. Let us point out however that there are still gaps that we do not understand here. For instance, why the system does not return an error when applying the command `solveSystem` in our case since, the initial polynomial system is not zero-dimensional? Our guess is that, when defining the three general linear combinations, there are small errors in the representation of complex numbers so that the initial polynomial system is zero-dimensional for MACAULAY2. This explanation and the related experiments support the following conjecture:

Conjecture A.2.2.

- *In $\mathbb{P}^3$, the answer $(\alpha, \beta) \in \mathbb{N}^2$ to our excess intersection question:*

$$26 = 6\alpha + 4\beta$$

is $(\alpha, \beta) = (3, 2)$.

- *In $\mathbb{P}^4$, the answer $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ to the analogous excess intersection problem:*

$$255 = 10\alpha + 10\beta + 5\gamma$$

is $(\alpha, \beta, \gamma) = (16, 8, 3)$.

- *In general, given $n \geqslant 2$, the answer $(\alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{n-1}$ is:*

$$n^n - 1 = \sum_{i=2}^{n} \binom{n+1}{i} \alpha_i$$

is $\alpha_i = n^{n-i} \times (i - 1)$ for $i \in \{2, \ldots, n\}$.

It seems to us that one can ask the same question for other type of Cremona maps in particular in the case of a monomial pfaffian Cremona map (but even for any monomial Cremona map). Namely the pfaffian Cremona map $q : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ associated to the matrix

$$\Phi_q = \begin{pmatrix} 0 & -x_1 & 0 & 0 & -x_0 \\ x_1 & 0 & -x_2 & 0 & 0 \\ 0 & x_2 & 0 & -x_3 & 0 \\ 0 & 0 & x_3 & 0 & -x_4 \\ x_0 & 0 & 0 & x_4 & 0 \end{pmatrix}$$

is a monomial pfaffian Cremona map (and in some sense very closed of what could be a standard pfaffian Cremona map) and its base locus if the union of 5 lines in $\mathbb{P}^4$ and has singular locus $p_1 = (1 : 0 : 0 : 0 : 0), \ldots, p_5 = (0 : 0 : 0 : 0 : 1)$.

Problem A.2.3. *What are the solution $(\alpha, \beta) \in \mathbb{N}^2$ to the analogous excess intersection question*

$$15 = 5\alpha + 5\beta$$

in the case of the previous monomial pfaffian Cremona map?

Let us precise that Algorithm A.2.1 do not provide any consistent answer in this latter case (the outputted numbers are not integer in mean).

Appendix B

Questions and perspectives

B.1 Generic composition of determinantal Cremona maps

When considering two Cremona maps $f = (f_0 : \ldots : f_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ and $g = (g_0 : \ldots : g_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$, a fundamental fact is that the composition $f \circ g$ is also a Cremona map. The base ideal $\mathrm{I}_{f\circ g}$ of $f \circ g$ however differs from the naive substitution ideal $\big(f_0(g_0, \ldots, g_n), \ldots, f_n(g_0, \ldots, g_n)\big)$ when the zero locus of this latter ideal has codimension 0 or 1, a basic example being the case of the standard Cremona $f = g = (x_1x_2 : x_0x_2 : x_0x_1) : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ since $\mathrm{I}_{f\circ g} = (x_0, x_1, x_2)$ is generated in degree 1. In more accurate term, the algebraic degree $d_{n-1}(f \circ g)$ may not be equal to the product $d_{n-1}(f) \times d_{n-1}(g)$ of the algebraic degrees of f and g. One can however avoid this latter situation by considering the following *generic composition*: take a generic element $h = (h_0 : \ldots : h_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}} \in \mathrm{PGl}_n(\mathrm{k})$, that is, for all $i \in \{0, \ldots, n\}$, $h_i = \lambda_{i,0}x_0 + \ldots + \lambda_{i,n}x_n$ is generic with respect to the polytope $\mathrm{Conv}\{(\underbrace{0, \ldots, 0}_{n}), (1, \underbrace{0, \ldots, 0}_{n-1}), \ldots, (\underbrace{0, \ldots, 0}_{n-1}, 1)\}$ and define the *generic composition* $f \hat{\circ} g = f \circ h \circ g$ (we point out that a generic composition is always defined even if g is not dominant).

Proposition B.1.1. *Let $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be a map whose base ideal I_f is the n-minors ideal of a $(n + 1) \times n$ matrix Φ_f and let $g = (g_0 : \ldots : g_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be any rational map.*

Then the generic composition $f \hat{\circ} g = f \circ h \circ g$, with $h = (h_0 : \ldots : h_n) : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ being a generic element of $\mathrm{PGl}_{n+1}(\mathrm{k})$, has projective degrees' vector:

$$d(f \hat{\circ} g) = \big(d_0(f)d_0(g), d_1(f)d_1(g), \ldots, d_n(f)d_n(g)\big). \qquad \text{(B.1.1)}$$

Moreover $f \hat{\circ} g$ is determinantal of Hilbert-Burch matrix $\Phi_{f\hat{\circ}g}$ equal to the matrix Φ_f in which the variables $x_0, \ldots, x_n$ have been substituted by $h_0(g_0, \ldots, g_n)$, ..., $h_n(g_0, \ldots, g_n)$.

Remark B.1.2. When considering Cremona maps, $d_0(f) = d_0(g) = d_n(f) = d_n(g) = 1$ and (B.1.1) can be rewritten as:

$$d(f \hat{\circ} g) = (1, d_1(f)d_1(g), \ldots, d_{n-1}(f)d_{n-1}(g), 1).$$

Let us also emphasize that even though $f\hat{\circ}g$ is determinantal, it may not be Koszul-determinantal even if f is Koszul-determinantal. It is for instance the case for the generic composition $\tau\hat{\circ}\tau$ where τ is the standard Cremona map of $\mathbb{P}^n_{\mathrm{k}}$, $n \geqslant 3$, in which case $\operatorname{codim} \mathbb{V}\big(\mathrm{I}_1(\Phi_{\tau\hat{\circ}\tau})\big) = 2 < n$. However, this restriction does not apply to the plane case $n = 2$ so Proposition B.1.1 provides a tool to build plane Koszul-determinantal Cremona maps starting from ones with smaller algebraic degree. Hence, this process leads to solutions of interpolation problem in higher degree.

Proof of Proposition B.1.1. Here we use the geometric computation of the projective degrees' vector $d(u) = (d_0(u), \dots, d_n(u))$ of a map $u : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ as already explained in Remark 1.1.3, namely:

$$[\Gamma_u] = \sum_{k=0}^{n} (\deg_{\mathbb{P}}^{n-k,k} \Gamma_u) \xi_{\mathbf{x}}^{n-k} \xi_{\mathbf{y}}^{k} \in A(\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}})$$

where $A(\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}) \simeq \mathbb{Z}[\xi_{\mathbf{x}}, \xi_{\mathbf{y}}]/(\xi_{\mathbf{x}}^{n+1}, \xi_{\mathbf{y}}^{n+1})$ is the Chow ring of $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}}$ [EH16, Theorem 2.10] and where $\xi_{\mathbf{x}}$ (resp. $\xi_{\mathbf{y}}$) is the pull-back of the hyperplane class $\Xi_{\mathbf{x}}$ (resp. $\Xi_{\mathbf{y}}$) of $\mathbb{P}^n_{\mathrm{k}}$ via the first (resp. second) projection map $\mathbb{P}^n_{\mathrm{k}} \times \mathbb{P}^n_{\mathrm{k}} \to \mathbb{P}^n_{\mathrm{k}}$.

Now, let us describe the generic composition $f\hat{\circ}g = f \circ h \circ g$ as $\mathbb{P}^n_3 \xrightarrow{f} \mathbb{P}^n_2 \xrightarrow{h\circ g} \mathbb{P}^n_1$ where for $i \in \{1, 2, 3\}$, Ξ_i is the hyperplane class of $\mathbb{P}^n_i$ in $A(\mathbb{P}^n_i) \simeq \mathbb{Z}[\Xi_i]/(\Xi_i^{n+1})$. The projective degrees of f, g, and $f\hat{\circ}g$, can then be expressed as follow [Dol11, 7.1.3]:

$$\begin{aligned}
\forall k \in \{0, \dots, n\},\ g^*(\Xi_2^{n-k}) &= d_k(g)\Xi_1^{n-k} \\
f^*(\Xi_3^{n-k}) &= d_k(f)\Xi_2^{n-k} \\
(f\hat{\circ}g)^*(\Xi_3^{n-k}) &= d_k(f\hat{\circ}g)\Xi_1^{n-k}
\end{aligned}$$

By Kleiman's theorem [EH16, Th. 1.7], since h is generic in $\mathrm{PGl}_{n+1}(\mathrm{k})$,

$$(f \circ h)^*(\Xi_3^{n-k}) = \big(d_k(f)\big)\Xi_2^{n-k}$$

for any $k \in \{0, \dots, n\}$ and where the equality is an equality in the Chow ring $A(\mathbb{P}^n_2)$ of $\mathbb{P}^n_2$. Hence

$$\begin{aligned}
(f \circ h \circ g)^*(\Xi_3^{n-k}) &= g^*\big((f \circ h)^*(\Xi_3^{n-k})\big) \\
&= d_k(f) \cdot g^*(\Xi_2^{n-k}) = d_k(g)d_k(f)\Xi_1^{n-k}
\end{aligned}$$

which shows (B.1.1) of Proposition B.1.1.

We show now that the base ideal $\mathrm{I}_{f\hat{\circ}g}$ is the n-minors ideal of a matrix $\Phi_{f\hat{\circ}g}$ equal to the matrix Φ_f in which the variables $x_0, \dots, x_n$ have been substituted by $h_0(g_0, \dots, g_n)$, …, $h_n(g_0, \dots, g_n)$. Let $\hat{\Phi}_f$ be equal to the matrix Φ_f in which the variables $x_0, \dots, x_n$ are substituted by $h_0(g_0, \dots, g_n)$, …, $h_n(g_0, \dots, g_n)$ and remark that there exists a complex

$$0 \longrightarrow \mathrm{R}^n \xrightarrow{\hat{\Phi}_f} \mathrm{R}^{n+1} \longrightarrow \mathrm{I}_{f\hat{\circ}g} \longrightarrow 0 \qquad \text{(B.1.2)}$$

and Hilbert-Burch theorem [Eis95, Theorem 20.15] states that (B.1.2) is exact if and only if $\operatorname{codim} \mathbb{V}(\mathrm{I}_2(\hat{\Phi}_f)) = 2$ in which case $\mathrm{I}_{f\hat{\circ}g} = \mathrm{I}_2(\hat{\Phi}_f)$.

We show that (B.1.2) is exact by contradiction. If (B.1.2) is not exact, it means that the n-minors of $\hat{\Phi}_f$ share a common factor (this last argument use the fact that the base ideal of a rational map $\mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ is essentially unique, see [Sim04, Proposition 1.1 and Definition 1.2]). Hence $\mathrm{I}_{f\hat{\circ}g}$ is generated in degree smaller than $d_{n-1}(f)d_{n-1}(g)$ which are the degrees of the n-minors of $\hat{\Phi}_f$. But the degree of the generators of $\mathrm{I}_{f\hat{\circ}g}$ is the algebraic degree $d_{n-1}(f\hat{\circ}g)$ of $f\hat{\circ}g$ which contradicts (B.1.1). □

Following Proposition B.1.1, the projective degrees of the generic composition of two Cremona maps is characterize and we wonder if this rigidity has an interpretation at the level of the bi-graded resolution of the graph of the considered map. Actually this question is not specific to determinantal maps.

Problem A.

Given $n \geqslant 2$, let $f : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ and $g : \mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^n_{\mathrm{k}}$ be two Cremona maps together with the bigraded free resolution $\mathbb{F}_f$ and $\mathbb{F}_g$ of their respective graph Γ_f and Γ_g. Can we express the bigraded free resolution $\mathbb{F}_{f\hat{\circ}g}$ of the graph $\Gamma_{f\hat{\circ}g}$ of the generic composition $f\hat{\circ}g$ as an operation between $\mathbb{F}_f$ and $\mathbb{F}_g$?

Problem A seems already challenging in the case of plane Cremona maps, but, via an experimental approach, a beginning of answer could start as follows:

Conjecture B.1.3. *Assume that $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ is a general determinantal map and let $g : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ be a (Cremona map) of $\mathbb{P}^2$ and let*

$$\mathbb{F}_g : \quad \cdots \longrightarrow \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ijk}} \to \cdots \to \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ij3}}$$

$$\to \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ij2}} \to \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ij1}} \to \mathrm{S}$$

be a bi-graded free resolution of Γ_g.

Then a bi-graded free resolution of $\Gamma_{f\hat{\circ}g}$ reads:

$$\mathbb{F}_{f\hat{\circ}g} : \quad \cdots \longrightarrow \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ijk}} \to \cdots$$

$$\longrightarrow \bigoplus_{i,j\geqslant 0} \mathrm{S}(-i,-j)^{\beta_{ij4}} \to \begin{array}{c} \mathrm{S}(-4,-d_1(g)) \\ \oplus \\ \bigoplus_{i,j\geqslant 0} \mathrm{S}(-2i,-j)^{\beta_{ij3}} \end{array}$$

$$\longrightarrow \begin{array}{c} \mathrm{S}(-2,-d_1(g))^3 \\ \oplus \\ \bigoplus_{i,j\geqslant 0} \mathrm{S}(-2i,-j)^{\beta_{ij2}} \end{array} \to \begin{array}{c} \mathrm{S}(-1,-d_1(g))^2 \\ \oplus \\ \bigoplus_{i,j\geqslant 0} \mathrm{S}(-2i,-j)^{\beta_{ij1}} \end{array} \to \mathrm{S}$$

Another direction in this thematic is to focus on the reciprocal of Proposition B.1.1: what conditions should we impose to the base locus of f and the base locus of g^{-1} in order that each projective degree $d_k(f\circ g)$ of the composition $f\circ g$ be the product $d_k(f)d_k(g)$ of the projective degrees of f and g? When $n=2$, the considered base locus are set-theoretically sets of points and one has that $d_1(f\circ g)=d_1(f)d_1(g)$ if and only if $\mathbb{V}(\mathrm{I}_{g^{-1}})\cap\mathbb{V}(\mathrm{I}_f)=\emptyset$ (a condition verified when considering a generic composition).

When $n\geqslant 3$, doing a generic $f\hat{\circ}g$ implies that

$$\operatorname{codim}\mathbb{V}(\mathrm{I}_f+\mathrm{I}_{(h\circ g)^{-1}})=\operatorname{codim}\mathbb{V}(\mathrm{I}_f)+\operatorname{codim}\mathbb{V}(\mathrm{I}_{(h\circ g)^{-1}}),$$

the maximum expected, and one could want sharper conditions. Let us emphasize that such sharper conditions might be difficult to determine, as illustrated by the following example.

Example B.1.4. Let $g=(-x_0x_1+x_0x_2,x_0x_3,-x_0x_1+x_1x_2,x_1x_3):\mathbb{P}^3_{\mathrm{k}}\dashrightarrow\mathbb{P}^3_{\mathrm{k}}$ and the determinantal map $f:\mathbb{P}^3_{\mathrm{k}}\dashrightarrow\mathbb{P}^3_{\mathrm{k}}$ defined by its Hilbert-Burch matrix

$$\Phi_f=\begin{pmatrix} x_0 & 0 & 0\\ 0 & x_1+x_2 & 0\\ -x_2 & -x_2 & -x_2\\ 0 & 0 & x_0+x_3\end{pmatrix}.$$

After computation with a computer system, one has that $\operatorname{codim}\mathbb{V}(\mathrm{I}_f+\mathrm{I}_{g^{-1}})=3<\operatorname{codim}\big(\mathbb{V}(\mathrm{I}_f)\big)+\operatorname{codim}\big(\mathbb{V}(\mathrm{I}_{g^{-1}})\big)=2+2=4$ but $d(f)=(1,3,3,1)$, $d(g)=(1,2,3,1)$ and $d(f\circ g)=(1,6,7,1)$.

Let us now outline one consequence of Proposition B.1.1.

Corollary B.1.5. *Let $g:\mathbb{P}^n_{\mathrm{k}}\dashrightarrow\mathbb{P}^n_{\mathrm{k}}$ be a rational map and let $\Phi_f=(\phi_{ij})_{\substack{0\leqslant i\leqslant n\\1\leqslant j\leqslant n}}\in\mathrm{R}^{(n+1)\times n}$ where for any $i\in\{0,\dots,n\}$ and $j\in\{1,\dots,n\}$,*

$$\phi_{ij}=\sum_{i=0}^{n}\lambda_k^{(ij)}g_k$$

is a general linear combination of $g_1,\dots,g_n$. Then the map u whose base ideal I_u is the n-minors ideal of Φ has projective degrees:

$$d(u)=(\binom{n}{0}d_0(g),\binom{n}{1}d_1(g),\binom{n}{2}d_2(g),\dots,\binom{n}{n-1}d_{n-1}(g),1).$$

Reciprocally, given a map u whose base ideal is the n-minors ideal of a matrix $\Phi=(\phi_{ij})_{\substack{0\leqslant i\leqslant n\\1\leqslant j\leqslant n}}$ were all the entries ϕ_{ij} have the same degree d. Then the map $g=(g_0:\dots:g_n):\mathbb{P}^n_{\mathrm{k}}\dashrightarrow\mathbb{P}^n_{\mathrm{k}}$, where for $i\in\{0,\dots,n\}$, $g_i=\sum\limits_{j=1}^{n}\lambda_{ij}\phi_{ij}$ is a general linear combination of the entries of the j-th column of Φ, has projective degrees

$$d(g)=(\frac{d_0(u)}{\binom{n}{0}}=,\frac{d_1(u)}{\binom{n}{1}},\dots,\dots,\frac{d_{n-1}(u)}{\binom{n}{n-1}},1)$$

Proof. From [GSP06, Theorem 2], any general determinantal map $\tilde{f}$ whose Hilbert-Burch matrix $\Phi_{\tilde{f}} = (\phi^{(f)}_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}})$ is filled by general linear polynomials has projective degrees' vector

$$d(\tilde{f}) = (1, \binom{n}{1}, \dots, \binom{n}{n-1}, 1).$$

The conclusion of Corollary B.1.5 then follows from Proposition B.1.1 by remarking that $u = \tilde{f} \circ h \circ g$ where h a general element of $\mathrm{PGl}_{n+1}(k)$. ☐

Corollary B.1.5 was a leading result to answer Problem 3 in the sense that it provides instances of composed map $f \circ g$ whose base ideal is the 3-minors ideal of a (4×3)-matrix but where the base ideal of its inverse do not satisfy Hilbert-Burch theorem.

Example B.1.6. Let $g = (x_0x_3, x_1x_3, x_2x_3, x_0^2 - x_1x_2) : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ $(d(g) = (1, 2, 2, 1))$ and the determinantal map $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ defined by its Hilbert-Burch matrix

$$\Phi_f = \begin{pmatrix} x_0 & x_3 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_1 & x_0 \\ x_3 & x_2 & x_1 \end{pmatrix}.$$

After computation with a computer system, one has that $d(f \circ g) = (1, 6, 6, 1)$, $\mathrm{I}_{f\circ g}$ is the 3-minors ideal of Φ_f whose variables have been substituted by g_0, g_1, g_2, g_3. However $\mathrm{I}_{(f\circ g)^{-1}}$ has minimal free resolution:

$$0 \to \mathrm{R}(-9) \to \begin{matrix} \mathrm{R}(-8)^3 \\ \oplus \\ \mathrm{R}(-9) \end{matrix} \to \mathrm{R}(-6)^4 \to \mathrm{R}$$

Actually, Problem 3 has a negative answer even in restriction to Koszul-determinantal maps.

Example B.1.7. The Koszul-determinantal Cremona map $f : \mathbb{P}^3_{\mathrm{k}} \dashrightarrow \mathbb{P}^3_{\mathrm{k}}$ whose Hilbert-Burch matrix Φ_f is

$$\Phi_f = \begin{pmatrix} x_0 & x_2 & x_1^3x_3 + x_1^4 \\ x_1 & x_0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & (x_0^2 - x_1x_2)^2 \end{pmatrix}$$

has its inverse which is not determinantal as it can be computed by a computer system (and $d(f) = (1, 6, 3, 1)$).

Despite these previous examples, we still wonder if the inverse of a plane determinantal Cremona map is determinantal:

Problem B.

Is the inverse of a plane determinantal Cremona map $f : \mathbb{P}^2_{\mathrm{k}} \dashrightarrow \mathbb{P}^2_{\mathrm{k}}$ determinantal?

Let us precise that, as for Problem 3, Problem B could be answered by another elementary example of a plane determinantal Cremona map whose inverse is not determinantal. Our difficulty in the plane case is that we do not know many non determinantal plane Cremona map. For instance, we could not construct a non determinantal Cremona map f of algebraic degree $d_1(f)$ equal to 6 even if we know by [HS12, Th.2.12 (ii)] what would be the free resolution of its base ideal. In short, we've lacked examples of non determinantal plane Cremona maps to test Problem B.

B.2 Dimension of the families of determinantal Cremona maps

In addition to answer questions about the bi-graded free resolutions of glued maps such as Conjecture 1.2.4, we believe that it remains a more fundamental question about the dimension of the families of determinantal Cremona maps and let us briefly outline why. Given a 4×3 matrix Φ with homogeneous linear entries in $\mathrm{R} = \mathrm{k}[x_0, \ldots, x_3]$, and assuming that the 3-minors ideal of Φ defines an arithmetically Cohen-Macaulay subscheme correspond to a smooth point of the Hilbert scheme of all arithmetically Cohen-Macaulay subscheme of codimension 2 of $\mathbb{P}^3_{\mathrm{k}}$ which has dimension 24 [Ell75, Théorème 2]. Given a 5×5 skew-symmetric matrix Φ whose subdiagonal is filled with homogeneous linear entries in $\mathrm{R} = \mathrm{k}[x_0, \ldots, x_4]$, and assuming that the 4-pfaffian ideal of Φ defines a codimension 3 arithmetically Gorenstein subscheme of $\mathbb{P}^4_{\mathrm{k}}$ correspond to a smooth point of the Hilbert scheme of all codimension 3 arithmetically Gorenstein subscheme of $\mathbb{P}^4_{\mathrm{k}}$ which has dimension 25 [KMR98, Th.2.6, Rmk 2.8]. Hence, we can associate to any general determinantal map of $\mathbb{P}^3_{\mathrm{k}}$ (and more generally of $\mathbb{P}^n_{\mathrm{k}}$) and to any general pfaffian Cremona map of $\mathbb{P}^4_{\mathrm{k}}$ (and more generally to $\mathbb{P}^n_{\mathrm{k}}$ for $n \geqslant 4$ odd) a smooth point on a component of the Hilbert scheme of the associated base ideal.

Problem C.

Can we associate any family of determinantal Cremona maps we build to a Hilbert scheme? What are the dimension of these families?

Problem C is somehow also related to [DH16] and [DH17] where the two authors describe the determinantal families of cubo-cubic (resp. of quarto-quartics) in the set of all Cremona map of degree 3 (resp. degree 4) and give their respective dimension.

Among the other questions we still have, one concerns the non general pfaffian Cremona maps, see Example A.1.10 for an example of such a map. Our problem can be summed up as:

Problem D.

Does it exist a pfaffian Cremona map $q : \mathbb{P}^4_{\mathrm{k}} \dashrightarrow \mathbb{P}^4_{\mathrm{k}}$ whose base ideal I_q is not of linear type and such that its residual scheme $\mathbb{P}(\mathrm{I}_q)$ has codimension 4 in $\mathbb{P}^4_{\mathrm{k}} \times \mathbb{P}^4_{\mathrm{k}}$?

The conditions on $\mathbb{P}(\mathrm{I}_q)$ are conditions on the entries of the associated skew-symmetric matrix Φ_q and we could not figure out how to build a pfaffian Cremona map from them. An answer to Problem D should however shed more lights on the families of pfaffian Cremona maps of high degree.

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