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Knutson ideals

PhD Thesis

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Abstract

In this thesis, we begin the study of a new class of ideals, called *Knutson ideals*, which has several useful connections with different aspects of commutative algebra, such as F-singularity theory, Gröbner basis theory, determinantal rings theory and combinatorics.

First, we show that the main properties that this class has in polynomial rings over $\mathbb{F}_p$ are preserved when one introduces the definition of Knutson ideal also in polynomial rings over fields of characteristic zero. Then, we discuss the case of determinantal ideals of Hankel matrices and generic matrices, proving that both of them are Knutson ideals. In particular, in positive characteristic, they all define F-pure rings.

In the case of Hankel matrices, we also give a characterization of all the ideals belonging to the family. Interestingly, it turns out that they are all Cohen-Macaulay ideals.

In the case of generic matrices, we obtain a useful result about Gröbner bases of certain sums of determinantal ideals. More specifically, given $I = I_1 + \ldots + I_k$ a sum of ideals of minors on adjacent columns or rows, we prove that the union of the Gröbner bases of the I_j's is a Gröbner basis of I.

Lastly, we focus on the connection between Knutson ideals and binomial edge ideals associated to weakly closed graphs. Inspired by Matsuda's work on weakly closed graphs, we show that their binomial edge ideals are Knutson ideals (in particular, they are F-pure in positive characteristic). Furthermore, we conjecture that the converse is still true, i.e, the binomial edge ideals in $\mathcal{C}_f$ are exactly those associated to weakly closed graphs.

Contents

Chapter 1

Introduction

In this thesis we are going to investigate a recent class of ideals, called *Knutson ideals*, from several points of view. The motivation behind the study of these ideals is that they have several useful connections with different aspects of commutative algebra, such as F-singularity theory, Gröbner basis theory, determinantal rings theory and combinatorics.

The name "Knutson ideals" appears for the first time in [8] and it arises from Knutson's work on compatibly split ideals and subvarieties (see [17]). In the past 30 years these latter ideals have aroused interest both in commutative algebra and algebraic geometry since they play a crucial role in the study of rings and varieties in characteristic p. Actually, it has to be said that these "characteristic p techniques" are a powerful tool also for proving theorems for algebras and varieties over fields of characteristic zero by means of the "reduction modulo p", as we will see in Chapter 4.

In order to introduce the reader to the main subject of this thesis, it is worth spending a few words on Frobenius splittings and compatibly split ideals. These arguments will be covered in more detail in Chapter 3.

The notion of *F-splitting* dates back to the 1970s when Hochster and Roberts ([13]) gave a proof of the Cohen-Macaulayness of rings of invariants. In their work, the authors did not explicitly mention the term "F-split"; it will be formally used for the first time in [22] by Mehta and Ramanathan who used the Frobenius map in a more geometric setting to investigate global properties of Schubert varieties and other related objects. This paper has been the groundwork for many other works concerning remarkable properties of F-split varities, as for example the vanishing of cohomology of line bundles.

The definition of F-split ring is actually quite simple. Given a ring R of positive characteristic, we know that it comes equipped with a special endomorphism, called *Frobenius map*, defined as follows

$$\begin{aligned} F : R &\longrightarrow R \\ r &\longrightarrow r^p. \end{aligned}$$

This map encodes many algebraic properties of the ring R; for example, the injectivity of F is equivalent to R being reduced. Furthermore, it is a basic fact that an injective function admits a left inverse and we will be interested in studying these inverses. Indeed, roughly speaking, R is said to be F-split if the Frobenius map has a left-inverse in Mod_R. In such a case, this inverse is called a F-splitting of R.

Given an F-splitting φ of a ring R (of positive characteristic), it seems natural to consider those ideals I such that $\varphi(I) \subseteq I$ or equivalently, such that φ descends to a splitting of R/I. These ideals are said to be *compatibly split.*

For our purposes, it can be extremely useful to think a Frobenius splitting as a geometric object one might attach to a ring of positive characteristic. Thus, knowing an explicit Frobenius splitting would be handier rather than solely the knowledge of the existence, so that we can consider all the compatibly split ideals with respect to φ. One of the main features of compatibly split subvarieties is that they are reduced and closed under finite unions and intersections, hence they can be used to prove that certain intersections are reduced.

In 2009 a step forward in the study of compatibly split ideals was made; Kumar and Mehta in [16] and independently Schwede in [27], proved that, given an F-splitting φ of a Noetherian ring R (of positive characteristic), there are only finitely many compatibly split ideals with respect to φ. Furthermore, Knutson in [17] arose the problem of explicitly listing these compatibly split ideals for some classical splittings. A few years later, Katzman and Schwede found an algorithm for computing all of the compatibly split ideals (see [18]). This algorithm, which works for a fixed prime p and a fixed variety, is based on some ideas contained in [16] and [27] together with ideas coming from tight closure theory.

Knutson's paper on compatibly split ideals [17] has been the starting point for the development of this thesis.

Assume that R is a ring of positive characteristic which possesses a Frobenius splitting φ. The following were already known (see [3]):

- R is reduced
- If an ideal I of R is compatibly split, $\varphi(I) = I$ and I is radical.

- If I and J are compatibly split ideals, so are $I+J$ and $I \cap J$.
- If I is a compatibly split ideal, the colon $I : J$ is compatibly split for every ideal J.

In the case R is Noetherian, compatibly split ideals are a finite number, so the above properties lead to an algorithm to construct a family of radical ideals. This algorithm starts from a compatibly split ideal I and construct many more radical ideals by taking minimal primes, sums and intersections, then iterating. In some special cases, in this way we obtain all the possible φ-compatibly split ideals.

One of the main result contained in Knutson's work explores the relation between Frobenius splitting and degeneration. Under certain assumptions, the author proves that compatibly split ideals degenerate to compatibly split ideals, thus, stay radical. From this result one can infer other properties of compatibly split ideals; for instance the fact that if I and J are two compatibly split ideals of R with respect to a certain F-splitting, then the union of their Gröbner bases is a Gröbner basis of their sum.

Throughtout the thesis we will mainly focus our attention on the case R is a polynomial ring. To define *Knutson ideals*, we take f a polynomial in R with squarefree leading term and consider the F-splitting $\varphi := \mathrm{Tr}(f^{p-1}\bullet)$, where the trace map $\mathrm{Tr}(\bullet)$ is uniquely defined by its application to monomials m :

$$\mathrm{Tr}(m) := \begin{cases} \dfrac{\sqrt[p]{m\prod x_i}}{\prod x_i} & \text{if} \quad m\prod x_i \text{ is a } p\text{th-power} \\ 0 & \text{otherwise.} \end{cases}$$

A straightforward computation shows that the principal ideal (f) is compatibly split with respect to φ and we can apply the previous algorithm to construct many more compatibly split ideals starting from (f) and taking colons, sums and intersections.

We will deonte the family of ideals we obtain with this procedure by $\mathcal{C}_f$ and we will call its elements *Knutson ideals associated to f.*

Our main goal will be to extend Knutson's results about degeneration of compatibly split ideals to more general polynomial rings and use them to study determinantal ideals and binomial edge ideals.

Outline of the thesis

In this thesis we are going to generalize Knutson's results to the case of polynomial rings over any fields. Then we will apply these results to study determinantal ideals of Hankel matrices and generic matrices. Lastly, inspired by Matsuda (see [21]), we will study F-purity property of weakly closed graphs and their link with Knutson binomial edge ideals.

In Chapter 2 and Chapter 3 we collect some well known results about determinantal rings and F-singularities which we will use throughout the thesis in some more or less evident form. Thus, these chapters do not contain any original result but their purpose is to give the reader all the background information needed to understand the rest of the manuscript.

Chapter 4 is dedicated to extend Knutson's results to polynomial rings over fields of any characteristic. Once given the definition of *Knutson ideals* in polynomial rings over any field, we will show that the properties Knutson proved in the case of polynomial rings over $\mathbb{F}_p$ stay unchanged. To do so, we will first prove that every Knutson ideal has squarefree initial ideal (this generalizes Knutson's result [17, Theorem 2.(2)]) in the case of polynomial rings over fields of positive characteristic (see Proposition 4.1.1). Then we will use the achieved result together with reduction modulo p to prove that the same holds for polynomial rings over fields of characteristic 0 (see Proposition 4.2.1). From this property one can infer other two important properties of this family of ideals. Indeed, from Remark 4.2.4 it easly follows that every family of Knutson ideals is finite. Furthermore, using again Remark 4.2.4 and the fact that

$$\mathrm{in}_{\prec}(I \cap J) = \mathrm{in}_{\prec}(I) \cap \mathrm{in}_{\prec}(J) \Leftrightarrow \mathrm{in}_{\prec}(I + J) = \mathrm{in}_{\prec}(I) + \mathrm{in}_{\prec}(J)$$

we get that given $I = I_1 + \ldots + I_k$ a sum of Knutson ideals, the union of the Gröbner bases of the I_j's is a Gröbner basis of I.

In Chapter 5 we will focus our attention on determinantal ideals of Hankel matrices and generic matrices.

In Section 5.1 we will prove that determinatal ideal of Hankel matrices are Knutson ideals for a suitable choice of the polynomial f (see Theorem 5.1.1 and Theorem 5.1.2). As a consequence of these results, we will derive an alternative proof (see Corollary 5.1.3) of the F-purity of Hankel determinantal rings, a result recently proved by different methods in [6].

Moreover, proving these theorems, it comes out that determinantal ideals of certain submatrices of Hankel matrices are Knutson ideals. Being a family of Knutson ideals finite, it is natural to ask whether they are all the ideals belonging to the family or not. This leads to a characterization of all the ideals belonging to the family (see Section 5.2). Interestingly, they all define Cohen-Macaulay rings (see Remark 5.2.7).

In Section 5.3, we will show that also determinantal ideals of generic matrices are Knutson ideals. As a corollary we obtain an interesting result about Gröbner bases of certain sums of determinantal ideals on adjacent columns or rows (see Corollary 5.3.4).

Unlike in the case of Hankel matrices, a characterization of all the ideals belonging to the family has not been found yet. A first step towards this result would be to understand the primary decompostion of certain sums belonging to the family. Some known results (see [14], [23]) suggest what these primary decompositions might be and computer experiments (using Macaulay2) seem to confirm this guess. Finding this characterization could lead to interesting properties on the Gröbner bases of determinantal-like ideals and would also answer to a question asked by F.Mohammadi and J. Rhau in [23].

Chapter 6 investigates the relation between Knutson ideals and binomial edge ideals associated to weakly closed graphs.

We will apply previous results about generic matrices to study F-purity of binomial edge ideals. Inspired by Matsuda's work on weakly closed graphs, we will show that their binomial edge ideals are Knutson ideals (see Proposition 6.1.5); in particular, they are F-pure. Proving this we will also find a characterization of weakly closed graphs in terms of the minimal primes of their associated binomial edge ideals (see Proposition 6.1.6).

Eventually, we conjecture that the converse of Proposition 6.1.5 is still true, i.e, the binomial edge ideals in $\mathcal{C}_f$ are exactly those associated to weakly closed graphs.

Chapter 2

Determinantal rings theory

This chapter is meant to be a quick survey on the the theory of determinantal ideals. We collect some well known facts and results which we will use throughout the thesis. We state them in the case of polynomial rings over a field $\mathbb{K}$. However, it should be stressed that most of the theory of determinantal rings can be obtained using $\mathbb{Z}$ as base ring and then transferred to an arbitrary ring of coefficients B.

The study of determinantal rings has become a central topic in commutative algebra but it also has interesting connections with represention theory, invariant theory and combinatorics.

Determinatal ideals also have a nice geometric interpretation. Consider a $m \times n$ matrix of indeterminates X and define I_t to be the ideal generated by the t-minors of X. It is a basic fact that every linear map $f : \mathbb{K}^m \to \mathbb{K}^n$ can be represented by an $m \times n$ matrix. Thus $V(I_t)$ is the variety of $\mathbb{K}$-linear maps of rank less than t and $\mathbb{K}[X]/I_t$ is its coordinate ring.

2.1 Determinantal ideals of generic matrices

Let m, n be two positive integers. We will denote by X_{mn} the generic matrix of size $m \times n$ with entries x_{ij}, that is

$$X = X_{mn} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}.$$

Given a generic matrix, we can consider its entries as variables of a polynomial ring $S = \mathbb{K}[X] = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ over an arbitrary field $\mathbb{K}$. The aim of the theory is to investigate the properties of the ideal generated by the t-minors of X, with $1 \leq t \leq \min(m, n)$, which we will denote by $I_t(X)$.

Adopting a simplicistic approach, one might try to study $I_t(X)$ by studying directly its generators. However, since determinants are extremly complicated polynomial, this approach would be unsuccesful. This led to the introduction of new powerful tools which have become the most commonly used in the investigation of determinatal ideals: standard bitableaux and straightening law.

Roughly speaking, the idea underlying this new approach is to think of minors as generators of the algebra $\mathbb{K}[X]$, so that products of minors are products in these generators (so we can regard them as "monomials"). As a consequence, we need to choose some of them to construct a $\mathbb{K}$-basis of $\mathbb{K}[X]$. These "monomials"are known as *standard bitableuax* and the *straightening law* tells us how to write product of minors as linear combination of standard bitableaux.

Remark 2.1.1. Even if we won't deepen this topic here, we want to point out that this new approach to the theory of determinantal ideals can be generalized to a larger class of algebras, called *ASL algebras* (algebras with straightening law). For more details, the interest reader is referred to [4].

2.1.1 Minors and determinantal ideals

Given two sequences of positive integers $a_1 < a_2 < \ldots < a_t \leq m$ and $b_1 < b_1 < \ldots < b_t \leq n$, we define

$$[a_1, \ldots, a_t | b_1 \ldots b_t]$$

to be the t-minor of X with row indices $a_1, \ldots, a_t$ and column indices $b_1, \ldots, b_t$. Thus

$$I_t(X) = \{[a_1, \ldots, a_t | b_1 \ldots b_t] \mid a_1 < a_2 < \ldots < a_t \leq m, b_1 < b_1 < \ldots < b_t \leq n\}.$$

By convention, $[|] = 1 \in \mathbb{K}$ (empty minor).

From now on, unless otherwise stated, we will assume $m \leq n$. In this case, the maximal minors of X are the minors of size m and their row indices are authomatically fixed to be $1, \ldots, m$. This allows us to simplify our notation:

$$[b_1, \ldots, b_m] := [1, \ldots, m | b_1, \ldots, b_m].$$

Example 2.1.2. Let

$$X = X_{23} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}.$$

Then $I_2(X) = ([12], [13], [23])$, where

$$\begin{aligned}
[12] &= x_{11}x_{22} - x_{12}x_{21} \\
[13] &= x_{11}x_{23} - x_{13}x_{21} \\
[23] &= x_{12}x_{23} - x_{13}x_{23}.
\end{aligned}$$

2.1.2 Standard bitableaux and straightening law

Denote by $\mathcal{M}(X)$ the set of non-empty minors of X. We can define on $\mathcal{M}$ the partial order $\preceq$ given by

$$[a_1, \dots, a_t | b_1 \dots b_t] \preceq [c_1, \dots, c_s | d_1 \dots d_s] \Leftrightarrow t \geq s, \quad a_i \leq c_i \text{ and } b_i \leq d_i \quad \forall i.$$

In particular, among maximal minors this order becomes:

$$[a_1 \dots a_m] \preceq [b_1, \dots, b_m] \Leftrightarrow a_i \leq b_i \quad \forall i.$$

Note that $(\mathcal{M}, \preceq)$ is a distributive lattice.

Remark 2.1.3. Obviously, $\mathcal{M}$ generates $\mathbb{K}[X]$ as a $\mathbb{K}$-algebra, then products of minors become "monomials "in these new variables.

Given this partial order, we can define a special subset of product of minors.

Definition 2.1.4. Let $\Delta = \delta_1 \cdots \delta_w$ be a product of minors of X. We say that Δ is a *standard bitableau* if $\delta_1 \preceq \delta_2 \preceq \dots \preceq \delta_w$. We will denote by Σ the set of standard bitableaux.

The name "bitableu"comes from the combinatorial interpretation of this ring-theoretic object. Suppose that

$$\Delta = \delta_1 \cdots \delta_w, \qquad \delta_i = [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}].$$

Then we can represent Δ by a pair of Young tableaux:

$$\begin{array}{|c|cccc|c|c|}
\hline
a_{1t_1} & \dots & \dots & \dots & a_{12} & a_{11} \\
\hline
 & a_{2t_2} & \dots & \dots & a_{22} & a_{21} \\
\hline
 & & & & \vdots & \\
 & & & a_{wt_w} & \dots & a_{w1} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|cc|c|}
\hline
b_{11} & b_{12} & b_{13} & \dots & \dots & b_{1t_1} \\
\hline
b_{21} & b_{22} & b_{23} & \dots & b_{2t_2} & \\
\hline
 & \vdots & & & & \\
b_{w1} & \dots & b_{wt_w} & & & \\
\hline
\end{array}$$

With this interpretation, being standard means that each column of the bitableau has non-decreasing indices from the top to the botton.

Example 2.1.5. Let $m = 4$ and $n = 6$ and consider the following product of minors of X_{46}:

$$\Delta = \underbrace{[134|235]}_{\delta_1}\underbrace{[123|146]}_{\delta_2}\underbrace{[2|3]}_{\delta_3}.$$

The correspondent bitableaux is

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 3 & 2 & 1 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 1 & 4 & 6 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Note that Δ is non-standard: $b_{11} > b_{21}$.

In general, the product of standard bitableaux could be non-standard. However Doubilet, Rota and Stein in [9] have proved that the set of standard bitableaux Σ is a $\mathbb{K}$-vector basis for the polynomial ring $\mathbb{K}[X]$. In particular they have shown that every element of $\mathbb{K}[X]$ can be written in a unique way as a $\mathbb{K}$-linear combination of standard bitableaux. This theorem (see e.g. [2, Theorem 7.2.7] or [4, Section 4]) is known as *Fundamental straightening law of Doubilet-Rota-Stein* and states the followings:

Theorem 2.1.6. *(a) The standard bitableaux are a $\mathbb{K}$-vector basis of $\mathbb{K}[X]$.*

(b) If the product $\gamma\delta$ of minors is not a standard bitableau, then it has a representation

$$\gamma\delta = \sum x_i \epsilon_i \eta_i \qquad x_i \in \mathbb{K} \setminus \{0\}$$

where $\epsilon_i\eta_i$ is a standard bitableau, $\epsilon_i \prec \gamma, \delta \prec \eta_i$.

(c) If Δ is an arbitrary bitableau, its representation as a linear combination of standard bitableaux Σ can be found applying recursively the straightening relations described in (b).

(d) Let Δ be a bitableau and $\Sigma = \sigma_1 \cdots \sigma_w$ be a standard bitableau appearing in its standard representation. Then $\sigma_1 \preceq \delta$ for all factors δ of Δ.

To prove the theorem, one first proves its restriction to the subalgebra $\mathbb{K}[\mathcal{M}_m]$ generated by maximal minors from which the general case can be inferred.

Indeed the algebra $\mathbb{K}[\mathcal{M}_m]$ is the homogeneous coordinate ring of a Grassmann variety, so the m-mionors must satisfy the well known Plücker relations:

Proposition 2.1.7. [4, Lemma 4.4] (Plücker relations) *For all indices* $a_i, \ldots, a_p, b_q, \ldots, b_m, c_1, \ldots, c_s \in \{1, \ldots, n\}$ *such that* $s = m - p + q - 1 > m$ *and* $t = m - p > 0$ *one has*

$$\sum_{\substack{i_1 < \ldots < i_t \\ i_{t+1} < \ldots < i_s \\ \{i_1, \ldots, i_s\} = \{1, \ldots, s\}}} \operatorname{sign}(i_1 \ldots i_s)[a_1 \ldots a_p c_{i_1} \ldots c_{i_t}][c_{i_{t+1}} \ldots c_{i_s} b_q \ldots b_m] = 0.$$

Example 2.1.8. Let $m = 3$ and $n = 6$. Consider the product of maximal minors $\Delta = [146][235]$. This is a non-standard bitableaux that can be written using the fundamental straightening law (more specifically, Plücker relations) as:

$$[146][235] = -[123][456] - [125][346] + [135][246]$$

where each product of minors in the right hand side is a standard bitableaux.

2.1.3 Determinantal rings and Gröbner basis

The straightening law, together with induction on the size of minors, allows to prove basic properties of the determinantal ideal $I_t(X)$ without much effort.

One of the first results is about the $\mathbb{K}$-basis of determinantal ideals and determinantal rings.

Theorem 2.1.9. [4, Section 5] *The standard bitableaux* $\Gamma = \gamma_1 \cdots \gamma_u$ *such that* $|\gamma_1| \geq t$ *form a* $\mathbb{K}$*-basis of* I_t.

The standard bitableaux $\Gamma' = \delta_1 \cdots \delta_v$ *such that* $|\delta_j| \leq t - 1$ *for all* j *form a* $\mathbb{K}$*-basis of* $\mathbb{K}[X]/I_t$.

A second important property of determinantal rings and ideals that we will use often in the rest of the thesis is the following.

Theorem 2.1.10. [2, Theorem 7.3.1] *The ring* $\mathbb{K}[X]/I_t(X)$ *is a domain of Krull dimension* $(m+n-t+1)(t-1)$. *In particular, the determinantal ideal* $I_t(X)$ *is a prime ideal and*

$$\operatorname{ht}(I_t(X)) = (n - t + 1)(m - t + 1).$$

Another standard method to understand determinantal rings is to derive their properties from the analogous properties of their initial ideals via Gröbner basis. One of the advantages is that $\mathbb{K}[X]/\operatorname{in}_{\prec}(I_t(X))$ is the Stanley-Reisner ring of a simplicial complex and can be investigated using combinatorial tools.

For generic matrices the description of Gröbner basis of determinantal ideals was given separately by Sturmfels in [29] and by Herzog and Trung in [15].

An effortless computation using Buchberger's algorithm shows that the 2-minors of a generic matrix X form themselves a Gröbenr basis for $I_2(X)$ with respect to a diagonal term order. Although this result can be generalized to every size of the minors, Buchberger's algorithm becomes complicated and less effective for $t > 2$. One of the reason is that the map that assigns to every standard bitableaux its initial monomial fails to be injective.

Sturmfels overcame this problem by means of the Knuth-Robinson-Schensted correspondence and he proved the following.

Theorem 2.1.11. [29, Theorem 1] *Fix a diagonal term order $\prec$ on $\mathbb{K}[X]$, that is a monomial term order such that the initial term of each minor is given by the product of its diagonal terms. Then the t-minors of X form a Gröbner basis for $I_t(X)$.*

Remark 2.1.12. For maximal minors, a stronger result holds; they form a *universal* Gröbner basis for $I_m(X)$ (i.e. they are a Gröbner basis for every monomial order). This is not true for t-minors with $1 < t < m$, but for $t = 2$ a universal Gröbner basis is known.

2.2 Determinantal ideals of Hankel matrices

Inspired by the case of generic matrices, Conca developed a standard monomial theory for generic Hankel matrices (see [7]).

In contrast with the case of generic matrices, the polynomial ring $\mathbb{K}[X]$ endowed with this standard monomial basis is not properly an ASL algebra. Nonetheless, it still has a quadratic straightening law with similar properties to the ones we had for generic matrices.

2.2.1 Standard monomials

Let $x_1, \ldots, x_n$ be indeterminates over an arbitrary field $\mathbb{K}$. For $j = 1, \ldots, n$, we denote by X_j the generic Hankel matrix with j rows and entries $x_1, \ldots, x_n$, that is

$$X_j = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_{n-j+1} \\ x_2 & x_3 & x_4 & \dots & x_{n-j+2} \\ x_3 & x_4 & x_4 & \dots & x_{n-j+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_j & x_{j+1} & x_{j+2} & \dots & x_n \end{bmatrix}.$$

Sometimes, in order to specify the variables we are considering, we will write $X_j^{(l,n)}$ to denote the generic Hankel matrix with j rows and entries $x_l, \dots, x_n$, that is

$$X_j^{(l,n)} = \begin{bmatrix} x_l & x_{l+1} & x_{l+2} & \dots & x_{n-j+1} \\ x_{l+1} & x_{l+2} & x_{l+3} & \dots & x_{n-j+2} \\ x_{l+2} & x_{l+3} & x_{l+4} & \dots & x_{n-j+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{l+j-1} & x_{l+j} & x_{l+j-1} & \dots & x_n \end{bmatrix}.$$

Given two sequences of positive integers $a_1 < a_2 < \dots < a_s$ and $b_1 < b_2 < \dots < b_s$ such that $a_i + b_i \le n+1$, we define

$$[a_1, \dots, a_s | b_1 \dots b_s]$$

to be the s-minor of X with row indices $a_1, \dots, a_s$ and column indices $b_1, \dots, b_s$. We will call a minor of the form $[1, \dots, s | b_1, \dots, b_s]$ maximal minor or maximal s-minor and we will write shortly $[b_1, \dots, b_s]$.

Definition 2.2.1. We say that a sequence of integers $a_1, \dots, a_s$ is a $<_1$-chain if $a_i + 1 < a_{i+1}$ for $i = 1, \dots, s-1$. Ananalogously, we say that a monomial of the form $x_{a_1} \cdots x_{a_s}$ is a $<_1$-chain if its indices form a $<_1$-chain.

Define on $\mathbb{K}[X]$ a diagonal term order (e.g. degree lexicographic term order). Then it is easy to see that a monomial is a $<_1$-chain if and only if it is the intial monomial of a minor of X with respect to this term order. Obviously, this minor could not be unique but if we restrict ourselves to maximal minors, we get the following bijective correspondence:

$$\begin{aligned} \varphi : \{<_1 \textit{-chains of } \mathbb{K}[X]\} &\longleftrightarrow \{\textit{maximal minors of } X\} \\ x_{a_1} \cdots x_{a_s} &\longmapsto [a_1, a_2 - 1, \dots, a_s - s + 1] \end{aligned}$$

We want to extend this map to any monomial of $\mathbb{K}[X]$. For this aim, one observes that each monomial m can be decomposed into a product of $<_1$-chains and this decomposition is unique. We are going to illustrate this with an example:

Example 2.2.2. Let $m = x_2^2x_3^2x_4x_6^2x_7$. Consider the maximal $<_1$-chain with respect to the lexicographic order

$$m_1 = x_2x_4x_6.$$

Dividing m by m_1 we get $\tilde{m_1} = x_2x_3^2x_6x_7$. Now the maximal $<_1$-chain of $\tilde{m_1}$ is given by

$$m_2 = x_2x_6.$$

Dividing $\tilde{m_1}$ by m_2, we get $\tilde{m_2} = x_3^2x_7$. The maximal $<_1$ chain of $\tilde{m_2}$ is

$$m_3 = x_3x_7.$$

Dividing again we obtain

$$\tilde{m_3} = m_4 = x_3.$$

We have found the following canonical decomposition of m:

$$m = (x_2x_4x_6)(x_2x_6)(x_3x_7)(x_3).$$

Let $m = m_1m_2 \dots m_k$ be the canonical decomposition of m and denote by s_i the degree of m_i. The sequence of integers $s_1, \dots, s_k$ is called *shape* of m and it is non-increasing. Using the decomposition into $<_1$-chains, m can be represented by a tableau where each row is given by the $<_1$-chain column indices of the factors of m.

Using the canonical decomposition we can extend φ in this way

$$\begin{aligned} \phi : \{\textit{monomials of } \mathbb{K}[X]\} &\longleftrightarrow \{\textit{products of maximal minors of } X\} \\ m = m_1 \dots m_k &\longmapsto \varphi(m_1)\dots\varphi(m_k). \end{aligned}$$

Note that by construction $\mathrm{in}_\prec(\phi(m)) = m$, therefore ϕ is injective. Now we define the set of *standard monomials* of X as the image of the map ϕ and we have the following bijection

$$\phi : \{\textit{monomials of } \mathbb{K}[X]\} \longleftrightarrow \{\textit{standard monomials of } X\}.$$

Remark 2.2.3. In terms of tableaux, a standard monomial is represeneted by a single tableau $A = (a_{ij})$ of shape $s_1, \dots, s_k$ which is standard in the classical sense (i.e. it has increasing rows and non-decreasing columns from the top to the bottom) and that satisfies other additional properties.

Example 2.2.4. Let $m = m = x_2^2x_3^2x_4x_6^2x_7$ be a monomial as in the previous example. We already know that its canonical decomposition is $m = (x_2x_4x_6)(x_2x_6)(x_3x_7)(x_3)$. Using tableaux, we can represent m as

2	4	6
2	6	
3	7	
3		

and the correspondent standard tableau is

2	3	4
2	5	
3	6	
3		

.

As for generic matrices, one can prove the following.

Theorem 2.2.5. [7, Theorem 1.2] *The standard monomials form a $\mathbb{K}$ basis for the polynomial ring $\mathbb{K}[X]$.*

Remark 2.2.6. Assume that μ_1, μ_2, μ_3 are maximal minors such that $\mu_1\mu_2$ and $\mu_2\mu_3$ are standard monomials. Then the product $\mu_1\mu_2\mu_3$ might not be a standard monomial. Hence $\mathbb{K}[X]$ is not an ASL algebra.

A product of maximal minors $\mu_1\mu_2 \dots \mu_k$ is a standard monomial if and only if $\mu_i\mu_j$ is standard for every pair of indices $1 \leq i < j \leq k$.

As a consequence of Theorem 2.2.5, every polynomial $f \in \mathbb{K}[X]$ can be written in a unique way as a $\mathbb{K}$-linear combination of standard monomials. This expression is called *standard representation* or *straightening law* of f.

2.2.2 Determinantal ideals and relations between minors

Let $X = X_j^{(1,n)}$ be a Hankel matrix and let $t \leq \min(j, n-j+1)$, we denote by $I_t(X)$ the determinantal ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by all the t-minors of X.

Thanks to the symmetric structure of Hankel matrices, the minors have intersting relations which will be useful for the proof of our results.

First of all, one can note that

$$[a_1, \dots, a_s | b_1, \dots, b_s] = [b_1, \dots, b_s | a_1, \dots, a_s]$$

and

$$[a_1+1,\dots,a_s+1|b_1,\dots,b_s]=[a_1,\dots,a_s|b_1+1,\dots,b_s+1].$$

Conca stated other important relations; in particular one has the followings (e.g. see [7, Corollary 2.2]).

Theorem 2.2.7. [7, Corollary 2.2]

(a) If $j>t$, then every t-minor of X_j is a linear combination of t-minors of X_{j-1}.

(b) $I_t(X_j)=I_t(X_t)$ for all $t\le j\le n+1-t$.

(c) Every t-minor of X is a linear combination of maximal t-minors.

Remark 2.2.8. Note that part (b) of the previous theorem says that $I_t(X_j^{(1,n)})$ does not depend on j but only on t and n.

Conca used these relations in order to prove that similar properties to those we have seen for determinatal ideals of generic matrices still hold in the case of Hankel matrices (e.g. the shape of standard monomials appearing in the straightening law).

Moreover, an analogous of Theorem 2.1.10 is also known.

Theorem 2.2.9. [1, Section 6] *Let X_j be a Hankel matrix. The determinantal ideal I_t is prime and*

$$\operatorname{ht} I_t = n-2t+2.$$

In particular, the minors of X_j are irreducible polynomials.

For further details about determinantal ideals of Hankel matrices, the reader is referred to [7] and [30].

Chapter 3

Frobenius splittings and F-singularities

In this chapter we recall some basic definitions and results about Frobenius theory, focusing in particular on the notion of F-splitting. Indeed, the work by Knutson (see [17]) about Frobenius splitting and compatibly split ideals has been the starting point of this thesis.

In this chapter, unless otherwise stated, we assume that all rings are Noetherian and contain $\mathbb{F}_p$. This latter assumption makes the Frobenius map be an endomorphism of ring.

The Frobenius endomorphism is a powerful tool because it encodes important properties of the ring R we want to study. In other words, it allows us to detect algebraic properties of a prime characteristic ring R by studying the algebraic and geometric properties of the Frobenius; this leads to the notion of F-*singularity*.

3.1 Frobenius homomorphism

Every ring R of positive characteristic p is equipped with a ring endomorphism, called *Frobenius endomorphism* and defined as follows:

$$\begin{aligned} F : R &\longrightarrow R \\ r &\longrightarrow r^p. \end{aligned}$$

For each $e \in \mathbb{N}$, we can iterate the above endomorphism e times and we obtain the eth Frobenius endomorphism:

$$\begin{aligned} F^e : R &\longrightarrow R \\ r &\longrightarrow r^{p^e}. \end{aligned}$$

Note that the Frobenius endomorphism gives R a structure of module over itself with the non-standard action defined by $r.x = r^p x$. To avoid confusion, we will denote by $F_* R$ this R-module.

To understand the importance of the Frobenius in the study of rings of prime characteristic, we begin with this simple example.

Remark 3.1.1. R is reduced if and only if F is injective. In fact, assume that F is injective and let $x^n = 0$. Then $x^{p^e} = 0$ for some integer e sucht that $p^e \geq n$. But F^e is injective (beacause F is injective), so $x = 0$. This proves that R is reduced. Viceversa, let $F(x) = x^p = 0$ and assume that R is reduced. Then $x = 0$, so F is injective.

Note that F is almost never surjective. A ring R of positive characteristic such that its Frobenius map is an isomorphism is said to be *perfect*.

Remark 3.1.2. If R is a perfect Noetherian domain of characteristic p, then R is a field.

If R is reduced, denoting by R^p the ring of pth-powers of the elements of R, we get the isomorphism

$$\begin{aligned} R &\longrightarrow R^p \\ r &\longrightarrow r^p \end{aligned}$$

Hence F factors through the inclusion $R^p \hookrightarrow R$:

$$\begin{tikzcd} R \arrow[rr, "F"] \arrow[dr, "\cong"'] & & R \\ & R^p \arrow[ur, hook, "\iota"'] & \end{tikzcd}.$$

We can therefore identify F with this inclusion and view $F_* R$ as an R^p -module.

By the same reasoning, if we denote by $R^{1/p}$ the ring of pth-roots of all elements of R (inside the algebraic closure of its total ring of fractions $\mathrm{Frac}(R)$), the map

$$\begin{aligned} R^{1/p} &\longrightarrow R \\ r^{1/p} &\longrightarrow r. \end{aligned}$$

is an isomorphism. So we can factor the Frobenius into an inclusion followed by an isomorphism

$$\begin{tikzcd} R \arrow[rr, "F"] \arrow[dr, hook, "\iota"'] & & R \\ & R^{1/p} \arrow[ur, "\cong"'] & \end{tikzcd}.$$

This means that we can also think the Frobenius map as the inclusion $R \subset R^{1/p}$.

Once R has been given the structure of a module over R^p, one may ask if this module, that we have called F_*R, has some interesting properties (e.g. finitness, flatness). This leads to the introduction of the notion of F-singularity, that we are going to investigate in the following sections.

3.2 F-finiteness and Kunz's theorem

The first nice property one may ask for the module F_*R is to be finite.

Definition 3.2.1. R is called *F-finite* if for some (or equivalently, every) $e > 0$, the Frobenius map is a finite morphism, i.e., the target R is finitely generated as a module over the source R.

A simple argument shows that F-finitness can be checked by passing to the reduced ring, i.e., R is F-finite if and only if $R/\sqrt{0}$ is F-finite.

Example 3.2.2. Let $R = \mathbb{F}_p[x]$. Then R is a free R-module of rank p with basis $1, x, \ldots, x^{p-1}$. Equivalently, $R^{1/p}$ is a finitely generated R-module with basis $1, x^{1/p}, x^{2/p}, \ldots, x^{p-1/p}$. More generally, this holds for polynomial rings $\mathbb{K}[x]$ over a perfect field $\mathbb{K}$, that is $\mathbb{K} = \mathbb{K}^p$.

Moreover, it is easy to check that F-finitness is preserved under classical operations.

Proposition 3.2.3. *Let R be an F-finite ring. Then*

1) R/I is F-finite for every ideal I.

2) $S^{-1}R$ is F-finite for every S multiplicative subset of R.

3) $R[x]$ and $R[[x]]$ are F-finite.

As a consequence, we get

Corollary 3.2.4. *If $(R, \mathfrak{m})$ be a complete local ring, R is F-finite if and only if $R/\mathfrak{m}$ is F-finite.*

Using Proposition 3.2.3, we can generalize Example 3.2.2 to polynomial rings in several variables.

Example 3.2.5. Let $R = \mathbb{F}_p[x_1, \dots, x_n]$. Then R is a free R^p-module of rank p^n with basis $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq p-1\}$.

The F-finitness guarantees that the rings we are dealing with are reasonably good:

Proposition 3.2.6. *If R is F-finite, then*

1) [20, Theorem 2.5] *R is excellent; in particular it is universally catenary.*

2) [11, Remark 13.6] *R is a homomorphic image of a regular ring.*

Furthermore, Kunz proved that regularity can be detected by means of the Frobenius.

Theorem 3.2.7. [19, Theorem 2.1] *R is regular if and only if the Frobenius map $F^e : R \to R$ is flat for some (or equivalently, for all) $e > 0$.*

3.3 F-splitting and F-purity

The central topic of this thesis was inspired by the work of Knutson about Frobenius splitting of polynomial rings and compatibly split ideals (see [17]).

Frobenius splitting, toghether with F-purity and other Frobenius techniques, has attracted many researchers from commutative algebra, algebraic geometry and representation theory. Both the notions of F-splitting and F-purity were originally suggested in the 1970s by Hochster and Roberts ([13]). The term Frobenius split, however, was formally introduced by Mehta and Ramanathan in [22], where they transferred these ideas in a projective setting.

In [13] the authors proved that the ring of invariants of a linearly reductive affine linear group acting on a regular ring is Cohen-Macaulay by means of F-singularities. The key idea of the proof is that the ring of invariants (of a linearly reductive group) turns out to be a direct summand of a polynomial ring, and so it inherits a strong form of Frobenius splitting which implies Cohen-Macaulayness.

We have already seen that being reduced for R is equivalent to F being injective; in particular, this means that F has a one-side inverse. We are going to study these inverses.

Definition 3.3.1. R is said to be *F-split* if the Frobenius splits in the category of R-modules, i.e., if there exists a homomorphism $\varphi : F_*R \to R$ of R-modules such that $\varphi \circ F = 1_R$. Such a φ is called an F-splitting of R.

Remark 3.3.2. The above definition is equivalent to require that there exists $\varphi \in \mathrm{Hom}_R(F_*R, R)$ such that $\varphi(1) = 1$.

Example 3.3.3. $R = \mathbb{F}_p[x_1, \dots, x_n]$ is F-split. We have already seen that R is a free R^p-module with basis $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \le a_i \le p-1\}$. We can define φ to be the projection onto the summand generated by $1 = x_1^0 \cdots x_n^0$ and this is a Frobenius splitting for R.

If R is F-split, then F is obviously injective and, for a reduced ring, this means that R^p is a direct summand of R. This fact is particularly interesting beacause many nice algebraic properties are inherited when passing to direct summands. In addition, the F-splitting property itself passes on to direct summands.

Proposition 3.3.4. *Let R be a direct summand of an F-split ring S. Then R is F-split.*

Moreover, what we have seen in Example 3.3.3 for polynomial rings can be generalized as follows.

Proposition 3.3.5. *Every F-finite regular ring is F-split.*

It should be observed that by Kunz's theorem, if R is an F-finite regular local ring, then R is a free R^p-module. So, in this case, regularity can be thought as the condition that R completely decomposes into direct sum of copies of R^p, whereas F-split means that R contains *at least one* direct sum copy of R^p.

Hochster and Roberts introduced a weaker notion called *F-purity.*

Definition 3.3.6. R is said to be *F-pure* if F is a pure map, meaning that for all R-module M the map $F \otimes 1 : R \otimes_R M \to F_*R \otimes_R M$ is injective.

Obviously, F- split always implies F-pure but the converse does not hold in general. Nonetheless these notions are equivalent in many cases, for esample under the hypotesis of F-finiteness or if R is a complete local ring.

In general, it is quite difficult to identify those rings which are F-pure. However Fedder gave a useful criterion to check whether an ideal defines a F-pure ring or not.

Proposition 3.3.7. [10, Theorem 1.12] *Let $(R, \mathfrak{m})$ be a regular local ring and let $I \subseteq R$ be an ideal. Then R/I is F-pure if and only if $(I^{[p]} : I) \nsubseteq \mathfrak{m}^{[p]}$ where $I^{[p]}$ denotes the ideal generated by the pth powers of elements of I.*

Note that Fedder's criterion does not require F-finiteness and it can be easily tested by means of a computer algebra system.

Example 3.3.8. Let $R = \mathbb{K}[x, y, z]/(x^3 + y^3 + z^3)$. By Fedder's criterion R is F-pure if and only if $(I^{[p]} : I) = ((x^3 + y^3 + z^3)^{p-1}) \nsubseteq (x^p, y^p, z^p)$. The terms of the monomial expansion of the left hand side are of the form $\binom{p-1!}{i!j!k!}x^{3i}y^{3j}z^{3k}$ where $i + j + k = p - 1$. If $p \equiv 1 \pmod 3$, then there is a non zero term in the monomial expansion that is a multiple of $(xyz)^{p-1}$. Hence, R is F-pure. If $p \equiv 2$ mod 3, say $p = 3h + 2$ for some integer h, then one of i, j, k must be $\geq h + 1$ and when we multiply by 3 we get an exponent $\geq p$. So we can infer that R is not F-pure.

Example 3.3.9. Let R be a Stanley-Reisner ring, i.e. the quotient of a polynomial ring by a squarefree monomial ideal. If f is a squarefree monomial, then $(x_1 \cdots x_n)^{p-1}f \in (f^p)$. Hence $(x_1 \cdots x_n)^{p-1} \in (I^{[p]} : I)$ if I is a squarefree monomial ideal but $(x_1 \cdots x_n)^{p-1} \notin \mathfrak{m}^p$. Then, by Fedder's criterion R is F-pure. Actually, squarefree monomial ideals are the only F-pure monomial ideals. In fact, if I is a non squarefree monomial ideal, then I is not reduced and so it cannot be F-pure.

3.4 Compatibly split ideals

In this last section of Chapter 3, we will discuss the case of polynomial rings. More explicitly, we will describe their F-splittings and we will introduce the notion of *compatibly split ideal* whose properties motivate the definition of a new class of ideals which will be the main subject of this thesis .

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the standard graded polynomial ring with coefficients in a perfect field of prime characteristic p. We already know (see Example 3.2.5) that F_*S is a free S-module with basis $B = \{x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq$

$i_k \leq p-1\}$. In particular, S is F-split. Our aim is to give a description of all F-splittings of S and, more generally, to understand the structure of $\mathrm{Hom}_S(F_*S, S)$ as an F_*S-module. This latter module is clearly free as an S-module and it is generated by the dual basis $B^* = \{\varphi_{i_1,\ldots,i_n}\}$, where $\varphi_{i_1,\ldots,i_n}$ is the S-linear map defined as follows:

$$\varphi_{i_1,\ldots,i_n}(x_1^{j_1}\cdots x_n^{j_n}) = \begin{cases} 1 & (j_1,\ldots,j_n) = (i_1,\ldots,i_n) \\ 0 & (j_1,\ldots,j_n) \neq (i_1,\ldots,i_n). \end{cases}$$

Note that for each tuple $(i_1,\ldots,i_n)$, $\varphi_{i_1,\ldots,i_n}$ defines a splitting of S.

In order to understand the structure of $\mathrm{Hom}_S(F_*S, S)$ as a F_*S-module, we define $\mathrm{Tr} := \varphi_{p-1,\ldots,p-1} \in \mathrm{Hom}_S(F_*S, S)$. It is straightforward to prove that the following is an isomorphism of F_*S-modules:

$$\begin{aligned} \Phi : F_*S &\longrightarrow \mathrm{Hom}_S(F_*S, S) \\ f &\mapsto \mathrm{Tr}(f\bullet) : g \mapsto \mathrm{Tr}(fg) \end{aligned}$$

Clearly Φ is an injection: if $\mathrm{Tr}(f\bullet) = 0$ then $f\,\mathrm{Tr}(\bullet) = \mathrm{Tr}(f^p\bullet) = 0$. So $f = 0$. For surjectivity, it suffices to note that $\varphi_{i_1,\ldots,i_n} = \mathrm{Tr}(x_1^{p-i_1-1}\cdots x_n^{p-i_n-1}\bullet)$. This says that $\mathrm{Hom}_S(F_*S, S)$ is generated by Tr as an F_*S-module.

Remark 3.4.1. It should be noted that $\mathrm{Tr}(f\bullet)$ is not an F-splitting in general. In order to be an F-splitting, we need that $\mathrm{Tr}(f) = 1$. In other words, $\mathrm{Tr}(f\bullet)$ is an F-splitting if and only if the following two conditions hold:

(i) $x_1^{p-1}\cdots x_n^{p-1} \in \mathrm{Supp}(f)$ and it has coefficient equal to 1.

(ii) If $x_1^{u_1}\cdots x_n^{u_n} \in \mathrm{Supp}(f)$ with $u_1 \equiv \ldots \equiv u_n \equiv -1$ then $u_1 = \ldots = u_n = p-1$.

We have already discussed, both in the introduction and in the previous section, the crucial role that Frobenius splittings play in algebraic geometry and commutative algebra. Thus, one may ask whether the property of being F-split is preserved when passing to the quotient or not.

Given an F-split ring R, those ideals I such that the F-splitting of R descends to an F-splitting of R/I are said to be *compatibly split.* In this last part of Chapter 3, we are going to summarize some well known results about these ideals, with particular attention to Knutson's theorems on Gröbner degenerations of compatibly split ideals (see [17]).

We start off with the formal definition of compatibly split ideals.

Definition 3.4.2. Let R be an F-split ring and let ϕ be a Frobenius splitting of R. An ideal $I \subset R$ is said to be *compatibily split* (with respect to ϕ) if $\phi(I) \subset I$.

For the sake of completeness, we will resume some basic properties of these ideals. The interested reader is referred to [3, Section 1.2] for further details.

Proposition 3.4.3. *Let R be an F-split ring and let I and J be two ideals of R. Then the followings hold:*

1. *R is reduced*
2. *If I is compatibly split, then I is radical and $\phi(I) = I$ (equivalently, R/I is F-split).*
3. *If I and J are both compatibly split ideals, then so are $I + J$ and $I \cap J$. In particular, they are radical.*
4. *If I is compatibly split and J is an arbitrary ideal, then $I : J$ is compatibily split. In particular the prime components of I are compatibly split.*

Remark 3.4.4. Note that the property of being compatibly split is much stronger than being radical. In fact, the sum of two radical ideals is usually not radical.

Proof. (1) If R is not reduced, then there exists a non-zero nilpotent r. Assume that m is the first integer such that $r^m \neq 0$ and $r^{m+1} = 0$. Then $0 = \varphi((r^m)^p) = r^m$, a contradiction.
(2) If I is compatibly split , then φ induces an F-splitting $\overline{\varphi}$ on R/I. So, the quotient R/I is F-split. In particular, R/I is reduced, equivalently I is radical. Furthermore, $\varphi(I) = I$. Indeed, since $I^{[p]} \subset I$, the inclusion $I \subset \phi(I)$ holds for any F-splitting ϕ.
(3) Easily follows from the additivity of the F-splitting φ and from the fact that $\varphi(I \cap J) \subset \varphi(I) \cap \varphi(J)$.
(4) Let $r \in I : J$, we need to prove that $\varphi(r) \in I : J$.

$$\begin{aligned} r \in I : J \Leftrightarrow \forall j \in J, rj \in I \Rightarrow \forall j \in J, rj^p \in I \Rightarrow \forall j \in J, \varphi(rj^p) \in \varphi(I) \subset I \\ \Leftrightarrow \forall j \in J, \varphi(r)j \in I \Leftrightarrow \varphi(r) \in I : J \end{aligned}$$

□

This proposition suggests an algorithm to construct a family of radical ideals.

Corollary 3.4.5. [17, Corollary 1] *Let I be a compatibly split ideal in an F-split ring R. Starting from I and taking prime components, sums and intersections, we obtain many more compatibly split ideals. In particular, they will be radical.*

Interestingly, it has been proved (see [27], [16]) that in Noetherian F-split rings there are a finite number of compatibly split ideals with respect to a fixed splitting.

By Remark 3.4.1, if we take $f = x_1^{p-1} \cdots x_n^{p-1}$, then $\mathrm{Tr}(f\bullet) \in \mathrm{Hom}_S(F_*S, S)$ is an F-splitting of S, called the *fundamental splitting* of S and the compatibly split ideals with respect to it are known:

Proposition 3.4.6. [17, Lemma 1] *The compatibly split ideals of S with respect to the fundamental splitting $\mathrm{Tr}(x_1^{p-1} \cdots x_n^{p-1}\bullet)$ are exactly the squarefree monomial ideals of S.*

Knutson in [17] investigates the relation between Frobenius splitting and degeneration. One first result in this direction is the following:

Theorem 3.4.7. [17, Theorem 2] *Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a perfect field $\mathbb{K}$ of positive characteristic and let $f \in S$ be a polynomial of degree at most n. Assume that $\mathrm{in}_\prec(f) = \prod_i x_i$.*

(i) $\mathrm{Tr}(f^{p-1}) = \mathrm{Tr}(\mathrm{in}_\prec(f)^{p-1})$. *In paticular,* $\mathrm{Tr}(f^{p-1}\bullet)$ *defines a Frobenius splitting if and only if* $\mathrm{Tr}(\mathrm{in}_\prec(f)^{p-1}\bullet)$ *does.*

(ii) Assume that $\mathrm{Tr}(f^{p-1}\bullet)$ *and* $\mathrm{Tr}(\mathrm{in}_\prec(f)^{p-1}\bullet)$ *are F-splitting. If I is compatibly split with respect to* $\mathrm{Tr}(f^{p-1}\bullet)$*, then* $\mathrm{in}_\prec(I)$ *is compatibly split with respect to* $\mathrm{Tr}(\mathrm{in}_\prec(f)^{p-1}\bullet)$.

Actually, Knutson proved the theorem for polynomial rings over $\mathbb{F}_p$, but this result can be extended to polynomial rings with coefficients in a perfect ring $\mathbb{K}$ of characteristic p.

Indeed, the first part of Theorem 3.4.7 relies on the following lemma which holds for polynomial rings over perfect field of positive characteristic.

Lemma 3.4.8. [17, Lemma 5] *Let $f \in S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial of degree at most n over a perfect field $\mathbb{K}$ of characteristic p. Then* $\mathrm{Tr}(f^{p-1})$ *is the pth-root of the coefficient of $\prod x_i^{p-1}$ in f^{p-1}.*

Second part of Theorem 3.4.7 is instead based on [17, Lemma 2]:

Lemma 3.4.9. [17, Lemma 2] *Let $\omega = (\omega_1, \dots, \omega_n) \in (\mathbb{N}_{>0})^n$ be a weight vector. Then for any polynomial $g \in \mathbb{K}[x_1, \dots, x_n]$, either* $\mathrm{Tr}(\mathrm{in}_\omega(g)) = 0$ *or* $\mathrm{Tr}(\mathrm{in}_\omega(g)) = \mathrm{in}_\omega(\mathrm{Tr}(g))$.

The author gave a proof of the above lemma in the case $\mathbb{K} = \mathbb{F}_p$ which easily generalizes to perfect field of characteristic p; one has just to keep in mind that every element c of a perfect field $\mathbb{K}$ of characteristic p has a pth root in $\mathbb{K}$ (in the case $\mathbb{K} = \mathbb{F}_p$, $c^p = c$).

If we restrict our attention to more specific splittings, we can obtain much stronger results about compatibly split ideals.

Assume that f is a polynomial in S such that $\mathrm{in}(f)$ is a product of distinct variables (geometrically, if $\deg f = n$ this says that the hypersurface $f = 0$ is necessary singular). Then, it turns out that all the ideals obtained from the ideal (f) by taking colons, sums and intersections, and iterating, have squarefree initial ideal.

Theorem 3.4.10. [17, Theorem 4] *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{K}$ of positive characteristic. Let f be a polynomial in S such that $\mathrm{in}(f) = \prod_i x_i$ with respect to the lexicographic term order. Let I be one of the ideals constructed from the ideal (f) by taking colons, sums and intersections, and iterating. Then $\mathrm{in}\,(I)$ is a Stanley-Reisner ideal.*

Proof. Possibly multiplying by the product of the variables missing in $\mathrm{in}(f)$, we can always assume that $\deg(f) = n$ and $\mathrm{in}(f) = \prod_{i=1}^{n} x_i$. Thereby, we only enlarge the set of ideals obtained by the algorithm. Since $\mathrm{Tr}\,(\mathrm{in}(f)^{p-1}\bullet) = \mathrm{Tr}(x_1^{p-1} \cdots x_n^{p-1}\bullet)$ defines a splitting (specifically, the fundamental splitting), by the first part of Theorem 3.4.7 also $\mathrm{Tr}(f^{p-1}\bullet)$ is a splitting of S. Furthermore, by part (2) of Theorem 3.4.7, if I is a compatibly split ideal with respect to $\mathrm{Tr}(f^{p-1}\bullet)$ then $\mathrm{in}(I)$ is compatibly split with respect to $\mathrm{Tr}\,(\mathrm{in}(f)^{p-1}\bullet) = \mathrm{Tr}\left(\prod_{i=1}^{n} x_n\bullet\right)$, so it is a Stanley-Reisner ideal by Proposition 3.4.6. $\square$

The family of ideals we obtain with the procedure described in Theorem 3.4.10 will be the main subject of the following chapters.

The next simple observation comes directly from Theorem 3.4.10 and Corollary 3.4.5.

Remark 3.4.11. In the case $\mathrm{in}(f) = \prod_{i=1}^{n} x_i$, $\mathrm{Tr}(f^{p-1}\bullet)$ is an F-splitting and the principal ideal (f) is obviously compatibly split with respect to it. In fact, if $rf \in (f)$ then $\mathrm{Tr}(f^{p-1}rf) = \mathrm{Tr}(f^p r) = f\,\mathrm{Tr}(r) \in (f)$. Hence, we can apply Corollary 3.4.5 to construct many more compatibly split ideals starting from (f) and taking colons, sums and intersections. All these ideals have squarefree initial ideal.

We have already seen that under certain assumptions, compatibly split ideals have squarefree initial ideal. Using this fact, Knutson proved an interesting and useful result about Gröbner basis of compatibly split ideals. He first showed the following:

Proposition 3.4.12. [17, Corollary 2] *Let $\{I_i\}_i \in K$ be a finite set of polynomial ideals in $S = \mathbb{K}[x_1, \ldots, x_n]$ such that for any $K' \subset K$, $\operatorname{in}\left(\bigcap_{i \in K'} I_i\right)$ is radical. Then*

$$\operatorname{in}\left(\sum_{i \in K} I_i\right) = \sum_{i \in K} \operatorname{in}(I_i).$$

In other words, if $\mathcal{G}_{I_i}$ is a Gröbner basis for I_i, then $\bigcup_{i \in K} \mathcal{G}_{I_i}$ is a Gröbner basis for $\sum_{i \in K} I_i$.

Then he applied this proposition to Theorem 3.4.10 to prove that we can concatenate Gröbner basis of compatibly split ideals.

Theorem 3.4.13. [17, Theorem 6] *Let $f \in S = \mathbb{K}[x_1, \ldots, x_n]$ such that $\operatorname{in}(f) = \prod_i x_i$ with respect to the lexicographic term order and let I and J be two compatibly split ideals with respect to $\operatorname{Tr}(f^{p-1} \bullet)$. Then*

$$\mathcal{G}_{I+J} = \mathcal{G}_I \cup \mathcal{G}_J.$$

In the following chapter we are going to investigate the compatibly split ideals described in Remark 3.4.11, generalizing Knutson's theory to polynomial rings over any field.

Chapter 4

Knutson ideals in any characteristic

Motivated by Knutson's work on compatibly split ideals described in Chapter 3, Conca and Varbaro defined a new family of ideals (see [8]), namely "Knutson ideals", starting from a polynomial f with squarefree leading term and applying Corollary 3.4.5 to the principal ideal (f). In this chapter, we begin the study of this class of ideals whose properties allow us to prove interesting results on radicality and F-purity of certain ideals.

Assume for the moment that $\mathbb{K} = \mathbb{F}_p$ is the finite field with p elements and fix $f \in S = \mathbb{K}[x_1, \dots, x_n]$ a polynomial such that its leading term $\mathrm{in}_\prec(f)$ is a squarefree monomial for some term order $\prec$. We can define many more ideals starting from the principal ideal (f) and taking associated primes, intersections and sums. Thereby, if $\deg f = n$, by Remark 3.4.11 we obtain a family of ideals compatibly split with respect to $\mathrm{Tr}(f^{p-1}\bullet)$.

Geometrically this means that we start from the hypersurface defined by f and we construct a family of new subvarieties $\{Y_i\}_i$ by taking irreducible components, intersections and unions.

Definition 4.0.1 (Knutson ideals)**.** Let $f \in S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial such that its leading term $\mathrm{in}_\prec(f)$ is a squarefree monomial for some term order $\prec$. Define $\mathcal{C}_f$ to be the smallest set of ideals satisfying the following conditions:

1. $(f) \in \mathcal{C}_f$;
2. If $I \in \mathcal{C}_f$ then $I : J \in \mathcal{C}_f$ for every ideal $J \subseteq S$;
3. If I and J are in $\mathcal{C}_f$ then also $I + J$ and $I \cap J$ must be in $\mathcal{C}_f$.

If I is an ideal in $\mathcal{C}_f$, we say that I is a *Knutson ideal associated to* f. More generally, we say that I is a *Knutson ideal* if $I \in \mathcal{C}_f$ for some f.

As explained in the previous chapter, Knutson proved that if $\mathbb{K} = \mathbb{F}_p$, this class of ideals has some interesting properties, such as:

(I) Every $I \in \mathcal{C}_f$ has a squarefree initial ideal, so every Knutson ideal is radical.

(II) If two Knutson ideals are different their initial ideals are different. So $\mathcal{C}_f$ is finite.

(III) The union of the Gröbner bases of Knutson ideals associated to f is a Gröbner basis of their sum.

Remark 4.0.2. Actually, assuming that every ideal of C_f is radical, the second condition in Definition 4.0.1 can be replaced by the following:

2′. If $I \in \mathcal{C}_f$ then $\mathcal{P} \in \mathcal{C}_f$ for every $\mathcal{P} \in \operatorname{Min}(I)$.

In fact, let $I \in \mathcal{C}_f$, then

$$I = \sqrt{I} = P_1 \cap P_2 \cap \ldots \cap P_r$$

where P_i are the minimal primes of I. Fix $c \in (P_2 \cap \ldots \cap P_r) \setminus P_1$. Clearly $P_1 \subseteq (I : c) \subseteq P_1$, hence $P_1 = (I : c)$. The same holds for every P_i. Viceversa, it is easy to observe that if I is radical, then the minimal primes of $I : J$ are exactly the minimal primes of I that do not contain J.

This background has been the starting point of our work.

In this chapter we introduce the definition of Knutson ideals in polynomial rings over any field and we show that the properties listed in the previous discussion stay unchanged. To do so, we will first generalize Knutson's results (Theorem 3.4.7 and Theorem 3.4.10) to fields of positive characteristic (see Proposition 4.1.1) and then we use the achieved result together with reduction modulo p to prove that the same holds for polynomial rings over fields of characteristic 0 (see Proposition 4.2.1). The remaining two properties can be inferred from the first one. Indeed, the finitness of the family C_f is an easy consequence of Remark 4.2.4, while the last property can be deduced again by Remark 4.2.4 using the fact that

$$\operatorname{in}_\prec(I \cap J) = \operatorname{in}_\prec(I) \cap \operatorname{in}_\prec(J) \Leftrightarrow \operatorname{in}_\prec(I + J) = \operatorname{in}_\prec(I) + \operatorname{in}_\prec(J).$$

In the case of homogeneous ideals, the latter equivalence comes from the usual short exact sequence

$$0 \longrightarrow S/(I \cap J) \longrightarrow S/I \oplus S/J \longrightarrow S/(I+J) \longrightarrow 0$$

using the fact that the Hilbert function does not change when passing to the inital ideal. If I and J are not homogeneous, the equivalence is still true but the proof requires more work.

4.1 Fields of characteristic $p > 0$

Let $\mathbb{K}$ be a field of characteristic $p > 0$ and let $f \in S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial such that $\text{in}_\prec(f)$ is squarefree for some term order $\prec$. As in the case of $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$, we can construct the family C_f as the smallest set of ideals such that:

- $(f) \in C_f$
- $I \in C_f, \quad J \subseteq S \Rightarrow I : J \in C_f$
- $I, J \in C_f \Rightarrow I + J, \quad I \cap J \in C_f.$

We want to prove the following result.

Proposition 4.1.1. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and let f be a polynomial in $S = \mathbb{K}[x_1, \ldots, x_n]$ such that* $\text{in}_\prec(f)$ *is squarefree for some term order $\prec$. If $I \in C_f$ then* $\text{in}_\prec(I)$ *is squarefree.*

A first step toward this result has already been done in Chapter 3, observing that Lemma 3.4.9, and so Theorems 3.4.7 and 3.4.10, hold also for polynomial rings over perfect fields of characteristic p.

To prove Proposition 4.1.1, we reduce to the case of perfect fields of positive characteristic so that we can apply Theorem 3.4.10.

Let $\mathbb{K} \hookrightarrow \overline{\mathbb{K}}$ be the extension of $\mathbb{K}$ to its algebraic closure $\overline{\mathbb{K}}$. Since $\text{char}(\mathbb{K}) = p$, then $\overline{\mathbb{K}}$ is a perfect field of characteristic p.

Let $\bar{S} = \overline{\mathbb{K}}[x_1, \ldots, x_n]$ and consider the natural extension:

$$\iota : S \longrightarrow \overline{S}.$$

So $\overline{f} := \iota(f)$ is a polynomial in $\overline{S}$ (we regard f as a polynomial with coefficients in $\overline{K}$). Again we can construct the family $\overline{C_f} := C_{\overline{f}}$ in $\overline{S}$.

First of all, one can show that $I \in C_f \Rightarrow I\overline{S} \in \overline{C_f}$. To prove this we will use these well known facts.

Fact 1. ([21, p.46]) The extension of polynomial rings

$$S = \mathbb{K}[x_1, \ldots, x_n] \hookrightarrow \overline{S} = \overline{\mathbb{K}}[x_1, \ldots, x_n]$$

is a flat extension.

Fact 2. ([21, Theorem 7.4]) Let $\pi : A \longrightarrow B$ be a flat ring extension and I and J two ideals of A. Then:

(i) $(I \cap J)B = IB \cap JB$

(ii) If J is finitely generated then $(I : J)B = IB : JB$.

Proof. **(i)** Consider the exact sequence:

$$\begin{array}{ccccc} I \cap J & \hookrightarrow & A & \longrightarrow & A/I \oplus A/J \\ x & \mapsto & x & \mapsto & (x + I, x + J) \end{array}$$

Since B is A-flat, tensoring with B we get another exact sequence

$$(I \cap J)B \hookrightarrow B \xrightarrow{\pi} B/IB \oplus B/JB$$

By exactness we have $(I \cap J)B = IB \cap JB$.

(ii) First suppose $J = (a) \subseteq A$ and consider the exact sequence

$$\begin{array}{ccccc} I : (a) & \overset{\iota}{\hookrightarrow} & A & \xrightarrow{\cdot a} & A/I \\ x & \mapsto & x & \mapsto & ax + I \end{array}$$

Since B is A-flat, tensoring with B we get the following exact sequence:

$$(I : (a))B \overset{\iota}{\hookrightarrow} B \xrightarrow{\cdot a} B/IB$$

and by exactness again, we get

$$\operatorname{Im} \iota = (I : (a))B = \ker(\cdot a) = \{b \in B \mid ba \in IB\} = IB : (a)B$$

If $J = (a_1, \ldots, a_m)$ is finitely generated, then

$$I : J = I : (a_1, \ldots, a_m) = \bigcap I : (a_i).$$

So we can reduce to the case of a principal ideal and use identity (i). $\square$

Coming back to our original problem, we want to show that if $I \in C_f$ then $I\overline{S} \in \overline{C_f}$.

Since, by Fact 1, $S \longrightarrow \overline{S}$ is a flat extension, we can use Fact 2 to get the following equalities:

- $(I + J)\overline{S} = I\overline{S} + J\overline{S}$ (always true)
- $(I \cap J)\overline{S} = I\overline{S} \cap J\overline{S}$ (true for flat extensions)
- $(I : J)\overline{S} = I\overline{S} : J\overline{S}$ (true for flat extensions and J finitely generated).

Consider $(f) \in C_f$, then $(f)\overline{S} = (\overline{f}) \subseteq \overline{S}$ and $(\overline{f}) \in \overline{C_f}$ by definition.

Now let $I, J \in C_f$ such that $I\overline{S}, J\overline{S} \in \overline{C_f}$. By definition $I + J, I \cap J \in C_f$. Using previous identities, we get

$$(I + J)\overline{S} = \underbrace{I\overline{S}}_{\in \overline{C_f}} + \underbrace{J\overline{S}}_{\in \overline{C_f}} \in \overline{C_f}$$

$$(I \cap J)\overline{S} = \underbrace{I\overline{S}}_{\in \overline{C_f}} \cap \underbrace{J\overline{S}}_{\in \overline{C_f}} \in \overline{C_f}.$$

Eventually, let's consider $I \in C_f$ and $J \subseteq S$ an arbitrary ideal. By definition $I : J \in C_f$. Suppose $I\overline{S} \in \overline{C_f}$. Since J is finitely generated, then

$$(I : J)\overline{S} = \underbrace{I\overline{S}}_{\in \overline{C_f}} : \underbrace{J\overline{S}}_{\subseteq \overline{S}} \in \overline{C_f}.$$

So we have proved that if $I \in C_f$ then $I\overline{S} \in \overline{C_f}$. Using this result, we can now prove Proposition 4.1.1.

Proof of Proposition 4.1.1. Let $I \in C_f$. Then $I\overline{S} \in \overline{C_f}$ and by Theorem 3.4.10 $\mathrm{in}_\prec(I\overline{S})$ is squarefree beacause we are working in a polynomial ring over a perfect field of characteristic $p > 0$. But since Buchberger's algorithm is "stable" under base extensions, we have

$$\mathrm{in}_\prec(I\overline{S}) = \mathrm{in}_\prec(I)\overline{S}.$$

So $\mathrm{in}_\prec(I)$ is squarefree. $\square$

4.2 Fields of characteristic 0

Let $\mathbb{K}$ be a field of characteristic 0 and let $f \in S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial such that $\mathrm{in}_\prec(f)$ is squarefree for some term order $\prec$.

As in the previous cases, we can construct the family C_f as the smallest set of ideals such that:

- $(f) \in C_f$
- $I \in C_f, \quad J \subseteq S \Rightarrow I : J \in C_f$
- $I, J \in C_f \Rightarrow I + J, \quad I \cap J \in C_f$.

We want to prove the analogous of Proposition 4.1.1 in the case of polynomial rings over fields of characteristic 0.

Proposition 4.2.1. *Let $\mathbb{K}$ be a field of characteristic 0 and let f be a polynomial in $S = \mathbb{K}[x_1, \dots, x_n]$ such that $\mathrm{in}_\prec(f)$ is squarefree for some term order $\prec$. If $I \in C_f$ then $\mathrm{in}_\prec(I)$ is squarefree.*

We know that this holds in polynomial rings over fields of characteristic $p > 0$. Using Proposition 4.1.1, we will show that the same holds if we are working over fields of characteristic 0.

4.2.1 Reduction modulo p and initial ideals

Let $\mathbb{K}$ be a field of characteristic 0 and define $S := \mathbb{K}[x_1, \dots, x_n]$. Consider an ideal $I \subseteq S$. Since I is finitely generated, it is always possible to construct a finitely generated $\mathbb{Z}$-algebra $A \subset \mathbb{K}$ such that if $I' := I \cap A[x_1, \dots, x_n]$ then $I'S = I$. To do so it suffices to take $A = \mathbb{Z}[\alpha_1, \dots, \alpha_s]$ where the α_i are the coefficients of the generators of I which are not integers.

Example 4.2.2. If $I = (\sqrt{2}x - \pi y) \subset \mathbb{R}[x, y]$, we can take $A = \mathbb{Z}[\sqrt{2}, \pi]$.

Let p be a prime number which is not invertible in A and fix $P \in \mathrm{Min}(pA)$. The quotient ring A/P is an integral domain of characteristic $p > 0$ and we can define I'_p to be the image of I' under the projection map

$$\pi \colon A[x_1, \dots, x_n] \longrightarrow {}^{A}\!/\!_{P}[x_1, \dots, x_n].$$

Since A/P is a domain we can construct its fraction field $\mathrm{Frac}\left({}^{A}\!/\!_{P}\right)$ and we define $S_p := \mathrm{Frac}\left({}^{A}\!/\!_{P}\right)[x_1, \dots, x_n]$. So we can consider the extended ideal $I(p) := I'_p S_p$ in the polynomial ring S_p.

This is what we call a *reduction modulo* $p \in \mathbb{N}$. Although the notation might be confusing, the ideal $I(p)$ does not depend only on p and I but also on the choice of $P \in \operatorname{Min}(pA)$.

To summarize, we have constructed the following diagram:

$$\begin{array}{ccc}
A[x_1,\dots,x_n] & \longrightarrow & S = \mathbb{K}[x_1,\dots,x_n] \qquad\qquad \text{char} = 0 \\
\big\downarrow \pi & & \\
\left(A/P\right)[x_1,\dots,x_n] & \longrightarrow & \operatorname{Frac}\left(A/P\right)[x_1,\dots,x_n] \qquad \text{char} = p > 0
\end{array}$$

Note that the lower map in the diagram is flat.

The next lemma states that taking initial ideals commutes with reduction modulo p for all sufficently large p.

Lemma 4.2.3. *Let $\mathbb{K}$ be a field of characteristic 0 and $S = \mathbb{K}[x_1,\dots,x_n]$ the polynomial ring over $\mathbb{K}$ with a fixed term order $\prec$. Take $I_1,\dots,I_m$ ideals in S. Then for all $p \gg 0$ there exists a reduction modulo p such that*

$$\operatorname{in}_\prec(I_j(p)) = \operatorname{in}_\prec(I_j)(p) \qquad \forall j = 1,\dots,m.$$

Proof. It suffices to prove the result for $m = 1$. If $m > 1$ we can always choose p greater than the maximum of the p_j such that the result is true for I_j and we are done.

Consider an ideal $I = (f_1,\dots,f_r) \subseteq S$ and construct the finitely generated $\mathbb{Z}$-algebra $A = \mathbb{Z}[\alpha_1,\dots,\alpha_t] \subset \mathbb{K}$ where the α_i are the coefficients of the generators of I which are not integers. Note that since A is a finitely generated $\mathbb{Z}$-algebra, all the polynomial rings we are dealing with are Noetherian.

Accordingly with the previous notation, we set

$$\begin{aligned}
I' &:= I \cap A[x_1,\dots,x_n] \\
I'' &:= I' \operatorname{Frac}(A)[x_1,\dots,x_n] = I \cap \operatorname{Frac}(A)[x_1,\dots,x_n].
\end{aligned}$$

Using Buchberger's algorithm we can compute a Gröbner basis for I''. Let $G_{I''} = \{g_1,\dots,g_s\}$ be this Gröbner basis. Since Buchberger's algorithm is "stable" under base extensions we get $G_{I''} = G_I$, that is $G_{I''}$ is also a Gröbner basis for I in S.

Observe that, possibly multiplying by an element of A, we can assume that $g_1,\dots,g_s$ are polynomials in $A[x_1,\dots,x_n]$.

In the computation of the Gröbner basis, no new coefficients appear but we need to invert some elements $\lambda_1, \dots, \lambda_t \in A$ to compute S-polynomials. If we find a prime number p and a minimal prime $P \in \operatorname{Min}(pA)$ such that $\lambda_1, \dots, \lambda_t \notin P$, then $\lambda_1, \dots, \lambda_t$ are invertible in $\operatorname{Frac}\left({}^{A}\!/\!_{P}\right)$, so the algorithm is exactly the same also when we reduce modulo p. This will imply that $\overline{G_I} = \{\overline{g}_1, \dots, \overline{g}_s\}$ is a Gröbner basis for $I(p)$, hence

$$\operatorname{in}_\prec(I(p)) = (\operatorname{in}_\prec(\overline{g}_1), \dots, \operatorname{in}_\prec(\overline{g}_s)).$$

Since we are working in Noetherian domains, the principal ideal (pA) has finitely many minimal primes and by Krulls Hauptidealsatz if $P \in \operatorname{Min}(pA)$ then $\operatorname{ht}(P) = 1$. Moreover it's easy to see that if p and q are two different prime numbers, then $\operatorname{Min}(pA) \cap \operatorname{Min}(qA) = \emptyset$. Assume that there exists a prime ideal $Q \in \operatorname{Min}(pA) \cap \operatorname{Min}(qA)$, then $Q \supseteq (pA), (qA)$. In particular $p, q \in Q$ and they are coprime. This would imply that $1 \in Q$, a contradiction. Similarly the ideal $(\lambda_i A)$ has finitely many minimal primes of height 1, therefore there exists a prime number $\overline{p_i}$ such that

$$\forall p > \overline{p_i},\ \forall P \in \operatorname{Min}(pA) \quad \lambda_i \notin P.$$

Taking $\overline{p} := \max \overline{p_i}$, we get that

$$\forall p > \overline{p},\ \forall P \in \operatorname{Min}(pA) \quad \lambda_1, \dots, \lambda_t \notin P.$$

This proves that $\operatorname{in}_\prec(I(p)) = (\operatorname{in}_\prec(\overline{g}_1), \dots, \operatorname{in}_\prec(\overline{g}_s))$ for $p > \overline{p}$.

Using a similar, but easier, argument we can prove that there exists a prime number $\tilde{p} > 0$ such that

$$\operatorname{in}_\prec(I)(p) = (\overline{\operatorname{in}_\prec(g_1)}, \dots, \overline{\operatorname{in}_\prec(g_s)}) = (\operatorname{in}_\prec(\overline{g}_1), \dots, \operatorname{in}_\prec(\overline{g}_s)) \qquad \forall p > \tilde{p}.$$

So we can conclude that $\operatorname{in}_\prec(I)(p) = \operatorname{in}_\prec(I(p))$ for $p \gg 0$. □

4.2.2 Knutson ideals in characteristic 0

We want to prove Proposition 4.2.1 in characteristic 0. To do so, we reduce to the case of fields of positive characteristic using previous results.

As in the case of fields of positive characteristic, we first need to show that if $I \in C_f$ then $I(p) \in C_f(p) := C_{f(p)}$ for all prime numbers large enough.

Remark 4.2.4. Note that if C is a family of ideals closed under intersections and such that $\operatorname{in}_\prec(I)$ is squarefree for every $I \in C$, then C is a finite set. In fact, it is easy to check that

$$\operatorname{in}_\prec(I \cap J) \subseteq \operatorname{in}_\prec(I) \cap \operatorname{in}_\prec(J) \subseteq \sqrt{\operatorname{in}_\prec(I \cap J)}.$$

Since C is closed under intersections, $I \cap J \in C$ and therefore $\mathrm{in}_\prec(I \cap J) = \sqrt{\mathrm{in}_\prec(I \cap J)}$. So, from the previous chain of subsets, we get

$$\mathrm{in}_\prec(I \cap J) = \mathrm{in}_\prec(I) \cap \mathrm{in}_\prec(J).$$

More generally, this holds for every finite intersection:

$$\mathrm{in}_\prec\left(\bigcap_i I_i\right) = \bigcap_i \mathrm{in}_\prec(I_i).$$

We claim that if $I, J \in C$ and $I \neq J$, then $\mathrm{in}_\prec(I) \neq \mathrm{in}_\prec(J)$ and since $\mathrm{in}_\prec(I)$ is squarefree for every $I \in C$, these initial ideals are a finite number. Hence C is finite. To prove the claim, assume that $\mathrm{in}_\prec(I) = \mathrm{in}_\prec(J)$. Then

$$\mathrm{in}_\prec(I \cap J) = \mathrm{in}_\prec(I) \cap \mathrm{in}_\prec(J) = \mathrm{in}_\prec(I) = \mathrm{in}_\prec(J).$$

Considering that $I \cap J \subseteq I, J$, we get $I = I \cap J = J$. This completes the proof of the claim.

The following result simplifies our proof, allowing us to prove the result for a single ideal at time using 4.2.3.

Lemma 4.2.5. *TFAE:*

1. $\exists \tilde{p} \gg 0$ *s.t.* $I \in C_f \Rightarrow I(p) \in C_f(p) \quad \forall p \geq \tilde{p}$.

2. $\forall I \in C_f \quad \exists \tilde{p}_I \gg 0$ *s.t.* $I(p) \in C_f(p) \quad \forall p \geq \tilde{p}_I$.

Proof. 1. $\Rightarrow$ 2. Obvious.
2. $\Rightarrow$ 1. If 2 holds, then $\mathrm{in}_\prec(I(p))$ is squarefree $\forall I \in C_f$ and for $p \geq \tilde{p}_I$. But we know from Lemma 4.2.3 that $\mathrm{in}_\prec(I(p)) = \mathrm{in}_\prec(I)(p)$ for p large enough, so $\mathrm{in}_\prec(I)$ is squarefree for every $I \in C_f$. By the previous remark, we get that C_f is finite. Once we know that C_f is finite, we can take $\tilde{p} = \max p_I$ and we are done.

$\square$

Proof of Proposition 4.2.1. We begin by proving that if $I \in C_f$ then there exists a prime number $\tilde{p}_I$ such that $I(p) \in C_f(p) = C_{f(p)}$ for all $p \geq \tilde{p}_I$. By Lemma 4.2.5, this is equivalent to prove that there exists a prime number $\tilde{p}$ which does not depend on the choice of the ideal, such that if $I \in C_f$ then $I(p) \in C_f(p)$ for all $p \geq \tilde{p}$.

Consider $(f) \in C_f$, then $(f)(p) = (f(p)) \subseteq S_p$ and as we already explained $(f(p)) \in C_f(p) = C_{f(p)}$ for all $p \gg 0$.

Now let $I, J \in C_f$ such that $I(p), J(p) \in C_f(p)$. By definition of C_f, $I+J, I \cap J \in C_f$ and we need to prove that $(I+J)(p), (I \cap J)(p) \in C_f(p)$. Obviously

$$(I+J)(p) = \underbrace{I(p)}_{\in C_f(p)} + \underbrace{J(p)}_{\in C_f(p)} \in C_f(p).$$

Now consider the intersection ideal $I \cap J \in S$. It is clear that $(I \cap J)(p) \subseteq I(p) \cap J(p)$. If we show that they have the same initial ideal, we get

$$(I \cap J)(p) = \underbrace{I(p)}_{\in C_f(p)} \cap \underbrace{J(p)}_{\in C_f(p)} \in C_f(p).$$

Using elimination theory and Buchberger's algorithm, we can compute a Gröbner basis of $I \cap J$. Inndeed, it is a well know fact (see e.g. [5, Theorem 11, p.187]) that

$$I \cap J = (tI + (1-t)J) \cap S$$

where $(tI + (1-t)J)$ is an ideal in $S[t]$ that we are contracting back to S.

In other words, a Gröbner basis of $I \cap J$ is obtained from a Gröbner basis of $tI + (1-t)J$ by dropping the elements of the basis that contain the variable t (the so called first elimination ideal with respect to a suitable term order).

Therefore

$$(I \cap J)(p) = ((tI + (1-t)J) \cap S)(p)$$
$$I(p) \cap J(p) = (tI(p) + (1-t)J(p)) \cap S_p.$$

By Lemma 4.2.3, $\mathrm{in}_\prec(tI + (1-t)J)(p) = \mathrm{in}_\prec(tI(p) + (1-t)J(p))$ for all $p \gg 0$ and we can conclude that

$$\mathrm{in}_\prec(I \cap J)(p) = \mathrm{in}_\prec(I(p) \cap J(p)).$$

A similar argument works for $I : J$ with $I \in C_f(p)$ and $J = (f_1, \ldots, f_l) \subset S$. In fact it is known (see e.g. [5, Theorem 11, p.196]) that

$$I : J = \left(\frac{1}{f_1}(I \cap (f_1))\right) \cap \left(\frac{1}{f_2}(I \cap (f_2))\right) \cap \ldots \cap \left(\frac{1}{f_l}(I \cap (f_l))\right).$$

Thus, we can use again elimination theory to compute these intersections and arguing as we have done before, we get that

$$(I : J)(p) = \underbrace{I(p)}_{\in C_f(p)} : \underbrace{J(p)}_{\subseteq S_p} \in C_f(p).$$

In conclusion, we have proved that if $I \in C_f$ then $I(p) \in C_f(p) = C_{f(p)}$ for all p large enough.

Now let $I \in C_f$. Then $I(p) \in C_f(p)$ for $p \gg 0$ and by Proposition 4.1.1 $\mathrm{in}_\prec(I(p))$ is squarefree beacause we are working in a polynomial ring over a field of positive characteristic. But we know from Lemma 4.2.3 that

$$\mathrm{in}_\prec(I(p)) = \mathrm{in}_\prec(I)(p) \qquad \forall p \gg 0.$$

So $\mathrm{in}_\prec(I)$ is squarefree.

$\square$

Chapter 5

Knutson determinantal ideals

In this chapter we are going to discuss the case of determinantal ideals of Hankel matrices and generic matrices.

In Section 5.1 we prove that determinantal ideals of Hankel matrices are Knutson ideals for a suitable choice of f (see Theorem 5.1.1 and Theorem 5.1.2). As a consequence of these results, one can derive an alternative proof (see Corollary 5.1.3) of the F-purity of Hankel determinantal rings, a result recently proved by different methods in [6].

Actually, in the case of Hankel matrices, we prove even more, giving a characterization of all the ideals belonging to the family (see Section 5.2). Interestingly, they all define Cohen-Macaulay rings (see Remark 5.2.7).

In Section 5.3, we are going to show that also determinantal ideals of generic matrices are Knutson ideals. As a corollary we obtain an interesting result about Gröbner bases of certain sums of determinantal ideals. More specifically, given $I = I_1 + \ldots + I_k$ a sum of ideals of minors on adjacent columns or rows, we prove that the union of the Gröbner bases of the I_j's is a Gröbner basis of I (see Corollary 5.3.4).

Unlike in the case of Hankel matrices, a characterization of all the ideals belonging to the family has not been found yet. Finding this characterization could lead to interesting properties on the Gröbner bases of determinantal-like ideals.

5.1 Knutson ideals of Hankel matrices

Let $X_m^{(l,n)}$ be the generic Hankel matrix with m rows and entries $x_l, \ldots, x_n$ (see Section 2.2). Note that once we have fixed m, l and n, the number of columns of $X_m^{(l,n)}$ is $n - m - l + 2$.

In particular, we are interested in square Hankel matrices of size m and rectangular Hankel matrices of size $m \times (m+1)$. In these cases, if we fix m then n is uniquely determined.

Assume for simplicity that $l = 1$:

$$X_m^{(1,n)} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_m \\ x_2 & x_3 & x_4 & \dots & x_{m+1} \\ x_3 & x_4 & x_5 & \dots & x_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_m & x_{m+1} & x_{m+2} & \dots & x_n \end{bmatrix}, \qquad \begin{array}{c}\text{square Hankel matrix:} \\ n = 2m-1\end{array}$$

$$X_m^{(1,n)} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_m & x_{m+1} \\ x_2 & x_3 & x_4 & \dots & x_{m+1} & x_{m+2} \\ x_3 & x_4 & x_5 & \dots & x_{m+2} & x_{m+3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ x_m & x_{m+1} & x_{m+2} & \dots & x_{n-1} & x_n \end{bmatrix}, \qquad \begin{array}{c}\text{Hankel matrix of size} \\ m \times (m+1): \\ n = 2m.\end{array}$$

Let $X = X_m^{(1,n)}$ be a Hankel matrix and let $t \leq \min(m, n-m+1)$, we want to show that $I_t(X)$ is a Knutson ideal. To do so, by Theorem 2.2.7, it is sufficient to consider the cases where X is a square Hankel matrix or an almost-square Hankel matrix of size $m \times (m+1)$.

We start by proving that determinantal ideals of a generic square Hankel matrix are Knutson ideals for a suitable choice of f.

Theorem 5.1.1. *Let $X = X_m^{(1,n)}$ be the square Hankel matrix of size m with entries $x_1, \dots, x_n$, where $n = 2m-1$ and consider the polynomial*

$$f = \det X \cdot \det X_{m-1}^{(2,n-1)} \in S = \mathbb{K}[x_1, \dots, x_n].$$

Then $I_t(X) \in \mathcal{C}_f$ for $t = 1, \dots, m$.

Proof. Fix a diagonal term order $\prec$ on S (that is a monomial term order such that the initial term of each minor is given by the product of its diagonal terms). Then

$$\begin{aligned} \mathrm{in}_\prec(f) &= \mathrm{in}_\prec(\det X) \cdot \mathrm{in}_\prec(\det X_{m-1}^{(2,n-1)}) = \\ &= (x_1 \cdot x_3 \cdots x_n) \cdot (x_2 \cdot x_4 \cdots x_{n-1}) = \prod_{i=1}^n x_i. \end{aligned}$$

Hence $\mathrm{in}_{\prec}(f)$ is squarefree and we can construct the Knutson family of ideals associated to f.

For simplicity of notation, we define

$P_1 := X_{m-1}^{(1,n-1)}$: rectangular matrix obtained by dropping the last row of X

$P_2 := X_{m}^{(2,n)}$: rectangular matrix obtained by dropping the first column of X

$Q := X_{m-1}^{(2,n-1)}$: square matrix obtained by dropping the last row and the first column of X.

By Definition 4.0.1, $(f) \in \mathcal{C}_f$ and $(f) : J \in \mathcal{C}_f$ for every ideal $J \subseteq S$. Choosing $J = (\det X)$ and $J = (\det Q)$, we get

$$\begin{aligned} (f) : (\det X) = (\det Q) \in \mathcal{C}_f \\ (f) : (\det Q) = (\det X) \in \mathcal{C}_f. \end{aligned} \tag{5.1}$$

In particular, $I_m(X) = (\det X) \in \mathcal{C}_f$. This proves the theorem in the case $t = m$.

Now let $t = m-1$. By Theorem 2.2.9, we know that every determinanatal ideal of a generic Hankel matrix H is prime and its height is given by the following formula:

$$\mathrm{ht}(I_s(H)) = n - 2s + 2 \tag{5.2}$$

where n is the number of variables. In this case

$$\mathrm{ht}(I_t(X)) = 2m - 1 - 2(m-1) + 2 = 3.$$

From equalities (5.1), taking the sum, we get

$$I_m(X) + I_{m-1}(Q) = (\det X, \det Q) \in \mathcal{C}_f.$$

Moreover

$$\mathrm{in}_{\prec}(I_m(X) + I_{m-1}(Q)) = (x_1x_3\cdots x_n, x_2x_4\cdots x_{n-1})$$

is a complete intersection of height 2, so $I_m(X) + I_{m-1}(Q)$ is a complete intersection of height 2 as well.

Now observe that

$$\mathrm{ht}(I_t(P_1)) = \mathrm{ht}(I_t(P_2)) = n - 1 + 2 - 2t = 2m - 1 - 1 + 2 - 2(m-1) = 2$$

and

$$I_t(P_1), I_t(P_2) \supseteq (\det X, \det Q) = I_{t+1}(X) + I_t(Q) \in \mathcal{C}_f.$$

This means that $I_t(P_1)$ and $I_t(P_2)$ must be minimal primes over the ideal $(\det X, \det Q) \in C_f$. Thus, they and their sum must be in $\mathcal{C}_f$ by definition. Hence $I_t(X)$ is a prime ideal of height 3 and it contains the sum of two distinct prime ideals of height 2, namely $I_t(P_1) + I_t(P_2)$. This shows that $I_t(X) \in \mathcal{C}_f$, since it is a minimal prime over $I_t(P_1) + I_t(P_2)$ which is in $\mathcal{C}_f$. The same argument can be used in general to prove that $I_t(X) \in \mathcal{C}_f$ for every $t = 1, \dots, m$.

Suppose by induction that $I_t(X), I_t(P_1), I_t(P_2), I_t(Q) \in \mathcal{C}_f$; we want to prove that the same holds for $t-1$.

By (5.7), we know that

$$\mathrm{ht}(I_{t-1}(Q)) = n - 2 - 2(t-1) + 2 = n - 2t + 2.$$

and

$$\mathrm{ht}(I_t(P_1)) = \mathrm{ht}(I_t(P_2)) = n - 1 - 2t + 2 = n - 2t + 1.$$

Moreover

$$I_{t-1}(Q) \supseteq I_t(P_1) + I_t(P_2)$$

and $I_t(P_1) + I_t(P_2) \in \mathcal{C}_f$ by induction. So $I_{t-1}(Q)$ must be minimal over $I_t(P_1) + I_t(P_2)$. This proves that $I_{t-1}(Q) \in \mathcal{C}_f$.

As a consequence we get that $I_t(X) + I_{t-1}(Q) \in \mathcal{C}_f$. This ideal is the sum of two distinct prime ideals of height $n - 2t + 2$ and it is contained in $I_{t-1}(P_1)$ and $I_{t-1}(P_2)$ which are two prime ideals of height one more, that is $n - 2t + 3$. Hence we have that $I_{t-1}(P_1)$ and $I_{t-1}(P_2)$ are miniaml primes over the sum $I_t(X) + I_{t-1}(Q)$ which is in $\mathcal{C}_f$ and so they must be in $\mathcal{C}_f$. It remains to show that $I_{t-1}(X) \in \mathcal{C}_f$. To do so, one can observe that

$$I_{t-1}(X) \supseteq I_{t-1}(P_1) + I_{t-1}(P_2) \in \mathcal{C}_f.$$

Hence $I_{t-1}(X)$ is a prime ideal of height $n - 2t + 4$ that contains the sum of two distinct prime ideals in $\mathcal{C}_f$ of height $n - 2t + 3$. Thus $I_{t-1}(X)$ must be a minimal prime over $I_{t-1}(P_1) + I_{t-1}(P_2) \in \mathcal{C}_f$. By definition, we get $I_{t-1}(X) \in \mathcal{C}_f$. This completes the proof.

□

A similar result holds for Hankel matrices of size $m \times (m+1)$.

Theorem 5.1.2. *Let $X = X_m^{(1,n)}$ be the rectangular Hankel matrix of size $m \times (m+1)$ with entries $x_1, \dots, x_n$, where $n = 2m$ and let f be the polynomial $f = \det X_m^{(1,n-1)} \cdot \det X_m^{(2,n)}$ in $S = \mathbb{K}[x_1, \dots, x_n]$. Then $I_t(X) \in \mathcal{C}_f$ for $t = 1, \dots, m$.*

Proof. In this case we define

$P_1 = X_m^{(1,n-1)}$: square matrix obtained by dropping the last column of X
$P_2 = X_m^{(2,n)}$: square matrix obtained by dropping the first column of X
$Q = X_m^{(2,n-1)}$: rectangular matrix obtained by dropping the first and the last column of X.

Then the proof is similar to that of the case of square Hankel matrices. □

From the previous theorems, we can derive an alternative proof of [6, Theorem 4.1].

Corollary 5.1.3. *Let H be a generic Hankel matrix of size $r \times s$. Then*
(a) *$I_t(H)$ is a Knutson ideal for every $t \leq \min(r, s)$.*
(b) *If $\mathbb{K}$ is a field of positive characteristic, then $S/I_t(H)$ is F-pure.*

Proof. (a) Using Remark 2.2.8, we may assume that the Hankel matrix H has the right size (that is $m \times m$ or $m \times m+1$), so we can apply Theorem 5.1.1 or Theorem 5.1.2.

(b) We may assume that $\mathbb{K}$ is a perfect field of positive characteristic. In fact, we can always reduce to this case by tensoring with the algebraic closure of $\mathbb{K}$ and the F-purity property descends to the non-perfect case. Using Remark 3.4.11, we know that the ideal (f) is compatibly split with respect to the Frobenius splitting defined by $\mathrm{Tr}(f^{p-1}\bullet)$ (where f is taken to be as in the prevoious theorems). Thus all the ideals belonging to C_f are compatibly split with respect to the same splitting, in particular $I_t(H)$. This implies that such Frobenius splitting of S provides a Frobenius splitting of $S/I_t(H)$. Being $S/I_t(H)$ F-split, it must be also F-pure. □

5.2 Characterization of all Knutson ideals of Hankel matrices

Proving Theorem 5.1.1, it comes out that determinantal ideals of certain submatrices of Hankel matrices are Knutson ideals. Since we know that $\mathcal{C}_f$ is finite, it is natural to ask whether they are all the ideals belonging to the family or not.

The only way to construct new ideals in C_f starting from two ideals belonging to the family is taking their sums, their intersections and their minimal primes. So we have to control that in the algorithm we used to prove Theorem 5.1.1 we take all possible sums, intersections and minimal primes of ideals in C_f.

The previous algorithm proceeds according to the scheme below:

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{ht}=2 \arrow[dl, bend right] & I_{m-1}(P_1), I_{m-1}(P_2) \arrow[r, dash, "\substack{\text{minimal}\\ \text{primes}}"] \arrow[d, dash, "+"] & I_m(X)+I_{m-1}(Q) \\
 & \mathrm{ht}=3 \arrow[dl, bend right] & I_{m-1}(X), I_{m-2}(Q) \arrow[r, dash, "\substack{\text{minimal}\\ \text{primes}}"] \arrow[d, dash, "+"] & I_{m-1}(P_1)+I_{m-1}(P_2) & \\
\mathrm{ht}=4 & I_{m-2}(P_1), I_{m-2}(P_2) \arrow[r, dash, "\substack{\text{minimal}\\ \text{primes}}"] \arrow[d, dash, "+"] & I_{m-1}(X)+I_{m-2}(Q) & & \\
 & \vdots & & &
\end{tikzcd}$$

Since two ideals of different height in the scheme are always contained one into the other, if we take their intersection or sum we do not obtain a new ideal. Moreover all the ideals of type $I_t(P_1), I_t(P_2), I_t(X), I_t(Q)$ are prime ideals, so they are (the only) minimal primes over themselves. If we show that at each step there are no other minimal primes, it turns out that the ideals given by the above procedure are all the possible ideals belonging to the family C_f, that is:

Theorem 5.2.1. *Let $X = X_m^{(1,n)}$ be the square Hankel matrix of size m with entries $x_1, \dots, x_n$ and let $f = \det X \cdot \det X_{m-1}^{(2,n-1)} \in S = \mathbb{K}[x_1, \dots, x_n]$. Then the only ideals belonging to $\mathcal{C}_f$ are those of the form*

$$I_t(P_1), I_t(P_2), I_t(X), I_t(Q), I_t(X)+I_{t-1}(Q), I_{t-1}(P_1)+I_{t-1}(P_2).$$

By the above discussion, to prove Theorem 5.2.1, it is enough to prove the following:

Proposition 5.2.2. *With the notation introduced before, we get the following primary decompositions:*

1. $I_t(X)+I_{t-1}(Q) = I_{t-1}(P_1) \cap I_{t-1}(P_2)$
2. $I_{t-1}(P_1)+I_{t-1}(P_2) = I_{t-1}(X) \cap I_{t-2}(Q)$.

The inclusion $\subseteq$ is obvious in both cases. It remains to prove the reverse inclusion. To do so we will apply the following result which is a consequence of [2, Corollary 4.6.8].

Lemma 5.2.3. *Let I, J be two ideals in a polynomial ring S such that the following conditions hold:*

1. $\operatorname{ht}(I) = \operatorname{ht}(J) =: h$
2. $I \subseteq J$
3. $\operatorname{ht}(P) = h \quad \forall P \in \operatorname{Ass}(I)$

then

$$e(S/I) = e(S/J) \Rightarrow I = J.$$

Furthermore, in the proof of Proposition 5.2.2 we will need to apply recursively a result by Peskine e Szpiro (see Proposition 5.2.4) to prove that the ideals $I_{t-1}(P_1) + I_{t-1}(P_2)$ and $I_t(X) + I_{t-1}(Q)$ are Gorenstein for every $t = 1, \dots, m$ and that

$$\operatorname{ht}(I_{t-1}(P_1) + I_{t-1}(P_2)) = \operatorname{ht}(I_t(X) + I_{t-1}(Q)) + 1.$$

By the purity of Macaulay, this will imply that all the three conditions of Lemma 5.2.3 are staisfied.

Proposition 5.2.4. *[26, Remark 1.4] Let I and J be two homogeneous ideals in a polynomial ring S with no associated primes in common and suppose that $S/(I \cap J)$ is Gorenstein. Then:*

1. *S/I is Cohen-Macaulay if and only if S/J is Cohen-Macaulay.*
2. *If S/I is Cohen-Macaulay, then ${}^{S}\!/_{(I+J)}$ is Gorenstein and*

$$\operatorname{ht}(I + J) = \operatorname{ht}(I) + 1.$$

Collecting together all these results, we can prove Proposition 5.2.2.

Proof of Proposition 5.2.2. For $k \geq 1$ we define:

- $I_k := I_{m-h}(X)$ and $J_k := I_{m-h-1}(Q)$, if $k = 2h + 1$
- $I_k := I_{m-h}(P_1)$ and $J_k := I_{m-h}(P_2)$, if $k = 2h$.

We want to show that $I_k + J_k = I_{k+1} \cap J_{k+1}$ for every $1 \leq k \leq 2(m-2)+1$. We proceed by induction on k, following the usual scheme.

First of all observe that all these ideals are different homogenous prime ideals of height k in S (in particular, they have no associated prime ideals in common) and since they are determinantal ideals, they are also Cohen-Macaulay.

Assume $k = 1$. Then $I_1 + J_1 = I_m(X) + I_{m-1}(Q) = (\det X, \det Q)$ and $I_2 \cap J_2 = I_{m-1}(P_1) \cap I_{m-1}(P_2)$. We know that $I_1 + J_1 \subseteq I_2 \cap J_2$ and that $I_1 + J_1$ is a complete intersection of height 2. In particular it is Gorenstein and by the purity of Macaualy, all its associated primes P have the same height, namely $\mathrm{ht}(P) = \mathrm{ht}(I_1 + J_1) = 2$. Moreover $\mathrm{ht}(I_2 \cap J_2) = 2 = \mathrm{ht}(I_1 + J_1)$. Hence $I_1 + J_1$ and $I_2 \cap J_2$ satisfy all the hypothesies of Lemma 5.2.3. If we show that they have the same multiplicity, we get the desired equality.

Since I_1+J_1 is a complete intersection, we have that $e(I_1+J_1) = m(m-1)$. Moreover the h-vector of the determinantal ring of a Hankel matrix H of size $t \times s$ is well known. In fact, being $\mathrm{ht}(I_t(H)) = n - 2t + 2$ and using Remark 2.2.8, the Eagon-Northcott complex provides a minimal free resolution of $S/I_t(H)$. In particular $S/I_t(H)$ is Cohen-Macaulay and has linear resolution. Therefore:

$$h^{S/I_t(H)} = \left(1, (s-t+1), \binom{s-t+2}{2}, \cdots, \binom{s-1}{t-1}\right) \tag{5.3}$$

and its multiplicity is

$$e(S/I_t(H)) = 1 + (s-t+1) + \binom{s-t+2}{2} + \cdots + \binom{s-1}{t-1}. \tag{5.4}$$

Using this formula we get:

$$\begin{aligned} e(I_2 \cap J_2) &= e(I_{m-1}(P_1)) + e(I_{m-1}(P_2)) = 2e(I_{m-1}(P_1)) \\ &= 2\left(1 + (m-m+1+1) + \binom{3}{2} + \binom{4}{3} + \cdots + \binom{m-1}{m-2}\right) \\ &= 2(1+2+3+4+\cdots+(m-1)) \\ &= 2\binom{m}{2} = m(m-1). \end{aligned}$$

Hence $e(I_1 + J_1) = e(I_2 \cap J_2)$ and by Lemma 5.2.3 we get $I_1 + J_1 = I_2 \cap J_2$. Furthermore, using Lemma, 5.2.4 we get that $I_2 + J_2$ is Gorestein and $\mathrm{ht}(I_2 + J_2) = \mathrm{ht}(I_{m-1}(P_1)) + 1 = 3$.

Now assume $k = 2$. Then $I_2 + J_2 = I_{m-1}(P_1) + I_{m-1}(P_2)$ and $I_3 \cap J_3 = I_{m-1}(X) \cap I_{m-2}(Q)$. From the previous case, we know that $I_2 + J_2$ is Gorenstein and it has height 3. As a consequence of the purity theorem of Macaulay we have that $\mathrm{ht}(P) = \mathrm{ht}(I_2 + J_2)$ for all the associated primes P of $I_2 + J_2$. In addition we know that $I_2 + J_2 \subseteq I_3 \cap J_3$ and that they have the same height. Again $I_2 + J_2$ and $I_3 \cap J_3$ satisfy all the hypothesies of Lemma 5.2.3. If we show that they have the same multiplicity, we get the desired equality.

Iterating this procedure, we get the thesis. More generally, let $k \geq 2$. By induction we may assume that $I_k \cap J_k = I_{k-1} + J_{k-1}$ is Gorenstein and that

$\mathrm{ht}(I_k + J_k) = k+1 = \mathrm{ht}(I_{k+1} \cap J_{k+1})$. Since $I_k + J_k \subseteq I_{k+1} \cap J_{k+1}$, if we show that $e(I_k + J_k) = e(I_{k+1} \cap J_{k+1})$, by lemma 5.2.3 we get $I_k + J_k = I_{k+1} \cap J_{k+1}$ and using Lemma 5.2.4, we obtain that $I_{k+1} + J_{k+1}$ is Gorenstein of height $(k+1)+1$.

Therefore it is enough to show that $e(I_k + J_k) = e(I_{k+1} \cap J_{k+1})$ for every k. In other words, we need to prove the following equalities:

- $e(I_t(P_1) + I_t(P_2)) = e(I_t(X) \cap I_{t-1}(Q))$
- $e(I_t(X) + I_{t-1}(Q)) = e(I_{t-1}(P_1) \cap I_{t-1}(P_2))$.

To compute the multiplicity of these ideals, we first compute their h-vectors. Let $I := I_t(P_1)$ and $J := I_t(P_2)$ and consider the following exact sequence:

$$0 \longrightarrow S/(I \cap J) \longrightarrow S/I \oplus S/J \longrightarrow S/(I+J) \longrightarrow 0.$$

By additivity of Hilbert series on short exact sequence, we get:

$$HS_{S/(I+J)}(t) = HS_{S/I \oplus S/J}(t) - HS_{S/(I \cap J)}(t). \tag{5.5}$$

From the previous discussion we already know that

$$\mathrm{ht}(I_t(P_1) + I_t(P_2)) = h + 1$$

where $h := \mathrm{ht}(I_t(P_1)) = \mathrm{ht}(I_t(P_2)) = \mathrm{ht}(I_t(P_1) \cap I_t(P_2))$.

This implies that

$$\dim S/(I_t(P_1) \cap I_t(P_2)) = \dim S/I_t(P_1) = \dim I_t(P_2) = n - h =: d$$

and

$$\dim S/(I_t(P_1) + I_t(P_2)) = d - 1.$$

Using the well known fact that the Hilbert series is a rational function (see e.g. [BH, Corollary 4.1.8]), we get

$$\frac{h^{S/(I+J)}(z)}{(1-z)^{d-1}} = \frac{h^{S/I \oplus S/J}(z) - h^{S/(I \cap J)}(z)}{(1-z)^d}, \tag{5.6}$$

hence

$$h^{S/(I+J)}(z) = \frac{h^{S/I \oplus S/J}(z) - h^{S/(I \cap J)}(z)}{(1-z)}.$$

It is straightforward to see that S/I and S/J have the same h-vector, namely:

$$h^{S/I} = h^{S/J} = (h_0^{S/I}, h_1^{S/I}, \ldots, h_{t-1}^{S/I})$$

where $h_i^{S/I} = \binom{2m-2t+i-1}{i}$ for $i \leq t-1$. As a consequence, we have that $e(I \cap J) = e(I) + e(J) = 2e(I)$.

Let $\overline{S}/\overline{I \cap J}$ be the Artinian reduction of $S/(I \cap J)$. Since I and J are generated in degree t, for $i < t$ we have

$$h_i^{S/(I\cap J)} = h_i^{\overline{S}/\overline{I\cap J}} = \dim \overline{S}_i = \binom{2m-2t+i-1}{i} = h_i^{S/I}.$$

Thus $h_i^{S/(I\cap J)} = h_i^{S/I}$ for $i < t$. But $I \cap J$ is a Gorenstein ideal, so its h-vector must be symmetric and we already know that $e(I \cap J) = 2e(I)$. This implies that

$$h^{S/I\cap J} = (h_0^{S/I}, h_1^{S/I}, \ldots, h_{t-2}^{S/I}, h_{t-1}^{S/I}, h_{t-1}^{S/I}, h_{t-2}^{S/I}, \ldots, h_1^{S/I}, h_0^{S/I}).$$

Substituting in (5.6), we get:

$$\begin{aligned} h^{S/(I+J)}(z) &= \frac{2h^{S/I}(z) - h^{S/(I\cap J)}(z)}{(1-z)} = \\ &= \frac{2\sum_{i=0}^{t-1} h_i^{S/I} z^i - \sum_{i=0}^{t-1} h_i^{S/I} z^i - \sum_{i=t}^{2t-1} h_{2t-1-i}^{S/I} z^i}{1-z} = \\ &= \frac{h_0^{S/I} + h_1^{S/I} z + \cdots + h_{t-1}^{S/I} z^{t-1} - h_{t-1}^{S/I} z^t - \cdots - h_1^{S/I} z^{2t-2} - h_0^{S/I} z^{2t-1}}{1-z}. \end{aligned}$$

Dividing by $1-z$, we finally obtain

$$h^{S/(I+J)} = \left(h_0^{S/I}, h_0^{S/I} + h_1^{S/I}, \ldots, \sum_{i=0}^{t-2} h_i^{S/I}, \sum_{i=0}^{t-1} h_i^{S/I}, \sum_{i=0}^{t-2} h_i^{S/I}, \ldots, h_0^{S/I} + h_1^{S/I}, h_0^{S/I} \right).$$

Note that a similar argument shows that if we consider $I = I_t(X)$ and $J = I_{t-1}(Q)$, then

$$h^{S/(I+J)} = \left(h_0^{S/I}, h_0^{S/I} + h_1^{S/I}, \ldots, \sum_{i=0}^{t-2} h_i^{S/I}, \sum_{i=0}^{t-2} h_i^{S/I}, \ldots, h_0^{S/I} + h_1^{S/I}, h_0^{S/I} \right).$$

Now we can compute the multiplicities.

In fact, using the relation $\binom{j}{k} + \binom{j}{k+1} = \binom{j+1}{k+1}$ and the identity (5.3), we get the relation

$$h_i^{S/I_t(P_1)} = h_i^{S/I_t(P_2)} = h_i^{S/I_t(X)} - h_{i-1}^{S/I_t(X)}.$$

From the h-vector of $S/_{(I+J)}$, using the fact that $h_i^{S/I_t(X)} = h_i^{S/I_{t-1}(Q)}$, we have:

$$\begin{aligned}
e(I_t(P_1)+I_t(P_2)) =& h_0^{S/I_t(P_1)} + \left(h_0^{S/I_t(P_1)} + h_1^{S/I_t(P_1)}\right) + \cdots \\
&\cdots + \sum_{i=0}^{t-2} h_i^{S/I_t(P_1)} + \sum_{i=0}^{t-1} h_i^{S/I_t(P_1)} + \sum_{i=0}^{t-2} h_i^{S/I_{t-1}(P_1)} + \cdots \\
&\cdots + \left(h_0^{S/I_t(P_1)} + h_1^{S/I_t(P_1)}\right) + h_0^{S/I_t(P_1)} = \\
&= \underbrace{h_0^{S/I_t(X)} + h_1^{S/I_t(X)} + \cdots + h_{t-2}^{S/I_t(X)} + h_{t-1}^{S/I_t(X)}}_{=e(I_t(X))} + \\
&\quad + \underbrace{h_{t-2}^{S/I_t(X)} + \cdots + h_1^{S/I_t(X)} + h_0^{S/I_t(X)}}_{=e(I_{t-1}(Q))} = \\
&= e(I_t(X)) + e(I_{t-1}(Q)) = e(I_t(X) \cap I_{t-1}(Q)).
\end{aligned}$$

So the first equality has been proved.

For the second equality, one can argue in a similar way observing that

$$h_i^{S/I_t(X)} = h_i^{S/I_{t-1}(P_1)} - h_{i-1}^{S/I_{t-1}(P_1)}.$$

Computing the multiplicity of $I_t(X) + I_{t-1}(Q)$ from its h-vector, we get

$$\begin{aligned}
e(I_t(X)+I_{t-1}(Q)) =& h_0^{S/I_t(X)} + \left(h_1^{S/I_t(X)} + h_0^{S/I_t(X)}\right) + \cdots + \sum_{i=0}^{t-2} h_i^{S/I_t(X)} + \\
&+ \sum_{i=0}^{t-2} h_i^{S/I_t(X)} + \cdots + \left(h_0^{S/I_t(X)} + h_1^{S/I_t(X)}\right) + h_0^{S/I_t(X)} = \\
&= 2\left(h_0^{S/I_{t-1}(P_1)} + h_1^{S/I_{t-1}(P_1)} + \cdots + h_{t-2}^{S/I_{t-1}(P_1)}\right) = \\
&= 2\left(e(I_{t-1}(P_1)) = e(I_{t-1}(P_1) \cap I_{t-1}(P_2)).\right.
\end{aligned}$$

□

It is worth noticing that, while proving Proposition 5.2.2, we also found out the following property:

Proposition 5.2.5. *The ideals $I_t(X) + I_{t-1}(Q) = I_{t-1}(P_1) \cap I_{t-1}(P_2)$ and $I_{t-1}(P_1) + I_{t-1}(P_2) = I_{t-1}(X) \cap I_{t-2}(Q)$ are Gorenstein ideals for every t.*

Remark 5.2.6. In the proof of Proposition 5.2.2 we have computed the following h-vectors:

(a) Let $I = I_t(P_1)$ and $J = I_t(P_2)$. Then:

$$h^{S/(I+J)} = \left(h_0^{S/I}, h_0^{S/I} + h_1^{S/I}, \dots, \sum_{i=0}^{t-2} h_i^{S/I}, \sum_{i=0}^{t-1} h_i^{S/I}, \sum_{i=0}^{t-2} h_i^{S/I}, \dots, h_0^{S/I} + h_1^{S/I}, h_0^{S/I} \right).$$

(b) Let $I = I_t(X)$ and $J = I_{t-1}(Q)$. Then:

$$h^{S/(I+J)} = \left(h_0^{S/I}, h_0^{S/I} + h_1^{S/I}, \dots, \sum_{i=0}^{t-2} h_i^{S/I}, \sum_{i=0}^{t-2} h_i^{S/I}, \dots, h_0^{S/I} + h_1^{S/I}, h_0^{S/I} \right).$$

Note that these h-vectors are unimodal, as $h_i^{S/I}$ is non-negative for every i. It should be stressed that this is expected by the g-conjecture since we have proved that $I + J$ is always Gorenstein.

Remark 5.2.7. We have shown that if $f = \det X \det Q$ then S/I is Cohen-Macaulay for every ideal $I \in C_f$. We want to point out that this fact is proper of this specific choice of f: if we consider for example $f = x_1 \cdots x_n$ then C_f is the family of all the squarefree monomial ideals of S and most of them are not Cohen-Macaulay.

5.3 Knutson ideals of generic matrices

Let m, n be two positive integers with $m < n$, we will denote by X_{mn} the generic matrix of size $m \times n$ with entries x_{ij}, that is

$$X_{mn} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}.$$

Moreover, for any $1 \leq i < j \leq n$ and $1 \leq k < l \leq m$, we denote by $X_{[i,j]}^{[k,l]}$ the submatrix of X_{mn} with column indices $i, i+1, \dots, j$ and row indices $k, k+1, \dots, l$. In the case $[k,l] = [1,m]$, to shorten the notation, we omit the superscript and we simply write $X_{[i,j]}$. Simmetrically, if $[i,j]=[1,n]$, we write $X^{[k,l]}$ instead of $X_{[1,n]}^{[k,l]}$.

Given a generic matrix X_{mn} and an integer $t \leq \min(m,n)$, we are going to prove that $I_t(X)$ is a Knutson ideals for a suitable choice of the polynomial f.

Theorem 5.3.1. *Let $X = X_{mn}$ be the generic matrix of size $m \times n$ with entries x_{ij} and $m < n$. Consider the polynomial*

$$f = \prod_{k=0}^{m-2} \left(\det X_{[1,k+1]}^{[m-k,m]} \cdot \det X_{[n-k,n]}^{[1,k+1]} \right) \cdot \prod_{k=1}^{n-m+1} \left(\det X_{[k,m+k-1]} \right)$$

in $S = \mathbb{K}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. Then $I_t(X_{[a,b]})$ and $I_t(X^{[c,d]})$ are Knutson ideals associated to f for any $t \in \{1,\dots,m\}$ and for every $1 \leq a \leq b \leq n$ and $1 \leq c \leq d \leq m$.

In particular, $I_t(X) \in \mathcal{C}_f$ for $t = 1,\dots,m$.

A first step towards the proof of Theorem 5.3.1 is showing that all the ideals generated by the t-minors on t adjacent columns or rows are in $\mathcal{C}_f$.

This fact is formally stated in the lemma below.

Lemma 5.3.2. *Let $X = X_{m\times n}$ be the generic square matrix of size $m \times n$ with entries x_{ij} and let f to be as in Theorem 5.3.1. If we fix $t \leq m$, then:*

$$I_t(X_{[i,t+i-1]}) \in \mathcal{C}_f \quad \forall i = 1,\dots,n-t+1$$

and

$$I_t(X^{[i,t+i-1]}) \in \mathcal{C}_f \quad \forall i = 1,\dots,m-t+1.$$

Proof. By Theorem 2.1.10 we know that every determinanatal ideal of a generic matrix $X = X_{m\times n}$ is prime and its height is given by the following formula:

$$\operatorname{ht}(I_t(X)) = (n-t+1)(m-t+1). \tag{5.7}$$

Therefore

$$\operatorname{ht}(I_t(X_{[i,t+i-1]})) = (t+i-1-i+1-t+1)(m-t+1) = m-t+1.$$

We have three possibilities for i:

1st CASE: $m-t+1 \leq i \leq n-m+1$. Then

$$I_t(X_{[i,t+i-1]}) \supseteq \Big(\det X_{[i,i+m-1]}, \det X_{[i-1,i+m-2]}, \dots, \det X_{[i-m+t,i+t-1]} \Big).$$

2nd CASE: $i \leq m-t$. Then

$$I_t(X_{[i,t+i-1]}) \supseteq \Big(\det X_{[1,m]}, \det X_{[2,m+1]}, \dots, \det X_{[i,i+m-1]}, \det X^{[m-t-i+2,m]}_{[1,t+i+1]}, \det X^{[m-t-i+1,m]}_{[1,t+i]} \dots, \det X^{[2,m]}_{[1,m-1]} \Big).$$

3rd CASE: $i \geq n-m+2$. Then

$$I_t(X_{[i,t+i-1]}) \supseteq \Big(\det X_{[n-m+1,n]}, \det X_{[n-m,n-1]}, \dots, \det X_{[t+i-m,t+i-1]}, \det X^{[1,n-i+1]}_{[i,n]}, \det X^{[1,n-i+2]}_{[i-1,n]} \dots, \det X^{[1,m-1]}_{[n-m+2,n]} \Big).$$

Define H to be the right hand side ideal for each of the previous cases. Note that the initial ideal of H is given by some of the diagonals of the matrix X. Since these monomials are coprime, this ideal, and thus H, is a complete intersection and

$$\mathrm{ht}(H) = m-t+1$$

in each of the above mentioned cases. So $I_t(X_{[i,t+i-1]})$ is minimal over H.

By Definition 4.0.1, $(f) : J \in \mathcal{C}_f$ for every ideal $J \subseteq S$. Taking J to be the principal ideal generated by the product of some of the factors of f, we have that all the principal ideals generated by one of the factors of f are Knutson ideal associated to f. Being H a sum of these ideals, $H \in \mathcal{C}_f$.
In conclusion, we get that $I_t(X_{[i,t+i-1]})$ is a minimal prime over an ideal of $\mathcal{C}_f$. So it is in $\mathcal{C}_f$.

By symmetry, one can prove that $I_t(X^{[i,t+i-1]}) \in \mathcal{C}_f$ and this concludes the proof. □

Using Lemma 5.3.2, we can then prove Theorem 5.3.1.

Proof. Fix $t \in \{1, \dots, m\}$. We want to prove that $I_t(X_{[a,b]}), I_t(X^{[c,d]}) \in \mathcal{C}_f$ for every $1 \leq a \leq b \leq n$ and $1 \leq c \leq d \leq m$. From this, in particular, we will get that $I_t(X) \in \mathcal{C}_f$.

By Lemma 5.3.2, we know that $I_t(X_{[1,t]}), I_t(X_{[2,t+1]}) \in \mathcal{C}_f$ and so their sum. We claim that that the minimal prime decomposition of the sum is given by

$$I_t(X_{[1,t]}) + I_t(X_{[2,t+1]}) = I_t(X_{[1,t+1]}) \cap I_{t-1}(X_{[2,t]}).$$

To simplify the notation, we set $I_1 := I_t(X_{[1,t]})$, $I_2 := I_t(X_{[2,t+1]})$, $P_1 := I_t(X_{[1,t+1]})$ and $P_2 := I_{t-1}(X_{[2,t]})$. With this notation, we need to prove the following:

$$I_1 + I_2 = P_1 \cap P_2.$$

We already know that $I_1 + I_2 \subseteq P_1 \cap P_2$. Passing to the correspondent algebraic varieties, we get the reverse inclusion

$$\mathcal{V}(I_1 + I_2) \supseteq \mathcal{V}(P_1 \cap P_2)$$

If we prove that $\mathcal{V}(I_1 + I_2) \subseteq \mathcal{V}(P_1 \cap P_2)$, then

$$\mathcal{V}(I_1 + I_2) = \mathcal{V}(P_1 \cap P_2)$$

and this is equivalent to say that $\sqrt{I_1 + I_2} = \sqrt{P_1 \cap P_2}$. Since $I_1 + I_2 \in \mathcal{C}_f$, it is radical and $P_1 \cap P_2$ is radical because P_1 and P_2 are both radical ideals, then

$$I_1 + I_2 = P_1 \cap P_2$$

and we are done.
For this aim, let $\mathbf{X} \in \mathcal{V}(I_1 + I_2) = \mathcal{V}(I_1) \cap \mathcal{V}(I_2)$. This means that $\mathbf{X}_{[1,t]}$ and $\mathbf{X}_{[2,t+1]}$ have rank less or equal than $t-1$. Now we consider two cases:

CASE 1. Suppose that $\mathbf{X}_{[2,t]}$ has rank less or equal than $t-2$. This implies that all the $(t-1) \times (t-1)$-minors corresponding to this interval vanish on $\mathbf{X}$. So $\mathbf{X} \in \mathcal{V}(P_2)$.

CASE 2. Suppose that $\mathbf{X}_{[2,t]}$ has full rank, namely $t-1$. Then it generates a vector space V of dimension $t-1$. But by assumption, $\mathbf{X}_{[1,t]}$ and $\mathbf{X}_{[2,t+1]}$ have rank less or equal than $t-1$, so they also generate the vector space V. Consequently, $\mathbf{X}_{[1,t+1]}$ generates the vector space V and this means that all the $t \times t$- minors of our matrix X vanish on $\mathbf{X}$. Therefore we have proved that $\mathbf{X} \in \mathcal{V}(P_1)$.

This proves the claim and shows that $I_t(X_{[1,t+1]}) \in \mathcal{C}_f$, being a minimal prime over a Knutson ideal.

In the same way, simply shifting the submatrices, we get that $I_t(X_{[k,t+k]}) \in \mathcal{C}_f$ for every $k = 1, \ldots, n-t$.

In particular $I_t(X_{[2,t+2]}) \in \mathcal{C}_f$; therefore the sum $I_t(X_{[1,t+1]}) + I_t(X_{[2,t+2]})$ belongs to $\mathcal{C}_f$.

Using a similar argument to that used to prove the claim, it can be shown that the primary decomposition of the latter sum is given by

$$I_t(X_{[1,t+1]}) + I_t(X_{[2,t+2]}) = I_t(X_{[1,t+2]}) \cap I_{t-1}(X_{[2,t+1]}).$$

Therefore $I_t(X_{[1,t+2]})$ is a Knutson ideal associated to f.

Again, simply shifting the submatrices, we get that $I_t(X_{[k,t+k+1]}) \in \mathcal{C}_f$ for every $k = 1, \ldots, n-t-1$.

Iterating this procedure we get $I_t(X_{[a,b]}) \in \mathcal{C}_f$ for every $1 \leq a < b \leq n$ such that $b - a \geq t-1$.

Simmetrically, one can prove that $I_t(X^{[c,d]}) \in \mathcal{C}_f$ for every $1 \leq c < d \leq n$ such that $d - c \geq t - 1$.

In particular, this argument shows that $I_t(X) \in \mathcal{C}_f$ for every $t \in \{1, \ldots, m\}$ and we are done.

□

As an immediate consequence of the previous theorem, we get an alternative proof of F-purity of determinantal ideals of generic matrices in positive characteristic.

Corollary 5.3.3. *Assume that $\mathbb{K}$ is a field of positive characteristic and let X be a generic matrix of size $m \times n$. Then $S/I_t(X)$ is F-pure.*

Proof. We may assume that $\mathbb{K}$ is a perfect field of positive characteristic. In fact, we can always reduce to this case by tensoring with the algebraic closure of $\mathbb{K}$ and the F-purity property descends to the non-perfect case. From Theorem 3.4.10 and Remark 3.4.11, we know that the ideal (f) is compatibly split with respect to the Frobenius splitting defined by $\mathrm{Tr}(f^{p-1}\bullet)$ (where f is taken to be as in the previous theorems). Thus all the ideals belonging to $\mathcal{C}_f$ are compatibly split with respect to the same splitting, in particular $I_t(X)$. This implies that such Frobenius splitting of S provides a Frobenius splitting of $S/I_t(X)$. Being $S/I_t(X)$ F-split, it must be also F-pure. □

Theorem 5.3.1 also yields an interesting result about the behaviour of Gröbner bases of sums of determinantal ideals.

Corollary 5.3.4. *Let X be a generic matrix of size $m \times n$ and let I be a sum of ideals, say $I = I_1 + I_2 + \ldots + I_k$, where each I_i is of the form either $I_{t_i}(X_{[a_i,b_i]})$ or $I_{t_i}(X^{[a_i,b_i]})$. Then*

$$\mathcal{G}_I = \mathcal{G}_{I_1} \cup \mathcal{G}_{I_2} \cup \ldots \cup \mathcal{G}_{I_k}$$

where $\mathcal{G}_J$ denotes a Gröbner basis of the ideal J.

Furthermore, if $\mathbb{K}$ has positive characteristic, I is also F-pure.

Proof. By Theorem 5.3.1, we know that $I_{t_i}(X_{[a_i,b_i]})$ and $I_{t_i}(X^{[a_i,b_i]})$ are Knutson ideals. From property (III) of Knutson ideals, we get the thesis. □

This result comes in handy in many situations; here are some examples.

Example 5.3.5. Let $X = (x_{ij})$ be the generic square matrix of size 6 and consider the ideal

$$J = I_3(X_{[1,3]}) + I_3(X^{[1,3]})$$

in the polynomial ring $S = \mathbb{K}[X]$. Then J is the ideal generated by the 3-minors of the following highlighted ladder

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} \end{bmatrix}.$$

From Corollary 5.3.4, we get that set of 3-minors that generate J is a Gröbner basis of J with respect to any diagonal term order. Actually, this result was already known for ladder determinantal ideals (see [24, Corollary 3.4]).

Nonetheless, Corollary 5.3.4 can be applied to more general sums of ideals. Consider for instance the ideal

$$J = I_2(X_{[12]}) + I_2(X^{[12]}) + I_2(X_{[56]}) + I_2(X^{[56]})$$

that is the ideal generated by the 2-minors inside the below coloured region of X

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} \end{bmatrix}.$$

In this case, J is not a ladder determinantal ideal but we can use Corollary 5.3.4 to prove that the 2-minors that generate J form a Gröbner basis for the ideal J with respect to any diagonal term order. In fact, J is a sum of ideals of the form $I_t(X_{[a,b]})$ or $I_t(X^{[c,d]})$ which are Knutson ideals from Theorem 5.3.1. Then a Gröbner basis for J is given by the union of their Gröbner bases.

Furthermore, we can also consider sums of ideals of minors of different sizes, such as

$$J = I_2(X_{[2,4]}) + I_3(X^{[2,5]}).$$

In this case, J is generated by the 2-minors of the blue rectangular submatrix and the 3-minors of the red rectangular submatrix illustrated below

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} \end{bmatrix}.$$

Again from Corollary 5.3.4, being $I_2(X_{[2,4]})$ and $I_3(X^{[2,5]})$ Knutson ideals, the union of their Gröbner bases is a Gröbner basis for J. So, a Gröbner basis of J is given by the 2-minors of $X_{[2,4]}$ and the 3-minors of $X^{[2,5]}$.

Chapter 6

Knutson binomial edge ideals

In this chapter we continue the study about Knutson ideals, focusing on their connection with binomial edge ideals associated to weakly closed graphs.

We apply previous results to study F-purity of binomial edge ideals. Inspired by Matsuda's work on weakly closed graphs, we show that their binomial edge ideals are Knutson ideals (in particular, they are F-pure in positive characteristic). Furthermore, we conjecture that the converse is still true, i.e, the binomial edge ideals in $\mathcal{C}_f$ are exactly those associated to weakly closed graphs.

6.1 Weakly closed graphs

In this section we are going to apply the results obtained in Section 5.3, in order to investigate F-purity of binomial edge ideals associated to certain graphs.

Let $R = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let G be a graph on n vertices. We will write J_G to denote the binomial edge ideal associated to G, that is

$$J_G = ([i,j] := x_i y_j - x_j y_i \mid \{i,j\} \in E(G)).$$

In other words, J_G is the ideal generated by the 2-minors of the matrix

$$X_{2n} = \begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ y_1 & y_2 & y_3 & \ldots & y_n \end{bmatrix}.$$

whose column indices are given by the edges of G.

In [21], the author introduces the notion of weakly closed graph as a generalization of closed graphs, previously defined by Herzog, Hibi, Hreindóttir, Kahle and Rauh in [12].

Definition 6.1.1. Let G be a simple graph on $[n]$. G is said to be *weakly closed* if there exists a labeling of the vertices that satisfies the following condition: for all integers $1 \leq i < j < k \leq n$, if $\{i, k\} \in E(G)$ then $\{i, j\} \in E(G)$ or $\{j, k\} \in E(G)$.

Matsuda also gave a characterization of this new family of graphs in terms of comparability graphs (see Theorem 6.1.3).

For those who are not familiar with graph theory, we first recall the definition of comparability graph.

Definition 6.1.2. Let $P = ([n], \prec)$ be a partially ordered set and define $G(P)$ to be the graph on the vertex set $[n]$ such that $\{i, j\} \in E(G(P))$ with $i < j$ if and only if $i \prec j$. We say that G is *comparability* if there exists a partially ordered set P such that $G = G(P)$.

Theorem 6.1.3. *[21, Theorem 1.9] A graph G is weakly closed if and only if it is co-comparability, i.e. the complement of G is comparability.*

As a consequence of this theorem he got that complete multipartite graphs and interval graphs are weakly closed and that weakly closed graphs are perfect (see [21] for more details).

Furthermore, assuming that $\mathbb{K}$ has positive characteristic, Matsuda generalized Othani's theorem about F-purity of binomial edge ideals associated to complete multipartite graphs (see [25, Theorem 3.1]).

Theorem 6.1.4. *[21, Theorem 2.3] Let G be a weakly closed graph and let J_G be the binomial edge ideal associated to G. Then R/J_G is F-pure.*

We will derive an alternative proof of Matsuda's theorem using Knutson ideals.

Proposition 6.1.5. *Let G be a weakly closed graph on $[n]$, then $J_G \in \mathcal{C}_f$ where f is taken to be as in Theorem 5.3.1. In particular, if $\mathbb{K}$ has positive characteristic then R/J_G is F-pure.*

Proof. For each subset $S \subset [n]$ and $T = [n] \setminus S$, define $G_1, \ldots, G_{c(S)}$ to be the connected components of G_T (i.e. the restriction of G to T) and let $\widetilde{G}_1, \ldots, \widetilde{G}_{c(S)}$ be the corresponding complete graphs on their vertices. Set

$$P_S := \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\widetilde{G}_{c(S)}} \right).$$

Clearly, P_S is a prime ideal and it has been shown (see [12]) that the primary decomposition of the binomial edge ideal associated to G is given by

$$J_G = \bigcap_{S \subseteq [n]} P_S.$$

If we prove that P_S is a Knutson ideal of f for each S, then we get that $J_G \in \mathcal{C}_f$.

By lemma 5.3.2, we know that $(x_i, y_i) \in \mathcal{C}_f$. It remains to prove that $J_{\widetilde{G}_i} = I_2(X_{[j_1,\dots,j_{t_i}]}) \in \mathcal{C}_f$.

If $j_{k+1} = j_k + 1$ for every $k = 1, \dots, t_i - 1$, that is if the vertices of G_i are consecutive, then $J_{\widetilde{G}_i} \in \mathcal{C}_f$ follows easily from 5.3.2.

Now assume that $j_k - j_{k-1} > 1$ for some k and let l be a vertex such that $j_{k-1} < l < j_k$. Since G_i is connected, there exist $m, n \in V(G_i)$ such that $m \leq j_{k-1}$, $n \geq j_k$ and $\{m, n\} \in E(G_i) \subset E(G)$. Furthermore, we have $\{m, l\} \in E(G)$ or $\{l, n\} \in E(G)$, because G is weakly closed. Assume that $l \notin S$. This would imply that $l \in V(G_i)$, a contradiction. This shows that if there is a 'gap'in the vertices of the connected component G_i, every vertex l in this gap must be in S. Hence, if we set $\overline{J}_{\widetilde{G}_i} := I_2(X_{[j_1,j_1+1,\dots,j_{t_i}-1,j_{t_i}]})$, we get

$$P_S := \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\widetilde{G}_1}, \dots, J_{\widetilde{G}_{c(S)}} \right) = \left(\bigcup_{i \in S} \{x_i, y_i\}, \overline{J}_{\widetilde{G}_1}, \dots, \overline{J}_{\widetilde{G}_{c(S)}} \right).$$

By 5.3.2 $\overline{J}_{\widetilde{G}_i} \in \mathcal{C}_f$ and $\{x_i, y_i\} \in \mathcal{C}_f$, so $P_S \in \mathcal{C}_f$ for every $S \subset [n]$.

□

With the previous notation, set $\overline{P}_S := \left(\bigcup_{i \in S} \{x_i, y_i\}, \overline{J}_{\widetilde{G}_1}, \dots, \overline{J}_{\widetilde{G}_{c(S)}} \right)$. It has just been shown that if G is a weakly-closed graph then $P_S = \overline{P}_S$ for every minimal primes of J_G. Actually, the other direction is still true and we have the following characterization of weakly closed graphs.

Proposition 6.1.6. *Let G be a connected graph and let $J_G = \bigcap_{S \subseteq [n]} P_S$ be the primary decomposition of the binomial edge ideal associated to G. The following are equivalent*

(1) G is weakly closed

(2) there exists a labeling of the vertices of G such that $P_S = \overline{P}_S$ for every minimal prime of J_G.

Proof. (1) $\Rightarrow$ (2) has already been proved.

(2) $\Rightarrow$ (1) Let us assume by contradiction that G is not weakly closed. We will show that it is possible to find $S \subseteq [n]$ such that $P_S \neq \overline{P}_S$ and we are done.

Since G is not weakly closed, for each labeling of the vertices there exist $k, l, m \in V(G)$ with $k < l < m$ such that $\{k, m\} \in E(G)$ and $\{k, l\}, \{l, m\} \notin E(G)$. Nonetheless, there are a finite number of paths of length ≥ 2 connecting k to l and l to m. We will denote them as follows:

$$\begin{aligned}
p_1 : &\quad k, p_{11}, p_{12}, \dots, l \\
p_2 : &\quad k, p_{21}, p_{22}, \dots, l \\
&\quad \vdots \\
p_r : &\quad k, p_{r1}, p_{r2}, \dots, l \\
q_1 : &\quad l, q_{11}, q_{12}, \dots, m \\
q_2 : &\quad l, q_{21}, q_{22}, \dots, m \\
&\quad \vdots \\
q_t : &\quad l, q_{t1}, q_{t2}, \dots, m.
\end{aligned}$$

Now, take S to be the set of the first vertices of the previous paths, that is

$$S = \{p_{11}, p_{21}, \dots, p_{r1}, q_{11}, q_{21}, \dots, q_{s1}\}.$$

We claim that $P_S \neq \overline{P}_S$.

By definition of S, $\{k, m\}$ and l do not belong to the same connected component of $G_{[n]\setminus S}$. Without loss of generality we can assume that G_1 is the connected component of $\{k, m\}$ and G_2 is the connected component of l. Observe that $x_l, y_l \notin \bigcup_{i \in S}\{x_i, y_i\}$. Thus, it is straightforward to see that

$$\left(\bigcup_{i\in S}\{x_i, y_i\}, J_{\tilde{G}_1}, J_{\tilde{G}_2}\right) \neq \left(\bigcup_{i\in S}\{x_i, y_i\}, \overline{J}_{\tilde{G}_1}, \overline{J}_{\tilde{G}_2}\right).$$

This shows that $P_S \neq \overline{P}_S$. $\square$

6.2 Characterization of Knutson binomial edge ideals

Using this characterization, we would like to prove that the inverse of Proposition 6.1.5 is still true, that is :

Conjecture 6.2.1. *G is weakly closed* $\Leftrightarrow J_G \in \mathcal{C}_f$

So far, we have the following:

$$\begin{tikzcd}
G \text{ weakly closed} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & P_S = \overline{P}_S, \ \forall P_S \in \mathrm{Min}(J_G) \\
J_G \in \mathcal{C}_f \arrow[r, Rightarrow] & J_G \ F\text{-pure}
\end{tikzcd}$$

Remark 6.2.2. If I, J, K are Knutson ideals, then sum distributes over intersection:

$$I + (J \cap K) = (I + J) \cap (I + K).$$

This fact easily follows from Remark 4.2.4 and property (III) of Knutson ideals described in the introduction of Chapter 4.

By definition of $\mathcal{C}_f$ the sum of two Knutson ideals is again a Knutson ideal. Since we know that binomial edge ideals of weakly closed graphs are in $\mathcal{C}_f$, so are their minimal primes. The sum of two prime ideals could be non prime, hence one of the first thing we need to check in order to prove Conjecture 6.2.1 is the following:

Lemma 6.2.3. *Assume that P and Q are minimal primes of the binomial edge ideals of two weakly closed graphs, so that $P = \overline{P}$ and $Q = \overline{Q}$. Then every minimal prime L of the sum $P + Q$ has the property $L = \overline{L}$.*

To prove the lemma, we first need the following observation about the structure of binomial edge ideals with exactly two associated primes.

Remark 6.2.4. Let P be a minimal prime of a binomial edge ideal on n vertices, then it has the form

$$P = \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{G_1}, \ldots, J_{G_t} \right)$$

where each G_i is a complete graph with vertex set V_i. Denote by $\tilde{V}$ the set of vertices that do not appear in P. Then $I_2(X_{[1,n]}) \cap P$ is the primary decomposition of the binomial edge ideal of the graph

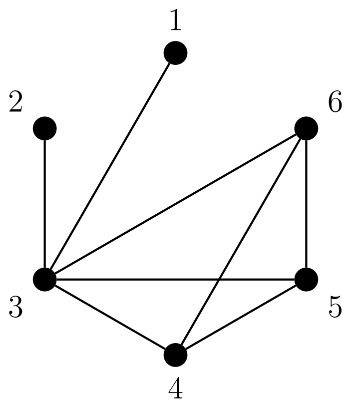
$$G = K_S \cup K_{S,\tilde{V}} \cup K_{S,V_1} \cup \ldots \cup K_{S,V_t} \cup G_1 \cup \ldots \cup G_t$$

where K_S denote the complete graph on S and K_{S,V_i} denote the complete bipartite graph on S and V_i.

Actually, in [28] Sharifan proved that if G is a connected graph on n, then $\mathrm{Ass}\,|(J_G)| = 2$ if and only if G is the join of a complete graph G_1 and a graph G_2 which is a disjoint union of complete graphs.

Take for example $P = (x_3, y_3, I_2(X_{[4,6]}))$. Then $S = \{3\}$, $V_1 = \{4, 5, 6\}$ and $\tilde{V} = \{1, 2\}$. Hence,

$$G = K_{3,V_1} \cup K_{3,\tilde{V}} \cup K_{[4,6]}$$



and $P \cap I_2(X_{[1,6]}) = I_G = ([3, 6], [3, 5], [3, 4], [2, 3], [1, 3], [5, 6], [4, 6], [4, 5])$.

Now, we give the proof of Lemma 6.2.3.

Proof. By assumption, there exist G_1 and G_2 weakly closed graphs such that $P \in \mathrm{Min}(I_{G_1})$ and $Q \in \mathrm{Min}(I_{G_2})$. Let $P + Q = L_1 \cap \ldots \cap Ł_t$ be the minimal primary decomposition of $P + Q$. We want to prove that the L_i are minimal prime ideals of the binomial edge ideal of a weakly closed graph, so that $L_i = \overline{L_i}$ for every i.

Note that we can always choose n big enough such that $L_i \not\supseteq I_2(X_{[1,n]})$ for every $i \in \{1, \ldots, t\}$. Let $X = X_{[1,n]}$.

Assume for the moment that $P, Q \subseteq I_2(X)$, then P and Q must not contain variables, that is

$$P = I_{\overline{G}_P}$$
$$Q = I_{\overline{G}_Q}$$

where $\overline{G}_P$ and $\overline{G}_Q$ are unions of disjoint complete graphs on consecutive vertices (because G_1 and G_2 are weakly closed). Hence

$$P + Q = I_{\overline{G}_P} + I_{\overline{G}_Q} = I_{\overline{G}_P \cup \overline{G}_Q}.$$

Being $\overline{G}_P \cup \overline{G}_Q$ a weakly closed graph, $L = \overline{L}$ for every ideal $L \in \mathrm{Min}(P + Q)$.

Now assume without loss of generality that $P \not\subseteq I_2(X)$. By Proposition 6.1.5, we know that $I_{G_1}, I_{G_2} \in \mathcal{C}_f$ and so are P and Q. Hence $P + Q \in \mathcal{C}_f$. Now we consider the intersections

$$\begin{gathered} I_2(X) \cap P \\ I_2(X) \cap Q. \end{gathered}$$

By the previous remark, we know that these are binomial edge ideals. Furthermore, being $P = \overline{P}$ and $Q = \overline{Q}$, these intersections are binomial edge ideals of two weakly closed graphs, say $\tilde{G}_1$ and $\tilde{G}_2$. Again by Proposition 6.1.5, $I_{\tilde{G}_1}$ and $I_{\tilde{G}_2}$ are Knutson ideals of f, so

$$I_{\tilde{G}_1 \cup \tilde{G}_2} = I_{\tilde{G}_1} + I_{\tilde{G}_2} \in \mathcal{C}_f.$$

By Remark 6.2.2

$$\begin{aligned} I_{\tilde{G}_1 \cup \tilde{G}_2} = I_{\tilde{G}_1} + I_{\tilde{G}_2} &= (I_2(X) \cap P) + (I_2(X) \cap Q) \\ &= I_2(X) \cap (I_2(X) + Q) \cap (I_2(X) + P) \cap (P + Q) \\ &= I_2(X) \cap (P + Q) \\ &= I_2(X) \cap (L_1 \cap \ldots \cap L_t) \end{aligned}$$

If this were the primary decomposition of $I_{\tilde{G}_1 \cup \tilde{G}_2}$, then $L_1, \ldots, L_t$ would be minimal prime ideals of a binomial edge ideal of a weakly closed graph, hence the thesis.

Since we have assumed that $P \not\subseteq I_2(X)$, clearly $I_2(X) \not\supseteq L_1 \cap \ldots \cap L_t = P + Q$. Moreover $L_i \not\supseteq I_2(X) \cap L_1 \cap \ldots \cap L_{i-1} \cap L_{i+1} \cap \ldots \cap L_t$, otherwise L_i would contain either $I_2(X)$ or L_j for some $j \neq i$, but this is impossible by the choice of X and the fact that the L_i are minimal primes of $P + Q$. This shows that $I_2(X) \cap (L_1 \cap \ldots \cap L_t)$ is the minimal primary decomposition of $I_{\tilde{G}_1 \cup \tilde{G}_2}$ and we are done. $\square$

In order to prove Conjecture 6.2.1, note that if we start from the ideal $(f) = (x_n[1,2] \cdots [n-1,n]y_1)$ and we take its minimal primes, we obtain the following ideals:

$$(x_n), ([1,2]), ([2,3]), \ldots, ([n-1,n]), (y_1).$$

Among them the only binomial edge ideals are those of the form $([i, i+1])$, which corresponds to the graph with exactly one edge, namely $\{i, i+1\}$, which is clearly weakly closed.

If we take the sum of these binomial edge ideals, we obtain binomial edge ideals of (union of) paths on consecutive vertices. We can then consider their associated primes and the sum of these primes. Iterating this procedure, by Lemma 6.2.3, the prime ideals we obtain are always of the form

$$P_S := \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\widetilde{G}_{c(S)}} \right)$$

with $P_S = \overline{P}_S$. So if the intersection of these primes is a binomial edge ideal J_G, G must be weakly closed by Proposition 6.1.6.

It remains to investigate the case when we start from an ideal that contains (x_n) or (y_1).

Some computational experiments suggest that in this case we obtain prime ideals which can be written as the sum of an ideal generated by variables and an ideal L_S with $L_S = \overline{L}_S$. This would prove Conjecture 6.2.1 that the only binomial edge ideals we can obtain in $\mathcal{C}_f$ are those associated to weakly closed graphs (by Proposition 6.1.6). However, this is an ongoing project that will be object of further investigation in the future.

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