Mathematical Structures in Computer Science (2021), 1–30
doi:10.1017/S0960129521000207

CAMBRIDGE UNIVERSITY PRESS

PAPER

Doctrines, modalities and comonads

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(Received 28 November 2020; revised 2 August 2021; accepted 4 August 2021)

Abstract
Doctrines are categorical structures very apt to study logics of different nature within a unified environment: the 2-category **Dtn** of doctrines. Modal interior operators are characterised as particular adjoints in the 2-category **Dtn**. We show that they can be constructed from comonads in **Dtn** as well as from adjunctions in it, and we compare the two constructions. Finally we show the amount of information lost in the passage from a comonad, or from an adjunction, to the modal interior operator.
The basis for the present work is provided by some seminal work of John Power.

Keywords: Modal operator; doctrine; adjunction; comonad; temporal logics; linear logic

1. Introduction

The approach to logic proposed by F.W. Lawvere via hyperdoctrines has proved very fruitful as it provides an extremely suitable environment where to analyse both syntacic aspects of logic and semantic aspects as well as compare one with the other, see Lawvere (1969, 1970). The suggestion is to see a logic as a functor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ from the opposite of a category to the category of posets and monotone functions where the category $\mathcal{C}$ collects the 'types' of the logic and terms in context, a poset $P(c)$ presents the 'properties' of the type c with the order relation describing their 'entailments'. The reader is referred to Section 2 for the precise details, but may just keep in mind, for the present discussion, that the contravariant powerset functor $\mathcal{P}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$ is an instance of a doctrine.

One of the main points of Lawvere's structural approach to logic is that all the logical operators are obtained from adjunctions. That view in itself is very powerful and contributes to unifying many different aspects in logic. In the present paper, we show that also a wide class of modal operators, namely, those satisfying axioms T and 4 as in Definition 2.1, is obtained from adjunctions.

Typically, modalities are unary logical operators, which are quite well understood in the context of propositional logic. However, their meaning is less clear in a typed logical formalism. In this setting, there are various semantics which are interrelated, and we show that many of these are instances of the general situation of an adjunction between two homomorphisms of doctrines.

Since they are structured categories, doctrines get swiftly organised in a 2-category. And, as we learned also from the works of John Power, in a 2-category one can develop a very productive theory of monads and comonads, extending the elementary case of the 2-category $\mathcal{C}at$ of small categories, functors and natural transfomations.

Doctrines are a rather simple categorical framework for logic, but still capable to cover a large range of examples. We could have considered more general settings such as indexed preorders (equivalently, faithful fibrations) or even arbitrary fibrations, but we preferred to keep things at a very simple level as already there one finds many interesting examples. Yet, after this first step our plan is to extend results to general fibrations in future work.

We show that an adjunction in the 2-category of doctrines gives rise to a doctrine with a modal operator. An adjunction between doctrines is very much like an adjunction between categories: roughly, it consists of two doctrines $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ and two homomorphisms of doctrines connecting them, which should be thought of as an interpretation of P in Q (the left adjoint) and an interpretation of Q in P (the right adjoint). Such a situation can be summarised by a modal logic which uses the logic Q to describe properties of types in $\mathcal{C}$ (the base category of P) and the modal operator to recover (an image of) properties described by P. In a sense, we extend the logic P through the adjunction to a richer logic and use a modal operator to keep memory of the original logic. As we said, many standard approaches to the semantics of modal logic are instances of such construction.

Taking a slightly different perspective, we show that also a comonad in the 2-category of doctrines determines a doctrine with a modal operator, this time on the category of coalgebras for the comonad. Intuitively, we get a logic where types have a dynamics, given by the coalgebra structure, and the modal operator specifies when a property is invariant for such dynamics.

These two constructions are tightly related. Relying on results in Blackwell et al. (1989), we show that every comonad in the 2-category of doctrines determines an adjunction, hence, also a modal operator. In fact, the construction starting from comonads is defined in this way. On the other hand, every adjunction determins a comonad, hence a modal operator. However, the two construction starting from an adjunction do not coincide, but we show they can be canonically compared by a homomorphism of doctrines preserving the modal operator.

We further our analysis measuring in a categorical form how the passage to a modal operator hides part of the structure that generated it.

In Section 2 we introduce *interior operators* on doctrines, which are the class of modal operators we are interested in. In Section 3 we recall basic notions about comonads and adjunctions in a general 2-category. In Section 4 we define the 2-categories of doctrines and doctrines with interior operators that are at the core of our analysis. In Section 5 we show how to construct an interior operator starting from an adjunction between doctrines, while in Section 6 we describe the analogous construction starting from a comonad on a doctrine. Finally, in Section 7 we compare the two constructions showing they are part of local adjunctions, in the sense of Betti and Power (1988), between the 2-category of doctrines with modal operator and, respectively, the 2-category of adjunctions and that of comonads in the 2-category of doctrines. In Appendix A we sketch an example on how to use our construction to obtain models of the bang modality of linear logic.

2. Interior Operators and Doctrines

A simple semantic approach to propositional standard modal logic (satisfying axioms T and 4) would consider an ***interior operator*** on a poset $(H, \leq)$, *i.e.* a monotone function $\mathrm{j}\colon H \to H$ such that, for all $x \in H$, $\mathrm{j}(x) \leq x$ and $\mathrm{j}(x) \leq \mathrm{j}(\mathrm{j}(x))$, see *e.g.* Esakia (2004). The intuition is that the elements of the poset are an interpretation of (some kind of) formulas, the order relation realises the entailment between them, and the interior operator $\mathrm{j}\colon H \to H$ acts as a modality on formulas.

From a similar semantic point of view, one could consider a many-sorted logic to be a ***doctrine*** $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$, *i.e.* a (contra)variant functor from a category $\mathcal{C}$ to the category $\mathcal{P}os$ of posets and monotone functions. Such a functor is often called an ***indexed poset*** in consonancy with the more general notion of indexed category.

The intuition for a doctrine is that the objects of the category provide the interpretations of the sorts in the logic and the arrows interpret terms between sorts. For an object X in $\mathcal{C}$, the poset PX

gives the interpretations for the formulas expressing the properties of 'arbitrary elements' of X – although no set-theoretic determination of X may have been provided, see Lawvere (1969, 1970), but also Jacobs (1999); Maietti and Rosolini (2013*a*).

Conjoining these two semantic approaches it is quite natural to consider interior operators on a doctrine as an extension to many-sorted logic, of the propositional modal logic satisfying axioms T and 4, like the $\Box$-modality, a.k.a. *necessity* modality, of S4 modal logic.

Definition 2.1. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ *be a doctrine. An* ***interior*** *modal operator on* P *is a natural transformation* $\Box\colon P \xrightarrow{\cdot} P$ *such that, for each object* X *in* $\mathcal{C}$, *the following inequalities hold:*

(i) $\Box_X \leq_X \mathrm{id}_{PX}$
(ii) $\Box_X \leq_X \Box_X \circ \Box_X$

Note that standard axioms of the S4 modal operator, see *e.g.* Awodey et al. (2014), require further structure. But here we consider the very simple structure of a poset on the fibres because we want to focus mainly on the comonadic structure of the modality.

In the following, an element $\alpha \in PX$ of the form $\alpha = \Box_X \beta$ for some $\beta \in PX$ will be called $\Box$-***stable***. An immediate consequence of Definition 2.1, obtained combining the two requirements on $\Box$, is that $\Box_X = \Box_X \circ \Box_X$. Hence $\Box$-stable elements are the fixed points of $\Box_X$, that is, those elements $\alpha \in PX$ such that $\Box_X \alpha = \alpha$.

Examples 2.2. Let $\mathrm{j}\colon H \to H$ be an interior operator on the poset $(H, \leq)$, *i.e.* a monotone function such that, for all $x \in H$, $\mathrm{j}(x) \leq x$ and $\mathrm{j}(x) \leq \mathrm{j}(\mathrm{j}(x))$. Given this, we can consider two examples of doctrines with an interior operator:

(1) Let $\hat{H}\colon \mathbf{1}^{\mathrm{op}} \to \mathcal{P}os$ be the functor defined on the category with a single object $\star$ and a single arrow $\mathrm{id}_\star$ as $\hat{H}(\star) = H$. Then j is an interior operator on $\hat{H}$.
(2) The functor $H^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$, which maps a set X to H^X with the pointwise order and a function $t\colon X \to Y$ to the monotone function $- \circ t\colon H^Y \to H^X$, is a doctrine. The natural transformation $\mathrm{j} \circ -\colon H^{(-)} \xrightarrow{\cdot} H^{(-)}$ given by postcomposition with j is an an interior operator on $H^{(-)}$.

Note that the example in (a) is obtained from that in (b) by precomposing the doctrine $H^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$ with the (opposite of the) functor $\star \mapsto \{0\}\colon \mathbf{1} \to \mathcal{S}et$ which maps the one object $\star$ to a(ny) singleton set.

Example 2.3. Consider the category *Opn* of topological spaces and open continuous maps. Define $P\colon Opn^{\mathrm{op}} \to \mathcal{P}os$ as $P(X, \tau) = \mathcal{P}(X)$, the powerset of the set X, and $Pt = t^{-1}$, the inverse image along the open continuous function $t\colon (X, \tau) \to (Y, \sigma)$ Let (X, τ) be a topological space, then τ is the set of fixed points of the interior operator $\mathrm{int}_\tau\colon \mathcal{P}(X) \to \mathcal{P}(X)$, which maps a subset $A \subseteq X$ to its topological interior. Since $\mathrm{int}_\tau(A) \subseteq A$ and $\mathrm{int}_\tau(A) \subseteq \mathrm{int}_\tau(\mathrm{int}_\tau(A))$, for each $A \subseteq X$, to get an an interior operator on P we need to prove that int_τ is natural. Indeed, consider an open continuous map $t\colon (X, \tau) \to (Y, \sigma)$, and a subset $B \subseteq Y$. So $t^{-1}(\mathrm{int}_\sigma(B)) \subseteq \mathrm{int}_\tau(t^{-1}(B))$ by continuity of t. But also $t(\mathrm{int}_\tau(t^{-1}(B))) \subseteq \mathrm{int}_\sigma(B)$ since the set $t(\mathrm{int}_\tau(t^{-1}(B))) \subseteq B$ is open by openness of t. So $t^{-1}(\mathrm{int}_\sigma(B)) = \mathrm{int}_\tau(t^{-1}(B))$ which proves that $\mathrm{int}\colon P \xrightarrow{\cdot} P$.

Example 2.4. A ***Kripke frame*** is a pair $K = (W, R)$ where W is the set of ***possible worlds*** and $R \subseteq W \times W$ is the ***accesibility relation***. On the poset $\mathcal{P}(W)$ ordered by set inclusion, consider the monotone function $\mathrm{j}_R\colon \mathcal{P}(W) \to \mathcal{P}(W)$ defined as

$$\mathrm{j}_R(A) = \left\{ w \in W \,\middle|\, R(w) \subseteq A \right\}$$

where $R(w) = \{v \in W \mid (w, v) \in R\}$. When R is reflexive and transitive (*i.e.* a preorder on W), for any $w \in W$, we have $w \in R(w) = R(R(w))$. Hence j_R is an interior operator.

(a) As a particular instance of Example 2.2(b), postcomposition with the interior operator $\mathrm{j}_R \circ -\colon \mathcal{P}(W)^{(-)} \stackrel{\cdot}{\to} \mathcal{P}(W)^{(-)}$ endows the doctrine $\mathcal{P}(W)^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$ with an an interior operator. Intuitively, given a 'formula' $\alpha \in \mathcal{P}(W)^D$, for an element x of D, the set $\alpha(x) \subseteq W$ consists of those worlds where x satisfies α. Indeed, one can see the data consisting of the Kripke frame K and the set D as a constant domain skeleton as in Definition 1 in Braüner and Ghilardi (2007), where the fibres $\mathcal{P}(W)^{D^n}$ enlist all possible interpretations for predicates as n varies.

(b) Another doctrine with an interior operator built from a Kripke frame K with a reflexive and transitive accessibility relation can be obtained via W-indexed families. Consider the category $W\text{-}\mathcal{F}am$ whose

objects are W-indexed families of sets, that is, pairs $X = (\overline{X}, (X_w)_{w\in W})$, where $X_w \subseteq \overline{X}$, for all $w \in W$, and where

an arrow $t\colon X \to Y$ is a function $t\colon \overline{X} \to \overline{Y}$ such that, for each $w \in W$, $X_w \subseteq t^{-1}(Y_w)$.

Consider the subobject functor $\mathsf{Sub}_{W\text{-}\mathcal{F}am}\colon W\text{-}\mathcal{F}am^{\mathrm{op}} \to \mathcal{P}os$ mapping a W-indexed family to the poset $\mathsf{Sub}_{W\text{-}\mathcal{F}am}(X)$ of its subfamilies, *i.e.* a family A such that $\overline{A} \subseteq \overline{X}$ and $A_w \subseteq X_w$ for each $w \in W$, ordered by pointwise inclusion. The action on arrows is defined pointwise by inverse image. For each W-indexed family X the function $\square_X\colon \mathsf{Sub}_{W\text{-}\mathcal{F}am}(X) \to \mathsf{Sub}_{W\text{-}\mathcal{F}am}(X)$

$$(\square_X A)_w = \bigcap_{v\in R(w)} A_v$$

is clearly monotone; and it satisfies conditions (i) and (ii) in Definition 2.1 for the same reason as in the previous example. Moreover, it is natural in X since, for each function $t\colon Y \to X$, we have

$$t^{-1}\big((\square_X A)_w\big) = t^{-1}\left(\bigcap_{v\in R(w)} A_v\right) = \bigcap_{v\in R(w)} t^{-1}(A_v) = (\square_Y t^{-1}(A))_w$$

for any $w \in W$. Though surprising, we shall see in Example 4.2 that this example is a universal completion of the previous one in (a).
Intuitively, given a W-indexed family D, for each $w \in W$, the subset D_w consists of those elements of $\overline{D}$ which are present at the world w, and, given a 'formula' $\alpha \in \mathsf{Sub}_{W\text{-}\mathcal{F}am}(D)$, for each world $w \in W$, the set α_w consist of those elements x which are present and satisfy α at w. Indeed, one can see the data consisting of the Kripke frame K and the w-indexed family D as a varying domain skeleton as in Definition 7 in Braüner and Ghilardi (2007), with few additional requirements, where the fibres $\mathsf{Sub}_{W\text{-}\mathcal{F}am}(D^n)$ enlist all possible interpretations for predicates as n varies.

(c) Yet another possibility is to consider a doctrine over the category of presheaves on the preorder K; we shall discuss this in Example 5.12, as a particular case of a more general construction.

3. Adjunctions and Comonads in a 2-Category

In this section, we recall basic notions which can be introduced in an arbitrary 2-category with the purpose to use them in the particular case of the 2-category of doctrines.

Given a (strict) 2-category $\mathcal{K}$, we denote 0-cells as $A, B, C, \ldots$, which we shall refer to also as ***objects*** of $\mathcal{K}$; a 1-cell, also referred to as ***1-arrow***, from A to B will be written as $a\colon A \to B$ while a 2-cell, or ***2-arrow***, from the 1-cell a to the 1-cell b will be written as $\alpha\colon a \Rightarrow b$. Composition of 1-cells and horizontal composition of 2-cells is denoted as $\circ$, and often omitted – we shall use it mainly to emphasise the composition of functions and functors. The identity 1-cell on the object A is denoted by e_A and the identity 2-cell on the 1-cell a is denoted by 1_a. A horizontal composition with a 2-identity cell 1_a will be written simply as αa. Vertical composition of 2-cells is denoted as $\cdot$. So, for instance, the defining property of vertical composition of natural transformations would be written as something like $(\psi \cdot \phi)_C = \psi_C \circ \phi_C$.

Many well-known concepts from standard category theory can be transferred to an arbitrary 2-category $\mathcal{K}$; a basic reference is Street (1972).

Definition 3.1. *Let $\mathcal{K}$ be a 2-category.*

(i) *An **adjunction** $\mathbb{A}$ in $\mathcal{K}$ consists of the following data: two objects C and D, two 1-arrows $l\colon C \to D$ and $r\colon D \to C$, and two 2-arrows $\eta\colon e_C \Rightarrow rl$ and $\epsilon\colon lr \Rightarrow e_D$, such that the following triangles of 2-arrows commute*

$$
\begin{tikzcd}
l \arrow[r, Rightarrow, "l\eta"] \arrow[dr, Rightarrow, "1_l"'] & lrl \arrow[d, Rightarrow, "\epsilon l"] \\
 & l
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[r, Rightarrow, "\eta r"] \arrow[dr, Rightarrow, "1_r"'] & rlr \arrow[d, Rightarrow, "r\epsilon"] \\
 & r.
\end{tikzcd}
\tag{1}
$$

(ii) *A **comonad** $\mathbb{c}$ in $\mathcal{K}$ consists of an object A, a 1-arrow $c\colon A \to A$, and two 2-arrows $\nu\colon c \Rightarrow e_A$ and $\mu\colon c \Rightarrow cc$, such that the following diagrams of 2-arrows commute*

$$
\begin{tikzcd}
 & c \arrow[dl, Rightarrow, "1_c"'] \arrow[d, Rightarrow, "\mu"] \arrow[dr, Rightarrow, "1_c"] & \\
c & cc \arrow[l, Rightarrow, "c\nu"] \arrow[r, Rightarrow, "\nu c"'] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "\mu"'] & cc \arrow[d, Rightarrow, "c\mu"] \\
cc \arrow[r, Rightarrow, "\mu c"] & ccc.
\end{tikzcd}
\tag{2}
$$

(iii) *In line with Power and Watanabe (2002); Street (1972), one says that $\mathcal{K}$ **admits the Eilenberg–Moore construction** for the comonad (A, c, μ, ν) if there is a universal representation of the following 2-problem: given an object B in $\mathcal{K}$, objects are pairs (x, ξ) with*

$$
\begin{tikzcd}
 & A \arrow[d, "c"] \\
B \arrow[ur, "x"] \arrow[dr, "x"'] \arrow[r, phantom, "\xi"] & A
\end{tikzcd}
\tag{3}
$$

and such that the diagrams of 2-arrows

$$
\begin{tikzcd}
x \arrow[r, Rightarrow, "\xi"] \arrow[d, Rightarrow, "\xi"'] & cx \arrow[d, Rightarrow, "\mu x"] \\
cx \arrow[r, Rightarrow, "c\xi"'] & ccx
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, Rightarrow, "\xi"] \arrow[dr, Rightarrow, "1_x"'] & cx \arrow[d, Rightarrow, "\nu x"] \\
 & x
\end{tikzcd}
\tag{4}
$$

commute; an arrow $\gamma\colon (x,\xi)\to(y,\zeta)$ *is a 2-arrow* $\gamma\colon x\Rightarrow y$ *such that the following diagram commutes*

$$\begin{tikzcd} x \arrow[r, Rightarrow, "\xi"] \arrow[d, Rightarrow, "\gamma"'] & cx \arrow[d, Rightarrow, "c\gamma"] \\ y \arrow[r, Rightarrow, "\zeta"] & cy. \end{tikzcd} \tag{5}$$

Spelling out the data for an Eilenberg–Moore construction for the comonad $\mathbb{c}=(A,c,\mu,\nu)$, it requires that there is an object $A^{\mathbb{C}}$ in $\mathcal{K}$ together with a 1-arrow and a 2-arrow as in

$$\begin{tikzcd} & A \arrow[d, "c"] \\ A^{\mathbb{C}} \arrow[ur, "u^{\mathbb{C}}"] \arrow[r, "u^{\mathbb{C}}"'] \arrow[ur, phantom, shift right=4ex, "\omega^{\mathbb{C}}\,\Nearrow"] & A \end{tikzcd}$$

which satisfy the commutative diagrams in (4). Moreover, for any object B in $\mathcal{K}$, every pair (x,ξ) as in (3) satisfying (4) can be obtained by precomposition

$$\begin{tikzcd} & A \arrow[d, "c"] \\ B \arrow[ur, "x"] \arrow[r, "x"'] \arrow[ur, phantom, shift right=4ex, "\xi\,\Nearrow"] & A \end{tikzcd} = \begin{tikzcd} & & A \arrow[d, "c"] \\ B \arrow[r, "x'"] & A^{\mathbb{C}} \arrow[ur, "u^{\mathbb{C}}"] \arrow[r, "u^{\mathbb{C}}"'] \arrow[ur, phantom, shift right=4ex, "\omega^{\mathbb{C}}\,\Nearrow"] & A \end{tikzcd}$$

for a unique 1-arrow $x'\colon B\to A^{\mathbb{C}}$, and similarly for arrows $\gamma\colon(x,\xi)\to(y,\zeta)$ between pairs:

$$\begin{tikzcd} B \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\gamma"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, bend left, "x'"] \arrow[r, bend right, "y'"'] \arrow[r, phantom, "\Downarrow\gamma'"] & A^{\mathbb{C}} \arrow[r, "u^{\mathbb{C}}"] & A \end{tikzcd}$$

for a unique 2-arrow $\gamma'\colon x'\Rightarrow y'$ in $\mathcal{K}$.

In case the universality condition is verified for each comonad in $\mathcal{K}$, it can be restated in terms of a 2-adjunction after introducing the appropriate[1] 2-category $\mathsf{Adj}(\mathcal{K})$ of adjunctions in $\mathcal{K}$ and the 2-category $\mathsf{Cmd}(\mathcal{K})$ of comonads in $\mathcal{K}$. Since we can safely refer the reader to Power and Watanabe (2002) for a very clear presentation of the general setup, we limit ouselves to recapping the main diagram of 2-adjunctions:

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "\mathrm{Inc}"] \arrow[r, phantom, "\perp"] & \mathsf{Cmd}(\mathcal{K}) \arrow[l, bend left, "\mathrm{EM}"] \arrow[r, bend right, "\mathrm{EMA}"'] \arrow[r, phantom, "\perp"] & \mathsf{Adj}(\mathcal{K}) \arrow[l, bend right, "\mathrm{Cmd}"'] \end{tikzcd} \tag{6}$$

where the 2-functor Inc sends an object A in $\mathcal{K}$ to the identity comonad $(A, e_A, 1_{e_A}, 1_{e_A})$ on A, and the 2-functor EM sends a comonad $\mathbb{c}=(A,c,\mu,\nu)$ to its Eilenberg–Moore object $A^{\mathbb{C}}$; while the 2-functor Cmd sends an adjunction $\mathbb{A}=(C,D,l,r,\eta,\epsilon)$ to the associated comonad $(D, lr, l\eta r, \epsilon)$, and the 2-functor EMA sends a comonad $\mathbb{c}$ to the Eilenberg–Moore adjunction between A and $A^{\mathbb{C}}$.

Example 3.2. Although the terminology already suggests clearly the kind of generalisation adopted, we hasten to point out that in the 2-category *Cat* of (small) categories, functors and natural transfomations, the definitions in (i) and (ii) instantiate exactly to the usual notions of (standard) adjunction between categories $l\dashv r$ – where η and ϵ are the unit and the counit of the adjunction – and to comonads. Clearly, *Cat* admits the Eilenberg–Moore construction for every comonad.

In the next sections we shall characterise adjunctions and comonads in the 2-category **Dtn** of doctrines.

4. The 2-Category of Doctrines

The 2-category **Dtn** of ***doctrines*** consists of the following data:

objects are doctrines, *i.e.* a functor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ from the opposite of a category $\mathcal{C}$ to the category $\mathcal{P}os$ of posets and monotone functions – in the nomenclature of indexed categories, the category $\mathcal{C}$ is named the ***base*** of the doctrine, for X an object in $\mathcal{C}$ the poset $P(X)$ is the ***fibre over*** X, and for $t\colon X \to Y$ an arrow in $\mathcal{C}$, the monotone function $Pt\colon PY \to PX$ is called ***reindexing along*** t;[2]

a 1-arrow $(F, f)\colon P \to Q$ from the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ to the doctrine $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ is a pair where the first component $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and the second component $f\colon P \stackrel{\cdot}{\to} QF^{\mathrm{op}}$ is a natural transformation;

a 2-arrow $\theta\colon (F, f) \Rightarrow (F', f')$ is a natural transformation $\theta\colon F \stackrel{\cdot}{\to} F'$ such that, for each object X in $\mathcal{C}$, $f_X \leq_X (Q\theta^{\mathrm{op}})_X \circ f'_X$.

Composition of 1-arrows $(G, g)\colon (\mathcal{B}, M) \to (\mathcal{C}, P)$ and $(F, f)\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ is (essentially) pairwise $(FG, (fG^{\mathrm{op}}) \cdot g)\colon (\mathcal{B}, M) \to (\mathcal{D}, Q)$.

Composition of 2-arrows $\theta\colon (F, f) \Rightarrow (F', f')$ and $\zeta\colon (F', f') \Rightarrow (F'', f'')$ is the natural transformation $(\zeta_X \circ \theta_X)_{X \in \mathcal{C}_0}\colon (F, f) \Rightarrow (F'', f'')$ since, for any object X in $\mathcal{C}$,

$$f_X \leq_X Q(\theta^{\mathrm{op}}{}_X) \circ f'_X \leq_X Q(\theta^{\mathrm{op}}{}_X) \circ Q(\zeta^{\mathrm{op}}{}_X) \circ f''_X \leq_X Q((\zeta \circ \theta)^{\mathrm{op}}{}_X) \circ f''_X.$$

There is an obvious forgetful 2-functor $\mathbf{Dtn} \to \mathbf{Cat}$ to the 2-category of categories, functors and natural transformations, which maps a doctrine $(\mathcal{C}, P)$ to its base category $\mathcal{C}$, and acts similarly on the arrows. Note that such a 2-functor is actually a 2-fibration, in the sense of Hermida (1999), where cartesian 1-arrows are 'change of base', that is, arrows of the form (F, id), while vertical 1-arrows are arrows of the form $(\mathrm{Id} f)$.[3]

We define also the 2-category $\Box$-**Dtn** of doctrines endowed with an interior operator as follows:

objects are pairs $(P, \Box)$ where P is a doctrine and $\Box$ is an interior operator on P;

a 1-arrow from $(P, \Box)$ to $(Q, \Box')$ is a 1-arrow $(F, f)\colon P \to Q$ in **Dtn** such that, for each object X in the base category of P, we have $f_X \circ \Box_X \leq \Box'_{FX} \circ f_X$;

a 2-arrow from (F, f) to (G, g) is a 2-arrow $\theta\colon (F, f) \Rightarrow (G, g)$ in **Dtn**.

Compositions are inherited from those of the 2-category **Dtn**.

It is easy to verify that the requirement on the component f of a 1-arrow in $\Box$-**Dtn** is equivalent to the condition that $\Box'_{FX} \circ f_X \circ \Box_X = f_X \circ \Box_X$, *i.e.* f_X maps $\Box$-stable elements to $\Box$-stable elements.

Example 4.1. Consider the forgetful functor $U\colon \mathcal{O}pn \to \mathcal{S}et$, and for a topological space (X, τ) let $u_X = \mathrm{id}_{\mathcal{P}(X)}\colon \mathcal{P}(X) \to \mathcal{P}(X)$. If (P, int) is as in Example 2.3, then $(U, u)\colon (P, \mathrm{int}) \to (\mathcal{P}, \mathrm{Id}_{\mathcal{P}})$ is a 1-arrow in $\Box$-**Dtn**.

Example 4.2. For a Kripke frame $K = (W, R)$ where R is reflexive and transitive, the pairs $(\mathcal{P}(W)^{(-)}, \mathrm{j}_R \circ -)$ and $(\mathsf{Sub}_{W\text{-}\mathcal{F}am}, \Box)$, introduced in Example 2.4, are objects of $\Box$-**Dtn**.

Consider the functor $C\colon \mathcal{S}et \to W\text{-}\mathcal{F}am$ which maps a set S the pair $(S, (S)_{w\in W})$ where the second component is the constant family of value S. Also, for $\alpha \in \mathcal{P}(W)^S$, consider the W-indexed family given by

$$(c_S(\alpha))_w := \left\{ s \in S \;\middle|\; w \in \alpha(s) \right\}.$$

Then $(C, c)\colon (\mathcal{P}(W)^{(-)}, \mathrm{j}_R \circ -) \to (\mathsf{Sub}_{W\text{-}\mathcal{F}am}, \Box)$ is a 1-arrow in $\Box$-**Dtn**.

One can show that the 1-arrow $(C, c)\colon \mathcal{P}(W)^{(-)} \to \mathsf{Sub}_{W\text{-}\mathcal{F}am}$ is the comprehension completion of the doctrine $\mathcal{P}(W)^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$, and that the interior operator $\Box$ is the canonical extension of the other operator $\mathrm{j}_R \circ -$, see Maietti and Rosolini (2013*b*); Streicher (1991).

Remark 4.3. There is a forgetful 2-functor $\Box$-**Dtn** $\to$ **Dtn** which deletes the interior operator. It has a right 2-adjoint, which sends a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ to (P, id) and is the identity both on 1-arrows and 2-arrows. Indeed, for any object $(P, \Box)$ in $\Box$-**Dtn** the inequality $\Box_X \le \mathrm{id}_{PX}$ holds; so for any 1-arrow $(F, f)\colon P \to Q$ in **Dtn** we have $f_X \circ \Box_X \le f_X$ by monotonicity of f_X.

5. Interior Modalities from Adjunctions

The main goal of this section is to connect interior operators as in Definition 2.1 and adjunctions in **Dtn**. First we characterise the general 2-categorical notion of adjunction, as introduced in Section 3, for the particular case of the 2-category **Dtn** in terms of the functors and natural transformations involved.

Proposition 5.1. *An adjunction in the 2-category* **Dtn** *in the sense of Definition 3.1(i) is completely determined by an octuple $(P, Q, L, \lambda, R, \rho, \eta, \epsilon)$, where $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ are doctrines, $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ are functors, $\lambda\colon P \xrightarrow{\cdot} QL^{\mathrm{op}}$, $\rho\colon Q \xrightarrow{\cdot} PR^{\mathrm{op}}$, $\eta\colon \mathrm{Id}_{\mathcal{D}} \xrightarrow{\cdot} RL$ and $\epsilon\colon LR \xrightarrow{\cdot} \mathrm{Id}_{\mathcal{D}}$ are natural transformations such that*

(i) *$(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ is an adjunction in* **Cat**;
(ii) *$(L, \lambda)\colon P \to Q$ and $(R, \rho)\colon Q \to P$ are 1-arrows in* **Dtn**;
(iii) *$\eta\colon (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_P) \Rightarrow (RL, (\rho L^{\mathrm{op}})\lambda)$ and $\epsilon\colon (LR, (\lambda R^{\mathrm{op}})\rho) \Rightarrow (\mathrm{Id}_{\mathcal{D}}, \mathrm{id}_Q)$ are 2-arrows in* **Dtn**.

Proof. If $(P, Q, l, r, \eta, \epsilon)$ is an adjunction in **Dtn**, applying the forgetful functor **Dtn** $\to$ **Cat** one gets immediately i where L and R are the first components of l and r respectively. The rest of the proof is plain bookkeeping. $\Box$

As for any 2-category, one can consider the 2-category Adj(**Dtn**) of adjunctions in **Dtn**. The following proposition is just as straightforward as the previous one.

Proposition 5.2. *The 2-category Adj(**Dtn**) of adjunctions in* **Dtn** *has objects which are adjunctions $\mathbb{A} = (P^{\mathbb{A}}, Q^{\mathbb{A}}, L^{\mathbb{A}}, \lambda^{\mathbb{A}}, R^{\mathbb{A}}, \rho^{\mathbb{A}}, \eta^{\mathbb{A}}, \epsilon^{\mathbb{A}})$ as in Proposition 5.1, where $P^{\mathbb{A}}\colon (\mathcal{C}^{\mathbb{A}})^{\mathrm{op}} \to \mathcal{P}os$ and $Q^{\mathbb{A}}\colon (\mathcal{D}^{\mathbb{A}})^{\mathrm{op}} \to \mathcal{P}os$.*

*A 1-arrow $(F, f, G, g, \theta)\colon \mathbb{A} \to \mathbb{B}$ in Adj(**Dtn**) consists of two 1-arrows $(F, f)\colon P^{\mathbb{A}} \to P^{\mathbb{B}}$ and $(G, g)\colon Q^{\mathbb{A}} \to Q^{\mathbb{B}}$, and a 2-arrow $\theta\colon (FR^{\mathbb{A}}, (f(R^{\mathbb{A}})^{\mathrm{op}})\rho^{\mathbb{A}}) \Rightarrow (R^{\mathbb{B}}G, (\rho^{\mathbb{B}}G^{\mathrm{op}})g)$ in* **Dtn** *such that the triple (F, G, θ) is a homomorphism of adjunctions in* **Cat**, *and the two natural transformations $(g(L^{\mathbb{A}})^{\mathrm{op}})\lambda^{\mathbb{A}}\colon P^{\mathbb{A}} \xrightarrow{\cdot} Q^{\mathbb{B}}(GL^{\mathbb{A}})^{\mathrm{op}}$ and $(\lambda^{\mathbb{B}}F^{\mathrm{op}})f\colon P^{\mathbb{A}} \xrightarrow{\cdot} Q^{\mathbb{B}}(L^{\mathbb{B}}F)^{\mathrm{op}}$ coincide (note that $GL^{\mathbb{A}} = L^{\mathbb{B}}F$ by the first condition).*

*A 2-arrow $(\alpha, \beta)\colon (F, f, G, g, \theta) \Rightarrow (F', f', G', g', \theta')$ in Adj(**Dtn**) consists of two 2-arrows $\alpha\colon (F, f) \Rightarrow (F', f')$ and $\beta\colon (G, g) \Rightarrow (G', g')$ in* **Dtn** *such that (α, β) is a 2-cell from the adjunction homomorphism (F, G, θ) to the adjunction homomorphism (F', G', θ') in* **Cat**.

Remark 5.3. To elucidate the conditions in Proposition 5.2 in terms of some diagrams, consider first that the forgetful 2-functor **Dtn** $\to$ **Cat** extends to a 2-functor $\mathsf{Adj}(\mathbf{Dtn}) \to \mathsf{Adj}(\mathbf{Cat})$. Hence the condition that the triple (F, G, θ) is a homomorphism of adjunctions in **Cat** requires that the diagram of functors

$$\begin{tikzcd} \mathcal{C}^{\mathbb{A}} \arrow[r, "F"] \arrow[d, "L^{\mathbb{A}}"'] & \mathcal{C}^{\mathbb{B}} \arrow[d, "L^{\mathbb{B}}"] \\ \mathcal{D}^{\mathbb{A}} \arrow[r, "G"] & \mathcal{D}^{\mathbb{B}} \end{tikzcd}$$

commutes as well as (either of) the diagrams of natural transformations

$$\begin{tikzcd} F \arrow[r, "F\eta^{\mathbb{A}}"] \arrow[d, "\eta^{\mathbb{B}}F"'] & FR^{\mathbb{A}}L^{\mathbb{A}} \arrow[d, "\theta L^{\mathbb{A}}"] \\ R^{\mathbb{B}}L^{\mathbb{B}}F \arrow[r, equal] & R^{\mathbb{B}}GL^{\mathbb{A}} \end{tikzcd} \qquad \begin{tikzcd} L^{\mathbb{B}}FR^{\mathbb{A}} \arrow[r, "L^{\mathbb{B}}\theta"] \arrow[d, equal] & L^{\mathbb{B}}R^{\mathbb{B}}G \arrow[d, "\epsilon^{\mathbb{B}}G"] \\ GL^{\mathbb{A}}R^{\mathbb{A}} \arrow[r, "G\epsilon^{\mathbb{A}}"'] & G \end{tikzcd}$$

as the two commutativity conditions are equivalent. For instance, if we assume the first commutes, postcomposing it with $L^{\mathbb{B}}$ and precomposing it with $R^{\mathbb{A}}$, and using the naturality of θ and $\epsilon^{\mathbb{B}}$ and the triangular identities of adjunctions, we get the second as depicted in the following diagram:

$$\begin{tikzcd} L^{\mathbb{B}}FR^{\mathbb{A}} \arrow[r, "L^{\mathbb{B}}F\eta^{\mathbb{A}}R^{\mathbb{A}}"] \arrow[d, "L^{\mathbb{B}}\eta^{\mathbb{B}}FR^{\mathbb{A}}"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dd, bend right=60, "\mathrm{id}"'] & L^{\mathbb{B}}FR^{\mathbb{A}}L^{\mathbb{A}}R^{\mathbb{A}} \arrow[r, "L^{\mathbb{B}}FR^{\mathbb{A}}\epsilon^{\mathbb{A}}"] \arrow[d, "L^{\mathbb{B}}\theta L^{\mathbb{A}}R^{\mathbb{A}}"] & L^{\mathbb{B}}FR^{\mathbb{A}} \arrow[d, "L^{\mathbb{B}}\theta"] \\ L^{\mathbb{B}}R^{\mathbb{B}}L^{\mathbb{B}}FR^{\mathbb{A}} \arrow[r, equal] \arrow[d, "\epsilon^{\mathbb{B}}L^{\mathbb{B}}FR^{\mathbb{A}}"] & L^{\mathbb{B}}R^{\mathbb{B}}GL^{\mathbb{A}}R^{\mathbb{A}} \arrow[r, "L^{\mathbb{B}}R^{\mathbb{B}}G\epsilon^{\mathbb{A}}"] & L^{\mathbb{B}}R^{\mathbb{B}}G \arrow[d, "\epsilon^{\mathbb{B}}G"] \\ L^{\mathbb{B}}FR^{\mathbb{A}} \arrow[r, equal] & GL^{\mathbb{A}}R^{\mathbb{A}} \arrow[r, "G\epsilon^{\mathbb{A}}"] & L^{\mathbb{B}}R^{\mathbb{B}}G \end{tikzcd}$$

The condition that the pair (α, β) is a 2-cell from the adjunction homomorphism (F, G, θ) to the adjunction homomorphism (F', G', θ') in **Cat** translates into commutativity of the following diagrams of natural transformations:

$$\begin{tikzcd} L^{\mathbb{B}}F \arrow[r, "L^{\mathbb{B}}\alpha"] \arrow[d, equal] & L^{\mathbb{B}}F' \arrow[d, equal] \\ GL^{\mathbb{A}} \arrow[r, "\beta L^{\mathbb{A}}"] & G'L^{\mathbb{A}}. \end{tikzcd} \qquad \begin{tikzcd} FR^{\mathbb{A}} \arrow[r, "\alpha R^{\mathbb{A}}"] \arrow[d, "\theta"'] & F'R^{\mathbb{A}} \arrow[d, "\theta'"] \\ R^{\mathbb{B}}G \arrow[r, "R^{\mathbb{B}}\beta"] & R^{\mathbb{B}}G' \end{tikzcd}$$

From now on, when referring to an adjunction in the 2-category **Dtn**, we shall take advantage of Proposition 5.1 and write it as an octuple $(P, Q, L, \lambda, R, \rho, \eta, \epsilon)$.

Example 5.4. Examples are many as any adjunction between categories with pullbacks gives rise to an adjunction between the doctrines of subobjects. In details, given a category with pullbacks $\mathcal{C}$, one can define a functor $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ taking advantage of the fact that pulling back preserves monos. The functor maps an object to the poset of its subobjects and reindexing along $f\colon X' \to X$

is as follows: a subobject $[A \overset{\alpha}{\hookrightarrow} X]$, determined by the isomorphism class of the mono α, is taken to the subobject determined by the mono α' obtained as a pullback

$$\begin{tikzcd} A' \arrow[r, hook, "\alpha'"] \arrow[d] & X' \arrow[d, "f"] \\ A \arrow[r, hook, "\alpha"'] & X. \end{tikzcd}$$

Let $\mathcal{D}$ be also a category with pullbacks, and consider an adjunction $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ where $L: \mathcal{C} \to \mathcal{D}$ preserves pullbacks (as a right adjoint, the functor $R: \mathcal{D} \to \mathcal{C}$ preserves all existing limits). Between the doctrines $\mathsf{Sub}_{\mathcal{C}}: \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $\mathsf{Sub}_{\mathcal{D}}: \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ there are 1-arrows of **Dtn** $(L, \lambda): \mathsf{Sub}_{\mathcal{C}} \to \mathsf{Sub}_{\mathcal{D}}$ and $(R, \rho): \mathsf{Sub}_{\mathcal{D}} \to \mathsf{Sub}_{\mathcal{C}}$, where for X in $\mathcal{C}$ and Y in $\mathcal{D}$

$$\lambda_X([A \overset{\alpha}{\hookrightarrow} X]) = [LA \overset{L\alpha}{\hookrightarrow} LX] \qquad \rho_X([B \overset{\beta}{\hookrightarrow} Y]) = [RA \overset{R\beta}{\hookrightarrow} RY].$$

The naturality of λ and ρ follows since reindexing is given by pulling back, and L and R preserve pullbacks. To see that $(\mathsf{Sub}_{\mathcal{C}}, \mathsf{Sub}_{\mathcal{D}}, L, \lambda, R, \rho, \eta, \epsilon)$ is an adjunction in **Dtn** there remains to check that $\eta: (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_{\mathsf{Sub}_{\mathcal{C}}}) \Rightarrow (RL, (\rho L^{\mathrm{op}})\lambda)$ and $\epsilon: (LR, (\lambda R^{\mathrm{op}})\rho) \Rightarrow (\mathrm{Id}_{\mathcal{D}}, \mathrm{id}_{\mathsf{Sub}_{\mathcal{D}}})$ are 2-arrows of **Dtn**: in other words, for any $[A \overset{\alpha}{\hookrightarrow} X]$ and $[B \overset{\beta}{\hookrightarrow} Y]$, we have

$$[\alpha] \le \mathsf{Sub}_{\mathcal{C}}\,(\eta_X)\,[RL(\alpha)] \quad \text{and} \quad [LR(\beta)] \le \mathsf{Sub}_{\mathcal{D}}\,(\epsilon_Y)\,[\beta].$$

But this follows from naturality of η and ϵ together with the reindexing pullbacks

$$\begin{tikzcd} A \arrow[dr, dashed] \arrow[drr, hook, bend left, "\alpha"] \arrow[ddr, bend right, "\eta_A"'] & & \\ & P \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[d, "\eta_X"] \\ & RLA \arrow[r, hook, "RL(\alpha)"'] & RLX \end{tikzcd} \qquad \begin{tikzcd} LRB \arrow[dr, dashed] \arrow[drr, hook, bend left, "LR(\beta)"] \arrow[ddr, bend right, "\epsilon_B"'] & & \\ & P \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & LRY \arrow[d, "\epsilon_Y"] \\ & B \arrow[r, hook, "\beta"'] & Y \end{tikzcd}$$

We now put to use the characterisation in Proposition 5.1 to construct an interior operator starting from an adjunction of doctrines. We begin the process performing the construction for a very specific type of adjunctions: adjunctions between vertical 1-arrows.

Proposition 5.5. *Let $P: \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q: \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ be doctrines, and suppose the octuple $(P, Q, \mathrm{Id}_{\mathcal{C}}, \lambda, \mathrm{Id}_{\mathcal{C}}, \rho, \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}, \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}})$ is an adjunction in* **Dtn**. *Then*

(i) *for each object X in $\mathcal{C}$, the following adjunction holds between the fibres*

$$\begin{tikzcd} PX \arrow[r, bend left, "\lambda_X"] \arrow[r, phantom, "\perp"] & QX \arrow[l, bend left, "\rho_X"] \end{tikzcd}, \qquad X \in \mathcal{C}_0;$$

(ii) *$\square = \lambda \cdot \rho$ is an interior operator on Q.*

Proof. By Proposition 5.1, the hypothesis ensures that $\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}: (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \rho\lambda)$ and $\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}: (\mathrm{Id}_{\mathcal{C}}, \lambda\rho) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \mathrm{id})$ are 2-arrows in **Dtn**. From this, the conclusion follows directly. $\square$

Example 5.6. Recall from Rosenthal (1990) that a *commutative quantale* is a complete lattice endowed with further structure $(V, \bigvee, \leq, \otimes, 1)$ where $(V, \bigvee, \leq)$ is a complete lattice, $(V, \otimes, 1)$ is a commutative monoid such that the operation $\otimes$ distributes over sups:

$$x \otimes \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \otimes x_i)$$

for elements x and families $(x_i)_{i\in I}$ in V – note that this yields that $\otimes$ is monotone in its two arguments.

Let $R_V = \{x \in V \mid x \leq 1 \text{ and } x \leq x \otimes x\} \subseteq V$. It is easy to check that $1 \in R_V$ and R_V is closed with respect to $\otimes$ and $\bigvee$. Hence $(R_V, \bigvee, \leq, \otimes, 1)$ is a commutative quantale. Let $\iota\colon R_V \to V$ be the inclusion function which clearly preserves sups. Its right adjoint $r\colon V \to R_V$ is determined as $r(x) = \bigvee\{y \in R_V \mid y \leq x\}$.

Consider the doctrine $V^{(-)}\colon \mathcal{S}et^{\text{op}} \to \mathcal{P}os$ and $R_V^{(-)}\colon \mathcal{S}et^{\text{op}} \to \mathcal{P}os$ mapping a set X to the sets of functions V^X and R_V^X, ordered pointwise, and acting on functions by precomposition. And the 1-arrow $(\mathrm{Id}, \iota \circ -)\colon R_V^{(-)} \to V^{(-)}$ has a right adjoint given by $(\mathrm{Id}, r \circ -)\colon V^{(-)} \to R_V^{(-)}$. Hence, by Proposition 5.5, there is an interior operator $!\colon V^{(-)} \stackrel{\cdot}{\to} V^{(-)}$ given by $!_X\alpha = \iota \circ r \circ \alpha$, for any set X and $\alpha \in V^X$.

Recall that the doctrine $V^{(-)}$ carries a much richer structure induced from that of the original quantale V: for any set X, $(V^X, \bigvee, \leq_X, \otimes_X, 1_X)$ is a commutative quantale with the pointwise structure and, for $\alpha, \beta \in V^X$, the operation $\alpha \multimap_X \beta := \bigvee\{\zeta \in V^X \mid \alpha \otimes_X \zeta \leq_X \beta\}$ determines an adjoint pair $- \otimes_X \alpha \dashv \alpha \multimap_X -$; *i.e.* for every $\gamma \in V^X$, one has that $\alpha \otimes_X \gamma \leq_X \beta$ if and only if $\gamma \leq_X \alpha \multimap_X \beta$. Furthermore, the interior operator $!\colon V^{(-)} \stackrel{\cdot}{\to} V^{(-)}$ enjoys additional properties: for any set X and $\alpha, \beta \in V^X$, we have $!_X\alpha \leq_X 1_X$ and $!_X\alpha \leq_X !_X\alpha \otimes_X !_X\alpha$, and $1_X \leq_X !_X 1_X$ and $!_X\alpha \otimes_X !_X\beta \leq_X !_X(\alpha \otimes_X \beta)$. Therefore, the indexed poset $V^{(-)}$ provides a model of first order intuitionistic linear logic, where $!$ is the linear exponential modality.

Examples 5.7. Let $P\colon \mathcal{C}^{\text{op}} \to \mathcal{P}os$ be a doctrine. The propositional connectives are defined in terms of adjunctions involving P and another doctrine defined from it where the adjoint functors between the base categories are the identity, see Lawvere (1969), see also Jacobs (1999); Maietti and Rosolini (2015). So Proposition 5.5 provides interior operators associated with each connectives. Two interesting instances are the following:

(1) Consider the doctrine $P^2\colon \mathcal{C}^{\text{op}} \to \mathcal{P}os$, defined by $P^2X = PX \times PX$ and $P^2f = Pf \times Pf$. Note that there is a 1-arrow $(\mathrm{Id}_\mathcal{C}, \Delta)\colon P \to P^2$ where $\Delta_X = (\mathrm{id}_{PX}, \mathrm{id}_{PX})$. Conjunction on P is determined by a right adjoint to $(\mathrm{Id}_\mathcal{C}, \Delta)$ in **Dtn**, that is the octuple $(\mathrm{Id}_\mathcal{C}, \Delta, \mathrm{Id}_\mathcal{C}, \wedge, \mathrm{id}_{\mathrm{Id}_\mathcal{C}}, \mathrm{id}_{\mathrm{Id}_\mathcal{C}})$ is an adjunction between P and P^2. Hence, by Proposition 5.5, there is an interior operator on P^2 given by $(\alpha, \beta) \mapsto (\alpha \wedge \beta, \alpha \wedge \beta)$, for $\alpha, \beta \in PX$.

(2) Assume further that $\mathcal{C}$ has finite products and consider an object X in $\mathcal{C}$. Consider the doctrine $P^X\colon \mathcal{C}^{\text{op}} \to \mathcal{P}os$, determined as $P^X(Y) = P(Y \times X)$ and $P^X(f) = P(f \times \mathrm{id}_X)$. There is a 1-arrow $(\mathrm{Id}_\mathcal{C}, p^X)\colon P \to P^X$ where $p^X_Y = P\pi_1$ and $\pi_1\colon Y \times X \to Y$ is the first projection. A universal quantifier $\forall_X$ on P over X is a right adjoint to $(\mathrm{Id}_\mathcal{C}, p^X)$ in **Dtn**, *i.e.* the octuple $(P, P^X, \mathrm{Id}_\mathcal{C}, p^X, \mathrm{Id}_\mathcal{C}, \forall_X, \mathrm{id}_{\mathrm{Id}_\mathcal{C}}, \mathrm{id}_{\mathrm{Id}_\mathcal{C}})$ is an adjunction in **Dtn**. Hence, by Proposition 5.5, there is an interior operator on P^X given as $\alpha \mapsto p^X(\forall_X\alpha)$ for $\alpha \in P^X(Y) = P(Y \times X)$.

We did not consider the other cases of connectives because the modality each of those induces is the identity as the next proposition explains in a more general context.

Proposition 5.8. *Let $P\colon \mathcal{C}^{\text{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{C}^{\text{op}} \to \mathcal{P}os$ be doctrines on the same base category. Suppose $(P, Q, \mathrm{Id}_\mathcal{C}, \lambda, \mathrm{Id}_\mathcal{C}, \rho, \mathrm{id}_{\mathrm{Id}_\mathcal{C}}, \mathrm{id}_{\mathrm{Id}_\mathcal{C}})$ is an adjunction. Then, for each object X in $\mathcal{C}$, the following hold:*

(i) $\lambda_X \cdot \rho_X \cdot \lambda_X = \lambda_X$ *and* $\rho_X \cdot \lambda_X \cdot \rho_X = \rho_X$*;*
(ii) $\lambda_X \cdot \rho_X = \mathrm{id}_{QX}$ *if and only if* ρ_X *is injective if and only if* λ_X *is surjective;*
(iii) $\rho_X \cdot \lambda_X = \mathrm{id}_{PX}$ *if and only if* λ_X *is injective if and only if* ρ_X *is surjective.*

Proof. (i) is immediate since the adjunction $\lambda_X \dashv \rho_X$ involves posetal categories. (ii) and (iii) follow directly from (i). □

The next step is an application of a remarkable result by Hermida (1994) about fibred adjunctions as it allows to show that any adjunction in **Dtn** can be ***factored*** as the composition of two adjunctions where one is the identity adjunction on the base categories. For this, recall that **Dtn** has a vertical/cartesian factorisation system, that is, any 1-arrow $(F, f)\colon P \to Q$ from the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ to the doctrine $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ can be factored by 'change of base' as $(F, \mathrm{id}_{QF^{\mathrm{op}}}) \circ (\mathrm{Id}_{\mathcal{C}}, f)$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[d, "\mathrm{Id}_{\mathcal{C}}^{\mathrm{op}}"'] \arrow[drr, bend left, "P"] \arrow[drr, phantom, shift right=2ex, "\swarrow f"] & & \\
\mathcal{C}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] \arrow[rr, "QF^{\mathrm{op}}"] & & \mathcal{P}os \\
\mathcal{D}^{\mathrm{op}} \arrow[urr, bend right, "Q"'] \arrow[urr, phantom, shift left=2ex, "\mathrm{id} \searrow"] & &
\end{tikzcd}$$

The factorisation of the adjunction follows this decomposition for the left adjoint. Recall Lemma 3.2 from Hermida (1994) in the case of doctrines.

Lemma 5.9. *Let* $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ *be an adjunction in* **Cat**. *If* $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ *is a doctrine, then there is an adjunction* $(QL^{\mathrm{op}}, Q, L, \mathrm{id}, R, Q\epsilon^{\mathrm{op}}, \eta, \epsilon)$ *in* **Dtn** *as depicted in the diagram*

$$\begin{tikzcd}
QL^{\mathrm{op}} \arrow[rr, bend left, "{(L, \mathrm{id})}"] & & Q. \arrow[ll, bend left, "{(R, Q\epsilon^{\mathrm{op}})}"]
\end{tikzcd} \tag{7}$$

Proof. We apply Proposition 5.1 to show $(QL^{\mathrm{op}}, Q, L, \mathrm{id}_{QL^{\mathrm{op}}}, R, Q\epsilon^{\mathrm{op}}, \eta, \epsilon)$ is an adjunction in **Dtn**. Since $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ is already an adjunction in **Cat**, it remains to check the natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \stackrel{\cdot}{\to} RL$ and $\epsilon\colon LR \stackrel{\cdot}{\to} \mathrm{Id}_{\mathcal{D}}$ determine 2-arrows in **Dtn** as follows

$$\eta\colon (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_{QL^{\mathrm{op}}}) \Rightarrow (RL, Q(\epsilon L)^{\mathrm{op}}) \qquad \epsilon\colon (LR, Q\epsilon^{\mathrm{op}}) \Rightarrow (\mathrm{Id}_{\mathcal{D}}, \mathrm{id}_Q).$$

In other words, the inequalities

$$\mathrm{id}_{QLX} \le QL\eta_X \circ Q\epsilon_{LX} \qquad Q\epsilon_Y \le Q\epsilon_Y$$

hold for each object X in $\mathcal{C}$ and Y in $\mathcal{D}$. They are in fact identities: the second is immediate, and the first follows from the triangular identity (1) for an adjunction

$$QL\eta_X \circ Q\epsilon_{LX} = Q(\epsilon_{LX} \circ L\eta_X) = Q\mathrm{id}_{LX} = \mathrm{id}_{QLX} \tag{8}$$

by functoriality of Q. □

Theorem 3.4 in Hermida (1994) restricted to the case of doctrines is the following.

Theorem 5.10. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ *and* $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ *be doctrines, and suppose the octuple* $(P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ *is an adjunction in* **Dtn**. *Then that adjunction factors through the adjunction in* (7) *as*

$$\begin{tikzcd} P \arrow[r, bend left, "(\mathrm{Id}_{\mathcal{C}},\lambda)"] & QL^{\mathrm{op}} \arrow[l, bend left, "(\mathrm{Id}_{\mathcal{C}},(P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}}))"] \arrow[r, bend left, "(L,\mathrm{id})"] & Q. \arrow[l, bend left, "(R,Q\epsilon^{\mathrm{op}})"] \end{tikzcd} \tag{9}$$

where the first one is $(P, QL^{\mathrm{op}}, \mathrm{Id}_{\mathcal{C}}, \lambda, \mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}}), \mathrm{id}, \mathrm{id})$.

Proof. We see the $(P, QL^{\mathrm{op}}, \mathrm{Id}_{\mathcal{C}}, \lambda, \mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}}), \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}, \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}})$ is an adjunction in **Dtn** as another application of Proposition 5.1. Obviously $(\mathcal{C}, \mathcal{C}, \mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}, \mathrm{id}, \mathrm{id})$ is the identity adjunction in **Cat**. To check the natural transformation $\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cdot} \mathrm{Id}_{\mathcal{C}}$ determines 2-arrows in **Dtn**

$$\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}\colon (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_P) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}})\lambda) \quad \text{and} \quad \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}\colon (\mathrm{Id}_{\mathcal{C}}, \lambda(P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}})) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_{QL^{\mathrm{op}}})$$

we must see that the inequalities

$$\mathrm{id}_{PX} \leq P\eta_X \circ \rho_{LX} \circ \lambda_X \quad \text{and} \quad \lambda_X \circ P\eta_X \circ \rho_{LX} \leq \mathrm{id}_{QLX}$$

hold for each object X in $\mathcal{C}$. The first inequality holds since $\eta\colon (\mathrm{Id}_{\mathcal{C}}, \mathrm{id}_P) \Rightarrow (RL, (\rho L^{\mathrm{op}})\lambda)$ is a 2-arrow in **Dtn**. For the second inequality, note that $\lambda_X \circ P\eta_X \circ \rho_{LX} = QL\eta_X \circ \lambda_{RLX} \circ \rho_{LX}$ since $\lambda\colon P \xrightarrow{\cdot} QL^{\mathrm{op}}$. Since $\epsilon\colon (LR, (\lambda R^{\mathrm{op}})\rho) \Rightarrow (\mathrm{Id}_{\mathcal{D}}, \mathrm{id}_Q)$ is a 2-arrow in **Dtn**, we have that $\lambda_{RLX} \circ \rho_{LX} \leq Q\epsilon_{LX}$. Now the result follows from (8).

To see that the composition of the two adjunctions gives the original adjunction, note that the top and bottom compositions in (9) give the top and bottom 1-arrow in

$$\begin{tikzcd} P \arrow[r, bend left, "(L,\lambda)"] & Q. \arrow[l, bend left, "(R,\rho)"] \end{tikzcd}$$

It is immediate to see that $(L, \mathrm{id}) \cdot (\mathrm{Id}_{\mathcal{C}}, \lambda) = (L, \lambda)$. For the other composition, the first components coincide trivially, and for the second components apply the commutativity of the following diagram of natural transformations

$$\begin{tikzcd} Q \arrow[r, "\rho"'] \arrow[rr, bend left, "\rho"] \arrow[d, "Q\epsilon^{\mathrm{op}}"] & PR^{\mathrm{op}} \arrow[r, "P((R\epsilon)(\eta R))^{\mathrm{op}}"'] \arrow[d, "PR^{\mathrm{op}}\epsilon^{\mathrm{op}}"'] & PR^{\mathrm{op}} \\ QL^{\mathrm{op}}R^{\mathrm{op}} \arrow[r, "\rho L^{\mathrm{op}}R^{\mathrm{op}}"] & PR^{\mathrm{op}}L^{\mathrm{op}}R^{\mathrm{op}} \arrow[ur, "P\eta^{\mathrm{op}}R^{\mathrm{op}}"'] & \end{tikzcd}$$

where the square commutes by naturality of ρ, the right-hand triangle by functoriality of P, and the top triangle by one of the triangular identities for adjunctions (1). Finally one sees immediately the compositions of the 2-arrows give the 2-arrows of the original adjunction. $\square$

Corollary 5.11. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ *and* $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ *be doctrines, and suppose the octuple* $(P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ *is an adjunction in* **Dtn**. *Then* $\square = \lambda \cdot (P\eta^{\mathrm{op}}) \cdot (\rho L^{\mathrm{op}})$ *is an interior operator on the doctrine* $QL^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$.

Proof. It follows immediately applying Proposition 5.5 to the first adjunction in (9). $\square$

Example 5.12. Let $\mathcal{C}$ and $\mathcal{D}$ be category with pullbacks, and let $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ be an adjunction where $L\colon \mathcal{C} \to \mathcal{D}$ preserves pullbacks. As in Example 5.4, there is an adjunction $(\mathsf{Sub}_{\mathcal{C}}, \mathsf{Sub}_{\mathcal{D}}, L, \lambda, R, \rho, \eta, \epsilon)$ on the doctrines of subobjects. By Corollary 5.11, there is an interior operator on the doctrine $\mathsf{Sub}_{\mathcal{D}}L^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$, defined as $\square_X\alpha = L\alpha'$, where $X \in \mathcal{C}_0$ and $\alpha \in \mathsf{Sub}_{\mathcal{D}}(LX)$ and $\alpha' \in \mathsf{Sub}_{\mathcal{C}}(X)$ is defined by the following pullback diagram

$$\begin{tikzcd} P \arrow[r, hook, "\alpha'"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[d, "\eta_X"] \\ RA \arrow[r, hook, "R\alpha"'] & RLX \end{tikzcd}$$

The construction is reminiscent of that of a modal operator from a geometric morphism between elementary toposes, see the original paper Ghilardi and Meloni (1988), or Section 10.1 in Braüner and Ghilardi (2007), and also Awodey and Birkedal (2003); Awodey et al. (2002); Reyes (1991). Indeed, a geometric morphism from the topos $\mathcal{E}$ to the topos $\mathcal{F}$ is an adjunction $(\mathcal{E}, \mathcal{F}, L, R, \eta, \epsilon)$ such that the left adjoint L preserves finite limits.

The paradigmatic example of a interior operator obtained from a geometric morphism is that offered by presheaves over a category $\mathcal{C}$. Recall that the category of presheaves over $\mathcal{C}$ is the functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$. If we let $\mathcal{C}_0$ be the discrete category of the objects of $\mathcal{C}$ and write $i\colon \mathcal{C}_0 \to \mathcal{C}$ the inclusion functor, postcomposition with it determines a functor $L = - \circ i^{\mathrm{op}}\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et] \to [\mathcal{C}_0{}^{\mathrm{op}}, \mathcal{S}et]$ which preserves all limits and colimits as these are computed pointwise – although $\mathcal{C}_0 = \mathcal{C}_0^{\mathrm{op}}$ we maintain the redundant notation $\mathcal{C}_0^{\mathrm{op}}$ just for mental hygiene. Since the functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$ is complete and has a generating set, L has a right adjoint $R\colon [\mathcal{C}_0{}^{\mathrm{op}}, \mathcal{S}et] \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$. Hence, $L \dashv R$ is a geometric morphism, thus it induces an interior operator on

$$\begin{tikzcd} {[\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]^{\mathrm{op}}} \arrow[rr, "\mathsf{Sub}_{[\mathcal{C}_0{}^{\mathrm{op}}, \mathcal{S}et]}L^{\mathrm{op}}"] \arrow[dr, "L^{\mathrm{op}}"'] & & \mathcal{P}os \\ & {[\mathcal{C}_0{}^{\mathrm{op}}, \mathcal{S}et]} \arrow[ur, "\mathsf{Sub}_{[\mathcal{C}_0{}^{\mathrm{op}}, \mathcal{S}et]}"'] & \end{tikzcd}$$

Finally, note that, if $K = (W, R)$ is a Kripke frame with R reflexive and transitive, taking $\mathcal{C} = K^{\mathrm{op}}$, the above geometric morphism provides another way to construct Kripke models categorically. In detail, a presheaf D over K^{op} specifies, for each world $w \in W$, a set $D(w)$, modelling individuals which exist at the world w, and, for each wRv, a function $D_{wv}\colon D_w \to D_v$, describing how individuals existing at the world w 'evolve' in the world v. A 'formula' α on D is a family of subsets, that is, for each world $w \in W$, $\alpha_w \subseteq D_w$, and the modal operator identifies those formulas which are subpresheaves of D, namely, those α such that, for all $w, v \in W$, if wRv then $\alpha_w \subseteq D_{wv}^{-1}(\alpha_v)$.

We conclude this section showing that the construction in Corollary 5.11 extends to a 2-functor $\mathrm{AM}\colon \mathsf{Adj}(\mathbf{Dtn}) \to \square\text{-}\mathbf{Dtn}$.

For an adjunction $\mathbb{A}$ in **Dtn** write

$$\square^{\mathbb{A}} := \lambda^{\mathbb{A}} \cdot (P^{\mathbb{A}}(\eta^{\mathbb{A}})^{\mathrm{op}}) \cdot (\rho^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}})$$

which is an interior operator by Corollary 5.11. Let $\mathrm{AM}(\mathbb{A}) = (Q^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}}, \square^{\mathbb{A}})$. For a 1-arrow $(F, f, G, g, \theta)\colon \mathbb{A} \to \mathbb{B}$, let

$$\mathrm{AM}((F, f, G, g, \theta)) := (F, g(L^{\mathbb{A}})^{\mathrm{op}}). \tag{10}$$

For a 2-arrow $(\alpha, \beta)\colon (F, f, G, g, \theta) \Rightarrow (F', f', G', g', \theta')$, let

$$\mathrm{AM}((\alpha, \beta)) := \alpha. \tag{11}$$

Proposition 5.13. *With the assignments above,* $\mathrm{AM}\colon \mathsf{Adj}(\mathbf{Dtn}) \to \square\text{-}\mathbf{Dtn}$ *is a 2-functor.*

Proof. We just have to check that the identities in (10) and (11) determine arrows in $\square$-**Dtn**, as the algebraic identities will then follow immediately. Since $g(L^{\mathbb{A}})^{\mathrm{op}}\colon Q^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}} \overset{\cdot}{\to} Q^{\mathbb{B}}(GL^{\mathbb{A}})^{\mathrm{op}}$ and $GL^{\mathbb{A}} = L^{\mathbb{B}}F$ by Proposition 5.2, in order to see that

$$(F, g(L^{\mathbb{A}})^{\mathrm{op}})\colon (Q^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}}, \square^{\mathbb{A}}) \to (Q^{\mathbb{B}}(L^{\mathbb{B}})^{\mathrm{op}}, \square^{\mathbb{B}})$$

ia a 1-arrow in $\square$-**Dtn** we are left to check that for every object X in the base category of $Q(L^{\mathbb{A}})^{\mathrm{op}}$, we have

$$g_{L^{\mathbb{A}}X} \cdot \square^{\mathbb{A}}_X \leq \square^{\mathbb{B}}_{FX} \cdot g_{L^{\mathbb{A}}X}.$$

In the diagram of natural transformations

$$\begin{tikzcd}
Q^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}} \arrow[r, "\rho^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}}"] \arrow[ddd, "g(L^{\mathbb{A}})^{\mathrm{op}}"] & P^{\mathbb{A}}(R^{\mathbb{A}}L^{\mathbb{A}})^{\mathrm{op}} \arrow[r, "P^{\mathbb{A}}(\eta^{\mathbb{A}})^{\mathrm{op}}"] \arrow[dd, "f(R^{\mathbb{A}}L^{\mathbb{A}})^{\mathrm{op}}"'] \arrow[ddr, phantom, "*"] & P^{\mathbb{A}} \arrow[r, "\lambda^{\mathbb{A}}"] \arrow[dd, "f"'] & Q^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}} \arrow[d, "g(L^{\mathbb{A}})^{\mathrm{op}}"] \\
 & & & Q^{\mathbb{B}}(GL^{\mathbb{A}})^{\mathrm{op}} \arrow[d, equal] \\
{} \arrow[r, phantom, "\geq"] & P^{\mathbb{B}}(FR^{\mathbb{A}}L^{\mathbb{A}})^{\mathrm{op}} \arrow[r, "P^{\mathbb{B}}(F\eta^{\mathbb{A}})^{\mathrm{op}}"] & P^{\mathbb{B}}F^{\mathrm{op}} \arrow[r, "\lambda^{\mathbb{B}}F^{\mathrm{op}}"] & P^{\mathbb{B}}(L^{\mathbb{B}}F)^{\mathrm{op}} \\
Q^{\mathbb{B}}(L^{\mathbb{A}})^{\mathrm{op}} \arrow[r, "P^{\mathbb{B}}\theta(L^{\mathbb{A}})^{\mathrm{op}}"] \arrow[dr, "\rho^{\mathbb{B}}(L^{\mathbb{B}}F)^{\mathrm{op}}"'] & P^{\mathbb{B}}(R^{\mathbb{B}}GL^{\mathbb{A}})^{\mathrm{op}} \arrow[u, "P^{\mathbb{B}}(\theta L^{\mathbb{A}})^{\mathrm{op}}"] \arrow[ur, "P^{\mathbb{B}}(\theta L^{\mathbb{A}} \cdot F\eta^{\mathbb{A}})^{\mathrm{op}}"] & & \\
 & P^{\mathbb{B}}(R^{\mathbb{B}}L^{\mathbb{B}}F)^{\mathrm{op}} \arrow[uur, "P^{\mathbb{B}}(\eta^{\mathbb{B}}F)^{\mathrm{op}}"'] & &
\end{tikzcd}$$

the marked square commutes by naturality of f, the triangle by functoriality of $P^{\mathbb{B}}$, and all the other paths commutes (possibly up to inequality as shown) by Proposition 5.2.

Given now a 2-arrow $(\alpha, \beta)\colon (F, f, G, g, \theta) \Rightarrow (F', f', G', g', \theta')$ in $\mathsf{Adj}(\mathbf{Dtn})$ to see that $\alpha\colon (F, gL^{\mathbb{A}}) \Rightarrow (F', g'L^{\mathbb{A}})$ is a 2-arrow in $\square$-**Dtn**, we have to show that, for every object X in the base category of $Q(L^{\mathbb{A}})^{\mathrm{op}}$, it is the case that $g_{L^{\mathbb{A}}X} \leq Q'L^{\mathbb{B}}\alpha_X \cdot g'_{L^{\mathbb{A}}X}$. By Proposition 5.2, the equality $L^{\mathbb{B}}\alpha = \beta L^{\mathbb{A}}$ holds and, since $\beta\colon (G, g) \Rightarrow (G', g')$ in **Dtn**, we obtain that $g_{L^{\mathbb{A}}X} \leq Q'\beta_{L^{\mathbb{A}}X} \cdot g'_{L^{\mathbb{A}}X}$, as needed. □

Example 5.14. A particular example of interior operators is found in the categorical semantics of the linear exponential modality (a.k.a. bang modality) of propositional linear logic provided by linear–non-linear adjunctions. A ***linear–non-linear adjunction*** is a monoidal adjunction between a symmetric monoidal category and a cartesian category; the induced comonad on the symmetric monoidal category interprets the bang modality, see Benton (1994). The categorical notion swiftly extends to doctrines where the construction in Corollary 5.11 provides a model of the bang modality in a higher order setting. The role of the cartesian category is played by a ***primary doctrine***, see *e.g.* Emmenegger et al. (2020)), that is, a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ where $\mathcal{C}$ has finite products and, for each object X in $\mathcal{C}$, the fibre PX carries an inf-semilattice structure preserved by reindexing. The role of the symmetric monoidal category is played by a (*symmetric*) *monoidal doctrine*, which one defines following the work on monoidal indexed categories of Moeller and Vasilakopoulou (2020). We give some of the details in Appendix A, but shall develop fully the particular instance of interior operators in a subsequent paper.

6. Interior Modalities from Comonads

As is well known, there is a deep connection between comonads and adjunctions in a 2-category: every adjunction determines a comonad. Viceversa, when the 2-category admits the Eilenberg–Moore construction for comonads, a comonad generates an adjunction. This connection is particularly interesting when we consider a left exact comonad K on a topos $\mathcal{E}$: the category of coalgebras $\mathcal{E}^K$ is a topos and the Eilenberg–Moore adjunction between $\mathcal{E}^K$ and $\mathcal{E}$ is a geometric morphism, see *e.g.* Mac Lane and Moerdijk (1992). As we have seen in Example 5.12, geometric morphisms generate interior operators; hence, combining these two facts, we obtain that a left exact comonads on an elementary topos determines an interior operator.

In this section, we study the relationship between adjunctions and comonads in the 2-category **Dtn** of doctrines, showing how comonads generate adjunctions, as expected, and interior operators from those. We start by determining comonads in **Dtn**.

Proposition 6.1. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ be a doctrine. A comonad on P is completely determined by a quadruple $\mathsf{K} = (K, \kappa, \mu, \nu)$ where*

(i) *(K, μ, ν) is a comonad on $\mathcal{C}$;*
(ii) *$(K, \kappa)\colon P \to P$ is a 1-arrow in* **Dtn***;*
(iii) *$\mu\colon (K, \kappa) \Rightarrow (K^2, (\kappa K^{\mathrm{op}})\kappa)$ and $\nu\colon (K, \kappa) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \mathrm{id})$ are 2-arrows in* **Dtn**.

Proof. Straightforward. □

Remark 6.2. More explicitly, condition (ii) in Proposition 6.1 requires that $\mu\colon P \stackrel{\cdot}{\to} PK^{\mathrm{op}}$ and condition (iii) in Proposition 6.1 states that, for each object X in $\mathcal{C}$, the following inequalities hold

$$\kappa_X \leq P\mu_X \circ \kappa_{KX} \circ \kappa_X \qquad \text{and} \qquad \kappa_X \leq P\nu_X.$$

For abstract reasons, a comonad in **Dtn** always admits the Eilenberg–Moore construction, see Blackwell et al. (1989). Here we limit ourselves to present the direct computation of the Eilenberg–Moore object for a comonad $\mathsf{K} = (K, \kappa, \mu, \nu)$ on the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$. The Eilenbeerg-Moore object for K can be given on the doctrine $P^{\mathsf{K}}\colon \left(\mathcal{C}^K\right)^{\mathrm{op}} \to \mathcal{P}os$ defined as follows.

The category $\mathcal{C}^K$ is the category of coalgebras for the comonad (K, μ, ν) on $\mathcal{C}$, namely, objects are pairs (C, c) where C is an object in $\mathcal{C}$ and $c\colon C \to KC$ is an arrow in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
& C \arrow[r, "c"] \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "c"] & KC \arrow[d, "\mu_C"] \\
C & KC \arrow[l, "\nu_C"] \arrow[r, "Kc"'] & KKC
\end{tikzcd}$$

commutes, and an arrow $f\colon (C, c) \to (C', c')$ is an arrow $f\colon C \to C'$ in $\mathcal{C}$, such that

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "c"'] & C' \arrow[d, "c'"] \\
KC \arrow[r, "Kf"] & KC'.
\end{tikzcd}$$

With the intention to produce the doctrine $P^{\mathsf{K}}\colon \left(\mathcal{C}^K\right)^{\mathrm{op}} \to \mathcal{P}os$, for each coalgebra (C, c) let $P^{\mathsf{K}}(C, c)$ be the suborder of PC on the subset $\left\{\alpha \in PC \,\middle|\, \alpha \leq Pc(\kappa_C(\alpha))\right\}$.

Given an arrow $f\colon (C, c) \to (C', c')$ in $\mathcal{C}^{K}$ and $\beta \in P^{\mathsf{K}}(C', c')$, note that $\beta \leq Pc'(\kappa_{C'}(\beta))$ by definition of $P^{\mathsf{K}}(C', c')$. Thus

$$\begin{aligned} Pf(\beta) \leq Pf(Pc'(\kappa_{C'}(\beta))) = P(c'f)(\kappa_{C'}(\beta))) &= P(fKc)(\kappa_{C'}(\beta))) \\ &= Pc(PK(f)(\kappa_{C'}(\beta))) = Pc(\kappa_C(Pf(\beta))). \end{aligned}$$

So Pf sends elements of $P^{\mathsf{K}}(C', c')$ to elements of $P^{\mathsf{K}}(C, c)$: let $P^{\mathsf{K}}f$ be the restriction of Pf. It follows immediately that P^{K} is a doctrine.

Remark 6.3. Note that the inequality $Pc(\kappa_C(\alpha)) \leq \alpha$ holds for every $\alpha \in PC$, by properties of c and ν_C. Hence the elements of $P^{\mathsf{K}}(C, c)$ are the fixpoints of $Pc \circ \kappa_C$. Furthermore, as we shall see, $Pc \circ \kappa_C$ is an idempotent on PC (it is a consequence of Proposition 6.6). Thus, as in $\mathcal{P}os$ idempotents split, one gets $P^{\mathsf{K}}(C, c)$ by splitting $Pc \circ \kappa_C$.

Next we introduce the ***forgetful*** 1-arrow $(U^{\mathsf{K}}, \iota^{\mathsf{K}})\colon P^{\mathsf{K}} \to P$ as follows: the functor $U^{\mathsf{K}}\colon \mathcal{C}^{K} \to \mathcal{C}$ is the actual forgetful functor from the category of coalgebras; the natural transformation $\iota^{\mathsf{K}}\colon P^{\mathsf{K}} \stackrel{\cdot}{\to} P(U^{\mathsf{K}})^{\mathrm{op}}$ is given by the inclusion of $P^{\mathsf{K}}(C, c)$ into PC as (C, c) varies amongst the objects of $\mathcal{C}^{K}$. It is immediate to see the functor U^{K} is faithful and, for each object (C, c) in $\mathcal{C}^{K}$, the map $\iota^{\mathsf{K}}_{(C,c)}$ is injective.

Finally the universal 2-arrow $\varsigma^{\mathsf{K}}\colon (U^{\mathsf{K}}, \iota^{\mathsf{K}}) \Rightarrow (K, \kappa)(U^{\mathsf{K}}, \iota^{\mathsf{K}})$ as requested in (3) is given by the family ς^{K} given by

$$\varsigma^{\mathsf{K}}_{(C,c)} := c\colon C \to KC, \qquad \text{as } (C, c) \text{ varies amongst the objects in } \mathcal{C}^{K}.$$

One sees immediately that $\varsigma^{\mathsf{K}}\colon U^{\mathsf{K}} \stackrel{\cdot}{\to} KU^{\mathsf{K}}$. It determines an appropriate 2-arrow in **Dtn** because for any $\alpha \in P^{\mathsf{K}}(C, c)$, by definition of $P^{\mathsf{K}}(C, c)$ one has that

$$\alpha \leq Pc(\kappa_C(\alpha)) = \left(P\varsigma^{\mathsf{K}}_{(C,c)} \circ \kappa_{\left(U^{\mathsf{K}}\right)^{\mathrm{op}}_{(C,c)}}\right)(\alpha)$$

After introducing the dramatis personæ, we are ready to prove the characterisation of the Eilenberg–Moore construction for a comonad in **Dtn**.

Theorem 6.4. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ be a doctrine and K a comonad on P. Then*

$$\begin{tikzcd} & P \arrow[dd, "{(K,\kappa)}"] \\ P^{\mathsf{K}} \arrow[ur, "{(U^{\mathsf{K}},\iota^{\mathsf{K}})}"] \arrow[dr, "{(U^{\mathsf{K}},\iota^{\mathsf{K}})}"'] \arrow[r, phantom, "\Rightarrow \varsigma^{\mathsf{K}}"] & \\ & P \end{tikzcd}$$

is the Eilenberg–Moore construction for K *in* **Dtn**.

Proof. We begin the proof analysing the data for the 2-problem in Definition 3.1(iii): one has an arbitrary doctrine $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ and a diagram of 1-arrows and 2-arrows in **Dtn**

$$\begin{tikzcd} & P \arrow[dd, "{(K,\kappa)}"] \\ Q \arrow[ur, "{(X,x)}"] \arrow[dr, "{(X,x)}"'] \arrow[r, phantom, "\Rightarrow \xi"] & \\ & P \end{tikzcd} \tag{12}$$

where the pair $((X,x),\xi)$ satisfies the two commutativity conditions in (4). These translate precisely in the commutative diagrams of natural transformations

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "\xi"'] & KX \arrow[d, "\mu X"] \\ KX \arrow[r, "K\xi"'] & KKX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, "\mathrm{id}_X"'] & KX \arrow[d, "\nu X"] \\ & X \end{tikzcd} \tag{13}$$

while the condition on the 2-arrow in (12) requires that the natural transformation $\xi\colon X \stackrel{\bullet}{\to} KX$ is such that, for every object D in $\mathcal{D}$ and $\beta \in Q(D)$, we have

$$x_D(\beta) \le P(\xi_D)(\kappa_{X(D)}(x_D(\beta))). \tag{14}$$

In turn, the commutativity of the two diagrams (13) is equivalent to requiring that, for every object D in $\mathcal{D}$, there is a structure of coalgebra $(X(D),\xi_D)$ for the comonad (K,μ,ν) on the object $X(D)$ in the category $\mathcal{C}$, and that, for every arrow $f\colon D \to D'$ in $\mathcal{D}$, the arrow $X(f)\colon (X(D),\xi_D) \to (X(D'),\xi_{D'})$ is a homomorphism of coalgebras. At the same time, condition (14) is equivalent to requiring that the monotone function $x_D\colon Q(X(D)) \to P(X(D))$ factors through

$$\begin{tikzcd} Q(X(D)) \arrow[rr, "x_D"] \arrow[dr, "x_D"'] & & P(X(D)) \\ & P^{\mathsf{K}}(X(D),\xi_D) \arrow[ur, hook] & \end{tikzcd}$$

Hence the data for the 2-problem determine precisely a 1-arrow $(\overline{(X,\xi)},x)\colon Q \to P^{\mathsf{K}}$ ensuring uniqueness, and it is immediate to check that the required diagram commutes.

Similarly, for an arrow $\gamma\colon ((X,x),\xi) \to ((Y,y),\upsilon)$ of the 2-problem, that is, a 2-arrow $\gamma\colon (X,x) \Rightarrow (Y,y)$ in **Dtn**, the commutative diagram (5) determines precisely a natural transformation $\overline{\gamma}\colon \overline{(X,\xi)} \stackrel{\bullet}{\to} \overline{(Y,\upsilon)}$; the inequality encoded in the 2-arrow $\gamma\colon (X,x) \Rightarrow (Y,y)$ in **Dtn** is the same as that encoded in the 2-arrow $\overline{\gamma}\colon (\overline{(X,\xi)},x) \Rightarrow (\overline{(Y,\upsilon)},y)$ in **Dtn**. □

Corollary 6.5. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ be a doctrine and $\mathsf{K} = (K,\kappa,\mu,\nu)$ be a comonad on P. Then there is an adjunction $\mathbb{A}^{\mathsf{K}} = (P^{\mathsf{K}}, P, U^{\mathsf{K}}, \iota^{\mathsf{K}}, \hat{K}, \kappa, \eta^{\mathsf{K}}, \nu)$ between P^{K} and P.*

Proof. It follows from Theorem 6.4 and general results in Street (1972). But we make explicit each component of the adjunction as is obtained from the general case. Amongst the data determining the adjunction, only two may need to be described: the functor $\hat{K}\colon \mathcal{C} \to \mathcal{C}^K$ is the free coalgebra functor and gives, for an object X in $\mathcal{C}$, the free coalgebra $\hat{K}X = (KX,\mu_X)$. The natural transformation is the canonical embedding of a coalgebra into the free coalgebra $\eta^{\mathsf{K}}\colon \mathrm{Id}_{\mathcal{C}^K} \stackrel{\bullet}{\to} \hat{K}U^{\mathsf{K}}$ defined as $\eta^{\mathsf{K}}_{(X,c)} = c$. □

In fact, in the general 2-adjunction between comonads and adjunctions in a 2-category $\mathcal{K}$ when $\mathcal{K}$ admits the Eilenberg–Moore construction, as in diagram (6), we know that the Eilenberg–Moore construction gives the right 2-adjoint from the 2-category $\mathsf{Cmd}(\mathcal{K})$ of comonads in $\mathcal{K}$. So we briefly collect the data for the 2-category $\mathsf{Cmd}(\mathbf{Dtn})$ in order to apply that result in the present situation. The 2-category $\mathsf{Cmd}(\mathbf{Dtn})$ has

objects which are pairs (P,K) where P is a doctrine and K is a comonad on P;

1-arrows from (P,K) to (Q,J), with $\mathsf{K} = (K,\kappa,\mu^{\mathsf{K}},\nu^{\mathsf{K}})$ and $\mathsf{J} = (J,\psi,\mu^{\mathsf{J}},\nu^{\mathsf{J}})$, consist of a 1-arrow $(F,f)\colon P \to Q$ and a 2-arrow $\theta\colon (FK,(fK^{\mathrm{op}})\kappa) \Rightarrow (JF,(\psi F^{\mathrm{op}})f)$ in **Dtn** such that the following diagrams of functors and natural transformations commute:

$$\begin{tikzcd} FK \arrow[r, "\theta"] \arrow[dr, "F\nu^{\mathsf{K}}"'] & JF \arrow[d, "\nu^{\mathsf{J}}F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} FK \arrow[rr, "\theta"] \arrow[d, "F\mu^{\mathsf{K}}"'] & & JF \arrow[d, "\mu^{\mathsf{J}}F"] \\ FK^2 \arrow[r, "\theta K"'] & JFK \arrow[r, "J\theta"'] & J^2F \end{tikzcd}$$

2-arrows from $((F,f),\theta)$ to $((G,g),\zeta)$, which are 1-arrows from (P,K) to (Q,J), with $\mathsf{K}=(K,\kappa,\mu^{\mathsf{K}},\nu^{\mathsf{K}})$ and $\mathsf{J}=(J,\psi,\mu^{\mathsf{J}},\nu^{\mathsf{J}})$, consist of a 2-arrow $\alpha\colon(F,f)\Rightarrow(G,g)$ such that the following diagram of functors and natural transformations commutes

$$\begin{tikzcd} FK \arrow[r, "\alpha K"] \arrow[d, "\theta"'] & GK \arrow[d, "\zeta"] \\ JF \arrow[r, "J\alpha"'] & JG \end{tikzcd}$$

The instance of diagram (6) which we have been addressing is the following:

$$\begin{tikzcd} \mathbf{Dtn} \arrow[r, bend left, "\mathrm{Inc}"] \arrow[r, phantom, "\perp"] & \mathsf{Cmd}(\mathbf{Dtn}) \arrow[l, bend left, "\mathrm{EM}"] \arrow[r, bend right, "\mathrm{EMA}"'] \arrow[r, phantom, "\perp"] & \mathsf{Adj}(\mathbf{Dtn}) \arrow[l, bend right, "\mathrm{Cmd}"'] \end{tikzcd}$$

Since by Corollary 5.11 every adjunction between doctrines induces an interior operator, via EMA one obtains an interior operator also from a comonad.

Proposition 6.6. *Let $P\colon \mathcal{C}^{\mathrm{op}}\to\mathcal{P}os$ be a doctrine and $\mathsf{K}=(K,\kappa,\mu,\nu)$ a comonad on P. Then, the natural transformation $\square^{\mathsf{K}}\colon PU^{\mathsf{K}}\dot{\to}PU^{\mathsf{K}}$, defined, for each coalgebra (X,c) in $\mathcal{C}^{K}$, by $\square^{\mathsf{K}}_{(X,c)}=Pc\circ\kappa_X$, is an interior operator on $PU^{\mathsf{K}\mathrm{op}}\colon\mathcal{C}^{K}\to\mathcal{P}os$.*

Proof. By Corollary 6.5, $(U^{\mathsf{K}},\iota^{\mathsf{K}},\hat{K},\kappa,\eta^{\mathsf{K}},\nu)$ is an adjunction between P^{K} and P. By Corollary 5.11, $\square^{\mathsf{K}}=\iota^{\mathsf{K}}\cdot(P^{\mathsf{K}}\eta^{\mathsf{K}})\cdot(\kappa U^{\mathsf{K}})$ is an interior operator on $PU^{\mathsf{K}\mathrm{op}}\colon\mathcal{C}^{K\mathrm{op}}\to\mathcal{P}os$, but, for each coalgebra (X,c) in $\mathcal{C}^{K}$, $\eta^{\mathsf{K}}_{(X,c)}=c$ and $U^{\mathsf{K}}(X,c)=X$, $P^{\mathsf{K}}c=Pc$ by definition, and ι^{K} is an inclusion. $\square$

Example 6.7. An interesting case of Proposition 6.6 is that of toposes of presheaves as models of first order modal logic. We have already seen in Example 5.12 how one obtains an interior operator

$$\begin{tikzcd} {[\mathcal{C}^{\mathrm{op}},\mathcal{S}et]^{\mathrm{op}}} \arrow[rr, "\mathsf{Sub}_{[\mathcal{C}_0{}^{\mathrm{op}},\mathcal{S}et]}L^{\mathrm{op}}"] \arrow[dr, "L^{\mathrm{op}}"'] & & \mathcal{P}os \\ & {[\mathcal{C}_0{}^{\mathrm{op}},\mathcal{S}et]} \arrow[ur, "\mathsf{Sub}_{[\mathcal{C}_0{}^{\mathrm{op}},\mathcal{S}et]}"'] & \end{tikzcd}$$

on the category of presheaves $[\mathcal{C}^{\mathrm{op}},\mathcal{S}et]$ from the adjunction which is the geometric morphism

$$\begin{tikzcd} {[\mathcal{C}_0{}^{\mathrm{op}},\mathcal{S}et]} \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & {[\mathcal{C}^{\mathrm{op}},\mathcal{S}et]} \arrow[l, bend left, "-\circ i^{\mathrm{op}}"] \end{tikzcd} \tag{15}$$

where $\mathcal{C}_0$ denotes the discrete category of the objects of $\mathcal{C}$ and $i\colon\mathcal{C}_0\to\mathcal{C}$ is the inclusion functor. But the category of presheaves is exactly the category of coalgebras for the comonad determined by the adjunction (15), see Johnstone (2002); so Proposition 6.6 applies, and the modal operator obtained on a presheaf model is obtained directly from the subobject doctrine on $[\mathcal{C}_0{}^{\mathrm{op}},\mathcal{S}et]]$ and the geometric morphism that determines the presheaves as coalgebras.

7. The Global Picture

Proposition 5.13 produces a construction of an interior operator from adjunctions as a 2-functor AM: $\mathsf{Adj}(\mathbf{Dtn}) \to \square\text{-}\mathbf{Dtn}$. And Proposition 6.6 describes the action of the composition CM in the diagram

$$\begin{tikzcd}
\mathsf{Cmd}(\mathbf{Dtn}) \arrow[drr, "\mathrm{CM}"'] \arrow[rr, bend right, "\mathrm{EMA}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Adj}(\mathbf{Dtn}) \arrow[ll, bend right, "\mathrm{Cmd}"'] \arrow[d, "\mathrm{AM}"] \\
& & \square\text{-}\mathbf{Dtn}.
\end{tikzcd}$$

The goal of this section is to complete the above diagram, by showing that AM is part of a local adjunction, see Betti and Power (1988). Hence so is CM.

We start by comparing the 2-functor AM to the composite $\mathrm{CM} \circ \mathrm{Cmd}$, both constructing a doctrine with interior operator from an adjunction in **Dtn**. They do not coincide, but can be canonically compared by a 2-natural transformation. Recall that CM maps a comonad (P, K), for $\mathsf{K} = (K, \kappa, \mu^{\mathsf{K}}\ \nu^{\mathsf{K}})$, to the doctrine with an interior operator $(P(U^{\mathsf{K}})^{\mathrm{op}}, \square^{\mathsf{K}})$ where $\square^{\mathsf{K}}_{(X,c)} = Pc \cdot \kappa$.

Since AM is a 2-functor, its action on the unit of the 2-adjunction $\mathrm{Cmd} \dashv \mathrm{EMA}$ produces a natural comparison $\mathrm{AM}(\mathbb{A}) \to \mathrm{CM}(\mathrm{Cmd}(\mathbb{A}))$ for $\mathbb{A} = (P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ an adjunction in **Dtn**.

Indeed, let $\mathsf{K} := \mathrm{Cmd}(\mathbb{A}) = (LR, (\lambda R^{\mathrm{op}})\rho, L\eta R, \epsilon)$ be the induced comonad on Q. The component of the unit of the 2-adjunction on $\mathbb{A}$ is given by the 1-arrow $(K, k, \mathrm{Id}, \mathrm{id}, \mathrm{id})\colon \mathbb{A} \to \mathrm{EMA}(\mathsf{K})$, where $(K, k)\colon P \to Q^{\mathsf{K}}$ is the comparison 1-arrow given by the Eilenberg–Moore construction. The 1-arrow (K, k) is obtained by the universal property of Q^{K} applied to the following diagram:

$$\begin{tikzcd}
& Q \arrow[d, "{(LR, (\lambda R^{\mathrm{op}})\rho)}"] \\
P \arrow[ur, "{(L, \lambda)}"] \arrow[dr, "{(L, \lambda)}"'] \arrow[r, phantom, "\Nearrow L\eta"] & Q
\end{tikzcd}$$

More explicitly, (K, k) is defined as follows: $KX := (LX, L\eta_X)$, for each object X in the base category of P, $Kf := Lf$, for each arrow in the base category of P, and $k = \lambda$. This is well-defined thanks to the following chain of inequalities:

$$\lambda_X \leq \lambda_X \circ P\eta_X \circ \rho_{LX} \circ \lambda_X = Q(L\eta_X) \circ ((\lambda R^{\mathrm{op}}) \cdot \rho)_{LX} \circ \lambda_X.$$

Proposition 7.1. *Let $\mathbb{A} = (P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ be an adjunction in* **Dtn**, *and consider $\mathsf{K} := (LR, (\lambda R^{\mathrm{op}})\rho, L\eta R, \epsilon)$ the associated comonad on the doctrine Q. Let (K, k) be the comparison 1-arrow. Then, $(K, \mathrm{id})\colon (QL^{\mathrm{op}}, \square^{\mathbb{A}}) \to (Q(U^{\mathsf{K}})^{\mathrm{op}}, \square^{\mathsf{K}})$ is a 1-arrow in* **Dtn** *and $\square^{\mathbb{A}} = \square^{\mathsf{K}} K$.*

Proof. It is immediate since, for each object X, $\square^{\mathbb{A}}_X = \lambda_X P\eta_X \rho_{LX} = QL\eta_X \lambda_{RLX} \rho_{LX} = \square^{\mathsf{K}}_{KX}$. $\square$

Finally, let us note that this comparison 1-arrow is a component of a 2-natural transformation, obtained by postcomposition of the unit of the 2-adjunction $\mathrm{Cmd} \dashv \mathrm{EMA}$ with the 2-functor AM.

In order to show that AM is part of a local adjunction, We start by constructing a comonad from an object $(P, \square)$ in $\square$-**Dtn**.

Proposition 7.2. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ be a doctrine and $\square\colon P \xrightarrow{\cdot} P$ be an interior operator on P. Then, $(\mathrm{Id}_{\mathcal{C}}, \square, \mathrm{id}, \mathrm{id})$ is a comonad on P.*

Proof. There is only to check that $\mathrm{id}\colon (\mathrm{Id}_{\mathcal{C}}, \square) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \mathrm{id})$ and $\mathrm{id}\colon (\mathrm{Id}_{\mathcal{C}}, \square) \Rightarrow (\mathrm{Id}_{\mathcal{C}}, \square \cdot \square)$ are well-defined 2-arrows. But, for each object X in $\mathcal{C}$, $\square_X \leq \mathrm{id}_{PX}$ and $\square_X \leq \square_X \cdot \square_X$ hold by Definition 2.1. $\square$

In other words, Proposition 7.2 shows that an interior operator on a doctrine P is exactly a vertical comonad on it.

We introduce the 2-functor $\mathrm{MC}\colon \square\text{-}\mathbf{Dtn} \to \mathsf{Cmd}(\mathbf{Dtn})$ by letting, for $(P, \square)$ a doctrine with interior operator, $\mathrm{MC}((P,\square)) := (P, \mathrm{Id}, \square, \mathrm{id}, \mathrm{id})$, which is a comonad by Proposition 7.2; for a 1-arrow $(F,f)\colon (P,\square^P) \to (Q, \square^Q)$ $\mathrm{MC}((F,f)) := (F, f, \mathrm{id})$; for a 2-arrow $\theta\colon (F,f) \Rightarrow (G,g)$ $\mathrm{MC}(\theta) := \theta$.

Proposition 7.3. *With the assignments above,* $\mathrm{MC}\colon \square\text{-}\mathbf{Dtn} \to \mathsf{Cmd}(\mathbf{Dtn})$ *is a 2-functor.*

Proof. The proof is straightforward. The only interesting part is checking that it is well defined on the 1-arrows. Indeed, for each object X in the base category $\mathcal{C}$ of the doctrine P, we have $f_X \cdot \square^P_X \leq \square^Q_{FX} \cdot f_X$, by definition of 1-arrow in $\square$-**Dtn**. And this ensures that $\mathrm{id}\colon (F, f \cdot \square^P) \Rightarrow (F, (\square^Q F^{\mathrm{op}}) \cdot f)$ is a 2-arrow in **Dtn**. $\square$

It is easy to see that the 2-functor MC is full and faithful. Hence the 2-category $\square$-**Dtn** is isomorphic to the 2-category of vertical comonads in **Dtn**.

Now let $\mathrm{MA}\colon \square\text{-}\mathbf{Dtn} \to \mathsf{Adj}(\mathbf{Dtn})$ be the composition $\square\text{-}\mathbf{Dtn} \xrightarrow{\mathrm{MC}} \mathsf{Cmd}(\mathbf{Dtn}) \xrightarrow{\mathrm{EMA}} \mathsf{Adj}(\mathbf{Dtn})$ which sends an object $(P, \square)$ in $\square$-**Dtn** to the Eilenberg–Moore adjunction of the associated comonad $\mathrm{MC}(P,\square) = (\mathrm{Id}_{\mathcal{C}}, \square, \mathrm{id}, \mathrm{id})$

$$\begin{tikzcd} \square P \arrow[rr, bend left, "{(\mathrm{Id}_{\mathcal{C}}, \iota^{\mathsf{K}})}"] \arrow[rr, phantom, "\perp"] & & P \arrow[ll, bend left, "{(\mathrm{Id}_{\mathcal{C}}, \square)}"] \end{tikzcd}$$

where, from the general construction in (6), the Eilenberg–Moore object $\square P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ for the comonad induced by $\square$ is $\square PX = \left\{\alpha \in PX \,\middle|\, \alpha = \square_X \alpha\right\}$. Also $\square Pf = Pf$, and $\iota^{\mathsf{K}}\colon \square P \stackrel{\cdot}{\to} P$ is the inclusion.

Theorem 7.4. *There is a local adjunction* $\mathrm{MA} \dashv \mathrm{AM}$*, where*

- *the unit* $\Delta\colon \mathrm{Id}_{\square\text{-}\mathbf{Dtn}} \stackrel{\cdot}{\to} \mathrm{AM} \cdot \mathrm{MA}$ *is the identity lax 2-natural transformation, and*
- *the counit* $\nabla\colon \mathrm{MA} \cdot \mathrm{AM} \stackrel{\cdot}{\to} \mathrm{Id}_{\mathsf{Adj}(\mathbf{Dtn})}$ *is given, for an adjunction* $\mathbb{A} = (P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ *where* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ *and* $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$*, by* $\nabla_{\mathbb{A}} = (\mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}}) \cdot (\rho L^{\mathrm{op}}), L, \mathrm{id}, \eta)$*, as in the following diagram*

$$\begin{tikzcd} \square Q L^{\mathrm{op}} \arrow[rr, bend left, "{(\mathrm{Id}_{\mathcal{C}}, \iota^{\mathsf{K}})}"] \arrow[rr, phantom, "\perp"] \arrow[d, "{(\mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}}) \cdot (\rho L^{\mathrm{op}}))}"'] \arrow[drr, phantom, "\searrow\eta"] & & QL^{\mathrm{op}} \arrow[ll, bend left, "{(\mathrm{Id}_{\mathcal{C}}, \square^{\mathbb{A}})}"] \arrow[d, "{(L, \mathrm{id})}"] \\ P \arrow[rr, bend left, "{(L, \lambda)}"] \arrow[rr, phantom, "\perp"] & & Q \arrow[ll, bend left, "{(R, \rho)}"] \end{tikzcd}$$

and, for each 1-arrow $\phi\colon \mathbb{A} \to \mathbb{B}$*,* $\nabla_\phi = (\mathrm{id}, \mathrm{id})$.

Proof. The fact that Δ is a well-defined lax 2-natural transformation is straightforward, since $\mathrm{AM} \cdot \mathrm{MA} = \mathrm{Id}_{\square\text{-}\mathbf{Dtn}}$. We check that $\nabla_{\mathbb{A}}$ is a 1-arrow from $\mathrm{MA}((QL^{\mathrm{op}}, \square^{\mathbb{A}}))$ to $\mathbb{A}$. We have $(L \circ \mathrm{Id}_{\mathcal{C}}, \lambda \cdot (P\eta^{\mathrm{op}}) \cdot (\rho L^{\mathrm{op}})) = (L \circ \mathrm{Id}_{\mathcal{C}}, \mathrm{id} \cdot \iota^{\mathsf{K}})$, since, for each object X in $\mathcal{C}$ and $\alpha \in \square QL^{\mathrm{op}}X$, we have $\lambda_X(P\eta_X(\rho_{LX}(\alpha))) = \square^{\mathbb{A}}_X \alpha = \alpha$, by definition of $\square QL^{\mathrm{op}}$. Then, we have to check that $\eta\colon (\mathrm{Id}_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}}) \cdot (\rho L^{\mathrm{op}}) \cdot \square^{\mathbb{A}}) \Rightarrow (RL, (\rho L^{\mathrm{op}}) \cdot \mathrm{id})$ is a 2-arrow in **Dtn**, but this holds because

$\eta\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cdot} RL$ is a natural transformation and , for each object X in $\mathcal{C}$, $\square^{\mathbb{A}}_X \le \mathrm{id}_{QL^{\mathrm{op}}X}$, hence we get $P\eta_X \circ \rho_{LX} \circ \square^{\mathbb{A}}_X \le P\eta_X \circ \rho_{LX}$.

Now, consider a 1-arrow $\phi = (F, f, G, g, \theta)\colon \mathbb{A} \to \mathbb{B}$ in $\mathsf{Adj}(\mathbf{Dtn})$; hence, we have $\mathrm{MA}(\mathrm{AM}(\phi)) = (F, g(L^{\mathbb{A}})^{\mathrm{op}}, F, g(L^{\mathbb{A}})^{\mathrm{op}}, \mathrm{id})$, and we have to show that

$$\nabla_\phi = (\mathrm{id}, \mathrm{id})\colon (F, f, G, g, \theta) \circ \nabla_{\mathbb{A}} \Rightarrow \nabla_{\mathbb{B}} \circ (F, g(L^{\mathbb{A}})^{\mathrm{op}}, F, g(L^{\mathbb{A}})^{\mathrm{op}}, \mathrm{id})$$

is a 2-arrow in $\mathsf{Adj}(\mathbf{Dtn})$. To this end, it is enough to prove that

$$\mathrm{id}\colon (F, f \cdot (P^{\mathbb{A}}(\eta^{\mathbb{A}})^{\mathrm{op}}) \cdot (\rho^{\mathbb{A}}(L^{\mathbb{A}})^{\mathrm{op}})) \Rightarrow (F, (P^{\mathbb{B}}(\eta^{\mathbb{B}})^{\mathrm{op}}F^{\mathrm{op}}) \cdot (\rho^{\mathbb{B}}(L^{\mathbb{B}})^{\mathrm{op}}F^{\mathrm{op}}) \cdot (g(L^{\mathbb{A}})^{\mathrm{op}}))$$

and

$$\mathrm{id}\colon (GL^{\mathbb{A}}, g(L^{\mathbb{A}})^{\mathrm{op}}) \Rightarrow (L^{\mathbb{B}}F, g(L^{\mathbb{A}})^{\mathrm{op}})$$

are 2-arrows in $\mathbf{Dtn}$, since the other conditions are trivially satisfied as the two components are identities. The second is a 2-arrow since, by definition of 1-arrow in $\mathsf{Adj}(\mathbf{Dtn})$, the equality $GL^{\mathbb{A}} = L^{\mathbb{B}}F$ holds. To see that so is the first, consider the following inequalities for X an object in $\mathcal{C}$:

$$\begin{aligned}
f_X \circ P^{\mathbb{A}}\eta^{\mathbb{A}}_X \circ \rho^{\mathbb{A}}_{L^{\mathbb{A}}X} &= P^{\mathbb{B}}F\eta^{\mathbb{A}}_X \circ f_{R^{\mathbb{A}}L^{\mathbb{A}}X} \circ \rho^{\mathbb{A}}_{L^{\mathbb{A}}X} && f \text{ is natural}\\
&\le P^{\mathbb{B}}F\eta^{\mathbb{A}}_X \circ P^{\mathbb{B}}\theta_{L^{\mathbb{A}}X} \circ \rho^{\mathbb{B}}_{GL^{\mathbb{A}}X} \cdot g_{L^{\mathbb{A}}X} && \theta \text{ is a 2-arrow in } \mathbf{Dtn}\\
&= P^{\mathbb{B}}\eta^{\mathbb{B}}_{FX} \circ \rho^{\mathbb{B}}_{L^{\mathbb{B}}FX} \circ g_{L^{\mathbb{A}}X} && (\theta L^{\mathbb{A}})(F\eta^{\mathbb{A}}) = \eta^{\mathbb{B}}F \text{ and } GL^{\mathbb{A}} = L^{\mathbb{B}}F
\end{aligned}$$

Finally, we have the check the adjunction triangular laws: $(\mathrm{AM}\nabla)(\Delta\mathrm{AM}) = \mathrm{Id}_{\mathrm{AM}}$ and $(\nabla\mathrm{MA})(\mathrm{MA}\Delta) = \mathrm{Id}_{\mathrm{MA}}$. The former holds as $\mathrm{AM}(\nabla_{\mathbb{A}})$ is the identity on $\mathrm{AM}(\mathbb{A})$ for any adjunction $\mathbb{A}$. The latter holds because, for any object $(P, \square)$ in $\square$-$\mathbf{Dtn}$, $\nabla_{\mathrm{MA}((P,\square))}$ is the identity on $\mathrm{MA}((P, \square))$, since $\mathrm{MA}((P, \square))$ is the Eilenberg–Moore adjunction of the comonad $(\mathrm{Id}, \square, \mathrm{id}, \mathrm{id})$ on P. $\square$

Now recall that, by definition, we have $\mathrm{MA} = \mathrm{EMA} \cdot \mathrm{MC}$ and observe that $\mathrm{Cmd} \cdot \mathrm{EMA} = \mathrm{Id}_{\mathsf{Cmd}(\mathbf{Dtn})}$. Hence $\mathrm{MC} = \mathrm{Cmd} \circ \mathrm{MA}$. Therefore, $\mathrm{MC} \dashv \mathrm{CM}$ is a local adjunction, as stated in the following corollary.

Corollary 7.5. *There is a diagram of* (*lax*) *2-adjunctions*

$$\begin{tikzcd}
\mathsf{Cmd}(\mathbf{Dtn}) \arrow[r, bend right, "\mathrm{EMA}"'] \arrow[dr, bend left, "\mathrm{CM}"] & \mathsf{Adj}(\mathbf{Dtn}) \arrow[l, bend right, "\mathrm{Cmd}"'] \arrow[l, phantom, "\perp"] \arrow[d, bend right, "\mathrm{AM}"'] \arrow[d, phantom, "\vdash"] \\
& \square\text{-}\mathbf{Dtn} \arrow[u, bend right, "\mathrm{MA}"'] \arrow[ul, bend left, "\mathrm{MC}"] \arrow[ul, phantom, "\top"]
\end{tikzcd}$$

where the diagonal adjunction is the composite of the other two.

Finally we refine Theorem 5.10, providing a new factorisation through the doctrine $\square QL^{\mathrm{op}}$.

Theorem 7.6. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ be doctrines and consider an adjunction $(L, \lambda, R, \rho, \eta, \epsilon)$ between them. Then, the following diagram (of adjunctions)*

$$
\begin{tikzcd}
 & \Box QL^{\mathrm{op}} \arrow[dl, bend right, "{(\mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}}))}"'] \arrow[d, bend right, "{(\mathrm{Id}_{\mathcal{C}}, \iota^{\mathsf{K}})}"'] \arrow[dr, bend left, "{(L, \iota^{\mathsf{K}})}"'] & \\
P \arrow[ur, bend right, "{(\mathrm{Id}_{\mathcal{C}}, \lambda)}"'] \arrow[ur, phantom, "\top"] \arrow[dr, bend left, "{(\mathrm{Id}_{\mathcal{C}}, \lambda)}"] \arrow[dr, phantom, "\perp"] & & Q \arrow[ul, bend right, "{(R, \Box(Q\varepsilon^{\mathrm{op}}))}"'] \arrow[ul, phantom, "\top"] \arrow[dl, bend left, "{(R, Q\varepsilon^{\mathrm{op}})}"] \arrow[dl, phantom, "\perp"] \\
 & QL^{\mathrm{op}} \arrow[ul, bend left, "{(\mathrm{Id}_{\mathcal{C}}, (P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}}))}"] \arrow[u, bend right, "{(\mathrm{Id}_{\mathcal{C}}, \Box)}"'] \arrow[u, phantom, "\dashv"] \arrow[ur, bend right, "{(L, \mathrm{id})}"'] &
\end{tikzcd}
$$

commutes. Moreover $\lambda\colon P \stackrel{\bullet}{\to} \Box QL^{\mathrm{op}}$ is surjective and $(P\eta^{\mathrm{op}})(\rho L^{\mathrm{op}})\colon \Box QL^{\mathrm{op}} \stackrel{\bullet}{\to} P$ is injective.

Proof. The commutativity of the diagram follows immediately from the definition of $\Box$ and condition (i) in Proposition 5.8 and Theorem 5.10. The fact that, for each object X, the function $\lambda_X\colon PX \to \Box QL^{\mathrm{op}}X$ is surjective and $P\eta_X\rho_{LX}\colon \Box QL^{\mathrm{op}}X \to PX$ is injective, follows from condition (ii) in Proposition 5.8, noting that $\Box_X = \lambda_X \circ P\eta_X \circ \rho_{LX}$ is the identity on $\Box QL^{\mathrm{op}}X$ by definition. $\Box$

Example 7.7. Temporal Logics. Consider the standard powerset doctrine $\mathcal{P}\colon \mathcal{S}et^{\mathrm{op}} \to \mathcal{P}os$, sending a set X to the powerset $\mathcal{P}(X)$ and a function $t\colon X \to Y$ to the inverse image function $t^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$, and a 1-arrow $(F, f)\colon \mathcal{P} \to \mathcal{P}$. Suppose that $F\colon \mathcal{S}et \to \mathcal{S}et$ is an accessible functor, hence it admits a free comonad (cf. Ghani et al. (2001)) $K^F\colon \mathcal{S}et \to \mathcal{S}et$. We recall the construction in the following.

- Given a set A, let $K^FA = \nu X.A \times FX$ be the (underlying set of the) final coalgebra for the functor

$$
\begin{array}{ccc}
\mathcal{S}et & \xrightarrow{A \times F} & \mathcal{S}et \\
X & \longmapsto & A \times FX
\end{array}
$$

  and denote by $\zeta_A\colon K^FA \to A \times F(K^FA)$ the structure map of the final $A \times F$-coalgebra, which is an iso by the Lambek Lemma.
- Since $(\mathrm{id}, \mathrm{pr}_2 \circ \zeta_A)\colon K^FA \to K^FA \times F(K^FA)$ is a $K^FA \times F$-coalgebra, there is a unique $K^FA \times F$-coalgebra homomorphism $\mu^F_A\colon K^FA \to K^FK^FA$ such that the diagram

$$
\begin{tikzcd}
K^FA \arrow[r, "{(\mathrm{id}, \mathrm{pr}_2 \circ \zeta_A)}"] \arrow[d, "\mu^F_A"'] & K^FA \times F(K^FA) \arrow[d, "\mu^F_A \times \mathrm{id}"] \\
K^FK^FA \arrow[r, "\zeta_A"] & K^FK^F \times F(K^FK^FA)
\end{tikzcd}
$$

  commute.
- Let $\nu^F_A\colon K^FA \to A$ be $\nu^F_A = \mathrm{pr}_1 \circ \zeta_A$.

- Given a function $t\colon B \to A$, the function $\zeta_B\colon K^F B \to B \times F(K^F B)$ is a final $B \times F$-coalgebra; let $K^F t\colon K^F B \to K^F A$ be the unique $A \times F$-homomorphism such that the diagram

$$\begin{tikzcd}
K^F B \arrow[r, "\zeta_B"] \arrow[d, "K^F t"'] & B \times F(K^F B) \arrow[r, "t \times \mathrm{id}"] & A \times F(K^F B) \arrow[d, "K^F t"] \\
K^F A \arrow[rr, "\zeta_A"] & & A \times F(K^F A)
\end{tikzcd}$$

commutes.

We can also define a natural transformation $\kappa_f\colon \mathcal{P} \xrightarrow{\cdot} \mathcal{P}\, K^{F^{\mathrm{op}}}$ as follows. Consider a set A and a subset $\alpha \in \mathcal{P}(A)$. We define a function $\phi_\alpha\colon \mathcal{P}\big(K^F A\big) \to \mathcal{P}\big(K^F A\big)$ as $\phi_\alpha(\beta) = \zeta_A^*(\alpha \times f_{K^F A}(\beta))$, which is monotone by construction, hence, since $\mathcal{P}\big(K^F A\big)$ is a complete lattice, by the Knaster–Tarski theorem, ϕ_α has a greatest fixed point, given by $\nu\phi_\alpha = \bigcup \big\{\beta \in \mathcal{P}\big(K^F A\big) \,\big|\, \beta \subseteq \phi_\alpha(\beta)\big\}$.

Define $\kappa_A^f(\alpha)$ as $\nu\phi_\alpha$. This function is monotone, because, if $\alpha \subseteq \beta$, then $\nu\phi_\alpha = \zeta_A^*(\alpha \times f_{K^F A}(\nu\phi_\alpha)) \subseteq \zeta_A^*(\beta \times f_{K^F A}(\nu\phi_\alpha)) = \phi_\beta(\nu\phi_\alpha)$. Thus, by coinduction, we get $\nu\phi_\alpha \subseteq \nu\phi_\beta$, as needed. In order to prove that κ_A^f is natural in A, we have to check that, for each function $t\colon B \to A$ and $\alpha \in \mathcal{P}(A)$, it is the case that $(K^F t)^*(\nu\phi_\alpha) = \nu\phi_{t\,\mathrm{inf}\,(\alpha)}$. First, note that

$$\begin{aligned}
(K^F t)^*(\nu\phi_\alpha) &= (K^F t)^*(\zeta_A^*(\alpha \times f_{K^F A}(\nu\phi_\alpha))) \\
&= (\zeta_A \circ K^F t)^*(\alpha \times f_{K^F A}(\nu\phi_\alpha)) \\
&= ((\mathrm{id} \times FK^F t) \circ (t \times \mathrm{id}) \circ \zeta_B)^*(\alpha \times f_{K^F A}(\nu\phi_\alpha)) \\
&= \zeta_B^*(t^*(\alpha) \times (FK^F t)^*(f_{K^F A}(\nu\phi_\alpha))) \\
&= \zeta_B^*(t^*(\alpha) \times f_{K^F B}((K^F t)^*(\nu\phi_\alpha))) \\
&= \phi_{t^*(\alpha)}((K^F t)^*(\nu\phi_\alpha)).
\end{aligned}$$

Hence, by coinduction, we get $(K^F t)^*(\nu\phi_\alpha) \subseteq \nu\phi_{t^*(\alpha)}$. To prove the other inclusion, we just have to prove that $K^F t[\nu\phi_{t^*(\alpha)}] \subseteq \nu\phi_\alpha$, where $K^F t[\beta]$ denotes the direct image of $\beta \in \mathcal{P}\big(K^F B\big)$ along $K^F t$. To this end, we note that

$$\begin{aligned}
K^F t[\nu\phi_{t^*(\alpha)}] &\subseteq K^F t[\zeta_B^*(t^*(\alpha) \times f_{K^F B}(\nu\phi_{t^*(\alpha)}))] \\
&= K^F t[((t \times \mathrm{id}) \circ \zeta_B)^*(\alpha \times f_{K^F B}(\nu\phi_{t^*(\alpha)}))] \\
&\subseteq \zeta_A^*((\mathrm{id} \times FK^F t)[\alpha \times f_{K^F B}(\nu\phi_{t^*(\alpha)})]) \\
&= \zeta_A^*(\alpha \times FK^F t[f_{K^F B}(\nu\phi_{t^*(\alpha)})]) \\
&\subseteq \zeta_A^*(\alpha \times f_{K^F A}(K^F t[\nu\phi_{t^*(\alpha)}])) \\
&= \phi_\alpha(K^F t[\nu\phi_{t^*(\alpha)}]).
\end{aligned}$$

To check that $\mathsf{K}^F = (K^F, \kappa^F, \mu^F, \nu^F)$ is a comonad on $\mathcal{P}$, it is enough to show the following two inequalities: (1) $\kappa_A^F(\alpha) \subseteq (\nu_A^F)^*(\alpha)$ and (2) $\kappa_A^F(\alpha) \subseteq (\mu_A^F)^*(\kappa_{K^F A}^F(\kappa_A^F(\alpha)))$ for all $\alpha \in \mathcal{P}(A)$.

Ad (1) note that $\alpha \times f_{K^F A}(\nu\phi_\alpha) \subseteq \mathrm{pr}_1^*(\alpha)$. Hence $\nu\phi_\alpha = \zeta_A^*(\alpha \times f_{K^F A}(\nu\phi_\alpha)) \subseteq \zeta_A^*(\mathrm{pr}_1^*(\alpha)) = (\nu_A^F)^*(\alpha)$.

Ad (2) we show $\mu_A^F[\nu\phi_\alpha] \subseteq \nu\phi_{\nu\phi_\alpha}$. First of all, since $\alpha \times f_{K^F A}(\nu\phi_\alpha) \subseteq \mathrm{pr}_2^*(f_{K^F A}(\nu\phi_\alpha))$, we have $\nu\phi_\alpha = \zeta_A^*(\alpha \times f_{K^F A}(\nu\phi_\alpha)) \subseteq (\mathrm{pr}_2 \circ \zeta_A)^*(f_{K^F A}(\nu\phi_\alpha))$. Hence $\nu\phi_\alpha \subseteq \nu\phi_\alpha \cap (\mathrm{pr}_2 \circ$

$\zeta_A)^*(f_{K^F A}(\nu\phi_\alpha)) = (\mathrm{id}, \mathrm{pr}_2 \circ \zeta_A)^*(\nu\phi_\alpha \times f_{K^F A}(\nu\phi_\alpha))$. Therefore

$$\begin{aligned}
\mu^F_A[\nu\phi_\alpha] &\subseteq \mu^F_A[(\mathrm{id}, \mathrm{pr}_2 \circ \zeta_A)^*(\nu\phi_\alpha \times f_{K^F A}(\nu\phi_\alpha))] \\
&\subseteq \zeta^*_{K^F A}((\mathrm{id} \times F\mu^F_A)[\nu\phi_\alpha \times f_{K^F A}(\nu\phi_\alpha)]) \\
&= \zeta^*_{K^F A}(\nu\phi_\alpha \times F\mu^F_A[f_{K^F A}(\nu\phi_\alpha)]) \\
&\subseteq \zeta^*_{K^F A}(\nu\phi_\alpha \times f_{K^F K^F A}(\mu^F_A[\nu\phi_\alpha])) \\
&= \phi_{\nu\phi_\alpha}(\mu^F_A[\nu\phi_\alpha]).
\end{aligned}$$

Thus by coinduction we obtain (2).

Applying the construction in Proposition 6.6, we obtain a comonadic modal operator $\Box^{\mathsf{K}^F}$ on the indexed poset $Q\colon (\mathcal{S}et^{\mathsf{K}^F})^{\mathrm{op}} \to \mathcal{P}os$, mapping a coalgebra (A, c) for the comonad K^F to $\mathcal{P}(A)$ and a coalgebra morphism $t\colon (B, d) \to (A, c)$ to the inverse image function $t^*\colon \mathcal{P}(A) \to \mathcal{P}(B)$. Explicitly, given a coalgebra (A, c) and an element $\alpha \in \mathcal{P}(A)$, we have $\Box^{\mathsf{K}^F}_{(A,c)}\alpha = c^*(\kappa^F_A(\alpha)) = c^*(\nu\phi_\alpha)$.

This setting has a temporal interpretation: given the 1-arrow (F, f), the functor F represents the 'branching type', namely, the branching structure of time, and f lifts formulas to branches. The functor K^F models the whole time structure, that is, the present and all possible futures, generated by the branching type F, and κ^F lifts a formula to time structures, basically, universally quantifying over time, according to f, roughly saying that the formula holds in all possible future branches. Given a coalgebra (A, c) for the comonad K^F, for each $x \in A$, $c(x)$ represents the whole evolution of x along time, hence, for each $\alpha \in \mathcal{P}(A)$, we have $x \in \Box^{\mathsf{K}^F}_{(A,c)}\alpha$ if all future evolutions of x belongs to α. Therefore, roughly, $\Box^{\mathsf{K}^F}$ is a generic kind of 'always' modality, typical of temporal logics. In the following we consider two explicit instances of this situation.

Example 7.8. Linear time. Consider $(F, f) = (\mathrm{Id}, \mathrm{id})$, that is, each instant has exactly one possible future. The free comonad is the stream comonad $\mathsf{Str}A = \nu X.A \times X = A^\omega$, mapping a set A to the set A^ω of sequences of elements in A indexed over natural numbers. Given a sequence $a \in A^\omega$, we write s_i to denote the ith element of s, and $s[i..]$ to denote the sequence $r \in A^\omega$ such that $r_j = s_{j+i}$ for all $j \in \mathbb{N}$. Then, the counit maps s to s_0 (the first element, namely the present) and the comultiplication maps s to the sequence $(s[i..])_{i\in\mathbb{N}}$, namely the sequence of all suffixes of s.

Let $\alpha \in \mathcal{P}(A)$, we have $\kappa^F_A(\alpha) = \{s \in A^\omega \mid s_i \in \alpha \text{ for all } i \in \mathbb{N}\}$, namely, the set of sequences where all elements belongs to/satisfies α. Therefore, if (A, c) is a coalgebra for Str, $\Box^{\mathsf{Str}}_{(A,c)}\alpha = \{x \in A \mid c(x)_i \in \alpha \text{ for all } i \in \mathbb{N}\}$, that is, it is the set of all elements $x \in A$ such that all its future instances (including the present one) belongs to α.

Therefore, $\Box^{\mathsf{Str}}_{(A,c)}$ provides a model for the 'globally' (**G**) modality of Linear Temporal Logic (LTL) Baier and Katoen (2008) and, moreover, the modality on the free coalgebra $(\mathsf{Str}A, \mu^{\mathsf{Str}}_A)$ implements exactly the standard semantics of such a modality on infinite sequences.

Example 7.9. Finitely ordered branching time. Let $F\colon \mathcal{S}et \to \mathcal{S}et$ be the functor $FX = \bigcup_{n\in\mathbb{N}} X^n$. We can consider several natural transformations $f\colon \mathcal{P} \dot{\to} \mathcal{P}\, F^{\mathrm{op}}$ making (F, f) a 1-arrow. The two paradigmatic examples are the following: $f^\forall_A(\alpha) = \{(n, (x_1, \ldots, x_n)) \in FX \mid x_i \in \alpha \text{ for all } i \in 1..n\}$ and $f^\exists_A(\alpha) = \{(n, (x_1, \ldots, x_n)) \in FX \mid x_i \in \alpha \text{ for some } i \in 1..n\}$.

The free comonad is Tr, mapping a set A to the set of finitely branching and ordered trees labelled by A. Formally, such a tree is a partial function $t\colon \mathbb{N}^\star \rightharpoonup A$ with a non-empty and prefix-closed domain such that, if $(k_1, \ldots, k_n) \in \mathsf{dom}t$ and $k \le k_n$, then $(k_1, \ldots, k) \in \mathsf{dom}t$ (cf. Aczel et al. (2003); Courcelle (1983)). The counit maps a tree t to the label of its root, that is $t(\varepsilon)$, where ε is the empty sequence, and the comultiplication maps a tree t to $\mu^F_A(t)$ such that

$\mathsf{dom}\mu_A^T(t) = \mathsf{dom}t$ and $\mu_A^F(t)(u)$ is the subtree of t rooted at $u \in \mathsf{dom}t$. The behaviour of the natural transformation κ^F of course depends on f, for instance, for $f = f^\forall$, it maps $\alpha \in \mathcal{P}(A)$ to the set of trees where all nodes have label in α, while for $f = f^\exists$, it maps $\alpha \in \mathcal{P}(A)$ to the set of trees containing an infinite path starting from the root where all nodes have label in α.

Then, given a coalgebra (A, c) for the comonad Tr and $\alpha \in \mathcal{P}(A)$, we have $x \in \Box^{\mathsf{Tr}}_{(A,c)}\alpha$ if all nodes in $c(x)$ have label in α, when $f = f^\forall$, and if there is an infinite path in $c(x)$ where all nodes have label in α, when $f = f^\exists$. Therefore, $\Box^{\mathsf{Tr}}_{(A,c)}$ provides a model for the modalities 'invariantly' (**AG**) and 'potentially always' (**EG**) of Computation Tree Logic (CTL) Baier and Katoen (2008), depending on the choice of f.

Acknowledgements. The authors would like to thank Jacopo Emmenegger, Fabio Pasquali and Cosimo Perini Brogi for many helpful discussions on the subject, and the two referees for their useful comments.

Notes

1 There are many reasonable 2-categories whose objects are adjunctions in $\mathcal{K}$. In this paper, the 2-category $\mathsf{Adj}(\mathcal{K})$ we introduce is the one that gives rise to the 2-adjunction with $\mathsf{Cmd}(\mathcal{K})$.
2 In the following, we may sometime refer to a doctrine as a pair $(\mathcal{C}, P)$ in order to make the base $\mathcal{C}$ of the doctrine conspicous.
3 Many notions in this paper can be phrased using the language of 2-fibrations, but with the hope to keep the presentation more accessible, we shall just highlight the connection in a few important cases.

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Appendix A. Interior Operators from Linear–Non-Linear Adjunctions

A well-known approach to provide categorical semantics to the linear exponential modality ! – read as 'bang' – of propositional linear logic is by means of linear–non-linear adjunctions as in Benton (1994). A ***linear–non-linear adjuction*** is a monoidal adjunction beween a symmetric monoidal category and a cartesian category; the induced comonad on the symmetric monoidal category interprets the bang modality. This notion is easily extended to doctrines where the construction in Corollary 5.11 provides a model of the bang modality in a higher order setting.

In the present context, the role of the cartesian category is played by a ***primary doctrine***, that is, a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ where $\mathcal{C}$ has finite products and, for each object X in $\mathcal{C}$, the fibre PX carries an inf-semilattice structure preserved by reindexing, see *e.g.* Emmenegger et al. (2020). The symmetric monoidal category turns into a (*symmetric*) *monoidal doctrine*, which we define below, following the definition of monoidal indexed categories in Moeller and Vasilakopoulou (2020). We shall employ the 2-cartesian structure of the 2-category **Dtn**. So, in the following, given indexed posets $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$, we denote by $P \times Q\colon (\mathcal{C} \times \mathcal{D})^{\mathrm{op}} \to \mathcal{P}os$ the *product* doctrine mapping a pair of objects (X, Y) to the product (in $\mathcal{P}os$) $PX \times QY$ and acting similarly on arrows. Furthermore, we denote by $\mathbf{1}$ the *terminal* doctrine whose base is the terminal category and mapping its unique object to the singleton poset. We shall write $\alpha_{P_1,P_2,P_3}\colon P_1 \times (P_2 \times P_3) \to (P_1 \times P_2) \times P_3$, $\lambda_P\colon \mathbf{1} \times P \to P$, $\rho_P\colon \mathbf{1} \times P \to P$, and $\sigma_{P_1,P_2}\colon P_1 \times P_2 \to P_2 \times P_1$ for the usual 1-iso for associativity, left and right identity, and symmetry.

A (***symmetric***) ***monoidal doctrine*** consists of

- a doctrine $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$,
- two 1-arrows $(\otimes, \bullet)\colon Q \times Q \to Q$ and $(I, \iota)\colon \mathbf{1} \to Q$, and
- four invertible 2-arrows

$$a\colon (\otimes, \bullet) \circ ((\otimes, \bullet) \times (\mathrm{Id}, \mathrm{id})) \circ \alpha_{Q,Q,Q} \Rightarrow (\otimes, \bullet) \circ ((\mathrm{Id}, \mathrm{id}) \times (\otimes, \bullet))$$

$$l\colon (\otimes, \bullet) \circ ((I, \iota) \times (\mathrm{Id}, \mathrm{id})) \Rightarrow \lambda_Q \qquad r\colon (\otimes, \bullet) \circ ((\mathrm{Id}, \mathrm{id}) \times (I, \iota)) \Rightarrow \rho_Q$$

$$s\colon (\otimes, \bullet) \circ \sigma_{Q,Q} \Rightarrow (\otimes, \bullet)$$

such that $(\mathcal{D}, \otimes, I, a, l, r, s)$ is a symmetric monoidal category. As the 2-arrows a, l, r and s are invertible, the inequalities they induce on the fibres are actually equalities, namely, the following diagrams commute

$$\begin{tikzcd}
(QA \times QB) \times QC \arrow[r, "(\alpha_{Q,Q,Q})_{A,B,C}"] \arrow[d, "\bullet_{A,B} \times \mathrm{id}"'] & QA \times (QB \times QC) \arrow[r, "\mathrm{id} \times \bullet_{B,C}"] & QA \times Q(B \otimes C) \arrow[d, "\bullet_{A,B\otimes C}"] \\
Q(A \otimes B) \times QC \arrow[r, "\bullet_{A\otimes B,C}"] & Q((A \otimes B) \otimes C) \arrow[r, "Q(a_{A,B,C})"] & Q(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
Q1 \times A \arrow[r, "(\iota, \mathrm{id})"] \arrow[d, "(\lambda_Q)_A"'] & QI \times QA \arrow[d, "\bullet_{I,A}"] \\
QA \arrow[r, "Q(l_A)"] & Q(I \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
QA \times 1 \arrow[r, "(\mathrm{id}, \iota)"] \arrow[d, "(\rho_Q)_A"'] & QA \times QI \arrow[d, "\bullet_{A,I}"] \\
QA \arrow[r, "Q(r_A)"] & Q(A \otimes I)
\end{tikzcd}$$

$$\begin{tikzcd}
QA \times QB \arrow[r, "\bullet_{A,B}"] \arrow[d, "(\sigma_Q)_{A,B}"'] & Q(A \otimes B) \arrow[d, "Q(s_{A,B})"] \\
QB \times QA \arrow[r, "\bullet_{B,A}"] & Q(B \otimes A)
\end{tikzcd}$$

Note that a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ is a monoidal doctrine with $(\times, \sqcap)\colon P \times P \to P$ and $(1, \top_1)\colon \mathbf{1} \to P$, where 1 is the terminal object and $\top_1$ is the top element in $P1$, $\times$ is the binary product in the category and $\sqcap$ is defined, for all objects X, Y in $\mathcal{C}$, by $\sqcap_{X,Y} = \wedge_{X\times Y} \circ (P\pi_1 \times P\pi_2)$, where $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are the projections.

Now, consider a primary doctrine P and a monoidal doctrine Q. An adjunction $(P, Q, L, \lambda, R, \rho, \eta, \epsilon)$ is said to be *monoidal* if L and R are lax monoidal functors and η and ϵ are monoidal natural trasformations, that is, we have the following additional structure:

- two 2-arrows $u\colon (I, \iota) \Rightarrow (L\lambda) \circ (1, \top)$ and $\phi\colon (\otimes, \bullet) \circ ((L, \lambda) \times (L, \lambda)) \Rightarrow (L, \lambda) \circ (\times, \sqcap)$, that is, $u\colon I \to L1$ and, for all objects X, Y in $\mathcal{C}$, $\phi_{X,Y}\colon LX \otimes LY \to L(X \otimes Y)$ are arrows in $\mathcal{D}$, and
- two 2-arrows $v\colon (1, \top) \Rightarrow (R, \rho) \circ (I, \iota)$ and $\psi\colon (\times, \sqcap) \circ ((R, \rho) \times (R, \rho)) \Rightarrow (R, \rho) \circ (\times, \sqcap)$, that is, $v\colon 1 \to RI$ and, for all objects A, B in $\mathcal{D}$, $\psi_{A,B}\colon RA \times RB \to R(A \times B)$ are arrows in $\mathcal{C}$, and
- the following diagrams commute:

$$\begin{tikzcd}
X \times Y \arrow[rr, "\mathrm{id}_{X\times Y}"] \arrow[d, "\eta_X \times \eta_Y"'] & & X \times Y \arrow[d, "\eta_{X\times Y}"] \\
RLX \times RLY \arrow[r, "\psi_{LX,LY}"] & R(LX \otimes LY) \arrow[r, "R\phi_{X,Y}"] & RL(X \times Y)
\end{tikzcd}$$

$$\begin{tikzcd}
LRA \otimes LRB \arrow[r, "\phi_{RA,RB}"] \arrow[d, "\epsilon_A \otimes \epsilon_B"'] & L(RA \times RB) \arrow[r, "L\psi_{A,B}"] & LR(A \otimes B) \arrow[d, "\epsilon_{A\otimes B}"] \\
A \otimes B \arrow[rr, "\mathrm{id}_{A\otimes B}"] & & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "v"] \arrow[d, "\eta_1"'] & RI \arrow[r, "Ru"] & RL1 \\
RL1 \arrow[urr, "\mathrm{id}_{RL1}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "u"] \arrow[drr, "\mathrm{id}_I"'] & L1 \arrow[r, "Lv"] & LRI \arrow[d, "\epsilon_I"] \\
& & I
\end{tikzcd}$$

and the following inequalities on the fibres:

$$\begin{tikzcd}
QA \times QB \arrow[r, "\bullet_{A,B}"] \arrow[d, "\rho_A \times \rho_B"'] & Q(A \otimes B) \arrow[dd, "\rho_{A\otimes B}"] \\
PRA \times PRB \arrow[d, "\sqcap_{RA,RB}"'] \arrow[r, phantom, "\leq"] & \\
P(RA \times RB) & PR(A \otimes B) \arrow[l, "P\psi_{A,B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
PX \times PY \arrow[r, "\sqcap_{X,Y}"] \arrow[d, "\lambda_X \times \lambda_Y"'] & P(X \times Y) \arrow[dd, "\lambda_{X\times Y}"] \\
QLX \times QLY \arrow[d, "\bullet_{LX,LY}"'] \arrow[r, phantom, "\leq"] & \\
Q(LX \otimes LY) & QL(X \times Y) \arrow[l, "Q\phi_{X,Y}"']
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "\iota"] \arrow[d, "\top_1"'] \arrow[dr, phantom, "\leq"] & QI \arrow[d, "\rho_I"] \\
P1 & PRI \arrow[l, "Pv"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\top_1"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\leq"] & P1 \arrow[d, "\lambda_1"] \\
QI & QL1 \arrow[l, "Qu"]
\end{tikzcd}$$

From general results about monoidal adjunctions between categories, we know that u and ϕ are (natural) isos. Hence the inequalities on the left-hand side are equalities, that is, those diagrams commute.

Consider now the doctrine $QL^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$. By Corollary 5.11, there is an interior operator $!\colon QL^{\mathrm{op}} \xrightarrow{\cdot} QL^{\mathrm{op}}$ defined as $! = \lambda \cdot (P\eta^{\mathrm{op}}) \cdot \rho L^{\mathrm{op}}$. However, in this richer context, QL^{op} has a richer structure. First of all $\mathcal{C}$ has finite products, hence, for each object X in $\mathcal{C}$, there are arrows $\zeta\colon X \to 1$ and $\Delta_X\colon X \to X \times X$ natural in X. Then, we can define a monoid structure on $QL^{\mathrm{op}}X$ as the two composite arrows

$$\begin{tikzcd}
1 \arrow[r, "\iota"] \arrow[rrr, bend right=10, "e_X"'] & QI \arrow[r, "Qu^{-1}"] & Q(L1) \arrow[r, "QL\zeta_X"] & Q(LX)
\end{tikzcd}$$

$$\begin{tikzcd}
Q(LX) \times Q(LX) \arrow[r, "\bullet_{LX,LX}"] \arrow[rrr, bend right=5, "*_X"'] & Q(LX \otimes LX) \arrow[r, "Q\phi^{-1}_{X,X}"] & Q(L(X \times X)) \arrow[r, "QL\Delta_X"] & Q(LX).
\end{tikzcd}$$

It follows that $(QL^{\mathrm{op}}X, *_X, e_X)$ is a commutative monoid and that such structure is preserved by reindexing. This structure interprets the multiplicative conjunction of linear logic and its unit. To ensure that ! correctly interprets the ‘bang’ modality of linear logic, four properties, in addition to those of interior operators, are required to hold: for each object X in $\mathcal{C}$ and $\alpha, \beta \in Q(LX)$,

$$(1)\ !_X\alpha \leq e_X \qquad (2)\ !_X\alpha \leq !_X\alpha *_X !_X\alpha$$

$$(3)\ e_X \leq !_X e_X \qquad (4)\ !_X\alpha *_X !_X\beta \leq !_X(\alpha *_X \beta).$$

(1) Note that $P\eta_X(\rho_{LX}(\alpha)) \in PX$, which is an inf-semilattice with top element $\top_X$, hence $P\eta_X(\rho_{LX}(\alpha)) \leq \top_X = P\zeta_X(\top_1)$, because reindexing preserves the inf-semilattice structure. Therefore, we get $!_X\alpha = \lambda_X(P\eta_X(\rho_{LX}(\alpha))) \leq \lambda_X(P\zeta_X(\top_1)) = QL\zeta_X(\lambda_1(\top_1)) = e_X$, by naturality of λ and one of the diagrams above.

(2) Again, note that $P\eta_X(\rho_{LX}(\alpha)) \in PX$, which is an inf-semilattice, hence $P\eta_X(\rho_{LX}(\alpha)) \leq P\eta_X(\rho_{LX}(\alpha)) \wedge_X P\eta_X(\rho_{LX}(\alpha))$. Since $\pi_i \circ \Delta_X = \mathrm{id}_X$, using naturality of $\wedge$, we get

$$\begin{aligned} P\eta_X(\rho_{LX}(\alpha)) &\leq P\Delta_X(P\pi_1(P\eta_X(\rho_{LX}(\alpha))) \wedge_{X\times X} P\pi_2(P\eta_X(\rho_{LX}(\alpha)))) \\ &= P\Delta_X(P\eta_X(\rho_{LX}(\alpha)) \sqcap_{X,X} P\eta_X(\rho_{LX}(\alpha))) \end{aligned}$$

Therefore, applying λ_X and using one of the diagrams above we get

$$\begin{aligned} !_X\alpha &= \lambda_X(P\eta_X(\rho_{LX}(\alpha))) \\ &\leq \lambda_X(P\Delta_X(P\eta_X(\rho_{LX}(\alpha)) \sqcap_{X,X} P\eta_X(\rho_{LX}(\alpha))) \\ &= QL\Delta_X(\lambda_{X\times X}(P\eta_X(\rho_{LX}(\alpha)) \sqcap_{X,X} P\eta_X(\rho_{LX}(\alpha)))) \\ &= \lambda_X(P\eta_X(\rho_{LX}(\alpha))) *_X \lambda_X(P\eta_X(\rho_{LX}(\alpha))) \\ &= !_X\alpha *_X !_X\alpha \end{aligned}$$

(3) By one of the diagrams above, naturality of λ and the fact that reindexing in P preserves the inf-semilattice structure, we have $e_X = \lambda_X(\top_X)$. Furthermore, since $\eta\colon (\mathrm{Id}, \mathrm{id}) \Rightarrow (RL, (\rho L^{\mathrm{op}})\lambda)$ is a 2-arrow in **Dtn**, we get

$$e_X = \lambda_X(\top_X) \leq \lambda_X(P\eta_X(\rho_{LX}(\lambda_X(\top_X)))) = !_X e_X$$

(4) Using the diagrams above and the definitions of $*_X$ and $!_X$ we get

$$\begin{aligned} !_X\alpha *_X !_X\beta &= (\lambda_X(P\eta_X(\rho_{LX}(\alpha)))) *_X (\lambda_X(P\eta_X(\rho_{LX}(\beta)))) \\ &= QL\Delta_X(\lambda_{X\times X}(P\eta_X(\rho_{LX}(\alpha)) \sqcap_{X,X} P\eta_X(\rho_{LX}(\beta)))) \\ &= \lambda_X(P\Delta_X(P(\eta_X \times \eta_X)(\rho_{LX}(\alpha) \sqcap_{RLX,RLX} \rho_{LX}(\beta)))) \\ &\leq \lambda_X(P\Delta_X(P(\eta_X \times \eta_X)(P\psi_{LX,LX}(\rho_{LX\otimes LX}(\alpha \bullet_{LX,LX} \beta))))) \end{aligned}$$

From one of the diagrams above, we have $\psi_{LX,LX} \circ (\eta_X \times \eta_X) = R\phi^{-1}_{X,X} \circ \eta_{X\times X}$, hence we get

$$\begin{aligned} !_X\alpha *_X !_X\beta &\leq \lambda_X(P\Delta_X(P(\eta_X \times \eta_X)(P\psi_{LX,LX}(\rho_{LX\otimes LX}(\alpha \bullet_{LX,LX} \beta))))) \\ &= \lambda_X(P\Delta_X(P\eta_{X\times X}(PR\phi^{-1}_{X,X}(\rho_{LX\otimes LX}(\alpha \bullet_{LX,LX} \beta))))) \\ &= \lambda_X(P\eta_X(\rho_{LX}(QL\Delta_X(Q\phi^{-1}_{X,X}(\alpha \bullet_{LX,LX} \beta))))) \\ &= !_X(\alpha *_X \beta) \end{aligned}$$

Cite this article: Dagnino F and Rosolini G (2021). Doctrines, modalities and comonads. *Mathematical Structures in Computer Science*. https://doi.org/10.1017/S0960129521000207