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Weighted statistics of L-functions

PhD Thesis

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Abstract

This thesis consists of four chapters.

Chapter 1 and 2 are a general introduction to the Riemann zeta function, with a special focus on the theory of moments; no original results are contained here.

In Chapter 3 we study a weighted value distribution of the Riemann zeta function on the critical line. More specifically, assuming the Riemann Hypothesis, we investigate the distribution of $\log|\zeta(1/2+it)|$ *with respect to various tilted measures, proving several weighted analogues of Selberg's central limit theorem. Moreover we prove unconditionally the analogue results in the corresponding random matrix theory setting. These contents first appeared in [54] and [55].*

Chapter 4 is devoted to a weighted version of the one-level density of the non-trivial zeros of L-functions, tilted by a power of the L-function evaluated at the central point. First, we study this problem for the Riemann zeta function, both unconditionally and assuming the ratio conjecture. Then we generalize these ideas for specific families of L-functions with different symmetry types; in particular we consider a symplectic and an orthogonal family of L-functions and, under the relevant ratio conjecture, we study the weighted one-level density of non-trivial zeros of these L-functions.

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I will conclude with you, my family; thanks to my parents, Max and Gloria, and to my grandma Nuccia, I'll find a way to tell you why, directly in Italian. And finally thanks to you, Lisa, because the future is bright, right Pupi?

Contents

Chapter 1

Introduction

1.1 The Riemann zeta function

In the half-plane $\Re(s) > 1$, with $s \in \mathbb{C}$, the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

and represents a fundamental instrument in the understanding of the distribution of prime numbers. Indeed, as a consequence of the unique factorization theorem of integers, one has

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.2}$$

for $\Re(s) > 1$ (originally due to Euler [52], 1737), where the product over p has to be interpreted over all the prime numbers. This identity is called Euler product for the Riemann zeta function and expresses the arithmetical object on the right hand side (a product over primes) in terms of a function of complex variable s, which therefore becames one of our main tools on primes.

This connection is explicit in Riemann's memoir of 1860 [142], where the author proved the basic analytic properties of zeta and made several remarkable conjectures. Riemann showed that ζ, which is a holomorphic function in the half plane of convergence by definition (1.1), has a meromorphic continuation in the whole complex plane, with a unique simple pole at $s = 1$ with residue 1. Moreover it satisfies the functional equation

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) \tag{1.3}$$

giving a symmetry with respect to the central point $s = \frac{1}{2}$. This symmetry allows us to deduce the properties of $\zeta(s)$ for $\Re(s) < 0$ from its properties in the half plane of convergence $\Re(s) > 1$, where the Riemann zeta function is defined by the explicit expression (1.1). For instance, since $\zeta(s)$ has no zeros for $\Re(s) > 1$ by (1.2), then the only zeros of $\zeta(s)$ for $\Re(s) < 0$ are at the points $s = -2n$, $n \in \mathbb{Z}_+$, coming from the poles of the Γ-function. These are called trivial zeros of the Riemann zeta function. What remains mysterious is the so-called critical strip, i.e. the region of the complex plane for $0 \leq \Re(s) \leq 1$. Riemann conjectured that the number of zeros ρ in the critical strip (called non-trivial zeros) such that $0 < \Im(\rho) \leq T$ can be asymptotically evaluated as $T \to \infty$, being

$$\begin{aligned} N(T) :&= |\{\rho \in \mathbb{C} : 0 \leq \Re(\rho) \leq 1, 0 < \Im(\rho) \leq T, \zeta(\rho) = 0\}| \\ &= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \end{aligned} \tag{1.4}$$

This formula has been proved by von Mangoldt [117] in 1905 (after a first version in 1895). Moreover the famous Riemann Hypothesis (RH) speculates that all the non-trivial zeros lie on the critical line, i.e. on the line $\Re(s) = \frac{1}{2}$. Finally Riemann sketched a proof of a formula for the counting function of primes up to x in terms of the non-trivial zeros of ζ. In particular, provided that there are no zeros on the line $\Re(s) = 1$, this formula would imply

$$\pi(x) := \sum_{\substack{p \leq x \\ p \text{ prime}}} 1 \sim \frac{x}{\log x}. \tag{1.5}$$

as $x \to \infty$. Equation 1.5 is an asymptotic formulation of the prime number theorem, proven independently by de la Vallée Poussin [159] and Hadamard [68] in 1896. More details about Riemann's memoir can be found in Edward's [51] and Davenport's [46, Chapter 8] books, while for a full and detailed description of standard properties of the Riemann zeta function we refer to [155].

1.2 Montgomery's pair correlation

In 1973 Montgomery [121] investigated the vertical spacing of the non-trivial zeros of the Riemann zeta function. In particular, assuming RH so that $\rho = \frac{1}{2} + i\gamma$ ($\gamma \in \mathbb{R}$) denotes the generic non-trivial zero of zeta, he studied the distribution of the difference $\gamma - \gamma'$ between the zeros. He defined

$$F(\alpha) = F(\alpha, T) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma') \tag{1.6}$$

where α and $T \ge 2$ are real numbers, $w(u)$ is a weighting function defined by $w(u) = 4/(4+u^2)$ and proved the following result.

Theorem 1.1 (Montgomery)**.** *Assume RH and let $\varepsilon > 0$ be fixed. Then $F(\alpha)$ is real, even and non-negative. Furthermore, uniformly for $0 \le \alpha \le 1-\varepsilon$, as $T \to \infty$, we have*

$$F(\alpha) = \alpha + o(1) + (1+o(1))T^{-2\alpha}\log T.$$

Moreover he remarked (and then proved with Goldston in [62]) that the theorem holds uniformly for $0 \le \alpha \le 1$. The function $F(\alpha)$ allows to get information about sums involving $\gamma - \gamma'$, just by convolving with an appropriate kernel. Indeed, if we consider a test function $r(u) \in L^1$ such that its Fourier transform

$$\widehat{r}(\alpha) = \int_{-\infty}^{+\infty} r(u) e^{-2\pi i\alpha u} du \tag{1.7}$$

is again in L^1, then multiplying (1.6) by $\widehat{r}(\alpha)$ and integrating, we have

$$\sum_{0<\gamma,\gamma'\le T} r\Big((\gamma-\gamma')\frac{\log T}{2\pi}\Big) w(\gamma-\gamma') = \frac{T}{2\pi}\log T \int_{-\infty}^{+\infty} F(\alpha)\widehat{r}(\alpha) d\alpha. \tag{1.8}$$

Since Theorem 1.1 gives us information about $F(\alpha)$ in the range $-1 \le \alpha \le 1$, Montgomery's result allows us to evaluate asymptotically the right hand side of (1.8) provided that the Fourier transform of the test function is supported in $[-1, 1]$. With particular choices of $r(\alpha)$, Montgomery derived interesting consequences about the zeros, such as the result (conditional on RH) which asserts that at least $\frac{2}{3}$ of the non-trivial zeros are simple. In the range $\alpha > 1$, speculating about the uniform distribution of primes in arithmetic progression, Montgomery came to the following conjecture.

Conjecture 1.2 (Strong Pair Correlation)**.** *For any fixed M, then*

$$F(\alpha) = 1 + o(1), \quad \textit{uniformly for } 1 \le \alpha \le M.$$

By assuming Conjecture 1.2, together with Theorem 1.1, one can take full advantage of Equation (1.8), getting

Conjecture 1.3 (Pair Correlation)**.** *For any fixed $\alpha < \beta$, as $T \to \infty$ we have*

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\le T:\\ \frac{2\pi\alpha}{\log T}\le\gamma-\gamma'\le\frac{2\pi\beta}{\log T}}} 1 \sim \int_\alpha^\beta \left[1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right]du+\delta(\alpha,\beta)$$

where $\delta(\alpha,\beta) = 1$ if $0 \in [\alpha,\beta]$ and zero otherwise.

The pair correlation conjecture essentially explains how the presence of a zero at height T influences the presence of another zero nearby. More precisely, if we normalize the non-trivial zeros so that their mean spacing is one, we expect that on average, given a (scaled) zero $\widetilde{\gamma}$, the number of (scaled) zeros which are less than x away from $\widetilde{\gamma}$ is given by

$$\int_{-x}^{x} f(u)du + 1$$

with

$$f(u) := 1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2.$$

Thus the function above is (conjecturally) the pair correlation function of the zeros of the Riemann zeta function (note that the $+1$ is given by the zero $\widetilde{\gamma}$ itself). This was the first context where the connection between the theory of the Riemann zeta function and random matrix theory appeared. Indeed Dyson and Montgomery realized that the function $f(u)$ is also the pair correlation function of the eigenvalues of random complex hermitian matrices in the limit as the size of the matrix tends to infinity (see [50], Equations (6.13) and (9.61)). Moreover, the eigenvalues of these matrices (Gaussian Unitary Ensemble) have the same correlations as the phases of the eigenvalues of unitary matrices of size $N \times N$, scaled by $\frac{2\pi}{N}$, averaged over the Circular Unitary Ensemble, in the limit as $N \to \infty$ (see [50]). This conjectural analogy between non-trivial zeros of zeta and eigenvalues of unitary matrices is also supported numerically [128, 129], as well as by other heuristic methods [107]. Furthermore, this connection can be extended to the n-th correlation function, which has been rigorously computed for restricted ranges (by Hejhal [82] in the case $n = 3$ and by Rudnick and Sarnak [144] for any n) and heuristically calculated without restrictions [13, 14]. From all these speculations, statistical properties of the Riemann zeta function appear to be modeled by properties of characteristic polynomials averaged over the Circular Unitary Ensemble;

following this philosophy, random matrix theory calculations inspired numerous conjectures and theorems about value distribution and moments of ζ, see [105, 109, 150] and many subsequent works.

The connection between number theory and random matrix theory goes beyond, as Katz and Sarnak [105] extended this analogy to L-functions. They introduced the idea of symmetry type for various families of L-functions, which should govern the behaviour of the non-trivial zeros; in particular the statistical properties of the zeros of L-functions in certain families are conjectured to follow the statistics of the eigenvalues of one of the classical compact groups of random matrices, i.e. unitary $U(N)$, symplectic $USp(2N)$ or orthogonal $O(N)$, depending on the specific family considered. This spectral interpetation of the non-trivial zeros was supported by numerical evidence [143] and also by investigations in the function fields case [106]. In particular, this philosophy suggested that the distribution of the "low-lying zeros" (i.e. those which are close to the central point) of certain families of L-functions are governed by the symmetry type of the family, as Iwaniec, Luo and Sarnak [99] proved for a wide variety of families; we refer to Section 1.6 for a more comprehensive account about the low-lying zeros. Moreover, Keating-Snaith [110] and Conrey-Farmer [26] gave evidence that the symmetry type also controls the behaviour of mean values of the L-functions; in Chapter 2 we will analyze in details these ideas and its numerous applications.

1.3 ζ on the critical line

From the discussion in Section 1.1, it looks like the Riemann zeta function is unintelligible in the critical strip. Indeed, Equation (1.2) says that $\zeta(s)$ is equal to an Euler product in the half plane of convergence $\Re(s) > 1$, but this is no longer true to the left of 1 (and in particular on the critical line), since the non-trivial zeros do not reflect Euler product type behavior. Nevertheless, assuming RH, the influence of zeros can be controlled and we still expect that ζ behaves like an Euler product also in the critical strip. As explained by Harper [74, Principle 1.3]:

For many purposes (especially statistical questions not directly involving the zeros of zeta), for $\sigma \geq \frac{1}{2}$ the Riemann zeta function $\zeta(\sigma + it)$ "behaves like" an Euler product of suitable lenght $P = P(\sigma, t)$

$$\prod_{p \leq P} \left(1 - \frac{1}{p^{\sigma+it}}\right)^{-1}.$$

For instance, Gonek, Hughes and Keating [66] proved a formula reflecting the above principle, which under RH looks roughly like

$$\zeta(1/2+it) \approx \prod_{p\leq X}\left(1-\frac{1}{p^{1/2+it}}\right)^{-1} \prod_{|t-\gamma|\leq \frac{1}{\log X}} \left(i(t-\gamma)e^{\gamma}\log X\right) \tag{1.9}$$

with X a parameter, c a positive constant and γ the imaginary part of the non-trivial zeros. Equation (1.9) gives an approximation of the Riemann zeta function on the critical line in terms of a truncated Euler product and (essentially) a truncated Hadamard product. The parameter X governs how much each of the two factors counts (but still it has to be taken in an intermediate range, so that both terms contribute, see [66, page 8] for further details). Although it is very hard to prove rigorous statement corresponding to the above principle, an interesting example is due to Radziwiłł and Soundararajan [135]. In Proposition 3 and 4 of this paper devoted to a new proof of Selberg's central limit theorem, the authors prove that ζ can be written as the exponential of a sum over primes (i.e. an Euler product) if $\Re(s) = \frac{1}{2} + \alpha$, where the shift α is of size essentially of size $\asymp 1/(\log T)$ (actually slightly larger, being $\alpha = W/\log T$ with $W \to \infty$ slowly). In several applications (see e.g. [135] and [2]), such an approximation (just off the critical line) is sufficient to prove results on the critical line too.
Moreover, we remark that Equation (1.9) suggests that for a suitable $X = X(T)$ we have

$$\log|\zeta(1/2+it)| \approx \Re\sum_{p\leq X}\frac{1}{p^{1/2+it}} + (\text{contribution from zeros}) \tag{1.10}$$

which makes the Dirichlet polynomial on the right hand side of (1.10) crucial in the understanding of the behavior of the Riemann zeta function on the critical line. Indeed the contribution coming from the zeros can be bounded (under RH) in several important applications, as in Soundararajan's work [151] about upper bounds for the moments of zeta and Harper's sharp refinement [72]. We then consider the Dirichlet polynomial

$$P_X(t) = \sum_{p\leq X}\frac{1}{p^{1/2+it}} \tag{1.11}$$

and we notice that, for $T \leq t \leq 2T$ and $T \to \infty$, the function $p^{-it} = e^{-it\log p}$ is uniformly distributed on the unit circle, rotating around at speed $\log p$. In addition, all the variables in the collection $(p^{-it})_{p \text{ prime}}$ should behave as if they were independent, since the numbers $\log p$ are linearly independent over $\mathbb{Q}$. We remark that this is no longer true if we consider the functions $(n^{-it})_{n\in\mathbb{N}}$ (for example 6^{-it} is determined by 2^{-it} and 3^{-it}) and we refer to [123, Chapter 7] for the general case. Therefore one expects that, as t varies in $[T,2T]$, the average behaviour of $P_X(t)$ should be the same as that of

$$\sum_{p\le X}\frac{U_p}{p^{1/2}} \tag{1.12}$$

where U_p are independent random variable uniformly distributed on the complex unit circle.

1.4 Selberg's central limit theorem

A central question in the understanding of the Riemann zeta function on the critical line is about its typical size and therefore the value distribution of $|\zeta(1/2+it)|$, as t varies in $[T,2T]$. The discussion at the end of Section 1.3 suggests that $\log|\zeta(1/2+it)|$ should behave as the Dirichlet polynomial $\Re\sum_{p\le X}p^{-1/2-it}$, which can be seen as a sum of many uniform independent random variables. Thus we expect that our random variable $\log|\zeta(1/2+it)|$ converges in distribution to a Gaussian random variable. More specifically, Selberg proved a central limit theorem for $\log|\zeta(1/2+it)|$, showing that it is asymptotically distributed as a Gaussian with mean 0 and variance $\frac{1}{2}\log\log T$, as $T\to\infty$ (see [148] for the first results in this direction and Tsang's thesis [158] for an actual formulation of the central limit theorem).

Theorem 1.4 (Selberg)**.** *Let V be a fixed real number. Then as $T\to\infty$ we have*

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\left\{\frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}}\ge V\right\}=\int_V^\infty e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}+o(1).$$

Analogous statements hold also more generally, for example in the case of the imaginary part of $\log\zeta(1/2+it)$ or for other L-functions (see e.g. [149, 15, 114]). The proof of Selberg's central limit theorem, described in detail in Tsang's PhD thesis [158], is built in the two steps we mentioned before. First, one shows that $\log|\zeta(1/2+it)|$ can be approximated by the real part of the Dirichlet polynomial (1.11), then one proves that $\Re\sum_{p\le X}p^{-1/2-it}$ converges in distribution to a Gaussian random variable by the method of moments. While the second step is rather easy, as the moments of the Dirichlet polynomial can be computed pretty precisely, the first one is typically quite hard. Indeed, as we mentioned before, an approximation like

$$\log|\zeta(1/2+it)| \approx \Re \sum_{p\le X} \frac{1}{p^{1/2+it}} \tag{1.13}$$

cannot be true pointwise, because of the influence of the non-trivial zeros of zeta (the left hand side is not even always defined there!), then something like (1.13) has to be proven on average. To do so, in the classical proof a complicated manipulation invoking zero density estimates for ζ has been performed, whereas recently Radziwiłł and Soundararajan [135] found an easier and shorter way. Roughly speaking, they used the off-line approximation mentioned in Section 1.3 in order to bound the first moment of the difference $|\log|\zeta(1/2+it)| - \Re\sum_{p\le X} p^{-1/2-\alpha-it}|$ for a specific (but flexible) X, with α a small shift. This ensures that the distribution (but not the moments as in the original proof) of $\log|\zeta(1/2+it)|$ is asymptotically the same as that of the Dirichlet polynomial, which is Gaussian by the method of moments. We should also mention that this new method would need substantial modifications if one were interested in $\Im \log\zeta(1/2+it)$, while Tsang's approach works both for the real and imaginary part of log-zeta.

A fundamental question related to Theorem 1.4 is about the uniformity in V; in which range for V does the asymptotic

$$\frac{1}{T} \operatorname*{meas}_{t\in[T,2T]} \left\{ \frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}} \ge V \right\} \sim \int_V^\infty e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \tag{1.14}$$

hold if $V = V(T)$ is a function of the parameter T which goes to infinity as T grows? In other words we are wondering about the large deviation regime of $\log|\zeta(1/2+it)|$, i.e. the large values of ζ on the critical line. Of course this question is strictly related to the problem of moments of the Riemann zeta function, which will be addressed in the next chapter. Although we stated Selberg's central limit theorem only for bounded V, actually the proof in [158] shows the asymptotic (1.14) uniformly in the range $V \ll (\log\log\log T)^{1/2-\varepsilon}$, for some $\varepsilon > 0$. More recently Radziwiłł [133] introduced a new method that extended (1.14) to the large deviation range $V \ll (\log\log T)^{1/10-\varepsilon}$. Furthermore he conjectured that the largest range of uniformity should be $V = o(\sqrt{\log\log T})$, while if $V \sim k\sqrt{\log\log T}$ for some fixed $k > 0$ then (1.14) should fail because of a constant in front, since the left hand side is expected to be asymptotic to a standard Gaussian times a constant depending on k (related to the constant of moments, see Equations (2.3), (2.4) and (2.5)). Even though (1.14) should not be true if $V \gg \sqrt{\log\log T}$, Soundararajan [151] speculates that an upper bound like

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\left\{\frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}}\geq V\right\} \ll \frac{1}{V}\exp\left(-\frac{V^2}{2}\right) \tag{1.15}$$

should hold in a large range for V and he got a weaker form of this bound in the range $V = o(\sqrt{\log\log T}\log\log\log T)$, assuming RH. In particular, in the range $V = k\sqrt{2\log\log T}$ with $k \geq 0$, Soundararajan conditionally proved [151, Corollary B] that

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\big\{\log|\zeta(1/2+it)| \geq k\log\log T\big\} = (\log T)^{-k^2+o(1)} \tag{1.16}$$

while the expected sharp upper bound should be

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\big\{\log|\zeta(1/2+it)| \geq k\log\log T\big\} \ll_k \frac{(\log T)^{-k^2}}{\sqrt{\log\log T}}. \tag{1.17}$$

Here we should also mention that Jutila [101] proved unconditionally that

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\left\{\frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}}\geq V\right\} \ll \exp\left(-\frac{V^2}{2}\left(1+O\left(\frac{V}{\sqrt{\log\log T}}\right)\right)\right)$$

in the range $0 \leq V \leq \sqrt{2\log\log T}$, which immediately implies

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\left\{\frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}}\geq V\right\} \ll \exp\left(-\frac{V^2}{2}\right)$$

for $0 \leq V \ll (\log\log T)^{1/6}$. Note that this has to be compared with the shorter range $V \ll (\log\log T)^{1/10-\varepsilon}$ (then enlarged by Inoue [93, Theorem 4] to $V = o(\log\log T)^{1/6}$), where Radziwiłł [133, Theorem 1] obtained the sharper bound (1.15). Lastly we recall that Soundararajan also proved in [152, Theorem 1] a lower bound for the measure of the set of large values in a specific range.

1.5 The order of magnitude of $\zeta(1/2+it)$

Another central problem is the understanding of the maximal order of magnitude of the Riemann zeta function on the critical line, i.e. the largest values attained by $|\zeta(1/2+it)|$, with $t \in [T, 2T]$. The functional equation implies that [155, Equation (5.1.5)]

$$\zeta(1/2+it) \ll_\varepsilon t^{1/4+\varepsilon} \tag{1.18}$$

for any $\varepsilon > 0$ (which can be removed by using the approximate functional equation for zeta [155, Equation (4.12.4)]). Via Weyl method [162, 163] for a certain type of exponential sums, Hardy and Littlewood improved (1.18) to

$$\zeta(1/2+it) \ll t^{1/6}(\log t)^{3/2}.$$

After many small improvements of this bound [160, 156, 131, 157, 120, 16, 17, 90, 91] (see [155, Chapter 5] and the introduction of [20] for a more comprehensive account of the literature), the current best bound

$$\zeta(1/2+it) \ll_\varepsilon t^{13/84+\varepsilon}$$

(for every $\varepsilon > 0$) is due to Bourgain [20]. Conjecturally, the Lindelöf Hypothesis claims that

$$\zeta(1/2+it) \ll_\varepsilon t^{\varepsilon} \tag{1.19}$$

for all $\varepsilon > 0$, while, assuming RH, Littlewood [116] proved that for some positive constant C (which has been reduced later in [141, 151, 45] up to $C > \log 2/2$)

$$\zeta(1/2+it) \ll \exp\left(C\frac{\log t}{\log\log t}\right). \tag{1.20}$$

Moreover, also Ω-results are known; firstly Titchmarsh [155, Theorem 8.12] proved a lower bound for the maximum size of the Riemann zeta function on long intervals, showing that for any $\alpha < \frac{1}{2}$ and large T

$$\max_{t\in[0,T]} |\zeta(1/2+it)| \geq \exp((\log T)^\alpha)$$

which has been improved over years ([122] under RH and then [6, 152] unconditionally) to

$$\max_{t\in[0,T]} |\zeta(1/2+it)| \geq \exp\left(c\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right)$$

for some constant c and then again to

$$\max_{t\in[0,T]} |\zeta(1/2+it)| \geq \exp\left(c\frac{\sqrt{\log T \log\log\log T}}{\sqrt{\log\log T}}\right) \tag{1.21}$$

for any $c < 1/\sqrt{2}$ by Bondarenko and Seip [18] (see also [19, 44] for improvements in the value of the constant c). As (1.20) and (1.21) do not match, the asymptotic of the maximum of zeta over a long interval is still unclear and debated. Nevertheless, Farmer, Gonek and Hughes [57] conjectured that Equation (1.20) is closer to the truth, being

$$\max_{t\in[0,T]} |\zeta(1/2+it)| \sim \sqrt{\frac{1}{2}\log T \log\log T}. \tag{1.22}$$

Since the conjectural estimate above about the global maximum of zeta is still unproven and also questioned, Fyodorov, Hiary and Keating [58, 59] proposed the study of local maxima, i.e. the maximum order of $|\zeta(1/2+it)|$ in random short intervals of bounded length. By analogy to what happens in the random matrix theory setting, they also conjectured what the answer should be; namely, if t is a uniform random variable in the interval $[T, 2T]$, then

$$\max_{|u|\le 1} \log|\zeta(1/2+it+iu)| = \log\log T - \frac{3}{4}\log\log\log T + X_T \tag{1.23}$$

where the random variable X_T tends to a limiting distribution as $T\to\infty$. This conjecture has been studied intensively in the last years; assuming RH, Najnudel [126] proved the first order of (1.23), i.e. that as $T\to\infty$

$$\max_{|u|\le 1} \log|\zeta(1/2+it+iu)| \sim \log\log T$$

for all $t\in[T,2T]$ with the possible exception of a set of measure $o(T)$ (and also the analogue result for the imaginary part of the logarithm of zeta). This has been proved unconditionally by Arguin, Belius, Bourgade, Radziwiłł and Soundararajan [2] independently. Then Harper [73] got an upper bound for the second order of (1.23), showing that for $t\in[T,2T]$ and for any $g(T)\to\infty$ then

$$\max_{|u|\le 1} \log|\zeta(1/2+it+iu)| \le \log\log T - \frac{3}{4}\log\log\log T + \frac{3}{2}\log_4 T + g(T)$$

with the possible exception of a set of measure $o(T)$ (here $\log_4 T$ denotes $\log\log\log\log T$). Finally, very recently Arguin, Bourgade and Radziwiłł announced the proof of the precise conjectured asymptotic for the maximum of zeta in a typical short interval on the critical line (see [3] for the upper bound, the lower bound is to appear in a subsequent paper, which should be the part II of [3]).

1.6 The density conjecture

Linear statistics of the non-trivial zeros of the Riemann zeta function are crucial in number theory and have been investigated for many years; in this section we give an overview on the one-level density of the non-trivial zeros of the Riemann zeta function and other L-functions. To begin with, we denote the non-trivial zeros of zeta by $\frac{1}{2} + i\gamma$, with $\gamma \in \mathbb{C}$ (that is not assuming RH) and we look at $N(T)$, the number of non-trivial zeros up to height T. As mentioned in Section 1.1, we recall that the classical Riemann-von Mangoldt formula (1.4) studies the behavior of $N(T)$ as $T \to \infty$, that is

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\Big(\frac{1}{T}\Big)$$

with

$$S(T) = \frac{1}{\pi}\arg\zeta\Big(\frac{1}{2} + iT\Big) \ll \log T.$$

In particular, this implies that the density of the non-trivial zeros around T is $\frac{\log T}{2\pi}$. One classically considers a smooth localization of the counting function $N(T)$ at height t; namely, one takes f a real-valued and even function in the Schwartz space and studies for any $t \in [T, 2T]$ the sum

$$N_f(t) := \sum_{\gamma} f\Big(\frac{\log T}{2\pi}(\gamma - t)\Big).$$

Note that we re-scaled the argument, so that $N_f(t)$ is a counting function in intervals of size $\frac{2\pi}{\log T}$, i.e. the mean spacing of the non-trivial zeros. The quantity $N_f(t)$ can be evaluated asymptotically on average as t varies in the interval $[T, 2T]$, getting

$$\frac{1}{T}\int_T^{2T} \sum_{\gamma} f\Big(\frac{\log T}{2\pi}(\gamma - t)\Big)dt = \int_{-\infty}^{+\infty} f(x)dx + O\Big(\frac{1}{\log T}\Big) \tag{1.24}$$

for test functions f such that the support of $\widehat{f}$ is contained in [-2,2] (see e.g. [25]). The same result can be proved for any compactly supported $\widehat{f}$ either with a smooth average over t (see [87]) or under RH.

What we described above is the first example of a more general phenomenon, which is believed to hold in the Selberg class (see [149], [34], [102, 103, 104] and the related papers for an account on this class of L-functions). In Section 1.2 we discussed that it has been conjectured that the limiting statistical properties of eigenvalues of random matrices model the limiting properties of the zeros of L-functions; now we will see that this connection also occurs in computing the one-level density. Assuming GRH, that is the Riemann Hypothesis for the L-functions we are considering, we take $\mathcal{F}$ a "natural" family of L-functions ordered by log-conductor $c(L)$ and we are interested in

$$D(f,\mathcal{F},Y) := \sum_{\substack{L\in\mathcal{F}\\ c(L)\leq \log Y}} \sum_{\gamma_L} f(c(L)\gamma_L) \tag{1.25}$$

where γ_L denotes the imaginary part of a generic non-trivial zero of L. This quantity gives information about the so-called low-lying zeros, i.e. those which are close to the central point, since only the zeros within a distance $\ll 1/c(L)$ of the central point contribute significantly to the sum in (1.25), being f a Schwartz function. The density conjecture [105] is that

$$\lim_{Y\to\infty} \frac{D(f,\mathcal{F},Y)}{\sum_{L\in\mathcal{F},c(L)\leq\log Y} 1} = \int_{-\infty}^{+\infty} f(x) W_{\mathcal{F}}(x)dx \tag{1.26}$$

where $W_{\mathcal{F}}(x)$ is the kernel appearing in the analogous average in the corresponding random matrix theory setting; in particular the distribution of the low-lying zeros should depend only on the "symmetry type" of the family $\mathcal{F}$. For example the family $\{\zeta(s+ia) : a\in\mathbb{R}\}$ is modeled by the unitary group, indeed by (1.24) we know that $W_{\{\zeta\}}(x) = W_U(x) = 1$, which is the one-level density function for the scaled limit of $U(N)$. The same happens for the family of the Dirichlet L-functions modulo q, as for this specific case Hughes and Rudnick [88] proved (1.26) with $W_{\{L_\chi\}}(x) = 1$, for any f such that supp $\widehat{f} \subset [-2,2]$. The family of Dirichlet L-functions associated with real quadratic characters is instead conjectured to be symplectic, indeed Ozluk and Snyder [130] proved that the one-level density function for this family is $W_{USp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}$ for f such that supp $\widehat{f} \subset [-2,2]$, assuming GRH. Also higher degree cases have been studied intensively in literature, for example in [99].

1.7 Outline of this thesis

This thesis is composed of two main original projects, briefly described below.

1.7.1 Weighted value distributions of zeta on the critical line

Chapter 3 is devoted to a weighted analogue of Selberg's central limit theorem; more specifically, for any $m, k \in \mathbb{N}$, we consider the tilted measure

$$|\zeta^{(m)}(1/2+it)|^{2k} dt$$

and we study the value distribution of $\log|\zeta(1/2+it)|$ with respect to this measure. Assuming the Riemann Hypothesis and the asymptotic formula for the twisted and shifted moments of zeta (see Conjecture 2.6), we prove a central limit theorem also in this weighted case, showing that $\log|\zeta(1/2+it)|$ has an asymptotically Gaussian distribution with mean $k \log\log T$ and variance $\frac{1}{2}\log\log T$, as $t \in [T, 2T]$ and $T \to \infty$. In particular, if $m = 0$ and $k = 1$ or 2, our central limit theorem holds assuming the Riemann Hypothesis only, as the twisted moments of zeta are known in these cases.

The proof builds on the classical approximation of $\log|\zeta(1/2+it)|$ by a Dirichlet polynomial, with an error due to the contribution of the zeros of ζ. To control the mean value of the contribution of the zeros with respect the the weighted measure, we appeal to Kirila's and Milinovich's generalization of Gonek's result about the discrete moments of zeta, both conditional on the Riemann Hypothesis. Then, thanks to the assumption of the asymptotic formula for the twisted moments of zeta, we show by the method of moments that the Dirichlet polynomial has an approximately normal distribution with respect to the weighted measure.

Moreover we also tackle the analogous question in the random matrix theory side, where the corresponding weighted central limit theorem can be proved unconditionally.

1.7.2 Weighted one-level density of zeros of L-functions

In Chapter 4 we investigate the weighted one-level density of the non-trivial zeros of the Riemann zeta function and other L-functions. In particular, given a family $\mathcal{F}$ of L-functions, we consider the quantity

$$\sum_{\gamma_L} f(c(L)\gamma_L)$$

where γ_L denotes the imaginary part of a generic zero of L and f is a real-valued and even test function in the Schwartz space, as in the classical one-level density. In this case, however, we average over the family $\mathcal{F}$ tilting by the k-th power of $L(1/2)$, $k \in \mathbb{N}$. Namely, the quantity we consider is

$$\mathcal{D}_k^{\mathcal{F}}(f) = \mathcal{D}_k^{\mathcal{F}}(f, X) := \frac{1}{\sum_{\substack{L \in \mathcal{F} \\ c(L) \leq \log X}} V(L(\frac{1}{2}))^k} \sum_{\substack{L \in \mathcal{F} \\ c(L) \leq \log X}} \sum_{\gamma_L} f(c(L)\gamma_L) V(L(\tfrac{1}{2}))^k$$

as $X \to \infty$, where the function V depends on the symmetry type, being $V(z) = |z|^2$ in the unitary case, $V(z) = z$ for symplectic and orthogonal cases. For three specific families (one for each symmetry type), assuming the Riemann Hypothesis and the ratio conjecture for these families (i.e. the asymptotic formula for the moments of ratios of products of L-functions), we evaluate asymptotically the quantity $S_k^{\mathcal{F}}(f)$ for $k \leq 4$. More precisely we show that also in the weighted case the one-level density has the shape as in the density conjecture (1.26), being

$$S_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^{(k)}(x) dx + O\Big(\frac{1}{\log X}\Big) \tag{1.27}$$

where the kernels $W_G^{(k)}(x)$ only depend on k and on the symmetry type of the family $\mathcal{F}$. We remark that the superscript $^{(k)}$ indicates that we are weighting with the k-th power; in particular W_G^k is not the k-th power of W_G.

The philosophy of the proof follows Conrey-Snaith's work [43]. In the unitary case, for $k = 1$, we also prove (1.27) unconditionally for test functions f whose Fourier transform is supported in $(-\frac{1}{2}, \frac{1}{2})$, building on Hughes-Rudnick's strategy [87].

Chapter 2

The moments of $|\zeta(1/2+it)|$

2.1 The leading order

The average order on the critical line is a classical question in the theory of the Riemann zeta function. This problem, although it is easier than the pointwise one, has been solved only in its most basic forms. For any $k > 0$, we denote with

$$M_k(T) := \frac{1}{T}\int_T^{2T} |\zeta(1/2+it)|^{2k} dt \tag{2.1}$$

the $2k$-th moment of the Riemann zeta function and we look at the the behavior of $M_k(T)$ as $T \to \infty$. In the case $k = 1$, Hardy and Littlewood [71] proved the asymptotic formula for the second moment of zeta

$$M_1(T) \sim \log T \tag{2.2}$$

while the fourth moment case has been solved by Ingham [92], who showed that

$$M_2(T) \sim \frac{1}{2\pi^2}(\log T)^4.$$

For higher moments, such an asymptotic formula is still out of reach. Nevertheless, based on number theoretical arguments, Conrey and Ghosh [35] conjectured an asymptotic for the sixth moment

$$M_3(T) \sim \frac{42}{9!} a_3 (\log T)^9$$

(see (2.4) for the definition of a_k) while Conrey and Gonek [36] for the eighth

$$M_4(T) \sim \frac{24024}{16!} a_4 (\log T)^{16}.$$

Moreover for a general real number $k > 0$, it is believed that $M_k(T) \asymp (\log T)^{k^2}$ should be the correct order and many speculations have been made about the implied constant [31, 33, 79]. By analogy with the moments of characteristic polynomials of unitary matrices, Keating and Snaith [109] finally gave a precise conjecture for the constant of the moments, stating that

$$M_k(T) \sim g_k a_k (\log T)^{k^2} \tag{2.3}$$

where the "arithmetical factor" a_k is defined by

$$a_k := \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{m!\Gamma(k)} \right)^2 p^{-m} \right) \right) \tag{2.4}$$

and the "geometrical factor" g_k by

$$g_k := \frac{G(1+k)^2}{G(1+2k)} \tag{2.5}$$

and G denotes the Barnes G-function. The same conjecture was also later obtained by Diaconu, Goldfeld and Hoffstein [48], with a different approach based on multiple Dirichlet series.

Also the moments of the derivative of the Riemann zeta function have been studied intensively. Ingham [92] proved an asymptotic formula for the second moment, showing that

$$\frac{1}{T}\int_T^{2T} |\zeta'(1/2+it)|^2 dt \sim \frac{1}{3}(\log T)^3$$

and Conrey [22] proved that

$$\frac{1}{T}\int_T^{2T} |\zeta'(1/2+it)|^4 dt \sim \frac{61}{1680\pi^2}(\log T)^8.$$

Again from random matrix theory computation, Conrey, Rubinstein and Snaith [42] conjectured the asymptotic formula

$$\frac{1}{T}\int_T^{2T} |\zeta'(1/2+it)|^{2k} dt \sim a_k b_k (\log T)^{k^2+2k} \tag{2.6}$$

where a_k is the same as in (2.4) and b_k is an explicit constant (see [42, Equation (1.4)] for the precise definition). We note that (2.6) is consistent with Hejhal's paper [83], suggesting that the left hand side is of order

$\asymp (\log T)^{k^2+2k}$. More generally, one can consider the problem of mixed moments of $|\zeta(1/2+it)|$ and $|\zeta'(1/2+it)|$. After the work of Keating and Snaith [109, 110] and Hall [69], Hughes [85] formulated the conjecture

$$\frac{1}{T}\int_T^{2T} |\zeta(1/2+it)|^{2k-2h}|\zeta'(1/2+it)|^{2h}dt \sim c(h,k)(\log T)^{k^2+2h} \tag{2.7}$$

for a certain constant $c(h,k)$, for any h,k real numbers such that $0 \le h < k - \frac{1}{2}$. We recall that Conrey [22] proved (2.7) in the case $h = 1$ and $k = 2$, showing that $C(1,2) = \frac{2}{15\pi^2}$. Recently, Assiotis, Keating and Warren [4] proved the analogue of conjecture (2.7) on the random matrix theory side, establishing the asymptotic of the joint moments of the characteristic polynomial of a random unitary matrix and its derivative, with general exponents.

2.2 Upper and lower bounds

Being asymptotic formulas too hard, strenuous efforts have been made to get bounds for the moments of the Riemann zeta function. The sharp lower bound

$$M_k(T) \gg_k (\log T)^{k^2} \tag{2.8}$$

has been obtained in several cases; classically [155, Theorem 7.19] such a bound was known for $k \in \mathbb{N}$ with a specific smooth averaging over t, then Ramachandra [138] proved (2.8) for $2k \in \mathbb{N}$ and Heath-Brown [79] for all positive rationals k. Conditionally on RH, Ramachandra [137, 140] and independently Heath-Brown [79] proved (2.8) for any positive real number k. In 2013, after Rudnick and Soundararajan's works [146, 147], Radziwiłł and Soundararajan [134] finally proved (2.8) for every real $k > 1$ unconditionally, while the analogue result in the range $0 < k < 1$ is due to the very recent work of Heap and Soundararajan [77]. See also [31, 7, 33] and finally [153] for discussions about the implied constant in (2.8).

On the other hand, obtaining unconditional sharp upper bounds for $M_k(T)$ appears to be much more complicated. Indeed, the Lindelöf Hypothesis (1.19) is equivalent to $M_k(T) \ll_{k,\varepsilon} T^\varepsilon$ for all natural numbers k, which is far from the expected actual order of magnitude $M_k(T) \asymp (\log T)^{k^2}$. Assuming RH, the conditional bound (1.20) trivially gives that

$$M_k(T) \ll \exp\left(2kC\frac{\log T}{\log\log T}\right) \tag{2.9}$$

for some positive constant C. Around 1980, the sharp upper bound

$$M_k(T) \ll_k (\log T)^{k^2} \tag{2.10}$$

has been proved in the limited range $0 \le k \le 2$ by Ramachandra [138, 139] and Heath-Brown [79, 80] under the assumption of RH. Recently Heap, Radziwiłł and Soundararajan [76] removed the assumption of RH in the same range for k, by using the same method as in [136]; before this, (2.10) had been proved unconditionally only for $k = 1/n$ (due to Heath-Brown [79]) and for $k = 1 + 1/n$ (due to Bettin, Chandee and Radziwiłł [11]), $n \in \mathbb{N}$. Moreover Radziwiłł [132] showed (2.10) conditionally on RH for $2 < k < 2 + 2/11$, by using the fourth twisted moment of zeta, i.e. the moment of $|\zeta(1/2+it)|^4$ times the square of a short Dirichlet polynomial (see Section 2.4 for further details about the twisted moments). For a general real $k > 0$, as we mentioned before, we strongly need to rely on RH. The outstanding improvement upon (2.9) is due to Soundararajan [151], who proved that for every positive real number k and every $\varepsilon > 0$ we have

$$M_k(T) \ll_{k,\varepsilon} (\log T)^{k^2+\varepsilon} \tag{2.11}$$

under the assumption of RH. The proof of this result is the prototypical manifestation of what we discussed in Section 1.3. An approximation like (1.10) is the first step of Soundararajan's proof and, thanks to RH, the "contribution from zeros" can be quite well-managed; in particular, since this contribution is negative for every $t \in [T, 2T]$, if one is interested in upper bounds for the moments, then the term coming from the zeros in (1.10) can be ignored, as one gets (roughly speaking) that for any parameter $x \le t$ then

$$\log|\zeta(1/2+it)| \lesssim \Re \sum_{p \le x} \frac{1}{p^{1/2+it}} + O\left(\frac{\log t}{\log x}\right) \tag{2.12}$$

(see [151, Proposition] and [72, Proposition 1]). In addition, being the contribution from zeros also bounded for most $t \in [T, 2T]$, using the inequality (2.12) we don't lose too much, since the ignored term coming from the zeros can be absorbed into the implied constant of the bound for the moments. Therefore, as far as upper bounds for the moments are concerned, one can study $\log|\zeta(1/2+it)|$ just by studying a sum over primes, which is exactly what Harper's principle described in Section 1.3 predicts and inequality (2.12) is a "rigorous" statement corresponding to that principle. The second step of Soundararajan's proof is then getting bounds for the measure of the set of large values of the Dirichlet polynomial (with suitable length) approximating $\log|\zeta(1/2+it)|$, which is expected to show Gaussian behaviour, as explained in Section 1.4. This strategy allowed Soundararajan to prove the quasi-optimal upper bound (2.11) and also suggests that, as he explicitely remarks in [151], the dominant contribution to $M_k(T)$ comes from the small set of those t such that $|\zeta(1/2+it)| \asymp (\log T)^k$, whose measure is about $T/(\log T)^{k^2}$. Nevertheless, as Harper beautifully explains in [72], Soundararajan's technique cannot lead to the optimal bound (2.10) for a twofold reason. To begin with, from the error term in (2.12) it is clear that we have to select the length of the polynomial carefully, since only the first $O(W)$ moments of a Dirichlet polynomial of length $T^{1/W}$ can be bounded well; on one hand we then want W to be large, as we need to study high moments, on the other hand if the polynomial is too short (i.e. W too large) the error term in (2.12) is out of control. To avoid this issue, Harper understood that one should split the polynomial as a sum of many terms and then raise each term to a different power, depending on its length. The second reason is structural, since optimal bounds for frequency of large values cannot be recovered from the moments via Markov inequality [72, footnote, p.3]. Then, to prove the optimal bound (2.10), one should work directly with moments, without involving large deviation analyses. Thanks to these two modifications to Soundararajan's method, Harper [72] succeeded in removing the ε from (2.11), finally getting (2.10) under the assumption of RH.

Putting all the results in this section together, we have that, if we assume RH, then for any fixed $k > 0$

$$(\log T)^{k^2} \ll_k M_k(T) \ll_k (\log T)^{k^2}.$$

2.3 The recipe

Beyond the leading order term described in Section 2.1, in the case $k = 1, 2$ also other terms of the expansion of $M_k(T)$ are known. The first example is due to Ingham [92], who proved

$$M_1(T) = \log \frac{T}{2\pi} + 2\gamma - 1 + O(T^{-1/2+\varepsilon}) \tag{2.13}$$

where γ denotes the Euler-Mascheroni constant; the error term has been then improved by Heath-Brown and Huxley [81] to $O(T^{-15/22}(\log T)^{111/22})$. Moreover Heath-Brown [78] showed that

$$M_2(T) = P_4\left(\log \frac{T}{2\pi}\right) + O(T^{-1/8+\varepsilon}) \tag{2.14}$$

where

$$P_4(x) = \sum_{n=0}^{4} c_n x^n$$

is a polynomial of degree 4, with

$$c_4 = \frac{1}{2\pi^2} \quad \text{and} \quad c_3 = \frac{2}{\pi^2}(4\gamma - 1 - \frac{12}{\pi^2}\zeta'(2)).$$

Later Conrey [24] gave a precise formula for $P_4(x)$, computing also the values of c_0, c_1, c_2 (see also [94]). The general problem of finding an asymptotic expansion for $M_k(T)$ with a T-power saving error term is of course of great interest (and also of outstanding difficulty). Nevertheless, in the case with k an integer, from computations in random matrix theory (see e.g. [27, 28]), it is believed that

$$M_k(T) = \mathcal{P}_k(\log T) + O(T^{1/2+\varepsilon}) \tag{2.15}$$

for some polynomial $\mathcal{P}_k$ of degree k^2, with leading coefficient matching with the Keating-Snaith prediction, see (2.3), (2.4) and (2.5).

In 2005, Conrey, Farmer, Keating, Rubinstein and Snaith [28] found out a new number theoretical heuristic machinery, called "recipe", which conjecturally produces all the main terms for any integral moment of the Riemann zeta function (and several other families of L-functions). In broad strokes, if we define

$$\boldsymbol{\zeta}_\alpha(s) := \zeta(s+\alpha_1)\cdots\zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1})\cdots\zeta(1-s-\alpha_{2k}) \tag{2.16}$$

the recipe starts by considering the shifted integral moment

$$\int_{-\infty}^{+\infty} \boldsymbol{\zeta}_\alpha(1/2+it)g(t)dt \tag{2.17}$$

with g a suitable weight function (e.g. $g(t) = \chi_{[T,2T]}(t)$ the characteristic function would be admissible) and the shifts α_i small enough (tipically of size $1/\log T$). The shifts are due to make the structure of the moments more revealing, as they avoid high-order poles in our expressions. Then one replaces each copy of zeta in (2.16) with the two terms of its approximate functional equation

$$\zeta(s) = \sum_m \frac{1}{m^s} + \chi(s)\sum_n \frac{1}{n^{1-s}} + \cdots$$

(the error is ignored and so are the ranges of summation) and multiplies out, getting 2^{2k} terms. Among all these terms, we only retain the $\binom{2k}{k}$ ones that involve an equal number of $\chi(s+\alpha_j)$ and $\chi(1-s-\alpha_{k+j})$ factors, as all the others are rapidly oscillating. Next step is averaging the coefficients, only considering the "diagonal part", as if this averaging process behaves like a harmonic detection device. For instance, the first term (i.e. the one we get by taking always the first term of the approximate functional equations) is

$$\sum_{\substack{m_1,\dots,m_k \\ n_1,\dots,n_k}} \frac{1}{m_1^{1/2+\alpha_1}\cdots m_k^{1/2+\alpha_k} n_1^{1/2-\alpha_{k+1}}\cdots n_k^{1/2-\alpha_{2k}}}\left(\frac{n_1\cdots n_k}{m_1\cdots m_k}\right)^{it}$$

whose diagonal term is

$$\sum_{m_1\cdots m_k=n_1\cdots n_k} \frac{1}{m_1^{1/2+\alpha_1}\cdots m_k^{1/2+\alpha_k} n_1^{1/2-\alpha_{k+1}}\cdots n_k^{1/2-\alpha_{2k}}}.$$

Therefore, if we define

$$R(s,\alpha) := \sum_{m_1\cdots m_k=n_1\cdots n_k} \frac{1}{m_1^{s+\alpha_1}\cdots m_k^{s+\alpha_k} n_1^{s-\alpha_{k+1}}\cdots n_k^{s-\alpha_{2k}}} \tag{2.18}$$

then $R(\frac{1}{2},\alpha)$ (of course the sum in the definition of $R(s,\alpha)$ does not converge at $\frac{1}{2}$ but we appeal to [28, Theorem 2.4.1] for its analytic continuation) is the contribution in the integral 2.17 that, according to the recipe, comes from the first term. Treating all the other terms similarly (see [28, Section 2.2] for further details), one produces the following conjecture.

Conjecture 2.1 ([28], Equation (2.1.2))**.** *For all $\varepsilon > 0$, for any suitable weight function g, we have*

$$\int_{-\infty}^{+\infty} \boldsymbol{\zeta}_\alpha(1/2+it)g(t)dt = \int_{-\infty}^{+\infty} M_\alpha(1/2+it)\Big(1+O(t^{-1/2+\varepsilon})\Big)g(t)dt$$

where $\boldsymbol{\zeta}_\alpha(s)$ is defined in (2.16) *and for $z = x+iy$ with $x, y \in \mathbb{R}$*

$$M_\alpha(z) := \left(\frac{y}{2\pi}\right)^{\frac{1}{2}(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})} \sum_{\sigma\in\Xi} W_\alpha(z;\sigma)$$

with Ξ the set of permutations $\sigma \in S_{2k}$ such that $\sigma(1) < \cdots < \sigma(k)$, $\sigma(k+1) < \cdots < \sigma(2k)$ and

$$W_\alpha(z;\sigma) := \left(\frac{y}{2\pi}\right)^{\frac{1}{2}(\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\cdots-\alpha_{\sigma(2k)})} R(x;\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2k)})$$

and $R(z,\alpha)$ is defined in (2.18)*.*

Moreover, by various manipulations (see [28, Subsections 2.3-7]) of the right hand side of the above formula, as $\alpha_i \to 0$ and with the choice $g(t) = \chi_{[T,2T]}(t)$, one gets the following version of the previous conjecture.

Conjecture 2.2 ([28], Conjecture 1.5.1)**.** *For every $\varepsilon > 0$*

$$\int_T^{2T} |\zeta(1/2+it)|^{2k} dt = \int_T^{2T} P_k\Big(\log \frac{t}{2\pi}\Big) dt + O(T^{1/2+\varepsilon})$$

where P_k is the polynomial of degree k^2 given by

$$P_k(x) := \frac{(-1)^k}{k!^2(2\pi i)^{2k}} \oint \cdots \oint A_k(z_1, \ldots, z_{2k}) \prod_{i,j=1}^{k} \zeta(1+z_i-z_{k+j}) \\ \cdot \frac{\Delta^2(z_1,\ldots,z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} e^{\frac{x}{2}\sum_{j=1}^{k} z_j - z_{k+j}} dz_1 \cdots dz_{2k}$$

with the path of integration being small circles around $z_i = 0$, where A_k is the Euler product

$$A_k(z) := \prod_p \prod_{i,j=1}^{k} \Big(1 - \frac{1}{p^{1+z_i-z_{k+j}}}\Big) \int_0^1 \prod_{j=1}^{k} \Big(1 - \frac{e^{2\pi i\theta}}{p^{\frac{1}{2}+z_j}}\Big)^{-1} \Big(1 - \frac{e^{-2\pi i\theta}}{p^{\frac{1}{2}-z_{k+j}}}\Big)^{-1} d\theta$$

and

$$\Delta(z_1,\ldots,z_{2k}) := \prod_{1\le i<j\le 2k} (z_j - z_i).$$

First of all we note that this conjecture confirms the prior belief (2.15) and also gives an explicit (conjectural) formula for the polynomial $\mathcal{P}_k$, which is in accordance with (2.13) and (2.14). Moreover, this formula agrees with the random matrix theory analogue (see [27, 28]) and in particular with Keating-Snaith conjecture, as the leading coefficient of P_k is exactly what one expects from (2.3) (see [28, Subsection 2.7]). We recall that the recipe suggests analogue asymptotic formulas for integral moments of L-functions in many other cases, both for one L-function along its critical line (the so called t-aspect) and with respect to averages over families of L-functions.

An interesting generalization of (2.17) is the case of ratios of products of zeta or other L-functions. On the random matrix theory side, the analogue quantity for the characteristic polynomials of matrices in the classical compact (i.e. unitary, symplectic and orthogonal) groups has been studied by using the methods of supersymmetry [29, 84]. In number theory, Farmer [53] was the first to consider the quantity

$$R_\zeta(\alpha,\beta,\gamma,\delta) := \frac{1}{T}\int_T^{2T} \frac{\zeta(1/2+it+\alpha)\zeta(1/2-it+\beta)}{\zeta(1/2+it+\gamma)\zeta(1/2-it+\delta)}dt \qquad (2.19)$$

and he conjectured that if the shifts are $\ll 1/\log T$ then

$$R_\zeta(\alpha,\beta,\gamma,\delta) \sim 1 + (1 - T^{-\alpha-\beta})\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha+\beta)(\gamma+\delta)}.$$

This conjecture has many interesting consequences regarding the Riemann zeta function, including Montgomery pair correlation discussed in Section 1.2 (see [43] for further discussions). Moreover, an extension of the above formula also in the general case of ratios with many copies of zeta in the numerator and denominator would imply interesting statements about the zeros of zeta (see [30, Section 7]). Therefore, Conrey, Farmer and Zirnbauer [30] applied a modification of the recipe for integral moments to the case of ratios getting the following statement, called the ratio conjecture (here we state this conjecture in a slightly weaker form than in [30], as far as the shifts are concerned).

Conjecture 2.3 ([30], Conjecture 5.1)**.** *Let us denote $\chi(s)$ the explicit factor in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. For any positive integers K, L, Q, R and for any $\alpha_1, \ldots, \alpha_{K+L}, \gamma_1, \ldots, \gamma_Q, \delta_1, \ldots, \delta_R$ complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} \frac{\prod_{k=1}^K \zeta(s+\alpha_k)\prod_{l=K+1}^{K+L}\zeta(1-s-\alpha_l)}{\prod_{q=1}^Q \zeta(s+\gamma_q)\prod_{r=1}^R \zeta(1-s+\delta_r)}dt \\
&\qquad = \frac{1}{T}\int_T^{2T} \sum_{\sigma\in\Xi_{K,L}} \prod_{k=1}^K \frac{\chi(s+\alpha_k)}{\chi(s-\alpha_{\sigma(k)})} Y_U A_\zeta(...)dt + O(T^{1/2+\varepsilon}) \\
&\qquad \textit{with } (...) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(K)}; -\alpha_{\sigma(K+1)}, \ldots, -\alpha_{\sigma(K+L)}; \gamma; \delta)
\end{aligned}$$

where

$$Y_U(\alpha;\beta;\gamma;\delta) := \frac{\prod_{k=1}^K\prod_{l=1}^L \zeta(1+\alpha_k+\beta_l)\prod_{q=1}^Q\prod_{r=1}^R \zeta(1+\gamma_q+\delta_r)}{\prod_{k=1}^K\prod_{r=1}^R \zeta(1+\alpha_k+\delta_r)\prod_{l=1}^L\prod_{q=1}^Q \zeta(1+\beta_l+\gamma_q)}$$

and A_ζ is an Euler product, absolutely convergent for all of the variables *in small disks around 0, which is given by*

$$A_\zeta(\alpha;\beta;\gamma;\delta) := \prod_p \frac{\prod_{k=1}^K \prod_{l=1}^L (1-1/p^{1+\alpha_k+\beta_l}) \prod_{q=1}^Q \prod_{r=1}^R (1-1/p^{1+\gamma_q+\delta_r})}{\prod_{k=1}^K \prod_{r=1}^R (1-1/p^{1+\alpha_k+\delta_r}) \prod_{l=1}^L \prod_{q=1}^Q (1-1/p^{1+\beta_l+\gamma_q})}$$
$$\sum_{\sum a_k+\sum c_q=\sum b_l+\sum d_r} \frac{\prod \mu(p^{c_q}) \prod \mu(p^{d_r})}{p^{\sum(1/2+\alpha_k)a_k+\sum(1/2+\beta_l)b_l+\sum(1/2+\gamma_q)c_q+\sum(1/2+\delta_r)d_r}}$$

while $\Xi_{K,L}$ *denotes the subset of permutations* $\sigma \in S_{K+L}$ *of* $\{1,2,\ldots,K+L\}$ *for which* $\sigma(1) < \sigma(2) < \cdots < \sigma(K)$ *and* $\sigma(K+1) < \sigma(K+2) < \cdots < \sigma(K+L)$.

Moreover, we mention a new approach due to Conrey and Keating [37, 38, 39, 40, 41], which leads to conjecture all of the predictions about the moments of zeta from a number theoretical investigation, in particular from the study of the divisor correlations. See also [61], where this approach was undertook first by Goldston and Gonek, and Hamieh and Ng's work [70] for a proof of Conrey-Keating conjectures in special cases.

We end this section recalling that also the ratio conjecture can be generalized to families of L-functions (see e.g. [30, Conjecure 5.2, 5.3 and 5.4]) and that it agrees with the analogue results in random matrix theory (which are theorems, not only conjectures, see e.g. [30, Theorem 4.1]).

2.4 Twisted moments

An interesting tool regarding the moments of zeta, which is going to be of particular importance in the following, is the mean square of the product of the Riemann zeta function times a Dirichlet polynomial. After Iwaniec [95] had considered this problem first, Balasubramanian, Conrey and Heath-Brown [5] proved an asymptotic formula for the second twisted moment of zeta, for a suitably short Dirichlet polynomial. Specifically, for any

$$A(s) = \sum_{n\le T^\theta} \frac{a_n}{n^s} \tag{2.20}$$

of length θ and with slow-growing coefficients (i.e. $a_n \ll T^\varepsilon$ for all $\varepsilon > 0$), the authors showed that

$$\begin{aligned}\int_T^{2T} &|\zeta(1/2+it)|^2|A(1/2+it)|^2 dt \\ &= T \sum_{m,n\le T^\theta} \frac{a_n\overline{a_m}}{[n,m]}\left(\log\left(\frac{T(n,m)^2}{nm}\right)+c\right)+o(T)\end{aligned} \tag{2.21}$$

provided that $\theta < \frac{1}{2}$, denoting $c := 2\gamma + \log(2/\pi) - 1$. The length of the polynomial here plays a fundamental role in the applications (see also [23]), in particular if one were allowed to take $\theta = 1-\varepsilon$ (which is the largest conjectural admissible length) in (2.21) then the Lindelöf Hypothesis would follow. Thanks to a new estimate for trilinear forms of Kloosterman fractions [10], Bettin, Chandee and Radziwiłł [11] broke the "$\frac{1}{2}$-barrier" in (2.21), showing that if $\theta < \frac{1}{2} + \frac{1}{66}$ then

$$\begin{aligned}\int_{\mathbb{R}} |\zeta(1/2+it)|^2 |A(1/2+it)|^2 \phi\Big(\frac{t}{T}\Big) dt \\ = T \sum_{n,m \le T^\theta} \frac{a_n \overline{a_n}}{[n,m]} \int_{\mathbb{R}} \bigg(\log\bigg(\frac{t(n,m)^2}{2\pi nm}\bigg) + 2\gamma \bigg) \phi\Big(\frac{t}{T}\Big) dt \\ + O\Big(T^{\frac{3}{20}+\varepsilon} T^{\frac{33}{20}\theta} + T^{\frac{1}{3}+\varepsilon}\Big)\end{aligned} \tag{2.22}$$

for any $\phi(x)$ smooth function supported in $[1,2]$. We remark that while if $\theta < \frac{1}{2}$ only the diagonal terms (in the same sense as in the recipe, Section 2.3) contribute to the second shifted moment, for $\theta > \frac{1}{2}$ also the non-diagonal terms give a non negligible contribution. Finally we note that with exactly the same ideas as in [5, 11], one can straightforwardly state (2.21) in a more general way, which is going to be useful in the next chapters (see e.g. [8] for the easy modifications needed to account for the shifts).

Lemma 2.4. *Let $A(s) = \sum_{n \le T^\theta} a(n) n^{-s}$ and $B(s) = \sum_{m \le T^\sigma} b(m) m^{-s}$ be Dirichlet polynomials with $a(n) \ll n^\varepsilon$, $b(m) \ll m^\varepsilon$ for every $\varepsilon > 0$ and $\theta + \sigma < 1$. Then, denoting $c := 2\gamma + \log(2/\pi) - 1$, we have:*

$$\begin{aligned}\int_T^{2T} A(1/2+it)\overline{B(1/2+it)} |\zeta(1/2+it)|^2 dt \\ = T \sum_{m,n} \frac{a(n)\overline{b(m)}}{[n,m]} \bigg(\log\bigg(\frac{T(n,m)^2}{nm}\bigg) + c \bigg) + o(T).\end{aligned} \tag{2.23}$$

We refer to Appendix A for the proof of Lemma 2.4.

Also the fourth moment case has been studied with twists. In particular [95, 47, 161] proved upper bounds for

$$\int_T^{2T} |\zeta(1/2+it)|^4 |A(1/2+it)|^2 dt \tag{2.24}$$

in the case $\theta < \frac{1}{10}$, $\theta < \frac{1}{5}$ and $\theta < \frac{1}{4}$ respectively (θ here is the same as in (2.20)). After [60, 124], Hughes and Young [89] proved an asymptotic formula with all the main terms for a shifted version of (2.24) in the case when $\theta < \frac{1}{11}$, then improved by Bettin, Bui, Li and Radziwiłł [9] to $\theta < \frac{1}{4}$ (also in this case this result is expected to hold for $\theta < 1$).

Theorem 2.5 ([9], Theorem 1.2)**.** *Let $T > 2$ and let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha, \beta, \gamma, \delta \ll (\log T)^{-1}$. Furthermore, let $\phi(x)$ be a smooth function supported in $[1, 2]$ with derivatives $\phi^{(j)}(x) \ll_j T^{\varepsilon}$ for any $j \geq 0$. Consider*

$$A(s) = \sum_{a \leq T^\theta} \frac{\alpha_a}{a^s} \quad \text{and} \quad B(s) = \sum_{b \leq T^\theta} \frac{\beta_b}{b^s}$$

where $\alpha_a \ll a^\varepsilon$ and $\beta_b \ll b^\varepsilon$, and denote

$$I_{\alpha,\beta,\gamma,\delta}(T) := \int_{\mathbb{R}} \boldsymbol{\zeta}_{\alpha,\beta,\gamma,\delta}(t) A(1/2+it)\overline{B(1/2+it)}\phi\left(\frac{t}{T}\right)dt$$

with

$$\boldsymbol{\zeta}_{\alpha,\beta,\gamma,\delta}(t) := \zeta(1/2+it+\alpha)\zeta(1/2+it+\beta)\zeta(1/2-it+\gamma)\zeta(1/2-it+\delta).$$

Then

$$\begin{aligned} I_{\alpha,\beta,\gamma,\delta}(T) = \sum_{a,b \leq T^\theta} \alpha_a \overline{\beta_b} \int_{\mathbb{R}} \Bigg(& Z_{\alpha,\beta,\gamma,\delta,a,b}(t) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z_{-\gamma,-\delta,-\alpha,-\beta,a,b}(t) \\ & + \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z_{-\gamma,\beta,-\alpha,\delta,a,b}(t) + \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z_{-\delta,\beta,\gamma,-\alpha,a,b}(t) \\ & + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z_{\alpha,-\gamma,-\beta,-\delta,a,b}(t) + \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z_{\alpha,-\delta,\gamma,-\beta,a,b}(t) \Bigg) \phi\left(\frac{t}{T}\right) dt \\ & + O_\varepsilon(T^{1/2+2\theta+\varepsilon} + T^{3/4+\theta+\varepsilon}) \end{aligned}$$

where

$$Z_{\alpha,\beta,\gamma,\delta,a,b}(t) := \sum_{am_1m_2=bn_1n_2} \frac{1}{\sqrt{ab}\, m_1^{1/2+\alpha} m_2^{1/2+\beta} n_1^{1/2+\gamma} n_2^{1/2+\delta}} V\left(\frac{m_1m_2n_1n_2}{t^2}\right)$$

and

$$V(x) := \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} (2\pi)^{-2s} x^{-s} ds$$

with $G(s)$ an even entire function of rapid decay in any fixed strip $|\Re(s)| \leq C$ satisfying $G(0) = 1$ such that it is divisible by an even polynomial $Q_{\alpha,\beta,\gamma,\delta}(s)$[1]*, which is symmetric in the parameters $\alpha, \beta, \gamma, \delta$, invariant under the transformations $\alpha \to -\alpha$, $\beta \to -\beta$, etc. and zero at $s = -(\alpha+\gamma)/2$ (as well as other points by symmetry), and that $G(s)/Q_{\alpha,\beta,\gamma,\delta}(s)$ is independent of $\alpha, \beta, \gamma, \delta$.*

[1] i.e. such that $G(s)/Q_{\alpha,\beta,\gamma,\delta}(s)$ is an entire function.

The above theorem is stated in a form due to [9], where the arithmetic factors (Euler products) are not extrapolated; the reader may be also interested in the original statements [89, Theorem 1.1] and [9, Theorem 1.1]. Moreover, as $\alpha, \beta, \gamma, \delta \to 0$, with the choice $A = B$, Theorem 2.5 gives an asymptotic formula for (2.24).

In general, one cannot expect to prove an analogue of the previous theorem for higher powers of zeta, since not even the integer moments of zeta are known. Nevertheless, with a rather simple adaptation of the recipe, Hughes and Young [89] calculated, at least conjecturally, all the main terms of the twisted and shifted moments of the form

$$\int_T^{2T} \left(\frac{a}{b}\right)^{-it} \zeta(\tfrac{1}{2}+\alpha_1+it)\cdots\zeta(\tfrac{1}{2}+\alpha_k+it)\zeta(\tfrac{1}{2}+\beta_1-it)\cdots\zeta(\tfrac{1}{2}+\beta_k-it)dt$$

for suitably small twists a, b. Here we use the strategy of [9, Theorem 1.2] in order to re-write the original statement due to Hughes and Young (see [89, Conjecture 7.1]).

Conjecture 2.6 (Hughes-Young)**.** *Let T be a large parameter, $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k \ll (\log T)^{-1}$, Φ_j the set of subset of $\{\alpha_1, \dots, \alpha_k\}$ of cardinality j, for $j = 0, \dots, k$, and similarly Ψ_j the set of subset of $\{\beta_1, \dots, \beta_k\}$ of cardinality j. If $\mathcal{S} \in \Phi_j$ and $\mathcal{T} \in \Psi_j$ then write $\mathcal{S} = \{\alpha_{i_1}, \dots, \alpha_{i_j}\}$ and $\mathcal{T} = \{\beta_{l_1}, \dots, \beta_{l_j}\}$ where $i_1 < i_2 < \cdots < i_j$ and $l_1 < l_2 < \cdots < l_j$. Let $(\alpha_{\mathcal{S}}; \beta_{\mathcal{T}})$ be the tuple obtained from $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k)$ by replacing α_{i_r} with $-\beta_{l_r}$ and replacing β_{l_r} with $-\alpha_{i_r}$ for $1 \le r \le j$. Consider*

$$A(s) = \sum_{a \le T^\theta} \frac{f(a)}{a^s} \quad \text{and} \quad B(s) = \sum_{b \le T^\theta} \frac{g(b)}{b^s}$$

where $f(a) \ll a^\varepsilon$ and $g(b) \ll b^\varepsilon$ for any $\varepsilon > 0$. We then conjecture that there exists a $\delta > 0$, depending on k, such that if $\theta < \delta$ then

$$\int_T^{2T} \zeta(1/2+it+\alpha_1)\cdots\zeta(1/2+it+\alpha_k) \cdot \zeta(1/2-it+\beta_1)\cdots\zeta(1/2-it+\beta_k)A(1/2+it)\overline{B(1/2+it)}dt$$

equals

$$\sum_{a,b \le T^\theta} f(a)\overline{g(b)} \int_T^{2T} \sum_{0 \le j \le k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} Z_{\alpha_{\mathcal{S}};\beta_{\mathcal{T}},a,b}(t) \left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}} dt + O(T^{1-\eta})$$

for some $\eta > 0$, *where we have written* $(t/2\pi)^{-\mathcal{S}-\mathcal{T}}$ *for* $(t/2\pi)^{-\sum_{x\in\mathcal{S}} x - \sum_{y\in\mathcal{T}} y}$ *and*

$$Z_{\alpha;\beta,a,b}(t) := \sum_{am_1\cdots m_k = bn_1\cdots n_k} \frac{1}{\sqrt{ab}\, m_1^{1/2+\alpha_1}\cdots m_k^{1/2+\alpha_k} n_1^{1/2+\beta_1}\cdots n_k^{1/2+\beta_k}} \cdot V\left(\frac{m_1\cdots m_k n_1\cdots n_k}{t^k}\right)$$

with

$$V(x) := \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(2\pi)^{-ks} x^{-s} ds$$

where $G(s)$ *has the same properties as in Theorem 2.5.*

2.5 Discrete moments

If $\rho = \beta + i\gamma$ denotes a generic non-trivial zero of the Riemann zeta function, the discrete moments are defined by

$$J_k(T) := \frac{1}{N(T)} \sum_{0<\gamma\le T} |\zeta'(\rho)|^{2k} \tag{2.25}$$

with k a positive real number and $N(T)$ the normalizing factor given by (1.4), where the word "discrete" is due to the fact that $J_k(T)$ is by definition the $2k$-th moment of $|\zeta'(s)|$ on the (uniform) discrete probability space of the non-trivial zeros of zeta, whose imaginary part is positive and $\le T$. The definition (2.25) involves the derivative of the Riemann zeta function instead of zeta itself (otherwise J_k would have been trivially zero!); however, by Cauchy's integral formula, the quantity $J_k(T)$ is strictly related to

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} |\zeta(\rho+\alpha)|^{2k} \tag{2.26}$$

with α a small complex shift. In 1984, Gonek [63] intensively studied the second discrete moment of any derivative of the Riemann zeta function, proving under RH the following result (see also [64, Theorem 2]).

Theorem 2.7 ([63], Corollary 1 and 2)**.** *Suppose the Riemann Hypothesis is true. If* T *is sufficiently large and* $\mu \ge 1$, *then*

$$\sum_{0<\gamma\le T} |\zeta^{(\mu)}(\rho)|^2 \sim \frac{\mu^2}{(2\mu+1)(\mu+1)^2}\frac{T}{2\pi}(\log T)^{2\mu+2}$$

and in particular

$$J_1(T) \sim \frac{1}{12}(\log T)^3.$$

Moreover if α *is any real number satisfying* $|\alpha| \leq \frac{\log T}{2\pi}$*, then*

$$\sum_{0<\gamma\leq T}\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{2\pi\alpha}{\log T}\right)\right)\right|^2 = \left(1-\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}(\log T)^2+O(T\log T)$$

and the constant implicit in the O*-term is independent of* α*.*

For no other values of k an asymptotic formula for $J_k(T)$ is known, even conditionally on RH. Nevertheless, for any fixed $k \in \mathbb{R}$, Gonek [65] and Hejhal [83] independently conjectured that

$$J_k(T) \asymp (\log T)^{k(k+2)}, \tag{2.27}$$

which has been proved by Ng [127] in the case $k = 2$ on the assumption of RH. In addition, since very little is known in the general case, random matrix theory comes to rescue us, as usual; Hughes, Keating and O'Connel [86] conjectured that

$$J_k(T) \sim a_k f_k (\log T)^{k(k+2)} \tag{2.28}$$

for any fixed $k \in \mathbb{C}$ such that $\Re(k) > -3/2$, where a_k is the arithmetical factor (2.4) and

$$f_k := \frac{G(k+2)^2}{G(2k+3)}.$$

The analogy with Keating-Snaith conjecture (2.3) strikes the eye immediately. Morover, they gave a heuristic explanation suggesting that (2.27) should be false for $k \leq -3/2$.

Assuming RH, in the last ten years upper and lower bounds for $J_k(T)$ have been obtained. Milinovich and Ng [119] conditionally proved that

$$J_k(T) \gg_k (\log T)^{k(k+2)} \tag{2.29}$$

for any positive integer k, while Milinovich [118] obtained (on RH) the quasi optimal upper bound

$$J_k(T) \ll_{k,\varepsilon} (\log T)^{k(k+2)+\varepsilon} \tag{2.30}$$

with $k \in \mathbb{N}$ for any $\varepsilon > 0$. The proof of (2.30) builds upon the strategy that Soundararajan [151] used in order to get upper bounds for the frequency of large values of zeta. Very recently, Kirila [111] removed the ε in the exponent, by appyling the same ideas that allowed Harper [72] to improve on Soundararajan's bound for the moments of zeta, getting the following result.

Theorem 2.8 ([111], Theorems 1.1 and 1.2)**.** *Assume RH. Let $k > 0$, then*

$$J_k(T) \ll_k (\log T)^{k(k+2)}. \tag{2.31}$$

Moreover, for any $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $\Re(\alpha) \leq (\log T)^{-1}$, we have

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k} \ll_k (\log T)^{k^2}. \tag{2.32}$$

Chapter 3

Weighted value distributions of $\log|\zeta(1/2+it)|$

The work of this chapter was first published in [54] and [55]; more specifically see [54] for Section 3.1 and [55] for Section 3.2.

3.1 The weighted measure $|\zeta(1/2+it)|^2 dt$

In this chapter we discuss the value distribution of $\log|\zeta(1/2+it)|$ with respect to a certain weighted measure, which comes naturally out in the investigation of the large values of zeta. More specifically, here we are interested in conjecture (1.17) in the case $k=1$, i.e. in the expected upper bound

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\big\{\log|\zeta(1/2+it)|\geq\log\log T\big\}\ll\frac{1}{\log T\sqrt{\log\log T}}. \tag{3.1}$$

We recall that Soundararajan [151] essentially proved that the left hand side is $\ll(\log T)^{-1+o(1)}$, while the factor $\sqrt{\log\log T}$ on the denominator is the one which is still undetected. A possible approach to the above conjecture is by using the Mellin transform in order to get an integral representation for the characteristic function of positive reals, given by

$$\chi_{(0,+\infty)}(x)=\frac{1}{2\pi i}\int_{(c)}e^{wx}\frac{dw}{w} \tag{3.2}$$

for any $c>0$, where $\int_{(c)}$ denotes the integral over the vertical line $\Re(s)=c$. Writing the above expression for $x=\log|\zeta(1/2+it)|-\log\log T$ and with

$c=2$, averaging over $t\in[T,2T]$, we get

$$
\begin{aligned}
\frac{1}{T}&\operatorname*{meas}_{t\in[T,2T]}\big\{\log|\zeta(1/2+it)|\geq\log\log T\big\}\\
&=\frac{1}{T}\int_T^{2T}\frac{1}{2\pi i}\int_{(2)}e^{w(\log|\zeta(1/2+it)|-\log\log T)}\frac{dw}{w}dt\\
&=\frac{1}{\log T}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2+iu}\frac{1}{T\log T}\\
&\qquad\times\int_T^{2T}e^{iu(\log|\zeta(1/2+it)|-\log\log T)}|\zeta(1/2+it)|^2dt\,du
\end{aligned}
\tag{3.3}
$$

and hence the measure of the set we are interested in can be expressed in terms of the Fourier-Laplace transform of $\log|\zeta(1/2+it)|-\log\log T$, with respect to the Lebesgue measure tilted by the weight $|\zeta(1/2+it)|^2$. Therefore, the measure

$$|\zeta|^2dt:=|\zeta(1/2+it)|^2dt \tag{3.4}$$

becomes relevant in the understanding of the large values of zeta. This is not surprising, as the weighted measure $|\zeta|^2dt$ has the effect that in integrals we are giving more importance to the contribution of those t such that $|\zeta(1/2+it)|$ is large. We would like to point out that the factor $1/(T\log T)$ which appears in the third line of (3.3) is the natural normalization for the measure $|\zeta|^2dt$, in view of (2.2). Beside this, thanks to the asymptotic formula for the second twisted moment of zeta discussed in Section 2.4, we are able to compute the moments of a sufficiently short Dirichlet polynomial with respect to this weighted measure. For these reasons, we will be able to study the distribution of $\log|\zeta(1/2+it)|$ with respect to $|\zeta|^2dt$, which would be of help in the understanding of the large values of zeta; this is achieved in the following result.

Theorem 3.1. *Under the Riemann Hypothesis, as t varies in $T\leq t\leq 2T$, the distribution of $\log|\zeta(1/2+it)|$ is asymptotically Gaussian with mean $\log\log T$ and variance $\frac{1}{2}\log\log T$, with respect to the weighted measure $|\zeta|^2dt$.*

We note that this result is a manifestation of Girsanov's theorem from probability theory, which describes how a stochastic process changes under certain changes of measure. In the specific case we are interested in, Girsanov's theorem reduces to the simple fact that if we take X a Gaussian random variable of mean 0 and variance σ^2 with respect to the measure $d\nu$ and we tilt the measure against e^{yX} with $y\in\mathbb{R}$, then X is again Gaussian with respect to the resulting measure $d\tilde{\nu} := e^{yX}d\nu$, with the same variance but mean $y\sigma^2$ (it can be proved just by completing the square). Theorem 1 shows the same behavior for $\log|\zeta(1/2+it)|$ in the case $y=2$, reinforcing our expectation that $\log|\zeta(1/2+it)|$ behaves like a Gaussian in many respects (other interesting computations involving the Riemann zeta function inspired by Girsanov's theorem can be found in Harper's work [75], Section 3).

The general strategy to prove Theorem 3.1 builds on the ideas described in Section 1.3, where we discussed that, even though the Euler product formula only holds in the half-plane of convergence, for many purposes the Riemann zeta function behaves like an Euler product also on the critical line. Roughly speaking we expect that for a suitable $x = x(T)$ we have

$$\log|\zeta(1/2+it)| \approx \Re \sum_{p\le x} \frac{1}{p^{1/2+it}} + \text{(contribution from zeros)} \tag{3.5}$$

and that in several applications the contribution from the zeros can be controlled. This approximation also holds in our setting, as shown by the following proposition:

Proposition 3.2. *Let T be a large parameter. Denote $P(t) = \sum_{p\le x} p^{-1/2-it}$, where $x = T^{\varepsilon/k}$, $\varepsilon := (\log\log\log T)^{-1}$, k a positive integer. Under the Riemann Hypothesis, there exists a constant $C>0$ such that we have uniformly in k:*

$$\frac{1}{T\log T}\int_T^{2T} \big|\log|\zeta(1/2+it)| - \Re P(t)\big|^{2k} |\zeta|^2 dt \ll (Ck)^{4k}(\log\log\log T)^{2k+\frac{1}{2}}.$$

We remark that this is the only point where we rely on the assumption of RH. In fact, in order to estimate the contribution of the zeros that appears in (3.5) on average, we will end up bounding the sum over the non-trivial zeros $\sum_{0<\rho\le T}|\zeta(\rho+i\alpha)|^2$ with $|\alpha|\le 1$ a real parameter, which is known to be $\ll T(\log T)^2$ only conditionally on RH (see Theorem 2.7, due to Gonek).

Thanks to Proposition 3.2, at this point it suffices to show that the distribution of $\Re P(t)$ is approximately Gaussian with respect to the measure $|\zeta|^2dt$. This is achieved by the method of moments in the following result.

Proposition 3.3. *Let $P(t) = \sum_{p\le x} p^{-1/2-it}$, $x := T^{\varepsilon/k}$, $\varepsilon := \frac{1}{\log\log\log T}$. Denote $\mathcal{L} = \sum_{p\le x}\frac{1}{p}$. Then, for every fixed k integer*

$$\frac{1}{T\log T}\int_T^{2T} (\Re P(t) - \mathcal{L})^k |\zeta|^2 dt = \begin{cases} \big(\frac{\mathcal{L}}{2}\big)^{k/2}(k-1)!! + O_k\big(\mathcal{L}^{\frac{k-1}{2}}\big) & \text{if } k \text{ even} \\ O_k\big(\mathcal{L}^{(k-1)/2}\big) & \text{if } k \text{ odd.} \end{cases}$$

Note that by definition of x we know that $\log\log x = \log\log T - \log k + \log\varepsilon$, then for a fixed k we have $\log\log x = \log\log T + O(\log_4 T)$, where $\log_4$ denotes the fourth iterated natural logarithm. Hence by Mertens' theorem $\mathcal{L} = \log\log x + O(1) = \log\log T + O(\log_4 T)$. As a consequence, the right hand side in Proposition 3.3 matches with the moments of a normal of mean $\mathcal{L} \sim \log\log T$ and variance $\frac{\mathcal{L}}{2} \sim \frac{1}{2}\log\log T$. Putting together the two propositions one has that the moments of $\log|\zeta(1/2+it)|$ with respect to the measure $|\zeta|^2 dt$ are asymptotic to the moments of a Gaussian random variable of mean $\log\log T$ and variance $\frac{1}{2}\log\log T$, thus the theorem will follow, once we prove Propositions 3.2 and 3.3.

3.1.1 Proof of Proposition 3.2

Our proof of Proposition 3.2 is a modification of Theorem 5.1 in [158]. We recall that $P(t) = \sum_{p\le x} p^{-1/2-it}$ and $x = T^{\varepsilon/k}$ with $\varepsilon = (\log\log\log T)^{-1}$. Following Tsang's strategy, whose notations become easier under RH, we have (see [158], Equation (5.15)):

$$\log\zeta(1/2+it) - P(t) = S_1 + S_2 + S_3 + O(R) - L(t) \tag{3.6}$$

with

$$S_1 := \sum_{p\le x}\left(p^{-1/2-4/\log x} - p^{-1/2}\right)p^{-it}, \qquad S_2 := \sum_{\substack{p^r\le x\\ r\ge 2}} \frac{p^{-r(1/2+4/\log x+it)}}{r}$$

$$S_3 := \sum_{x<p\le x^3} \frac{\Lambda(n)}{\log n} n^{-1/2-4/\log x-it}, \quad R := \frac{5}{\log x}\left(\left|\sum_{n\le x^3}\frac{\Lambda(n)}{n^{1/2+4/\log x+it}}\right| + \log T\right)$$

$$L(t) := \sum_\rho \int_{1/2}^{1/2+4/\log x}\left(\frac{1}{2}+\frac{4}{\log x}-u\right)\frac{1}{u+it-\rho}\frac{1}{\frac{1}{2}+\frac{4}{\log x}-\rho}du,$$

where the sum in the definition of $L(t)$ is over all the non-trivial zeros of ζ. Hence

$$\log|\zeta(1/2+it)| - \Re(P(t)) = \Re(S_1)+\Re(S_2)+\Re(S_3)+O(R)-\Re(L(t)). \tag{3.7}$$

Thanks to the decomposition (3.7), we can bound the $2k$-th power of the modulus of the left hand side by

$$\begin{aligned}|\log|\zeta(1/2+it)| - \Re(P(t))|^{2k}&\\ &\le 5^{2k}(|S_1|^{2k} + |S_2|^{2k} + |S_3|^{2k} + |O(R)|^{2k} + |\Re(L(t))|^{2k}).\end{aligned}$$

So what remains to do is studying the moments of all these objects with respect to the weighted measure $|\zeta|^2dt$; to this aim we rely on Lemma 2.4.

Let's start with the first one:

$$\begin{aligned}
\frac{1}{T\log T}\int_T^{2T}|S_1|^{2k}|\zeta|^2dt &= \frac{1}{T\log T}\int_T^{2T}\left|\left(\sum_{p\le x}\frac{p^{-4/\log x}-1}{p^{1/2+it}}\right)^k\right|^2|\zeta|^2dt \\
&= \frac{1}{T\log T}\int_T^{2T}\left|\sum_{n\le x^k}\frac{1}{n^{1/2+it}}\sum_{\substack{p_1\cdots p_k=n\\ p_i\le x\,\forall i}}\prod_{i=1}^k\left(p_i^{-4/\log x}-1\right)\right|^2|\zeta|^2dt \\
&\ll \sum_{m,n\le x^k}\frac{(m,n)}{mn}\sum_{\substack{p_1\cdots p_k=n\\ q_1\cdots q_k=m\\ p_i,q_i\le x}}\prod_{i=1}^k|p_i^{-4/\log x}-1||q_i^{-4/\log x}-1| \\
&= \sum_{\substack{p_1,\dots,p_k\le x\\ q_1,\dots q_k\le x}}\frac{(p_1\cdots p_k,q_1\cdots q_k)}{p_1\cdots p_kq_1\cdots q_k}\prod_{i=1}^k|p_i^{-4/\log x}-1||q_i^{-4/\log x}-1|.
\end{aligned}$$

To make the GCD on the numerator explicit, we rewrite the primes $p_1,\dots,p_k$ highlighting the multiplicity of these primes:

$$\{p_1,\dots,p_k\}=\{p_1',\dots,p_l'\}$$

where the p_i''s are distinct and we denote $c_i\ge 1$ the multiplicity of p_i' in this set, so $c_1+\dots+c_l=k$. Now we do the same for the q_i's and we put in evidence if any q_i already appears among the p_i''s:

$$\{q_1,\dots,q_k\}=\{p_1',\dots,p_l'\}\cup\{q_1',\dots,q_m'\}$$

where the p_i''s and q_j''s are all distinct and we denote $e_i\ge 0$ and $d_i\ge 1$ the multiplicities of p_i' and q_i' respectively. Then we have $e_1+\dots+e_l+d_1+\dots+d_m=k$. In the following we drop the symbol $'$, just denoting the new primes with p_i,q_i. With these notations, the previous sum is

$$\begin{aligned}
\ll (k!)^2\sum_{\substack{l\le k\\ m\le k}}\sum_{\substack{c_1+\dots c_l=k\\ e_1+\dots e_l+d_1+\dots+d_m=k\\ c_i\ge1,\, d_i\ge1,\, e_i\ge0}}\prod_{i=1}^l\left(\sum_{p_i}\frac{|p_i^{-4/\log x}-1|^{c_i+e_i}}{p_i^{\max(c_i,e_i)}}\right)& \\
\cdot\prod_{i=1}^m\left(\sum_{q_i}\frac{|q_i^{-4/\log x}-1|^{d_i}}{q_i^{d_i}}\right)&
\end{aligned}$$

and if we ignore the equation for c_i, e_i, d_i we get

$$\begin{aligned}\ll (k!)^2 \sum_{\substack{l\le k\\ m\le k}} \prod_{i=1}^{l} \Bigg(\sum_{\substack{c_i\ge 1\\ e_i\ge 0}} \sum_{p_i\le x} \frac{|p_i^{-4/\log x}-1|^{c_i+e_i}}{p_i^{\max(c_i,e_i)}} \Bigg) \\ \cdot \prod_{i=1}^{m} \Bigg(\sum_{d_i\ge 1} \sum_{q_i\le x} \frac{|q_i^{-4/\log x}-1|^{d_i}}{q_i^{d_i}} \Bigg).\end{aligned} \tag{3.8}$$

Now we remark that only in the case $c_i = 1$ and $e_i \le 1$ the sum over p_i in the first parentheses gives an unbounded contribution. Indeed the remaining cases give

$$\begin{aligned}\sum_{\substack{c_i\ge 1,\, e_i\ge 0:\\ \max(c_i,e_i)\ge 2}} \sum_{p_i\le x} \frac{|p_i^{-4/\log x}-1|^{c_i+e_i}}{p_i^{\max(c_i,e_i)}} &\ll \sum_{\substack{c_i\ge 1,\, e_i\ge 0:\\ \max(c_i,e_i)\ge 2}} \sum_{p_i\le x} \frac{1}{p_i^{\max(c_i,e_i)-3/2}} \frac{1}{p_i^{3/2}} \\ &\ll \sum_{\substack{c_i\ge 1,\, e_i\ge 0:\\ \max(c_i,e_i)\ge 2}} \frac{1}{2^{\max(c_i,e_i)-3/2}} \sum_{p_i\le x} \frac{1}{p_i^{3/2}} \ll \sum_{\substack{c_i\ge 1,\, e_i\ge 0:\\ \max(c_i,e_i)\ge 2}} \frac{1}{2^{\max(c_i,e_i)}} \\ &\ll \sum_{\substack{c_i\ge 0\\ e_i\ge 0}} \frac{1}{2^{(c_i+e_i)/2}} \ll 1.\end{aligned}$$

We treat the second parentheses analogously, so that we get a bound for (3.8), which is

$$\ll (k!)^2 \sum_{l,m\le k} \Bigg(\sum_{p\le x} \frac{|p^{-4/\log x}-1|}{p} + \sum_{p\le x} \frac{|p^{-4/\log x}-1|^2}{p} + O(1) \Bigg)^{l+m}.$$

In order to bound the first sum we use the Taylor's approximation $e^{-z} = 1 + O(z)$ for $z \ll 1$, which yields

$$\sum_{p\le x} \frac{|p^{-4/\log x}-1|}{p} = \sum_{p\le x} \frac{|e^{-4\log p/\log x}-1|}{p} \ll \frac{1}{\log x} \sum_{p\le x} \frac{\log p}{\log x} \ll 1 \tag{3.9}$$

by Merten's first theorem. The second sum is $\ll 1$ too, being $|A-1|^2 \le |A-1|$ for $0 < A < 1$. Putting all together, the sum we are considering is:

$$\ll (k!)^2 \sum_{l,m\le k} 3^{l+m} \ll (k!)^2 C^{2k}$$

with C a sufficiently large positive constant. In conclusion, uniformly in k we have

$$\frac{1}{T\log T} \int_T^{2T} |S_1|^{2k} |\zeta|^2 dt \ll (Ck)^{2k}.$$

Now we focus on S_2. Using again Lemma 2.4 we have:

$$\frac{1}{T\log T}\int_T^{2T}|S_2|^{2k}|\zeta|^2dt \ll \sum_{\substack{p_1,\dots,p_k\le x\\ q_1,\dots,q_k\le x\\ r_1,\dots,r_k\ge 2\\ s_1,\dots,s_k\ge 2}} \frac{(p_1^{r_1}\cdots p_k^{r_k}, q_1^{s_1}\cdots q_k^{s_k})}{p_1^{r_1}\cdots p_k^{r_k}q_1^{s_1}\cdots q_k^{s_k}}.$$

We use the same decomposition of $\{p_1,\dots,p_k\}$ and $\{q_1,\dots,q_k\}$ as before, getting

$$\begin{aligned}
&\ll (k!)^2\sum_{m,l\le k}\sum_{\substack{p_1,\dots,p_l\le x\\ q_1,\dots,q_m\le x}}\sum_{\substack{a_1,\dots,a_l\ge 2\\ b_1,\dots,b_l\ge 0\\ f_1,\dots,f_m\ge 2}}\frac{1}{p_1^{\max(a_1,b_1)}\cdots p_l^{\max(a_l,b_l)}q_1^{f_1}\cdots q_m^{f_m}}\\
&=(k!)^2\sum_{m,l\le k}\prod_{i=1}^{l}\Bigg(\sum_{\substack{p_i\le x\\ a_i\ge 2\\ b_i\ge 0}}\frac{1}{p_i^{\max(a_i,b_i)}}\Bigg)\prod_{i=1}^{m}\Bigg(\sum_{\substack{q_i\le x\\ f_i\ge 2}}\frac{1}{q_i^{f_i}}\Bigg)\ll (k!)^2\sum_{m,l\le k}C^{m+l}\\
&\ll (Ck)^{2k}.
\end{aligned}$$

Let us investigate S_3 using the same approach. We have

$$\frac{1}{T\log T}\int_T^{2T}|S_3|^{2k}|\zeta|^2dt \ll \sum_{\substack{x<p_1^{r_1},\dots,p_k^{r_k}\le x^3\\ x<q_1^{s_1},\dots,q_k^{s_k}\le x^3}}\frac{(p_1^{r_1}\cdots p_k^{r_k}, q_1^{s_1}\cdots q_k^{s_k})}{p_1^{r_1}\cdots p_k^{r_k}q_1^{s_1}\cdots q_k^{s_k}}. \tag{3.10}$$

We begin studying the case when all the exponents r_i, s_i are equal to 1. We can implement the same technique as before, getting:

$$\begin{aligned}
&\ll (k!)^2\sum_{m,l\le k}\sum_{\substack{c_1,\dots,c_l\ge 1\\ e_1,\dots,e_l\ge 0\\ d_1,\dots,d_m\ge 1}}\sum_{\substack{x<p_1,\dots,p_l\le x^3\\ x<q_1,\dots,q_m\le x^3}}\frac{1}{p_1^{\max(c_1,e_1)}\cdots p_l^{\max(c_1,e_l)}q_1^{d_1}\cdots q_m^{d_m}}\\
&\qquad=(k!)^2\sum_{l,m\le k}\prod_{i=1}^{l}\Bigg(\sum_{c_i\ge 1}\sum_{e_i\ge 0}\sum_{x<p_i\le x^3}\frac{1}{p_i^{\max(c_i,e_i)}}\Bigg)\prod_{i=1}^{m}\Bigg(\sum_{d_i\ge 1}\sum_{x<q_i\le x^3}\frac{1}{q_i^{d_i}}\Bigg)\\
&\qquad=(k!)^2\sum_{l,m\le k}\prod_{i=1}^{l}\Bigg(2\sum_{x<p_i\le x^3}\frac{1}{p_i}+O(1)\Bigg)\prod_{i=1}^{m}\Bigg(\sum_{x<q_i\le x^3}\frac{1}{q_i}+O(1)\Bigg)\\
&\qquad\ll (k!)^2\sum_{l,m\le k}\big(2\log 3+C\big)^l\big(\log 3+C\big)^m\ll (Ck)^{2k}.
\end{aligned}$$

The contribution of the case where some exponents are larger than 1 in the right hand side of (3.10) is still $\ll (Ck)^{2k}$, by a combination of the previous computation and the argument we used in order to bound S_2.

Now we analyze the error term, which is

$$R \ll \frac{1}{\log x}\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big| + \frac{k}{\varepsilon}.$$

Hence

$$\begin{aligned}&\frac{1}{T\log T}\int_T^{2T}|R|^{2k}|\zeta|^2dt\\ &\ll \frac{1}{(\log x)^{2k}}\frac{1}{T\log T}\int_T^{2T}\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|^{2k}|\zeta|^2dt + \frac{k^{2k}}{\varepsilon^{2k}}\\ &\ll \frac{1}{(\log x)^{2k}}\frac{1}{T\log T}\int_T^{2T}\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|^{2k}|\zeta|^2dt + (\log\log\log T)^{2k}k^{2k}.\end{aligned}$$

We now study the first term, with the aim of proving

$$\frac{1}{T\log T}\int_T^{2T}\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|^{2k}|\zeta|^2dt \ll (Ck)^{2k}(\log x)^{2k}. \tag{3.11}$$

Using the usual approach we get:

$$\begin{aligned}&\frac{1}{T\log T}\int_T^{2T}\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|^{2k}|\zeta|^2dt\\ &\qquad\ll \sum_{\substack{p_1^{r_1},\dots,p_k^{r_k}\le x^3\\ q_1^{s_1},\dots,q_k^{s_k}\le x^3}}\frac{(p_1^{r_1}\cdots p_k^{r_k},q_1^{s_1}\cdots q_k^{s_k})}{p_1^{r_1}\cdots p_k^{r_k}q_1^{s_1}\cdots q_k^{s_k}}\log p_1\cdots\log p_k\log q_1\cdots\log q_k.\end{aligned} \tag{3.12}$$

Once again we start with the case where all the exponents are equal to 1 and we rewrite the sum in the usual way

$$(k!)^2\sum_{l,m\le k}\sum_{\substack{c_1,\dots,c_l\ge1\\ c_1+\dots+c_l=k}}\sum_{\substack{e_1,\dots,e_l\ge0\\ d_1,\dots,d_m\ge1\\ \sum e_i+\sum d_j=k}}\prod_{i=1}^{l}\Big(\sum_{p_i}\frac{(\log p_i)^{e_i+c_i}}{p_i^{\max(e_i,c_i)}}\Big)\prod_{i=1}^{m}\Big(\sum_{q_i}\frac{(\log q_i)^{d_i}}{q_i^{d_i}}\Big).$$

In the case $\max(e_i,c_i)>1$ (or $d_i>1$) the sum in the first (or second, respectively) parentheses is bounded because of the usual argument. The largest contribution comes from the case $0 \le e_i \le 1$, $c_i = 1$, $d_i = 1$, which gives

$$\begin{aligned}
(k!)^2 &\sum_{l,m\le k} \sum_{\substack{0\le e_1,\dots,e_l\le 1\\ e_1+\dots+e_l+m=k}} \prod_{i=1}^{l}\Big(\sum_{p_i}\frac{(\log p_i)^{1+e_i}}{p_i}\Big)\prod_{i=1}^{m}\Big(\sum_{q_i}\frac{\log q_i}{q_i}\Big)\\
&\le (k!)^2 \sum_{l,m\le k} \sum_{\substack{0\le e_1,\dots,e_l\le 1\\ e_1+\dots+e_l+m=k}} \prod_{i=1}^{l}(3\log x+O(1))^{1+e_i}\prod_{i=1}^{m}(3\log x+O(1))\\
&= (k!)^2 \sum_{l,m\le k} \sum_{\substack{0\le e_1,\dots,e_l\le 1\\ e_1+\dots+e_l+m=k}} (3\log x+O(1))^{l+\sum_{i=1}^{l} e_i+m}\\
&\ll (k!)^2 \sum_{l,m\le k} 2^l(4\log x)^{l+k} \ll (\log x)^{2k}(Ck)^{2k}.
\end{aligned}$$

As before, if some exponents among the r_i,s_j in (3.12) are larger than 1, then the contribution of this case in 3.12 is still $\ll (\log x)^{2k}(Ck)^{2k}$, by a combination of the previous computation and the technique we used to study S_2. This proves (3.11) and as a consequence we get

$$\frac{1}{T\log T}\int_T^{2T}\left(\frac{1}{\log x}\Big(\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|+\log T\Big)\right)^{2k}|\zeta|^2dt \qquad (3.13)$$
$$\ll (\log\log\log T)^{2k}(Ck)^{2k}.$$

What remains to investigate is the contribution of $L(t)$. Following Tsang ([158], Equation (5.21)) we have:

$$\Re L(t) \ll L_1(t) + L_2(t) \qquad (3.14)$$

where denoting with $\rho = \frac{1}{2} + i\gamma$ the non-trivial zeros of ζ

$$L_1(t) := \sum_\rho\Big(\frac{4}{\log x}\Big)^2\frac{1}{|\frac{4}{\log x}+i(t-\gamma)|^2}\int_{1/2}^{1/2+4/\log x}\frac{|u-\frac{1}{2}|}{(u-\frac{1}{2})^2+(t-\gamma)^2}du$$

$$L_2(t) := \Big(\frac{4}{\log x}\Big)^2\sum_\rho\frac{1}{|\frac{4}{\log x}+i(t-\gamma)|^2}$$

so we need to study the weighted moments of $L_1(t)$ and $L_2(t)$.
The latter is not difficult; indeed Selberg proved that (see [158], Equation (5.20))

$$\sum_\rho\frac{1}{|\frac{4}{\log x}+i(t-\gamma)|^2} \ll \log x\left(\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big|+\log T\right) \qquad (3.15)$$

hence in view of (3.13) we know that the $2k$-th moment of $L_2(t)$ is $\ll (\log\log\log T)^{2k}(Ck)^{2k}$. To deal with $L_1(t)$, we denote $\eta_t := \min_\rho |t-\gamma|$ and $\log^+ t := \max(\log t, 0)$. From Tsang's computation ([158], p.93) we know that:

$$L_1(t) \ll \frac{1}{\log x}\Bigg(\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big| + \log T\Bigg) + \frac{1}{\log x}\log^+\Big(\frac{1}{\eta_t\log x}\Big)\Bigg(\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big| + \log T\Bigg)$$

and the first term here is not a problem for the same reason as before. As a last step we study the $2k$-th moment of the second term. Applying the Cauchy-Schwarz inequality:

$$\frac{1}{(\log x)^{2k}}\int_T^{2T}\Big(\log^+\frac{1}{\eta_t\log x}\Big)^{2k}\Big(\Big|\sum_{n\le x^3}\frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big| + \log T\Big)^{2k}|\zeta|^2 dt$$
$$\ll \sqrt{T\log T}(\log\log\log T)^{2k}(Ck)^{2k}\sqrt{\int_T^{2T}\Big(\log^+\frac{1}{\eta_t\log x}\Big)^{4k}|\zeta|^2 dt}. \tag{3.16}$$

The proposition follows if we bound the remaining integral. Here the assumption of RH plays a central role, as it is needed in Theorem 2.7. For us the uniform upper bound $\ll T(\log T)^2$ for $|\alpha| \le \frac{\log T}{2\pi\log x}$ will be sufficient. Using this result we get:

$$\int_T^{2T}\Big(\log^+\frac{1}{\eta_t\log x}\Big)^{4k}|\zeta|^2 dt$$
$$\le \sum_{T-\frac{1}{\log x}\le\gamma\le 2T+\frac{1}{\log x}}\int_0^{1/\log x}\Big(\log^+\frac{1}{w\log x}\Big)^{4k}\big|\zeta\big(1/2+i(w+\gamma)\big)\big|^2 dw$$
$$= \sum_{T-\frac{1}{\log x}\le\gamma\le 2T+\frac{1}{\log x}}\int_0^1\Big(\log^+\frac{1}{t}\Big)^{4k}\Big|\zeta\Big(1/2+i\Big(\gamma+\frac{t}{\log x}\Big)\Big)\Big|^2\frac{dt}{\log x}$$
$$= \frac{1}{\log x}\int_0^1(\log t)^{4k}\sum_{T-\frac{1}{\log x}\le\gamma\le 2T+\frac{1}{\log x}}\Big|\zeta\Big(1/2+i\Big(\gamma+\frac{t}{\log x}\Big)\Big)\Big|^2 dt$$
$$\ll \frac{T(\log T)^2}{\log x}\int_0^1(\log t)^{4k}dt \ll T\log T k\varepsilon^{-1}(Ck)^{4k},$$

since $\int_0^1(\log t)^{4k}dt = \int_0^\infty e^{-t}t^{4k}dt = \Gamma(4k+1) = (4k)! \ll (4k)^{4k}$. Putting this into (3.16) one has that also the $2k$-th moment of $L_1(t)$ is bounded by $(Ck)^{4k}\varepsilon^{-1/2}(\log\log T)^{2k}$. Then the contribution of the zeros is under control, being

$$\frac{1}{T\log T}\int_T^{2T} |\Re L(t)|^{2k} dt \ll (Ck)^{4k}(\log\log\log T)^{2k+1/2}$$

and the proposition follows.

3.1.2 Proof of Proposition 3.3

Sketch of the proof

In order to prove Proposition 3.3, we need to perform a precise asymptotic analysis for the moments of $\Re P(t)$. First of all, since the polynomial is short ($n \le x = T^{\varepsilon/k} = T^{o(1/k)}$) one can easily compute its mean and variance by standard applications of Lemma 2.4. Indeed for any r, s integers one has that

$$\int_T^{2T} P(t)^r \overline{P(t)}^s |\zeta|^2 dt$$

equals

$$T \sum_{\substack{p_1,\ldots,p_r\le x\\ q_1,\ldots,q_s\le x}} \frac{(p_1\cdots p_r, q_1\cdots q_s)}{p_1\cdots p_r q_1\cdots q_s}\left(\log\left(\frac{T(p_1\cdots p_r, q_1\cdots q_s)^2}{p_1\cdots p_r q_1\cdots q_s}\right)+c\right) \tag{3.17}$$

up to an error $o(T)$, then, since $2\Re P(t) = P(t)+\overline{P(t)}$, the mean of $\Re P(t)$ is

$$\begin{aligned}\frac{1}{T\log T}\int_T^{2T} \Re P(t)|\zeta|^2 dt &= \frac{1}{\log T}\sum_{p\le x}\frac{\log T-\log p+c}{p} + o\Big(\frac{1}{\log T}\Big)\\ &= \mathcal{L} - \frac{\varepsilon}{k} + O\Big(\frac{\log\log T}{\log T}\Big) = \mathcal{L}+o(1).\end{aligned}$$

Similarly

$$\begin{aligned}
\int_T^{2T} (\Re P(t))^2|\zeta|^2 dt &= \int_T^{2T} \left(\frac{1}{4}P(t)^2 + \frac{1}{2}P(t)\overline{P(t)} + \frac{1}{4}\overline{P(t)}^2\right)|\zeta|^2 dt \\
&= \frac{T}{2}\Bigg[\sum_{p_1,p_2\le x} \frac{\log T - \log(p_1p_2) + c}{p_1p_2} \\
&\qquad + \sum_{p,q\le x} \frac{(p,q)}{pq}\left(\log\left(\frac{T(p,q)^2}{pq}\right) + c\right)\Bigg] + o(T) \\
&= \frac{T}{2}\Big(2\mathcal{L}^2\log T - 4\mathcal{L}\log x + \mathcal{L}\log T + O(\log T)\Big) \\
&= T\log T\left(\mathcal{L}^2 + \frac{\mathcal{L}}{2} - \frac{2\varepsilon\mathcal{L}}{k} + O(1)\right)
\end{aligned}$$

hence the variance is

$$\frac{1}{T\log T}\int_T^{2T} (\Re P(t) - \mathcal{L})^2|\zeta|^2 dt \sim \frac{\mathcal{L}}{2}.$$

To prove Proposition 3.3, we now have to compute the k-th moment of $\Re P(t) - \mathcal{L}$ with respect to $|\zeta|^2 dt$, for every k integer. Here we give a simplified sketch of the proof, leaving the rigorous one for the following section. First of all, since

$$\log\left(\frac{T(p_1\cdots p_r, q_1\cdots q_s)^2}{p_1\cdots p_r q_1\cdots q_s}\right) + c = \log T + \log\left(\frac{(p_1\cdots p_r, q_1\cdots q_s)^2}{p_1\cdots p_r q_1\cdots q_s}\right) + c \tag{3.18}$$

then expanding out the k-th power and using (3.17) one has

$$\begin{aligned}
&\frac{1}{T\log T}\int_T^{2T} (\Re P(t) - \mathcal{L})^k|\zeta|^2 dt \\
&\quad = \sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j 2^{-h}\sum_{r+s=h}\binom{h}{r}\sum_{\substack{p_1,\dots,p_r\le x\\ q_1,\dots,q_s\le x}} \frac{(p_1\cdots p_r, q_1\cdots q_s)}{p_1\cdots p_r q_1\cdots q_s} + \cdots
\end{aligned} \tag{3.19}$$

where the dots come from the contributions of the second and third terms in (3.18), which we are going to ignore in the following. Indeed the contribution of the constant c is clearly analogous but smaller than the one coming from $\log T$. Even though the second term in (3.18) is not negligible compared to the first one, its contribution in the right hand side of (3.19) can be computed in a similar way to the contribution of the first one, with

the important difference that in this case the main term will cancel out. Thus, we ignore it as well for now, focusing on the first term.
Let's suppose now that the primes $p_1, \dots, p_r$ are distinct and the primes $q_1, \dots, q_s$ are distinct as well. In order to compute explicitly the GCD, we fix an integer m, which is smaller than both r and s, and we suppose that m repetitions occur among the p_i and the q_j. Because of the previous assumptions, it can happen in $\binom{r}{m}\binom{s}{m}m!$ ways (selecting m primes among the p_i and m primes among the q_j, then permuting the two blocks multiplying by $m!$), hence

$$\sum_{\substack{p_1,\dots,p_r\le x \text{ distinct}\\ q_1,\dots,q_s\le x \text{ distinct}}} \frac{(p_1\cdots p_r, q_1\cdots q_s)}{p_1\cdots p_r q_1\cdots q_s}$$
$$= \sum_{m\le \min(r,s)} \binom{r}{m}\binom{s}{m} m! \sum_{\substack{p_1,\dots,p_{r+s-m}\le x\\ \text{distinct}}} \frac{1}{p_1\cdots p_{r+s-m}}.$$

We now drop the condition in the inner sum that the primes are distinct. As we will show in the following section, all these assumptions about distinct primes do not affect the asymptotic of the moment we are interested in. Indeed the errors coming from all these extra assumptions will all cancel out and give a contribution which is negligible with respect to the main term. With this assumption the previous sum becomes

$$\sum_{m\le \min(r,s)} \frac{1}{m!}\frac{r!}{(r-m)!}\frac{s!}{(s-m)!}\mathcal{L}^{r+s-m}.$$

Putting this into (3.19), recalling that $r!/(r-m)! = \partial_X^m [X^r]_{X=1}$, for k even we get:

$$\frac{1}{T\log T}\int_T^{2T} (\Re P(t) - \mathcal{L})^k |\zeta|^2 dt$$
$$\approx \sum_{j+h=k}\binom{k}{h}(-1)^j \mathcal{L}^j 2^{-h}\sum_{r+s=h}\binom{h}{r}\sum_{m\le \min(r,s)}\frac{1}{m!}\frac{r!}{(r-m)!}\frac{s!}{(s-m)!}\mathcal{L}^{r+s-m}$$
$$= \sum_{m\le \frac{k}{2}} \frac{\mathcal{L}^{k-m}}{m!}\partial_X^m\partial_Y^m\Big[\Big(\frac{X+Y}{2}-1\Big)^{k-2m}2^{-2m}\Big]_{X=Y=1}$$
$$= \sum_{m\le \frac{k}{2}} \frac{\mathcal{L}^{k-m}}{m!}\Big[\frac{k!}{(k-2m)!}\Big(\frac{X+Y}{2}-1\Big)^{k-2m}2^{-2m}\Big]_{X=Y=1}$$
$$= \sum_{m\le \frac{k}{2}} \frac{\mathcal{L}^{k-m}}{2^{2m}m!}\frac{k!}{(k-2m)!}\mathbb{1}_{2m=k} = \frac{k!}{2^k (k/2)!}\mathcal{L}^{k/2} = \Big(\frac{\mathcal{L}}{2}\Big)^{k/2}(k-1)!!$$

since $k! = 2^{k/2}(k/2)!(k-1)!!$ for any even k. Otherwise if k is odd, then the main term vanishes, being $m \le (k-1)/2$.

We now highlight the main difference from the classical case [135]. There one easily sees that $\int P(t)^r \overline{P(t)}^s dt$ is non negligible only if r equals s. Therefore just the diagonal term $r = s = k/2$ contributes to the main term of the k-th moment of $\Re P(t)$. On the other hand this is no longer true in the weighted case, since all the integrals $\int P(t)^r \overline{P(t)}^s |\zeta|^2 dt$ give a contribution of order $T \log T \mathcal{L}^{r+s}$. The main point is that in the classical case the mean of $\Re P(t)$ is 0, while with respect to the weighted measure $|\zeta|^2 dt$ the mean is $\sim \mathcal{L}$. Thus, even though in the weighted case the size of the k-th moment of $\Re P(t)$ is $\mathcal{L}^k$, the k-th moment of $\Re P(t) - \mathcal{L}$ has order $\mathcal{L}^{k/2}$. Showing this cancellation from k to $k/2$ is the bulk of the proof.

Rigorous proof

We now prove the result, following the line of the previous computation. Expanding out the k-th power and using $2\Re P(t) = P(t) + \overline{P(t)}$, one finds

$$\begin{aligned}\int_T^{2T} (\Re P(t) - \mathcal{L})^k |\zeta|^2 dt \\ = \sum_{j+h=k} \binom{k}{h} (-1)^j \mathcal{L}^j 2^{-h} \sum_{r+s=h} \binom{h}{r} \int_T^{2T} P(t)^r \overline{P(t)}^s |\zeta|^2 dt\end{aligned} \tag{3.20}$$

and the inner integral equals

$$T \sum_{\substack{p_1,\dots,p_r \le x \\ q_1,\dots,q_s \le x}} \frac{(p_1 \cdots p_r, q_1 \cdots q_s)}{p_1 \cdots p_r q_1 \cdots q_s} \left(\log \left(\frac{T (p_1 \cdots p_r, q_1 \cdots q_s)^2}{p_1 \cdots p_r q_1 \cdots q_s} \right) + c \right) + o(T)$$

in view of (3.17). Since $\log t = \partial_w [t^w]_{w=0}$, one gets

$$\int_T^{2T} P(t)^r \overline{P(t)}^s |\zeta|^2 dt = T(\log T + c) f_x(0) + T \partial_w [f_x(w)]_{w=0} + o(T) \tag{3.21}$$

where

$$f_x(w) := \sum_{\substack{p_1,\dots,p_r \le x \\ q_1,\dots,q_s \le x}} \frac{(p_1 \cdots p_r, q_1 \cdots q_s)^{2w+1}}{(p_1 \cdots p_r q_1 \cdots q_s)^{w+1}}.$$

In order to be able to compute explicitly the GCD, we put in evidence the possible repetitions among the primes, re-writing the p_i and the q_i as follows. First we put in evidence the repetitions among the primes p_i, writing

$$p_1,\dots,p_r \longrightarrow p_1,\dots,p_{r-v_1},{p'}_1^{\alpha_1},\dots,{p'}_{u_1}^{\alpha_{u_1}}$$

where $p_1,\dots,p_{r-v_1},{p'}_1,\dots,{p'}_{u_1}$ are all distinct, $\alpha_1+\dots+\alpha_{u_1}=v_1$, $\alpha_i\geq 2$ for every i. With this change of variable we need a normalization $\frac{r!}{(r-v_1)!}c_{\underline{\alpha}}$, where $c_{\underline{\alpha}}$ is a positive coefficient smaller than 1, which does not depend on r but just on the configuration $\alpha_1,\dots,\alpha_{u_1}$. Notice that if $v_1=0$, then $c_{\underline{\alpha}}=1$. Now we highlight the multiplicities of the primes q_j and we put in evidence those ones that already appear among the p'_i. Then we write

$$q_1,\dots,q_s \longrightarrow q_1,\dots,q_{s-v_2-a_2},p'_1,\dots,p'_{a_2},{q'}_1^{\beta_1},\dots,{q'}_{u_2}^{\beta_{u_2}},{p'}_1^{\gamma_1},\dots,{p'}_{u_1}^{\gamma_{u_1}}$$

with q_i distinct, q'_i distinct, $q'_i\neq {p'}_j$ for every i,j, $q_i\neq q'_j,p'_j$ for every i,j and $\beta_1+\dots+\beta_{u_2}+\gamma_1+\dots+\gamma_{u_1}+a_2=v_2$, $\beta_i\geq 2$, $\gamma_i\neq 1$ for every i. Also in this case the change of variable brings into play a normalization $\binom{s-v_2}{a_2}\binom{u_1}{a_2}a_2!\frac{s!}{(s-v_2)!}c_{\underline{\beta},\underline{\gamma}}$, where once again $c_{\underline{\beta},\underline{\gamma}}$ only depends on the configuration $\beta_1,\dots,\beta_{u_2},\gamma_1,\dots,\gamma_{u_1}$ and it is equal to 1 when $u_2=0$ and $\gamma_i=0$ for every i. The normalization coefficient comes from standard combinatorics as follows. We make the multiplicity of any q_i explicit, putting in evidence the $s-v_2$ ones which appear once. This can be done in $s!/(s-v_2)!$ ways times a coefficient described above, which does not depend on s. Moreover, in order to highlight the coincidences between the q_i and the p'_i (say we have a_2 coincidences), we select a_2 primes among $q_1,\dots,q_{s-v_2}$ ($\binom{s-v_2}{a_2}$ ways) and a_2 primes among $p'_1,\dots p'_{u_1}$ ($\binom{u_1}{a_2}$ ways) and then we permute the two blocks multiplying by $a_2!$. Then we have

$$f_x(w)=$$
$$\sum_{\substack{v_1,u_1\leq r\\ v_2,u_2\leq s}}\sum_{a_2\leq s-v_2}\binom{s-v_2}{a_2}\binom{u_1}{a_2}a_2!\sum_{\substack{\alpha_1+\dots+\alpha_{u_1}=v_1\\ \alpha_i\geq 2\ \forall i}}c_{\underline{\alpha}}\sum_{\substack{\beta_1+\dots+\beta_{u_2}+\gamma_1+\dots+\gamma_{u_1}=v_2\\ \beta_i\geq 2,\ \gamma_i\neq 1\ \forall i}}c_{\underline{\beta},\underline{\gamma}}$$
$$\frac{r!}{(r-v_1)!}\frac{s!}{(s-v_2)!}\sum_{\substack{p'_i,q'_j\\ \text{distinct}}}\frac{({p'}_1^{\alpha_1}\cdots{p'}_{u_1}^{\alpha_{u_1}},p'_1\cdots p'_{a_2}{p'}_1^{\gamma_1}\cdots{p'}_{u_1}^{\gamma_{u_1}})^{2w+1}}{({p'}_1^{\alpha_1}\cdots{p'}_{u_1}^{\alpha_{u_1}}p'_1\cdots p'_{a_2}{q'}_1^{\beta_1}\cdots{q'}_{u_2}^{\beta_{u_2}}{p'}_1^{\gamma_1}\cdots{p'}_{u_1}^{\gamma_{u_1}})^{w+1}}$$
$$\sum_{\substack{p_1,\dots,p_{r-v_1}\leq x\\ \text{distinct and }\neq p'_i\\ q_1,\dots,q_{s-v_2-a_2}\leq x\\ \text{distinct and }\neq p'_i,q'_j}}\frac{[(p_1\cdots p_{r-v_1},q_1\cdots q_{s-v_2-a_2})(p_1\cdots p_{r-v_1},{q'}_1^{\beta_1}\cdots{q'}_{u_2}^{\beta_{u_2}})]^{2w+1}}{(p_1\cdots p_{r-v_1}q_1\cdots q_{s-v_2-a_2})^{w+1}}.$$

For the sake of brevity let's denote $\underline{p'}$ and $\underline{q'}$ the product of p'_i and q'_i respectively with their exponents (for instance $\underline{p'}^{\underline{\alpha}}:={p'}_1^{\alpha_1}\cdots{p'}_{u_1}^{\alpha_{u_1}}$). To be able to compute the GCD between $\underline{p}$ and $\underline{q'}^{\underline{\beta}}$ in the inner sum, we now put in evidence the repetitions among the p_i and the q'_j. Let's say we have a_1 primes among the p_i which coincide with some q'_j. Then, denoting $r' := r - v_1 - a_1$ and $s' := s - v_2 - a_2$, we get

$$f_x(w) = \sum_{\substack{v_1,u_1\le r\\ v_2,u_2\le s}} \sum_{\substack{a_1\le r-v_1\\ a_2\le s-v_2}} \sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}} c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a}) \frac{r!}{(r-v_1-a_1)!} \frac{s!}{(s-v_2-a_2)!} \sum_{\substack{p'_i,q'_j\\ \text{distinct}}} \frac{(\underline{p'}^{\underline{\alpha}}, \underline{p'}\underline{p'}^{\underline{\gamma}})^{2w+1}(\underline{q'})^w}{(\underline{p'}^{\underline{\alpha}}\underline{p'}\underline{q'}^{\underline{\beta}}\underline{p'}^{\underline{\gamma}})^{w+1}} \sum_{\substack{p_1,\dots,p_{r'} \text{ distinct and } \neq p'_i,q'_j\\ q_1,\dots,q_{s'} \text{ distinct and } \neq p'_i,q'_j}} \frac{(p_1\cdots p_{r'}, q_1\cdots q_{s'})^{2w+1}}{(p_1\cdots p_{r'}q_1\cdots q_{s'})^{w+1}}$$

where $c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a})$ is a bounded coefficient which does not depend on r and s and it is equal to 1 when $u_i = v_i = a_i = 0$ for $i = 1,2$. Note that the sum over p'_i and q'_j is bounded when w is close to 0, since both β_i and $\max(\alpha_i, \gamma_i + 1)$ are ≥ 2. Lastly we want to put in evidence the repetitions among the p_i and the q_j, in order compute explicitly the last greatest common divisor $(p_1\cdots p_{r'}, q_1\cdots q_{s'})$ in the inner sum. If m repetitions occur, for any $m \le \min(r', s')$, we finally have $r' + s' - m$ distinct primes and the coefficient of normalization is $\binom{r'}{m}\binom{s'}{m}m!$. Therefore

$$f_x(w) = \sum_{\substack{v_1,u_1\le r\\ v_2,u_2\le s}} \sum_{\substack{a_1\le r-v_1\\ a_2\le s-v_2}} \sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}} c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a}) \sum_{\substack{p'_i,q'_j\\ \text{distinct}}} \frac{(\underline{p'}^{\underline{\alpha}}, \underline{p'}\underline{p'}^{\underline{\gamma}})^{2w+1}(\underline{q'})^w}{(\underline{p'}^{\underline{\alpha}}\underline{p'}\underline{q'}^{\underline{\beta}}\underline{p'}^{\underline{\gamma}})^{w+1}} \sum_{m\le\min(r',s')} \frac{r!}{(r'-m)!}\frac{s!}{(s'-m)!}\frac{1}{m!} \sum_{\substack{p_1,\dots,p_{r'+s'-2m}\\ q_1,\dots,q_m\\ \text{distinct and } \neq p'_i,q'_j}} \frac{1}{(p_1\cdots p_{r'+s'-2m})^{w+1}q_1\cdots q_m}. \tag{3.22}$$

After computing the GCD, we now remove the extra conditions in the inner sum, which force the primes p_i and q_j to be all distinct and $\neq p'_i, q'_j$. We get rid of the condition that forces the primes to be all distinct by using basic combinatorics and we remove the last condition $p_1,\dots,p_{r'+s'-2m}, q_1,\dots,q_m \neq p'_i, q'_j$, splitting the inner sums as

$$\sum_{\substack{p\le x\\ p\neq p'_i,q'_j}} \frac{1}{p^s} = \sum_{p\le x}\frac{1}{p^s} - \sum_{i=1}^{u_1}\frac{1}{p'^s} - \sum_{i=1}^{u_2}\frac{1}{q'^s}$$

and expanding out the powers by Newton's binomial formula. Hence we have (denote $h' := r' + s'$)

$$
\begin{aligned}
&f_x(w) = \\
&\sum_{\substack{v_1,u_1\le r\\ v_2,u_2\le s}}\sum_{\substack{a_1\le r-v_1\\ a_2\le s-v_2}}\sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}}\sum_{\substack{p'_i,q'_j\\ \text{distinct}}}\frac{(\underline{p'}^{\underline{\alpha}},\underline{p'}\underline{p'}^{\underline{\gamma}})^{2w+1}(\underline{q'})^w}{(\underline{p'}^{\underline{\alpha}}\underline{p'}\underline{q'}^{\underline{\beta}}\underline{p'}^{\underline{\gamma}})^{w+1}}\sum_{m\le\min(r',s')}\frac{r!}{(r'-m)!}\frac{s!}{(s'-m)!}\\
&\sum_{\substack{t_1\le h'-2m\\ t_2\le m\\ t_3\le t_1+t_2}}\sum_{\substack{\mathcal{P}\in Part\\ \mathcal{P}=\{R_1,\dots,R_{t_3}\}\\ \text{with } r_i:=\sum_{j\in R_i}a_j\ge 2}}\prod_{i=1}^{t_3}(\#R_i-1)!(-1)^{\#R_i-1}\Big(\sum_{p\ne\underline{p'},\underline{q'}}\frac{1}{p^{r_i}}\Big)c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a},\underline{t})\\
&\sum_{l_1\le h'-2m-t_1}\frac{(h'-2m)!}{l_1!(h'-2m-t_1-l_1)!}\Big(-\sum_{i=1}^{u_1}\frac{1}{p'^{1+w}}-\sum_{i=1}^{u_2}\frac{1}{q'^{1+w}}\Big)^{l_1}\\
&\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)^{h'-2m-t_1-l_1}\sum_{l_2\le m-t_2}\frac{1}{l_2!(m-t_2-l_2)!}\Big(-\sum_{i=1}^{u_1}\frac{1}{p'}-\sum_{i=1}^{u_2}\frac{1}{q'}\Big)^{l_2}\mathcal{L}^{m-t_2-l_2}
\end{aligned}
$$

where $c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a},\underline{t})$ is a bounded coefficient not depending on r,s,m, which is equal to 1 if the parameters v_i,u_i,t_i are all equal to 0 and $Part$ denotes the set of partitions of the set of the exponents of primes appearing in the inner sum in (3.22).

We are ready to plug the formula we got for $f_x(w)$ into the formula for the k-th moment of $\Re P(t)-\mathcal{L}$. Putting (3.20) and (3.21) together one has

$$
\begin{aligned}
\int_T^{2T}&(\Re P(t)-\mathcal{L})^k|\zeta|^2dt\\
&=(T(\log T+c))\Big[\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}\\
&\quad+\partial_w\Big[T\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}+o(T).
\end{aligned}
\tag{3.23}
$$

Now we exchange the order of summation, bringing the sum over j,h inside in order to appreciate the cancellation. By the explicit expression we got for $f_x(w)$ we have

$$\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j 2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)$$
$$=\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p_i',q_j'\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w)\frac{1}{(m-t_2-l_2)!}\mathcal{L}^{m-t_2-l_2}$$
$$\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j 2^{-h}\frac{(h'-2m)!}{(h'-2m-t_1-l_1)!}\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)^{h'-2m-t_1-l_1}$$
$$\sum_{r+s=h}\binom{h}{r}\frac{r!}{(r-v_1-a_1-m)!}\frac{s!}{(s-v_2-a_2-m)!} \tag{3.24}$$

where we denote $k':=k-v_1-v_2-a_1-a_2$ and

$$F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w):=$$
$$\frac{(\underline{p'}^{\underline{\alpha}},\underline{p'}\underline{p'}^{\underline{\gamma}})^{2w+1}(\underline{q'})^w}{(\underline{p'}^{\underline{\alpha}}\underline{p'}\underline{q'}^{\underline{\beta}}\underline{p'}^{\underline{\gamma}})^{w+1}}c(\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{a},\underline{t},\underline{l})\Big(\sum_{p\ne p_i',q_j'}\frac{1}{p^{r_i}}\Big)\prod_{i=1}^{t_3}(\#R_i-1)!$$
$$(-1)^{\#R_i-1}\Big(-\sum_{i=1}^{u_1}\frac{1}{p'^{1+w}}-\sum_{i=1}^{u_2}\frac{1}{q'^{1+w}}\Big)^{l_1}\Big(-\sum_{i=1}^{u_1}\frac{1}{p'}-\sum_{i=1}^{u_2}\frac{1}{q'}\Big)^{l_2}.$$

Note that the function $F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w)$ makes the sum over p_i,q_j in (3.24) converge. Moreover notice that in the trivial case $v_i=u_i=a_i=t_i=l_i=0$ for every i then we have that $F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w)=1$. Now we recall that the three quotients involving $r!$, $s!$ and $h'!$ can be expressed in terms of derivatives (for instance $r!/(r-v_1-a_1-m)!=\partial_X^{v_1+a_1+m}[X^r]_{X=1}$) then (3.24) becomes

$$\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p_i',q_j'\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w)\frac{1}{(m-t_2-l_2)!}\mathcal{L}^{m-t_2-l_2}$$
$$\partial_X^{v_1+a_1+m}\partial_Y^{v_2+a_2+m}\partial_Z^{t_1+l_1}\Big[\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)^{-v_1-v_2-a_1-a_2-2m-t_1-l_1}Z^{-v_1-v_2-a_1-a_2-2m}$$
$$\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}Z^h\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)^h(X+Y)^h\Big]_{X=Y=Z=1}.$$

Carrying out the computation straightforwardly, denoting $y=v_1+v_2+$

$a_1+a_2+t_1+l_1$, it yields

$$\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)$$
$$=\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p'_i,q'_j\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p'_i,q'_j;w)\frac{\mathcal{L}^{m-t_2-l_2}}{(m-t_2-l_2)!}$$
$$\frac{k(k-1)\cdots(k-y-2m+1)}{2^{y-t_1-l_1+2m}}\bigg(\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)-\mathcal{L}\bigg)^{k-y-2m}. \tag{3.25}$$

Now, recalling (3.23), we have to study the right hand side of (3.25) and its derivative at $w=0$. As we will see soon, only the former contributes to the main term of the k-th moment we are considering.

By definition of $\mathcal{L}:=\sum_{p\le x}\frac{1}{p}$, if $w=0$ then the expression in the parentheses on the right hand side of (3.25) vanishes. This forces its exponent to be zero, otherwise all the contribution vanishes. Hence we get

$$\Big[\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}$$
$$=\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p'_i,q'_j\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p'_i,q'_j;w)\frac{\mathcal{L}^{m-t_2-l_2}}{(m-t_2-l_2)!} \tag{3.26}$$
$$\frac{k(k-1)\cdots(k-y-2m+1)}{2^{y-t_1-l_1+2m}}\mathbb{1}_{2m=k-y}.$$

The main term is given by the largest m possible, i.e. $m=\frac{k}{2}$ if k is even. Since $2m=k-y$, then $y=0$ hence all the parameters that individuate the configuration vanish. Therefore

$$\Big[\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}=\frac{k!}{2^k(k/2)!}\mathcal{L}^{k/2}+O_k(\mathcal{L}^{k/2-1}) \tag{3.27}$$

which matches with the k-th moment of a Gaussian by basic properties of the double factorial, since $k!=2^{k/2}(k/2)!(k-1)!!$ for any even k. Note that the error term in (3.27) is given by the term $m=k/2-1$ hence it is $O_k(\mathcal{L}^{k/2-1})$. Of course if k is odd one can immediately see that the right hand side of (3.26) is $O_k(\mathcal{L}^{(k-1)/2})$.

Let's now analyze the derivative

$$\partial_w\Big[\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j 2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}$$
$$=\partial_w\Big[\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p_i',q_j'\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';w)\frac{1}{(m-t_2-l_2)!}\mathcal{L}^{m-t_2-l_2}$$
$$\cdot\frac{k(k-1)\cdots(k-y-2m+1)}{2^{y-t_1-l_1+2m}}\Big(\Big(\sum_{p\le x}\frac{1}{p^{1+w}}\Big)-\mathcal{L}\Big)^{k-y-2m}\Big]_{w=0}. \tag{3.28}$$

Recall that this term will be multiplied by a factor T in (3.23), while the other one by $T\log T$. When we compute the derivative using Leibniz's rule, the term where the derivative of F appears is trivially $O_k(\mathcal{L}^k/\log T)$, which is negligible. Indeed the sum over p_i', q_j' is still bounded because the exponents of the variables are larger that 2 and computing derivatives just $\log p_i$ or $\log q_j$ come out. We finally have to deal with the derivative of the inner term. Since

$$\partial_w\Big[\sum_{p\le x}\frac{1}{p^{1+w}}\Big]_{w=0}=-\sum_{p\le x}\frac{\log p}{p}\ll\log x=\frac{\varepsilon}{k}\log T$$

we get that the contribution coming from derivative of p^{-1-w} in (3.28) is

$$\ll\sum_{m\le\frac{k}{2}}\sum_{\substack{\underline{v},\underline{u},\underline{a}\\ \underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}\\ p_i',q_j'\text{ distinct}}}F_{\underline{v},\underline{u},\underline{a},\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{t},\mathcal{P},\underline{l}}(p_i',q_j';0)\frac{1}{(m-t_2-l_2)!}\mathcal{L}^{m-t_2-l_2}$$
$$\frac{k(k-1)\cdots(k-y-2m+1)}{2^{y-t_1-l_1+2m}}\partial_w\Bigg[\Big(\sum_{p\le x}\frac{1}{p^{1+w}}-\mathcal{L}\Big)^{k-y-2m}\Bigg]_{w=0}$$

which is $O_k\big(\varepsilon\log T\mathcal{L}^{(k-1)/2}\big)$ by the same argument as before. Hence

$$\partial_w\Big[\sum_{j+h=k}\binom{k}{h}(-1)^j\mathcal{L}^j 2^{-h}\sum_{r+s=h}\binom{h}{r}f_x(w)\Big]_{w=0}=O_k\big(\varepsilon\log T\mathcal{L}^{(k-1)/2}\big) \tag{3.29}$$

Putting both (3.27) and (3.29) into (3.23) the proof is complete.

3.2 The measures $|\zeta^{(m)}(1/2+it)|^{2k}dt$

In this section we generalize what we did in the previous one, investigating the value distribution of $\log|\zeta(1/2+it)|$ with respect to the measure

$$|\zeta^{(m)}(1/2+it)|^{2k}dt \tag{3.30}$$

for any fixed m, k non negative integers. The motivation is again due to the study of the large values of the Riemann zeta function and in particular to conjecture (1.17), since performing the same computation as in (3.3) we can express the quantity

$$\frac{1}{T}\operatorname*{meas}_{t\in[T,2T]}\Big\{\log|\zeta(1/2+it)|\geq k\log\log T\Big\} \tag{3.31}$$

in terms of the integral

$$\frac{1}{T(\log T)^{k^2}}\int_T^{2T} e^{iu(\log|\zeta(1/2+it)|-k\log\log T)}|\zeta(1/2+it)|^{2k}dt \tag{3.32}$$

for any fixed $k>0$. In analogy to what happened in Section 3.1, the frequency of large values (3.31) is then related to the distribution of $\log|\zeta(1/2+it)|$ with respect to the weighted measure (3.30), with $m=0$. Theorem 3.1 conditionally shows that, in the case $k=1$ and $m=0$, the Riemann zeta function behave log-normally; here, assuming RH, we prove that a central limit theorem for $\log|\zeta(1/2+it)|$ with respect to the measure $|\zeta^{(m)}(1/2+it)|^{2k}dt$ can be proved for every $m\in\mathbb{N}$, in both cases $k=1$ and $k=2$.

Corollary 3.4. *Assume the Riemann Hypothesis, let m be a non negative integer and $k=1$ or $k=2$. As t varies in $T\leq t\leq 2T$, the distribution of $\log|\zeta(1/2+it)|$ is asymptotically Gaussian with mean $k\log\log T$ and variance $\frac{1}{2}\log\log T$, with respect to the weighted measure $|\zeta^{(m)}(1/2+it)|^{2k}dt$.*

For all the other values of k, since not even the moments of zeta are known, we cannot expect to prove a central limit theorem, relying on RH only. However, if k is a positive integer, assuming the asymptotic formula for the twisted and shifted $2k$-th moments of the Riemann zeta function we can deal with the general case too, as shown in the following theorem.

Theorem 3.5. *Let $k, m\in\mathbb{N}$ and assume the Riemann Hypothesis and Conjecture 2.6 for k. As t varies in $T\leq t\leq 2T$, the distribution of $\log|\zeta(1/2+it)|$ is asymptotically Gaussian with mean $k\log\log T$ and variance $\frac{1}{2}\log\log T$, with respect to the weighted measure $|\zeta^{(m)}(1/2+it)|^{2k}dt$.*

In particular, being Conjecture 2.6 known in the cases $k=1$ and $k=2$ (see Lemma 2.4 and Theorem 2.5 respectively), we notice that Corollary 3.4 trivially follows from Theorem 3.5, without any further assumption than RH. Moreover, we remark that the proof of Theorem 3.5 (and so Corollary 3.4) differs from that of Theorem 3.1, being more direct and general. Nevertheless, the strategy we used in order to prove Theorem 3.1 leads to a stronger result, as it gives not only the distribution but also the moments (see Proposition 3.2).

We remark that Theorem 3.5 shows that the m-th derivative has no effect in the weighted distribution of $\log|\zeta(1/2+it)|$. This is consistent with the mixed moment conjecture ((2.7) and the nearby discussions), which indicates that $|\zeta'(1/2+it)|^{2h}$ amplifies the contribution coming from the large values in the same way as $|\zeta(1/2+it)|^{2h}$ would do (up to a normalization of $(\log T)^{2h}$). More generally, as far as moments are concerned, the m-th derivative of zeta should behave like zeta itself (see e.g. [32]), up to a normalization of $\log^m T$, in accordance with Theorem 3.5. We also note that, while in Selberg's classical case the mean is 0 because the contribution of the small values and that of the large values of zeta balance out, tilting with $|\zeta(1/2+it)|^{2k}$ the mean of $\log|\zeta(1/2+it)|$ moves to the right as k grows and this reflects the fact that the measure $|\zeta(1/2+it)|^{2k}dt$ gives more and more weight to the large values of the Riemann zeta function.

Moreover, we look at the shifted weighted measure $|\zeta(1/2+it+i\alpha)|^{2k}dt$ with α a real number such that $|\alpha|<1$. As we will see, the distribution of $\log|\zeta(1/2+it)|$ is quite sensitive to the parameter α. Indeed in computing the integral

$$\int_T^{2T} \log|\zeta(1/2+it)||\zeta(1/2+it+i\alpha)|^{2k}dt \tag{3.33}$$

one expects the same magnitude as in the unshifted case if $|\alpha|$ is smaller than $(\log T)^{-1}$, which is the typical scale for the Riemann zeta function. On the other hand, if $|\alpha|$ is larger than the two factors in the integral (3.33) start decorrelating, thus the size of the integral decreases. This phenomenon is shown in the following result, in which we use the notation

$$\widetilde{\mu}_\alpha := \begin{cases} \log\log T + O(1) & \text{if } |\alpha|\log T \leq 1 \\ -\log|\alpha| + O(1) & \text{if } |\alpha|\log T > 1 \end{cases}$$

for any $\alpha \in (-1,1)$.

Theorem 3.6. *Let $k \in \mathbb{N}$ and assume the Riemann Hypothesis and Conjecture 2.6 for k. As t varies in $T \le t \le 2T$, for any fixed and real α such that $|\alpha| < 1$, the distribution of $\log|\zeta(1/2+it)|$ is asymptotically Gaussian with mean $k\widetilde{\mu}_\alpha$ and variance $\frac{1}{2}\log\log T$, with respect to the measure $|\zeta(1/2+it+i\alpha)|^{2k}dt$.*

This theorem shows that the shift has no effect if it is smaller than $(\log T)^{-1}$. On the contrary, for larger values of the shift the mean gets smaller, for instance if

$$\alpha = \frac{(\log T)^\delta}{\log T}$$

with $\delta \in (0,1)$ then the mean is $\sim (1-\delta)k\log\log T$. In this shifted case too, if $k \le 2$ (and $k \in \mathbb{N}$ of course) then Theorem 3.6 holds assuming RH only.

Lastly, we show that in the random matrix theory setting, an analogous weighted central limit theorem can be proved unconditionally. We consider the characteristic polynomials

$$Z = Z(U,\theta) = \det(I - Ue^{-i\theta})$$

of $N \times N$ unitary matrices U and we investigate their distribution of values with respect to the circular unitary ensemble (CUE). As described in Section 1.2 it has been conjectured that the limiting distribution of the non-trivial zeros of the Riemann zeta function, on the scale of their mean spacing, is the same as that of the eigenphases θ_n of matrices in the CUE in the limit as $N \to \infty$. Then we consider a tilted version of the Haar measure and we have the following theorem.

Theorem 3.7. *As $N \to \infty$, the value distribution of $\log|Z|$ is asymptotically Gaussian with mean $k\log N$ and variance $\frac{1}{2}\log N$ with respect to the measure $|Z|^{2k}d_{Haar}$.*

As usual, the correspondence with Theorem 3.5 holds if we identify the mean density of the eigenangles θ_n, $N/2\pi$, with the mean density of the Riemann zeros at a height T up the critical line, $\frac{1}{2\pi}\log\frac{T}{2\pi}$, i.e. if

$$N = \log\frac{T}{2\pi}.$$

3.2.1 Proof of Theorems 3.5 and 3.6

To prove both the theorems, we introduce a set of shifts $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k$ and we denote for the sake of brevity

$$\boldsymbol{\zeta}_{\alpha,\beta}(t) := \zeta(1/2+\alpha_1+it)\cdots\zeta(1/2+\alpha_k+it)\zeta(1/2+\beta_1-it)\cdots\zeta(1/2+\beta_k-it).$$

The general strategy of the proof is similar to the one of Theorem 3.1, but here we avoid the detailed combinatorial analysis we performed in Section 3.1, by working with Euler products instead of Dirichlet series, inspired by [2, Proposition 5.1]. The first step is then approximating the logarithm of the Riemann zeta function with a suitable Dirichlet polynomial. With a slight change of notation with respect to the previous section, let's denote

$$\widetilde{P}(t) := \sum_{p\leq x} \frac{1}{p^{1/2+it}} \tag{3.34}$$

where $x := T^{\varepsilon}$, with $\varepsilon := (\log\log\log T)^{-1}$. Now, assuming RH, we show that $\log|\zeta(1/2+it)|$ has the same distribution as $\Re\widetilde{P}(t)$ with respect to the measure $\boldsymbol{\zeta}_{\alpha,\beta}(t)dt$, if the shifts are small enough. This is achieved in the following proposition by bounding the second moment of the difference. Note that, as mentioned before, this is weaker than what we did in Proposition 3.2, as it does not ensure that all the moments of $\log|\zeta(1/2+it)|$ are approximated by those of the Dirichlet polynomial, but still it is enough to deduce Theorem 3.5, once we have studied the distribution of the polynomial. With the same strategy, one should nonetheless be able the bound every moments of the difference in this general case too.

Proposition 3.8. *For k a non negative integer, assume Conjecture 2.6 and RH. Let T be a large parameter and $\alpha_1,\dots,\alpha_k$, $\beta_1,\dots,\beta_k\in\mathbb{C}$ such that $|\alpha_i|,|\beta_j|<1$, $|\Re(\alpha_i)|,|\Re(\beta_j)|\leq(\log T)^{-1}$ and $|\alpha_i-\beta_j|\ll(\log T)^{-1}$ for all $1\leq i,j\leq k$. Then we have:*

$$\int_T^{2T}\big|\log|\zeta(1/2+it)|-\Re\widetilde{P}(t)\big|^2\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\ll_k T(\log T)^{k^2}(\log\log\log T)^{5/2}.$$

Proof. The starting point is the same as in the proof of Proposition 3.2. We recall that, from Tsang's work [158, Equation (5.15)] we know that

$$\log\zeta(1/2+it)-\widetilde{P}(t)=S_1+S_2+S_3+O(R)-L(t) \tag{3.35}$$

with

$$S_1 := \sum_{p\leq x}\big(p^{-1/2-4/\log x}-p^{-1/2}\big)p^{-it}, \qquad S_2 := \sum_{\substack{p^r\leq x\\ r\geq 2}}\frac{p^{-r(1/2+4/\log x+it)}}{r}$$

$$S_3 := \sum_{x<n\leq x^3}\frac{\Lambda(n)}{\log n}n^{-1/2-4/\log x-it}, \quad R := \frac{5}{\log x}\left(\left|\sum_{n\leq x^3}\frac{\Lambda(n)}{n^{1/2+4/\log x+it}}\right|+\log T\right)$$

$$L(t) := \sum_\rho \int_{1/2}^{1/2+4/\log x} \left(\frac{1}{2}+\frac{4}{\log x}-u\right)\frac{1}{u+it-\rho}\frac{1}{\frac{1}{2}+\frac{4}{\log x}-\rho}du,$$

where the sum in the definition of $L(t)$ is over all the non-trivial zeros of ζ, then we have to bound the second moment of the terms on the right hand side of (3.35) with respect to weighted measure $\boldsymbol{\zeta}_{\alpha,\beta}(t)dt$, by using Conjecture 2.6 (note that we are allowed to apply the conjecture, since the shifts are small up to a change of variable, being $|\alpha_i-\beta_j| \ll (\log T)^{-1}$).

Let's start with $\int_T^{2T} |S_1|^2 \boldsymbol{\zeta}_{\alpha,\beta}(t)dt$ which is bounded by

$$\left|\int_T^{2T} \sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}} \left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}} \sum_{p,q\le x}(p^{-4/\log x}-1)(q^{-4/\log x}-1)Z_{\alpha_\mathcal{S},\beta_\mathcal{T},p,q}(t)dt\right|$$
$$\ll \int_T^{2T}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\left|\sum_{a,b\le x}\mathbb{1}_x(a)\mathbb{1}_x(b)(a^{-\frac{4}{\log x}}-1)(b^{-\frac{4}{\log x}}-1)Z_{\alpha_\mathcal{S},\beta_\mathcal{T},a,b}(t)\right|dt \tag{3.36}$$

where $\mathbb{1}_x(\cdot)$ is the indicator function of primes up to x. If we denote $\Omega(n)$ the function which counts the number of prime factors of n with multiplicity, then we have

$$\sum_{n\le x}\mathbb{1}_x(n)f(n) = \sum_{\substack{n:\\ p|n \implies p\le x}} \partial_z\left[z^{\Omega(n)}f(n)\right]_{z=0} \tag{3.37}$$

hence we get

$$\int_T^{2T}|S_1|^2\zeta_{\alpha,\beta}(t)dt$$
$$\ll \int_T^{2T}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\left|\partial_z\partial_w\left[\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{ks}I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z,w;s)ds\right]_{\substack{z=0\\ w=0}}\right|dt \tag{3.38}$$

where

$$I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z,w;s) := \sum_{\substack{a,b:\\ p|ab\implies p\le x}} z^{\Omega(a)}w^{\Omega(b)}g_x(a)g_x(b)\widetilde{Z}_{\alpha_\mathcal{S},\beta_\mathcal{T},a,b}(s) \tag{3.39}$$

with g_x the multiplicative function defined by $g_x(p^\alpha) = p^{-4\alpha/\log x} - 1$ and

$$\widetilde{Z}_{\alpha,\beta,a,b}(s) := \sum_{am_1\cdots m_k = bn_1\cdots n_k} \frac{1}{\sqrt{ab}\, m_1^{1/2+\alpha_1+s}\cdots m_k^{1/2+\alpha_k+s} n_1^{1/2+\beta_1+s}\cdots n_k^{1/2+\beta_k+s}}. \tag{3.40}$$

We now analyze the first term $I^0_{\alpha,\beta}(z,w;s)$ and then we will see how to apply the method to deal with all the others. Since the sum in the definition (3.39) is multiplicative we have

$$\begin{aligned} I^0_{\alpha,\beta}(z,w;s) &= \sum_{\substack{am_1\cdots m_k = bn_1\cdots n_k:\\ p|ab \implies p\le x}} \frac{z^{\Omega(a)} w^{\Omega(b)} g_x(a) g_x(b)}{\sqrt{ab}\, m_1^{\frac12+\alpha_1+s}\cdots m_k^{\frac12+\alpha_k+s} n_1^{\frac12+\beta_1+s}\cdots n_k^{\frac12+\beta_k+s}} \\ &= \prod_{p\le x} \sum_{\substack{a+m_1+\cdots+m_k=\\ =b+n_1+\cdots+n_k}} \frac{z^{\Omega(p^a)} w^{\Omega(p^b)} g_x(p^a) g_x(p^b)}{p^{\frac a2+\frac b2+m_1(\frac12+\alpha_1+s)+\cdots+m_k(\frac12+\alpha_k+s)+n_1(\frac12+\beta_1+s)+\cdots+n_k(\frac12+\beta_k+s)}} \\ &\quad\cdot \prod_{p>x} \sum_{\substack{m_1+\cdots+m_k=\\ =n_1+\cdots+n_k}} \frac{1}{p^{m_1(\frac12+\alpha_1+s)+\cdots+m_k(\frac12+\alpha_k+s)+m_1(\frac12+\beta_1+s)+\cdots+n_k(\frac12+\beta_k+s)}} \end{aligned} \tag{3.41}$$

and by putting in evidence the first terms in the Euler products, this is

$$\begin{aligned} &= A_{\alpha,\beta}(z,w;s) \prod_{p\le x}\left(1 + \frac{zwg_x(p)^2}{p} + \frac{zg_x(p)}{p^{1+\beta_1+s}} + \cdots + \frac{wg_x(p)}{p^{1+\alpha_k+s}}\right) \\ &\qquad \cdot \prod_{p\le x}\left(1 + \frac{1}{p^{1+\alpha_1+\beta_1+2s}} + \cdots + \frac{1}{p^{1+\alpha_k+\beta_k+2s}}\right) \\ &\qquad \cdot \prod_{p> x}\left(1 + \frac{1}{p^{1+\alpha_1+\beta_1+2s}} + \cdots + \frac{1}{p^{1+\alpha_k+\beta_k+2s}}\right) \end{aligned}$$

$$\begin{aligned} =&A^*_{\alpha,\beta}(z,w;s) \prod_{i,j=1}^{k} \zeta(1+\alpha_i+\beta_j+2s) \\ &\cdot \exp\left(\sum_{p\le x}\left\{\frac{zwg_x(p)^2}{p} + \frac{zg_x(p)}{p^{1+\beta_1+s}} + \cdots + \frac{wg_x(p)}{p^{1+\alpha_k+s}}\right\}\right) \end{aligned} \tag{3.42}$$

where $A_{\alpha,\beta}(z,w;s)$ and $A^*_{\alpha,\beta}(z,w;s)$ are arithmetical factors (Euler products) converging absolutely in a half-plane $\Re(s) > -\delta$ for some $\delta > 0$ uniformly for $|z|,|w| \le 1$, such that their derivatives with respect to z

and w at 0 also converge in the same half plane. We now have extracted the polar part, hence we are ready to shift the integral over s in (3.38) to the left of zero. To do so, it has been convenient to prescribe the same conditions for G as in [9, Remarks after Lemma 2.1], assuming that $G(s)$ vanishes at $s=-\frac{\alpha_i+\beta_j}{2}$ for all i,j, so that the only pole we pick in the contour shift is at $s=0$. Moreover we assume that the shifts are such that $|\alpha_i+\beta_j|\gg(\log T)^{-1}$ for every $1\le i,j\le k$, so that

$$\prod_{i,j=1}^{k}|\zeta(1+\alpha_i+\beta_j)|\ll_k(\log T)^{k^2}. \tag{3.43}$$

Hence by (3.41), (3.42) and (3.43), we get

$$\begin{aligned}&\partial_z\partial_w\left[\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{ks}I^0_{\alpha,\beta}(z,w;s)ds\right]_{\substack{z=0\\w=0}}\\&\ll_k(\log T)^{k^2}\left|\partial_z\partial_w\left[\exp\left(\sum_{p\le x}\left\{\frac{zwg_x(p)^2}{p}+\frac{zg_x(p)}{p^{1+\beta_1}}+\cdots+\frac{wg_x(p)}{p^{1+\alpha_k}}\right\}\right)\right]_{\substack{z=0\\w=0}}\right|\\&\ll_k(\log T)^{k^2}\left(\sum_{p\le x}\frac{|g_x(p)|}{p}\right)^2\ll_k(\log T)^{k^2}(\log\log\log T)^2\end{aligned}$$

being $|\Re(\alpha_i)|,|\Re(\beta_j)|\le(\log T)^{-1}$ and $\sum_{p\le x}|p^{-4/\log x}-1|/p\ll 1$ (see Equation (3.9)). All the other terms $I^{j,\mathcal{S},\mathcal{T}}_{\alpha,\beta}(z,w)$ can be treated exactly in the same way as $I^0_{\alpha,\beta}(z,w)$ by assuming that $|\alpha_i\pm\beta_j|\gg(\log T)^{-1}$ for every i,j, since they only differ from the first case by permutations and changes of signs of the shifts. Therefore we get

$$\partial_z\partial_w\left[\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{ks}I^{j,\mathcal{S},\mathcal{T}}_{\alpha,\beta}(z,w;s)ds\right]_{\substack{z=0\\w=0}}\ll_k(\log T)^{k^2}(\log\log\log T)^2$$

provided that

$$|\alpha_i\pm\beta_j|\gg(\log T)^{-1}\quad\text{for every } 1\le i,j\le k. \tag{3.44}$$

Plugging this into (3.38), we prove that

$$\int_T^{2T}|S_1|^2\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\ll_k T(\log T)^{k^2}(\log\log\log T)^2. \tag{3.45}$$

Moreover, since the left hand side in (3.45) is holomorphic in terms of the shifts, although we have proved the above for α_i,β_j such that (3.44) holds, the maximum modulus principle can be applied to obtain the bound to the enlarged domain we need.

We will treat similarly also the other pieces. As regards the second one, we have that $\int_T^{2T}|S_2|^2\boldsymbol{\zeta}_{\alpha,\beta}(t)dt$ is

$$\ll \int_T^{2T}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\Bigg|\sum_{\substack{p_1^{r_1},p_2^{r_2}\le x\\ r_1,r_2\ge 2}}\frac{p_1^{-\frac{4r_1}{\log x}}}{r_1}\frac{p_2^{-\frac{4r_2}{\log x}}}{r_2}Z_{\alpha_{\mathcal{S}},\beta_{\mathcal{T}},p_1^{r_1},p_2^{r_2}}(t)\Bigg|dt.$$

As before, we analyze the first term only since all the others are completely analogous. The term for $j=0$ is

$$\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\Big(\frac{t}{2\pi}\Big)^{ks} \cdot\sum_{\substack{p_1^{r_1}m_1\cdots m_k=p_2^{r_2}n_1\cdots n_k:\\ p_1^{r_1},p_2^{r_2}\le x,\ r_1,r_2\ge 2}}\frac{p_1^{-\frac{4r_1}{\log x}}p_2^{-\frac{4r_2}{\log x}}}{r_1r_2}\frac{1}{\sqrt{p_1^{r_1}p_2^{r_2}}\,m_1^{\frac12+\alpha_1+s}\cdots n_k^{\frac12+\beta_k+s}}ds$$

and this time, because of the condition $r_1,r_2\ge 2$, when we estimate the sum via the first terms of its Euler product we just get that the above is

$$\ll_k\prod_{i,j=1}^{k}|\zeta(1+\alpha_i+\beta_j)|$$

and, applying the same machinery as before, this yields

$$\int_T^{2T}|S_2|^2\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\ll_k T(\log T)^{k^2}. \tag{3.46}$$

We use the same approach in order to bound the second moment of S_3 as well, which is

$$\ll_k\int_T^{2T}\Bigg|\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\Big(\frac{t}{2\pi}\Big)^{ks} \cdot\sum_{\substack{am_1\cdots m_k=bn_1\cdots n_k\\ x<a,b<x^3}}\frac{\Lambda(a)\Lambda(b)}{\log a\log b}\frac{a^{-4/\log x}b^{-4/\log x}}{\sqrt{ab}\,m_1^{\frac12+\alpha_1+s}\cdots n_k^{\frac12+\beta_k+s}}\Bigg|dt$$

$$\ll_k T\Big(\sum_{x<p\le x^3}\frac{1}{p}\Big)^2\prod_{i,j=1}^{k}|\zeta(1+\alpha_i+\beta_j)|\ll_k T(\log T)^{k^2}$$

since the sum $\sum_{x<p\le x^3} p^{-1}$ is bounded.

We deal with R in the same way and we get that

$$\int_T^{2T} |R|^2 \boldsymbol{\zeta}_{\alpha,\beta}(t)dt \ll_k T(\log T)^{k^2}(\log\log\log T)^2 \tag{3.47}$$

where the extra factor with the triple log comes from the second term in the definition of R, while the first one can be treated analogously to S_3.

Finally we have to bound the second moment of $\Re L(t)$. To do so, in view of Equations (3.14) and (3.15), it suffices to study

$$\frac{1}{(\log x)^2}\int_T^{2T} \Big(\log^+ \frac{1}{\eta_t \log x}\Big)^2 \Big(\Big|\sum_{n\le x^3} \frac{\Lambda(n) n^{-4/\log x}}{n^{1/2+it}}\Big| + \log T\Big)^2 \boldsymbol{\zeta}_{\alpha,\beta}(t)dt \tag{3.48}$$

where $\eta_t := \min_\rho |t-\gamma|$ and $\log^+ t := \max(\log t, 0)$, with the aim of proving that this is $\ll_k T(\log T)^{k^2}(\log\log\log T)^{5/2}$. By applying the Cauchy-Schwarz inequality, the above is

$$\le \Big(\int_T^{2T} \Big(\frac{1}{\log x}\Big|\sum_{n\le x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}}\Big| + \frac{1}{\varepsilon}\Big)^4 \prod_{l=1}^k |\zeta(1/2+\beta_l - it)|^2 dt\Big)^{\frac{1}{2}}$$
$$\cdot \Big(\int_T^{2T} \Big(\log^+ \frac{1}{\eta_t \log x}\Big)^4 \prod_{l=1}^k |\zeta(1/2+\alpha_l+it)|^2 dt\Big)^{\frac{1}{2}} \tag{3.49}$$

and the first term can be treated as R above, to show that it is $\ll_k \sqrt{T(\log T)^{k^2}(\log\log\log T)^4}$. We now conclude the proof, bounding the second term in (3.49). If we denote $\tau := [T - \frac{1}{\log x}, 2T + \frac{1}{\log x}]$ then we have

$$\begin{aligned}
&\int_T^{2T} \Big(\log^+ \frac{1}{\eta_t \log x}\Big)^4 |\zeta(1/2+\alpha_1+it)|^2 \cdots |\zeta(1/2+\alpha_k+it)|^2 dt \\
&\quad \le \sum_{\gamma\in\tau} \int_{-1/\log x}^{1/\log x} \Big(\log^+ \frac{1}{|w|\log x}\Big)^4 \prod_{j=1}^k \big|\zeta\big(1/2+\alpha_j + i(w+\gamma)\big)\big|^2 dw \\
&\quad = \sum_{\gamma\in\tau} \int_{-1}^{1} \Big(\log^+ \frac{1}{|t|}\Big)^4 \prod_{j=1}^k \Big|\zeta\Big(1/2+\alpha_j + i\Big(\gamma + \frac{t}{\log x}\Big)\Big)\Big|^2 \frac{dt}{\log x} \\
&\quad \ll \frac{1}{\log x}\int_{-1}^{1} (\log|t|)^4 \prod_{j=1}^k \Big(\sum_{\gamma\in\tau}\Big|\zeta\Big(1/2+i\gamma+\alpha_j+i\frac{t}{\log x}\Big|^{2k}\Big)^{1/k} dt
\end{aligned} \tag{3.50}$$

by Hölder inequality. The remaining sum can be bounded under RH in view of Kirila's Theorem 2.8 about the discrete moments of zeta. Indeed, since the shifts $\alpha_j + i\frac{t}{\log x}$ in the sum over zeros in (3.50) have modulus ≤ 1 and real part $\le (\log T)^{-1}$ in absolute value, then we have

$$\sum_{\gamma\in\mathcal{T}} \Big|\zeta\Big(1/2+i\gamma+\alpha_j+i\frac{t}{\log x}\Big)\Big|^{2k} \ll_k T\log T(\log T)^{k^2}$$

for every $j = 1,\dots,k$ and putting this into (3.50) we get

$$\int_T^{2T} \Big(\log^+\frac{1}{\eta_t\log x}\Big)^4 \prod_{l=1}^{k} |\zeta(1/2+\alpha_l+it)|^2 dt \ll_k \frac{T\log T}{\log x}(\log T)^{k^2}. \quad (3.51)$$

Plugging (3.51) into (3.49) we prove that

$$\begin{aligned}\int_T^{2T} &|\Re L(t)|^2 \boldsymbol{\zeta}_{\alpha,\beta}(t)dt \\ &\ll_k \sqrt{T(\log T)^{k^2}(\log\log\log T)^4}\sqrt{T(\log T)^{k^2}\log\log\log T} \\ &\ll_k T(\log T)^{k^2}(\log\log\log T)^{5/2}\end{aligned} \quad (3.52)$$

concluding the proof of the proposition. □

The second step is getting rid of the small primes, showing that their contribution does not affect the distribution asymptotically. This simple fact will simplify the third and last step of the proof, as we will see in the following. Let's define

$$P(t) := \sum_{p\in X} \frac{1}{p^{1/2+it}} \quad (3.53)$$

where $X := (\log T, x]$ (we recall that $x = T^\varepsilon$ and $\varepsilon = (\log\log\log T)^{-1}$).

Proposition 3.9. *For k a non negative integer, assume Conjecture 2.6 and RH. Let T be a large parameter and $\alpha_1,\dots,\alpha_k, \beta_1,\dots,\beta_k \in \mathbb{C}$ such that $|\alpha_i|, |\beta_j| < 1$, $|\Re(\alpha_i)|, |\Re(\beta_j)| \le (\log T)^{-1}$ and $|\alpha_i - \beta_j| \ll (\log T)^{-1}$ for all $1 \le i, j \le k$. Then we have:*

$$\int_T^{2T} \big|\Re\widetilde{P}(t) - \Re P(t)\big|^2 \boldsymbol{\zeta}_{\alpha,\beta}(t)dt \ll_k T(\log T)^{k^2}(\log\log\log T)^2.$$

Proof. This can be proved with the same method used in Proposition 3.8. We recall that $\mathbb{1}_{\log T}(\cdot)$ denotes the indicator function of primes up to $\log T$. We start by studying

$$\frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}\Big(\frac{t}{2\pi}\Big)^{ks} \sum_{\substack{am_1\cdots m_k=\\ =bn_1\cdots n_k}} \frac{\mathbb{1}_{\log T}(a)\mathbb{1}_{\log T}(b)}{\sqrt{ab}\, m_1^{\frac12+\alpha_1+s}\cdots n_k^{\frac12+\beta_k+s}} ds \quad (3.54)$$

and, as usual, we estimate the sum with the first terms of its Euler product, extract the polar part, shift the integral over s to the left, getting that (3.54) is

$$\ll_k \prod_{i,j=1}^{k} |\zeta(1+\alpha_i+\beta_j)| \left| \partial_z \partial_w \left[\exp\left(\sum_{p\le \log T} \left\{ \frac{zw}{p} + \frac{z}{p^{1+\beta_1}} + \cdots + \frac{w}{p^{1+\alpha_k}} \right\} \right) \right]_{\substack{z=0\\w=0}} \right|$$

and by the same argument as in the proof of Proposition 3.8 the above is

$$\ll_k (\log T)^{k^2} \left(\sum_{p\le \log T} \frac{1}{p} \right)^2 \ll (\log T)^{k^2} (\log\log\log T)^2$$

and this concludes the proof. □

Finally we investigate the distribution of the polynomial $\Re P(t)$, which has the same distribution as $\log|\zeta(1/2+it)|$ thanks to Propositions 3.8 and 3.9. The most natural method to do so is studying the moments and this is achieved in the following result.

Proposition 3.10. *For k a non negative integer, assume Conjecture 2.6 and RH. Let T be a large parameter and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{C}$ such that $|\alpha_i|, |\beta_j| < 1$, $|\Re(\alpha_i)|, |\Re(\beta_j)| \le (\log T)^{-1}$ and $|\alpha_i - \beta_j| \ll (\log T)^{-1}$ for all $1 \le i, j \le k$. Denote $\mathcal{L} := \sum_{p\in X} \frac{1}{p} \sim \log\log T$ and $\mu \in \mathbb{R}$ such that $\mu \ll \log\log T$. Then for every fixed integer n we have*

$$\int_T^{2T} (\Re P(t) - k\mu)^n \boldsymbol{\zeta}_{\alpha,\beta}(t)dt = \int_T^{2T} \sum_{0\le j\le k} \sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}} \left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}} M_{\alpha_{\mathcal{S}},\beta_{\mathcal{T}}}(0)$$

$$\cdot\, \partial_z^n \left[e^{\frac{z^2}{4}\mathcal{L} - kz\mu} \exp\left(\frac{z}{2} \sum_{p\in X} \frac{g_p(\mathcal{S},\mathcal{T})}{p} \right) \right]_{z=0} dt$$

$$+ O_{k,n}\left(T(\log T)^{k^2-1+\varepsilon} \right)$$

where

$$g_p(\mathcal{S},\mathcal{T}) := \sum_{x_1\notin\mathcal{S}} p^{-x_1} + \sum_{x_2\notin\mathcal{T}} p^{-x_2} + \sum_{x_3\in\mathcal{S}} p^{x_3} + \sum_{x_4\in\mathcal{T}} p^{x_4}$$

and

$$M_{\alpha,\beta}(s) := \sum_{m_1\cdots m_k = n_1\cdots n_k} \frac{1}{m_1^{\frac{1}{2}+\alpha_1+s} \cdots m_k^{\frac{1}{2}+\alpha_k+s} n_1^{\frac{1}{2}+\beta_1+s} \cdots n_k^{\frac{1}{2}+\beta_k+s}}$$

so that $M_{\alpha,\beta}(0)$ is the first term of the moment of $\boldsymbol{\zeta}_{\alpha,\beta}$ predicted by the recipe; more precisely, with the notations of Conjecture 2.1, $M_{\alpha,\beta}(s) =$

$R(\frac{1}{2}+s,\alpha_1,\ldots,\alpha_k,-\beta_1,\ldots,-\beta_k)$, *where the sum which defines* $M_{\alpha,\beta}(s)$ *does not converge for* $s=0$ *so we appeal to [28, Theorem 2.4.1] for the analytic continuation.*

Proof. Expanding out the powers, since $2\Re(z)=z+\overline{z}$, we get

$$\begin{aligned}\int_T^{2T} &(\Re P(t)-k\mu)^n \boldsymbol{\zeta}_{\alpha,\beta}(t)dt \\ &= \sum_{j+h=n}\binom{n}{h}(-k\mu)^j 2^{-h}\sum_{r+s=h}\binom{h}{r}\int_T^{2T} P(t)^r\overline{P(t)}^s\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\end{aligned} \tag{3.55}$$

and using Conjecture 2.6, ignoring the error term which is negligible in this context, the inner integral in (3.55) is

$$\int_T^{2T}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}}\sum_{a,b}\mathbb{1}_X^{*r}(a)\mathbb{1}_X^{*s}(b)Z_{\alpha_{\mathcal{S}},\beta_{\mathcal{T}},a,b}(t)dt \tag{3.56}$$

where $\mathbb{1}_X^{*r}(a)$ denotes the indicator function of primes in the interval X, self-convoluted r times. If we define temporarily the multiplicative function g given by $g(p^n)=1/n!$, then the inner sum over a,b in (3.56) equals

$$\partial_z^r\partial_w^s\Bigg[\sum_{\substack{a,b:\\ p|ab\implies p\in X}} z^{\Omega(a)}w^{\Omega(b)}g(a)g(b)Z_{\alpha_{\mathcal{S}},\beta_{\mathcal{T}},a,b}(t)\Bigg]_{\substack{z=0\\ w=0}} \tag{3.57}$$

then plugging (3.56) and (3.57) into (3.55), recollecting together the powers we expanded before, we get

$$\int_T^{2T}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{ks}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}}\partial_z^n\left[e^{-k\mu z}I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z;s)\right]_{z=0}dsdt \tag{3.58}$$

with

$$I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z;s):=\sum_{\substack{a,b:\\ p|ab\implies p\in X}}\left(\frac{z}{2}\right)^{\Omega(a)+\Omega(b)}g(a)g(b)\widetilde{Z}_{\alpha_{\mathcal{S}},\beta_{\mathcal{T}},a,b}(s) \tag{3.59}$$

(see (3.40) for the definition of $\widetilde{Z}_{\alpha,\beta,a,b}(s)$). We study only the first term in (3.58), i.e. $j=0$, since all the other terms can be understood from the first one with a slight modification. Then we look at

$$\begin{aligned}I^0_{\alpha,\beta}(z;s) &= \sum_{\substack{am_1\cdots m_k=bn_1\cdots n_k\\ p|ab \implies p\in X}} \frac{(z/2)^{\Omega(a)+\Omega(b)}g(a)g(b)}{\sqrt{ab}\, m_1^{\frac12+\alpha_1+s}\cdots m_k^{\frac12+\alpha_k+s} n_1^{\frac12+\beta_1+s}\cdots n_k^{\frac12+\beta_k+s}}\\ &= \prod_{p\in X}\sum_{\substack{a+m_1+\cdots+m_k=\\ =b+n_1+\cdots+n_k}} \frac{(z/2)^{a+b}g(p^a)g(p^b)}{p^{\frac a2+\frac b2+m_1(\frac12+\alpha_1+s)+\cdots+n_k(\frac12+\beta_k+s)}}\\ &\qquad\cdot\prod_{p\notin X}\sum_{\substack{m_1+\cdots+m_k=\\ =n_1+\cdots+n_k}}\frac{1}{p^{m_1(\frac12+\alpha_1+s)+\cdots+n_k(\frac12+\beta_k+s)}}\\ &= \exp\left(\sum_{p\in X}\left\{\frac{(z/2)^2}{p}+\frac{z/2}{p^{1+\alpha_1+s}}+\cdots+\frac{z/2}{p^{1+\beta_k+s}}\right\}\right)F_{X,\alpha,\beta}(z;s)M_{\alpha,\beta}(s)\end{aligned} \tag{3.60}$$

where $F_{X,\alpha,\beta}(z;s)$ is an arithmetical factor (Euler product) converging absolutely in a product of half-planes containing the origin, such that $F_{X,\alpha,\beta}(z;0)$ is holomorphic at $z=0$, $F_{X,\alpha,\beta}(0,0)=1$ and all its derivatives at $z=0$ are small, i.e. $\partial_z^c[F_{X,\alpha,\beta}(z,0)]_{z=0} \ll (\log T)^{-1}$ for any positive integer c. Now we want to shift the integral over s in (3.58) to the left of zero, picking the contribution of the (unique) pole at $s=0$. To do so, we appeal to the meromorphic continuation of the function $M_{\alpha,\beta}(s)$, see [28, Theorem 2.4.1]; thus we can shift the path of integration to the vertical line (say) $\Re(s)=-\frac{1}{10}$, where the integral is trivially bounded by $\ll T^{1-1/10+\varepsilon}$ for any positive ε. Moreover, the contribution from the pole at $s=0$ gives

$$T\partial_z^n\left[e^{-k\mu z}e^{\frac{z^2}{4}\mathcal{L}}\exp\left(\frac z2\sum_{p\in X}\frac{p^{-\alpha_1}+\cdots+p^{-\beta_k}}{p}\right)F_{X,\alpha,\beta}(z;0)\right]_{z=0}M_{\alpha,\beta}(0).$$

Thanks to the bounds for μ, $\mathcal{L}$ and for the derivatives of $F_{X,\alpha,\beta}$, being $F_{X,\alpha,\beta}(0;0)=1$ and $M_{\alpha,\beta}(0)\ll(\log T)^{k^2}$ (again this is due to a similar argument as in the proof of Proposition 3.2, when we assume extra conditions on the shifts and then we appeal to the maximum modulus principle), we get that the term for $j=0$ in (3.58) equals

$$T\partial_z^n\left[e^{-k\mu z}e^{\frac{z^2}{4}\mathcal{L}}\exp\left(\frac z2\sum_{p\in X}\frac{p^{-\alpha_1}+\cdots+p^{-\beta_k}}{p}\right)\right]_{z=0}M_{\alpha,\beta}(0) \\ + O_{k,n}\Big(T(\log T)^{k^2-1+\varepsilon}\Big).$$

Analogously, the general term turns out to be

$$\int_T^{2T}\left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}}\partial_z^n\left[e^{\frac{z^2}{4}\mathcal{L}-k\mu z}\exp\left(\frac{z}{2}\sum_{p\in X}\frac{g_p(\mathcal{S},\mathcal{T})}{p}\right)\right]_{z=0}M_{\alpha_\mathcal{S},\beta_\mathcal{T}}(0)dt$$
$$+O_{k,n}\Big(T(\log T)^{k^2-1+\varepsilon}\Big)$$

and putting this into (3.56), i.e. summing over $j,\mathcal{S},\mathcal{T}$, we get the claim. □

Proof of Theorem 3.5

This proof follows easily from the three propositions we have proved above. If we take $\mu=\mathcal{L}$ and all the shifts small enough, i.e. $\alpha_i,\beta_j\ll(\log T)^{-1}$ for any i,j, then the exponent in the right hand side of the formula given by Proposition 3.10 becomes

$$\begin{aligned}\frac{z^2}{4}\mathcal{L}+\frac{z}{2}\sum_{p\in X}\frac{g_p(\mathcal{S},\mathcal{T})}{p}-zk\mathcal{L}\\ &=\frac{z^2}{4}\mathcal{L}+O\bigg(z\sum_{p\in X}\frac{|p^{-\alpha_1}-1|+\cdots+|p^{-\beta_k}-1|}{p}\bigg)\\ &=\frac{z^2}{4}\mathcal{L}+O_k\bigg(z\sum_{p\in X}\frac{|p^{-\alpha_1}-1|}{p}\bigg)=\frac{z^2}{4}\mathcal{L}+O_k(z)\end{aligned}$$

and does not depend on $\mathcal{S}$ and $\mathcal{T}$ asymptotically, indeed. Hence we can bring that factor outside and reconstruct the moment of $\boldsymbol{\zeta}_{\alpha,\beta}$, as follows

$$\begin{aligned}&\int_T^{2T}(\Re P(t)-k\mathcal{L})^n\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\\ &=\partial_z^n\left[e^{\frac{z^2}{4}\mathcal{L}+O_k(z)}\right]_{z=0}\int_T^{2T}\sum_{0\le j\le k}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}}M_{\alpha_\mathcal{S},\beta_\mathcal{T}}(0)dt\\ &\qquad+O_{k,n}\Big(T(\log T)^{k^2-1+\varepsilon}\Big)\\ &=\left(\partial_z^n\left[e^{\frac{z^2}{4}\mathcal{L}}\right]_{z=0}+O_{k,n}\Big((\log\log T)^{\frac{n-1}{2}}\Big)\right)\int_T^{2T}\boldsymbol{\zeta}_{\alpha,\beta}(t)dt\\ &\qquad+O_{k,n}\Big(T(\log T)^{k^2-1+\varepsilon}\Big).\end{aligned}$$

The claim follows by analyzing

$$\partial_z^n\left[e^{\frac{z^2}{4}\mathcal{L}}\right]_{z=0}=\left[\sum_{m_1+2m_2=n}\frac{n!}{m_1!m_2!}\left(\frac{2z\mathcal{L}}{4}\right)^{m_1}\left(\frac{1}{2!}\frac{\mathcal{L}}{2}\right)^{m_2}\right]_{z=0}. \tag{3.61}$$

If n is odd then the coefficient $\partial_z^n[e^{\frac{z^2}{4}\mathcal{L}}]_{z=0}$ vanishes, while if n is even then only the term for $m_1 = 0$ and $m_2 = \frac{n}{2}$ survives and gives $\frac{n!}{(n/2)!}(\frac{\mathcal{L}}{4})^{n/2} = (n-1)!!(\frac{\mathcal{L}}{2})^{n/2}$, i.e.

$$\int_T^{2T} (\Re P(t) - k\mathcal{L})^n \boldsymbol{\zeta}_{\alpha,\beta}(t)dt$$
$$= \begin{cases} (1+o_{k,n}(1))(n-1)!!(\frac{\mathcal{L}}{2})^{n/2} \int_T^{2T} \zeta_{\alpha,\beta}(t)dt & \text{if } n \text{ even} \\ o_{k,n}\big((\log\log T)^{n/2} \int_T^{2T} \zeta_{\alpha,\beta}(t)dt\big) & \text{if } n \text{ odd} \end{cases}.$$

This matches with the Gaussian coefficient then this proves that, in the limit $T \to \infty$, $\Re P(t)$ has Gaussian distribution, with mean $k \log\log T$ and variance $\frac{1}{2}\log\log T$ and then so does $\log|\zeta(1/2+it)|$, in view of Propositions 3.8 and 3.9. Theorem 3.5 follows by taking the derivatives with respect to the shifts.

Proof of Theorem 3.6

To derive Theorem 3.6, in Proposition 3.10 we set $\alpha_1 = \cdots = \alpha_k = i\alpha$ and $\beta_1 = \cdots = \beta_k = -i\alpha$, with $\alpha \in \mathbb{R}, |\alpha| < 1$ and we take μ as

$$\mu_\alpha := \sum_{p\in X} \frac{\cos(\alpha \log p)}{p} = \widetilde{\mu}_\alpha + O(1) = \begin{cases} \log\log T + O(1) & \text{if } |\alpha|\log T \le 1 \\ -\log|\alpha| + O(1) & \text{if } |\alpha|\log T > 1 \end{cases} \tag{3.62}$$

by partial summation. Then we get

$$\int_T^{2T} (\Re P(t) - k\mu_\alpha)^n |\zeta(1/2+i\alpha+it)|^{2k}dt$$
$$= \big(1+o_{k,n}(1)\big)\, \partial_z^n \Big[e^{\frac{z^2}{4}\mathcal{L}}\Big]_{z=0} \int_T^{2T} |\zeta(1/2+i\alpha+it)|^{2k}dt$$

since the quantity $\frac{z}{2}\sum_{p\in X}\frac{g_p(\mathcal{S},\mathcal{T})}{p} - zk\mu_\alpha$ vanishes for all $\mathcal{S}, \mathcal{T}$ for this choice of the shifts. The claim follows as in the proof of Theorem 3.5.

3.2.2 Proof of Theorem 3.7

Let us denote the moment generating function

$$M_N(s) := \langle |Z|^s \rangle = \sum_{j=0}^{\infty} \frac{\langle (\log|Z|)^j \rangle}{j!} s^j \tag{3.63}$$

(where the mean has to be considered over the group $U(N)$ with respect to the Haar measure) and the cumulants $Q_j = Q_j(N)$ by

$$\log M_N(s) = \sum_{j=1}^{\infty} \frac{Q_j}{j!} s^j. \tag{3.64}$$

In [109], among other things, Keating and Snaith studied the cumulants showing that

$$Q_n = \begin{cases} 0 & \text{if } n = 1 \\ \frac{1}{2}\log N + O(1) & \text{if } n = 2 \\ O(1) & \text{if } n \geq 3 \end{cases} \tag{3.65}$$

and deduced a central limit theorem proving that the limiting distribution of $\log|Z|$ is Gaussian with mean 0 and variance $\frac{1}{2}\log N$. Here, for any $k \in \mathbb{N}$, we study the distribution of random variable $\log|Z|$ with respect to the tilted measure $|Z|^{2k} d_{Haar}$. Before starting with our analysis, we recall that the moments of $|Z|$ are known also for non integer k (see [109] Equations (6) and (16)):

$$\begin{aligned} M_N(2k) = \langle |Z|^{2k} \rangle &= \exp\left(\sum_{j=1}^{\infty} \frac{(2k)^j}{j!} Q_j \right) \\ &= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \sim N^{k^2} \frac{G^2(1+k)}{G(1+2k)} \end{aligned} \tag{3.66}$$

where $k \in \mathbb{R}$ and G denotes the Barnes G-function. We also denote

$$\mathcal{M}_{2k} := \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}.$$

Now we are ready to consider the first moment

$$\langle |Z|^{2k} \log|Z| \rangle = \frac{d}{dx}\Big[\langle |Z|^{2k+x} \rangle \Big]_{x=0} = \frac{d}{dx}\left[\prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2} \right]_{x=0}$$

by (3.63) and (3.66). We compute the derivative by Leibniz's rule, writing

$$\begin{aligned} &\prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2} \\ &\qquad = \exp\left(\sum_{j=1}^{N} \Big\{ \log\Gamma(j) + \log\Gamma(j+2k+x) - 2\log\Gamma(j+k+x/2) \Big\} \right) \end{aligned}$$

and we get

$$\langle |Z|^{2k}\log|Z|\rangle = \prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}\sum_{j=1}^{N}\left\{\frac{\Gamma'}{\Gamma}(j+2k)-\frac{\Gamma'}{\Gamma}(j+k)\right\}. \quad (3.67)$$

Moreover an application of Stirling's formula yields

$$\frac{\Gamma'}{\Gamma}(j+2k)-\frac{\Gamma'}{\Gamma}(j+k) = \frac{k}{j}+O_k\left(\frac{1}{j^2}\right) \quad (3.68)$$

hence, by (4.55) and (3.68), the weighted mean of the random variable $\log|Z|$ is

$$\mu_{2k} := \frac{1}{\mathcal{M}_{2k}}\langle |Z|^{2k}\log|Z|\rangle = k\log N + O_k(1).$$

Then we study the weighted n-th moment of the random variable $\log|Z|$:

$$\begin{aligned}\langle |Z|^{2k}(\log|Z|-\mu_{2k})^n\rangle &= \sum_{h+j=n}\binom{n}{h}(-\mu_{2k})^j\langle |Z|^{2k}(\log|Z|)^h\rangle \\ &= \sum_{h+j=n}\binom{n}{h}\frac{d^j}{dx^j}\left[e^{-x\mu_{2k}}\right]_{x=0}\frac{d^h}{dx^h}\left[\prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2}\right]_{x=0} \quad (3.69)\\ &= \frac{d^n}{dx^n}\left[\exp\left(-x\mu_{2k}+\sum_{j=1}^{N}\log\left(\frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2}\right)\right)\right]_{x=0}.\end{aligned}$$

If we denote $f_j(x) = f_{N,k,j}(x) := \log\Gamma(j)+\log\Gamma(j+2k+x)-2\log\Gamma(j+k+\frac{x}{2})$, then we can carry on the computation in (3.69) by computing the derivative, getting

$$\begin{aligned}&\frac{d^n}{dx^n}\left[\exp\left(-x\mu_{2k}+\sum_{j=1}^{N}\log\left(\frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2}\right)\right)\right]_{x=0} \\ &= \sum_{m_1,\dots,m_n}\frac{n!}{m_1!\cdots m_n!}e^{\sum_{j=1}^{N}f_j(0)}\prod_{i=1}^{n}\left(\frac{1}{i!}\frac{d^i}{dx^i}\left[-x\mu_{2k}+\sum_{j=1}^{N}f_j(x)\right]_{x=0}\right)^{m_i} \\ &= \mathcal{M}_{2k}\sum_{m_1,\dots,m_n}\frac{n!}{m_1!\cdots m_n!}\left(-\mu_{2k}+\sum_{j=1}^{N}f_j'(0)\right)^{m_1} \\ &\qquad\cdot\left(\frac{1}{2}\sum_{j=1}^{N}f_j''(0)\right)^{m_2}\prod_{i=3}^{n}\left(\frac{1}{i!}\sum_{j=1}^{N}f_j^{(i)}(0)\right)^{m_i}\end{aligned} \quad (3.70)$$

where the sums in (3.70) are over the n-uple $(m_1, \dots, m_n)$ such that

$$m_1 + 2m_2 + \cdots + nm_n = n.$$

Using Stirling's approximation formula, one can easily estimate the derivatives of $f_j(x)$ and prove

$$\begin{aligned} \sum_{j=1}^{N} f_j'(0) &= \sum_{j=1}^{N} \Big\{ \frac{k}{j} + O_k(j^{-2}) \Big\} = k \log N + O_k(1); \\ \sum_{j=1}^{N} f_j''(0) &= \sum_{j=1}^{N} \Big\{ \frac{1}{j+2k} - \frac{1/2}{j+k} + O(j^{-2}) \Big\} = \frac{1}{2} \log N + O_k(1); \quad (3.71) \\ \sum_{j=1}^{N} f_j^{(i)}(0) &= \sum_{j=1}^{N} O(j^{-2}) = O(1) \quad \text{for all } i \geq 3. \end{aligned}$$

Putting together (3.69), (3.70) and (3.71) one has

$$\begin{aligned} \langle |Z|^{2k} (\log|Z| - \mu_{2k})^n \rangle =& \mathcal{M}_{2k} \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1! \cdots m_n!} \\ & \cdot \Big(\frac{1}{4} \log N + O_k(1) \Big)^{m_2} \Big(O_k(1) \Big)^{m_1+m_3+\cdots+m_n} \end{aligned}$$

then if n is even the asymptotic is given by $m_2 = n/2$ and $m_i = 0$ for $i \neq 2$, giving

$$\begin{aligned} \langle |Z|^{2k} (\log|Z| - \mu_{2k})^n \rangle &\sim_{k,n} \mathcal{M}_{2k} \frac{n!}{(n/2)!} \Big(\frac{1}{4} \log N \Big)^{n/2} \\ &= \mathcal{M}_{2k} (n-1)!! \Big(\frac{1}{2} \log N \Big)^{n/2} \end{aligned}$$

while if n is odd the n-th moment is surely $o_{k,n}(\mathcal{M}_{2k} (\log N)^{n/2})$.

3.3 Large values

Now that we have studied the distribution of the random variable $\log|\zeta(1/2+it)|$ with respect to the weighted measure $|\zeta|^{2k} dt := |\zeta(1/2+it)|^{2k} dt$, we go back to the large values of the Riemann zeta function, in view of (3.3). We recall that Radziwiłł formulated a precise conjecture about the frequency of large values in a specific range (we believe there is a small typo in the statement of [133, Conjecture 2], due to a factor of "2" missing in the range of Δ).

Conjecture 3.11 ([133], Conjecture 2)**.** *Let $k > 0$. If $\Delta \sim k\sqrt{2\log\log T}$, then*

$$\frac{1}{T}\underset{t\in[T,2T]}{\text{meas}}\left\{\frac{\log|\zeta(1/2+it)|}{\sqrt{\frac{1}{2}\log\log T}} \geq \Delta\right\} \sim C_k \int_{\Delta}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

where C_k denotes the constant of moments ($C_k = g_k a_k$, see (2.3)*).*

The above asymptotic formula specifies the implied constant of the expected sharp upper bound (1.17), speculating that the quantity

$$\mathcal{LV}_k := \frac{1}{T}\underset{t\in[T,2T]}{\text{meas}}\left\{\log|\zeta(1/2+it)| \geq k\log\log T\right\}$$

is asymptotic to the standard Gaussian integral $\int_{k\sqrt{2\log\log T}}^{+\infty} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ times the constant of moments C_k, i.e.

$$\mathcal{LV}_k \sim \frac{C_k}{2k\sqrt{\pi}}\frac{1}{(\log T)^{k^2}\sqrt{\log\log T}} \tag{3.72}$$

As mention in the previous chapters, via Mellin transform (3.2), one can write the left hand side of (3.72) in terms of the Fourier-Laplace trasform of $\log|\zeta(1/2+it)|$ with respect to the weighted measure $|\zeta|^{2k}dt$, getting

$$\begin{aligned}\mathcal{LV}_k =&\frac{1}{(\log T)^{k^2}}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2k+iu} \\ &\cdot\frac{1}{T(\log T)^{k^2}}\int_T^{2T} e^{iu(\log|\zeta(1/2+it)|-k\log\log T)}|\zeta|^{2k}dt\, du.\end{aligned} \tag{3.73}$$

By expanding the exponential in power series, one then relates the quantity $\mathcal{LV}_k$ to the weighted moments

$$\frac{1}{T(\log T)^{k^2}}\int_T^{2T}(\log|\zeta(1/2+it)| - k\log\log T)^n|\zeta|^{2k}dt.$$

We recall that Theorem 3.5 does not give an approximation for the weighted moments of $\log|\zeta(1/2+it)|$ but only for its weighted distribution, because of Proposition 3.8; this is not a main issue and may be solved by proving an analogue to Proposition 3.2 also in the case of the measure $|\zeta|^{2k}dt$. The crucial point is the error term that we have in Theorem 3.5, which is completely out of control once we plug it into (3.73).

If we ignore all the error terms and all the convergence problems, the following heuristic computation leads anyway to (3.72):

$$\begin{aligned}
\mathcal{LV}_k &\approx \frac{1}{(\log T)^{k^2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} \frac{1}{T(\log T)^{k^2}} \\
&\qquad\qquad \cdot \int_T^{2T} (\log|\zeta(1/2+it)| - k\log\log T)^n |\zeta|^{2k} dt \frac{du}{2k+iu} \\
&\approx \frac{1}{(\log T)^{k^2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{(iu)^n}{n!} \frac{1}{T(\log T)^{k^2}} \\
&\qquad\qquad \cdot C_k T(\log T)^{k^2} \frac{n!}{2^{n/2}(n/2)!} \left(\frac{1}{2}\log\log T\right)^{n/2} \frac{du}{2k+iu} \\
&= \frac{C_k}{(\log T)^{k^2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{\infty} \frac{(-u^2)^m}{(2m)!} \frac{(2m)!}{2^m m!} \left(\frac{1}{2}\log\log T\right)^m \frac{du}{2k+iu} \\
&= \frac{C_k}{(\log T)^{k^2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{4}\log\log T} \frac{du}{2k+iu} \\
&\sim \frac{C_k}{(\log T)^{k^2}} \frac{1}{\sqrt{\pi}2k\sqrt{\log\log T}} = \frac{C_k}{2k\sqrt{\pi}} \frac{1}{(\log T)^{k^2}\sqrt{\log\log T}}
\end{aligned}$$

where in the second line we used the approximation

$$\begin{aligned}
\frac{1}{C_k T(\log T)^{k^2}} &\int_T^{2T} (\Re P(t) - k\log\log T)^n |\zeta|^{2k} dt \\
&\approx \begin{cases} (n-1)!!(\frac{1}{2}\log\log T)^{n/2} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}
\end{aligned}$$

The above heuristic calculation is based on the classical idea of inverting the Fourier-Laplace transform (see e.g. [2, Proposition 5.5]) in order to get large deviation estimates. To make it rigorous, one has of course to truncate the sum over n of the exponential and then use the relevant result about moments; the higher one truncates the sum, the higher moments one has to study (and this means that we need to work with a shorter Dirichlet polynomial). Morover, if we know the moments with an error term which goes to infinity asymptotically, then its contribution when we re-construct the exponential (third line) goes completely out of control. For all these reason we are not able to derive Conjecture 3.11 from all the weighted distributional analyses we performed in the previous sections. Nevertheless, possible variations of Theorem 3.5 might perhaps make the computation above rigorous.

Chapter 4

Weighted one-level density of L-functions

The work presented in this chapter will be published in [56].

4.1 A weighted version of the one-level density

Let us assume the Riemann Hypothesis for all the L-functions that arise. As explained in Section 1.6, the classical one-level density considers a smooth localization at the central point of the counting function of non-trivial zeros of an L-function, averaged over a "natural" family of L-functions in the Selberg class[1]. More specifically, given an even and real-valued function f in the Schwartz space[2] and an L-function $L(s)$ in a family $\mathcal{F}$, we consider the quantity

$$\sum_{\gamma_L} f(c(L)\gamma_L) \tag{4.1}$$

where γ_L denotes the imaginary part of a generic non-trivial zero of L and $c(L)$ the log-conductor of $L(s)$ at the central point. We recall that $1/c(L)$ is the mean spacing of the non-trivial zeros of $L(s)$ around $s = \frac{1}{2}$. The one-level density for the family $\mathcal{F}$ is the average of the above quantity over

[1]We refer e.g. to [102] for the definitions and the basic properties of the Selberg class.

[2]In practice we will see that this condition can be weakened and a decaying like $f(x) \ll 1/(1+x^2)$ at infinity will suffice.

the family, i.e.

$$\lim_{X\to\infty} \frac{1}{\sum_{L\in\mathcal{F}_X} 1} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L), \tag{4.2}$$

with

$$\mathcal{F}_X := \{L \in \mathcal{F} : c(L) \leq \log X\}.$$

In literature this is also referred as the "low-lying zeros" density, as the sum (4.1) gives information on the distribution of the zeros of L which are close to the central point. Indeed if a zero is substantially more than $1/c(L)$ away from the central point, then it does not contribute significantly to the sum.

Katz and Sarnak [106] studied a wide variety of families and attached to each of these families of L-functions a symmetry type (i.e. unitary, symplectic or orthogonal, therefore identified by a group G), which should govern the one-level density of the considered family. Namely, the density conjecture predicts that

$$\lim_{X\to\infty} \frac{1}{\sum_{L\in\mathcal{F}_X} 1} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L) = \int_{-\infty}^{+\infty} f(x)W_{\mathcal{F}}(x)dx \tag{4.3}$$

where $W_{\mathcal{F}}$ equals the one-level density function W_G for the (scaled) limit of $G \in \{U(N), USp(2N), O(N), SO(2N), SO(2N+1)\}$, i.e. the kernel appearing in the analogous average in the corresponding random matrix theory setting. In particular, the kernel $W_{\mathcal{F}}$ is predicted to depend on G only. We recall that the function W_G is known for all the classical compact groups, being

$$\begin{aligned}
W_U(x) &= 1,\\
W_{USp}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_O(x) &= 1 + \frac{1}{2}\delta_0(x),\\
W_{SO^+}(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x},\\
W_{SO^-}(x) &= \delta_0(x) + 1 - \frac{\sin(2\pi x)}{2\pi x},
\end{aligned}$$

with δ_0 the Dirac δ-function centered at 0. Examples of one-level density theorems which prove (4.3) in specific cases can be found e.g. in [99, 87, 88, 25, 43].

In this chapter we investigate a weighted analogue of the one-level density. In particular we consider a tilted average over the family $\mathcal{F}$ of the quantity (4.1), multiplied by a power of L evaluated at the central point. This approach links the results on moments to those on the one-level density, making the connection between non-trivial zeros and the size of $L(\frac{1}{2})$ explicit. In the same spirit as in Chapter 3, the weighted average we consider amplifies the contribution coming from the L-functions that are large at the central point, near which zeros are expected to be rarer.

More specifically, given $k \in \mathbb{N}$, we are interested in

$$\mathcal{D}_k^{\mathcal{F}}(f) = \mathcal{D}_k^{\mathcal{F}}(f, X) := \frac{1}{\sum\limits_{L \in \mathcal{F}_X} V(L(\frac{1}{2}))^k} \sum_{L \in \mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L) V(L(\tfrac{1}{2}))^k \quad (4.4)$$

in the limit $X \to \infty$, where V depends on the symmetry type of the family; in particular $V(z) = |z|^2$ in the unitary case and $V(z) = z$ for the symplectic and orthogonal cases. The quantity $\mathcal{D}_k^{\mathcal{F}}(f)$ can be seen as a special case of

$$\frac{1}{\sum\limits_{L \in \mathcal{F}_X} V(L(\frac{1}{2}))^k} \sum_{L \in \mathcal{F}_X} g(L) V(L(\tfrac{1}{2}))^k \quad (4.5)$$

with $g(L)$ a function over the L-functions of a given family $\mathcal{F}$. In the unitary case, for example, we know from Soundararajan's work [151] that the dominant contribution to the $2k$-th moment comes from those L-functions such that the size of $|L(\frac{1}{2})|$ is about $(\log X)^{k+o(1)}$, which form a thin subset of size about $\#\mathcal{F}_X/(\log X)^{k^2+o(1)}$. Thus, if the function g has size 1, then only these L-functions contribute to the main term of the sum in (4.5). With the choice we made in (4.4), we have $g(L) = \sum_{\gamma_L} f(c(L)\gamma_L)$, which is not bounded but only $\ll c(L)$, by the Riemann-Von Mangoldt formula. However, the standard n-th level density [145] implies that $g(L) \ll c(L)^{\varepsilon}$ for all but $\#\mathcal{F}_X/(\log X)^A$ L-functions in the family, for every $A > 0$. Therefore, also in (4.4), we have that only the L-functions such that $|L(\frac{1}{2})| \asymp (\log X)^{k\pm\varepsilon}$ contribute significantly to the main term of the sum. For this reason, for unitary families, $\mathcal{D}_k^{\mathcal{F}}$ can be interpreted as a (weighted) one-level density for the thin subset $\{L \in F : (\log X)^{k-\varepsilon} \ll |L(\frac{1}{2})| \ll (\log X)^{k+\epsilon}\}$. Similarly, in the symplectic and orthogonal cases, $\mathcal{D}_k^{\mathcal{F}}$ is a weighted one-level density, focused on the L-functions in the family which are responsible to the k-th moment.

From the computations we perform throughout this chapter in specific cases, we speculate that the structure suggested by the density conjecture (4.3) holds also in the weighted case. Namely, we expect that

$$\mathcal{D}_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^k(x) dx + O\Big(\frac{1}{\log X}\Big) \tag{4.6}$$

where the weighted one-level density function W_G^k only depends on k and on the symmetry type of the family $\mathcal{F}$. Note that the superscript ^{k} is an index, indicating that we are weighting with the k-th power of $V(L(\frac{1}{2}))$; in particular W_G^k is not the k-th power of W_G.

This kind of weighting naturally appears also in other contexts, such as Kowalski, Saha and Tsimerman's paper [113], where the authors consider Siegel modular forms. Given a Siegel modular form F of genus 2, they compute the one-level density of the spinor L-function of F, with a weight ω^F which is essentially the modulus square of the first Fourier coefficient[3] of F. This family is expected to be orthogonal, but with this weight one does not obtain the usual kernel W_O. This discrepancy can be explained by Böcherer's conjecture [12, 49] (and in fact it supports it), which predicts that ω^F is proportional to the central value $L(\frac{1}{2}, F)$. To be more precise, it conjectures that $\omega^F \approx L(\frac{1}{2}, F)L(\frac{1}{2}, F \times \chi_4)$. Since $L(\frac{1}{2}, F \times \chi_4)$ is "uncorrelated" with $L(s, F)$ and with its zeros, then the kernel they obtained is indeed $W^1_{SO^+}$ (see e.g. Equation (4.100)[4] and note that weighting with $L(\frac{1}{2}, F)^k$ the odd part of the family does not contribute, if $k > 0$). Moreover, they notice that this kernel is the one that arises from symplectic symmetry types. Thus, the symmetry of the family jumps from O to USp, after weighting with ω^F. This transition can be seen as a particular case of Equation (4.8) below, which conjecturally predicts a relation between the weighted one-level density functions of different symmetry types.

4.2 Statement of main results

In the following, we focus on three specific families of L-functions, each with a different symmetry type; first, in Section 4.3, we consider the unitary family $\boldsymbol{\zeta} := \{\zeta(s + ia) : a \in \mathbb{R}\}$, i.e. the continuous family of the Riemann zeta function parametrized by a vertical shift. Then, in Section 4.4, we study the symplectic family $\boldsymbol{L}_\chi$ of quadratic Dirichlet L-functions. Finally Section 4.5 is devoted to the orthogonal family $\boldsymbol{L}_{\Delta,\chi}$ of the quadratic twists

[3] I.e. the Fourier coefficient corresponing to the identity matrix.

[4] In [113] the kernel is written as $1 - \frac{\delta_0}{2}$, which is equivalent to $W^1_{SO^+}$ for test functions whose Fourier transforms are supported in $[-1, 1]$, which is an assumption in [113].

of the L-function associated with the discriminant modular form Δ. For these families, under the assumption of the relevant Riemann Hypothesis and ratio conjecture, we perform an asymptotic analysis of $\mathcal{D}_k^{\mathcal{F}}(f)$. Our results confirm our prediction (4.6), for small values of k.

We start with the unitary family. Note that, since this is a continuous family, the average over the family in the definition of $\mathcal{D}_k^{\zeta}(f)$ is given by an integration over $t \in [T, 2T]$ instead of the sum in (4.4); we refer to Section 4.3 for further details. In this case, setting

$$W_U^0(x) := W_U(x) = 1,$$

$$W_U^1(x) := 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}$$

and

$$W_U^2(x) := 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4},$$

we prove the following theorem.

Theorem 4.1. *Let us assume the Riemann Hypothesis and the ratio conjecture (see Conjecture 2.3). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 2$, we have*

$$\mathcal{D}_k^{\zeta}(f) = \int_{-\infty}^{+\infty} f(x) W_U^k(x) dx + O\Big(\frac{1}{\log T}\Big).$$

For this unitary family, in the case $k = 1$, we also develop an alternative method built on Hughes-Rudnick's technique in [87], which allows to show (4.6) unconditionally[5] (see Theorem 4.10). This strategy works only for test functions whose Fourier transform's support is contained in $(-1, 1)$. The same ideas would apply also to the other cases, with appropriate modifications. Moreover, the analogous result can be proved in the random matrix theory setting without any assumption, since the formula for the ratios of characteristic polynomials averaged over the unitary group is known unconditionally (see [30, Theorem 4.1] and also [29, 84]). Therefore, denoting

$$Z = Z(A, \theta) := \det(I - Ae^{i\theta})$$

the characteristic polynomial of $N \times N$ matrices A, with the same proof as for Theorem 4.1, we prove the following result.

[5] Neither the Riemann Hypothesis nor the ratio conjecture is required.

Theorem 4.2. *Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 2$, we have*

$$\frac{1}{\int_{U(N)} |Z|^{2k} d_{Haar}} \int_{U(N)} \sum_{j=1}^{N} f\Big(\frac{N}{2\pi}\theta_j\Big) |Z|^{2k} d_{Haar} \overset{N\to\infty}{\longrightarrow} \int_{-\infty}^{+\infty} f(x) W_U^k(x) dx.$$

In the symplectic case, we compute the weighted one-level density functions for any non-negative integer $k \leq 4$. We set

$$\begin{aligned}
W_{USp}^0(x) &:= W_{USp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}, \\
W_{USp}^1(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}, \\
W_{USp}^2(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}, \\
W_{USp}^3(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3} \\
&\quad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}, \\
W_{USp}^4(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{10(1+\cos(2\pi x))}{(\pi x)^2} + \frac{90\sin(2\pi x)}{(\pi x)^3} \\
&\quad - \frac{15(3-31\cos(2\pi x))}{(\pi x)^4} - \frac{1470\sin(2\pi x)}{(\pi x)^5} \\
&\quad - \frac{315(1+9\cos(2\pi x))}{(\pi x)^6} + \frac{3150\sin(2\pi x)}{(\pi x)^7} - \frac{1575(1-\cos(2\pi x))}{(\pi x)^8}
\end{aligned}$$

and we prove the following result.

Theorem 4.3. *Let us assume the Riemann Hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_\chi$ (see Conjecture 4.16). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{USp}^k(x) dx + O\Big(\frac{1}{\log X}\Big).$$

Also in the symplectic case, with the same proof we also get the corresponding result in the random matrix theory setting unconditionally, as [30, Theorem 4.2] provides the analogue of Conjecture 4.16.

Theorem 4.4. *Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\frac{1}{\int_{USp(2N)} Z^k d_{Haar}} \int_{USp(2N)} \sum_{j=1}^{N} f\Big(\frac{N}{\pi}\theta_j\Big) Z^k d_{Haar} \overset{N\to\infty}{\longrightarrow} \int_{-\infty}^{+\infty} f(x) W^k_{USp}(x) dx.$$

Finally, for the (even) orthogonal family $\boldsymbol{L}_{\Delta,\chi}$, we denote

$$W^0_{SO^+}(x) := W_{SO^+}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x},$$
$$W^1_{SO^+}(x) := 1 - \frac{\sin(2\pi x)}{2\pi x},$$
$$W^2_{SO^+}(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},$$
$$W^3_{SO^+}(x) := 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},$$
$$W^4_{SO^+}(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3} - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.$$

Notice that there are strong similarities with the symplectic kernels; we will discussed these analogies in Section 4.2.1. Then we prove the following theorem.

Theorem 4.5. *Let us assume the Riemann Hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_{\Delta,\chi}$ (see Conjecture 4.21). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W^k_{SO^+}(x) dx + O\Big(\frac{1}{\log X}\Big).$$

Again the analogue result in random matrix theory is instead unconditional (relying on [30, Theorem 4.3] in place of Conjecture 4.21).

Theorem 4.6. *Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\frac{1}{\int_{SO(2N)} Z^k d_{Haar}} \int_{SO(2N)} \sum_{j=1}^{N} f\Big(\frac{N}{\pi}\theta_j\Big) Z^k d_{Haar} \overset{N\to\infty}{\longrightarrow} \int_{-\infty}^{+\infty} f(x) W^k_{SO^+}(x) dx.$$

4.2.1 A general Conjecture for W_G^k

Thanks to the explicit expressions we get for the kernels W_G^k in the range $k \le 4$, we can speculate about what happens for any $k \in \mathbb{N}$. First of all, we notice that the (distributional) Fourier transform of the kernels W_G^k exhibits a structure. From the explicit formulae for W_G^k we get in the range $k \le 4$, $\widehat{W_G^k}$ turns out to be an even function, supported on $[-1,1]$, uniquely determined by a polynomial on $[0,1]$. More precisely, we conjecture that

$$\widehat{W_G^k}(y) = \delta_0(y) + P_G^k(|y|)\chi_{[-1,1]}(y) \tag{4.7}$$

where P_G^k is a polynomial depending on k and G only. In particular, in the unitary case and with $k \ge 1$, we expect the degree of P_U^k to be $2k-1$ and $P_U^k(0) = -k$, $P_U^k(1) = 0$. For the symplectic family, if $k \ge 1$, we predict P_{USp}^k with degree $2k-1$ and $P_{USp}^k(0) = -(2k+1)/2$, $P_{USp}^k(1) = (-1)^{k+1}/2$. Finally for orthogonal symmetry type, we conjecture the degree of $P_{SO^+}^k$ to be $2k-3$ and $P_{SO^+}^k(0) = -(2k-1)/2$, $P_{SO^+}^k(1) = (-1)^k/2$ for any $k \ge 2$ (the case $k=1$ yields $\widehat{W}_{SO^+}^1(y) = \delta_0(y) - 1/2$). We collect into a table all the value of P_G^k we obtained for k small, which support our speculations. Note that the case $k=0$, corresponding to the first row in the table, was already known in literature, while all other results are new.

P_G^k	$G = U$	$G = USp$	$G = SO^+$
$k=0$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$k=1$	$y-1$	$2y-\frac{3}{2}$	$-\frac{1}{2}$
$k=2$	$-2y^3+4y-2$	$-4y^3+6y-\frac{5}{2}$	$2y-\frac{3}{2}$
$k=3$		$12y^5-20y^3+12y-\frac{7}{2}$	$-4y^3+6y-\frac{5}{2}$
$k=4$		$-40y^7+84y^5-60y^3+20y-\frac{9}{2}$	$12y^5-20y^3+12y-\frac{7}{2}$

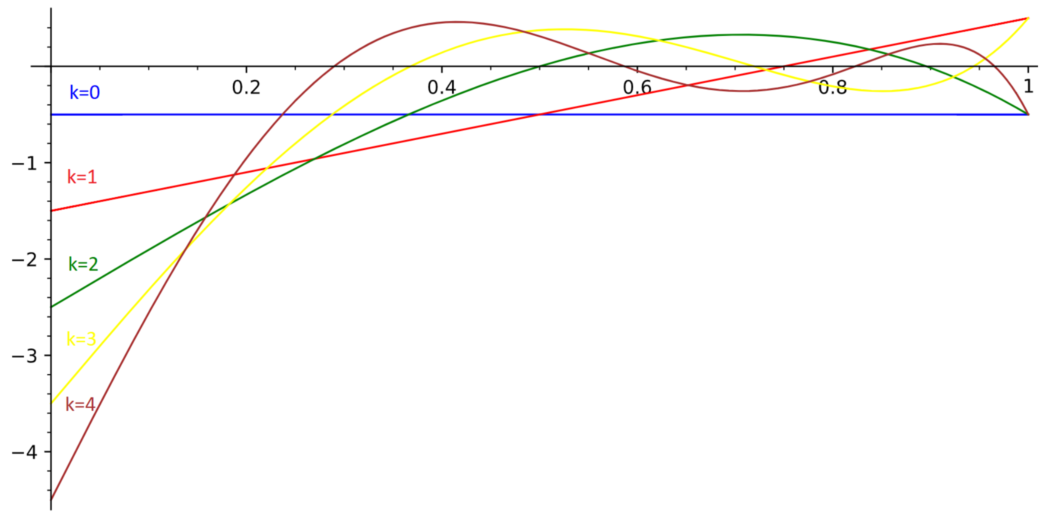

Figure 4.1: $P^k_{USp}(y)$, for $y \in [0, 1]$.

Looking at the table, we can detect relations between the weighted one-level density functions with different symmetry types. In particular, from the above discussion, it seems natural to expect that

$$W^k_{SO^+}(x) = W^{k-1}_{USp}(x) \tag{4.8}$$

for any $k \in \mathbb{Z}_+$. Moreover, the Fourier transforms of W^k_G suggest that the weighted one-level density function in the unitary case is the average of the symplectic and orthogonal cases; namely we conjecture that

$$W^k_U(x) = \frac{W^k_{USp}(x) + W^k_{SO^+}(x)}{2}. \tag{4.9}$$

We note that also the leading order moment coefficients $f_G(k)$ for the three compact groups U, USp, SO^+ satisfy relations linking them with each other, being (see [108], Equations (6.10) and (6.11))

$$f_{SO^+}(k) = 2^k f_{USp}(k-1) \quad \text{and} \quad 2^{k^2} f_U(k) = f_{USp}(k) f_{SO^+}(k).$$

Equations (4.8) and (4.9) can be seen as the analogue of the above formulae, in the context of the weighted one-level density.

Finally we conjecture an explicit formula for the polynomials P^k_G, which together with (4.7) provides a precise conjecture for the weighted kernels W^k_G. In view of Equations (4.8) and (4.9), it suffices to focus on the symplectic case only. Looking at what happens for $k \leq 4$, we speculate that

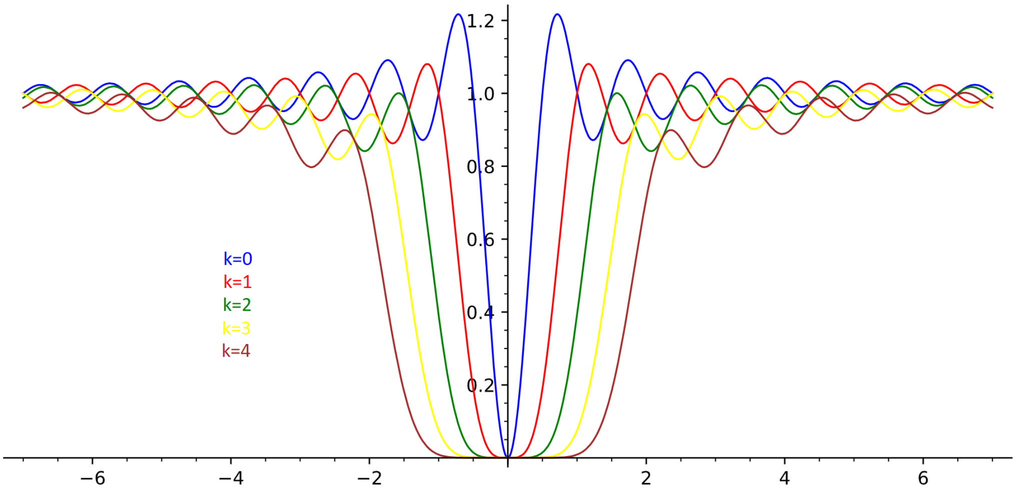

Figure 4.2: $W^k_{USp}(x)$ for $k = 0, \ldots, 4$.

for every positive integer k we have

$$P^k_{USp}(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1} \tag{4.10}$$

where the coefficient $c_{j,k}$ is defined by

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

We note that the sequence of the $c_{j,k}$'s appears in OEIS[6], as the number of diagonal dissections of a convex $(k+2)$-gon into j regions. By Fourier inversion, from (4.7) and (4.10), we get an explicit conjectural formula for W^k_{USp}, being

$$W^k_{USp}(x) = 1-(2k+1)\frac{\sin(2\pi x)}{2\pi x}+\sum_{j=1}^{k}\frac{k(k+1)}{2^{2j-2}\pi^{2j-1}}\frac{c_{j,k}}{2j-1}\frac{d^{2j-1}}{dx^{2j-1}}\left[\frac{1-\cos(2\pi x)}{2\pi x}\right].$$

From all these discussions, we can formulate the following conjecture.

Conjecture 4.7. *Let us consider a test function f, holomorphic in the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$*

[6]https://oeis.org/A033282.

as $x \to \infty$, then for any $k \in \mathbb{N}$. Given a family $\mathcal{F}$ of L-functions with symmetry type $G \in \{U, USp, SO^+\}$, we have

$$\mathcal{D}_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^k(x) dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where the weighted one-level density function W_G^k depends on k and G only. In addition the following relations hold

$$W_{SO^+}^k(x) = W_{USp}^{k-1}(x) \quad \text{and} \quad W_U^k(x) = \frac{W_{USp}^k(x) + W_{SO^+}^k(x)}{2}$$

for any $k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ respectively. Moreover, for every $k \in \mathbb{Z}_+$, in the symplectic case (the others can be recovered by the above relations), we have that

$$\widehat{W}_{USp}^k(y) = \delta_0(y) + P_{USp}^k(|y|)\chi_{[-1,1]}(y)$$

where P_{USp}^k is a polynomial of degree $2k-1$, given by

$$P_{USp}^k(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1},$$

with

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

4.2.2 An expression for $W_G^k(x)$ in terms of hypergeometric functions and its vanishing at $x = 0$

We now focus on the behaviour of the weighted kernels $W_G^k(x)$ at $x = 0$. For all symmetry types, it seems clear that the order of vanishing of $W_G^k(x)$ for $x \to 0$ increases as k grows. This phenomenon reflects the effect of the weight $V(L(1/2))^k$ in the average over the family, which gives more and more relevance to those L-functions that are large at the central point, as k increases. More precisely, for the unitary family we conjecture that

$$W_U^k(x) \sim \frac{\pi^{2k} x^{2k}}{(2k-1)!!(2k+1)!!} \tag{4.11}$$

as $x \to 0$, $k \in \mathbb{N}$. In particular, together with (4.6), this suggests that, on weighted average over the considered family, the number of normalized zeros which are less than ε away from the central point is typically $\asymp_k \varepsilon^{2k+1}$. Analogously, the asymptotic behaviour of the symplectic and orthogonal kernels can be deduced from (4.11) by Equations (4.8) and (4.9). For small values of k, the behaviour of $W_G^k(x)$ at $x = 0$ is outlined in the following table; the first row was already known in literature, all the others are new.

W_G^k $x \to 0$	$G = U$	$G = USp$	$G = SO^+$
$k = 0$	1	$\frac{2\pi^2 x^2}{3}$	2
$k = 1$	$\frac{\pi^2 x^2}{3}$	$\frac{2\pi^4 x^4}{45}$	$\frac{2\pi^2 x^2}{3}$
$k = 2$	$\frac{\pi^4 x^4}{45}$	$\frac{2\pi^6 x^6}{1575}$	$\frac{2\pi^4 x^4}{45}$
$k = 3$		$\frac{2\pi^8 x^8}{99225}$	$\frac{2\pi^6 x^6}{1575}$
$k = 4$		$\frac{2\pi^{10} x^{10}}{9823275}$	$\frac{2\pi^8 x^8}{99225}$

In the following conjecture, we condense all the speculations about the behaviour of the weighted kernels $W_G^k(x)$ as $x \to 0$.

Conjecture 4.8. *For $G \in \{U, USp, SO^+\}$, $k \in \mathbb{N}$, the weighted kernels W_G^k defined in Conjecture 4.7 satisfy the following asymptotic relations as $x \to 0$:*

$$W_U^k(x) \sim \frac{\pi^{2k} x^{2k}}{(2k-1)!!(2k+1)!!}$$
$$W_{USp}^k(x) \sim \frac{2\pi^{2(k+1)} x^{2(k+1)}}{(2k+1)!!(2k+3)!!}$$
$$W_{SO^+}^k(x) \sim \frac{2\pi^{2k} x^{2k}}{(2k-1)!!(2k+1)!!}.$$

Finally, assuming Conjecture 4.7, we obtain the expansion of $W_G^k(x)$ at $x = 0$. In particular, we show that the asymptotic behaviour of $W_G^k(x)$ can be deduced from the explicit formulae that we conjectured in Section 4.2.1. In view of Equations (4.8) and (4.9), it suffices to consider the symplectic case only.

Theorem 4.9. *Let us assume Conjecture 4.7. Then for any $k \in \mathbb{N}$ we have*

$$W_{USp}^k(x) = \sum_{m=1}^{\infty} \beta_{m,k} x^{2m}$$

with

$$\beta_{m,k} := (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m+1)!} \left((-1)^k + \frac{k(k+1)}{m+1} {}_3F_2 \left[\begin{matrix} 1-k,\ k+2,\ m+1 \\ m+2,\ 2 \end{matrix} ; 1 \right] \right)$$

where ${}_3F_2$ denotes the generalized hypergeometric function. Moreover, we have

$${}_3F_2 \left[\begin{matrix} 1-k,\ k+2,\ m+1 \\ m+2,\ 2 \end{matrix} ; 1 \right] = \begin{cases} \frac{(m+1)(-1)^{k+1}}{k(k+1)} & \text{if } 1 \leq m \leq k \\ \frac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!} \left(\binom{2k+1}{k+1} - 1 \right) & \text{if } m = k+1. \end{cases}$$

In particular, Conjecture 4.8 follows.

Proof. See Appendix B. □

4.3 The weighted one-level density for zeta

In this section we look closely at the family $\{\zeta(s + ia) : a \in \mathbb{R}\}$, which is an example of a family with unitary symmetry type. As explained in Section 1.6, one classically takes an even test function f and studies for any t the sum

$$N_f(t) := \sum_{\gamma} f\Big(\frac{\log T}{2\pi} (\gamma - t) \Big)$$

where $\gamma := -i(\rho - \frac{1}{2})$ and ρ runs over the non-trivial zeros of the Riemann zeta function. Of course, under RH, γ is real, being the imaginary part of a generic non-trivial zero of zeta. The above quantity can be evaluated asymptotically on average over $t \in [T, 2T]$, getting

$$\frac{1}{T} \int_T^{2T} N_f(t) dt = \int_{-\infty}^{+\infty} f(x) dx + O\Big(\frac{1}{\log T} \Big) \tag{4.12}$$

for test functions such that $\operatorname{supp} \widehat{f} \subset [-2, 2]$ (see e.g. [87]). Note that in this continuous example, the average over the family is given by an integration over t. Equation (4.12) proves that the family given by the Riemann zeta function parametrized by a vertical shift is a unitary family, in the sense of Katz-Sarnak [106]. In this section we want to compute the weighted analogue of the one-level density for zeta; as anticipated in Section 4.1, we first tilt the Lebesgue measure multiplying by $|\zeta(1/2+it)|^2$ and denote

$$\langle g\rangle_{|\zeta|^2} := \frac{1}{T\log T}\int_T^{2T} g(t)|\zeta(1/2+it)|^2 dt. \tag{4.13}$$

We then consider f an even test function whose Fourier transform

$$\widehat{f}(y) := \int_{-\infty}^{+\infty} f(x)e^{-2\pi ixy}dx$$

is compactly supported. If f is such that its Fourier transform's support[7] is small enough, then the weighted mean of $N_f(t)$ is treatable and this is achieved in the following theorem.

Theorem 4.10. *For any smooth, even and real-valued function f, such that $\widehat{f}$ is smooth and* $\operatorname{supp}\widehat{f}\subset(-1,1)$ *, we have*

$$\mathcal{D}_1^{\zeta}(f) := \langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)W_U^1(x)dx + O\Big(\frac{1}{\log T}\Big)$$

as $T\to\infty$, where

$$W_U^1(x) := 1-\operatorname{sinc}^2(x) = 1-\frac{\sin^2(\pi x)}{(\pi x)^2}.$$

We note that Theorem 4.10 is unconditional. Moreover, with an assumption about the moments of the Riemann zeta function we can remove the extra condition about the support of $\widehat{f}$. In particular, on RH, if we assume the ratio conjecture (see Conjecture 2.3), we can also handle the general case with f a decaying function, without any additional condition on its Fourier transorm's support:

Proposition 4.11. *Let us assume Conjecture 2.3 and the Riemann Hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)|<2$, real on the real line, even and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$. Then*

$$\mathcal{D}_1^{\zeta}(f) := \langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)W_U^1(x)dx + O\Big(\frac{1}{\log T}\Big)$$

[7]We define the support of f as the smallest closed set containing all points not mapped to zero by f. In particular the condition $\operatorname{supp} f\subset(-1,1)$ implies that $\operatorname{supp} f\subset[-a,a]$ for some $0\le a<1$.

with $W_U^1(x)$ as in Theorem 4.10.

The right hand side in Theorem 4.10 and Proposition 4.11 is revealing and it can be easily compared to the density conjecture in (4.3) and to the classical mean (4.12). Indeed, $W_U^1(x) \sim \frac{\pi^2}{3}x^2$ vanishes at $x = 0$ of order 2, showing a repulsion of zeros at height t which does not occur in the classical case. This repulsion can be explained by the fact that the measure $|\zeta(1/2+it)|^2 dt$ gives more weight to the large values of zeta, around which is more unlikely to find a zero.

In addition, with the same strategy as in the proof of Proposition 4.11 (but much longer computations, which can be done by using sage) we can also study the "fourth moment"case. Namely, we denote

$$\langle g\rangle_{|\zeta|^4} := \frac{1}{\frac{1}{2\pi^2}T(\log T)^4}\int_T^{2T} g(t)|\zeta(1/2+it)|^4 dt. \tag{4.14}$$

and we prove the following result.

Proposition 4.12. *Let us assume Conjecture 2.3 and the Riemann Hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then*

$$\mathcal{D}_2^\zeta(f) := \langle N_f\rangle_{|\zeta|^4} = \int_{-\infty}^{+\infty} f(x)W_U^2(x)dx + O\Big(\frac{1}{\log T}\Big) \tag{4.15}$$

with

$$W_U^2(x) := 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4}.$$

We note that Propositions 4.11 and 4.12 together imply Theorem 4.1. Finally we record the Fourier transform of the weighted kernels, being

$$\widehat{W_U^1}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ -y-1 & \text{if } -1 \le y < 0 \\ 0 & \text{if } y = 0 \\ y-1 & \text{if } 0 < y \le 1 \end{cases}$$

and

$$\widehat{W_U^2}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ 2y^3-4y-2 & \text{if } -1 \le y < 0 \\ -1 & \text{if } y = 0 \\ -2y^3+4y-2 & \text{if } 0 < y \le 1. \end{cases}$$

4.3.1 The explicit formula

The first tool we need in order to prove Theorem 4.10 is an explicit formula which allows us to treat the sum over zeros, due to Hughes and Rudnick. Since this lemma does not require RH, we recall that here $\gamma \in \mathbb{C}$.

Lemma 4.13 ([87], Lemma 2.1). *Let g be a smooth, compactly supported function and $h(r) = \int_{-\infty}^{+\infty} g(u)e^{iru}du$. Moreover we set $\Omega(r) = \frac{1}{2}\Psi(\frac{1}{4} + \frac{1}{2}ir) + \frac{1}{2}\Psi(\frac{1}{4} - \frac{1}{2}ir) - \log\pi$, where $\Psi(s) = \frac{\Gamma'}{\Gamma}(s)$ is the polygamma function, and we denote with $\Lambda(n)$ the von Mangoldt function. Then*

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\Omega(r)dr - \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(g(\log n) + g(-\log n)\Big) + h\Big(-\frac{i}{2}\Big) + h\Big(\frac{i}{2}\Big).$$

Applying Lemma 4.13 with $h(r) = f(\frac{\log T}{2\pi}(r-t))$ and $g(u) = \frac{e^{-itu}}{\log T}\widehat{f}(\frac{u}{\log T})$, we get:

$$N_f(t) = \overline{N_f(t)} + S_f(t) \tag{4.16}$$

where

$$\overline{N_f(t)} := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(r-t)\Big)\Omega(r)dr + f\Big(\frac{\log T}{2\pi}\Big(-\frac{i}{2}-t\Big)\Big) + f\Big(\frac{\log T}{2\pi}\Big(\frac{i}{2}-t\Big)\Big) \tag{4.17}$$

and

$$S_f(t) := -\frac{1}{\log T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\widehat{f}\Big(\frac{\log n}{\log T}\Big)(n^{-it} + n^{it}). \tag{4.18}$$

Since we are interested in the weighted mean of $N_f(t)$, we now compute $\langle\overline{N_f}\rangle_{|\zeta|^2}$ and we will treat $S_f(t)$ in the next section.

Proposition 4.14. *For any smooth, even and real-valued function f such that* $\operatorname{supp}\widehat{f} \subset [-a, a]$ *with $0 \le a < 1$, as $T \to \infty$ we have:*

$$\langle\overline{N_f}\rangle_{|\zeta|^2} = \widehat{f}(0) + O\Big(\frac{1}{\log T}\Big).$$

Proof. First of all, we notice that the first term on the right hand side of (4.17) can be simplified using [87, Lemma 2.2], which states that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(r-t)\Big)\Omega(r)dr = \frac{\Omega(t)}{\log T}\widehat{f}(0) + O\Big(\frac{(\log T)^{-2}}{1+|t|}\Big) + O\Big(\frac{\log(1+|t|)}{(\log T)^A}\Big) \tag{4.19}$$

for any $A > 1$. Moreover, since $\Omega(t) = \log(1+|t|) + O(1)$ for all t, one has

$$\Big\langle \frac{\Omega(t)}{\log T} \widehat{f}(0) \Big\rangle_{|\zeta|^2} = \widehat{f}(0) + O\Big(\frac{1}{\log T}\Big). \tag{4.20}$$

The tilted average of the errors in (4.19) is of course $O(1/\log T)$, then (4.19) and (4.20) yield

$$\begin{aligned}\langle \overline{N_f} \rangle_{|\zeta|^2} = \widehat{f}(0) &+ \Big\langle f\Big(\frac{\log T}{2\pi}\Big(-\frac{i}{2} - t\Big)\Big)\Big\rangle_{|\zeta|^2} \\ &+ \Big\langle f\Big(\frac{\log T}{2\pi}\Big(\frac{i}{2} - t\Big)\Big)\Big\rangle_{|\zeta|^2} + O\Big(\frac{1}{\log T}\Big).\end{aligned} \tag{4.21}$$

To complete the proof it suffices to show that also the average of the two remaining extra terms in (4.21) is $O(1/\log T)$. Let's bound the first one, the other is analogous. By Fourier inversion we have

$$\begin{aligned}&\Big\langle f\Big(\frac{\log T}{2\pi}\Big(-\frac{i}{2} - t\Big)\Big)\Big\rangle_{|\zeta|^2} \\ &\quad = \int_{-\infty}^{+\infty} \widehat{f}(y) T^{\frac{y}{2}} \frac{1}{T \log T} \int_T^{2T} T^{-ity} |\zeta(1/2+it)|^2 dt dy.\end{aligned} \tag{4.22}$$

We now recall the approximate functional equation for $|\zeta(1/2+it)|^2$ (see [115, Lemma 3])

$$|\zeta(1/2+it)|^2 = 2 \sum_{mn < T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \Big(\frac{m}{n}\Big)^{it} W\Big(\frac{2\pi mn}{t}\Big) + O(T^{-2/3})$$

with $\varepsilon > 0$ and

$$W(x) := \frac{1}{2\pi i} \int_{(\eta)} x^{-w} G(w) \frac{dw}{w}$$

with $\eta > 0$, $G(\cdot)$ an entire function with rapid decay along vertical lines (that is $G(x+iy) \ll |y|^{-A}$ for any fixed x and $A > 0$) and such that $G(-w) = G(w)$, $G(0) = 1$, $G(1/2) = 0$ and $G(w) = G(\overline{w})$. Then (4.22) equals

$$\frac{1}{T \log T} \int_{-\infty}^{+\infty} \widehat{f}(y) T^{\frac{y}{2}} \sum_{mn < T^{1+\varepsilon}} \frac{2}{\sqrt{mn}} \int_T^{2T} \Big(\frac{m}{nT^y}\Big)^{it} W\Big(\frac{2\pi mn}{t}\Big) dt dy$$

up to an error $O(T^{-\delta})$, for a suitable $\delta > 0$. Now we split the sum depending on a parameter $\Delta \asymp (\log T)^{-1}$ into the terms close to the diagonal, i.e. $|m - nT^y| \leq \Delta$, where we do not exploit the cancellation from the integal over t, and the remaining contribution with $|m - nT^y| > \Delta$, where we exploit the cancellation given by the inner integral. Thus we write

$$\Big\langle f\Big(\frac{\log T}{2\pi}\Big(-\frac{i}{2}-t\Big)\Big)\Big\rangle_{|\zeta|^2} = \mathcal{D} + \mathcal{O} + O(T^{-\delta}) \tag{4.23}$$

where

$$\mathcal{D} := \frac{1}{T\log T}\int_{-\infty}^{+\infty}\widehat{f}(y)T^{\frac{y}{2}}\sum_{\substack{mn<T^{1+\varepsilon}\\ |m-nT^y|\leq\Delta}}\frac{2}{\sqrt{mn}}\int_T^{2T}\Big(\frac{m}{nT^y}\Big)^{it}W\Big(\frac{2\pi mn}{t}\Big)dtdy$$

and

$$\mathcal{O} := \frac{1}{T\log T}\int_{-\infty}^{+\infty}\widehat{f}(y)T^{\frac{y}{2}}\sum_{\substack{mn<T^{1+\varepsilon}\\ |m-nT^y|>\Delta}}\frac{2}{\sqrt{mn}}\int_T^{2T}\Big(\frac{m}{nT^y}\Big)^{it}W\Big(\frac{2\pi mn}{t}\Big)dtdy.$$

To bound $\mathcal{D}$, we use the trivial bound for the inner integral over t (notice that $W(x) \ll_A \min(1, x^{-A})$ for $x > 0$), we bring the sum outside and we get

$$\mathcal{D} \ll \frac{1}{\log T}\sum_{mn<T^{1+\varepsilon}}\frac{1}{\sqrt{mn}}\int_{\substack{y\in[-a,a]\ s.t.\\ |m-nT^y|\leq\Delta}}\sqrt{T^y}dy.$$

With the change of variable $T^y = x$ we have

$$\begin{aligned}\mathcal{D} &\ll \frac{1}{\log T}\sum_{mn<T^{1+\varepsilon}}\frac{1}{\sqrt{mn}}\int_{\frac{m-\Delta}{n}}^{\frac{m+\Delta}{n}}\frac{dx}{\sqrt{x}\log T}\\ &\ll \frac{1}{(\log T)^2}\sum_{mn<T^{1+\varepsilon}}\frac{1}{\sqrt{mn}}\Bigg(\sqrt{\frac{m+\Delta}{n}}-\sqrt{\frac{m-\Delta}{n}}\Bigg)\end{aligned}$$

and since

$$\sqrt{\frac{m+\Delta}{n}}-\sqrt{\frac{m-\Delta}{n}} = \sqrt{\frac{m}{n}}\Bigg(\sqrt{1+\frac{\Delta}{m}}-\sqrt{1-\frac{\Delta}{m}}\Bigg) \ll \sqrt{\frac{m}{n}}\frac{\Delta}{m} = \frac{\Delta}{\sqrt{mn}}$$

then we get

$$\mathcal{D} \ll \frac{\Delta}{(\log T)^2}\sum_{mn<T^{1+\varepsilon}}\frac{1}{mn} \ll \Delta \asymp \frac{1}{\log T}.$$

For the term $\mathcal{O}$, we bound the inner integral by integrating by parts twice, getting

$$\int_T^{2T} \Big(\frac{m}{nT^y}\Big)^{it} W\Big(\frac{2\pi mn}{t}\Big) dt \ll \frac{\max_{t\in[T,2T]} |W(\frac{2\pi mn}{t})|}{|\log \frac{m}{nT^y}|} + \frac{\max_{t\in[T,2T]} |\frac{d}{dt} W(\frac{2\pi mn}{t})|}{|\log \frac{m}{nT^y}|^2} + \int_T^{2T} \frac{1}{|\log \frac{m}{nT^y}|^2} \Big| \frac{d^2}{dt^2} W\Big(\frac{2\pi mn}{t}\Big)\Big| dt \tag{4.24}$$

and since for any $l \in \mathbb{N}$ (see e.g. [11, Section 3.2])

$$\frac{d^l}{d^l} W\bigg(\frac{2\pi mn}{t}\bigg) \ll_{l,A} \frac{1}{t^l} \min\bigg(1, \Big(\frac{2\pi mn}{t}\Big)^{-A}\bigg)$$

then (4.24) yields

$$\int_T^{2T} \Big(\frac{m}{nT^y}\Big)^{it} W\Big(\frac{2\pi mn}{t}\Big) dt \ll \Big| \log \Big(\frac{m}{nT^y}\Big)\Big|^{-1} \ll \frac{nT^y}{|m - nT^y|}.$$

Therefore we have that

$$\begin{aligned} \mathcal{O} &\ll \frac{1}{T\log T} \int_{-a}^{a} T^{\frac{y}{2}} T^y \sum_{\substack{mn \le T^{1+\varepsilon} \\ |m-nT^y|>\Delta}} \sqrt{\frac{n}{m}} \frac{1}{|m - nT^y|} dy \\ &= \frac{1}{T\log T} \int_{-a}^{a} T^{\frac{y}{2}} T^y \sum_{n\le T^{1+\varepsilon}} \sqrt{n} \sum_{\substack{m \le \frac{T^{1+\varepsilon}}{n} \\ |m-nT^y|>\Delta}} \frac{1}{\sqrt{m}} \frac{1}{|m - nT^y|} dy. \end{aligned} \tag{4.25}$$

Now we split the sum over m as follows. If $1 \le m \le \frac{nT^y}{2}$ then $|m - nT^y| = nT^y - m \ge \frac{nT^y}{2}$, thus the contribution of these m in (4.25) is

$$\begin{aligned} &\ll \frac{1}{T\log T} \int_{-a}^{a} T^{\frac{y}{2}} \sum_{n\le T^{1+\varepsilon}} \frac{1}{\sqrt{n}} \sum_{m \le \frac{T^{1+\varepsilon}}{n}} \frac{1}{\sqrt{m}} dy \\ &\ll \frac{\sqrt{T^{1+\varepsilon}}}{T\log T} \int_{-a}^{a} T^{\frac{y}{2}} \sum_{n\le T^{1+\varepsilon}} \frac{1}{n} dy \ll T^{-\frac{1}{2}+\frac{a}{2}+\varepsilon} \end{aligned}$$

and that is $\ll O(1/\log T)$ being $a < 1$ (choosing $0 < \varepsilon < \frac{1}{2} - \frac{a}{2}$). For the m such that $\frac{nT^y}{2} < m \le nT^y - 1$, we have that $\frac{1}{\sqrt{m}} \ll \frac{1}{\sqrt{nT^y}}$, moreover the condition $mn < T^{1+\varepsilon}$ ensures that $n^2 < 2T^{-y}T^{1+\varepsilon}$; then the contribution of $\frac{nT^y}{2} < m \le nT^y - 1$ in (4.25) can be bounded by

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^y \sum_{n^2 \le T^{1-y+\varepsilon}} \sum_{\frac{nT^y}{2} < m \le nT^y - 1} \frac{1}{|m - nT^y|} dy.$$

In this range $|m - nT^y| = nT^y - m \in [1, \frac{nT^y}{2})$, then if we reparametrize the sum defining $l := [nT^y - m]$ ($[x]$ is the integer part of x) we get that the above is

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^y \sum_{n^2 \le T^{1-y+\varepsilon}} \sum_{1 \le l < nT^y} \frac{1}{l} dy \ll \frac{1}{T}\int_{-a}^{a} T^y \sum_{n^2 \le T^{1-y+\varepsilon}} 1\, dy$$

$$\ll \frac{1}{T}\int_{-a}^{a} T^y T^{\frac{1}{2}-\frac{y}{2}+\varepsilon} dy \ll T^{-\frac{1}{2}+\frac{a}{2}+\varepsilon} \ll \frac{1}{\log T}.$$

If $nT^y - 1 < m < nT^y + 1$ then $m \approx nT^y$ is fixed (at most two values of m satisfies this condition); in this range we use the condition $|m - nT^y| > \Delta \asymp (\log T)^{-1}$, so that we can bound the corresponding contribution in (4.25) with

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^{\frac{y}{2}} T^y \sum_{n \le T^{\frac{1}{2}-\frac{y}{2}+\varepsilon}} \sqrt{n}\frac{1}{\sqrt{nT^y}}\frac{1}{\Delta} dy \ll T^{-\frac{1}{2}+\frac{a}{2}+\varepsilon} \ll \frac{1}{\log T}.$$

For $nT^y + 1 \le m < 2nT^y$ we reparametrize defining $l := [m - nT^y]$ and, like before, the contribution of these m is

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^{\frac{y}{2}} T^y \sum_{n^2 T^y \le T^{1+\varepsilon}} \sqrt{n} \sum_{1 \le l \le nT^y} \frac{1}{\sqrt{nT^y}}\frac{1}{l} dy \ll \frac{1}{T}\int_{-a}^{a} T^y T^{\frac{1}{2}-\frac{y}{2}+\varepsilon} dy$$

which is $\ll (\log T)^{-1}$. Finally in the range $m \ge 2nT^y$ we have $|m - nT^y| = m - nT^y \ge nT^y$, then this last contribution is

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^{\frac{y}{2}} T^y \sum_{n \le T^{1+\varepsilon}} \sqrt{n} \sum_{m \le \frac{T^{1+\varepsilon}}{n}} \frac{1}{\sqrt{m}}\frac{1}{nT^y} dy$$

$$\ll \frac{1}{T\log T}\int_{-a}^{a} T^{\frac{y}{2}} \sum_{n \le T^{1+\varepsilon}} \frac{1}{\sqrt{n}}\sqrt{\frac{T^{1+\varepsilon}}{n}} dy$$

$$\ll \frac{T^{\frac{1}{2}+\varepsilon}}{T\log T}\int_{-a}^{a} T^{\frac{y}{2}} \sum_{n \le T^{1+\varepsilon}} \frac{1}{n} dy \ll T^{-\frac{1}{2}+\frac{a}{2}+\varepsilon} \ll \frac{1}{\log T}.$$

Plugging all these computations in (4.25) we get that $\mathcal{O} \ll \frac{1}{\log T}$ and the proposition follows. □

Thanks to Proposition 4.14 we proved that if $\operatorname{supp}\widehat{f} \subset (-1,1)$ then

$$\langle N_f\rangle_{|\zeta|^2} = \widehat{f}(0) + \langle S_f\rangle_{|\zeta|^2} + O\Big(\frac{1}{\log T}\Big). \tag{4.26}$$

In the following we study the remaining term $\langle S_f\rangle_{|\zeta|^2}$.

4.3.2 Proof of Theorem 4.10

In order to conclude the proof of Theorem 4.10 we have to handle the average of the sum over prime powers $S_f(t)$ and to perform this computation we rely on Lemma 2.4, which allows us to compute the moments of a sufficiently short Dirichlet polynomial with respect to $|\zeta(1/2+it)|^2dt$. We then prove the following proposition, which ends the proof of Theorem 4.10 together with (4.26).

Proposition 4.15. *For any smooth, even and real-valued function f such that* $\operatorname{supp}\widehat{f} \subset [-a,a]$ *with $0 \le a < 1$, as $T \to \infty$ we have:*

$$\langle S_f\rangle_{|\zeta|^2} = -\int_{-\infty}^{+\infty} \widehat{f}(y)(1-|y|)dy + O\Big(\frac{1}{\log T}\Big).$$

Proof. If we define the Dirichlet polynomial

$$P(t) := \sum_{n=1}^{\infty} \frac{g(n)}{n^{1/2+it}}, \qquad g(n) := \Lambda(n)\widehat{f}\Big(\frac{\log n}{\log T}\Big)$$

then by definition

$$\begin{aligned}\langle S_f\rangle_{|\zeta|^2} = -\frac{1}{T(\log T)^2}\int_T^{2T} P(t)|\zeta(1/2+it)|^2dt \\ -\frac{1}{T(\log T)^2}\int_T^{2T} P(-t)|\zeta(1/2+it)|^2dt.\end{aligned}$$

Since $g(n) = 0$ for $n > T^a$, then the Dirichlet polynomial is short enough so that we can apply Lemma 2.4 (this is the reason why we need $a < 1$), getting

$$\begin{aligned}\langle S_f\rangle_{|\zeta|^2} &= -\frac{2}{(\log T)^2}\sum_n \frac{g(n)}{n}(\log T - \log n + c) + o\bigg(\frac{1}{(\log T)^2}\bigg) \\ &= I_1 + I_2 + I_3 + o\bigg(\frac{1}{(\log T)^2}\bigg)\end{aligned} \tag{4.27}$$

with

$$I_1 := -\frac{2}{\log T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\widehat{f}\Big(\frac{\log n}{\log T}\Big)$$

$$I_2 := \frac{2}{(\log T)^2}\sum_{n=1}^{\infty}\frac{\Lambda(n)\log n}{n}\widehat{f}\Big(\frac{\log n}{\log T}\Big)$$

$$I_3 := -\frac{2c}{(\log T)^2}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\widehat{f}\Big(\frac{\log n}{\log T}\Big).$$

We now compute I_1; by partial summation

$$I_1 = -\frac{2}{\log T}\lim_{t\to\infty}\Bigg(\sum_{n\le t}\frac{\Lambda(n)}{n}\widehat{f}\Big(\frac{\log t}{\log T}\Big) - \int_1^t\sum_{n\le x}\frac{\Lambda(n)}{n}\widehat{f}\,'\Big(\frac{\log x}{\log T}\Big)\frac{dx}{x\log T}\Bigg)$$

and, since $\widehat{f}$ is compactly supported, the above equals

$$\frac{2}{\log T}\int_1^{\infty}\sum_{n\le x}\frac{\Lambda(n)}{n}\widehat{f}\,'\Big(\frac{\log x}{\log T}\Big)\frac{dx}{x\log T}.$$

Thus, with the change of variable $y=\frac{\log x}{\log T}$, we get

$$I_1 = \frac{2}{\log T}\int_0^{\infty}\sum_{n\le T^y}\frac{\Lambda(n)}{n}\widehat{f}\,'(y)dy.$$

and being

$$\sum_{n\le t}\frac{\Lambda(n)}{n} = \log t + O(1), \quad \text{as } t\to\infty$$

then

$$I_1 = 2\int_0^{\infty} y\widehat{f}\,'(y)dy + O\Big(\frac{1}{\log T}\Big).$$

as $T\to\infty$. Finally, integrating by parts, we have that

$$I_1 = -2\int_0^{+\infty}\widehat{f}(x)dx + O\Big(\frac{1}{\log T}\Big) = -\int_{\mathbb{R}}\widehat{f}(x)dx + O\Big(\frac{1}{\log T}\Big) \qquad (4.28)$$

since $\widehat{f}$ is even. This also proves that

$$I_3 = O\Big(\frac{1}{\log T}\Big). \qquad (4.29)$$

With the same technique we study I_2

$$I_2 = -\frac{2}{(\log T)^2}\int_1^\infty \sum_{n\le x}\frac{\Lambda(n)\log n}{n}\widehat{f}\,'\Big(\frac{\log x}{\log T}\Big)\frac{dx}{x\log T}.$$

Since

$$\sum_{n\le t}\frac{\Lambda(n)\log n}{n} = \frac{(\log t)^2}{2} + O(\log t), \quad \text{as } t\to\infty$$

with the change of variable $y = \frac{\log x}{\log T}$, the above is

$$= -\int_0^\infty y^2\widehat{f}\,'(y)dy + O\Big(\frac{1}{\log T}\Big).$$

Integrating by parts, we then get

$$I_2 = 2\int_0^\infty y\widehat{f}(y)dy + O\Big(\frac{1}{\log T}\Big) = \int_{\mathbb{R}} |y|\widehat{f}(y)dy + O\Big(\frac{1}{\log T}\Big) \tag{4.30}$$

thanks to the evenness of $\widehat{f}$. The claim follows putting together (4.27), (4.28), (4.30) and (4.29). □

Plugging Proposition 4.15 into (4.26) we get

$$\langle N_f\rangle_{|\zeta|^2} = \widehat{f}(0) - \int_{\mathbb{R}}\widehat{f}(y)(1-|y|)dy + O\Big(\frac{1}{\log T}\Big).$$

Then Theorem 4.10 easily follows by Plancherel identity (and because f is even), recalling that $\mathcal{F}(\frac{\sin^2(\pi\cdot)}{(\pi\cdot)^2})(y) = \max(1-|y|,0)$ which equals $1-|y|$ for $|y|<1$ (and this is guaranteed by the condition $\operatorname{supp}\widehat{f}\subset(-1,1)$).

4.3.3 Proof of Proposition 4.11

In order to remove the extra condition about the support of $\widehat{f}$ in Theorem 4.10, we rely on Conjecture 2.3, so that we can perform a similar computation as in Section 3 of [43] and prove Proposition 4.11. To begin with, we consider f a holomorphic function throughout the strip $|\Im(z)|<2$, which is real on the real line, even and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$. Then we introduce two parameters $\alpha,\beta\in\mathbb{R}$ of size $\asymp 1/\log T$, we denote

$$\boldsymbol{\zeta}_{\alpha,\beta}(t) := \zeta(1/2+\alpha+it)\zeta(1/2+\beta-it) \tag{4.31}$$

and we look at

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} := \frac{1}{T\log T}\int_T^{2T}\sum_\gamma f\Big(\frac{\log T}{2\pi}(\gamma-t)\Big)\boldsymbol{\zeta}_{\alpha,\beta}(t)dt \tag{4.32}$$

with $\gamma\in\mathbb{R}$ since we are assuming RH (we recall that $\rho=1/2+i\gamma$ are the non-trivial zeros of ζ). By the residue theorem we have that

$$\begin{aligned}\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} = \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\Big(\int_{(c)}-\int_{(1-c)}\Big)\frac{\zeta'}{\zeta}(s+it)\\ \cdot f\Big(\frac{-i\log T}{2\pi}(s-1/2)\Big)ds\ \boldsymbol{\zeta}_{\alpha,\beta}(t)dt\end{aligned} \tag{4.33}$$

where $c\in(\frac{1}{2},1)$ and $\int_{(c)}$ denotes the integral over the vertical line of those s such that $\Re(s)=c$ (note that this c is not the constant $2\gamma+\log(2/\pi)-1$ from Lemma 2.4). We select $c=\frac{1}{2}+\delta$ with $\delta\asymp(\log T)^{-1}$ and we first consider the integral over the c-line

$$\begin{aligned}\mathcal{I} &:= \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(c)}\frac{\zeta'}{\zeta}(s+it)f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)ds\ \boldsymbol{\zeta}_{\alpha,\beta}(t)dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(y-i\delta)\Big)\frac{d}{d\gamma}\Big[\frac{I(\alpha;\beta;\delta+iy+\gamma;\delta+iy)}{T\log T}\Big]_{\gamma=0}dy\end{aligned}$$

where

$$I(A;B;C;D) := \int_T^{2T}\frac{\zeta(1/2+A+it)\zeta(1/2+B-it)\zeta(1/2+C+it)}{\zeta(1/2+D+it)}dt. \tag{4.34}$$

Moments like (4.34) can be computed thanks to Conjecture 2.3 and it turns out to be

$$\begin{aligned}I(A;B;C;D) = \int_T^{2T}\Big\{&\frac{\zeta(1+A+B)\zeta(1+B+C)}{\zeta(1+B+D)}\\ &+\Big(\frac{t}{2\pi}\Big)^{-A-B}\frac{\zeta(1-A-B)\zeta(1-A+C)}{\zeta(1-A+D)}\\ &+\Big(\frac{t}{2\pi}\Big)^{-B-C}\frac{\zeta(1+A-C)\zeta(1-B-C)}{\zeta(1-C+D)}\Big\}dt\\ &+O(T^{1/2+\varepsilon})\end{aligned} \tag{4.35}$$

for suitable shifts A,B,C,D, i.e. with real part $\asymp(\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$, for every $\varepsilon>0$ (see e.g. [43, Section 2.1]). Notice that the arithmetical factor $A_\zeta(\alpha;\beta;\gamma;\delta)$ from Conjecture 2.3 equals 1 in our case, with $K=2$, $L=1$, $Q=1$, $R=0$, being

$$\begin{aligned}
&A_\zeta(A,B,C,D)\\
&=\prod_p \frac{(1-\frac{1}{p^{1+A+B}})(1-\frac{1}{p^{1+C+D}})}{(1-\frac{1}{p^{1+B+D}})}\sum_{a+c+d=b}\frac{\mu(p^d)}{p^{(1/2+A)a+(1/2+C)c+(1/2+B)b+(1/2+D)d}}\\
&=\prod_p \frac{(1-\frac{1}{p^{1+A+B}})(1-\frac{1}{p^{1+C+D}})}{(1-\frac{1}{p^{1+B+D}})}\sum_{a,c,d}\frac{\mu(p^d)}{p^{(1+A+B)a+(1+C+B)c+(1+D+B)d}}=1.
\end{aligned}$$

We now want to apply (4.35) with $A=\alpha$, $B=\beta$, $C=\delta+iy+\gamma$, $D=\delta+iy$ and to do so we need that the imaginary parts of all the shifts are $\ll_\varepsilon T^{1-\varepsilon}$. A standard technique to avoid this issue is splitting the integral over y in two pieces; the contribution to $\mathcal{I}$ coming from $|y|>T^{1-\varepsilon}$ is $\ll T^{-1+\varepsilon}$, thanks to the good decaying of f and to RH, since

$$\begin{aligned}
\frac{1}{T\log T}\int_T^{2T}|\zeta_{\alpha,\beta}(t)|\int_{|y|>T^{1-\varepsilon}}|f(y\log T)|\left|\frac{\zeta'}{\zeta}(1/2+\delta+iy+it)\right|dydt\\
\ll\frac{T^{\varepsilon/100}}{T}\int_T^{2T}\int_{|y|>T^{1-\varepsilon}}\frac{\log(|y|t)}{|y|^2}dydt\ll T^{-1+\varepsilon}.
\end{aligned}$$

Therefore we can truncate the integral over y at height $T^{1-\varepsilon}$, apply (4.35) and then re-extend the integration over y to infinity with a small error term. Thus, differentiating with respect to γ at $\gamma=0$, moving the path of integration to $\delta=0$ (we are allowed to so so since now the integral is regular at $\delta=0$) we get

$$\mathcal{I}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{1}{T\log T}\int_T^{2T}g_{\alpha,\beta}(y;t)dtdy+O(T^{-1/2+\varepsilon})\quad(4.36)$$

with

$$\begin{aligned}
g_{\alpha,\beta}(y;t):=&\frac{\zeta(1+\alpha+\beta)\zeta'(1+\beta+iy)}{\zeta(1+\beta+iy)}\\
&+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)\zeta'(1-\alpha+iy)}{\zeta(1-\alpha+iy)}\\
&-\Big(\frac{t}{2\pi}\Big)^{-\beta-iy}\zeta(1+\alpha-iy)\zeta(1-\beta-iy).
\end{aligned}\quad(4.37)$$

We notice that, when computing this derivative, it is useful to observe that if $f(z)$ is analytic at $z=0$, then (see [43, Equation (2.13)])

$$\frac{d}{d\gamma}\left[\frac{f(\gamma)}{\zeta(1-\gamma)}\right]_{\gamma=0}=-f(0).$$

Similarly we deal with the integral over the $(1-c)$-line in (4.33)

$$\begin{aligned}\mathcal{J} :&= \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(1-c)}\frac{\zeta'}{\zeta}(s+it)f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)ds\, \boldsymbol{\zeta}_{\alpha,\beta}(t)dt\\ &= \frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(1-s+it)\boldsymbol{\zeta}_{\alpha,\beta}(t)dtdy.\end{aligned}$$

Using the functional equation

$$\frac{\zeta'}{\zeta}(1-z) = \frac{X'}{X}(z) - \frac{\zeta'}{\zeta}(z)$$

where

$$\frac{X'}{X}(z) := \log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{z}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-z}{2}\Big)$$

we express $\mathcal{J}$ as a sum of two terms

$$\mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2 \tag{4.38}$$

with

$$\mathcal{J}_1 := \frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}(s-1/2)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)dtdy$$

and

$$\mathcal{J}_2 := \frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}(s-1/2)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)dtdy.$$

With $c = 1/2+\delta$, $\delta\to 0$, it is easy to see that

$$\begin{aligned}\mathcal{J}_1 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}(\tfrac{1}{2}+iy-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)dtdy\\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T+O(1)}{T\log T}\int_T^{2T}\boldsymbol{\zeta}_{\alpha,\beta}(t)dtdy\end{aligned}\tag{4.39}$$

since, using Stirling's approximation to estimate the gamma-factors, we have (again we can assume $y \ll T^{1-\varepsilon}$ because of the great decaying of f)

$$\begin{aligned}\frac{X'}{X}(1/2+iy-it) &= -\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iy}{2}-\frac{it}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}-\frac{iy}{2}+\frac{it}{2}\Big)+O(1)\\ &= -\frac{1}{2}\log\Big(-\frac{it}{2}\Big) - \frac{1}{2}\log\Big(\frac{it}{2}\Big)+O(1) = -\log T+O(1).\end{aligned}$$

Moreover, with the same choice of c as before, if we set $\alpha = \beta$ we get

$$\mathcal{J}_2 = \mathcal{I}. \tag{4.40}$$

Then (4.33), (4.38) and (4.40) imply that

$$\langle N_f \rangle_{|\zeta|^2}^{\alpha,\alpha} = \mathcal{I} + \mathcal{J} = -\mathcal{J}_1 + 2\mathcal{I} \tag{4.41}$$

and the function $\mathcal{J}_1 = \mathcal{J}_1(\alpha)$ is regular at $\alpha = 0$, then we can take the limit in (4.39), getting

$$\begin{aligned}\lim_{\alpha\to 0} \mathcal{J}_1 &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T + O(1)}{T\log T}\int_T^{2T} |\zeta(1/2+it)|^2 dt\, dy \\ &= -\frac{\log T + O(1)}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)dy = -\int_{-\infty}^{+\infty} f(x)dx + O\Big(\frac{1}{\log T}\Big)\end{aligned} \tag{4.42}$$

with the change of variable $\frac{\log T}{2\pi}y = x$. Lastly, we study the remaining term $\mathcal{I}$, from (4.36) and (4.37). We set $\alpha = \beta = a/\log T$ with $0 < a < 1$, we perform the same change of variable $\frac{\log T}{2\pi}y = x$ as before and we get

$$\mathcal{I} = \int_{-\infty}^{+\infty} f(x)\frac{1}{T(\log T)^2}\int_T^{2T} g_{\frac{a}{\log T},\frac{a}{\log T}}\Big(\frac{2\pi x}{\log T};t\Big)dtdx + O(T^{1/2+\varepsilon})$$

and since $\log\frac{t}{2\pi} = \log T + O(1)$ as $t \in [T, 2T]$, then

$$\begin{aligned}\mathcal{I} = \Big(\frac{1}{(\log T)^2} + O\Big(\frac{1}{(\log T)^3}\Big)\Big)\int_{-\infty}^{+\infty} f(x)\Big(&\zeta(1+\tfrac{2a}{\log T})\frac{\zeta'}{\zeta}(1+\tfrac{a+2\pi ix}{\log T}) \\ &+ e^{-2a}\zeta(1-\tfrac{2a}{\log T})\frac{\zeta'}{\zeta}(1-\tfrac{a-2\pi ix}{\log T}) \\ &- e^{-a-2\pi ix}\zeta(1+\tfrac{a-2\pi ix}{\log T})\zeta(1-\tfrac{a+2\pi ix}{\log T})\Big)dx\end{aligned} \tag{4.43}$$

where the error term is uniform in a. Now, we will prove that the above expression is regular at $a = 0$, showing that

$$\lim_{a\to 0}\mathcal{I} = \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)dx + O\Big(\frac{1}{\log T}\Big) \tag{4.44}$$

as $T \to \infty$, where

$$\mathcal{P}(x) := \frac{-1 + 2\pi ix + e^{-2\pi ix}}{4\pi^2x^2}$$

Intuitively, if we replace each zeta function with its leading term in the expansion at the point 1 given by $\zeta(1+z) \sim \frac{1}{z}$, we have

$$\begin{aligned}\mathcal{I} &\approx \frac{1}{(\log T)^2}\int_{-\infty}^{+\infty} f(x)\Big(\frac{-(\log T)^2}{2a(a+2\pi ix)} + e^{-2a}\frac{-(\log T)^2}{2a(a-2\pi ix)} \\ &\qquad\qquad + e^{-a-2\pi ix}\frac{(\log T)^2}{(a-2\pi ix)(a+2\pi ix)}\Big)dx \\ &= \int_{-\infty}^{+\infty} f(x)\Big(-\frac{1}{2a(a+2\pi ix)} - \frac{e^{-2a}}{2a(a-2\pi ix)} + \frac{e^{-a-2\pi ix}}{(a-2\pi ix)(a+2\pi ix)}\Big)dx\end{aligned}$$

and the function inside the parentheses above equals

$$\frac{-a(1+e^{-2a}) + 2\pi ix(1-e^{-2a}) + 2ae^{-a-2\pi ix}}{2a(a^2+4\pi^2x^2)} = \frac{-1+2\pi ix + e^{-2\pi ix} + O(a)}{4\pi^2x^2 + O(a^2)}$$

and then tends to $\mathcal{P}(x)$ as $a \to 0$.
To show (4.44) rigorously, we split the integral over x into two parts. We start with the case $x \ll \log T$; from Taylor approximation $f(1+s\pm y) = f(1+s) \pm yf'(1+s) + O_s(y^2)$ we get

$$\begin{aligned}\frac{\zeta'}{\zeta}\Big(1+\frac{2\pi ix}{\log T} \pm \frac{a}{\log T}\Big) &= \frac{\zeta'}{\zeta}\Big(1+\frac{2\pi ix}{\log T}\Big) \pm \frac{a}{\log T}\Big(\frac{\zeta'}{\zeta}\Big)'\Big(1+\frac{2\pi ix}{\log T}\Big) + O_T(a^2) \\ &=: c_1(x) \pm \frac{a}{\log T}c_2(x) + O_T(a^2)\end{aligned}$$

and

$$\zeta\Big(1-\frac{2\pi ix}{\log T} \pm \frac{a}{\log T}\Big) = \zeta\Big(1-\frac{2\pi ix}{\log T}\Big) + O_T(a) =: k(x) + O_T(a)$$

as $a \to 0$, with the notations $c_1(x) = c_1(x,T) := \frac{\zeta'}{\zeta}(1+\frac{2\pi ix}{\log T})$, $c_2(x) = c_1(x,T) := (\frac{\zeta'}{\zeta})'(1+\frac{2\pi ix}{\log T})$ and $k(x) = k(x,T) := \zeta(1-\frac{2\pi ix}{\log T})$. Moreover we use the asymptotic expansion

$$\zeta(1+z) = \frac{1}{z} + \gamma + O(z) \qquad z \to 0, \tag{4.45}$$

and we get

$$\begin{aligned}
&\int_{x\ll\log T} f(x)\Big(\zeta(1+\tfrac{2a}{\log T})\frac{\zeta'}{\zeta}(1+\tfrac{a+2\pi ix}{\log T}) \\
&\qquad + e^{-2a}\zeta(1-\tfrac{2a}{\log T})\frac{\zeta'}{\zeta}(1-\tfrac{a-2\pi ix}{\log T}) \\
&\qquad - e^{-a-2\pi ix}\zeta(1+\tfrac{a-2\pi ix}{\log T})\zeta(1-\tfrac{a+2\pi ix}{\log T})\Big)dx \\
&= \int_{x\ll\log T} f(x)\bigg(\Big[\frac{\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)+\frac{a}{\log T}c_2(x)+O_T(a^2)\Big] \\
&\qquad + e^{-2a}\Big[\frac{-\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)-\frac{a}{\log T}c_2(x)+O_T(a^2)\Big] \\
&\qquad - e^{-a-2\pi ix}\Big[k(x)+O_T(a)\Big]^2\bigg)dx
\end{aligned}$$

whose limit as $a\to 0$ is

$$\begin{aligned}
&\int_{x\ll\log T} f(x)\bigg\{\lim_{a\to 0}\bigg(\Big[\frac{c_1(x)\log T}{2a}+\frac{c_2(x)}{2}+\gamma c_1(x)\Big] \\
&\qquad + e^{-2a}\Big[\frac{-c_1(x)\log T}{2a}+\frac{c_2(x)}{2}+\gamma c_1(x)\Big]\bigg) - e^{-2\pi ix}k(x)^2\bigg\}dx \\
&= \int_{x\ll\log T} f(x)\bigg\{\lim_{a\to 0}\bigg(c_1(x)\log T\Big(\frac{1-e^{-2a}}{2a}\Big)+c_2(x)\Big(\frac{1+e^{-2a}}{2}\Big) \\
&\qquad + \gamma c_1(x)(1+e^{-2a})\bigg) - e^{-2\pi ix}k(x)^2\bigg\}dx \\
&= \int_{x\ll\log T} f(x)\Big\{c_1(x)\log T+c_2(x)+2\gamma c_1(x)-e^{-2\pi ix}k(x)^2\Big\}dx.
\end{aligned}$$

By definition of $c_1(x), c_2(x), k(x)$, the asymptotic expansion (4.45) yields $c_1(x) = -\frac{\log T}{2\pi ix}+O(1)$, $c_2(x)=\frac{(\log T)^2}{(2\pi ix)^2}+O(1)$ and $k(x)=\frac{(\log T)^2}{(2\pi ix)^2}-\frac{2\gamma\log T}{2\pi ix}+O(1)$, uniformly for $x\ll\log T$. Then the above is

$$= \int_{x\ll\log T} f(x)\Big\{-\frac{(\log T)^2}{2\pi ix}+\frac{(\log T)^2}{(2\pi ix)^2}-e^{-2\pi ix}\frac{(\log T)^2}{(2\pi ix)^2}+O(\log T)\Big\}dx$$

(note that the sum $2\gamma c_1(x)-e^{-2\pi ix}k(x)^2$ gives the third term in the parentheses with an error $O(\log T)$, a possible pole at $x=0$ cancels out), which is

$$= (\log T)^2\int_{x\ll\log T} f(x)\frac{-1+2\pi ix+e^{-2\pi ix}}{4\pi^2x^2}dx+O(\log T).$$

Finally we can re-extend the range of integration with a small error term (being $f(x) \ll 1/(1+x^2)$ and $\mathcal{P}(x)$ bounded), getting that the contribution of $x \ll \log T$ in the integral over x in (4.43), in the limit as $a \to 0$, equals

$$= (\log T)^2 \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)dx + O(\log T).$$

To prove (4.44), we finally have to bound the contribution of $x \gg \log T$ in the integral on the right hand side (4.43), as $a \to 0$; to do so, we use the bounds $\zeta(1+iy) \ll \log y$ (see [155, Theorem 3.5]) and $\frac{\zeta'}{\zeta}(1+iy) \ll \log y$ (see [155, Equation (3.11.9)]) for $y \gg 1$, thus the contribution coming from $x \gg \log T$ is

$$\begin{aligned}
&= \lim_{a\to 0} \int_{x\gg \log T} f(x)\bigg(\Big[\frac{\log T}{2a} + O(1)\Big]\Big[\frac{\zeta'}{\zeta}(1+\tfrac{2\pi i x}{\log T}) + O\Big(\frac{a}{\log T}\Big)\Big] \\
&\qquad + e^{-2a}\Big[-\frac{\log T}{2a} + O(1)\Big]\Big[\frac{\zeta'}{\zeta}(1+\tfrac{2\pi i x}{\log T}) + O\Big(\frac{a}{\log T}\Big)\Big] + O\big((\log x)^2\big)\bigg)dx \\
&= \int_{x\gg \log T} f(x)\bigg(\log T \frac{\zeta'}{\zeta}(1+\tfrac{2\pi i x}{\log T}) \lim_{a\to 0}\Big[\frac{1-e^{-2a}}{2a}\Big] + O\big((\log x)^2\big)\bigg)dx \\
&\ll \int_{x\gg \log T} |f(x)|\Big(\log T \log x + (\log x)^2\Big)dx \ll \log T \int_{x\gg \log T} \frac{\log x}{x^2}dx,
\end{aligned}$$

then (4.44) follows, being $\int_{x\gg \log T} \frac{\log x}{x^2}dx \ll 1$. Finally, if we decompose $\mathcal{P}(x)$ in even and odd part

$$\mathcal{P}(x) = -\frac{1}{2}\frac{\sin^2(\pi x)}{(\pi x)^2} - \frac{i\big(\sin(2\pi x) - 2\pi x\big)}{4\pi^2 x^2} \tag{4.46}$$

since f is even and $\mathcal{P}(x)$ bounded, we have

$$\lim_{a\to 0} \mathcal{I} = -\frac{1}{2}\int_{-\infty}^{+\infty} f(x)\frac{\sin^2(\pi x)}{(\pi x)^2}dy + O\Big(\frac{1}{\log T}\Big). \tag{4.47}$$

Putting together (4.41), (4.42) and (4.47) we finally get

$$\langle N_f \rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin^2(\pi x)}{(\pi x)^2}\Big)dx + O\Big(\frac{1}{\log T}\Big)$$

as $T \to \infty$ and the theorem has been proved.

4.3.4 Proof of Proposition 4.12

This proof builds on the same ideas as that of Proposition 4.11, even though we have to handle longer computations; to begin with, we introduce four parameters $\alpha, \beta, \nu, \eta \in \mathbb{R}$ of size $1/\log T$, we denote

$$\boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t) := \zeta(1/2+\alpha+it)\zeta(1/2+\beta+it)\zeta(1/2+\nu-it)\zeta(1/2+\eta-it)$$

and we look at

$$\langle N_f \rangle_{|\zeta|^4}^{\alpha,\beta,\nu,\eta} := \frac{1}{\frac{1}{2\pi^2}T(\log T)^4} \int_T^{2T} \sum_\gamma f\Big(\frac{\log T}{2\pi}(\gamma - t)\Big) \boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t) dt \tag{4.48}$$

with $\gamma \in \mathbb{R}$ since we are assuming RH. In analogy to Equation (4.41), the residue theorem yields

$$\langle N_f \rangle_{|\zeta|^4}^{\alpha,\beta,\alpha,\beta} = -\mathcal{J}_1 + 2\mathcal{I} \tag{4.49}$$

with

$$\begin{aligned} \mathcal{J}_1 = \mathcal{J}_1(\alpha,\beta) &= -\int_{-\infty}^{+\infty} f(x) \frac{\log T + O(1)}{\frac{1}{2\pi^2}T(\log T)^5} \int_T^{2T} \boldsymbol{\zeta}_{\alpha,\beta,\alpha,\beta}(t) dt dx \\ &\overset{\alpha,\beta\to 0}{\longrightarrow} -\int_{-\infty}^{+\infty} f(x) dx + O\Big(\frac{1}{\log T}\Big) \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} \mathcal{I} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(y - i\delta)\Big) \frac{d}{d\gamma}\Bigg[\frac{I(\alpha;\beta;\gamma;\delta+iy;\alpha;\beta)}{\frac{1}{2\pi^2}T(\log T)^4}\Bigg]_{\gamma=\delta+iy} dy \\ &= \int_{-\infty}^{+\infty} f\Big(x - \frac{i\delta \log T}{2\pi}\Big) \frac{d}{d\gamma}\Bigg[\frac{I(\frac{a}{\log T};\frac{b}{\log T};\gamma;\delta+\frac{2\pi i x}{\log T};\frac{a}{\log T};\frac{b}{\log T})}{\frac{1}{2\pi^2}T(\log T)^5}\Bigg]_{\gamma=\delta+\frac{2\pi i x}{\log T}} dx \end{aligned}$$

where $a, b \asymp 1$, $\delta \asymp 1/\log T$ and $I(A;B;C;D;F;G)$ is defined by

$$\int_T^{2T} \frac{\zeta(\frac12+A+it)\zeta(\frac12+B+it)\zeta(\frac12+C+it)\zeta(\frac12+F-it)\zeta(\frac12+G-it)}{\zeta(\frac12+D+it)} dt.$$

If the shifts satisfy the conditions prescribed by Conjecture 2.3 then such an integral can be evaluated by using to the ratio conjecture. According to the recipe, up to an error $O(T^{1/2+\varepsilon})$, the above moment is a sum of ten pieces, the first being

$$\int_T^{2T} \frac{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+C+F)\zeta(1+C+G)}{\zeta(1+D+F)\zeta(1+D+G)} \mathcal{A}_{A,B,C,D,F,G}\, dt$$

where

$$\mathcal{A}_{A,B,C,D,F,G} = \prod_p (1 - \frac{1}{p^{1+A+F}})(1 - \frac{1}{p^{1+A+G}})(1 - \frac{1}{p^{1+B+F}})(1 - \frac{1}{p^{1+B+G}})$$
$$(1 - \frac{1}{p^{1+C+F}})(1 - \frac{1}{p^{1+C+G}})(1 - \frac{1}{p^{1+D+F}})^{-1}(1 - \frac{1}{p^{1+D+G}})^{-1}$$
$$\sum_{a+b+c+d=f+g} \frac{\mu(p^d)}{p^{(\frac{1}{2}+A)a+(\frac{1}{2}+B)b+(\frac{1}{2}+C)c+(\frac{1}{2}+D)d+(\frac{1}{2}+F)f+(\frac{1}{2}+G)g}}.$$

It will be useful to notice that if all the shifts equal zero, then

$$\mathcal{A} := \mathcal{A}_{0,0,0,0,0,0} = \frac{1}{\zeta(2)}; \tag{4.51}$$

indeed, since $\mu(p^d)$ forces d to be either 0 or 1, we have that

$$\mathcal{A} = \prod_p \left(1 - \frac{1}{p}\right)^4 \left(\sum_{a+b+c=f+g} \frac{1}{p^{\frac{a+b+c+f+g}{2}}} + \sum_{a+b+c+1=f+g} \frac{-1}{p^{\frac{a+b+c+1+f+g}{2}}} \right)$$
$$= \prod_p \left(1 - \frac{1}{p}\right)^4 \sum_{n=0}^{\infty} \frac{1}{p^n} \Bigg[\sum_{a+b+c=n} 1 \cdot \sum_{f+g=n} 1 - \sum_{a+b+c+1=n} 1 \cdot \sum_{f+g=n} 1 \Bigg].$$

We recall that the number of representations of an integer n as a sum of k nonnegative integers is $\binom{n+k-1}{k-1}$ thus the sum over n above equals

$$\sum_{n=0}^{\infty} \frac{1}{p^n} \left[\frac{(n+2)(n+1)^2}{2} - \frac{n(n+1)^2}{2} \right] = \sum_{n=0}^{\infty} \frac{(n+1)^2}{p^n}.$$

Differentiating the closed formula for the geometric series, one easily gets that $\sum_n (n+1)^2 x^n = \frac{1+x}{(1-x^3)}$ for $|x| < 1$, therefore

$$\mathcal{A} = \prod_p \left(1 - \frac{1}{p}\right)^4 \frac{1 + \frac{1}{p}}{(1 - \frac{1}{p})^3} = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}$$

and (4.51) is proven.

All the other nine terms can be recovered from the first one just by swapping the shifts as prescribed by the recipe; doing so yields a formula for $I(A; B; C; D; F; G)$ and differentiating with respect to C at $C = D$ we get

$$\frac{d}{dC}\big[I(A;B;C;D;F;G)\big]_{C=D}$$
$$= \int_T^{2T}\Big(R_1 + (\tfrac{t}{2\pi})^{-A-F}R_2 + (\tfrac{t}{2\pi})^{-A-G}R_3 + (\tfrac{t}{2\pi})^{-B-F}R_4 + (\tfrac{t}{2\pi})^{-B-G}R_5$$
$$+ (\tfrac{t}{2\pi})^{-D-F}R_6 + (\tfrac{t}{2\pi})^{-D-G}R_7 + (\tfrac{t}{2\pi})^{-A-B-F-G}R_8$$
$$+ (\tfrac{t}{2\pi})^{-A-D-F-G}R_9 + (\tfrac{t}{2\pi})^{-B-D-F-G}R_{10}\Big)dt + O(T^{1/2+\varepsilon}) \tag{4.52}$$

with

$$R_1 = R_1(A,B,D,F,G) = \tfrac{\mathcal{A}_{A,B,D,D,F,G}}{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+D+F)\zeta(1+D+G)}$$
$$\cdot\Big[\tfrac{\zeta'}{\zeta}(1+D+F) + \tfrac{\zeta'}{\zeta}(1+D+G) + \frac{\mathcal{A}'_{A,B,D,D,F,G}}{A_{A,B,D,D,F,G}}\Big]$$
$$R_2 = R_1(-F,B,D,-A,G)$$
$$R_3 = R_1(-G,B,D,F,-A)$$
$$R_4 = R_1(A,-F,D,-B,G)$$
$$R_5 = R_1(A,-G,D,F,-B)$$
$$R_6 = R_6(A,B,D,F,G)$$
$$= -\tfrac{\zeta(1+A-D)\zeta(1+A+G)\zeta(1+B-D)\zeta(1+B+G)\zeta(1-F-D)\zeta(1-F+G)}{\zeta(1+D+G)}\mathcal{A}_{A,B,D,D,F,G}$$
$$R_7 = R_6(A,B,D,G,F)$$
$$R_8 = R_1(-F,-G,D,-A,-B)$$
$$R_9 = R_6(-F,B,D,G,-A)$$
$$R_{10} = R_6(A,-F,D,G,-B).$$

If the shifts A,B,D,F,G are $\ll 1/\log T$ the above formula simplifies a lot, since we have

$$R_1 = \frac{(-2D-F-G)\mathcal{A}}{(A+F)(A+G)(B+F)(B+G)(D+F)(D+G)} + O\Big(\frac{(\log T)^5}{\log T}\Big)$$

and

$$R_6 = \frac{-(D+G)\mathcal{A}}{(A-D)(A+G)(B-D)(B+G)(-F-D)(-F+G)} + O\Big(\frac{(\log T)^5}{\log T}\Big).$$

As in the proof of Proposition 4.11, by a truncation of the integral over x and Taylor approximations, we can use (4.52) to evaluate $\mathcal{I}$; one can use sage to carry out this massive computation, getting

$$\lim_{\substack{a\to 0\\ b\to 0}} \mathcal{I} = \int_{-\infty}^{+\infty} f(x) \frac{(\log T)^5 \mathcal{A}}{\frac{1}{2\pi^2}(\log T)^5} h(2\pi i x) dx + O\Big(\frac{1}{\log T}\Big) \tag{4.53}$$

with

$$h(y) := -\frac{y^3 - 2y^2 + 6 - e^{-y}(y^2 + 6y + 6)}{6y^4}.$$

Note that, as in the last section, we moved the path of integration over x to $\delta = 0$, being the integral regular at $\delta = 0$. Therefore, putting together (4.49), (4.50) and (4.53), we get that

$$\begin{aligned}\langle N_f \rangle_{|\zeta|^4} &= \int_{-\infty}^{+\infty} f(x)\Bigg(1 + 2\frac{2\pi^2}{\zeta(2)} h(2\pi i x)\Bigg) dx + O\Big(\frac{1}{\log T}\Big)\\ &= \int_{-\infty}^{+\infty} f(x)\Bigg(1 + 24 h(2\pi i x)\Bigg) dx + O\Big(\frac{1}{\log T}\Big)\\ &= \int_{-\infty}^{+\infty} f(x) W_U^2(x) dx + O\Big(\frac{1}{\log T}\Big)\end{aligned}$$

since f is even.

4.4 A symplectic example

In this section, we generalize the ideas that lead to Propositions 4.11 and 4.12 to deal with a symplectic family of L-functions. In particular we consider the symplectic family of Dirichlet L-functions $L(s, \chi_d)$ associated with real Dirichlet characters χ_d and, assuming the ratio conjecture for these L-functions, we investigate the weighted one-level density for this family.

4.4.1 The family of quadratic Dirichlet L-functions

To begin with, for any $q \in \mathbb{N}$, a Dirichlet character modulo q is a completely multiplicative and q-periodic function $\chi : \mathbb{Z} \to \mathbb{C}$, i.e. such that $\chi(mn) = \chi(m)\chi(n)$ and $\chi(m + q) = \chi(m)$ for every m, n. Moreover, $\chi(m) = 0$ if $(m, q) > 1$ and $\chi(m) \neq 0$ if $(m, q) = 1$. We consider only primitive characters, i.e. those which are not *induced* by any other character (see [1, Section 8.7] for the formal and complete definition). For each primitive character χ mod q, the associated Dirichlet L-function is defined as

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$$

in the half-plane $\Re(s)>1$. A Dirichlet character χ is even if $\chi(-1)=1$ and, in this case, $L(s,\chi)$ satisfies the functional equation

$$\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi)=\varepsilon_\chi\left(\frac{\pi}{q}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(1-s,\overline{\chi}) \tag{4.54}$$

with $\varepsilon_\chi:=\tau(\chi)/\sqrt{q}$ and the Gauss sum $\tau(\chi):=\sum_{n=1}^{q}\chi(n)e^{2\pi in/q}$; moreover the Dirichlet characters satisfy an orthogonality relation, being

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(m)\overline{\chi(n)}=\begin{cases}1 & \text{if } m=n \bmod q\\ 0 & \text{if } m\neq n \bmod q\end{cases}$$

if $(mn,q)=1$, with $\varphi(q)$ the number of positive integers $n\leq q$ for which n and q are coprime. The family $\{L(1/2,\chi):\chi \pmod q \text{ primitive}\}$ is again a unitary family.

We are mainly interested in quadratic Dirichlet characters, defined by the Kronecker symbol $\chi_d(n)=(\frac{d}{n})$, which are primitive with modulus $|d|$ and real (χ_d takes on the values $-1,0,+1$). We call d a fundamental discriminant (*f.d.*) if χ_d is a quadratic character and this forces the integer d to be either squarefree and congruent to 1 (mod 4) or 4 times a squarefree integer congruent to 2 or 3 (mod 4). The sequence of fundamental discriminants d is

$$\ldots,-20,-19,-15,-11,-8,-7,-4,-3,1,5,8,12,13,17,21,24,28,29,\ldots$$

and now we focus on the positive ones. If $d>1$, then χ_d is even and $L(s,\chi_d)$ is entire and satisfies the functional equation

$$\left(\frac{\pi}{d}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_d)=\left(\frac{\pi}{d}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(1-s,\chi_d) \tag{4.55}$$

while for $d=1$ then $L(s,\chi_d)=\zeta(s)$. The harmonic detector for the positive discriminants family is given by the orthogonality relation

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{0<d\leq X}\chi_d(n)=\begin{cases}a(n) & \text{if } n \text{ is a perfect square}\\ 0 & \text{if } n \text{ is not a perfect square}\end{cases}$$

where $a(n) = \prod_{p|n} \frac{p}{p+1}$ and $X^* \sim \frac{1}{2\zeta(2)} X$ is the number of fundamental discriminants below X.

The family $\{L(1/2, \chi_d) : d > 0, f.d.\}$ is a symplectic family, in the sense that it can be modeled by characteristic polynomials of symplectic matrices in the group $USp(2N)$, if we identify $2N \approx \log \frac{d}{\pi}$; indeed $\frac{d}{\pi}$ is the analytic conductor of $L(s, \chi_d)$, thus $\log \frac{d}{\pi}$ (i.e. the density of zeros) plays the role of $2N$ in the random matrix theory setting (see [28, Conjecture 1.5.3] and Comments below for some clarification concerning the "conductor"). Analogous considerations can be done working with negative fundamental discriminants.

4.4.2 The ratio conjecture for $L(s, \chi_d)$

We consider the moments of quadratic Dirichlet L-functions at the critical point $s = \frac{1}{2}$, i.e. the mean value

$$\sum_{d \leq X} L(\tfrac{1}{2}, \chi_d)^k \tag{4.56}$$

in the limit $X \to \infty$, where the summation over d has to be interpreted as the sum over all the positive fundamental discriminants d below X, here and in the following. Jutila [100] proved asymptotic formulae for the first moment, showing that

$$\sum_{d \leq X} L(\tfrac{1}{2}, \chi_d) \sim \frac{\mathcal{A}}{2} \frac{1}{2\zeta(2)} X \log X \tag{4.57}$$

where

$$\mathcal{A} = \prod_p \left(1 - \frac{1}{p(p+1)}\right) \tag{4.58}$$

and also for $k = 2$, proving

$$\sum_{d \leq X} L(\tfrac{1}{2}, \chi_d)^2 \sim \frac{\mathcal{B}}{24} \frac{1}{2\zeta(2)} X (\log X)^3 \tag{4.59}$$

with

$$\mathcal{B} = \prod_p \left(1 - \frac{4p^2 - 3p + 1}{p^3(p+1)}\right). \tag{4.60}$$

It is believed that

$$\sum_{d \leq X} L(\tfrac{1}{2}, \chi_d)^k \sim C_k X (\log X)^{k(k+1)/2} \tag{4.61}$$

and using analogies with random matrix theory, Keating and Snaith [110] also conjectured a precise value for the constant C_k. Moreover, the recipe described in Section 2.3 produces a conjectural asymptotic formula with all the main terms for the moments (4.56) with k integer and also for ratios of products of quadratic Dirichlet L-functions (see [30]), which is a symplectic analogue of Conjecture 2.3.

Conjecture 4.16 ([30], Conjecture 5.2)**.** *Let K, Q two positive integers, $\alpha_1, \dots, \alpha_K$ and $\gamma_1, \dots, \gamma_Q$ complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\sum_{d \le X} \frac{\prod_{k=1}^K L(1/2+\alpha_k, \chi_d)}{\prod_{q=1}^Q L(1/2+\gamma_q, \chi_d)}$$
$$= \sum_{d \le X} \sum_{\epsilon \in \{-1,1\}^K} \Big(\frac{d}{\pi}\Big)^{\frac{1}{2}\sum_k (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K g_S\Big(\frac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\Big) Y_S \mathcal{A}_S(\cdots)$$
$$+ O(X^{1/2+\varepsilon})$$

with $(\cdots) = (\epsilon_1\alpha_1, \dots, \epsilon_K \alpha_K; \gamma)$

where

$$Y_S(\alpha;\gamma) := \frac{\prod_{j \le k \le K} \zeta(1+\alpha_j+\alpha_k) \prod_{q<r\le Q} \zeta(1+\gamma_q+\gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q \zeta(1+\alpha_k+\gamma_q)}$$

and $\mathcal{A}_S$ is an Euler product, absolutely convergent for all of the variables in small disks around 0, which is given by

$$\mathcal{A}_S(\alpha;\gamma) := \prod_p \frac{\prod_{j\le k\le K}(1-1/p^{1+\alpha_j+\alpha_k}) \prod_{q<r\le Q}(1-1/p^{1+\gamma_q+\gamma_r})}{\prod_{k=1}^K \prod_{q=1}^Q (1-1/p^{1+\alpha_k+\gamma_q})}$$
$$\Bigg(1 + (1+1/p)^{-1} \sum_{0<\sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_q \mu(p^{c_q})}{p^{\sum_k a_k(1/2+\alpha_k) + \sum_q c_q(1/2+\gamma_q)}}\Bigg)$$

while

$$g_S(s) := \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}.$$

In particular, for our applications to the weighted one-level density, we are interested in the cases $K=2, Q=1$ and $K=3, Q=1$.

The case K=2, Q=1.

We start with

$$\sum_{d\le X}\frac{L(1/2+A,\chi_d)L(1/2+C,\chi_D)}{L(1/2+D,\chi_d)} \tag{4.62}$$

with A, C, D shifts, which satisfy the hypotheses prescribed by Conjecture 4.16; by the ratio conjecture, up to a negligible error $O(X^{1/2+\varepsilon})$, this is a sum of four terms and the first is

$$\sum_{d\le X}\frac{\zeta(1+2A)\zeta(1+2C)\zeta(1+A+C)}{\zeta(1+A+D)\zeta(1+C+D)}\mathcal{A}(A,C;D)$$

where

$$\begin{aligned}&\mathcal{A}(A,C;D)\\ &=\prod_p(1-\tfrac{1}{p^{1+2A}})(1-\tfrac{1}{p^{1+2C}})(1-\tfrac{1}{p^{1+A+C}})(1-\tfrac{1}{p^{1+A+D}})^{-1}(1-\tfrac{1}{p^{1+C+D}})^{-1}\\ &\qquad\cdot\Bigg(1+\tfrac{p}{p+1}\sum_{0<a+c+d\text{ even}}\frac{\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\Bigg).\end{aligned}$$

By using $\sum_{m+n\text{ even}}x^my^n=\frac{1+xy}{(1-x^2)(1-y^2)}$ and $\sum_{m+n\text{ odd}}x^my^n=\frac{x+y}{(1-x^2)(1-y^2)}$, since only the cases $d=0$ and $d=1$ contribute, the last factor equals

$$\begin{aligned}&1+\frac{p}{p+1}\Big[\sum_{0<a+c\text{ even}}\frac{1}{p^{a(1/2+A)+c(1/2+C)}}+\sum_{a+c\text{ odd}}\frac{-1}{p^{a(1/2+A)+c(1/2+C)+(1/2+D)}}\Big]\\ &=1+\frac{p}{p+1}\left[\frac{1+\frac{1}{p^{1+A+C}}}{(1-\frac{1}{p^{1+2A}})(1-\frac{1}{p^{1+2C}})}-1-\frac{\frac{1}{p^{1+A+D}}+\frac{1}{p^{1+C+D}}}{(1-\frac{1}{p^{1+2A}})(1-\frac{1}{p^{1+2C}})}\right]\end{aligned}$$

then

$$\begin{aligned}&\mathcal{A}(A,C;D)\\ &\quad=\prod_p(1-\tfrac{1}{p^{1+2A}})(1-\tfrac{1}{p^{1+2C}})(1-\tfrac{1}{p^{1+A+C}})(1-\tfrac{1}{p^{1+A+D}})^{-1}(1-\tfrac{1}{p^{1+C+D}})^{-1}\\ &\qquad\Bigg(1+\frac{p}{p+1}\Big[(1+\tfrac{1}{p^{1+A+C}})(1-\tfrac{1}{p^{1+2A}})^{-1}(1-\tfrac{1}{p^{1+2C}})^{-1}-1\\ &\qquad\qquad-(\tfrac{1}{p^{1+A+D}}+\tfrac{1}{p^{1+B+C}})(1-\tfrac{1}{p^{1+2A}})^{-1}(1-\tfrac{1}{p^{1+2C}})^{-1}\Big]\Bigg).\end{aligned}$$

In the following, it will be relevant to notice that for small values of the shifts, then the arithmetical coefficient $\mathcal{A}(A,C;D)$ tends to $\mathcal{A}$, defined in (4.58); namely, if $A, C, D \to 0$ then $\mathcal{A}(A, C; D) \sim \mathcal{A}(0,0;0)$ where

$$\mathcal{A}(0,0;0) = \prod_p (1-\tfrac{1}{p})^{3-2}\Big(1+\tfrac{p}{p+1}\Big[(1+\tfrac{1}{p})(1-\tfrac{1}{p})^{-2}-1-\tfrac{2}{p}(1-\tfrac{1}{p})^{-2}\Big]\Big)$$
$$= \prod_p (1-\tfrac{1}{p})\Big(1+\tfrac{p^2}{(p-1)(p+1)}-\tfrac{p}{p+1}\Big) = \prod_p \Big(\frac{p^2+p-1}{p(p+1)}\Big) = \mathcal{A}.$$

All the other terms can be easily recovered from the first one, just by changes of sign of the shifts, as the recipe suggests. This yields a formula for (4.62), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\sum_{d\le X} \frac{L'}{L}(1/2+D,\chi_d)L(1/2+A,\chi_d)$$
$$= \sum_{d\le X}\Big(Q_1 + (\tfrac{d}{\pi})^{-A} g_S(\tfrac12+A)Q_2 + (\tfrac{d}{\pi})^{-D} g_S(\tfrac12+D)Q_3 + \quad (4.63)$$
$$(\tfrac{d}{\pi})^{-A-D} g_S(\tfrac12+A+D)Q_4\Big) + O(X^{1/2+\varepsilon})$$

with

$$Q_1 = \mathcal{A}(A,D;D)\tfrac{\zeta(1+2A)}{\zeta(1+A+D)}\Big(\tfrac{2\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)} + \tfrac{\zeta'(1+A+D)\zeta(1+2D)}{\zeta(1+2D)}$$
$$- \tfrac{\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)}\Big) + \mathcal{A}'(A,D;D)\zeta(1+2A)$$
$$= \mathcal{A}(A,D;D)\frac{\zeta(1+2A)}{\zeta(1+A+D)}\Big(\tfrac{\zeta'}{\zeta}(1+2D)\zeta(1+A+D) + \zeta'(1+A+D)\Big)$$
$$+ \mathcal{A}'(A,D;D)\zeta(1+2A)$$

$$Q_2 = \mathcal{A}(-A,D;D)\frac{\zeta(1-2A)}{\zeta(1-A+D)}\Big(\tfrac{\zeta'}{\zeta}(1+2D)\zeta(1-A+D) + \zeta'(1-A+D)\Big)$$
$$+ \mathcal{A}'(-A,D;D)\zeta(1-2A)$$

$$Q_3 = -\mathcal{A}(A,-D;D)\frac{\zeta(1+2A)\zeta(1-2D)\zeta(1+A-D)}{\zeta(1+A+D)}$$

$$Q_4 = -\mathcal{A}(-A,-D;D)\frac{\zeta(1-2A)\zeta(1-2D)\zeta(1-A-D)}{\zeta(1-A+D)}.$$

Moreover, we notice that if the shifts are $\ll (\log X)^{-1}$, then we can approximate the formula (4.63), getting

$$\begin{aligned}\sum_{d\le X}&\frac{L'}{L}(1/2+D,\chi_d)L(1/2+A,\chi_d)\\ &=\mathcal{A}X^*\bigg(\frac{-A-3D}{(2A)(2D)(A+D)}+X^{-A}\frac{A-3D}{(-2A)(2D)(-A+D)}\\ &\quad+X^{-D}\frac{A+D}{(2A)(2D)(A-D)}+X^{-A-D}\frac{-A+D}{(-2A)(2D)(-A-D)}\bigg)\\ &\quad+O(\log X)\end{aligned}\tag{4.64}$$

being $\mathcal{A}(\pm A,\pm D,D)=\mathcal{A}+O(1/\log X)$ and $\zeta(1+z)=\frac{1}{z}+O(1)$ as $z\to 0$.

The case K=3, Q=1.

Now we study in details

$$\sum_{d\le X}\frac{L(1/2+A,\chi_d)L(1/2+B,\chi_d)L(1/2+C,\chi_d)}{L(1/2+D,\chi_d)}\tag{4.65}$$

with A,B,C,D as prescribed by Conjecture 4.16. This time, the asymptotic formula suggested by recipe is a sum of eight terms; the first is

$$\sum_{d\le X}\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+2C)\zeta(1+A+B)\zeta(1+A+C)\zeta(1+B+C)}{\zeta(1+A+D)\zeta(1+B+D)\zeta(1+C+D)}\mathcal{A}(A,B,C;D)$$

where the (rather horrible) arithmetical coefficient, which can be recovered by noticing that $\sum_{m+n+h \text{ even}}x^my^nz^h=\frac{1+xy+zx+zy}{(1-x^2)(1-y^2)(1-z^2)}$ and similarly $\sum_{m+n+h \text{ odd}}x^my^nz^h=\frac{x+y+z+xyz}{(1-x^2)(1-y^2)(1-z^2)}$, is given by

$$\begin{aligned}&\mathcal{A}(A,B,C;D)\\ &=\prod_p(1-\tfrac{1}{p^{1+2A}})(1-\tfrac{1}{p^{1+2B}})(1-\tfrac{1}{p^{1+2C}})(1-\tfrac{1}{p^{1+A+B}})(1-\tfrac{1}{p^{1+A+C}})\\ &\quad(1-\tfrac{1}{p^{1+B+C}})(1-\tfrac{1}{p^{1+A+D}})^{-1}(1-\tfrac{1}{p^{1+B+D}})^{-1}(1-\tfrac{1}{p^{1+C+D}})^{-1}\\ &\quad\bigg(1+\tfrac{p}{p+1}\Big[(1+\tfrac{1}{p^{1+A+B}}+\tfrac{1}{p^{1+A+C}}+\tfrac{1}{p^{1+B+C}})(1-\tfrac{1}{p^{1+2A}})^{-1}(1-\tfrac{1}{p^{1+2B}})^{-1}\\ &\quad(1-\tfrac{1}{p^{1+2C}})^{-1}-1-(\tfrac{1}{p^{1+A+D}}+\tfrac{1}{p^{1+B+D}}+\tfrac{1}{p^{1+C+D}}+\tfrac{1}{p^{2+A+B+C+D}})\\ &\quad(1-\tfrac{1}{p^{1+2A}})^{-1}(1-\tfrac{1}{p^{1+2B}})^{-1}(1-\tfrac{1}{p^{1+2C}})^{-1}\Big]\bigg).\end{aligned}$$

We notice that the arithmetical coefficient is convergent if all the variables are in small disk around 0, being

$$\mathcal{A}(\underline{0}) = \prod_p (1-\tfrac{1}{p})^3\Big(1+\tfrac{p}{p+1}\Big[(1+\tfrac{3}{p})(1-\tfrac{1}{p})^{-3}-1-(\tfrac{3}{p}+\tfrac{1}{p^2})(1-\tfrac{1}{p})^{-3}\Big]\Big)$$
$$= \prod_p (1-\tfrac{1}{p})^3\Big(1+\tfrac{p^2}{(p-1)^2}-\tfrac{p}{p+1}\Big) = \prod_p\Big(1-\tfrac{4p^2-3p+1}{p^3(p+1)}\Big) = \mathcal{B},$$

see (4.60) for the definition of $\mathcal{B}$. As in the previous example, this gives a formula for (4.65) with all the main terms and error $O(X^{1/2+\varepsilon})$. Differentiating this formula with respect to C at $C=D$, we get

$$\begin{aligned}\sum_{d\le X}\frac{L'}{L}(1/2+D,\chi_d)L(1/2+A,\chi_d)L(1/2+B,\chi_d)\\ = \sum_{d\le X}\Big(R_1 + (\tfrac{d}{\pi})^{-A}g_S(\tfrac12+A)R_2 + (\tfrac{d}{\pi})^{-B}g_S(\tfrac12+B)R_3\\ + (\tfrac{d}{\pi})^{-D}g_S(\tfrac12+D)R_4 + (\tfrac{d}{\pi})^{-A-B}g_S(\tfrac12+A+B)R_5\\ + (\tfrac{d}{\pi})^{-A-D}g_S(\tfrac12+A+D)R_6 + (\tfrac{d}{\pi})^{-B-D}g_S(\tfrac12+B+D)R_7\\ + (\tfrac{d}{\pi})^{-A-B-D}g_S(\tfrac12+A+B+D)R_8\Big) + O(X^{1/2+\varepsilon})\end{aligned} \tag{4.66}$$

with

$$\begin{aligned}R_1 = R_1(A,B,D) &= \mathcal{A}(A,B,D;D)\tfrac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)}{\zeta(1+A+D)\zeta(1+B+D)}\\ &\Big(\tfrac{2\zeta'(1+2D)\zeta(1+A+D)\zeta(1+B+D)+\zeta(1+2D)\zeta'(1+A+D)\zeta(1+B+D)}{\zeta(1+2D)}\\ &+\tfrac{\zeta(1+2D)\zeta(1+A+D)\zeta'(1+B+D)-\zeta(1+A+D)\zeta(1+B+D)\zeta'(1+2D)}{\zeta(1+2D)}\Big)\\ &+\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\mathcal{A}'(A,B,D;D)\\ R_2 &= R_1(-A,B,D)\\ R_3 &= R_1(A,-B,D)\\ R_4 = R_4(A,B,D) &= -\tfrac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\zeta(1-2D)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)}\\ &\cdot\mathcal{A}(A,B,-D;D)\\ R_5 &= R_1(-A,-B,D)\\ R_6 &= R_4(-A,B,D)\\ R_7 &= R_4(A,-B,D)\\ R_8 &= R_4(-A,-B,D).\end{aligned}$$

If $A, B, D \ll (\log X)^{-1}$ the above formula simplifies a lot, since in this case

$$\begin{aligned} R_1 &= \frac{-AB - 3AD - 3BD - 5D^2}{(2A)(2B)(2D)(A+B)(A+D)(B+D)}\mathcal{B} + O\Big(\frac{(\log X)^6}{(\log X)^3}\Big) \\ &=: f(A,B,D)\mathcal{B} + O\Big((\log X)^3\Big) \end{aligned}$$

and

$$\begin{aligned} R_4 &= \frac{-(A+D)(B+D)}{(2A)(2B)(-2D)(A+B)(A-D)(B-D)}\mathcal{B} + O\Big(\frac{(\log X)^6}{(\log X)^3}\Big) \\ &=: g(A,B,D)\mathcal{B} + O\Big((\log X)^3\Big) \end{aligned}$$

giving

$$\begin{aligned} &\sum_{d\le X} \frac{L'}{L}(1/2+D,\chi_d)L(1/2+A,\chi_d)L(1/2+B,\chi_d) \\ &= \mathcal{B}X^*\Big(f(A,B,D) + X^{-A}f(-A,B,D) + X^{-B}f(A,-B,D) \\ &+ X^{-D}g(A,B,D) + X^{-A-B}f(-A,-B,D) + X^{-A-D}g(-A,B,D) \\ &+ X^{-B-D}g(A,-B,D) + X^{-A-B-D}g(-A,-B,D)\Big) + O\Big((\log X)^3\Big) \end{aligned} \tag{4.67}$$

Analogous (but longer) formulae can be obtained also in the cases $K = 4, Q = 1$ and $K = 5, Q = 1$.

4.4.3 The weighted one-level density for $\{L(\frac{1}{2}, \chi_d)\}_d$

In this section, we want to perform similar computations as in Sections 4.3.3 and 4.3.4, investigating the weighted one-level density of the non-trivial zeros of quadratic Dirichlet L-functions. To warm up the engines, we sketch what happens in the classical case; the one-level density for the symplectic family of quadratic Dirichlet L-functions has been studied originally by Ozluk and Snyder [130], who proved that

$$\lim_{X\to\infty} \frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) = \int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x}\Big)dx \tag{4.68}$$

under GRH, for any f such that $\operatorname{supp}\widehat{f} \subset (-2,2)$. Moreover, Conrey and Snaith [43] showed (4.68) (also with lower order terms) with no constraint on the support of $\widehat{f}$, under the assumption of the ratio conjecture; namely, they consider f a test function, holomorphic throughout the strip $|\Im z| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$ and they study

$$\mathcal{D}_0^{\boldsymbol{L}_\chi}(f) := \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) \tag{4.69}$$

in the limit $X \to \infty$. By Cauchy theorem and the functional equation for $L(s, \chi_d)$ in the form

$$\frac{L'}{L}(1-s, \chi_d) = \frac{X'}{X}(s, \chi_d) - \frac{L'}{L}(s, \chi_d) \tag{4.70}$$

with

$$\frac{X'}{X}(s, \chi_d) = -\log\frac{d}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-s}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big),$$

for $\frac{1}{2} + \frac{1}{\log X} < c < \frac{3}{4}$, they write $\mathcal{D}_0^{\boldsymbol{L}_\chi}(f)$ as

$$\frac{1}{2\pi i}\int_{(c)} f\Big(\frac{-i\log X}{2\pi}(s-\tfrac{1}{2})\Big)\frac{1}{X^*}\sum_{d\le X}\Big\{-\frac{X'}{X}(s,\chi_d)+2\frac{L'}{L}(s,\chi_d)\Big\}ds. \tag{4.71}$$

Now they choose $c = \frac{1}{2}+\delta+iy$, $\delta \asymp (\log X)^{-1}$ and they perform the change of variable $\frac{\log X}{2\pi}y = x$. Thus the $-\frac{X'}{X}$ term in (4.71) is

$$\begin{aligned} &= \int_{-\infty}^{+\infty} f(x)\frac{1}{X^*\log X}\sum_{d\le X}\frac{-X'}{X}\Big(\frac{1}{2}+\frac{2\pi i x}{\log X},\chi_d\Big)dx + O\Big(\frac{1}{\log X}\Big) \\ &= \int_{-\infty}^{+\infty} f(x)dx + O\Big(\frac{1}{\log X}\Big) \end{aligned} \tag{4.72}$$

where the last equality easily follows by splitting the integral over x in two parts. If $x \ll \log X$ then $-\frac{X}{X}(\frac{1}{2}+\frac{2\pi i x}{\log X}) = \log d + O(1)$, while the contribution of $x \gg \log X$ is negligible because of the great decaying of f (we use Stirling's formula to bound the $\frac{\Gamma'}{\Gamma}$ factors). Putting together (4.71) and (4.72), they have that $\mathcal{D}_0^{\boldsymbol{L}_\chi}(f)$ equals

$$\int_{-\infty}^{+\infty} f(x)dx + \frac{2}{X^*\log X}\int_{-\infty}^{+\infty} f\Big(x - \frac{i\delta\log X}{2\pi}\Big)\sum_{d\le X}\frac{L'}{L}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)dx$$

up to an error $O((\log X)^{-1})$. After a truncation of the integral over x, they use the ratio conjecture to estimate the remaining sum over d, which gives

$$\sum_{d\le X}\frac{L'}{L}\Big(\frac{1}{2}+A,\chi_d\Big) = X^*\Big(\frac{-1}{2A} - X^{-A}\frac{-1}{2A}\Big) + O\Big(\frac{1}{\log X}\Big)$$

for $A \asymp (\log X)^{-1}$. We would stress that, in their paper [43], Conrey and Snaith performed a much more precise computation, also taking account of lower order terms. Note that the right hand side of the above formula is regular at $A = 0$, then shifting the integral to $\delta = 0$ is now safe. As $X \to \infty$ then

$$\begin{aligned}\mathcal{D}_0^{\boldsymbol{L}_\chi}(f) &= \frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\\ &= \int_{-\infty}^{+\infty} f(x)dx - \frac{2}{\log X}\int_{-\infty}^{+\infty} f(x)\Big(\frac{\log X}{4\pi i x} - e^{-2\pi i x}\frac{\log X}{4\pi i x}\Big)dx + O\Big(\frac{1}{\log X}\Big)\\ &= \int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{1}{2\pi i x} + \frac{e^{-2\pi i x}}{2\pi i x}\Big)dx + O\Big(\frac{1}{\log X}\Big)\end{aligned}$$

and, by evenness of f, this is asymptotic to

$$\int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x}\Big)dx$$

which matches with the one-level density for the eigenvalues of the matrices from the symplectic group $USp(2N)$. In particular, we notice that the one-level density function $1 - \frac{\sin(2\pi x)}{2\pi x}$ vanishes of order 2 at $x = 0$, being $\sim \frac{2\pi^2}{3}x^2$ as $x \to 0$.

Similarly to what we did in Section 4.3, we now want to compute the weighted one-level density in the symplectic case, tilted by $L(\frac{1}{2}, \chi_d)$. We note that, differently from what happens in the Riemann zeta function case, here we are allowed to consider the first power as well, as $L(\frac{1}{2}, \chi_d)$ is real. The analogue of (4.13) in this context is

$$\mathcal{D}_1^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(\frac{1}{2}, \chi_d)}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L(\tfrac{1}{2}, \chi_d) \tag{4.73}$$

and via ratio conjecture in the form of Equation (4.63) this can be studied asymptotically, as shown in the following result.

Proposition 4.17. *Assume GRH and Conjecture 4.16 for $K = 2, Q = 1$. For any test function f, holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1 + x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_1^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x)W^1_{USp}(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, *where*

$$W^1_{USp}(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. We start looking at

$$\frac{1}{\frac{\mathcal{A}}{2} X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L(\tfrac{1}{2}+\alpha, \chi_d) \tag{4.74}$$

with $\alpha \asymp (\log X)^{-1}$; note that, as $\alpha \to 0$, $\sum_{d\le X} L(\frac{1}{2}+\alpha,\chi_d)$ tends to $\frac{1}{2\zeta(2)}\frac{\mathcal{A}}{2} X \log X$ which is the normalization $\frac{\mathcal{A}}{2}X^* \log X$ we have in (4.74). As usual, we use Cauchy theorem and the functional equation (4.70) to write

$$\frac{1}{\frac{\mathcal{A}}{2} X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L(\tfrac{1}{2}+\alpha, \chi_d) = -\mathcal{J}(\alpha) + 2\mathcal{I}(\alpha) + O\Big(\frac{1}{\log X}\Big) \tag{4.75}$$

where

$$\begin{aligned}
\mathcal{J}(\alpha) :&= \frac{2}{\mathcal{A}X^*\log X}\frac{1}{2\pi}\int_{-\infty}^{+\infty} (-\log X) f\Big(\frac{y\log X}{2\pi}\Big) \sum_{d\le X} L(\tfrac{1}{2}+\alpha,\chi_d)dy \\
&= \frac{-2}{\mathcal{A}X^*\log X}\int_{-\infty}^{+\infty} f(x) \sum_{d\le X}\Big(L(\tfrac{1}{2},\chi_d) + O(\tfrac{1}{\log X})\Big)dx \\
&= -\int_{-\infty}^{+\infty} f(x)dx + O\Big(\frac{1}{\log X}\Big)
\end{aligned}$$

and

$$\mathcal{I}(\alpha) := \frac{2}{\mathcal{A}X^*(\log X)^2}\int_{-\infty}^{+\infty} f(x)\sum_{d\le X}\frac{L'}{L}(\tfrac{1}{2}+\delta+\tfrac{2\pi i x}{\log X},\chi_d)L(\tfrac{1}{2}+\alpha,\chi_d)dx$$

with $\delta \asymp (\log X)^{-1}$. Now we rely on the assumption of the ratio conjecture (in particular Equation (4.63) and (4.64)) to compute the sum over d; in particular, in the same way as in the the proof of Theorem 4.11, by a truncation of the integral over x and Taylor approximations, we get

$$\mathcal{I}(\alpha) = \frac{2}{(\log X)^2}\int_{-\infty}^{+\infty} f(x) g_X(\alpha, \delta + \tfrac{2\pi i x}{\log X})dx + O\Big(\frac{1}{\log X}\Big)$$

where

$$\begin{aligned}
g_X(\alpha, w) :=& \frac{-\alpha-3w}{(2\alpha)(2w)(\alpha+w)} + X^{-\alpha}\frac{\alpha-3w}{(-2\alpha)(2w)(-\alpha+w)} \\
&+ X^{-w}\frac{\alpha+w}{(2\alpha)(2w)(\alpha-w)} + X^{-\alpha-w}\frac{-\alpha+w}{(-2\alpha)(2w)(-\alpha-w)}.
\end{aligned}$$

The integral is regular at $\delta = 0$ then, if we denote $\alpha = \frac{a}{\log X}$, we get

$$\mathcal{I}(\tfrac{a}{\log X}) = \frac{2}{(\log X)^2} \int_{-\infty}^{+\infty} f(x) g_X(\tfrac{a}{\log X}, \tfrac{2\pi i x}{\log X}) dx + O\Big(\frac{1}{\log X}\Big)$$

which is regular at $a = 0$; indeed, if we take the limit as $a \to 0$ we get

$$\mathcal{I}(0) = \lim_{a \to 0} \mathcal{I}(\tfrac{a}{\log X}) = 2 \int_{-\infty}^{+\infty} f(x) g(2\pi i x) dx + O\Big(\frac{1}{\log X}\Big)$$

where

$$\begin{aligned} g(w) := \lim_{a \to 0} \Bigg(& \frac{-a - 3w}{(2a)(2w)(a+w)} + e^{-a} \frac{a - 3w}{(-2a)(2w)(-a+w)} \\ & + e^{-w} \frac{a+w}{(2a)(2w)(a-w)} + e^{-a-w} \frac{-a+w}{(-2a)(2w)(-a-w)} \Bigg) \\ = & \frac{-we^{-w} - 3w - 4e^{-w} + 4}{4w^2}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{D}_1^{\boldsymbol{L}_\chi}(f) &= \lim_{\alpha \to 0} \frac{1}{\frac{A}{2} X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big) L(\tfrac{1}{2} + \alpha, \chi_d) \\ &= -\mathcal{J}(0) + 2\mathcal{I}(0) + O\Big(\frac{1}{\log X}\Big) \\ &= \int_{-\infty}^{+\infty} f(x) \Big(1 + 4g(2\pi i x)\Big) dx + O\Big(\frac{1}{\log X}\Big) \end{aligned}$$

and since f is even the main term above equals

$$\int_{-\infty}^{+\infty} f(x) \Big(1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}\Big) dx.$$

□

Analogously, we can compute the weighted one-level density, tilted by the second power of $L(\frac{1}{2}, \chi_d)$, i.e.

$$\mathcal{D}_2^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d \le X} L(\frac{1}{2}, \chi_d)^2} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big) L(\tfrac{1}{2}, \chi_d)^2 \tag{4.76}$$

under the assumption of Conjecture 4.16, in the case $K = 3, Q = 1$.

Proposition 4.18. *Assume GRH and Conjecture 4.16 for $K = 3, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_2^{\boldsymbol{L}_X}(f) = \int_{-\infty}^{+\infty} f(x) W_{USp}^2(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W_{USp}^2(x) := 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. The proof works like that of Proposition 4.17; first, for $\alpha = \frac{a}{\log X} \asymp (\log X)^{-1}$ and $\beta = \frac{b}{\log X} \asymp (\log X)^{-1}$, we analyze

$$\frac{1}{\frac{\mathcal{B}}{24}X^*(\log X)^3} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L(\tfrac{1}{2}+\alpha, \chi_d) L(\tfrac{1}{2}+\beta, \chi_d) \tag{4.77}$$

which can be written as

$$-\mathcal{J}(\alpha,\beta) + 2\mathcal{I}(\alpha,\beta) + O\Big(\frac{1}{\log X}\Big) \tag{4.78}$$

where

$$\begin{aligned}\mathcal{J}(\alpha,\beta) &:= \frac{-24\log X}{\mathcal{B}X^*(\log X)^3}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{y\log X}{2\pi}\Big)\sum_{d\le X} L(\tfrac{1}{2}+\alpha,\chi_d)L(\tfrac{1}{2}+\beta,\chi_d) \\ &= -\int_{-\infty}^{+\infty} f(x)dx + O\Big(\frac{1}{\log X}\Big)\end{aligned} \tag{4.79}$$

and

$$\begin{aligned}\mathcal{I}(\alpha,\beta) :=& \frac{24}{\mathcal{B}X^*(\log X)^4}\int_{-\infty}^{+\infty} f(x) \\ &\times \sum_{d\le X}\frac{L'}{L}(\tfrac{1}{2}+\delta+\tfrac{2\pi i x}{\log X},\chi_d)L(\tfrac{1}{2}+\alpha,\chi_d)L(\tfrac{1}{2}+\beta,\chi_d)dx\end{aligned}$$

where $\delta \asymp (\log T)^{-1}$, as usual. With the usual machinery, the ratio conjecture (see Equations (4.66) and (4.67)) allows us to evaluate the sum over d; the resulting quantity is regular at $\delta = 0$ and at $\alpha = \frac{a}{\log X} = 0$, $\beta = \frac{b}{\log X} = 0$, thus taking the limit we get

$$\mathcal{I}(0,0) = 24\int_{-\infty}^{+\infty} f(x)h(2\pi i x)dx \tag{4.80}$$

with

$$h(y) := \frac{y^3e^{-y} - 5y^3 + 12y^2e^{-y} + 12y^2 + 48ye^{-y} + 48e^{-y} - 48}{48y^4}.$$

Putting all together, from (4.77), (4.78), (4.79) and (4.80), we finally get

$$\begin{aligned}\mathcal{D}_2^{\boldsymbol{L}_X} &= -\mathcal{J}(0,0) + 2\mathcal{I}(0,0) + O\Big(\frac{1}{\log X}\Big)\\ &= \int_{-\infty}^{+\infty} f(x)\Big(1 + 48h(2\pi i x)\Big)dx + O\Big(\frac{1}{\log X}\Big).\end{aligned}$$

Moreover, since f is even, the main term equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}\Big)dx.$$

□

In the same way, we study

$$\mathcal{D}_3^{\boldsymbol{L}_X}(f) := \frac{1}{\sum_{d\le X} L(\frac{1}{2},\chi_d)^3} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L(\tfrac{1}{2},\chi_d)^3 \qquad (4.81)$$

assuming Conjecture 4.16, in the case $K = 4, Q = 1$.

Proposition 4.19. *Assume GRH and Conjecture 4.16 for $K = 4, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_X}(f) = \int_{-\infty}^{+\infty} f(x) W_{USp}^3(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$\begin{aligned}W_{USp}^3(x) := 1 &+ \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\ &- \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.\end{aligned}$$

Proof. We consider $\alpha, \beta, \nu \in \mathbb{R}$ of size $\asymp 1/\log X$, we denote

$$\boldsymbol{L}_{\alpha,\beta,\nu}(\tfrac{1}{2},\chi_d) := L(\tfrac{1}{2}+\alpha,\chi_d)L(\tfrac{1}{2}+\beta,\chi_d)L(\tfrac{1}{2}+\nu,\chi_d)$$

and we look at

$$\frac{1}{\sum_{d\le X} L(\frac12,\chi_d)^3}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu}(\tfrac12,\chi_d).$$

With the usual machinery we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\bigg(1+2\frac{1}{\sum_{d\le X} L(\frac12,\chi_d)^3}\sum_{d\le X}\frac{L'}{L}(\tfrac12+\delta+\tfrac{2\pi i x}{\log X},\chi_d)\boldsymbol{L}_{\alpha,\beta,\nu}(\tfrac12,\chi_d)\bigg)dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. The remaining sum over d can be evaluated asymptotically by using the ratio conjecture (i.e. Conjecture 4.16 for $K=4, Q=1$). This can be done by using sage to carry out the easy but very long computations. Doing so, letting $\alpha,\beta,\nu\to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_X}(f)=\int_{-\infty}^{+\infty} f(x)\Big(1+2\cdot 2880\cdot h(2\pi i x)\Big)dx+O\Big(\frac{1}{\log X}\Big)$$

with

$$\begin{aligned} h(y) :=&\frac{-7y^5+24y^4-240y^2+2880}{5760y^6}\\ &+\frac{e^{-y}(-y^5-24y^4-240y^3-1200y^2-2880y-2880)}{5760y^6}. \end{aligned}$$

The claim follows, since f is even. □

Finally, we look at

$$\mathcal{D}_4^{\boldsymbol{L}_X}(f):=\frac{1}{\sum_{d\le X} L(\frac12,\chi_d)^4}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)L(\tfrac12,\chi_d)^4 \tag{4.82}$$

assuming Conjecture 4.16, in the case $K=5, Q=1$.

Proposition 4.20. *Assume GRH and Conjecture 4.16 for $K=5, Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_X}(f)=\int_{-\infty}^{+\infty} f(x)W_{USp}^4(x)dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$\begin{aligned} W_{USp}^4(x) := 1 &-\frac{\sin(2\pi x)}{2\pi x}-\frac{10(1+\cos(2\pi x))}{(\pi x)^2}+\frac{90\sin(2\pi x)}{(\pi x)^3}\\ &-\frac{15(3-31\cos(2\pi x))}{(\pi x)^4}-\frac{1470\sin(2\pi x)}{(\pi x)^5}\\ &-\frac{315(1+9\cos(2\pi x))}{(\pi x)^6}+\frac{3150\sin(2\pi x)}{(\pi x)^7}-\frac{1575(1-\cos(2\pi x))}{(\pi x)^8}. \end{aligned}$$

Proof. The proof works in the same way as the previous ones. We consider $\alpha, \beta, \nu, \eta \in \mathbb{R}$ of size $\asymp 1/\log X$, we denote

$$\boldsymbol{L}_{\alpha,\beta,\nu,\eta}(\tfrac{1}{2}, \chi_d) := L(\tfrac{1}{2}+\alpha, \chi_d)L(\tfrac{1}{2}+\beta, \chi_d)L(\tfrac{1}{2}+\nu, \chi_d)L(\tfrac{1}{2}+\eta, \chi_d)$$

and we look at

$$\frac{1}{\sum_{d\leq X} L(\frac{1}{2}, \chi_d)^4} \sum_{d\leq X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) \boldsymbol{L}_{\alpha,\beta,\nu,\eta}(\tfrac{1}{2}, \chi_d).$$

By the usual manipulations, the above equals

$$\int_{-\infty}^{+\infty} f(x)\bigg(1+2\frac{1}{\sum_{d\leq X} L(\frac{1}{2}, \chi_d)^4} \sum_{d\leq X} \frac{L'}{L}(\tfrac{1}{2}+\delta+\tfrac{2\pi i x}{\log X}, \chi_d)\boldsymbol{L}_{\alpha,\beta,\nu,\eta}(\tfrac{1}{2}, \chi_d)\bigg)dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. Thanks to Conjecture 4.16 with $K = 5, Q = 1$, the above can be computed asymptotically. As $\alpha, \beta, \nu, \eta \to 0$, with the help of sage, we then obtain

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x)\Big(1 + 2\cdot 4838400 \cdot h(2\pi i x)\Big)dx + O\Big(\frac{1}{\log X}\Big)$$

with

$$\begin{aligned} h(y) :=& \frac{-9y^7 + 40y^6 - 720y^4 + 20160y^2 - 403200}{9676800y^8} \\ &+ \frac{e^{-y}(y^7 + 40y^6 + 720y^5 + 7440y^4)}{9676800y^8} \\ &+ \frac{e^{-y}(47040y^3 + 181440y^2 + 403200y + 403200)}{9676800y^8}. \end{aligned}$$

Again, being f even, the claim follows. □

Propositions 4.17−4.20 prove Theorem 4.3. We also record the Fourier transforms of the weighted kernels, for $k \leq 4$. In the classical case, it is well-known that

$$\widehat{W}^0_{USp}(y) = \delta_0(y) - \frac{1}{2}\chi_{[-1,1]}(y);$$

moreover we have:

$$\widehat{W}^1_{USp}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ -2y - \frac{3}{2} & \text{if } -1 \leq y < 0 \\ -\frac{1}{2} & \text{if } y = 0 \\ 2y - \frac{3}{2} & \text{if } 0 < y \leq 1 \end{cases}$$

$$\widehat{W}^2_{USp}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ 4y^3 - 6y - \frac{5}{2} & \text{if } -1 \leq y < 0 \\ -\frac{3}{2} & \text{if } y = 0 \\ -4y^3 + 6y - \frac{5}{2} & \text{if } 0 \leq y \leq 1 \end{cases}$$

$$\widehat{W}^3_{USp}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ -12y^5 + 20y^3 - 12y - \frac{7}{2} & \text{if } -1 \leq y < 0 \\ -\frac{5}{2} & \text{if } y = 0 \\ 12y^5 - 20y^3 + 12y - \frac{7}{2} & \text{if } 0 < y \leq 1 \end{cases}$$

and

$$\widehat{W}^4_{USp}(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ 40y^7 - 84y^5 + 60^3 - 20y - \frac{9}{2} & \text{if } -1 \leq y < 0 \\ -\frac{7}{2} & \text{if } y = 0 \\ -40y^7 + 84y^5 - 60^3 + 20y - \frac{9}{2} & \text{if } 0 < y \leq 1. \end{cases}$$

4.5 An orthogonal example

As a last example, we analyze the orthogonal case of the family of quadratic twists of the L-functions associated with the discriminant modular form Δ.

4.5.1 Quadratic twists of $L_\Delta(s)$

We start by denoting with $SL_2(\mathbb{Z})$ the modular group, that is the group of 2×2 matrices with integer coefficients and determinant 1; let k be a positive integer, then a modular form of weight k is a complex-valued function f on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ which is holomorphic on $\mathbb{H}$ and "at the cusp"(i.e. as $z \to i\infty$), such that for any $z \in \mathbb{H}$ we have

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^k f(z) \quad \text{for any} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Since $f(z+1) = f(z)$, a modular form is also periodic with period 1, then it has a Fourier series of the form $f(z) = \sum a_f(n) q^n$, with $q = e^{2\pi i z}$. Moreover a modular form is called cusp form if $a_f(0) = 0$. For any cusp form f, the Dirichlet generating series for the Fourier coefficients of f is defined as

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{s+k/2}}$$

(see [98, Chapter 14] for a complete account on the definition of Hecke L-functions and their standard properties). Here we are interested in the discriminant modular form Δ, which is the unique normalized cusp form of weight 12; its Fourier coefficients define the Ramanujan tau function $\tau(n)$, being

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$$

with $q = e^{2\pi iz}$. Thus the L-function associated with Δ is defined by

$$L_\Delta(s) := \sum_{n=1}^{\infty}\frac{\tau^*(n)}{n^s}$$

where $\tau^*(n) = \tau(n)/n^{11/2}$. The family we want to describe is the collection of the quadratic twists of L_Δ, that are

$$L_\Delta(s,\chi_d) = \sum_{n=1}^{\infty}\frac{\chi_d(n)\tau^*(n)}{n^s} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right)^{-1}$$

and, for $d>0$, they satisfy the functional equation

$$\left(\frac{d^2}{4\pi^2}\right)^{\frac{s}{2}}\Gamma(s+11/2)L_\Delta(s,\chi_d) = \left(\frac{d^2}{4\pi^2}\right)^{\frac{1-s}{2}}\Gamma(1-s+11/2)L_\Delta(1-s,\chi_d).$$

Finally we also record that

$$\frac{1}{L_\Delta(s,\chi_d)} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right) =: \sum_{n=1}^{\infty}\frac{\chi_d(n)\mu_\Delta(n)}{n^s}$$

where μ_Δ is the multiplicative function defined by $\mu_\Delta(p) = -\tau^*(p)$, $\mu_\Delta(p^2) = 1$ and $\mu_\Delta(p^\alpha) = 0$ if $\alpha \geq 3$.

The family $\{L_\Delta(1/2,\chi_d) : d > 0, f.d.\}$ is an even orthogonal family, modeled by the group $SO(2N)$ with the identification $2N \approx \log\frac{d^2}{4\pi^2}$.

4.5.2 The ratio conjecture for $L_\Delta(s,\chi_d)$

The moments at the central value of L-functions associated with quadratic twists of a modular form have been studied extensively in recent years, but only the first moment [21, 96, 125] and partially the second [154, 136] have been obtained. It is known that such a family can be either symplectic or orthogonal, depending on the specific L-function we twist; in particular, if we start with the L-function associated with the discriminant modular form Δ, then we are in the latter case. For an orthogonal family $\mathcal{F}$, ordered by the conductor $C(f)$, Conrey-Farmer [26] and Keating-Snaith [110] predict that

$$\frac{1}{X^*} \sum_{\substack{f\in\mathcal{F} \\ C(f)\leq X}} L_f(1/2)^k \sim \frac{f_O(k)}{2} a(k)(\log X^A)^{k(k-1)/2} \tag{4.83}$$

where the above sum is over the X^* elements of the family $\mathcal{F}$ such that $C(f) \leq X$; $f_O(k)$ is the leading order coefficient of the moments of characteristic polynomials of matrices in $SO(2N)$; $a(k)$ is a constant depending on the particular family involved; A is a constant depending on the functional equation satisfied by the L-functions in the family, in particular on the degree of the relevant parameter in the functional equation for $L_f(s)$ (see [26, Equation (1.3)] for further details and examples). Moreover, also in this case the recipe [28] provides a precise formula with all the main terms for any integral moment, extended by [30] to ratios. The ratio conjecture for the orthogonal family of quadratic twists of the discriminant modular form can be stated as follows.

Conjecture 4.21 ([30], Conjecture 5.3)**.** *Let K, Q two positive integers, $\alpha_1, \ldots, \alpha_K$ and $\gamma_1, \ldots, \gamma_Q$ complex shifts with real part $\asymp (\log X)^{-1}$ and imaginary part $\ll_\varepsilon X^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\begin{aligned}\sum_{d\leq X} &\frac{\prod_{k=1}^K L_\Delta(1/2+\alpha_k, \chi_d)}{\prod_{q=1}^Q L_\Delta(1/2+\gamma_q, \chi_d)} \\ &= \sum_{d\leq X} \sum_{\epsilon\in\{-1,1\}^K} \Big(\frac{d^2}{4\pi^2}\Big)^{\frac{1}{2}\sum_k(\epsilon_k\alpha_k-\alpha_k)} \prod_{k=1}^K g_O\Big(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\Big) Y_O \mathcal{A}_O(\cdots) \\ &\qquad + O(X^{1/2+\varepsilon}) \\ &\textit{with } (\cdots) = (\epsilon_1\alpha_1, \ldots, \epsilon_K\alpha_K; \gamma)\end{aligned}$$

where

$$Y_O(\alpha;\gamma) := \frac{\prod_{j<k\leq K}\zeta(1+\alpha_j+\alpha_k)\prod_{q<r\leq Q}\zeta(1+\gamma_q+\gamma_r)\prod_{q\leq Q}\zeta(1+2\gamma_q)}{\prod_{k=1}^K\prod_{q=1}^Q\zeta(1+\alpha_k+\gamma_q)}$$

and $\mathcal{A}_O$ is an Euler product, absolutely convergent for all of the variables *in small disks around 0, which is given by*

$$\mathcal{A}_O(\alpha;\gamma) := \prod_p \frac{\prod_{j<k\le K}(1-\frac{1}{p^{1+\alpha_j+\alpha_k}})\prod_{q<r\le Q}(1-\frac{1}{p^{1+\gamma_q+\gamma_r}})\prod_{q\le Q}(1-\frac{1}{p^{1+2\gamma_q}})}{\prod_{k=1}^K\prod_{q=1}^Q(1-\frac{1}{p^{1+\alpha_k+\gamma_q}})}$$
$$\left(1+(1+1/p)^{-1}\sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_k \tau^*(p^{a_k})\prod_q \mu_\Delta(p^{c_q})}{p^{\sum_k a_k(1/2+\alpha_k)+\sum_q c_q(1/2+\gamma_q)}}\right)$$

while

$$g_O(s) := \frac{\Gamma(\frac{1}{2}-s+6)}{\Gamma(s-\frac{1}{2}+6)}.$$

In the following, we will analyze the applications of this conjecture to the weighted one-level density, as we did in Section 4.4.3 for a symplectic family. To do so, we first look at what Conjecture 4.21 gives in a few specific examples.

The case K=1, Q=0.

This is the easiest situation possible, corresponding to the first moment of $L_\Delta(1/2,\chi_d)$; for A a complex number which satisfies the hypotheses prescribed by Conjecture 4.21, the ratio conjecture yields

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta(1/2+A,\chi_d) = \mathcal{A}(A)+\left(\frac{d}{2\pi}\right)^{-2A}\frac{\Gamma(6-A)}{\Gamma(6+A)}\mathcal{A}(-A)+O(X^{-1/2+\varepsilon})$$

with

$$\mathcal{A}(A) := \prod_p\left(1+\frac{p}{p+1}\left[-1+\sum_{m=0}^{\infty}\frac{\tau^*(p^{2m})}{p^{(\frac{1}{2}+A)2m}}\right]\right)$$

We note that $\mathcal{A}(A)$ is regular at $A=0$; indeed the $m=0$ and $m=1$ terms give 1 and $\tau^*(p^2)p^{-1-2A}$ respectively, therefore an approximation for $\mathcal{A}(A)$ would be $\prod_p(1+\frac{\tau^*(p^2)}{p^{1+2A}}+\cdots)$. Differently from the unitary and symplectic cases, where the first term in the corresponding Euler products gives the polar factor $\zeta(1+2A)$, here we would have $L_\Delta(\mathrm{sym}^2,1+2A)$ the symmetric square of L_Δ, which is well-known to be regular and nonzero at 1 (see [97, Chapter 13] for a complete overview about the symmetric square and its properties). However, for the sake of brevity, we prefer not to factor out $L_\Delta(\mathrm{sym}^2,1+2A)$ and we leave the contribution of the symmetric square encoded in the arithmetical factor $\mathcal{A}(A)$, which converges in a small disk around 0. Thus, for $A=0$, we immediately get

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta(1/2,\chi_d) = 2\mathcal{A}+O(X^{-1/2+\varepsilon}) \tag{4.84}$$

where

$$\mathcal{A} := \mathcal{A}(0) = \prod_p \left(1 + \frac{p}{p+1}\left[-1 + \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{(\sqrt{p})^{2m}}\right]\right). \tag{4.85}$$

The sum over m in the above formula will recur often in the following, thus we denote it by $\mathcal{P}$; a closed formula for $\mathcal{P}$ is given by [43, Equation (2.49)]

$$\mathcal{P} := \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{(\sqrt{p})^{2m}} = \frac{1}{2}\left\{\left(1 - \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1} + \left(1 + \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1}\right\}.$$

The case K=2, Q=1.

We consider

$$\sum_{d \le X} \frac{L_\Delta(1/2 + A, \chi_d) L_\Delta(1/2 + C, \chi_D)}{L_\Delta(1/2 + D, \chi_d)} \tag{4.86}$$

with A, C, D shifts satisfying the usual hypotheses prescribed by the ratio conjecture; by Conjecture 4.21, up to a negligible error, this is a sum of four terms and the first is

$$\sum_{d \le X} \frac{\zeta(1 + A + C)\zeta(1 + 2D)}{\zeta(1 + A + D)\zeta(1 + C + D)} \mathcal{A}(A, C; D)$$

where

$$\begin{aligned} &\mathcal{A}(A, C; D) \\ &= \prod_p (1 - \tfrac{1}{p^{1+A+C}})(1 - \tfrac{1}{p^{1+2D}})(1 - \tfrac{1}{p^{1+A+D}})^{-1}(1 - \tfrac{1}{p^{1+C+D}})^{-1} \\ &\qquad \cdot \left(1 + \tfrac{p}{p+1} \sum_{0 < a+c+d \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\right) \end{aligned}$$

As usual, we note that $\mathcal{A}(A, C; D) \sim \mathcal{A}(0, 0; 0) = \mathcal{A}$ defined in (4.85) as $A, C, D \to 0$; indeed

$$\mathcal{A}(0, 0; 0) = \prod_p \left(1 + \tfrac{p}{p+1} \sum_{0 < a+c+d \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)\mu(p^d)}{p^{(a+c+d)/2}}\right)$$

and, since $\mu_\Delta(p^d)$ limits the choices for d to 0 or 1 or 2, the sum above equals

$$-1 + \sum_{a+c \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)}{(\sqrt{p})^{a+c}} - \frac{\tau^*(p)}{\sqrt{p}} \sum_{a+c \text{ odd}} \frac{\tau^*(p^a)\tau^*(p^c)}{(\sqrt{p})^{a+c}} + \frac{1}{p} \sum_{a+c \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)}{(\sqrt{p})^{a+c}}.$$

We recall the notation

$$\mathcal{P} := \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{(\sqrt{p})^{2m}} = \frac{1}{2}\left\{\left(1 - \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1} + \left(1 + \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1}\right\} \quad (4.87)$$

and we also set

$$\mathcal{D} := \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m+1})}{(\sqrt{p})^{2m+1}} = \frac{1}{2}\left\{\left(1 - \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1} - \left(1 + \frac{\tau^*(p)}{\sqrt{p}} + \frac{1}{p}\right)^{-1}\right\} \quad (4.88)$$

(see [43], Equations (2.49), (2.50)). With these notations, it is easy to prove that

$$\sum_{a+c \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)}{(\sqrt{p})^{a+c}} = \mathcal{P}^2 + \mathcal{D}^2$$

while

$$\sum_{a+c \text{ odd}} \frac{\tau^*(p^a)\tau^*(p^c)}{(\sqrt{p})^{a+c}} = 2\mathcal{P}\mathcal{D}.$$

Putting all together we have

$$\mathcal{A}(0,0;0) = \prod_p \left(1 + \frac{p}{p+1}\left[-1 + \left(1 + \frac{1}{p}\right)(\mathcal{P}^2 + \mathcal{D}^2) - \frac{\tau^*(p)}{\sqrt{p}} 2\mathcal{P}\mathcal{D}\right]\right)$$

and, since the multiplicative law of the Ramanujan function $\tau^*(p^{m+1})\tau^*(p) = \tau^*(p^{2m+2}) + \tau^*(p^{2m})$ implies that $\frac{\tau^*(p)}{\sqrt{p}}\mathcal{D} = -1 + \mathcal{P} + \frac{1}{p}\mathcal{P}$, then

$$\begin{aligned}\mathcal{A}(0,0;0) &= \prod_p \left(1 + \frac{p}{p+1}\left[-1 + (1 + \tfrac{1}{p})(\mathcal{P}^2 + \mathcal{D}^2) - 2\mathcal{P}(-1 + \mathcal{P} + \tfrac{1}{p}\mathcal{P})\right]\right) \\ &= \prod_p \left(1 + \frac{p}{p+1}\left[-1 + \left(1 + \tfrac{1}{p}\right)(\mathcal{D}^2 - \mathcal{P}^2) + 2\mathcal{P}\right]\right) = \mathcal{A}\end{aligned}$$

where the last equality can be elementary obtained by (4.87) and (4.88). All the other terms can be easily recovered from the first one, then we get a formula for (4.86), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\begin{aligned}&\sum_{d \le X} \frac{L'_\Delta}{L_\Delta}(1/2 + D, \chi_d) L(1/2 + A, \chi_d) \\ &\quad = \sum_{d \le X} \Big(Q_1 + (\tfrac{d}{2\pi})^{-2A} g_O(\tfrac{1}{2} + A) Q_2 + (\tfrac{d}{2\pi})^{-2D} g_O(\tfrac{1}{2} + D) Q_3 + \quad (4.89) \\ &\qquad (\tfrac{d}{2\pi})^{-2A-2D} g_O(\tfrac{1}{2} + A + D) Q_4 \Big) + O(X^{1/2+\varepsilon})\end{aligned}$$

with

$$Q_1 = \mathcal{A}(A, D; D)\Big(\tfrac{\zeta'}{\zeta}(1 + A + D) - \tfrac{\zeta'}{\zeta}(1 + 2D)\Big) + \mathcal{A}'(A, D; D)$$

$$Q_2 = \mathcal{A}(-A, D; D)\Big(\tfrac{\zeta'}{\zeta}(1 - A + D) - \tfrac{\zeta'}{\zeta}(1 + 2D)\Big) + \mathcal{A}'(-A, D; D)$$

$$Q_3 = -\mathcal{A}(A, -D; D)\frac{\zeta(1 + A - D)\zeta(1 + 2D)}{\zeta(1 + A + D)}$$

$$Q_4 = -\mathcal{A}(-A, -D; D)\frac{\zeta(1 - A - D)\zeta(1 + 2D)}{\zeta(1 - A + D)}$$

Moreover, we notice that if the shifts are of order $\asymp (\log X)^{-1}$, then we can approximate the formula (4.89), getting

$$\begin{aligned}\sum_{d\le X} &\frac{L'_\Delta}{L_\Delta}(1/2 + D, \chi_d)L(1/2 + A, \chi_d)\\ &= \mathcal{A}X^*\Bigg(\frac{A - D}{(A + D)2D} + X^{-2A}\frac{-A - D}{(-A + D)2D}\\ &\quad + X^{-2D}\frac{-A - D}{(A - D)2D} + X^{-2A-2D}\frac{A - D}{(-A - D)2D}\Bigg) + O(1)\end{aligned} \tag{4.90}$$

being $\mathcal{A}(\pm A, \pm D, D) = \mathcal{A} + O(1/\log X)$ and $\zeta(1+z) = \frac{1}{z} + O(1)$ as $z \to 0$.

The case K=2, Q=0.

We now analyze closely the second moment of $L_\Delta(1/2, \chi_d)$; we take two complex shifts A, B such that $A, B \asymp (\log X)^{-1}$ and we look at

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta(1/2 + A, \chi_d)L_\Delta(1/2 + B, \chi_d).$$

By Conjecture 4.21, ignoring the negligible error term $O(X^{1/2+\varepsilon})$, the above is

$$f(A, B) + X^{-2A}f(-A, B) + X^{-2B}f(A, -B) + X^{-2A-2B}f(-A - B) \tag{4.91}$$

with

$$f(A, B) := \zeta(1+A+B)\prod_p\left(1-\frac{1}{p^{1+A+B}}\right)\left(1+\frac{p}{p+1}\sum_{\substack{m+n>0\\ \text{even}}}\frac{\tau^*(p^m)\tau^*(p^n)}{p^{(\frac{1}{2}+A)m+(\frac{1}{2}+B)n}}\right)$$

Since $A, B \asymp (\log X)^{-1}$, we set $a = A \log X \asymp 1$ and $b = B \log X \asymp 1$, so that (4.91) becomes

$$\mathcal{B} \log X \left(\frac{1}{a+b} + \frac{e^{-2a}}{-a+b} + \frac{e^{-2b}}{a-b} + \frac{e^{-2a-2b}}{-a-b} \right) \left(1 + O\Big(\frac{1}{\log X} \Big) \right) \tag{4.92}$$

where

$$\begin{aligned} \mathcal{B} := & \prod_p \left(1 - \frac{1}{p} \right) \left(1 + \frac{p}{p+1} \sum_{\substack{m+n>0 \\ \text{even}}} \frac{\tau^*(p^m)\tau^*(p^n)}{p^{(m+n)/2}} \right) \\ = & \prod_p \left(1 - \frac{1}{p} \right) \left(1 + \frac{p}{p+1} (-1 + \mathcal{P}^2 + \mathcal{D}^2) \right) \end{aligned} \tag{4.93}$$

with $\mathcal{P}$ and $\mathcal{D}$ defined in (4.87) and (4.88) respectively. The expression in (4.92) is regular at $a = 0$ and $b = 0$, since the limit of the first parentheses as $a, b \to 0$ equals 4, therefore we finally get

$$\frac{1}{X^*} \sum_{d \le X} L_\Delta(1/2, \chi_d)^2 \sim 4\mathcal{B} \log X. \tag{4.94}$$

The case K=3, Q=1.

Finally we look at

$$\sum_{d \le X} \frac{L_\Delta(1/2 + A, \chi_d) L_\Delta(1/2 + B, \chi_d) L_\Delta(1/2 + C, \chi_d)}{L_\Delta(1/2 + D, \chi_d)} \tag{4.95}$$

with A, B, C, D as Conjecture 4.21 prescribes. The first of the eight terms given by the recipe is

$$\sum_{d \le X} \frac{\zeta(1 + 2D)\zeta(1 + A + B)\zeta(1 + A + C)\zeta(1 + B + C)}{\zeta(1 + A + D)\zeta(1 + B + D)\zeta(1 + C + D)} \mathcal{A}(A, B, C; D)$$

where

$$\begin{aligned} A(A, B, C; D) = & \prod_p \frac{(1 - \frac{1}{p^{1+2D}})(1 - \frac{1}{p^{1+A+B}})(1 - \frac{1}{p^{1+A+C}})(1 - \frac{1}{p^{1+B+C}})}{(1 - \frac{1}{p^{1+A+D}})(1 - \frac{1}{p^{1+B+D}})(1 - \frac{1}{p^{1+C+D}})} \\ & \left(1 + \frac{p}{p+1} \sum_{\substack{0 < a+b+c+d \\ \text{even}}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)\mu_\Delta(p^d)}{p^{(\frac{1}{2}+A)a + (\frac{1}{2}+B)b + (\frac{1}{2}+C)c + (\frac{1}{2}+D)d}} \right) \end{aligned}$$

is the arithmetical coefficient, absolutely convergent in small disks around 0, such that

$$\begin{aligned}
A(0,0,0;0) &= \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{p}{p+1}\sum_{\substack{0<a+b+c+d\\ \text{even}}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)\mu_\Delta(p^d)}{p^{(a+b+c+d)/2}}\right)\\
&= \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{p}{p+1}\left(\left(1-\frac{1}{p}\right)\sum_{\substack{a+b+c\\ \text{even}}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)}{(\sqrt{p})^{a+b+c}}\right.\right.\\
&\qquad\qquad \left.\left. - \frac{\tau^*(p)}{\sqrt{p}} \sum_{\substack{a+b+c\\ \text{odd}}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)}{(\sqrt{p})^{a+b+c}} - 1\right)\right)\\
&= \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{p}{p+1}(-1+\mathcal{P}^2+\mathcal{D}^2)\right) = \mathcal{B}
\end{aligned}$$

where from the second line to the third, we used the formulae

$$\sum_{a+b+c \text{ even}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)}{p^{a/2}p^{b/2}p^{c/2}} = \mathcal{P}(\mathcal{P}^2+3\mathcal{D}^2)$$

$$\sum_{a+b+c \text{ odd}} \frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)}{p^{a/2}p^{b/2}p^{c/2}} = \mathcal{D}(\mathcal{D}^2+3\mathcal{P}^2)$$

and the multiplicative law of the Ramanujan tau function. As in all the previous examples, this gives a formula for (4.95) with all the main terms and error $O(X^{1/2+\varepsilon})$ and differentiating this formula with respect to C at $C = D$, we get

$$\begin{aligned}
&\sum_{d\le X} \frac{L'_\Delta}{L_\Delta}(1/2+D,\chi_d)L_\Delta(1/2+A,\chi_d)L_\Delta(1/2+B,\chi_d)\\
&\quad= \sum_{d\le X}\Bigg(R_1 + (\tfrac{d}{2\pi})^{-2A} g_O(\tfrac12+A)R_2 + (\tfrac{d}{2\pi})^{-2B} g_O(\tfrac12+B)R_3\\
&\qquad + (\tfrac{d}{2\pi})^{-2D} g_O(\tfrac12+D)R_4 + (\tfrac{d}{2\pi})^{-2A-2B} g_O(\tfrac12+A+B)R_5\\
&\qquad + (\tfrac{d}{2\pi})^{-2A-2D} g_O(\tfrac12+A+D)R_6 + (\tfrac{d}{2\pi})^{-2B-2D} g_O(\tfrac12+B+D)R_7\\
&\qquad + (\tfrac{d}{2\pi})^{-2A-2B-2D} g_O(\tfrac12+A+B+D)R_8\Bigg) + O(X^{1/2+\varepsilon})
\end{aligned} \tag{4.96}$$

with

$$
\begin{aligned}
R_1 = R_1(A,B,D) &= \mathcal{A}(A,B,D;D)\zeta(1+A+B)\Big(\tfrac{\zeta'}{\zeta}(1+A+D) \\
&\qquad + \tfrac{\zeta'}{\zeta}(1+B+D) - \tfrac{\zeta'}{\zeta}(1+2D)\Big) + \zeta(1+A+B)\mathcal{A}'(A,B,D;D) \\
R_2 &= R_1(-A,B,D) \\
R_3 &= R_1(A,-B,D) \\
R_4 = R_4(A,B,D) &= -\frac{\zeta(1+2D)\zeta(1+A+B)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)} \\
&\qquad \cdot \mathcal{A}(A,B,-D;D) \\
R_5 &= R_1(-A,-B,D) \\
R_6 &= R_4(-A,B,D) \\
R_7 &= R_4(A,-B,D) \\
R_8 &= R_4(-A,-B,D).
\end{aligned}
$$

If $A,B,D \asymp (\log X)^{-1}$ the above formula simplifies a lot, since

$$
\begin{aligned}
R_1 &= \frac{AB-AD-BD-3D^2}{2D(A+B)(A+D)(B+D)}\mathcal{B} + O\Big(\frac{(\log X)^4}{(\log X)^3}\Big) \\
&=: f(A,B,D) + O(\log X)
\end{aligned}
$$

and

$$
\begin{aligned}
R_4 &= \frac{(A+D)(B+D)}{(-2D)(A+B)(A-D)(B-D)}\mathcal{B} + \Big(\frac{(\log X)^4}{(\log X)^3}\Big) \\
&=: g(A,B,D) + O(\log X)
\end{aligned}
$$

giving

$$
\begin{aligned}
\sum_{d\le X} &\frac{L'_\Delta}{L_\Delta}(1/2+D,\chi_d)L_\Delta(1/2+A,\chi_d)L_\Delta(1/2+B,\chi_d) \\
&= \mathcal{B}X^*\Big(f(A,B,D) + X^{-2A}f(-A,B,D) + X^{-2B}f(A,-B,D) \\
&+ X^{-2D}g(A,B,D) + X^{-2A-2B}f(-A,-B,D) + X^{-2A-2D}g(-A,B,D) \\
&+ X^{-2B-2D}g(A,-B,D) + X^{-2A-2B-2D}g(-A,-B,D)\Big) + O(\log X)
\end{aligned}
\tag{4.97}
$$

Analogous formulae can be obtained in the cases $K=4, Q=1$ and $K=5, Q=1$. With exactly the same ideas (but much longer computations) also the case $K>5$, $Q=1$ can be dealt.

4.5.3 The weighted one-level density for $\{L_\Delta(\frac{1}{2}, \chi_d)\}_d$

In analogy to what we did in Section 4.4.3, we now compute the weighted one-level density for the orthogonal family of quadratic twists of L_Δ. We assume the Riemann Hypothesis for the L-functions we are considering and we denote with $\gamma_{\Delta,d}$ the imaginary part of a generic zero of $L_\Delta(s, \chi_d)$. In the classical case, assuming the ratio conjecture, Conrey and Snaith [43] proved that

$$\lim_{X\to\infty} \frac{1}{X^*} \sum_{d\leq X} \sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) = \int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{\sin(2\pi x)}{2\pi x}\Big)dx \qquad (4.98)$$

for any test function f, satisfying the usual properties as in Theorem 4.5. We now use the formulae of the previous section to derive the weighted one-level density; we denote

$$\mathcal{D}_1^{\boldsymbol{L}_{\Delta,\chi}}(f) := \frac{1}{\sum_{d\leq X} L_\Delta(\frac{1}{2}, \chi_d)} \sum_{d\leq X} \sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) L_\Delta(\tfrac{1}{2}, \chi_d) \qquad (4.99)$$

and we prove the following result.

Proposition 4.22. *Assume GRH and Conjecture 4.21 for $K = 2, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1 + x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_1^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W^1_{SO^+}(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W^1_{SO^+}(x) := 1 - \frac{\sin(2\pi x)}{2\pi x}. \qquad (4.100)$$

Proof. The strategy of the proof is the same as in the unitary and symplectic cases, thus we will just sketch how the proof works, underlining the differences with the other cases. For $a \asymp 1/\log X$ a real parameter, we consider the quantity

$$\frac{1}{2\mathcal{A}X^*} \sum_{d\leq X} \sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) L_\Delta(\tfrac{1}{2} + \tfrac{a}{\log X}, \chi_d)$$

which can be written as ($\delta \asymp (\log X)^{-1}$)

$$\frac{\log(X^2)}{2\pi} \int_{-\infty}^{+\infty} f\Big(\frac{\log X}{\pi} y\Big)dy$$
$$+ 2\frac{1}{2\pi} \int_{-\infty}^{+\infty} f\Big(\frac{\log X}{\pi} y\Big) \frac{1}{2\mathcal{A}X^*} \sum_{d\leq X} \frac{L'_\Delta}{L_\Delta}(\tfrac{1}{2} + \delta + iy) L_\Delta(\tfrac{1}{2} + \tfrac{a}{\log X}, \chi_d)dy$$

with an error $O((\log X)^{-1})$, by using Cauchy's theorem and the functional equation $\frac{L'_\Delta}{L_\Delta}(1-s,\chi_d) = \frac{X'_\Delta}{X_\Delta}(s,\chi_d) - \frac{L'_\Delta}{L_\Delta}(s,\chi_d)$ with $\frac{X'_\Delta}{X_\Delta}(s,\chi_d) = -\log d^2 + O(1)$ (note that the square here is due to the conductor of $L_\Delta(s,\chi_d)$, which is $\frac{d^2}{4\pi^2}$). With the change of variable $\frac{\log X}{\pi} y = x$ the above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{1}{2\mathcal{A}X^* \log X}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+\delta+\frac{\pi i x}{\log X}\Big)L_\Delta(\tfrac{1}{2}+\tfrac{a}{\log X},\chi_d)\Big)dx.$$

Now we use the assumption of the ratio conjecture in the form of (4.89) and (4.90) to evaluate the sum over d, getting

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{2\log X}h_X\Big(\frac{a}{\log X},\frac{\pi i x}{\log X}\Big)\Big)dx + O\Big(\frac{1}{\log X}\Big)$$

with

$$\begin{aligned} h_X(\alpha,w) := \frac{\alpha - w}{(\alpha+w)2w} &+ X^{-2\alpha}\frac{-\alpha-w}{(-\alpha+w)2w} \\ &+ X^{-2w}\frac{-\alpha-w}{(\alpha-w)2w} + X^{-2\alpha-2w}\frac{\alpha-w}{(-\alpha-w)2w}. \end{aligned}$$

Letting $a \to 0$, being $h_X(0,w) = \frac{X^{-2w}-1}{w}$, we get

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{e^{-2\pi i x}-1}{2\pi i x}\Big)dx + O\Big(\frac{1}{\log X}\Big).$$

Putting all together, since f is even, we finally have

$$\mathcal{D}_1^{\boldsymbol{L}_{\Delta,X}}(f) = \int_{-\infty}^{+\infty} f(x)\Big(1-\frac{\sin(2\pi x)}{2\pi x}\Big)dx + O\Big(\frac{1}{\log X}\Big).$$

□

Similarly we compute the analogue of (4.98), tilting by the second power of $L_\Delta(\frac{1}{2},\chi_d)$, i.e.

$$\mathcal{D}_2^{\boldsymbol{L}_{\Delta,X}}(f) := \frac{1}{\sum_{d\le X} L_\Delta(\frac{1}{2},\chi_d)^2}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta(\tfrac{1}{2},\chi_d)^2 \quad (4.101)$$

under the assumption of Conjecture 4.21, in the case $K=3, Q=1$. This is achieved in the following proposition.

Proposition 4.23. *Assume GRH and Conjecture 4.21 for $K = 3, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W_{SO^+}^2(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W_{SO^+}^2(x) := 1 + \frac{\sin(2\pi x)}{\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. Again we start with

$$\frac{1}{4\mathcal{B}X^* \log X} \sum_{d \le X} \sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) L_\Delta(\tfrac{1}{2} + \tfrac{a}{\log X}, \chi_d) L_\Delta(\tfrac{1}{2} + \tfrac{b}{\log X}, \chi_d)$$

and with the usual machinery we write it as

$$\int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{1}{4\mathcal{B}X^*(\log X)^2}\mathcal{I}\Big(\frac{a}{\log X}, \frac{b}{\log X}, \delta + \frac{\pi i x}{\log X}\Big)\Big)dx + O\Big(\frac{1}{\log X}\Big)$$

with

$$\mathcal{I}(\alpha, \beta, w) := \sum_{d \le X} \frac{L'_\Delta}{L_\Delta}(\tfrac{1}{2} + w, \chi_d) L_\Delta(\tfrac{1}{2} + \alpha, \chi_d) L_\Delta(\tfrac{1}{2} + \beta, \chi_d).$$

Thanks to the assumption of the ratio conjecture (see Equations (4.96) and (4.97)) we are able to evaluate asymptotically the above sum, which is regular at $\alpha, \beta, \delta = 0$. More specifically we have that

$$\lim_{\substack{a \to 0 \\ b \to 0 \\ \delta \to 0}} \mathcal{I}\Big(\frac{a}{\log X}, \frac{b}{\log X}, \delta + \frac{y}{\log X}\Big) = \mathcal{B}X^*(\log X)^2 h(y) + O(\log X)$$

with

$$h(y) := \frac{-2ye^{-2y} - 6y - 4e^{-2y} + 4}{y^2}.$$

Then we get

$$\begin{aligned}\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f) &= \int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{1}{4}h(\pi i x)\Big)dx + O\Big(\frac{1}{\log X}\Big) \\ &= \int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{\sin^2(\pi x)}{(\pi x)^2}\Big)dx + O\Big(\frac{1}{\log X}\Big)\end{aligned}$$

since f is even. □

We go on and define

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f) := \frac{1}{\sum_{d\leq X} L_\Delta(\frac{1}{2},\chi_d)^3} \sum_{d\leq X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) L_\Delta(\tfrac{1}{2},\chi_d)^3,$$

analyzing the third-moment case.

Proposition 4.24. *Assume GRH and Conjecture 4.21 for $K = 4, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W^3_{SO^+}(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W^3_{SO^+}(x) := 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. We introduce the usual real parameters α, β, ν of size $\asymp 1/\log X$, we denote

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}(\tfrac{1}{2},\chi_d) := L_\Delta(\tfrac{1}{2}+\alpha,\chi_d)L_\Delta(\tfrac{1}{2}+\beta,\chi_d)L_\Delta(\tfrac{1}{2}+\nu,\chi_d)$$

and we consider

$$\frac{1}{\sum_{d\leq X} L_\Delta(\frac{1}{2},\chi_d)^3} \sum_{d\leq X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) \boldsymbol{L}_\Delta^{\alpha,\beta,\nu}(\tfrac{1}{2},\chi_d).$$

With the usual strategy we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\bigg(1 + \frac{1}{\sum_{d\leq X} L(\frac{1}{2},\chi_d)^3} \sum_{d\leq X} \frac{L'_\Delta}{L_\Delta}(\tfrac{1}{2}+\delta+\tfrac{2\pi i x}{\log X},\chi_d)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}(\tfrac{1}{2},\chi_d)\bigg)dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. We evaluate asymptotically the remaining sum over d thanks to Conjecture 4.21 for $K = 4, Q = 1$), using sage to carry out the computations. Doing so, letting $\alpha, \beta, \nu \to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x)\Big(1 + \frac{1}{2}h(\pi i x)\Big)dx + O\Big(\frac{1}{\log X}\Big)$$

with

$$h(y) := \frac{-5y^3 + 6y^2 - 6 + e^{-2y}(y^3 + 6y^2 + 12y + 6)}{y^4}.$$

The claim follows, since f is even. $\square$

Finally, in the following result we study the case $k = 4$, given by

$$\mathcal{D}_4^{\boldsymbol{L}_\Delta,x}(f) := \frac{1}{\sum_{d\le X} L_\Delta(\frac12,\chi_d)^4} \sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big) L_\Delta(\tfrac12,\chi_d)^4.$$

Proposition 4.25. *Assume GRH and Conjecture 4.21 for $K = 5, Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_\Delta,x}(f) = \int_{-\infty}^{+\infty} f(x) W_{SO^+}^4(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$\begin{aligned} W_{SO^+}^4(x) := 1 &+ \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3} \\ &- \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}. \end{aligned}$$

Proof. As usual, if we set

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}(\tfrac12,\chi_d) := L_\Delta(\tfrac12+\alpha,\chi_d)L_\Delta(\tfrac12+\beta,\chi_d)L_\Delta(\tfrac12+\nu,\chi_d)L_\Delta(\tfrac12+\eta,\chi_d),$$

then we express $\mathcal{D}_4^{\boldsymbol{L}_\Delta,x}(f)$ as the limit for $\alpha,\beta,\nu,\eta \to 0$ of

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{\sum_{d\le X} L(\frac12,\chi_d)^4}\sum_{d\le X}\frac{L_\Delta'}{L_\Delta}(\tfrac12+\delta+\tfrac{2\pi i x}{\log X},\chi_d)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}(\tfrac12,\chi_d)\Big)dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. The above can be evaluated asymptotically (again sage is of help in carrying out the computation) and we get

$$\mathcal{D}_4^{\boldsymbol{L}_\Delta,x}(f) = \int_{-\infty}^{+\infty} f(x)\Big(1+\frac12 h(\pi i x)\Big)dx + O\Big(\frac{1}{\log X}\Big)$$

with

$$\begin{aligned} h(y) :=& \frac{-7y^5+12y^4-30y^2+90}{y^6} \\ &- \frac{e^{-2y}(y^5+12y^4+60y^3+150y^2+180y+90)}{y^6}. \end{aligned}$$

Since f is even the claim follows. □

Theorem 4.5 follows by Propositions 4.22−4.25.

Appendix A

On Lemma 2.4

We recall that Lemma 2.4 gives the asymptotic formula for the second moment of zeta times suitably short Dirichlet polynomials. More specifically, let

$$A(s) = \sum_{n \le T^\theta} \frac{a(n)}{n^s} \quad \text{and} \quad B(s) = \sum_{m \le T^\sigma} \frac{b(m)}{m^s}$$

be Dirichlet polynomials with $a(n) \ll n^\varepsilon$, $b(m) \ll m^\varepsilon$ for every $\varepsilon > 0$. Then, if $\theta + \sigma < 1$, we have:

$$\begin{aligned} \int_T^{2T} & A(1/2+it)\overline{B(1/2+it)}|\zeta(1/2+it)|^2 dt \\ &= T \sum_{m,n} \frac{a(n)\overline{b(m)}}{[n,m]} \left(\log\left(\frac{T(n,m)^2}{nm} \right) + c \right) + o(T). \end{aligned} \tag{A.1}$$

where $c := 2\gamma + \log 4 - \log 2\pi - 1$.

In the case $A = B$, the asymptotic formula (A.1) is due to Balasubramanian, Conrey and Heath-Brown [5] and it is now classical. However, throughout this thesis, we need the slightly more general version (A.1). The proof, which is reported below for completeness, does not require any new idea and builds on well-known techniques [5, 11].

Nevertheless, we note that in the specific case $B(s) = 1$, (A.1) allows to evaluate asymptotically the moments of $|\zeta(1/2+it)|^2$ times a Dirichlet polynomial $A(s)$ of length T^θ, with $\theta < 1$, i.e. the quantity

$$\int_T^{2T} A(1/2+it)|\zeta(1/2+it)|^2 dt.$$

This is particularly useful in Chapter 4 (see Proposition 4.15).

Finally we note that with the same ideas as in [11], one should be able to break the "$\frac{1}{2}$-barrer" also in this more general case, proving (A.1) in wider range for θ, σ.

Proof (of Lemma 2.4). This proof is a simplified version of that of Theorem 1 in [11], hence we will just sketch the main changes and refer to [11] for further details. Let us denote $N_1 = T^\theta$ and $N_2 = T^\sigma$; then $N_1 N_2 \leq T^{1-\eta}$ for some $\eta > 0$. The left-hand side of (A.1) can be written as

$$\mathcal{D} + \mathcal{O} + O(T^{1-\delta})$$

for some $\delta > 0$, where the diagonal term is[1]

$$\mathcal{D} := T \sum_{m,n} \frac{a(n)\overline{b(m)}}{[n,m]} \left(\log\left(\frac{T(n,m)^2}{nm}\right) + c \right) + O(T^{1-\delta})$$

and the off-diagonal term is

$$\mathcal{O} := 2 \sum_{\substack{m_1,m_2,n_1,n_2 \\ \Delta = m_1 m_2 - m_2 n_1 \neq 0}} \frac{a(n_1)\overline{b(n_2)}}{\sqrt{m_1 m_2 n_1 n_2}} \int_T^{2T} \Big(\frac{m_1 n_2}{m_2 n_1}\Big)^{it} W\Big(\frac{2\pi m_1 m_2}{t}\Big) dt$$

where the summation is over $m_1 m_2 < T^{1+\varepsilon}$, $\varepsilon > 0$, and $W(x)$ is defined as in [11, Lemma 1].

Now it suffices to show that $\mathcal{O} \ll T^{1-\delta}$ for some $\delta > 0$. First we notice that if $\Delta \ll \frac{m_2 n_1}{T^{1-\varepsilon}}$ then we get no contribution. Indeed, if $\Delta \neq 0$, then

$$m_2 n_1 \gg T^{1-\varepsilon}, \tag{A.2}$$

which yields $m_1 \ll n_1 T^{2\varepsilon}$, being $m_1 m_2 < T^{1+\varepsilon}$. Then, since $\Delta = m_1 n_2 - m_2 n_1$, we get

$$m_2 n_1 \ll n_1 n_2 T^{2\varepsilon} \leq N_1 N_2 T^{2\varepsilon} \leq T^{1-\eta+2\varepsilon}. \tag{A.3}$$

For ε small enough with respect to η (say $4\varepsilon < \eta$), the condition (A.3) is incompatible with (A.2), then the case $\Delta \ll \frac{m_2 n_1}{T^{1-\varepsilon}}$ does not contribute.

Now we bound the contribution from $\Delta \gg \frac{m_2 n_1}{T^{1-\varepsilon}}$, with an integration by parts. Since $W(\frac{2\pi m_1 m_2}{t}) \ll 1$ and $\frac{d}{dt} W(\frac{2\pi m_1 m_2}{t}) \ll \frac{1}{t}$ (see [11, page 9]), then this contribution is[2]

$$\ll T^{\varepsilon/100} \sum_{\substack{m_1,m_2,n_1,n_2 \\ \Delta = m_1 m_2 - m_2 n_1 \gg \frac{m_2 n_1}{T^{1-\varepsilon}}}} \frac{1}{\sqrt{m_1 m_2 n_1 n_2}} \frac{m_2 n_1}{|m_1 n_2 - m_2 n_1|}. \tag{A.4}$$

[1]See [11], Equation (3.1) and the computation at page 14 for further details.

[2]This is also standard, noticing that $\frac{m_1 n_2}{m_2 n_1} = 1 + \frac{\Delta}{m_2 n_1} = 1 + \frac{m_1 n_2 - m_2 n_1}{m_2 n_1}$.

Now we split the range of summation as follows. For $1 \le m_1 n_2 \le \frac{m_2 n_1}{2}$ (and the case $m_1 n_2 \ge 2 m_2 n_1$ is completely analogous), we have that $|m_1 n_2 - m_2 n_1| \gg m_2 n_1$. Thus the contribution of $1 \le m_1 n_2 \le \frac{m_2 n_1}{2}$ in (A.4) is

$$\ll T^{\varepsilon} \sum_{\substack{m_1, m_2, n_1, n_2 \\ m_1 m_2 < T^{1+\varepsilon} \\ n_1 \le N_1, n_2 \le N_2}} \frac{1}{\sqrt{m_1 m_2 n_1 n_2}} \ll T^{\frac{1}{2}+2\varepsilon} \sqrt{N_1 N_2} \ll T^{1+2\varepsilon - \frac{\eta}{2}}$$

which is negligible for ε small enough, compared to η. In the case $\frac{m_2 n_1}{2} < m_1 n_2 \le m_2 n_1 - 1$ (and analogously $m_2 n_1 + 1 \le m_1 n_2 < 2 m_2 n_1$) we have that $\frac{1}{\sqrt{m_1 n_2}} \ll \frac{1}{\sqrt{m_2 n_1}}$; moreover, the condition $m_1 m_2 < T^{1+\varepsilon}$ implies $m_2 \ll T^{\frac{1}{2}+\frac{\varepsilon}{2}} \sqrt{\frac{N_2}{n_1}}$. Then, if we reparametrize the sum defining $l := [m_2 n_1 - m_1 n_2]$ we get that the contribution coming from $\frac{m_2 n_1}{2} < m_1 n_2 \le m_2 n_1 - 1$ in (A.4) is bounded by

$$T^{\varepsilon} \sum_{n_1 \le N_1} \sum_{m_2 \ll T^{\frac{1}{2}+\frac{\varepsilon}{2}} \sqrt{\frac{N_2}{n_1}}} \sum_{1 \le l \le \frac{m_2 n_1}{2}} \frac{\tau(l)}{l} \ll T^{\frac{1}{2}+2\varepsilon} \sqrt{N_2} \sum_{n_1 \le N_1} \frac{1}{\sqrt{n_1}} \ll T^{1+2\varepsilon - \frac{\eta}{2}}$$

which negligible if ε is small. Finally if $m_2 n_1 - 1 < m_1 n_2 < m_2 n_1 + 1$ then $m_1 n_2 \approx m_2 n_1$ is fixed; in this range we use the condition $|m_1 n_2 - m_2 n_1| \gg \frac{m_2 n_1}{T^{1-\varepsilon}}$, getting

$$T^{\varepsilon/50} \sum_{\substack{m_2 < T^{1+\varepsilon} \\ n_1 \le N_1}} \frac{T^{1-\varepsilon}}{m_2 n_1} \ll T^{1-\frac{\varepsilon}{2}}$$

and this concludes the proof. □

Appendix B

Proof of Theorem 4.9

By Fourier inversion, we have that

$$W^k_{USp}(x) = \int_{-\infty}^{+\infty} \widehat{W}^k_{USp}(y) e^{2\pi i x y} dy = \sum_{n=0}^{\infty} \left(\frac{(2\pi i)^n}{n!} \int_{-\infty}^{+\infty} \widehat{W}^k_{USp}(y) y^n dy \right) x^n.$$

Moreover, since $\widehat{W}^k_{USp}$ is even, then $\int_{-\infty}^{+\infty} \widehat{W}^k_{USp}(y) y^n dy = 0$ if n is odd. Hence, by definition of $\widehat{W}^k_{USp}$

$$W^k_{USp}(x) = \sum_{m=0}^{\infty} \beta_{m,k} x^{2m}$$

with

$$\beta_{m,k} = \frac{(2\pi i)^{2m}}{(2m)!} \int_{-\infty}^{+\infty} \Bigg(\delta_0(y) + \chi_{[-1,1]}(y) \bigg(- \frac{2k+1}{2} \\ - k(k+1) \sum_{j=1}^{k} (-1)^j c_{j,k} \frac{|y|^{2j-1}}{2j-1} \bigg) \Bigg) y^{2m} dy.$$

By computing the integral, being $\int_{-1}^{1} y^{2m} dy = \frac{2}{2m+1}$ and $\int_{-1}^{1} y^{2m} |y|^{2j-1} dy = \frac{1}{m+j}$, the above yields

$$\beta_{m,k} := \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!} \left[\frac{2k+1}{2m+1} + k(k+1) \sum_{j=1}^{k} \frac{(-1)^j}{(2j-1)(j+m)} c_{j,k} \right].$$

Since

$$c_{j,k} = \frac{1}{j} \binom{k-1}{j-1} \binom{k+j}{j-1} = \frac{j}{k(k+1)} \binom{k+j}{j} \binom{k}{j}$$

then we get

$$\beta_{m,k} = \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!}\left[\frac{2k+1}{2m+1} + S_m(1)\right] \tag{B.1}$$

where

$$S_m(x) = S_{m,k}(x) := \sum_{j=1}^{\infty} \frac{(-1)^j j}{(2j-1)(j+m)} \binom{k+j}{j}\binom{k}{j} x^j.$$

Now we write the factors in the above sum in terms of the Pochhammer symbol, defined as

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) \quad \text{for } n \geq 1;$$

namely

$$\begin{aligned}
\binom{k+j}{j} &= \frac{(k+1)_j}{j!} \\
(-1)^j \binom{k}{j} &= \frac{(-k)_j}{(1)j} \\
\frac{j}{(2j-1)(j+m)} &= \frac{1}{2m+1}\left(\frac{1}{2j-1} + \frac{m}{j+m}\right) \\
&= \frac{1}{2m+1}\left(-\frac{(-1/2)_j}{(1/2)_j} + \frac{(m)_j}{(m+1)_j}\right),
\end{aligned}$$

so that we have

$$S_m(x) = \frac{1}{2m+1}\sum_{j=1}^{\infty}\left(-\frac{(-1/2)_j}{(1/2)_j} + \frac{(m)_j}{(m+1)_j}\right)\frac{(-k)_j(k+1)_j}{(1)_j}\frac{x^j}{j!}. \tag{B.2}$$

Reparametrising the sum and using $(a)_{j+1} = a(a+1)_j$, this gives

$$S_m(x) = S_m^1(x) + S_m^2(x),$$

where

$$S_m^1(x) := \frac{-k(k+1)x}{2m+1}\sum_{j=0}^{\infty}\frac{(1/2)_j(-k+1)_j(k+2)_j}{(3/2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}$$

and

$$S_m^2(x) := \frac{-mk(k+1)x}{(2m+1)(m+1)}\sum_{j=0}^{\infty}\frac{(m+1)_j(-k+1)_j(k+2)_j}{(m+2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}.$$

By writing

$$\frac{1}{j+1} = 2\left(1 - \frac{2j+1}{2j+2}\right)$$

and

$$\frac{(1/2)_j}{(3/2)_j} = \frac{1}{2j+1}$$

then we get

$$S_m^1(x) = -\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ \frac{1}{2}\\ \frac{3}{2},\ 2\end{matrix};x\right] - \frac{{}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1},$$

where ${}_pF_q$ denotes the generalized hypergeometric function, defined as

$${}_pF_q\left[\begin{matrix}a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix};x\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!}.$$

Similarly, since

$$-\frac{1}{j+1} = \frac{1}{m} - \frac{j+m+1}{(j+1)m}$$

then we have

$$S_m^2(x) = \frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right] + \frac{{}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1}.$$

Therefore, substituting in (B.2) yields

$$\begin{aligned}S_m(x) = &-\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ \frac{1}{2}\\ \frac{3}{2},\ 2\end{matrix};x\right]\\ &+\frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right].\end{aligned} \tag{B.3}$$

Plugging (B.3) in (B.1), we obtain

$$\begin{aligned}\beta_{m,k} = \delta_0(m) - \frac{(-1)^m(2\pi)^{2m}}{(2m+1)!}\Bigg(&2k+1-2k(k+1)\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ \frac{1}{2}\\ \frac{3}{2},\ 2\end{matrix};1\right]\\ &+\frac{k(k+1)}{(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]\Bigg).\end{aligned} \tag{B.4}$$

Now we need a few lemmas, in order to be able to compute the remaining hypergeometric functions.

Lemma B.1. *For any $k \in \mathbb{N}$ we have*

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ \frac{1}{2}\\ \frac{3}{2},\ 2\end{matrix};1\right] = \begin{cases}\frac{1}{k+1} & \text{if } k \text{ even}\\ \frac{1}{k} & \text{if } k \text{ odd.}\end{cases} $$

Proof. We recall the reduction formula for the generalized hypergeometric function (see e.g. [67], Equation (17) in the case $n = 1$), being

$$ {}_{A+1}F_{B+1}\left[\begin{matrix}a_1,\ \ldots,\ a_A,\ c+n\\ b_1,\ \ldots,\ b_B,\ c\end{matrix};x\right] = \sum_{j=0}^{n}\binom{n}{j}\frac{1}{(c)_j}\frac{\prod_{i=1}^{A}(a_i)_j}{\prod_{i=1}^{B}(b_i)_j}\,{}_AF_B\left[\begin{matrix}a_1+j,\ \ldots,\ a_A+j\\ b_1+j,\ \ldots,\ b_B+j\end{matrix};x\right] $$

for any A, B positive integers, $n \in \mathbb{N}$. The left hand side can be then written as

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ \frac{1}{2},\ k+2\\ \frac{3}{2},\ 2\end{matrix};1\right] &= \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(\frac{1}{2})_j}{(\frac{3}{2})_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ \frac{1}{2}+j\\ \frac{3}{2}+j\end{matrix};1\right]\\ &= \sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(\frac{1}{2})_j}{(\frac{3}{2})_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ \frac{1}{2}+j\\ \frac{3}{2}+j\end{matrix};1\right] \end{aligned} \tag{B.5} $$

as $(1-k)_k = 0$. The remaining hypergeometric function can be computed by applying Gauss' summation theorem (see e.g. [112], Equation (3.1)), i.e. the formula

$$ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c) > \Re(a+b). $$

We recall that if $a = -n$, $n \in \mathbb{N}$, this is the Chu-Vandermonde identity (see again [112], immediately below Equation (3.1))

$$ {}_2F_1\left[\begin{matrix}-n,\ b\\ c\end{matrix};1\right] = \frac{(c-b)_n}{(c)_n}. $$

This yields

$$ {}_2F_1\left[\begin{matrix}1-k+j,\ \frac{1}{2}+j\\ \frac{3}{2}+j\end{matrix};1\right] = \frac{(k-j-1)!(j+\frac{1}{2})!}{(k-\frac{1}{2})!} \tag{B.6} $$

for $k > j$. Plugging Equation (B.6) into (B.5), we get

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1-k,\ \frac{1}{2},\ k+2 \\ \frac{3}{2},\ 2 \end{matrix};1\right] &= \frac{(k-1)!}{2(k-\frac{1}{2})!}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\frac{(j-\frac{1}{2})!}{(j+1)!} \\ &= \frac{(k-1)!}{2(k-\frac{1}{2})!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-\frac{1}{2})!}{(j+1)!} - \frac{(-1)^k}{2k(k+1)}. \end{aligned} \tag{B.7}$$

Moreover, since $(\frac{1}{2})_j = \frac{1}{\sqrt{\pi}}(j-\frac{1}{2})!$, $(-1)^j\binom{k}{j} = (-k)_j/j!$ and $(2)_j = (j+1)!$, we have

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-\frac{1}{2})!}{(j+1)!} = \sqrt{\pi}\ {}_2F_1\left[\begin{matrix} -k,\ \frac{1}{2} \\ 2 \end{matrix};1\right] = 2\frac{(k+\frac{1}{2})!}{(k+1)!}$$

by applying the Chu-Vandermonde identity. Putting this into (B.7), we finally get

$${}_3F_2\left[\begin{matrix} 1-k,\ \frac{1}{2},\ k+2 \\ \frac{3}{2},\ 2 \end{matrix};1\right] = \frac{k+\frac{1}{2}}{k(k+1)} - \frac{(-1)^k}{2k(k+1)}$$

and the claim follows. □

By using Lemma B.1, Equation (B.4) becomes

$$\beta_{m,k} = \delta_0(m) - \frac{(-1)^m(2\pi)^{2m}}{(2m+1)!}\left((-1)^k + \frac{k(k+1)}{(m+1)}{}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ m+1 \\ m+2,\ 2 \end{matrix};1\right]\right). \tag{B.8}$$

The coefficient $\beta_{0,k}$ can be then computed thanks to the following lemma.

Lemma B.2. *For any $k \in \mathbb{N}$ we have*

$${}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ 1 \\ 2,\ 2 \end{matrix};1\right] = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{2}{k(k+1)} & \text{if } k \text{ odd.} \end{cases}$$

Proof. By definition we have

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ 1 \\ 2,\ 2 \end{matrix};1\right] &= \sum_{j=0}^{\infty}\frac{(1-k)_j(k+2)_j(1)_j}{(2)_j(2)_j}\frac{1}{j!} \\ &= -\frac{1}{k(k+1)}\sum_{j=0}^{\infty}\frac{(-k)_{j+1}(k+1)_{j+1}}{(1)_{j+1}}\frac{1}{j!} \end{aligned}$$

since $(1)_j/(2)_j = 1/(j+1)$, $(1-k)_j = (-k)_{j+1}/(-k)$ and $(2)_j = (1)_{j+1}$. Reparametrising the series with $l = j+1$, the above yields

$$-\frac{1}{k(k+1)}\left(\sum_{l=0}^{\infty}\frac{(-k)_l(k+1)_l}{(1)_l}\frac{1}{l!}-1\right)=\frac{1-{}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\1\end{smallmatrix};1\right]}{k(k+1)}.$$

The claim is then proven, by noticing that ${}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\1\end{smallmatrix};1\right] = (-1)^k$ thanks to the Chu-Vandermonde identity. □

This implies that $\beta_{0,k} = 0$ for any $k \in \mathbb{N}$, proving the first part of Theorem 4.9. To complete our proof, we need to show that

$$\beta_{k+1,k} = \frac{2\pi^{2(k+1)}}{(2k+1)!!(2k+1)!!} \tag{B.9}$$

is the first nonzero coefficient. As a first step, the following lemma shows that $\beta_{i,k} = 0$ for all $1 \le i \le k$.

Lemma B.3. *For any $k \in \mathbb{N}$ and for any $1 \le m \le k$ we have*

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\m+2,\ 2\end{matrix};1\right]=\frac{(m+1)(-1)^{k+1}}{k(k+1)}.$$

Proof. We begin by applying the reduction formula, which yields ${}_3F_2\left[\begin{smallmatrix}1-k,\ k+2,\ m+1\\m+2,\ 2\end{smallmatrix};1\right] =$

$$\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(m+2)_j}{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\m+2+j\end{matrix};1\right]. \tag{B.10}$$

Moreover, the Chu-Vandermonde identity gives

$${}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\m+2+j\end{matrix};1\right]=\frac{(m-k)_{k-j-1}}{(m+j+2)_{k-j-1}}$$

Since $(m-k)_{k-j-1} = 0$ for all $j < m-1$, only the term $j = m-1$ survives in the sum in Equation (B.10). Hence we get

$$\begin{aligned}{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\m+2,\ 2\end{matrix};1\right]&=\frac{(1-k)_{m-1}}{(2)_{m-1}}\frac{(k+2)_{m-1}}{(m+2)_{m-1}}\frac{(m-k)_{k-m}}{(2m+1)_{k-m}}\\&=\frac{(-1)^{m-1}(k-1)!}{(k-m)!m!}\frac{(k+m)!(m+1)!}{(k+1)!(2m)!}\frac{(-1)^{k-m}(k-m)!(2m)!}{(m+k)!}\\&=\frac{(-1)^{k-1}(k-1)!}{m!}\frac{(m+1)!}{(k+1)!}=\frac{(-1)^{k-1}(m+1)}{k(k+1)},\end{aligned} \tag{B.11}$$

where in the first line we applied the equalities $(m+2)_{m-1} = (2m)!/(m+1)!$, $(k+2)_{m-1} = (k+m)!/(k+1)!$ and $(1-k)_{m-1} = (-1)^{m-1}(k-1)!/(k-m)!$. Similarly also $(m-k)_{k-m} = (-1)^{k-m}(k-m)!$ and $(2m+1)_{k-m} = (k+m)!/(2m)!$. □

Finally, with the following lemma, we can also compute $\beta_{k+1,k}$.

Lemma B.4. *For any $k \in \mathbb{N}$ we have*

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right] = \frac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right). $$

Proof. The idea of the proof is similar the the one of Lemma B.3. First we apply the reduction formula in order to write ${}_3F_2\left[\begin{smallmatrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{smallmatrix};1\right]$ as a finite sum of terms involving ${}_2F_1$, namely

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right] \\ &= \sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(k+3)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right]. \end{aligned} \tag{B.12} $$

Note that the term $j=k$ vanishes, as $(1-k)_k = 0$. Now we use Gauss's summation theorem and compute the remaining hypergeometric function, i.e.

$$ {}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right] = \frac{(k+j+2)!}{(2k+1)!}. $$

Plugging this into Equation (B.12), since $(-1)^j\binom{k}{j} = (-k)_j/j!$ and $(k+2)_j = (k+j+1)!/(k+1)!$, we have

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right] &= \frac{(k-1)!(k+2)!}{(2k+1)!}\sum_{j=0}^{k-1}\frac{(-k)_j(k+2)_j}{(2)_j}\frac{1}{j!} \\ &= \frac{(k-1)!(k+2)!}{(2k+1)!}\left({}_2F_1\left[\begin{matrix}-k,\ k+2\\ 2\end{matrix};1\right] - \frac{(-k)_k(k+2)_k}{(2)_k}\frac{1}{k!}\right). \end{aligned} \tag{B.13} $$

Therefore, since ${}_2F_1\left[\begin{smallmatrix}-k,\ k+2\\ 2\end{smallmatrix};1\right] = \frac{(-1)^k}{k+1}$, we get

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right] &= \frac{(k-1)!(k+2)!}{(2k+1)!}\left(\frac{(-1)^k}{k+1} - \frac{(-1)^k(2k+1)!}{((k+1)!)^2}\right) \\ &= \frac{2(k-1)!(k+2)!(-1)^k}{(2k+2)!}\left(1 - \frac{(2k+1)!}{(k+1)!k!}\right) \end{aligned} $$

and the claim follows. □

To conclude the proof of Theorem 4.9, we just combine Equation (B.8) with Lemma B.4, getting

$$\begin{aligned}\beta_{k+1,k} &= \frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{k(k+1)}{k+2}\frac{2(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right)\right)\\&= \frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2[(k+1)!]^2}{(2k+2)!}\left(\frac{(2k+1)!}{(k+1)!k!}-1\right)\right)\\&= \frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2(k+1)}{2k+2}+\frac{2[(k+1)!]^2}{(2k+2)!}\right)=\frac{(2\pi)^{2k+2}}{(2k+3)!}\frac{(k+1)!k!}{(2k+1)!}.\end{aligned}$$

Equation (B.9) follows by the identities $(2k+1)! = 2^k k!(2k+1)!!$ and $(2k+3)! = 2^{k+1}(k+1)!(2k+3)!!$.

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