University of Genoa

PhD's Program in Mathematics and Applications

Structure of covariant uniformly continuous Quantum Markov Semigroups

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Chapter 1

Introduction

Quantum Markov Semigroups (QMS) are a tool originally introduced in the realm of physics in order to be able to study open quantum systems (hence the name), and more specifically their time evolution. Specifically, when considering open quantum systems the physical system itself is let free to interact with the environment allowing for the study of a wider range of problems. Such problems are of great interest since they are a closer representation of real world application such as quantum optics or quantum computation, the latter a field that is gaining a lot of attention in recent years and where the interaction is one of the focal point of the theory. From a mathematical point of view, a QMS represents the natural generalization of a classical Markov semigroup justifying much of the interest gathered from the mathematical community. The main aim of this Thesis is not to study a QMS in and of itself, but we are rather going to focus on their behavior and their structure in the presence of some kind of symmetry. The concept of symmetry is again very central in the realm of physics, and one the recurs very frequently and which make their study quite interesting. In order to give some more details in this introduction we need to briefly present some of the basic mathematical tools we are going to go over in greater detail in later Chapters. In the theory of QMS the algebraic approach to quantum physics is used, therefore we are going to use as von Neumann algebra $\mathcal{A}$ (whose self-adjoint elements represent the observable physical properties of the underlying physical system) and a QMS is described by a semigroup on the algebra itself. More precisely, a QMS $\mathcal{T} = \{\mathcal{T}_t\}_{t\geq 0}$ is, in the more general case, is a weakly* continuous semigroup of linear, bounded, normal, completely positive and identity preserving maps

$$\mathcal{T} \colon \mathcal{A} \to \mathcal{A} \qquad \forall t \geq 0.$$

In this work we are going to consider uniformly continuous QMS defined on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. One of the most important tool in the theory, and one that we are going to use extensively throughout this Thesis, is the *generator* $\mathcal{L}$ of a QMS, which is defined as

$$\mathcal{L}(a) = \lim_{t\to 0^+} \frac{\mathcal{T}_t - \mathbb{1}}{t}.$$

which is equivalent to $\mathcal{T}_t = e^{t\mathcal{L}}$. On the other hand, a symmetry is described as a group G with an associated (unitary) representation π acting on the Hilbert space $\mathcal{H}$. With this notation, the property that we are going to focus on in this work is called *covariance* and mathematically expressed as

$$\mathcal{T}_t(\pi(g)^* a \pi(g)) = \pi(g)^* \mathcal{T}_t(a) \pi(g)$$

or, as we are going to see later, equivalently as

$$\mathcal{L}(\pi(g)^* a \pi(g)) = \pi(g)^* \mathcal{L}(a) \pi(g)$$

Such property is been extensively studied by Holevo in a series of papers (see [1, 2, 3, 4, 5]) in which he gave a comprehensive analysis of the structure of a generator satisfying the covariance property. Moreover, there are also been some application to specific symmetries like translation invariance in [6] and [7] to name a few. Our goal is then to expand on these results giving a characterization of the so called *decoherence free subalgebra* $\mathcal{N}(\mathcal{T})$ of a uniformly continuous QMS in the presence of a symmetry. This object is defined as the biggest subalgebra on which the time evolution acts as a *-automorphism or, in other words, as a set of operators on which the semigroup acts in a unitary way, resembling the simpler case of a reversible time evolution. $\mathcal{N}(\mathcal{T})$ has been studied widely in the literature and its structure has been described, in the non covariant case, by for example in [8, 9, 10]. More specifically, whenever the decoherence free subalgebra is *atomic*, that is when there exists a (at most countable) family of mutually orthogonal projection $\{p_i\}_{i\in I}$ that are minimal in the center of $\mathcal{N}(\mathcal{T})$ and that satisfy $\sum_{i\in I} p_i = \mathbb{1}$, allowing to decompose the subalgebra as

$$\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in I} p_i \mathcal{N}(\mathcal{T}) p_i$$

were each $p_i \mathcal{N}(\mathcal{T}) p_i$ is a type I factor, they showed there exist two sequences of Hilbert spaces $\{\mathcal{K}_i\}_{i\in I}$ and $\{\mathcal{M}_i\}_{i\in I}$ such that the above decomposition of $\mathcal{N}(\mathcal{T})$ can be rewritten as

$$\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in I} \left(\mathcal{B}(\mathcal{K}_i) \otimes \mathbb{1}_{\mathcal{M}_i}\right).$$

In the first place, we are interested in studying the relationship between an atomic decoherence-free subalgebra and the symmetry property satisfied by the QMS with respect to an irreducible representation, in particular which, if any, properties can be imposed on the sequences of Hilbert spaces $\{\mathcal{K}_i\}_{i\in I}$ and $\{\mathcal{M}_i\}_{i\in I}$. Subsequently, we try and extend these in the case of a generic representation, with particular focus on the intertwining of the atomic decomposition and the Peter-Weyl decomposition of the representation. The atomicity property of $\mathcal{N}(\mathcal{T})$ is limiting in the study of covariance, indeed we know that there exists non-atomic covariant semigroups for which we are interested in extending our techniques. In order to do so we introduce the concept of *direct integral*, which by itself is a generalization of the direct sum, allowing us to consider more general decomposition of the decoherence-free subalgebra. This last section of our work is limited in scope and we reserve the possibility of studying such problem more in depth in future works. For example, one possible direction of development is to study if there is any connection between the direct integral decomposition of the decoherence-free subalgebra and the one of the symmetry representation.

The structure of the Thesis is the following:

Chapter 1 We introduce the main mathematical concepts needed to our study. Specifically we recall the basic properties of completely positive maps and their decompositions such as the Stinespring Theorem and the Kraus decomposition. We then move to describe the QMS theory in its basic definitions, specializing our selves to the case of uniformly continuous QMSs and their generators. We fully describe their characterization and we recall the crucial GKSL representation the generator $\mathcal{L}$. We conclude the Chapter with the classification of projections according to the action of a QMS $\mathcal{T}$ which are going to be crucial in proving our main results.
Chapter 2 This Chapter represents the biggest original contribution of this Thesis. After recalling the results from the series of papers by Holevo on the structure of a covariant QMS generator, we proceed to prove a few simple results that are directly implied by such papers in Corollary 3.1.1, in which we show that the operators of any GKSL representation of $\mathcal{L}$ intertwines any unitary representation implementing the covariance property. Also Proposition 3.1.1, in which we show that the covariance property implies a simplified representation for $\mathcal{L}$, and Proposition 3.1.2 which provides the conditions under which an equivalent GKSL representation to a covariant one preserve the covariance property. Finally, we conclude the first section showing that $\mathcal{N}(\mathcal{T})$ and the set of fixed points w.r.t. a QMS $\mathcal{T}$ are invariant under conjugation by unitary representation. In the subsequent Section we temporarily specialize to the case of a covariant QMS w.r.t. an irreducible representation π which allows us to prove the most important result of this work. Namely, in Proposition 3.2.2 we prove that the action of the representation on the atomic decomposition of $\mathcal{N}(\mathcal{T})$ is to permute the factors $p_i\mathcal{N}(\mathcal{T})p_i$ of the decomposition itself, which in turn directly implies that such factors must be all isomorphic to each other as stated in Proposition 3.2.3. These two results are then put together in Theorem 3.2.1 that proves that the covariance property implies the following simplified structure for the decoherence free subalgebra

$$\mathcal{N}(\mathcal{T}) = (\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{M}})^d$$

where d is number of factors in the original atomic decomposition. At the same time, we manage to provide an extremely simplified structure for the GKSL representation whenever the covariance property is satisfied w.r.t. an irreducible representation. We conclude the Section proving a strong connection between the topology of the group and the property of $\mathcal{N}(\mathcal{T})$. Strikingly, we prove in Theorem 3.2.4 that the number of connected components of the symmetry group G and the number of factors in the decomposition of $\mathcal{N}(\mathcal{T})$ are interdependent, and as a side result we proved in Corollary 3.2.1 that $\mathcal{N}(\mathcal{T})$ can be a factor if and only if G is a connected group. In the following Section we generalize the previous results to the case of a generic (i.e. non irreducible) representation π for the symmetry group G. In order to do so we leverage the Peter-Weyl Theorem to show that for each subspace on which the representation acts irreducibly, the results of the previous Section hold true. We then, conclude the Chapter with come result to help the reader get a better understanding of this Chapter results, namely we analyze the $U(2)$ symmetry group, the so called *circulant QMS*, and finally we show an example of a covariant QMS for which $\mathcal{N}(\mathcal{T})$ is type II_1 factor, thus justifying the need for a more general theory we try to

introduce in the next Chapter.
Chapter 4 In the first Section of this Chapter we recall the basic results about direct integrals, how is possible to decompose Hilbert spaces, operators acting on such Hilbert spaces, and finally how to decompose von Neumann algebras which are our main focus. In the following two Sections we prove that $\mathcal{N}(\mathcal{T})$ is actually decomposable as a direct integral in Theorem 4.2.1, that any GKSL representation of a QMS can be decomposed in Proposition 4.3.2 and finally how from is possible to obtain the specific decomposition of a QMS $\mathcal{T}$ and its decoherence free subalgebra $\mathcal{N}(\mathcal{T})$ in Corollary 4.3.1.

Chapter 2

Preliminary material on QMS

In this Chapter we collect all the main definitions that are needed in the following Chapters.

2.1 Completely positive linear maps

Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras with unit $\mathbb{1}_{\mathcal{A}}$ and $\mathbb{1}_{\mathcal{B}}$ respectively. We will denote the unit with simply $\mathbb{1}$ whenever it is clear from the context whether it belongs to $\mathcal{A}$ or $\mathcal{B}$.

Definition 2.1.1. A linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is said to be

1. *n-positive* if for every $a_1, \ldots, a_n$ in $\mathcal{A}$ and every $b_1, \ldots, b_n$ in $\mathcal{B}$ the following holds
$$\sum_{i,j=1}^{n} b_i^* \phi(a_i^* a_j) b_j \geq 0;$$
2. *completely positive* if it is n-positive for every integer $n \geq 1$;
3. *Schwartz* if for every $a \in \mathcal{A}$
$$\phi(a)^* \phi(a) \leq \phi(a^* a)$$

The following Proposition follows easily from this definition.

Proposition 2.1.1. *If a linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, then it is completely positive.*

Proof. Let $a_1, \ldots, a_n$ in $\mathcal{A}$ and $b_1, \ldots, b_n$ in $\mathcal{B}$ then we have

$$\sum_{i,j=1}^{n} b_i^* \phi(a_i^* a_j) b_j = \sum_{i,j=1}^{n} (\phi(a_i) b_i)^* (\phi(a_j) b_j) \geq 0$$

since ϕ is a *-homomorphism. Finally, this expression holds for every n, thus we have the thesis. $\square$

To further characterize these properties of linear maps between von Neumann algebras, we introduce the set of $n \times n$ matrices, that we denote by $\mathcal{M}_n$, and its elements E_{ij} defined as

$$(E_{ij})_{hk} = \begin{cases} 1, & \text{if } i = h \text{ and } k = j \\ 0, & \text{otherwise} \end{cases}$$

for every $i, j = 1, \dots, n$. Particularly useful to us are the algebraic tensor products $\mathcal{A} \otimes \mathcal{M}_n$ and $\mathcal{B} \otimes \mathcal{M}_n$, which can be represented as the $n \times n$ matrices with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively. Elements of such tensor products admit a simple representation; indeed, every element $x \in \mathcal{A} \otimes \mathcal{M}_n$ can be written as

$$x = \sum_{i,j=1}^{n} x_{ij} \otimes E_{ij} \tag{2.1}$$

for some $x_{ij} \in \mathcal{A}$. Similarly, we can extend a linear map $\phi \colon \mathcal{A} \to \mathcal{B}$ to a map $\phi^{(n)} \colon \mathcal{A} \otimes \mathcal{M}_n \to \mathcal{B} \otimes \mathcal{M}_n$ for every $n \geq 1$ by defining it on a decomposable element such as $a \otimes E_{ij}$ as follows

$$\phi^{(n)}(a \otimes E_{ij}) = \phi(a) \otimes E_{ij} \tag{2.2}$$

These extended map allow us to give a more useful characterization of complete positivity which is equivalent to the previous one. In order to do so, we need first to prove the following Proposition

Proposition 2.1.2. *Let $\mathcal{A}$ be a von Neumann algebra and $x \in \mathcal{A} \otimes \mathcal{M}_n$ with $x = \sum_{i,j=1}^{n} x_{ij} \otimes E_{ij}$. Then, the following conditions are equivalent*

1. *x is positive;*
2. *x can be written as a finite sum of matrices of the form*

$$\sum_{i,j=1}^{n} a_i^* a_j \otimes E_{ij}$$

   where $a_1, \dots, a_n$ in $\mathcal{A}$;
3. *for every $a_1, \dots, a_n$ in $\mathcal{A}$ the following holds*

$$\sum_{i,j=1}^{n} a_i^* x_{ij} a_j \geq 0$$

Proof. $1 \implies 2$ Since x is a positive element it can be written as $y^* y$ for some $y \in \mathcal{A} \otimes \mathcal{M}_n$ (see, [11, Theorem 2.2.10]). Writing y as in (2.1) we obtain

$$x = \sum_{l=1}^{n} \sum_{i,j=1}^{n} y_{l_i}^* y_{l_j} \otimes E_{ij}$$

$2 \implies 3$ Trivial.
$3 \implies 1$ The von Neumann algebra $\mathcal{A}$ can be represented as a sub-algebra of all bounded operators on some Hilbert space $\mathcal{H}$ (see [11, Theorem 2.1.10]).

Such representation admits a cyclic vector v, and therefore Condition 3 directly implies

$$\sum_{i,j=1}^{n} \langle a_i v, x_{ij} a_j v\rangle \geq 0$$

Since v is cyclic its associated cyclic subspace is dense in $\mathcal{H}$ and therefore $w_i = a_i v \in \mathcal{H}$ for every $i = 1, \dots, n$. Thus we conclude $\sum_{i,j=1}^{n} \langle w_i, x_{ij} w_j\rangle \geq 0$ for every $w_1, \dots, w_n$. □

With this result we are ready to recast the complete positivity condition for a linear map ϕ in terms of its extension $\phi^{(n)}$.

Proposition 2.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras and let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a linear map. Then the following conditions are equivalent*

1. *ϕ is completely positive;*
2. *for every integer $n \geq 1$ the map $\phi^{(n)}$ defined in Equation (2.2) is positive.*

Proof. 2 $\implies$ 1 Consider the element $x = \sum_{i,j=1}^{n} a_i^* a_j \otimes E_{ij} \in \mathcal{A} \otimes \mathcal{M}_n$, by Item 2 of Proposition 2.1.2 we know it is positive and therefore also $\phi^{(n)}(x) = \sum_{i,j=1}^{n} \phi(a_i^* a_j) \otimes E_{ij}$ is positive. We conclude by Item 3 of Proposition 2.1.2.
1 $\implies$ 2 Since ϕ is completely positive $\sum_{i,j=1}^{n} \phi(a_i^* a_j) \otimes E_{ij}$ is positive and therefore $\phi^{(n)}$ is positive by the equivalence of Item 1 and 2 of Proposition 2.1.2. □

This condition can be further simplified whenever we suppose more structure on the target algebra $\mathcal{B}$.

Proposition 2.1.4. *Let $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a linear map between a von Neumann algebra $\mathcal{A}$ and $\mathcal{B}(\mathcal{K})$ the algebra of all bounded operators on a Hilbert space $\mathcal{K}$. Then ϕ is completely positive if and only if for every integer $n \geq 1$, for every $a_1, \dots, a_n \in \mathcal{A}$ and for every $v_1, \dots, v_n \in \mathcal{K}$ the following holds*

$$\sum_{i,j=1}^{n} \langle v_i, \phi(a_i^* a_j) u_j\rangle \geq 0$$

Proof. First of all notice that the algebraic tensor product $\mathcal{B}(\mathcal{K}) \otimes \mathcal{M}_n$ can be represented as the algebra of bounded operators on $\bigoplus_{i=1}^{n} \mathcal{K}$. Thus the above conditions is equivalent to requiring positivity of $\phi^{(n)}$ for every positive integer, which in turn is equivalent to complete positivity of ϕ by Proposition 2.1.3. □

In order to find the relationships that exist between 2-positivity, complete positivity, Markov and Schwarz, we give the following characterization of 2-positive linear maps.

Proposition 2.1.5. *Let $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a linear 2-positive map. Then the following properties are all satisfied*

1. *if $\phi(\mathbb{1})$ is invertible in $\mathcal{B}$, the we have the Schwarz inequality for all $a \in \mathcal{A}$*

$$\phi(a^*)\phi(\mathbb{1})^{-1}\phi(a) \leq \phi(a^* a)$$

2. *for all* $a \in \mathcal{A}$ *we have*

$$\phi(a^*)\phi(a) \leq \|\phi(\mathbb{1})\|\phi(a^*a)$$

3. ϕ *is continuous and satisfies*

$$\|\phi\| = \|\phi(\mathbb{1})\|$$

Proof. Given $\epsilon > 0$, consider the following operator in $\mathcal{B}(\mathcal{K}) \otimes \mathcal{M}_2$

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*) \\ \phi(a) & \phi(\mathbb{1}) + \epsilon\mathbb{1} \end{pmatrix} = \phi^{(2)} \begin{pmatrix} a^*a & a^* \\ a & \mathbb{1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \epsilon\mathbb{1} \end{pmatrix}$$

Such operator is positive for every $\epsilon > 0$ since it is the sum of two positive operators. Thus, for every $v, w \in \mathcal{K}$ we have

$$\langle v, \phi(a^*a)v\rangle + \langle v, \phi(a^*)w\rangle + \langle w, \phi(a)v\rangle + \langle w, (\phi(\mathbb{1}) + \epsilon\mathbb{1})\, w\rangle \geq 0$$

Now, since $\phi(\mathbb{1})$ is positive the operator $\phi(\mathbb{1}) + \epsilon\mathbb{1}$ has bounded inverse and therefore we can take $w = -(\phi(\mathbb{1}) + \epsilon\mathbb{1})^{-1}\phi(a)v$ which yields the following inequality

$$\left\langle w, \phi(a^*)(\phi(\mathbb{1}) + \epsilon\mathbb{1})^{-1}\phi(a)w \right\rangle \leq \langle w, \phi(a^*a)w\rangle$$

for every $w \in \mathcal{K}$. If we suppose $\phi(\mathbb{1})$ to be invertible then we directly obtain Item 1 of the Proposition by taking the limit $\epsilon \to 0$. Moving to Item 2, first of all we note that

$$\mathbb{1} \leq \|\phi(\mathbb{1}) + \epsilon\mathbb{1}\|(\phi(\mathbb{1}) + \epsilon\mathbb{1})^{-1}$$

thus we have

$$\phi(a^*)\phi(a) \leq \|\phi(\mathbb{1}) + \epsilon\mathbb{1}\|\phi(a^*)(\phi(\mathbb{1}) + \epsilon\mathbb{1})^{-1}\phi(a) \leq \|\phi(\mathbb{1}) + \epsilon\mathbb{1}\|\phi(a^*a)$$

taking again the limit $\epsilon \to 0$ we obtain Item 2. Finally, recalling that for any element x of some C^* algebra we have the property $\|x^*x\| = \|x\|^2$, together with Item 2 we have

$$\begin{aligned} \|\phi(a)\|^2 &= \|\phi(a^*)\phi(a)\| \\ &\leq \|\phi(\mathbb{1})\|\|\phi(a^*a)\| \\ &\leq \|\phi(\mathbb{1})\|\|\phi(\|a\|^2\mathbb{1})\| \\ &\leq \|a\|^2\|\phi(\mathbb{1})\|^2 \end{aligned}$$

thus, dividing by $\|a\|^2$ and taking the sup over $a \in \mathcal{A}$ we have Item 3. □

Corollary 2.1.1. *Proposition 2.1.5 directly implies the following result. If* $\phi\colon \mathcal{A} \to \mathcal{B}$ *is a completely positive linear map and is also Markov, i.e.* $\phi(\mathbb{1}) = \mathbb{1}$, *then it is Schwarz.*

Finally, we give a last definition closely related to complete positivity that we are going to need in the rest of the Thesis.

Definition 2.1.2. A linear map ϕ on a von Neumann algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called *conditionally positive* if given $au = 0$ with $aa^* \in D(\phi)$ and $u \in \mathcal{H}$ implies $\langle u, \phi(aa^*)u\rangle \geq 0$.

Now that we have recollected the basic properties of completely positive maps we are ready to give the following Theorem which describe any completely positive map in terms of representations of C^*-algebras.

Theorem 2.1.1 (Stinespring). *Let $\mathcal{A}$ be a C^*-algebra with unit and $\mathcal{B}$ a sub C^*-algebra of the algebra of all bounded operators $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Then a linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is completely positive if and only if it can be written as*

$$\phi(a) = V^*\pi(a)V \tag{2.3}$$

where $(\pi, \mathcal{H})$ is a representation of $\mathcal{A}$ on $\mathcal{H}$ for some Hilbert space $\mathcal{H}$ and $V\colon \mathcal{H} \to \mathcal{K}$ is a bounded operator.

Proof. Let ϕ as in Equation (2.3) and $(a_{ij})_{i,j=1,\ldots,n}$ be a positive element in $\mathcal{A} \otimes \mathcal{M}_n$. Then such a map is completely positive; indeed, for all vectors $v_1, \ldots, v_n \in \mathcal{K}$ we have that

$$\sum_{i,j=1}^{n} \langle v_i, \phi(a_{ij})v_j\rangle = \sum_{i,j=1}^{n} \langle Vv_i, \pi(a_{ij})Vv_j\rangle \geq 0$$

by Proposition 2.1.1 since π is a *-homomorphism, and this condition is equivalent to complete positivity by Proposition 2.1.4. We now prove the converse. Consider a completely positive map ϕ and the vector space given by the algebraic tensor product $\mathcal{A} \otimes \mathcal{K}$. Given any two elements $x = \sum_i a_i \otimes v_i$ and $y = \sum_i b_i \otimes w_i$ in $\mathcal{A} \otimes \mathcal{K}$, we can define a bilinear form $(\cdot\,,\cdot)$ as follows

$$(x, y) = \sum_{i,j} \langle v_i, \phi(a_i^* b_j)w_j\rangle$$

By the complete positivity of ϕ we immediately get

$$(x, x) = \sum_{i,j} \langle v_i, \phi(a_i^* a_j)v_j\rangle \geq 0$$

and therefore the bilinear form we just defined is positive. Let π_0 be the algebra homomorphism between $\mathcal{A}$ and the linear operators on $\mathcal{A} \otimes \mathcal{K}$ defined as

$$\pi_0(a)\left(\sum_i a_i \otimes v_i\right) = \sum_i (aa_i) \otimes v_i$$

then for any $x, y \in \mathcal{A} \otimes \mathcal{K}$ we have

$$(x, \pi_0(a)y) = (\pi_0(a^*)x, y).$$

Thus, the linear map

$$\omega\colon \mathcal{A} \ni a \mapsto (x, \pi_0(a)x) \in \mathbb{C}$$

is a positive linear functional over $\mathcal{A}$. Then, by [11, Proposition 2.3.11] we have that

$$\|\pi_0(a)x\|^2 = (x, \pi_0(a^*a)x) \leq \|a^*a\|\omega(\mathbb{1}) = \|a\|^2\|x\|^2$$

Let $\mathcal{N}$ be the kernel of the bilinear form we just defined. By the last inequality we proved, it follows that $\mathcal{N}$ is invariant under the action of $\pi_0(a)$ for every

$a \in \mathcal{A}$. We can therefore consider the pre-Hilbert space given by the quotient $\mathcal{A} \otimes \mathcal{K}/\mathcal{N}$ with pre-scalar product defined as

$$(x + \mathcal{N}, x + \mathcal{N}) = (x, x)$$

If we let $\mathcal{H}$ be the Hilbert space obtained by completion, the *-homomorphism π_0 extends to a representation π of $\mathcal{A}$ on $\mathcal{H}$ defined as

$$\pi(a)(x + \mathcal{N}) = \pi_0(a)x + \mathcal{N}$$

for every $a \in \mathcal{A}$ and every $x \in \mathcal{A} \otimes \mathcal{K}$. Finally, defining the operator V as

$$Vv = \mathbb{1} \otimes v + \mathcal{N}$$

is bounded, indeed

$$\|Vv\|^2 = \langle v, \phi(\mathbb{1})v\rangle \leq \|\phi(\mathbb{1})\|\|v\|^2$$

therefore we have the thesis. □

In the remaining of this Thesis we will call the pair π, V a *Stinespring representation* of the completely positive map ϕ. With the following Proposition we show that Stinespring representations are not unique.

Proposition 2.1.6. *Let π_1 and π_2 be two representation of $\mathcal{A}$ on the Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, and let $V_i \colon \mathcal{H} \to \mathcal{K}_i$ be two bounded operators such that*

$$\{\, \pi_i(a)V_i v \mid a \in \mathcal{A},\ v \in \mathcal{H} \,\}$$

is total in $\mathcal{K}_i$ for $i = 1, 2$ and such that

$$\phi(a) = V_i^* \pi_i(a) V_i$$

for $i = 1, 2$. Then there exists a unitary map $U \colon \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$UV_1 = V_2 \qquad \text{and} \qquad U\pi_1(a) = \pi_2(a)U$$

for every $a \in \mathcal{A}$.

Proof. We start by defining the linear map $U \colon \mathcal{K}_1 \to \mathcal{K}_2$ as

$$U\left(\sum_{j=1}^{n} \pi_1(a_j)V_1 v_j\right) = \sum_{j=1}^{n} \pi_2(a_j)V_2 v_j$$

for every integer $n \geq 1$, every $a_1, \dots a_n \in \mathcal{A}$ and every $v_1, \dots v_n$. By a direct computation we get

$$\langle U\pi_1(b)V_1 v, U\pi_1(a)V_1 w\rangle_{\mathcal{K}_2} = \langle v, \phi(b^* a)w\rangle_{\mathcal{H}} = \langle V_1 v, \pi_1(b^* a)V_1 w\rangle_{\mathcal{K}_1}$$

for all $a, b \in \mathcal{A}$ and $v, w \in \mathcal{H}$. Therefore U is an isometry, and similarly it can be shown that $U^* \colon \mathcal{K}_2 \to \mathcal{K}_1$ is an isometry, and therefore U is unitary. Concluding, since

$$UV_1 v = U\pi_1(\mathbb{1})V_1 v = \pi_2(\mathbb{1})V_2 v = V_2 v \quad \text{and} \quad U\pi_1(a)V_1 v = \pi_2(a)V_2 v$$

for all $v \in \mathcal{H}$, the thesis follows. □

We complete this section by a characterization of completely positive maps between von Neumann algebras as given by Kraus in [12]. Before stating the main Theorem, we need the following Lemma.

Lemma 2.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. A normal completely map $\phi\colon \mathcal{A} \to \mathcal{B}$ can be written as*

$$\phi(a) = V^*\pi(a)V$$

where V is a bounded operator from $\mathcal{K}$ to a Hilbert space $\mathcal{K}_1$ and π is a normal representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{K}_1)$.

Proof. Let π, V be a minimal Stinespring representation of ϕ, where $V : \mathcal{K} \to \mathcal{K}_1$. The only thing left to prove is that π is normal. Let $(x_i)_i$ be an increasing net of operators in $\mathcal{A}$ such that $\sup_i x_i = x \in \mathcal{A}$. Then $(x_i)_i$ converges to x in the σ-weak topology. For all vector $u, v \in \mathcal{K}$ and all operators $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\lim_i \langle \pi(b)Vv, \pi(x_i)\pi(a)Vu\rangle &= \lim_i \langle Vv, \pi(b^*x_ia)Vu\rangle \\
&= \lim_i \langle v, \phi(b^*x_ia)u\rangle \\
&= \langle v, \phi(b^*xa)u\rangle \\
&= \langle \pi(b)Vv, \pi(x)\pi(a)Vu\rangle
\end{aligned}$$

since ϕ is normal, therefore so is π. □

Theorem 2.1.2 (Kraus)**.** *Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}$ be another Hilbert space. A linear map $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is normal and completely positive if and only if it can be represented in the form*

$$\phi(a) = \sum_{i=1}^{\infty} V_i^* a V_i \tag{2.4}$$

where the operators $V_i\colon \mathcal{K} \to \mathcal{H}$ are linear bounded operators for every $i \geq 1$ such that the series $\sum_{i=1}^{\infty} V_i^* a V_i$ converges strongly for all $a \in \mathcal{A}$.

Proof. Consider a completely positive map ϕ as in Equation (2.4). Such a map is normal; indeed, consider a non-decreasing net $(x_i)_i$ of positive operators in $\mathcal{A}$ converging strongly to $x \in \mathcal{A}$, then for every $u \in \mathcal{K}$ we have

$$\begin{aligned}
\sup_i \langle u, \phi(x_i)u\rangle &= \sum_j \sup_i \langle V_j u, x_i V_j u\rangle \\
&= \sum_j \langle V_j u, x V_j u\rangle \\
&= \langle u, \phi(x)u\rangle
\end{aligned}$$

We now show the converse. Consider the representation of a completely positive map $\phi(a) = V^*\pi(a)V$ with a normal map π as in Lemma 2.1.1. We just need to recast such representation as in Equation (2.4). By decomposing $\mathcal{K}_1$ into cyclic orthogonal subspaces we can suppose that there exists a cyclic vector $w \in \mathcal{K}_1$ for $\pi(\mathcal{A})$. The state on $\mathcal{A}$

$$a \mapsto \langle w, \pi(a)w\rangle$$

is normal because so is π. Therefore (see [11, Theorem 2.4.21]) there exists a sequence $(u_i)_{i\geq 1}$ of vectors in $\mathcal{H}$ such that

$$\sum_{i=1}^{\infty} \|u_i\|^2 = 1 \qquad \text{and} \qquad \langle w, \pi(a)w\rangle = \sum_{i=1}^{\infty} \langle u_i, au_i\rangle$$

for every $a \in \mathcal{A}$. Therefore there exists contractions $V_i \colon \mathcal{K}_1 \to \mathcal{K}$ such that

$$V_i\pi(x)w = xu_i.$$

We also have for all $x \in \mathcal{A}$

$$\begin{aligned}\langle \pi(x)w, \pi(a)\pi(x)w\rangle &= \langle w, \pi(x^*ax)w\rangle \\ &= \sum_{i=1}^{\infty} \langle u_i, x^*axu_i\rangle \\ &= \sum_{i=1}^{\infty} \langle V_i\pi(x)w, aV_i\pi(x)w\rangle \\ &= \sum_{i=1}^{\infty} \langle \pi(x)w, V_i^*aV_i\pi(x)w\rangle\end{aligned}$$

and since w is cyclic for π this concludes the proof. □

2.2 Quantum Markov Semigroups

Having discussed the basics on the representation of completely positive maps, we can now discuss the main concepts about Quantum Markov Semigroups (QMS).

2.2.1 Definitions & Basic Results

Let $\mathcal{A}$ be a von Neumann algebra with unit $\mathbb{1}$ acting on a Hilbert space $\mathcal{H}$. The following definition completely characterize the concept of a Quantum Markov Semigroup.

Definition 2.2.1. A Quantum Markov Semigroup (QMS) on $\mathcal{A}$ is a family of bounded operators $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{A}$ satisfying the following properties

1. $\mathcal{T}_0(a) = a$ for all $a \in \mathcal{A}$;
2. $\mathcal{T}_{t+s}(a) = \mathcal{T}_t(\mathcal{T}_s(a))$ for all $t, s \geq 0$ and all $a \in \mathcal{A}$;
3. $\mathcal{T}_t$ is completely positive for all $t \geq 0$
4. $\mathcal{T}_t$ is σ-weakly continuous in $\mathcal{A}$ for all $t \geq 0$;
5. the map $t \mapsto \mathcal{T}_t(a)$ is continuous with respect to the σ-weak topology on $\mathcal{A}$ for every $a \in \mathcal{A}$;
6. $\mathcal{T}$ is Markov, i.e. $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$.

A QMS $\mathcal{T}$ admits what is called an *infinitesimal generator* which can be seen as a generalization of the time derivative of the QMS at $t = 0$. This fuzzy idea is formalized in the following definition.

Definition 2.2.2. Let $\mathcal{T}$ be a QMS. Then we define its *infinitesimal generator* as the operator $\mathcal{L}$ on $\mathcal{A}$ whose action is defined as

$$\mathcal{L}(a) \coloneqq w^* - \lim_{t\to 0} \frac{\mathcal{T}_t(a) - a}{t}$$

for every element $a \in \mathcal{A}$ for which the limits exists in the σ-weak topology.

The definitions we just gave describe time evolution for observables only, but these definitions can be moved onto state by duality as stated in the following.

Definition 2.2.3. The *predual* semigroup of a QMS $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$ is the semigroup denoted by $\mathcal{T}_*$ and defined on $\mathcal{A}_*$ as

$$(\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a))$$

for every $a \in \mathcal{A}$ and for every $\omega \in \mathcal{A}_*$.

The continuity hypothesis on $\mathcal{T}$ induce continuity properties on $\mathcal{T}_*$. Indeed, since $\mathcal{T}$ is continuous w.r.t. the σ-weak topology on $\mathcal{A}$, then the semigroup $\mathcal{T}_*$ is continuous w.r.t. the weak topology on the Banach space $\mathcal{A}_*$. Therefore $\mathcal{T}_*$ is a strongly continuous semigroup on $\mathcal{A}_*$ (see [11, Corollart 3.1.8]). Finally, since $\mathcal{T}$ is Markov, both $\mathcal{T}$ and $\mathcal{T}_*$ are semigroups of contractions by Proposition 2.1.5. The Markovian property allows us to also prove that $\mathcal{T}_*$ maps states into states. Indeed, given a normal state $\omega \in \mathcal{A}_*$, $\mathcal{T}_{*t}(\omega)$ is positive and

$$\|\mathcal{T}_{*t}(\omega)\| = (\mathcal{T}_{*t}(\omega))(\mathbb{1}) = \omega(\mathcal{T}_t(\mathbb{1})) = \omega(\mathbb{1}) = 1$$

which directly implies that $\mathcal{T}_{*t}(\omega)$ is again a state. Now that we have introduced an idea of time evolution on both the states and the observables of a quantum system, we can also define what it means to be invariant to such evolution.

Definition 2.2.4. A normal functional $\omega \in \mathcal{A}_*$ is said to *invariant* w.r.t. $\mathcal{T}$ (or $\mathcal{T}$-invariant for short) if ω is a fixed point of $\mathcal{T}_*$, i.e. $(\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a)) = \omega(a)$ for all $a \in \mathcal{A}$ and all $t \geq 0$. We denote with $\mathcal{F}(\mathcal{T}_*)$ the set of all such invariant functionals, and by $\mathcal{F}(\mathcal{T}_*)_1$ the set of all invariant states. Analogously, we can define the set of invariant observables as

$$\mathcal{F}(\mathcal{T}) \coloneqq \{\, a \in \mathcal{A} \mid \mathcal{T}_t(a) = a \;\forall\, t \geq 0 \,\}$$

Unfortunately, in general $\mathcal{F}(\mathcal{T})$ is not an algebra but it can be one with additional hypothesis. In particular we need the existence of a faithful family of normal invariant states, that is a family $\mathcal{G}$ of invariant states for which $a \in \mathcal{A}_+$ together with $\omega(a) = 0$ for all $\omega \in \mathcal{G}$ implies $a = 0$. Before proceeding, we need the following Lemma.

Lemma 2.2.1. *Let $\phi\colon \mathcal{A} \to \mathcal{A}$ be a completely positive map such that $\phi(\mathbb{1}) = \mathbb{1}$. Then, defining*

$$D(x, y) \coloneqq \phi(x^* y) - \phi(x)^* \phi(y)$$

is a positive sesquilinear form on $\mathcal{A}$. Moreover, for every $x, y \in \mathcal{A}$ we have

$$D(x, x) = 0 \quad \Longleftrightarrow \quad D(x, y) = 0$$

Proof. D is sesquilinear directly by its own definition. Also, D is positive by Schwarz inequality which is satisfied due to Corollary 2.1.1. Finally, applying the Cauchy-Schwarz inequality on the sesquilinear form $\omega(D(\cdot\,,\cdot))$ for some $\omega \in \mathcal{A}_*$, we obtain that $D(x,x) = 0$ implies $\omega(D(x,y)) = 0$ for all $\omega \in \mathcal{A}_*$ and all $y \in \mathcal{A}$ and therefore we have $D(x,y) = 0$ for all $y \in \mathcal{A}$. The converse is trivial and therefore we have the thesis. □

Proposition 2.2.1. *Suppose there exists a family $\mathcal{G}$ of faithful normal invariant states on $\mathcal{A}$. Then $\mathcal{F}(\mathcal{T})$ is a von Neumann sub-algebra of $\mathcal{A}$.*

Proof. Given an element $x \in \mathcal{F}(\mathcal{T})$ we have for all $t \geq 0$

$$\mathcal{T}_t(x^*x) = x^*x \tag{2.5}$$

Indeed, $\omega(\mathcal{T}_t(x^*x) - x^*x) = 0$ for all $\omega \in \mathcal{G}$ and all $t \geq 0$, but Schwarz inequality implies

$$\mathcal{T}_t(x^*x) - x^*x = \mathcal{T}_t(x^*x) - \mathcal{T}_t(x)^*\mathcal{T}_t(x) \geq 0 \quad \forall\, t \geq 0$$

and by the faithfulness of $\mathcal{G}$ we obtain Equation (2.5). Defining D_t as the sesquilinear form in Lemma 2.2.1 with $\phi = \mathcal{T}_t$, we have $D_t(b,b) = 0$ for all $b \in \mathcal{F}(\mathcal{T})$ and for all $t \geq 0$ by Equation (2.5). Then, by Lemma 2.2.1 $D_t(x,b) = 0$ for all $x \in \mathcal{A}$ and for all $t \geq 0$. In particular, given $a, b \in \mathcal{F}(\mathcal{T})$ we have $D_t(a^*, b) = 0$ implies

$$\mathcal{T}_t(ab) = \mathcal{T}_t(a)\mathcal{T}_t(b) = ab \qquad \forall\, t \geq 0$$

i.e. $ab \in \mathcal{F}(\mathcal{T})$. Thus, $\mathcal{F}(\mathcal{T})$ is a weakly* closed sub-algebra of $\mathcal{A}$, that is $\mathcal{F}(\mathcal{T})$ is a von Neumann sub-algebra of $\mathcal{A}$. □

Remark 2.2.1. It is important to note that, under the same hypotheses of Proposition 2.2.1, it possible to show (see [10, Proposition 17]) the predual of the algebra of fixed points of a QMS $\mathcal{T}$ is isomorphic to the algebra of fixed points of the predual of the QMS itself. More concisely, the following holds true

$$\mathcal{F}(\mathcal{T}_*) = \mathcal{F}(\mathcal{T})_*$$

2.2.2 Uniformly continuous QMSs

In this Thesis we focus mainly on a specific family of QMSs that presents a particular regularity property described in the following Definition.

Definition 2.2.5. A QMS $\mathcal{T}$ is said to be *uniformly continuous* if

$$\lim_{t\to 0} \|\mathcal{T}_t - \mathbb{1}\| = 0$$

Such QMSs are interesting to us since they have additional properties thanks to their regularity as detailed in the following Proposition.

Proposition 2.2.2. *Let $\mathcal{T}$ be a semigroup of bounded operators on a Banach space E. Then the following conditions are equivalent:*

1. *the map $t \mapsto \mathcal{T}_t$ is uniformly continuous;*
2. *the map $t \mapsto \mathcal{T}_t$ is uniformly differentiable;*

3. *the infinitesimal generator $\mathcal{L}$ is a bounded operator such that the series*

$$\mathcal{T}_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}^n$$

is uniformly convergent for every $t \in \mathbb{R}$.

If any of these conditions are satisfied, then $\mathcal{T}$ can be extended to a uniformly continuous group of operators on E such that

$$\|\mathcal{T}_t\| \leq e^{|t|\|\mathcal{L}\|}$$

See [11, Proposition 3.1.1] for a proof. The regularity of QMS can also be linked to regularity of its infinitesimal generator as stated in the following Proposition.

Proposition 2.2.3. *Let $\mathcal{T}$ be a uniformly continuous semigroup of bounded operators $\mathcal{T}_t$ on a von Neumann algebra $\mathcal{A}$, and let $\mathcal{L}$ be its infinitesimal generator. Then the following are equivalent:*

1. *$\mathcal{T}_t$ is σ-weakly continuous for every $t \geq 0$;*

2. *$\mathcal{L}$ is σ-weakly continuous.*

In a similar way as we did to define complete positivity starting from positivity, we can define the following property associated to conditional positivity.

Definition 2.2.6. A bounded linear operator $\mathcal{L}$ on a von Neumann algebra $\mathcal{A}$ is called *conditionally completely positive* if for every integer $n \geq 1$, the linear map $\mathcal{L}^{(n)}$ defined on $\mathcal{A} \otimes \mathcal{M}_n$ as $\mathcal{L}^{(n)}(a \otimes E_{ij}) = \mathcal{L}(a) \otimes E_{ij}$ with $i, j = 1, \ldots, n$ satisfies the inequality

$$\mathcal{L}^{(n)}(x^*x) - x^*\mathcal{L}^{(n)}(x) - \mathcal{L}^{(n)}(x^*)x + x^*\mathcal{L}^{(n)}(\mathbb{1})x \geq 0 \tag{2.6}$$

for every $x \in \mathcal{A} \otimes \mathcal{M}_n$.

The following Lemma shows a connection between conditional completely positivity and conditional positivity.

Lemma 2.2.2. *Let $\mathcal{L}$ be conditionally completely positive operator on a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$. Then for every integer $n \geq 1$, every $a_1, \ldots, a_n$ in $\mathcal{A}$ and $u_1, \ldots, u_n$ in $\mathcal{H}$ such that*

$$\sum_{i=1}^{n} a_i u_i = 0$$

we have

$$\sum_{i,j=1}^{n} \langle u_i, \mathcal{L}(a_i^* a_j) u_j \rangle \geq 0$$

Proof. Consider the element $x = \sum_{j=1}^{n} a_j \otimes E_1^j \in \mathcal{A} \otimes \mathcal{M}_n$ and the vector $u = (u_1, \ldots, u_n) \in \oplus_{i=1}^{n} \mathcal{H}$, then we have

$$\begin{aligned} 0 &\le \left\langle u, \left(\mathcal{L}^{(n)}(x^*x) - x^*\mathcal{L}^{(n)}(x) - \mathcal{L}^{(n)}(x^*)x + x^*\mathcal{L}^{(n)}(\mathbb{1})x\right) u\right\rangle \\ &= \sum_{i,j=1}^{n} \langle u_i, (\mathcal{L}(a_i^*a_j) - a_i^*\mathcal{L}(a_j) - \mathcal{L}(a_i^*)a_j + a_i^*\mathcal{L}(\mathbb{1})a_j)\, u_j\rangle \\ &= \sum_{i,j=1}^{n} \langle u_i, \mathcal{L}(a_i^*a_j)u_j\rangle \end{aligned}$$

which concludes the proof. □

Proposition 2.2.4. *Let $\mathcal{T}$ be a uniformly continuous semigroup on a von Neumann algebra $\mathcal{A}$ with infinitesimal generator $\mathcal{L}$. Then $\mathcal{T}_t$ is completely positive for every $t \ge 0$ if and only if $\mathcal{L}$ is conditionally completely positive and $\mathcal{L}(a^*) = \mathcal{L}(a)^*$ for every $a \in \mathcal{A}$. Let $\mathcal{H}$ be an Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$. Then a linear map $\mathcal{L}$ on $\mathcal{B}(\mathcal{H})$ such that $\mathcal{L}(a^*) = \mathcal{L}(a)^*$ for every $a \in \mathcal{B}(\mathcal{H})$ is conditionally completely positive if and only if there exists a completely positive map ϕ on $\mathcal{B}(\mathcal{H})$ and an element $G \in \mathcal{B}(\mathcal{H})$ such that*

$$\mathcal{L}(a) = \phi(a) + G^*a + aG \tag{2.7}$$

for every $a \in \mathcal{B}(\mathcal{H})$. Moreover, G is also such that

$$G + G^* \le \mathcal{L}(\mathbb{1})$$

Note that the choice of the operator G in Equation (2.7) is not unique. Indeed, defining $G' = G - c\mathbb{1}$ for any $c > 0$ it is possible to decompose $\mathcal{L}$ as

$$\mathcal{L}(a) = (\phi + 2c\mathbb{1})(a) + G'^*a + aG'$$

which is an admissible choice since $\phi + 2c$ is again completely positive.

We can now state the most important decomposition and characterization on the infinitesimal generator $\mathcal{L}$ of a QMS $\mathcal{T}$ due to Lindblad [13].

Theorem 2.2.1 (Lindblad)**.** *Let $\mathcal{T}$ be a uniformly continuous semigroup on $\mathcal{B}(\mathcal{H})$. Then $\mathcal{T}$ is a QMS if and only if there exists a complex separable Hilbert space $\mathcal{K}$, a bounded operator $L\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ and an operator G on $\mathcal{H}$ such that its (bounded) generator is given by*

$$\mathcal{L}(a) = L^*(a \otimes \mathbb{1})L + G^*a + aG \tag{2.8}$$

for all $a \in \mathcal{B}(\mathcal{H})$. The operator L can be chosen so that the set

$$\{\, (a \otimes \mathbb{1})Lu \mid a \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H} \,\}$$

is total in $\mathcal{H} \otimes \mathcal{K}$.

Proof. It $\mathcal{T}$ is a QMS then the infinitesimal generator $\mathcal{L}$ is conditionally completely positive by the first part of Proposition 2.2.4 and can therefore be represented in the same form as in Equation (2.7) by the second part of Proposition 2.2.4, and is also σ-weakly continuous by Proposition 2.2.3. Since the map $a \mapsto G^*a + aG$ is σ-weakly continuous, by Kraus' Theorem 2.1.2 applied to the map ϕ in Equation 2.7 we obtain the desired representation of $\mathcal{L}$. Conversely, if an operator $\mathcal{L}$ can be written as in Equation 2.8 then it is σ-weakly continuous and conditionally completely positive by Proposition 2.2.4. Thus we can say that is the infinitesimal generator of a uniformly continuous QMS by Proposition 2.2.3 and Proposition 2.2.4. □

The following corollary summarizes the properties and the characterization of the infinitesimal generator of a uniformly continuous QMS obtained so far.

Corollary 2.2.1. *Let $\mathcal{T}$ be a uniformly continuous QMS on a von Neumann algebra $\mathcal{A}$, and let $\mathcal{L}$ be a its infinitesimal generator. Then the following statements hold:*

1. *$\mathcal{L}$ is σ-weakly continuous;*
2. *$\mathcal{L}$ is conditionally completely positive;*
3. *$\mathcal{L}(a^*) = \mathcal{L}(a)^*$ for all $a \in \mathcal{A}$.*

Moreover, if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then there exist $G \in \mathcal{B}(\mathcal{H})$ and a sequence of operators $(L_l)_{l \geq 1}$ in $\mathcal{B}(\mathcal{H})$ such that

$$\mathcal{L}(a) = \sum_{l=1}^{\infty} L_l^* a L_l + G^* a + aG \tag{2.9}$$

for all $a \in \mathcal{B}(\mathcal{H})$ and

$$\sum_{l=1}^{\infty} L_l^* L_l + G^* + G = 0$$

Proof. We already prove everything in the previous Propositions and Theorems, we are only left to prove that the decomposition of the infinitesimal generator $\mathcal{L}$ given in Equation (2.7) can be rewritten as in Equation (2.9). Consider the orthonormal basis $(e_i)_{i=1}^{\dim \mathcal{K}}$ of the Hilbert space $\mathcal{K}$ introduced in Theorem 2.2.1 and defined the operators $L_l = \pi_l \circ L$ where

$$\begin{aligned} \pi_l \colon \mathcal{H} \otimes \mathcal{K} &\longrightarrow \mathcal{H} \\ u \otimes e_i &\longmapsto u\delta_{il} \end{aligned}$$

for every $l \geq 1$. Then the action of L can be decomposed as $Lu = \sum_{i=l}(L_l u) \otimes e_l$ for every $u \in \mathcal{H}$ so that $L^* a L = \sum_{l=1}^{\infty} L_l^* a L_l$ for all $a \in \mathcal{B}(\mathcal{H})$ and therefore Equation 2.9 follows directly. Finally, since $\mathcal{T}$ is Markov, we have

$$\mathcal{L}(\mathbb{1}) = \sum_{l=1}^{\infty} L_l^* L_l + G^* + G = 0$$

□

To conclude this subsection, note that whenever $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the operator G can be decomposed as

$$G = -\frac{1}{2} \sum_l L_l^* L_l - iH$$

for some bounded self-adjoint operator on $\mathcal{H}$. Therefore we can rewrite what is usually called the Lindblad form of $\mathcal{L}$ as

$$\mathcal{L}(a) = -\frac{1}{2} \left(\sum_l L_l^* L_l a + \sum_l a L_l^* L_l - 2 \sum_l L_l^* a L_l \right) + i[H, a] \tag{2.10}$$

Theorem 2.2.2. *Let $\mathcal{L}$ be the generator of a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$. Then there exists a bounded self-adjoint operator H a sequence $(L_k)_{k\geq 1}$ of elements in $\mathcal{B}(\mathcal{H})$ such that*

1. *$\sum_{k\geq 1} L_k^* L_k$ is strongly convergent;*
2. *if $\sum_{k\geq 1} |c_k|^2 < \infty$ and $c_0 \mathbb{1} + \sum_{k\geq 1} c_k L_k = 0$ for some sequence of scalars $(c_k)_{k\geq 0}$, then $c_k = 0$ for every $k \geq 0$;*
3. *$\mathcal{L}(a) = -\frac{1}{2}\left(\sum_l L_l^* L_l a + \sum_l a L_l^* L_l - 2\sum_l L_l^* a L_l\right) + i[H,a]$ for every $x \in \mathcal{B}(\mathcal{H})$.*

Moreover, if H' and $(L'_k)_{k\geq 1}$ is another set of bounded operators in $\mathcal{B}(\mathcal{H})$ with H' self-adjoint, then it satisfies Item 1 – 3 if and only if the length of the sequences $(L_k)_{k\geq 1}$ and $(L'_k)_{k\geq 1}$ are is the same and there exists a sequence of complex numbers $(\alpha_k)_{k\geq 1}$ with $\sum_{k\geq 1} \alpha_k < \infty$ and $\beta \in \mathbb{R}$ such that

$$H' = H + \beta\mathbb{1} + \frac{1}{2i}(S - S^*) \qquad L'_K = \sum_{j\geq 1} u_{kj} L_j + \alpha_k \mathbb{1} \tag{2.11}$$

for some unitary matrix $U = (u_{kj})_{kj}$ and where $S \coloneqq \sum_{j,k\geq 1} \bar{\alpha}_k u_{kj} L_j$.

Definition 2.2.7. Let $\mathcal{T}$ be a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$, and let $\mathcal{L}$ be its generator. Then a set of operators $\{(L_k)_{k\geq 1}, H\}$ such that Theorem 2.2.2 holds true is called a *GKSL* representation of $\mathcal{L}$.

2.3 Subharmonic projections

In this Section we are going to show the strong connection between the structure and properties of a QMS and its action on the projections contained in the von Neumann algebra. In order to do so, we first need to give a characterization of operators in the von Neumann algebra depending on the way a QMS $\mathcal{T}$ acts on them as in the following Definition.

Definition 2.3.1. Let $\mathcal{T}$ be a QMS on a von Neumann algebra. Then a positive operator $a \in \mathcal{A}$ is said to be

1. *subharmonic* if $\mathcal{T}_t(a) \geq a$;
2. *harmonic* if $\mathcal{T}_t(a) = a$;
3. *superharmonic* if $\mathcal{T}_t(a) \leq a$

for all $t \geq 0$.

Subharmonic projections will be fundamental in the proving the main results of this thesis. For this reason will be now stating some of the most important properties about them.

Lemma 2.3.1. *Let $a, p \in \mathcal{A}$, and p a projection. If $a \geq 0$ and $p^\perp a p^\perp = 0$, then $a = pap$.*

Proof. Let $u \in p\mathcal{H}^\perp$ and $v \in p\mathcal{H}$. Since a is positive, we have

$$\langle \lambda u + v, a(\lambda u + v)\rangle = 2\,\mathrm{Re}\,\langle \lambda u, av\rangle + \langle v, av\rangle \geq 0 \quad \lambda \in \mathbb{C} \tag{2.12}$$

But, if $\lambda \in \mathbb{R}$, Equation (2.12) implies $\mathrm{Re}\,\langle u, av\rangle = 0$, while if $\lambda \in i\mathbb{R}$ the same Equation implies $\mathrm{Im}\,\langle u, av\rangle = 0$. Therefore $\langle u, va\rangle = 0$ for all $u \in p\mathcal{H}^\perp$ and $v \in p\mathcal{H}$, i.e. $p^\perp ap = 0$; similarly it possible to prove $pap^\perp = 0$ which implies $a = pap$. □

Definition 2.3.2. Let ω a linear positive functional on $\mathcal{A}$. If $\omega(x^*x) \neq 0$ for every non zero $x \in \mathcal{A}$, then ω is said to be *faithful*.

Definition 2.3.3. Let ω be a normal, positive linear functional on $\mathcal{A}$ and define

$$L \coloneqq \{\, a \in \mathcal{A} \mid \omega(a^*a) = 0 \,\}$$

Then L is weakly closed left ideal, and therefore $L = \mathcal{A}p$ for some projection $p \in \mathcal{A}$ (see [14, Definition 1.14.2]). The orthogonal projection $p^\perp$ is called the *support* of ω and is denoted by $s(\omega)$.

Proposition 2.3.1. *Let ω be a positive normal functional on $\mathcal{A}$ and $s(\omega)$ its support, then ω is faithful on $s(\omega)\mathcal{A}s(\omega)$.*

Proof. Let $p = s(\omega)$ and assume $\omega(a) = 0$ for some $a \in p\mathcal{A}p_+$. Denoting with q_n the spectral decomposition of a in the interval $(1/n, \|a\|]$ with $n \geq 1$, we have

$$\omega(q_n) \leq n\omega(a) = 0 \qquad \forall n \geq 1$$

which implies $q_n \leq p^\perp$ for all $n \geq 1$. Since $q_n \leq na \leq n\|a\|p$, this means that $q_n = 0$ for all $n \geq 1$, which in turn implies that $q = \sup_n q_n = 0$. But given that q is the projection onto the closure of the range of a, we conclude that $a = 0$. □

Theorem 2.3.1. *The support projection of a normal invariant state for a QMS is subharmonic.*

Proof. Let ω be a normal invariant state on $\mathcal{A}$, $p = s(\omega)$, and fix $t \geq 0$. By the invariance of ω we get

$$\omega(p - p\mathcal{T}_t(p)p) = \omega(p - \mathcal{T}_t(p)) = 0$$

that implies $p\mathcal{T}_t(p)p = p$ because $p\mathcal{T}_t(p)p \leq p$ and ω is faithful on $p\mathcal{A}p$ according to Proposition 2.3.1. Therefore the projection $p^\perp$ satisfies $p\mathcal{T}_t(p^\perp)p = 0$, so that $\mathcal{T}_t(p^\perp) = p^\perp\mathcal{T}_t(p^\perp)p^\perp$ by Lemma 2.3.1. It follows that $\mathcal{T}_t(p^\perp) \leq p^\perp$ and consequently $\mathcal{T}_t(p) \geq p$. □

Proposition 2.3.2. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$ and p a projection in $\mathcal{A}$. Then the following properties are equivalent:*

1. *$p\mathcal{A}p$ is $\mathcal{T}$-invariant;*
2. *$\mathcal{T}_t(p) \leq p$ for all $t \geq 0$;*
3. *for every normal state ω in $\mathcal{B}(\mathcal{H})$ such that $p^\perp\rho p^\perp = \rho$, we have that $\mathrm{tr}\,[\rho\mathcal{T}_t(p)] = 0$ for all $t \geq 0$.*

Proof. 1 $\implies$ 2 — If Item 1 holds then we have

$$\mathcal{T}_t(p) = p\mathcal{T}_t(p)p \leq p\mathcal{T}_t(\mathbb{1})p = p$$

for all $t \geq 0$.
2 $\implies$ 3 · The inequality $\mathcal{T}_t(p) \leq p$ implies $p^\perp \mathcal{T}_t(p)p^\perp = 0$ and

$$0 \leq \operatorname{tr}\left[\rho\mathcal{T}_t(p)\right] = \operatorname{tr}\left[p^\perp \rho p^\perp \mathcal{T}_t(p)\right] = \operatorname{tr}\left[\rho p^\perp \mathcal{T}_t(p)p^\perp\right] = 0$$

for all $t \geq 0$.
3 $\implies$ 1 — For every ρ such that $p^\perp \rho p^\perp = \rho$ for all $t \geq 0$ we have

$$0 = \operatorname{tr}\left[\rho\mathcal{T}_t(p)\right] = \operatorname{tr}\left[p^\perp \rho p^\perp \mathcal{T}_t(p)\right] = \operatorname{tr}\left[\rho p^\perp \mathcal{T}_t(p)p^\perp\right]$$

It follows then that $p^\perp \mathcal{T}_t(p)p^\perp = 0$ and that $\mathcal{T}_t(p) = p\mathcal{T}_t(p)p$ with $\mathcal{T}_t(p)$ is positive. Finally, for any positive $x \in p\mathcal{A}p$ it holds that

$$0 \leq \mathcal{T}_t(x) \leq \|x\|\mathcal{T}_t(p) = \|x\|p$$

and therefore each $\mathcal{T}_t(x)$ belongs to $p\mathcal{A}p$, i.e. is $\mathcal{T}$-invariant. □

Definition 2.3.4. A QMS on $\mathcal{A}$ is said to be *irreducible* if there exists no non-trivial superharmonic projections.

Proposition 2.3.3. *Let $\mathcal{T}$ be an irreducible QMS on $\mathcal{A}$. If ω is a normal invariant state, then it is faithful.*

Proof. Let $p = s(\omega)$, since ω is invariant then p is subharmonic by Theorem 2.3.1. Therefore we have $p = \mathbb{1}$ or $p = 0$ because $\mathcal{T}$ is irreducible. But $p = 0$ implies $\omega(\mathbb{1}) = 0$, i.e. $\omega = 0$ that is a contradiction, hence $p = \mathbb{1}$ which means ω is faithful. □

Lemma 2.3.2. *Let p a projection in $\mathcal{A}$ and ω a normal state on $\mathcal{A}$. Then $\omega \in p\mathcal{A}_* p$ if and only if $s(\omega) \leq p$.*

Proof. Supposing that $\omega \in p\mathcal{A}_* p$, then it is clear that $\omega(p^\perp) = 0$, and therefore $s(\omega) \leq p$. Conversely, suppose that $s(\omega) \leq p$, then

$$p\omega p(a) = \omega(pap) = \omega(s(\omega)paps(\omega)) = \omega(s(\omega)as(\omega)) = \omega(a)$$

for all $a \in \mathcal{A}$, that is $\omega = p\omega p \in p\mathcal{A}p$. □

Proposition 2.3.4. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$ and p a projection in $\mathcal{A}$. Then the following properties hold*

1. *p is subharmonic;*
2. *$p\mathcal{A}_* p$ is $\mathcal{T}_*$-invariant;*
3. *$p\mathcal{T}_t(p)p = p\mathcal{T}_t(pap)p$ for all $a \in \mathcal{A}$ and $t \geq 0$.*

Proof. 1 $\implies$ 2 — If ω is a state in $p\mathcal{A}_* p$ then

$$\mathcal{T}_{*t}(\omega)p^\perp = \omega(\mathcal{T}_t(p^\perp)) \leq \omega(p^\perp) = 0$$

i.e. $\mathcal{T}_{*t}(\omega) \in p\mathcal{A}_* p$ by Lemma 2.3.2.
2 $\implies$ 3 — Let $\omega \in \mathcal{A}_*$ and $a \in \mathcal{A}$, since $\mathcal{T}_{*t}(p\omega p)$ is in $p\mathcal{A}_* p$, we have that

$$\omega(p\mathcal{T}_t(pap)p) = (p\mathcal{T}_{*t}(p\omega p)p)(a) = \mathcal{T}_{*t}(p\omega p)(a) = \omega(p\mathcal{T}_t(a)p)$$

for all $t \geq 0$, and since ω is arbitrary we have Item 3.
3 $\implies$ 1 — Taking $a = \mathbb{1}$ in Item 3, we get $p = \mathcal{T}_t(p)p$ and therefore $p\mathcal{T}_(p^\perp)p = 0$ which implies that p is subharmonic by Lemma 2.3.1. □

Given a QMS $\mathcal{T}$ on $\mathcal{A}$ we can define a *reduced* semigroup thanks a subharmonic projection $p \in \mathcal{A}$. Indeed, since $p\mathcal{A}_* p$ is $\mathcal{T}_*$-invariant we can restrict $\mathcal{T}_*$ to a weakly continuous semigroup on $p\mathcal{A}_* p$. Denoting by $\{\mathcal{T}_t^p\}_{t\geq 0}$ the dual of the restriction, for every $a \in p\mathcal{A}p = (p\mathcal{A}_* p)^*$ and $\omega \in p\mathcal{A}_* p$ we have

$$((\mathcal{T}_{*t}|_{p\mathcal{A}_* p})^*(a))(\omega) = (\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a)) = \omega(p\mathcal{T}_(a)p)$$

for all $t \geq 0$, i.e. we have

$$\mathcal{T}_t^p(a) = p\mathcal{T}_t(a)p \tag{2.13}$$

for all $a \in p\mathcal{A}p$ and $t \geq 0$. More interestingly the restricted QMS $\mathcal{T}_t^p$ is a QMS. Indeed, is normal, completely positive and $\mathcal{T}_t^p(p) = p$ since

$$p = p\mathcal{T}_t(\mathbb{1})p \geq p\mathcal{T}_t(p)p \geq p$$

Chapter 3

Covariant QMS

Now that we introduced all the needed basic concepts about QMSs we are ready to study the main property we set off to study in this thesis, i.e. covariant QMSs. To this purpose we first need to define one last object that will play a pivotal role in this chapter.

Definition 3.0.1. Let $\mathcal{T}\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a uniformly continuous QMS. Then the *decoherence-free* (DF) subalgebra of $\mathcal{T}$, denoted by $\mathcal{N}(\mathcal{T})$, is defined by

$$\mathcal{N}(\mathcal{T}) = \left\{ x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \mathcal{T}_t(x^*x) = \mathcal{T}_t(x)^*\mathcal{T}_t(x), \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x)^* \;\forall\, t \geq 0 \right\}. \tag{3.1}$$

The following Proposition gives some basics properties of $\mathcal{N}(\mathcal{T})$.

Proposition 3.0.1. *Let $\mathcal{T}$ be a uniformly continuous QMS and $\mathcal{N}(\mathcal{T})$ the associated DF algebra, then we have*

1. *$\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant for all $t \geq 0$;*
2. *the equalities $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x)^*\mathcal{T}_t(y)$ and $\mathcal{T}_t(yx^*) = \mathcal{T}_t(y)\mathcal{T}_t(x)^*$ hold for all $x \in \mathcal{N}(\mathcal{T})$, $y \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$;*
3. *$\mathcal{N}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$.*

Proof. (1) Let $x \in \mathcal{N}(\mathcal{T})$ and $t > 0$. For all $s > 0$ we have

$$\mathcal{T}_s(\mathcal{T}_t(x^*x)) = \mathcal{T}_{s+t}(x^*x) = \mathcal{T}_{s+t}(x^*)\mathcal{T}_{s+t}(x) = \mathcal{T}_s(\mathcal{T}_t(x^*))\mathcal{T}_s(\mathcal{T}_t(x)).$$

Moreover, by exchanging x and x^*, we obtain $\mathcal{T}_s(\mathcal{T}_t(xx^*)) = \mathcal{T}_s(\mathcal{T}_t(x))\mathcal{T}_s(\mathcal{T}_t(x^*))$ which proves that $\mathcal{T}_t(x) \in \mathcal{N}(\mathcal{T})$.
(2) For all $t \geq 0$ and $x, y \in \mathcal{B}(\mathcal{H})$ define $D_t(x,y) = \mathcal{T}_t(x^*y) - \mathcal{T}_t(x^*)\mathcal{T}_t(y)$. Clearly, if $x \in \mathcal{N}(\mathcal{T})$, then $D_t(x,x) = 0$ for all $t \geq 0$, therefore $D_t(x,y) = 0$ for all $x \in \mathcal{N}(\mathcal{T})$, $y \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$ by Lemma 2.2.1, which yields the Thesis for Item (2).
(3) $\mathcal{N}(\mathcal{T})$ is a vector space by Item (2). Given $x, y \in \mathcal{N}(\mathcal{T})$ we can prove that $xy \in \mathcal{N}(\mathcal{T})$ by a direct computation

$$\mathcal{T}_t((xy)^*(xy)) = \mathcal{T}_t(y^*)\mathcal{T}_t(x^*)\mathcal{T}_t(x)\mathcal{T}_t(y) = \mathcal{T}_t((xy)^*)\mathcal{T}_t(xy) \quad \forall\, t \geq 0$$

The invariance under conjugation is trivial. Finally, consider any net x_γ of elements in $\mathcal{N}(\mathcal{T})$ converging σ-strongly to $x \in \mathcal{B}(\mathcal{H})$, then we have

$$\mathcal{T}_t(x^*x) = \lim_\gamma \mathcal{T}_t(x^*x_\gamma) = \lim_\gamma \mathcal{T}_t(x^*)\mathcal{T}_t(x_\gamma) = \mathcal{T}_t(x^*)\mathcal{T}_t(x)$$

which proves that x belongs to $\mathcal{N}(\mathcal{T})$ and therefore Item (3) is proved. □

The DF algebra is closely related to the set of operators $x \in \mathcal{B}(\mathcal{H})$ on which the action of $\mathcal{T}$ is unitary as shown in the following.

Proposition 3.0.2. *For any self-adjoint H in any GKSL representation of the generator $\mathcal{L}$ of uniformly continuous QMS $\mathcal{T}$ we have*

$$\mathcal{N}(\mathcal{T}) \subseteq \left\{\, x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \mathcal{T}_t(x) = e^{itH}xe^{-itH} \ \forall t \geq 0 \,\right\}$$

Proof. It is trivial to show that if $\mathcal{T}_t(x) = e^{itH}xe^{-itH}$ then $x \in \mathcal{N}(\mathcal{T})$. Suppose now conversely that $x \in \mathcal{N}(\mathcal{T})$, then by differentiating $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x)^*\mathcal{T}_t(x)$ we get $\mathcal{L}(x^*x) = x^*\mathcal{L}(x) + \mathcal{L}(x^*)x$. But since we a also have for any $y \in \mathcal{B}(\mathcal{H})$ that

$$\mathcal{L}(y^*y) - y^*\mathcal{L}(y) - \mathcal{L}(y^*)y = \sum_{l\geq 1} [L_l, y]^*[L_l, y] \tag{3.2}$$

we conclude that $[L_l, x] = 0$ for all $x \in \mathcal{N}(\mathcal{T})$. Moreover, we have $x^* \in \mathcal{N}(\mathcal{T})$ given that $\mathcal{N}(\mathcal{T})$ is a *-algebra and therefore $[L_l, x^*] = 0$ which implies $[L_l^*, x] = 0$ from which we conclude that $\mathcal{L}(x) = i[H, x]$ for all $x \in \mathcal{N}(\mathcal{T})$. Finally, let $t > 0$ and $x \in \mathcal{N}(\mathcal{T})$, then for all $0 \leq s \leq t$ we have $\mathcal{T}_s(x) \in \mathcal{N}(\mathcal{T})$ and

$$\begin{aligned}\frac{d}{ds} e^{i(t-s)H}\mathcal{T}_s(x)e^{-i(t-s)H} &= ie^{i(t-s)H}[H, \mathcal{T}_s(x)]e^{-i(t-s)H} \\ &\quad - ie^{i(t-s)H}H\mathcal{T}_s(x)e^{-i(t-s)H} - ie^{i(t-s)H}\mathcal{T}_s(x)e^{-i(t-s)H} \\ &= 0\end{aligned}$$

Thus, the function $s \mapsto e^{i(t-s)H}\mathcal{T}_s(x)e^{-i(t-s)H}$ is constant on $[0, t]$ and therefore taking $s = 0$ we get $\mathcal{T}_t(x) = e^{itH}xe^{-itH}$. □

Proposition 3.0.3. *$\mathcal{N}(\mathcal{T})$ is the biggest von Neumann sub-algebra on $\mathcal{B}(\mathcal{H})$ on which every operator $\mathcal{T}_t$ is a *-automorphism.*

Proof. Thank to Proposition 3.0.2 it is clear that the restriction of $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is injective. We are therefore left to prove that given $x \in \mathcal{N}(\mathcal{T})$ there exists a $t \in \mathcal{N}(\mathcal{T})$ such that $x = \mathcal{T}_t(y)$. Note first that, since $\mathcal{T}$ is a uniformly continuous QMS, it can be extended to a uniformly continuous group $\mathcal{T}_t$ with $t \in \mathbb{R}$ which, by analyticity, can be expressed as $\mathcal{T}_{-t}(x) = e^{-itH}xe^{itH}$. Thus, we only need to prove that $\mathcal{T}_{-t}(x) \in \mathcal{N}(\mathcal{T})$ for all $t > 0$ and $x \in \mathcal{N}(\mathcal{T})$. Fix then $s, t > 0$ and $x \in \mathcal{N}(\mathcal{T})$, clearly also $xx^* \in \mathcal{N}(\mathcal{T})$ and $\mathcal{T}_s(x^*x) \in \mathcal{N}(\mathcal{T})$. Moreover we have

$$\begin{aligned}\mathcal{T}_s(\mathcal{T}_{-t}(x)^*\mathcal{T}_{-t}(x)) &= \mathcal{T}_s(e^{-itH}x^*xe^{itH}) \\ &= \mathcal{T}_{s-t}(x^*x) \\ &= \mathcal{T}_{-t}(\mathcal{T}_s(x^*x)) \\ &= e^{-i(t-s)H}x^*xe^{i(t-s)H} \\ &= e^{isH}e^{-itH}x^*e^{itH}e^{-isH}e^{isH}e^{-itH}xe^{itH}e^{-isH} \\ &= \mathcal{T}_s(\mathcal{T}_{-t}(x^*))\mathcal{T}_s(\mathcal{T}_{-t}(x))\end{aligned}$$

that is $\mathcal{T}_{-t}(x) \in \mathcal{N}(\mathcal{T})$. □

Finally, we have one more characterization of the DF algebra through the operators of a GKSL representation of $\mathcal{T}$. Indeed, let H and L_l be the operators of a GKSL representation of the generator $\mathcal{L}$ of a QMS $\mathcal{T}$, then we can define the iterated commutators $\delta_H^n(x)$ for any $x \in \mathcal{B}(\mathcal{H})$ as $\delta_H^0(x) = x$, $\delta_H^1(x) = [H, x]$ and $\delta_H^{n+1}(x) = [H, \delta_H^n(x)]$. With these definitions we have the following Proposition.

Proposition 3.0.4. *The DF subalgebra of a uniformly continuous QMS $\mathcal{T}$ can be expressed as*

$$\mathcal{N}(\mathcal{T}) = \left\{ \delta_H^n(L_l), \delta_H^n(L_l)^* \;\middle|\; n \geq 0, l \geq 1 \right\}'$$

Proof. We are going to prove the Thesis by induction. Let $x \in \mathcal{N}(\mathcal{T})$, then $\mathcal{T}_t(x) \in \mathcal{N}(\mathcal{T})$ by Proposition 3.0.1 and consequently $\mathcal{L}(x) = \lim_{t \to 0} t^{-1}(\mathcal{T}_t(x) - x) \in \mathcal{N}(\mathcal{T})$. Moreover, recalling the proof of Proposition 3.0.2, we have $[L_l, x] = [L_l^*, x] = 0$ and $\mathcal{L}(x) = i\delta_H^1(x) \in \mathcal{N}(\mathcal{T})$. We start the induction argument by noting that all elements of $\mathcal{N}(\mathcal{T})$ commute with $\delta_H^0(L_l) = L_l$ and $\delta_H^0(L_l^*) = L_l^*$. Suppose then that they commute with $\delta_H^n(L_l)$ and $\delta_H^n(L_l^*)$ for some positive n, by the Jacobi identity we get

$$[x, \delta_H^{n+1}(L_l)] = -[H, [\delta_H^n(L_l), x]] - [\delta_H^n(L_l), [x, H]] = 0$$

since $[x, H] = i\mathcal{L}(x) \in \mathcal{N}(\mathcal{T})$. Therefore we have that all elements of $\mathcal{N}(\mathcal{T})$ commute with $\delta_H^{n+1}(L_l)$ and $\delta_H^{n+1}(L_l^*) = -\delta_H^{n+1}(L_l)^*$, where the last equality follows from the fact that $\mathcal{N}(\mathcal{T})$ is a *-algebra. With this have proved that $\mathcal{N}(\mathcal{T}) \subset \left\{ \delta_H^n(L_l), \delta_H^n(L_l)^* \;\middle|\; n \geq 0, l \geq 1 \right\}'$. To prove the converse, suppose that $x \in \left\{ \delta_H^n(L_l), \delta_H^n(L_l)^* \;\middle|\; n \geq 0, l \geq 1 \right\}'$, then it commutes with both L_l and L_l^* so that $\mathcal{L}(x) = i\delta_H^1(x)$. Therefore also $\delta_H^1(x)$ commutes with both L_l and L_l^* by the Jacobi identity

$$[L_l, \delta_H^1(x)] = -[H, [x, L_l]] - [x, \delta_H^1(L_l)] = 0$$

and similarly for $[L_l^*, \delta_H^1(x)]$. Suppose now by induction that $\mathcal{L}^n(x) = i^n \delta_H^n(x)$ and $\delta_H^k(x)$ commute with $\delta_H^{n-k}(L_l)$ and $\delta_H^{n-k}(L_l^*)$ for some n and all $k \leq n$. Then $\mathcal{L}^{n+1}(x) = i^n \mathcal{L}(\delta_H^n(x))$ so that

$$\begin{aligned}
\mathcal{L}^{n+1}(x) &= i^{n+1}\delta_H^{n+1}(x) + \frac{1}{2}\sum_{l \geq 1}(L_L^*[\delta_H^n(x), L_l] + [L_l^*, \delta_H^n(x)]L_l) \\
&= i^{n+1}\delta_H^{n+1}(x)
\end{aligned}$$

By a repeated use of the Jacobi identity we obtain

$$\begin{aligned}
[\delta_H^k(x), \delta_H^{n-k+1}(L_l)] &= -[[\delta_H^{k-1}(x), \delta_H^{n+1-k}(L_l)], H] - [[\delta_H^{n-k+1}(L_l), H], \delta_H^{k-1}(x)] \\
&= [\delta_H^{k-1}(x), \delta_H^{n+2-k}(L_l)] \\
&= \dots \\
&= [x, \delta_H^{n+1}(L_l)] \\
&= 0
\end{aligned}$$

and $[\delta_H^k(x), \delta_H^{n-k+1}(L_l^*)] = 0$ analogously. It follows that $\mathcal{L}(x)^n = i^n \delta_H^n(x)$ for all $n \geq 0$ so that $\mathcal{T}_t(x) = e^{itH} x e^{-itH}$ and thus $x \in \mathcal{N}(\mathcal{T})$ by Proposition 3.0.2. $\square$

When $\mathcal{N}(\mathcal{T})$ is atomic (i.e. for every non-zero projection $p \in \mathcal{N}(\mathcal{T})$ there exists a non-zero minimal projection $q \in \mathcal{N}(\mathcal{T})$ such that $q \leq p$), we obtain some additional information on the structure of the semigroup. In particular, in [8] the following result has been proved:

Theorem 3.0.1. *$\mathcal{N}(\mathcal{T})$ is an atomic algebra if and only if there exist two countable sequences of Hilbert spaces $(\mathcal{K}_i)_{i\in I}$, $(\mathcal{M}_i)_{i\in I}$ such that $\mathcal{H} = \oplus_{i\in I}(\mathcal{K}_i \otimes \mathcal{M}_i)$ and $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I}(\mathcal{B}(\mathcal{K}_i) \otimes \mathbb{1}_{\mathcal{M}_i})$ up to a unitary isomorphism. In this case:*

1. *for every GKSL representation of $\mathcal{L}$ by means of operators $H, (L_\ell)_{\ell\geq 1}$, up to a unitary isomorphism we have*

$$L_\ell = \oplus_{i\in I}\left(\mathbb{1}_{\mathcal{K}_i} \otimes M_\ell^{(i)}\right)$$

for a collection $(M_\ell^{(i)})_{\ell\geq 1}$ of operators in $\mathcal{B}(\mathcal{M}_i)$, such that the series $\sum_{\ell\geq 1} M_\ell^{(i)*} M_\ell^{(i)}$ strongly convergent for all $i \in I$, and

$$H = \oplus_{i\in I}\left(K_i \otimes \mathbb{1}_{\mathcal{M}_i} + \mathbb{1}_{\mathcal{K}_i} \otimes M_0^{(i)}\right)$$

for self-adjoint operators $K_i \in \mathcal{B}(\mathcal{K}_i)$ and $M_0^{(i)} \in \mathcal{B}(\mathcal{M}_i)$, $i \in I$,

2. *we have $\mathcal{T}_t(x \otimes y) = e^{itK_i} x e^{-itK_i} \otimes \mathcal{T}_t^{\mathcal{M}_i}(y)$ for all $x \in \mathcal{B}(\mathcal{K}_i)$ and $y \in \mathcal{B}(\mathcal{M}_i)$, where $\mathcal{T}_t^{\mathcal{M}_i}$ is the QMS on $\mathcal{B}(\mathcal{M}_i)$ generated by*

$$\begin{aligned} \mathcal{L}^{\mathcal{M}_i}(y) = i[M_0^{(i)}, y] + \\ -\frac{1}{2}\sum_{\ell\geq 1}\left((M_\ell^{(i)})^* M_\ell^{(i)} y - 2(M_\ell^{(i)})^* y M_\ell^{(i)} + y(M_\ell^{(i)})^* M_\ell^{(i)}\right). \end{aligned} \tag{3.3}$$

Note that, setting p_i the orthogonal projection onto $\mathcal{K}_i \otimes \mathcal{M}_i$ for all $i \in I$, we get a family of mutually orthogonal projections which are minimal in the center of $\mathcal{N}(\mathcal{T})$ and such that $\sum_i p_i = \mathbb{1}$.

Remark 3.0.1. Note that whenever there exists a normal faithful invariant state, then $\mathcal{N}(\mathcal{T})$ is atomic (see [15]) and we can therefore always consider the decomposition $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I}(\mathcal{B}(\mathcal{K}_i) \otimes \mathbb{1}_{\mathcal{M}_i})$.

We conclude this Section showing the relationship between the DF subalgebra and the set of fixed point with respect to $\mathcal{T}$. Recall the definition of the set of fixed point

$$\mathcal{F}(\mathcal{T}) = \{\, x \in \mathcal{B}(\mathcal{H}) \mid T_t(x) = x \ \forall\, t \geq 0 \,\} = \{\, x \in \mathcal{B}(\mathcal{H}) \mid \mathcal{L}(x) = 0 \,\}. \tag{3.4}$$

then we have the following results about projection in $\mathcal{F}(\mathcal{T})$.

Lemma 3.0.1. *An orthogonal projection $p \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{F}(\mathcal{T})$ if and only if it commutes with the operators L_l and H of any GKSL representation of the generator $\mathcal{L}$.*

Proof. Clearly, if p commutes with both L_l and H then $\mathcal{L}(p) = 0$ and thus $\mathcal{T}_t(p) = p$. Conversely, if $p \in \mathcal{F}(\mathcal{T})$ then $\mathcal{L}(p) = 0$ and multiplying on both sides by $p^\perp = \mathbb{1} - p$ we obtain

$$0 = p^\perp \mathcal{L}(p) p^\perp = p^\perp \sum_{l \geq 1} L_l^* p L_l p^\perp$$

and therefore $pL_l p^\perp = 0$. In an analogous way, from $\mathcal{L}(p^\perp) = \mathcal{L}(\mathbb{1} - p) = \mathcal{L}(\mathbb{1}) - \mathcal{L}(p) = 0$ we obtain $p^\perp L_l p = 0$. Moreover, by taking the conjugate we get $p^\perp L_l^* p = p^\perp L_l^* p = 0$ and thus p commutes with L_l and L_l^*. In this way we have that the action of the generator is reduced to $\mathcal{L}(p) = i[H, p] = 0$ from which we see that p also commutes with H. □

In general $\mathcal{F}(\mathcal{T})$ is not an algebra unlike $\mathcal{N}(\mathcal{T})$ as shown in the following example.

Example 3.0.1. Let $\mathcal{H} = \mathbb{C}^3$ and $\{e_i\}_{i=0,1,2}$ its canonical basis. We can define a generator $\mathcal{L}$ through its GKSL generator as $L = |e_0\rangle\langle e_2|$ and $H = L^*L = |e_0\rangle\langle e_0|$. For any matrix $a \in M_3(\mathbb{C})$ we have

$$\begin{aligned}\mathcal{L}(a) &= (a_{00} - a_{22}) \,|e_2\rangle\langle e_2| - \left(\frac{1}{2} + i\right) (a_{02} \,|e_0\rangle\langle e_2| + a_{12} \,|e_1\rangle\langle e_2|) \\ &= \left(\frac{1}{2} - i\right) (a_{20} \,|e_2\rangle\langle e_0| + a_{21} \,|e_2\rangle\langle e_1|)\end{aligned}$$

From this we can see that a is a fixed point for the QMS associated to $\mathcal{L}$ if and only if $a_{00} = a_{22}$ and $a_{02} = a_{20} = a_{12} = a_{21}$. Therefore, given a fixed point a it satisfies $\mathcal{L}(a^*a) = 0$ if and only if a commutes with L by Equation (3.2), which is equivalent to $a_{10} = 0$. By Proposition 3.0.4 we have that a matrix a belongs to $\mathcal{N}(\mathcal{T})$ if and only if it commutes with all the iterated commutators, but since in this example it is easy to show that $\delta_H^n(L) = L$ and $\delta_H^n(L^*) = L^*$ for all $n \geq 0$ it is sufficient for a to commute with L and L^* for it to belong to $\mathcal{N}(\mathcal{T})$. By a simple computation, it is possible to see that $[a, L] = 0$ and $[a, L^*] = 0$ if and only if $a_{ij} = 0$ for $i \neq j$ and $a_{22} = a_{00}$. Therefore we can conclude that for this example $\mathcal{N}(\mathcal{T}) \subseteq \mathcal{F}(\mathcal{T})$ and that $\mathcal{F}(\mathcal{T})$ is not an algebra.

Despite not being an algebra in general, there are still some cases in which $\mathcal{F}(\mathcal{T})$ can be one. The following Proposition gives a full characterization of the situations in which $\mathcal{F}(\mathcal{T})$ is an algebra.

Proposition 3.0.5. *The following statements hold for every uniformly continuous QMS $\mathcal{T}$:*

1. *the set of fixed points $\mathcal{F}(\mathcal{T})$ is a *-algebra if and only if it is contained in the DF subalgebra $\mathcal{N}(\mathcal{T})$;*

2. *if the QMS $\mathcal{T}$ has a faithful invariant state, then $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$;*

3. *if $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, then it coincides with $\{\, L_l, L_l^*, H \mid L \geq 1 \,\}'$.*

Proof. (1) If $\mathcal{F}(\mathcal{T})$ is contained in $\mathcal{N}(\mathcal{T})$ then for every $x \in \mathcal{F}(\mathcal{T})$ we have $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$ which means that $x^*x \in \mathcal{F}(\mathcal{T})$ and therefore $\mathcal{F}(\mathcal{T})$ is a *-algebra. Conversely, if $\mathcal{F}(\mathcal{T})$ is a *-algebra, then for all $x \in \mathcal{F}(\mathcal{T})$ we have $x^*x \in \mathcal{F}(\mathcal{T})$ and therefore $\mathcal{T}_t(x^*x) = x^*x = \mathcal{T}_t(x^*)\mathcal{T}_t(x)$ and therefore $x \in \mathcal{N}(\mathcal{T})$.
(2) Let ρ be a faithful invariant state for $\mathcal{T}$. If x is a fixed point, then by the complete positivity of the QMS we have $x^*x = \mathcal{T}_t(x^*)\mathcal{T}_t(x) \leq \mathcal{T}_t(x^*x)$ and $\operatorname{tr}[\rho(\mathcal{T}_t(x^*x) - x^*x)] = 0$ by invariance of ρ. Thus, $\mathcal{T}_t(x^*x) = x^*x$ for all $t \geq 0$ since ρ is faithful and therefore $x^*x \in \mathcal{F}(\mathcal{T})$.
(3) If $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ it is generated by its projections which belong to $\{ L_l, L_l^*, H \mid L \geq 1 \}'$ by Lemma 3.0.1 and therefore $\mathcal{F}(\mathcal{T}) \in \{ L_l, L_l^*, H \mid L \geq 1 \}'$. Conversely, every $x \in \{ L_l, L_l^*, H \mid L \geq 1 \}'$ satisfy $\mathcal{L}(x) = 0$ and so $\mathcal{T}_t(x) = x$. □

3.1 Structure of the generator of a covariant QMS

The concept of *covariance* in physics describes a property of the observables under the action of the set of transformations associated to some symmetry. More precisely, a symmetry is represented mathematically as a group G while its action by a suitable representation π of G on a Hilbert space $\mathcal{H}$, so that a law of physics is said to be covariant if its predictions are not affected by the action of $\pi(g)$ for any $g \in G$. The following definition specifies what we mean by this vague description in the context of QMSs and time evolution of open quantum systems.

Definition 3.1.1. Let G be a locally compact group and $\pi\colon g \mapsto \pi(g)$ a continuous unitary representation of G on an Hilbert space $\mathcal{H}$. Then a uniformly continuous QMS $\mathcal{T}$ on $\mathcal{B}(\mathcal{H})$ is said to be *covariant* with respect to the representation π if

$$\mathcal{T}_t(\pi(g)^* x\, \pi(g)) = \pi(g)^* \mathcal{T}_t(x)\pi(g) \tag{3.5}$$

for all $x \in \mathcal{B}(\mathcal{H})$, $g \in G$ and $t \geq 0$. Equivalently, if $\mathcal{L}$ is the generator of $\mathcal{T}$ the covariance property reads

$$\mathcal{L}(\pi(g)^* x \pi(g)) = \pi(g)^* \mathcal{L}(x)\pi(g) \tag{3.6}$$

for all $x \in \mathcal{B}(\mathcal{H})$ and $g \in G$.

To give some intuition about this definition think of G as the group describing the change of reference frame in physical system (i.e. the Galilean group or the Poincaré group), then Equation (3.5) simply states that the we are free to compute the time evolution of any x according to $\mathcal{T}$ before or after changing the reference frame without any change in the result.

From now on we will study covariant QMSs and whether this additional hypothesis allows us to give a richer structure to the associated generators and DF subalgebras. First of all we note that the structure of the generator of a covariant uniformly continuous QMS was fully characterized by Holevo in [1], Section 2, in the case of amenable locally compact groups. When $\mathcal{T}$ is uniformly continuous and G is compact, the result can be restated as follows.

Theorem 3.1.1. *Let G be a compact group, $\pi\colon g \mapsto \pi(g)$ a continuous unitary representation of G on $\mathcal{H}$. If $\mathcal{T}$ is a uniformly continuous covariant QMS* *on $\mathcal{B}(\mathcal{H})$, then there exists a GKSL representation of $\mathcal{L}$ given by operators $\{H, L_k : k \geq 1\}$ satisfying:*

1. $\sum_k L_k^* \pi(g)^* x \pi(g) L_k = \sum_k \pi(g)^* L_k^* x L_k \pi(g)$ *for all $x \in \mathcal{B}(\mathcal{H})$ and $g \in G$,*
2. $H \in \{ \pi(g) \mid g \in G \}'$.

Moreover, the condition in Item 1 is equivalent to

$$\pi(g)^* L_j \pi(g) = \sum_k v(g)_{jk} L_k \tag{3.7}$$

for all $g \in G$, where $V(g) = (v(g)_{jk})_{jk}$ is a unitary matrix. In particular, for all $g \in G$, the operators $\left\{ H,\ \pi(g)^* L_k \pi(g) \;\middle|\; k \geq 1 \right\}$ give the same GKSL representation of $\mathcal{L}$.

Proof. Consider the algebraic tensor product $\mathcal{H} \otimes \mathcal{B}(\mathcal{H})$ generated by elements of the form $u \otimes X$ for some $u \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. We can define an inner product on $\mathcal{H} \otimes \mathcal{B}(\mathcal{H})$ as follows

$$\langle u \otimes X, v \otimes Y \rangle_{\mathcal{H} \otimes \mathcal{B}(\mathcal{H})} = \langle u, \mathrm{D}\mathcal{L}[X, Y] v \rangle_{\mathcal{H}}$$

where we have put

$$\mathrm{D}\mathcal{L}[X, Y] \coloneqq \mathcal{L}(X^* Y) - X^* \mathcal{L}(Y) - \mathcal{L}(X)^* Y$$

thus we can render $\mathcal{H} \otimes \mathcal{B}(\mathcal{H})$ an Hilbert space by completing w.r.t. this inner product. We will also need the following maps: a $*$-representation ψ of $\mathcal{B}(\mathcal{H})$ in $\mathcal{H} \otimes \mathcal{B}(\mathcal{H})$ defined as

$$\psi(Y)(u \otimes X) = u \otimes XY - Xu \otimes Y;$$

a continuous unitary representation of G on $\mathcal{H} \otimes \mathcal{B}(\mathcal{H})$ defined as

$$\rho(g)(u \otimes X) = \pi(g) u \otimes \pi(g) X \pi(g)^*;$$

and finally a linear map $B\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{H}))$ defined as

$$B(X) u = u \otimes X$$

which clearly satisfies

$$B(XY) = \psi(Y) B(X) + B(Y) X \tag{3.8}$$

$$B(X)^* B(Y) = \mathrm{D}\mathcal{L}[X, Y]. \tag{3.9}$$

Indeed we have for any $v \in \mathcal{H}$

$$\begin{aligned} (\psi(Y) B(X) + B(Y) X)\, v &= \psi(Y)(v \otimes X) + Xv \otimes Y \\ &= v \otimes XY \\ &= B(XY) v \end{aligned}$$

While for the second property, given any $v, w \in \mathcal{H}$ we have

$$\begin{aligned} \left\langle v, B(X)^* B(Y) w \right\rangle_{\mathcal{H}} &= \langle B(X) v, B(Y) w \rangle_{\mathcal{H}} \\ &= \langle v \otimes X, w \otimes Y \rangle_{\mathcal{H} \otimes \mathcal{B}(\mathcal{H})} \\ &= \langle v, \mathrm{D}\mathcal{L}[X, Y] w \rangle_{\mathcal{H}} \end{aligned}$$

By the definition of ψ and B respectively, and the covariance of the generator $\mathcal{L}$ we have immediately that

$$\psi(\pi(g)^* X \pi(g)) = \rho(g)^* \psi(X) \rho(g) \tag{3.10}$$
$$B(\pi(g)^* X \pi(g)) = \rho(g)^* B(X) \pi(g) \tag{3.11}$$

Moreover, in [16] is shown that a map satisfying Equation (3.8) can be written as

$$B(X) = \psi(X)C - CX \tag{3.12}$$

for some $C \in \overline{\{ B(X)Y \mid X, Y \in \mathcal{B}(\mathcal{H}) \}}^{w*}$. Given C we can define the map $A\colon G \to \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{H}))$

$$A(g) \coloneqq \rho(g)^* C \pi(g) - C$$

which, intertwines $\psi(X)$ with X by Equation (3.10) and (3.11) as follows

$$\psi(X) A(g) = A(g) X \tag{3.13}$$

Indeed, by a direct computation

$$\begin{aligned}
\psi(X)A(g) &= \psi(X)\rho(g)^* C\pi(g) - \psi(X)C \\
&= \rho(g)^* \psi(\pi(g) X \pi(g)^*) C \pi(g) - B(X) - CX \\
&= \rho(g)^* B(\pi(g) X \pi(g)^*) \pi(g) + \rho(g)^* C \pi(g) X - B(X) - CX \\
&= B(X) + \rho(g)^* C \pi(g) X - B(X) - CX \\
&= (\rho(g)^* C \pi(g) - C) X \\
&= A(g) X
\end{aligned}$$

The map $A(g)$ satisfies also the following cocycle relation for every $g, h \in G$

$$A(gh) = \rho(h)^* A(g) \pi(h) + A(h) \tag{3.14}$$

Since G is compact we can integrate over it with respect to the Haar measure μ. Therefore we can define the operator

$$\bar{A} = \int_G A(g) \, \mathrm{d}\mu(g)$$

such that averaging Equation (3.13) we obtain

$$\psi(X) \bar{A} = \bar{A} X$$

while averaging (3.14) we get

$$\int_G A(gh) \, \mathrm{d}\mu(g) = \int_G \rho(h)^* A(g) \pi(h) \, \mathrm{d}\mu(g) + \int_G A(h) \, \mathrm{d}\mu(g)$$

So by the left-invariance of the Haar measure μ we get

$$\bar{A} = \rho(h)^* \bar{A} \pi(h) + A(h)$$

and finally, by reordering we get

$$A(h) = -\rho(h)^* \bar{A} \pi(h) + \bar{A}$$

respectively. We can now define yet another operator as $A_0 = C + \bar{A}$ that satisfies

$$B(X) = \psi(X)A_0 - A_0X$$

which is easily provable recalling that $\psi(X)\bar{A} = \bar{A}X$. Defining the map

$$\phi(X) = A_0^*\psi(X)A_0$$

we can prove that $\mathrm{D}\mathcal{L}(X,Y) = \mathrm{D}\phi(X,Y)$, where

$$\mathrm{D}\phi(X,Y) = \phi(X^*Y) - X^*\phi(Y) - \phi(X^*)Y + X^*\phi(\mathbb{1})Y$$

First of all note that $\phi(X) = A_0^*B(X)+A_0^*A_0X$, and since Equation (3.8) implies $B(\mathbb{1}) = 0$, we immediately see that $\phi(\mathbb{1}) = A_0^*A_0$. Plugging these relation into the definition of $\mathrm{D}\phi$ we get

$$\begin{aligned}
\mathrm{D}\phi(X,Y) &= A_0^*B(X^*Y) + A_0^*A_0X^*Y - X^*A_0B(Y) - X^*A_0^*A_0Y \\
&\quad - A_0^*B(X^*)Y - A_0^*A_0X^*Y + X^*A_0^*A_0Y \\
&= A_0^*\psi(X^*)B(Y) + A_0^*B(X^*)Y - X^*A_0^*B(Y) - A_0^*B(X^*)Y \\
&= \left(A_0^*\psi(X^*) - X^*A_0^*\right)B(Y) \\
&= B(X)^*B(Y) \\
&= \mathrm{D}\mathcal{L}(X,Y)
\end{aligned}$$

In [16, Theorem 3.1] it is been proven that this equality together with Equation (3.9) imply that the generator can be written as follows

$$\mathcal{L}(x) = \phi(x) - \frac{1}{2}\left(\phi(\mathbb{1})x + x\phi(\mathbb{1})\right) + i[H,x]$$

where H is some self-adjoint operator in $\mathcal{B}(\mathcal{H})$. We now need to prove that the newly found map ϕ is covariant. Let's start by showing the following transformation property of A_0

$$\begin{aligned}
\rho(g)A_0 &= \rho(g)C + \rho(g)\bar{A} \\
&= C\pi(g) - \rho(g)A(g) + \rho(g)A(g) + \bar{A}\pi(g) \\
&= (C + \bar{A})\pi(g) \\
&= A_0\pi(g)
\end{aligned}$$

which directly implies the covariance of ϕ

$$\begin{aligned}
\pi(g)^*\phi(x)\pi(g) &= \pi(g)^*A_0^*\psi(x)A_0\pi(g) \\
&= A_0^*\rho(g)^*\psi(x)\rho(g)A_0 \\
&= A_0^*\psi(\pi(g)^*x\pi(g))A_0 \\
&= \phi(\pi(g)^*x\pi(g))
\end{aligned}$$

Again by the results of Subsection 2.2.2 and since ϕ is a completely positive map, we know there exists a family of operators $(L_k)_{k\geq 0}$ such that $\phi(x) = \sum_k L_k^*xL_k$ which gives Item 1 of the Theorem. Indeed, we can show that ϕ is a completely positive map by a direct computation. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be operators in $\mathcal{B}(\mathcal{H})$, the we have

$$\begin{aligned}\sum_{i,j=1}^{n} b_i^* \phi(a_i^* a_j) b_j &= \sum_{i,j=1}^{n} b_i^* A_0^* \psi(a_i^* a_j) A_0 b_j \\ &= \sum_{i,j=1}^{n} b_i^* A_0^* \psi(a_i^*) \psi(a_j) A_0 b_j \\ &= \sum_{i,j=1}^{n} (\psi(a_i) A_0 b_i)^* (\psi(a_j) A_0 b_j) \geq 0\end{aligned}$$

where the second equality holds since ψ is a *-representation. Therefore ϕ is completely positive since this relations holds for every n. Moreover, since both $\mathcal{L}$ and ϕ are covariant, this implies that also $[H, x]$ must be so and therefore $[\pi(g)^* H \pi(g) - H, x] = 0$ for every $x \in \mathcal{B}(\mathcal{H})$. So, if similarly to what we did for A, we define

$$H(g) = \pi(g)^* H \pi(g) - H$$

we know that $H(g) \in \mathcal{B}(\mathcal{H})'$ for every $g \in G$. Once again, $H(g)$ satisfies the cocycle relation

$$H(gh) = \pi(g)^* H \pi(g) + H(h)$$

and furthermore, if we average over G we obtain an operator $\bar{H} \in \mathcal{Z}(\mathcal{B}(\mathcal{H}))$ such that

$$H(h) = -\pi(g)^* \bar{H} \pi(g) + \bar{H}$$

Finally, let $H_0 = H + \bar{H}$, which satisfies $\pi(g)^* H_0 \pi(g) = H_0$ for every $g \in G$, that is $H_0 \in \{\, \pi(g) \mid g \in G \,\}'$, and also $[H_0, x] = [H, x]$ for every $x \in \mathcal{B}(\mathcal{H})$ implying $H \in \{\, \pi(g) \mid g \in G \,\}'$ and thus concluding the proof. □

Definition 3.1.2. We say that a special representation of $\mathcal{L}$ by means of operators H and $(L_k)_{k \geq 1}$ is *covariant* if H commutes with every $\pi(g)$ and L_k's satisfies Equation (3.7).

Theorem 3.1.1 directly implies the following corollary that we will widely use in the reminder of this Chapter.

Corollary 3.1.1. *Let H, $(L_k)_k$ and π as in Theorem 3.1.1. Then H and $\sum_k L_k^* L_k$ intertwine the representation π.*

Proof. By Theorem 3.1.1 we already know that $\pi(g)H = H\pi(g)$ for all $g \in G$. To conclude the proof is enough to note that by Item 1 of the same Theorem and the unitarity of π we have

$$\pi(g)^* \left(\sum_k L_k^* L_k \right) \pi(g) = \sum_k L_k^* (\pi(g)^* \pi(g)) L_k = \sum_k L_k^* L_k.$$

Therefore $\pi(g) \left(\sum_k L_k^* L_k\right) = \left(\sum_k L_k^* L_k\right) \pi(g)$ for all $g \in G$. □

Whenever the representation π is irreducible it is possible to further specify the structure of the generator $\mathcal{L}$.

Proposition 3.1.1. *Let G be a compact group, $\pi\colon g \mapsto \pi(g)$ an irreducible unitary representation of G on a finite dimensional Hilbert space $\mathcal{H}$. Let also $H, (L_k)_k$ operators in a covariant GKSL representation of $\mathcal{L}$. Then $\mathcal{L}$ can be written as*

$$\mathcal{L}(x) = \sum_k L_k^* x L_k - \epsilon x \tag{3.15}$$

where ϵ is a real positive constant such that $\sum_k L_k^* L_k = \epsilon \mathbb{1}$.

Proof. By Corollary 3.1.1 both H and $\sum_k L_k^* L_k$ intertwine the representation π, therefore we have that H, $\sum_k L_k^* L_k \in \mathbb{C}\mathbb{1}$ by the irreducibility of π and Schur's Lemma. Moreover, since $\sum_k L_k^* L_k$ is a positive operator, there exists a constant $\epsilon > 0$ such that $\sum_k L_k^* L_k = \epsilon \mathbb{1}$. Equation (3.15) directly follows being H proportional to the identity matrix. □

In Theorem 3.1.1 we characterized the behavior of the operators $\{ H, (L_k)_{k\geq 1} \}$ of a specific GKSL representation of a covariant generator $\mathcal{L}$. We also know that a GKSL representation is not unique as stated in Theorem 2.2.2, indeed any other representation $\{ H', (L'_k)_{k\geq 1} \}$ related to the original according to Equation 2.11 is still a valid representation of the same generator $\mathcal{L}$. A natural question, that we are going to address in the following Proposition, is therefore whether the transformation between equivalent GKSL representations preserves the covariance properties stated in Theorem 3.1.1 or not.

Proposition 3.1.2. *Let $\mathcal{T}$ be a uniformly continuous QMS w.r.t. a unitary representation π of a compact group G, and let $\{ H,\ (L_k)_{k\geq 1} \}$ be a GKSL representation of the generator $\mathcal{L}$ satisfying Theorem 3.1.1. Then another GKSL representation $\{ H',\ (L'_k)_{k\geq 1} \}$ satisfies Theorem* (3.1.1) *if and only if it is connected to the former by Equation* (2.11) *with*

$$\alpha_k = \sum_h (UV(g)U^*)_{kh} \alpha_h, \tag{3.16}$$

where $V(g) = (v(g)_{ij})_{ij}$ is the unitary matrix that appears in Equation (3.7)*.*

Proof. We begin proving that Equation (3.16) ensure the existence of a unitary matrix $V'(g)$ such that Equation (3.7) holds also for the operators $(L'_k)_{k\geq 1}$. Since $\{ H,\ (L_k)_{k\geq 1} \}$ and $\{ H',\ (L'_k)_{k\geq 1} \}$ are two equivalent GKSL representations of $\mathcal{L}$, by Equation (2.11) and Equation (3.7) applied to $(L_k)_{k\geq 1}$ we have

$$\begin{aligned}
\pi(g)^* L'_k \pi(g) &= \sum_h u_{kh} \pi(g)^* L_h \pi(g) + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g))_{kh} L_h + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g)U^*)_{kh} (L'_h - \alpha_h \mathbb{1}) + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g)U^*)_{kh} L'_h + \left(\alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right) \mathbb{1}.
\end{aligned}$$

If condition (3.16) holds, then $(L'_k)_k$ clearly satisfies the covariance condition (3.7) with respect to the unitary matrix $V'(g) = UV(g)U^*$. On the other hand, since $\mathcal{L}$ is covariant, there exists a unitary matrix $W(g) = (w(g)_{kh})_{kh}$ such that $\pi(g)^* L'_k \pi(g) = \sum_h w(g)_{kh} L'_h$ which implies

$$\sum_h (UV(g)U^* - W(g))_{kh} L'_h + \left(\alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right) \mathbb{1} = 0.$$

Since $\{ \mathbb{1}, (L'_k)_{k\geq 1} \}$ are linearly independent and we have both

$$\sum_k |(UV(g)U^* - W(g))_{kh}|^2 \leq \sum_k |(UV(g)U^*)_{kh}|^2 + \sum_k |(W(g))_{kh}|^2 = 2 < \infty$$

and

$$\sum_k \left| \alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right|^2 \leq \sum_k |\alpha_k|^2 + \sum_k \left| \sum_h (UV(g)U^*)_{kh} \alpha_h \right|^2 < \infty,$$

We immediately have $UV(g)U^* = W(g)$ and $\alpha_k = \sum_h (UV(g)U^*)_{kh} \alpha_h$, i.e. Equation (3.16) is fulfilled. This condition is also sufficient to ensure $[H', \pi(g)] = 0$ to hold for all $g \in G$. First of all we note that

$$[H', \pi(g)] = [H + \beta \mathbb{1} + \frac{1}{2i}(S - S^*), \pi(g)] = \frac{1}{2i}[S - S^*, \pi(g)]$$

for all $g \in G$ and $\beta \in \mathbb{R}$. Moreover, since $[S^*, \pi(g)] = -[S, \pi(g^{-1})]^*$, it is enough to prove that $[S, \pi(g)] = 0$ for all $g \in G$ to conclude the proof. Indeed we have

$$\begin{aligned}
S\pi(g) &= \sum_{i,j} \overline{\alpha}_i u_{ij} L_j \pi(g) \\
&= \pi(g) \sum_{i,j} \overline{\alpha}_i (UV(g))_{ij} L_j \\
&= \pi(g) \sum_{h,j} \overline{\alpha}_h (UV(g)^* U^* UV(g))_{hj} L_j \\
&= \pi(g) S
\end{aligned}$$

for all $g \in G$ and therefore $[H', \pi(g)] = 0$ for all $g \in G$. $\square$

Before moving on to study how the results we just found about covariant generators affect the overall structure of the associated QMSs, we give a simple result describing the direct action of a representation π on $\mathcal{N}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T})$.

Proposition 3.1.3. *Let $\mathcal{T}$ be a uniformly continuous QMS covariant w.r.t. a unitary representation π of a compact group G. Then*

$$\pi(g)^* \mathcal{N}(\mathcal{T}) \pi(g) = \mathcal{N}(\mathcal{T}) \quad \textit{and} \quad \pi(g)^* \mathcal{F}(\mathcal{T}) \pi(g) = \mathcal{F}(\mathcal{T})$$

for all $g \in G$.

Proof. Let $g \in G$ and x be in $\mathcal{N}(\mathcal{T})$. We have to prove that $y := \pi(g)^* x \, \pi(g)$ belongs to $\mathcal{N}(\mathcal{T})$. The covariance of $\mathcal{T}$ gives

$$\begin{aligned}
\mathcal{T}(y^* y) &= \mathcal{T}_t(\pi(g)^* x^* x \, \pi(g)) = \pi(g)^* \mathcal{T}_t(x^* x) \pi(g) \\
&= (\pi(g)^* \mathcal{T}_t(x^*) \, \pi(g)) \, (\pi(g)^* \mathcal{T}_t(x) \pi(g)) = \mathcal{T}_t(\pi(g)^* x^* \pi(g)) \mathcal{T}_t(\pi(g)^* x \, \pi(g)) \\
&= \mathcal{T}_t(y^*) \mathcal{T}_t(y).
\end{aligned}$$

In the same way we can show the equality $\mathcal{T}_t(y^* y) = \mathcal{T}_t(y) \mathcal{T}_t(y^*)$, i.e. $y \in \mathcal{N}(\mathcal{T})$. For the set of fixed points $\mathcal{F}(\mathcal{T})$ the proof is very similar. $\square$

3.1.1 Example – Circulant QMS

In this subsection we present as simple but explicit example of a covariant QMS in order to clarify the very abstract results given up until now. Let $p \geq 2$, $\mathcal{H} = \mathbb{C}^p$ and consider the von Neumann algebra of $p \times p$ matrices with complex entries $M_p(\mathbb{C})$. Instead of indexing the canonical basis $\{ e_k \}$ of $\mathbb{C}^p$ with the set $\{ 1, \dots, p \}$, we use $\mathbb{Z}_p$ which is the cyclic group of order p. This choice naturally gives rise to a group to consider for the covariance while leaving the index untouched. In order to define the QMS generator, we introduce the *primary permutation matrix* defined as

$$J = \sum_{k \in \mathbb{Z}_p} |e_k\rangle\langle e_{k+1}|$$

with this we can give the GKSL representation of the generator $\mathcal{L}$ for every $x \in M_p(\mathbb{C})$ as

$$\mathcal{L}(x) = \sum_{k=1}^{p-1} \gamma(p-k) J^{*k} x J^k - x \tag{3.17}$$

where $\gamma \in \mathbb{C}^{p-1}$ such that its component $\gamma(k) > 0$ for all $k = 1, \dots, p-1$ and $\sum_{k=1}^{p-1} \gamma(k) = 1$. In the usual GKSL notation (as defined in Theorem 2.2.2), we have that $L_k = \sqrt{\gamma(p-k)} J^k$ for all $k = 1, \dots, p-1$ and $H = \mathbb{1}$. Therefore, for every $\gamma \in \mathbb{C}^{p-1}$ satisfying the above conditions we have a valid circulant generator, which in turns gives rise to a uniformly continuous QMS $\mathcal{T}$. From Proposition 3.0.4 we can obtain a characterization of the decoherence free subalgebra associated to $\mathcal{T}$. Indeed, in the general case we have that $\mathcal{N}(\mathcal{T}) = \{ \delta_H^n(L_k), \delta_H^n(L_k)^* \mid n \geq 0, k \geq 1 \}'$, but in this case we can greatly simplify this expression by noting that $\delta_H^n(L_k) = \delta_H^n(L_k^*) = 0$ for all $n \geq 1$ since $H = \mathbb{1}$ and that $\delta_H^0(L_k) = L_k$ and $\delta_H^0(L_k^*) = L_k^*$. Moreover, by Item 3 of Proposition 3.0.5 we have that $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ and they are both given by

$$\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) = \left\{ J^k, J^{*k} \mid k \in \mathbb{Z}_p \right\}'$$

Moreover, noticing that $J^* = J^{p-1}$, and that, for any $x \in M_p(\mathbb{C})$, if $[J, x] = 0$ then $[J^k, x] = 0$ for all $k \in \mathbb{Z}_p$, we can simplify the previous expression to

$$\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) = \{ J \}'$$

The structure of matrix that commutes with matrices like J, that is a permutation matrix associated to a p-cycle, is been widely studied and can be for example seen in [17]. Now that we have introduced the QMS we can move on to study its covariance with respect to $\mathbb{Z}_p$. Let $\pi \colon \mathbb{Z}_p \to M_p(\mathbb{C})$ defined as $\pi(k) = J^k$. To prove that the circulant generator is covariant under this representation directly prove the condition in Equation (3.6)

$$\begin{aligned}
\mathcal{L}(\pi(n)^* x \pi(n)) &= \sum_{k=1}^{p-1} \gamma(p-k) J^{*k} \pi(n)^* x \pi(n) J^k - \pi(n)^* x \pi(n) \\
&= \sum_{k=1}^{p-1} \gamma(p-k) J^{*k} J^{*n} x J^n J^k - \pi(n)^* x \pi(n) \\
&= \sum_{k=1}^{p-1} \gamma(p-k) J^{*n} J^{*k} x J^k J^n - \pi(n)^* x \pi(n) \\
&= \pi(n)^* \left(\sum_{k=1}^{p-1} \gamma(p-k) J^{*k} x J^k \right) \pi(n) - \pi(n)^* x \pi(n) \\
&= \pi(n)^* \left(\sum_{k=1}^{p-1} \gamma(p-k) J^{*k} x J^k - x \right) \pi(n) \\
&= \pi(n)^* \mathcal{L}(x) \pi(n)
\end{aligned}$$

for all $n \in \mathbb{Z}_p$ and all $x \in M_p(\mathbb{C})$. Moreover, Theorem 3.1.1 translates the covariance condition in the context of a GKSL representation of $\mathcal{L}$. To adapt the results of this general Theorem to this example it suffices to note that

$$\pi(n)^* L_j \pi(n) = \pi(n)^* J^j J^{*n} = \pi(n)^* J^{*n} J^j = \pi(n)^* \pi(n) L_j = L_j$$

and therefore Item (1) follows immediately, while Equation (3.7), which we recall in the general case reads as

$$\pi(g)^* L_j \pi(g) = \sum_{k \geq 1} v(g)_{jk} L_k$$

also follows by taking $V(n) = \mathbb{1}$ for all $n \in \mathbb{Z}_p$. Finally, Item (2) follows immediately since $H = \mathbb{1}$. From this we can see that the Lindblad operators are invariant under conjugation by the representation, while in the general case such conjugation would yield a linear combination of Lindblad operators. This result has direct implications on the results of Proposition 3.1.2. Indeed, recall that the Proposition states that, given one covariant GKSL representation of a generator $\mathcal{L}$, all the other equivalent GKSL representations of the same generator are not necessarily covariant. An equivalent GKSL representation preserves covariance if the associated transformation satisfied the condition of Equation (3.16), which for circulant QMSs is always trivially satisfied simply because $V(k) = \mathbb{1}$ for all $k \in \mathbb{Z}_p$, which in turn implies that all equivalent GKSL representations of circulant QMS are covariant. Another important result for covariant QMSs is given by Proposition 3.1.3, which for circulant QMSs is equivalent to ask

$$\pi(j)^* \{ J \}' \pi(j) = \{ J \}'$$

for all $j \in \mathbb{Z}_p$. To prove this for circulant QMSs, we fix any $x \in \mathcal{N}(\mathcal{T})$ and we compute

$$\begin{aligned}
[\pi(j)^* x \pi(j), J] &= J^{*j}[x J^j, J] + [J^{*j}, J] x J^j \\
&= J^{*j}[x, J] J^j + J^{*j} x [J^j, J] + [J^{*j}, J] x J^j
\end{aligned}$$

Since $x \in \mathcal{N}(\mathcal{T})$ we have $[x, J] = 0$, also obviously $[J^j, J] = 0$ and finally recalling that $J^{*j} = J^{p-j}$ as noted before, we conclude that $[\pi(j)^* x\pi(j), J] = 0$. This allows us to conclude that $\pi(j)^* x\pi(j) \in \mathcal{N}(\mathcal{T})$ for all $j \in \mathbb{Z}_p$ and $x \in \mathcal{N}(\mathcal{T})$. In particular we can see that, since any $x \in \mathcal{N}(\mathcal{T})$ commutes with both J^k and J^{*k} for all $k \in \mathbb{Z}_p$ by construction, then it is invariant under conjugation by the representation, i.e. $\pi(n)^* x\pi(n) = x$ for all $x \in \mathcal{N}(\mathcal{T})$ and $n \in \mathbb{Z}_p$. Finally we conclude this Section noting that the result involving irreducible representations do not apply in this example since the chosen representation π of $\mathbb{Z}_p$ is not irreducible. To see this simply note that, fixed $v = (1, \dots, 1) \in \mathbb{C}^p$ we have

$$Jv = v \implies J^k v = v \quad \forall\, k \in \mathbb{Z}_p$$

which implies that $\mathrm{span}(v)$ is an invariant subspace for π.

3.2 Irreducible representations

At the start of this Chapter we saw how, in the atomic case, the DF subalgebra $\mathcal{N}(\mathcal{T})$ associated to a QMS uniformly continuous $\mathcal{T}$ can be decomposed as a direct sum inducing a decomposition of the underlying Hilbert space $\mathcal{H}$. At the same time, a representation π of a compact group G can be decomposed into the sum of irreducible representation (see Appendix A) inducing another decomposition of $\mathcal{H}$. In this section we start studying the interplay between these two decomposition of the underlying Hilbert space starting from the simplest case, i.e. when π is an irreducible representation of a compact group G. In the following Proposition we begin recalling some known results.

Proposition 3.2.1. *If π is an irreducible representation, the following facts hold.*

1. *$\mathcal{H}$ is finite-dimensional.*
2. *$\mathcal{N}(\mathcal{T})$ is an atomic algebra.*
3. *The cardinality of the index set I enumerating the terms of the atomic decomposition of $\mathcal{N}(\mathcal{T})$ is finite.*

Proof. 1. See Theorem A.2.3 in the Appendix.
2. It is a well know fact that every finite-dimensional algebra is atomic (see e.g. [18], Theorem 4.4.1). Last statement is simply a characterization of atomic algebras.
3. It is clear, since $\mathcal{H}$ is a finite-dimensional space. □

In Theorem 3.0.1 we saw that there exists family of mutually orthogonal projections $(p_i)_{i\in I}$ minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ such that $\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in I} p_i \mathcal{N}(\mathcal{T}) p_i$. Therefore, a natural way to study how π acts on the decomposition of $\mathcal{N}(\mathcal{T})$ is to study the relationship between π and the family $(p_i)_{i\in I}$ as shown in the following Proposition.

Proposition 3.2.2. *Let $\mathcal{T}$ be a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$ covariant w.r.t. an irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$. Then for each $g \in G$ there exists a unique permutation σ_g of I such that*

$$p_i = \pi(g)^* p_{\sigma_g(i)}\, \pi(g), \qquad \forall\, i \in I. \tag{3.18}$$

Moreover, if $\mathcal{N}(\mathcal{T})$ is not a factor, there is at least one $g \in G$ such that σ_g is not trivial.

Proof. If $\mathcal{N}(\mathcal{T})$ is a factor then $\operatorname{card} I = 1$ which implies that the permutation σ_g must be trivial for all $g \in G$, and therefore Equation (3.18) is trivially satisfied. Assume then that $\mathcal{N}(\mathcal{T})$ is not a factor so that $\operatorname{card} I > 1$, and let $i \in I$ and $g \in G$. Since $\pi(g)$ is unitary and $\pi(g)^*\mathcal{N}(\mathcal{T})\pi(g) = \mathcal{N}(\mathcal{T})$ by Proposition 3.1.3, there exists a unique projection $q_i \in \mathcal{N}(\mathcal{T})$ depending on g, such that

$$p_i = \pi(g)^* q_i \pi(g). \tag{3.19}$$

As a first step we claim that q_i is minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ since so is p_i. Let's start by showing that $q_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$. Consider any $x \in \mathcal{N}(\mathcal{T})$, then we have

$$\pi(g)^* q_i x \pi(g) = (\pi(g)^* q_i \pi(g))\, (\pi(g)^* x \pi(g)) = p_i\, (\pi(g)^* x \pi(g))$$

but since $\pi(g)^* x \pi(g) \in \mathcal{N}(\mathcal{T})$ and $p_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ they commute and so we get

$$\pi(g)^* q_i x\, \pi(g) = (\pi(g)^* x\, \pi(g))\, p_i = (\pi(g)^* x\, \pi(g))\, (\pi(g)^* q_i \pi(g))\,.$$

Finally, the unitarity of $\pi(g)$ implies $q_i x = x q_i$, which is the thesis. Suppose now that there exists a non zero projection $q_i' \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ satisfying $q_i' \le q_i$. Clearly $\pi(g)^* q_i' \pi(g)$ belongs to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ by Proposition 3.1.3 and therefore

$$\pi(g)^* q_i' \pi(g) \le \pi(g)^* q_i \pi(g) = p_i,$$

but by the minimality of p_i we either have $\pi(g)^* q_i' \pi(g) = 0$ or $\pi(g)^* q_i' \pi(g) = p_i$. The first case implies $q_i' = 0$, which is a contradiction since we assumed q_i' to be non zero, therefore we are left with the following chain of equalities

$$\pi(g)^* q_i' \pi(g) = \pi(g)^* q_i \pi(g) = p_i$$

But, again by the unitarity of π, this means $q_i' = q_i$, i.e. q_i is minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$, proving the claim. Now, since item 2 of Proposition 3.2.1 gives

$$q_i = \sum_{j \in I} q_{j(i)} \otimes \mathbb{1}_{\mathcal{M}_{j(i)}}$$

for some projection $q_{j(i)} \in \mathcal{B}(\mathcal{K}_{j(i)})$, we immediately obtain $q_i \ge q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}}$ for all $l(i) \in I$. But q_i is minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$, to which also $q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}}$ belongs, and so we either have $q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}} = 0$ or $q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}} = q_i$. Now, if $q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}} = 0$ for all $l(i) \in I$, we have $q_i = 0$ contradicting the assumption $p_i \ne 0$. Hence, there exists a unique $l(i) \in I$ satisfying $q_{l(i)} \otimes \mathbb{1}_{\mathcal{M}_{l(i)}} = q_i$, and moreover we can define a map $\sigma\colon I \to I$ as $\sigma(i) = l(i)$. The uniqueness of $l(i)$ is a consequence of the the fact that $\{q_i\}_{i \in I}$ is a set of orthogonal projections. It follows that

$$p_i = \pi(g)^* \left(q_{\sigma(i)} \otimes \mathbb{1}_{\mathcal{M}_{\sigma(i)}} \right) \pi(g) \le \pi(g)^* \left(p_{\sigma(i)} \right) \pi(g),$$

being $q_{\sigma(i)} \otimes \mathbb{1}_{\mathcal{M}_{\sigma(i)}}$ a projection in $\mathcal{B}(\mathcal{K}_{\sigma(i)} \otimes \mathcal{M}_{\sigma(i)})$ and $p_{\sigma(i)}$ the unit of this space. Since $p_{\sigma(i)}$ is minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ we get $p_i = \pi(g)^* p_{\sigma(i)} \pi(g)$, i.e. equality (3.18) holds. The only step left is to prove that σ is a permutation. Indeed $\sigma(i) = \sigma(j)$ with $i \ne j$ implies

$$p_i = \pi(g)^* p_{\sigma(i)} \pi(g) = \pi(g)^* p_{\sigma(j)} \pi(g) = p_j,$$

which is impossible by construction. Finally, since $\mathcal{N}(\mathcal{T})$ is not a factor, we have that $p_i \neq \mathbb{1}_{\mathcal{H}}$ for all $i \in I$. If for all $g \in G$ there exists a permutation σ_g of I such that $\sigma_g(i) = i$, then

$$p_i = \pi(g)^* p_i \, \pi(g) \quad \forall\, g \in G.$$

Therefore, Schur's Lemma implies $p_i \in \mathbb{C}\mathbb{1}_{\mathcal{H}}$, contradicting the assumption. □

Given the complexity of the proof, some clarification about the meaning of this last Proposition is due. What we found is that, given the decomposition $\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i \otimes \mathcal{M}_i)$, whenever we conjugate one of the blocks $\mathcal{B}(\mathcal{K}_i \otimes \mathcal{M}_i)$ by $\pi(g)$ we map it into one of the other blocks $\mathcal{B}(\mathcal{K}_j \otimes \mathcal{M}_j)$ according to the permutation σ_g, that is $\sigma_g(i) = j$. It's also worth noting that some of the blocks can remain fixed, i.e. $\sigma_g(i) = i$, since we are ensured that this does not happen for all $g \in G$ and therefore $\mathcal{K}_i \otimes \mathcal{M}_i$ does not represent a invariant subspace of $\mathcal{H}$ w.r.t. π, since that would contradict the irreducibility of the representation. From this picture should be clear that this result imposes strong limitations on the decomposition of $\mathcal{N}(\mathcal{T})$, indeed in the next Proposition we prove that all the blocks are isomorphic to each other.

Proposition 3.2.3. *Let $\mathcal{T}$ be a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$ covariant w.r.t an irreducible representation π of a compact group G, and let $\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i \otimes \mathcal{M}_i)$ be the atomic decomposition of the DF subalgebra with* $\operatorname{card} I = d$*. Then the following statements hold:*

1. *For all $i, j \in I$ there exists $g_{ij} \in G$ such that $p_i = \pi(g_{ij})^* p_j \pi(g_{ij})$, i.e.*

$$\pi(g_{ij})^* (\mathcal{K}_j \otimes \mathcal{M}_j) = \mathcal{K}_i \otimes \mathcal{M}_i.$$

   In particular, all the spaces $\mathcal{K}_j \otimes \mathcal{M}_j$ are unitarily isomorphic and

$$\pi(g_{ij}) \mathcal{B}(\mathcal{K}_j \otimes \mathcal{M}_j) \pi(g_{ij})^* = \mathcal{B}(\mathcal{K}_i \otimes \mathcal{M}_i). \tag{3.20}$$

2. *Fixed one element $i \in I$ and set $\mathcal{K}_i \otimes \mathcal{M}_i = \mathcal{K} \otimes \mathcal{M}$, the space $\mathcal{H}$ is isometrically isomorphic to $(\mathcal{K} \otimes \mathcal{M})^d$ through the unitary operator*

$$U \colon \mathcal{H} \to \bigoplus_{j\in I} (\mathcal{K} \otimes \mathcal{M}) = (\mathcal{K} \otimes \mathcal{M})^d$$

   given by $U := \bigoplus_{j\in I} \pi(g_{ij})^*$. Moreover we have

$$U\mathcal{B}(\mathcal{H})U^* = \mathcal{B}\left((\mathcal{K} \otimes \mathcal{M})^d\right) = M_d(\mathcal{K} \otimes \mathcal{M}).$$

Proof. 1. Fix $i, j \in I$. If $i = j$ the statement is trivial taking $g_{ii} = e$, the unit of G. Assume then $i \neq j$. Since the operator

$$P = \int_G \pi(g)\, p_i \pi(g)^* \mathrm{d}g$$

intertwines π (see Lemma A.2.4 in the Appendix) and π is irreducible, Schur's Lemma gives $P = \lambda \mathbb{1}_{\mathcal{H}}$, with $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, by equation (3.18), we obtain

$$\lambda p_j = p_j \int_G \pi(g) p_i \pi(g)^* \mathrm{d}g = \int_G p_j p_{\sigma_g(i)} \, \mathrm{d}g.$$

Since the projections $(p_k)_{k\in I}$ are mutually orthogonal and $p_j \neq 0$, this implies that there exists a $g_{ij} \in G$ such that $p_j = p_{\sigma_{g_{ij}}(i)}$, i.e. $p_i = \pi(g_{ij})^* p_j \pi(g_{ij})$. This means that $\pi(g_{ij})^*(\mathcal{K}_j \otimes \mathcal{M}_j) = \mathcal{K}_i \otimes \mathcal{M}_i$.
2. It is a trivial consequence of equation (3.20). □

As we have seen in Theorem 3.0.1, from the atomic decomposition of $\mathcal{N}(\mathcal{T})$ it is possible to characterize the associated QMS $\mathcal{T}$ and its generator $\mathcal{L}$. Since we have been able to determine the atomic decomposition in the covariant case, we are now ready to give a new Theorem in which we detail the structure of a covariant generator putting together Theorem 3.1.1 and Proposition 3.2.3.

Theorem 3.2.1. *Let $\mathcal{T}$ be a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$ covariant w.r.t. an irreducible representation π of a compact group G, then, following the notations of Proposition 3.2.3, (up to a unitary isomorphism) any covariant GKSL representation of the generator $\mathcal{L}$ is given by operators*

$$\left\{ \mathbb{1}_{(\mathcal{K}\otimes\mathcal{M})^d},\, (\mathbb{1}_{\mathcal{K}} \otimes M_l)^d \,:\, l \geq 1 \right\}$$

for a collection $(M_l)_l$ of operators in $\mathcal{B}(\mathcal{M})$ such that $\sum_l M_l^* M_l$ is strongly convergent, specifically there exists $\epsilon > 0$ such that $\sum_l M_l^* M_l = \epsilon \mathbb{1}_{\mathcal{M}}$. Moreover, we have

$$\mathcal{N}(\mathcal{T}) = \left(\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{M}}\right)^d. \tag{3.21}$$

Proof. First of all we recall that, since $\mathcal{N}(\mathcal{T}) = \oplus_{j\in I} \left(\mathcal{B}(\mathcal{K}_j) \otimes \mathbb{1}_{\mathcal{M}_j}\right)$ is an atomic algebra, in any GKSL representation of $\mathcal{L}$ we have

$$H = \oplus_{j\in I} \left(K_j \otimes \mathbb{1}_{\mathcal{M}_j} + \mathbb{1}_{\mathcal{K}_j} \otimes M_0^{(j)}\right), \quad L_l = \oplus_{j\in I} \left(\mathbb{1}_{\mathcal{K}_j} \otimes M_l^{(j)}\right)$$

for self-adjoint operators $K_j \in \mathcal{B}(\mathcal{K}_j)$, $M_0^{(j)} \in \mathcal{B}(\mathcal{M}_j)$, and $M_l^{(j)} \in \mathcal{B}(\mathcal{M}_l)$ such that $\sum_l (M_l^{(j)})^* M_l^{(j)}$ is strongly convergent. Moreover, the covariance of $\mathcal{T}$ gives in particular by Theorem 3.1.1

$$H = \lambda \mathbb{1} = \oplus_{j\in I} (\lambda \mathbb{1}_{\mathcal{K}_j} \otimes \mathbb{1}_{\mathcal{M}_j})$$

for some $\lambda \in \mathbb{C}$, so that

$$p_i H p_i = \lambda \mathbb{1}_{\mathcal{K}_i} \otimes \mathbb{1}_{\mathcal{M}_i}.$$

Now, we fix $i \in I$ and set $\mathcal{K}_i = \mathcal{K}$, $\mathcal{M}_i = \mathcal{M}$, $M_0^{(i)} = M_0$, $M_h^{(i)} = M_h$ for all $h \geq 1$, and given $j \in I$, we choose $g_{ij} \in G$ such that $\pi(g_{ij}) p_i = p_j \pi(g_{ij})$ as of item 1 of Proposition 3.2.3. Therefore, multiplying on the right and on the left by p_i both sides of Equation (3.7), we obtain both

$$p_i \pi(g_{ij})^* L_h \pi(g_{ij}) p_i = \sum_l \alpha_{hl}^{(g_{ij})} p_i L_l p_i = \sum_l \alpha_{hl}^{(g_{ij})} (\mathbb{1}_{\mathcal{K}} \otimes M_l)$$

and

$$p_i \pi(g_{ij})^* L_h \pi(g_{ij}) p_i = \pi(g_{ij})^* p_j L_h p_j \pi(g_{ij}) = \pi(g_{ij})^* \left(\mathbb{1}_{\mathcal{K}_j} \otimes M_h^{(j)}\right) \pi(g_{ij}).$$

So that by putting this two relations together we get

$$\pi(g_{ij})^* \left(\mathbb{1}_{\mathcal{K}_j} \otimes M_h^{(j)}\right) \pi(g_{ij}) = \sum_l \alpha_{hl}^{(g_{ij})} (\mathbb{1}_{\mathcal{K}} \otimes M_l),$$

i.e. $\{\mathbb{1}_{\mathcal{K}_j\otimes\mathcal{M}_j}, \pi(g_{ij})^*(\mathbb{1}_{\mathcal{K}_j}\otimes M_h^{(j)})\pi(g_{ij}) \,:\, h\geq 1\}$ and $\{\mathbb{1}_{\mathcal{K}\otimes\mathcal{M}}, \mathbb{1}_{\mathcal{K}}\otimes M_h \,:\, h\geq 1\}$ induce, by Theorem 2.2.2, the same GKSL representation of $\mathcal{L}^{\mathcal{K}\otimes\mathcal{M}}$, the generator of the QMS $\mathcal{T}^{\mathcal{K}\otimes\mathcal{M}} = \mathcal{T}^{\mathcal{K}_i\otimes\mathcal{M}_i}$ on $\mathcal{B}(\mathcal{K}\otimes\mathcal{M})$ obtained by restriction. Since equation (3.20) holds, we can conclude that

$$\pi(g_{ij})^*\, \mathcal{L}^{\mathcal{K}_j\otimes\mathcal{M}_j}\left(\pi(g_{ij})x\pi(g_{ij})^*\right)\pi(g_{ij}) = \mathcal{L}^{\mathcal{K}\otimes\mathcal{M}}(x) \quad \forall\, x\in\mathcal{K}\otimes\mathcal{M},$$

that is all QMSs $\mathcal{T}^{\mathcal{K}_j\otimes\mathcal{M}_j}$ are isometrically isomorphic to $\mathcal{T}^{\mathcal{K}\otimes\mathcal{M}}$. Finally, recalling the definition of U in item 2 in Proposition 3.2.3, we have

$$\begin{aligned} UHU^* &= \sum_{j\in I}(\lambda\mathbb{1}_{\mathcal{K}\otimes\mathcal{M}}) = \lambda\mathbb{1}_{(\mathcal{K}\otimes\mathcal{M})^d} \\ UL_hU^* &= \oplus_{j\in I}\left(\mathbb{1}_{\mathcal{K}}\otimes M_h\right) = \oplus_{d\text{ times}}\left(\mathbb{1}_{\mathcal{K}}\otimes M_h\right). \end{aligned}$$

and by a direct application of Proposition 3.1.1 we get $\sum_l M_l^* M_l = \epsilon\mathbb{1}_{\mathcal{M}}$. In order to conclude the proof we have to prove Equation (3.21): so, let $x\in\mathcal{B}(\mathcal{K}_j)$ and consider the product

$$U(x\otimes\mathbb{1}_{\mathcal{M}_j})U^* = \pi(g_{ij})^*(x\otimes\mathbb{1}_{\mathcal{M}_j})\pi(g_{ij}).$$

Since $x\otimes\mathbb{1}_{\mathcal{M}_j}$ belongs to $\mathcal{N}(\mathcal{T})$, by Proposition 3.1.3 also $\pi(g_{ij})^*(x\otimes\mathbb{1}_{\mathcal{M}_j})\pi(g_{ij})$ is in this algebra. On the other hand it belongs to $\mathcal{B}(\mathcal{K}\otimes\mathcal{M})$ for item 1 in Proposition 3.2.3, and then $\pi(g_{ij})^*(x\otimes\mathbb{1}_{\mathcal{M}_j})\pi(g_{ij})\in\mathcal{B}(\mathcal{K})\otimes\mathbb{1}_{\mathcal{M}}$. This happens for all $j\in J$, allowing us to conclude that

$$\begin{aligned} U\mathcal{N}(\mathcal{T})\,U^* &= \oplus_{j\in J}\pi(g_{ij})^*\left(\mathcal{B}(\mathcal{K}_j)\otimes\mathcal{M}_j)\right)\pi(g_{ij}) \\ &= \oplus_{j\in J}(\mathcal{B}(\mathcal{K})\otimes\mathcal{M}) \\ &= (\mathcal{B}(\mathcal{K})\otimes\mathcal{M})^d. \end{aligned}$$ □

Up until now we avoided any hypothesis on the group G other than compactness and we focused solely on its representation π. In the following we will show that the topology of G actually plays a big role on the properties of the DF subalgebra for a covariant QMS. More precisely there is a strong connection between the number of connected components of G and the number of factors that appear in the decomposition of $\mathcal{N}(\mathcal{T})$.

Theorem 3.2.2. *If G is connected and $\mathcal{T}$ is a uniformly continuous QMS covariant with respect to an irreducible representation π, then $\mathcal{N}(\mathcal{T})$ is a type I factor.*

Proof. Let I be the index set in the decomposition of $\mathcal{N}(\mathcal{T})$, assume it to have a cardinality strictly greater than 1, and let $i\in I$. Given $v\in p_i(\mathcal{H})$, $\|v\|=1$, using the continuity at e of the map $G\ni g\mapsto\pi(g)v\in\mathcal{H}$, there exists a open neighborhood U of e such that

$$\|v-\pi(g)v\| < 1 \quad \forall\, g\in U. \tag{3.22}$$

Taken $g\in U\setminus\{e\}$, by Proposition 3.2.2 we have $\pi(g)p_i = p_j\pi(g)$ for some $j\in I$; now, if $j\neq i$, then

$$\|v-\pi(g)v\|^2 = \|v\|^2+\|\pi(g)v\|^2 = 2,$$

since v and $\pi(g)v$ are orthogonal (recall that $v \in p_i(\mathcal{H})$ and $\pi(g)v \in p_j(\mathcal{H})$). This clearly contradicts inequality (3.22) giving $\pi(g)p_i = p_i\pi(g)$ for all $g \in U$. Since G is connected and U is an open neighborhood of e, we have $G = \cup_{n\geq 1} U^n$ (see [19]), and so we can write any $h \in G$ as $h = g_1 \cdots g_n$ for some $n \geq 1$ and $g_1, \dots, g_n \in U$. Hence, we obtain

$$\pi(h)p_i = \pi(g_1)\cdots\pi(g_n)p_i = \pi(g_1)\cdots p_i\pi(g_n) = \ldots = p_i\pi(g_1)\cdots\pi(g_n) = p_i\pi(h)$$

i.e. p_i intertwines π, and then $p_i = \mathbb{1}$ by Schur's Lemma. This means that I has cardinality 1, and consequently, $\mathcal{N}(\mathcal{T})$ is a type I factor. □

This result, despite being very easy to prove, is very powerful, since it tells us that by just assuming that the group is connected and compact we completely fix the decomposition of the DF subalgebra $\mathcal{N}(\mathcal{T})$ (namely there is no decomposition at all). Moreover, we can further extend this result to not connected groups and show that the structure of the group has to be very rich in order for the decomposition of $\mathcal{N}(\mathcal{T})$ to be comprised of many elements. To be able to prove such result we first recall that, if G is not connected, it can be written as disjoint union of its connected components $\{G_\alpha\}_{\alpha\in A}$ for some set A, i.e.

$$G = \cup_{\alpha\in A} G_\alpha, \tag{3.23}$$

where G_{α_e}, $\alpha_e \in A$, contains the identity e. In addition G_{α_e} is a (normal) connected subgroup of G, so that, for all open neighborhood V of e, we have

$$G_{\alpha_e} = \cup_{n\geq 1} V^n. \tag{3.24}$$

Following these notations we can prove the following results.

Theorem 3.2.3. *Let $\mathcal{T}$ be a uniformly continuous QMS covariant w.r.t. an irreducible representation π of G on $\mathcal{H}$, then the following facts hold:*

1. *$\pi(g)p_l = p_l\pi(g)$ for all $g \in G_{\alpha_e}$ and $l \in I$,*
2. *given $i, j \in I$ with $i \neq j$, there exists $\alpha = \alpha_{ij} \in A$, $\alpha \neq \alpha_e$ such that:*
   (i) $\pi(g)p_i = p_j\pi(g)$,
   (ii) $\pi(g)^*(\mathcal{K}_j \otimes \mathcal{M}_j) = \mathcal{K}_i \otimes \mathcal{M}_i$,
   (iii) $\pi(g)^*\mathcal{B}(\mathcal{K}_j \otimes \mathcal{M}_j)\pi(g) = \mathcal{B}(\mathcal{K}_i \otimes \mathcal{M}_i)$

   for all $g \in G_\alpha$,
3. *the number of connected components of G is greater or equal than the cardinality of I, i.e.* $\mathrm{card}(A) \geq \mathrm{card}(I)$.

Proof. If G is connected, we can immediately conclude by Theorem 3.2.2. Assume then that G is not connected, so that we can express G as in equation (3.23), and let $i \in I$. Since G_{α_e} is a connected group, in a similar way to the proof of Theorem 3.2.2 (applied to G_{α_e}), we obtain $\pi(g)p_i = p_i\pi(g)$ for all $g \in G_{\alpha_e}$. Since the same argument holds for all $i \in I$ simply changing the choice of the open neighborhood U, we can conclude that

$$\pi(g)p_i = p_i\pi(g) \quad \forall\, g \in G_{\alpha_e},\ i \in I, \tag{3.25}$$

proving statement 1. 2. Let $i, j \in I$, $i \neq j$. By Proposition 3.2.3 there exists an element $g_0 := g_{ij} \in G$ such that $\pi(g_0)p_i = p_j\pi(g_0)$. This means that g_0 has to belong to a connected component different from G_{α_e} (by equation (3.25)), i.e. there exists $\alpha = \alpha_{ij} \neq \alpha_e$ such that $g_0 \in G_\alpha$ Now, since the map $a\colon G_\alpha \times G_\alpha \to G$, defined as $a(g,h) = gh^{-1}$ is continuous, its image is a connected subspace containing $e = a(g_0, g_0)$, and so it is in G_{α_e}. Therefore, for all $g \in G_\alpha$ with $g \neq g_0$, we have $gg_0^{-1} \in G_{\alpha_e}$, so that

$$p_j\pi(g)\pi(g_0)^* = p_j\pi(gg_0^{-1}) = \pi(gg_0^{-1})p_j = \pi(g)\pi(g_0)^*p_j = \pi(g)p_i\pi(g_0)^*.$$

We then obtain

$$p_j\pi(g) = \pi(g)p_i \quad \forall\, g \in G_\alpha. \tag{3.26}$$

Statement (ii) and (iii) trivially follow. 3. The arbitrariness of $j \neq i$ implies that the cardinality of $A \setminus \{\alpha_e\}$ is greater than the number of indexes in I different from i: indeed, if we have $\pi(g_{ik})p_i = p_k\pi(g_{ik})$ for some $g_{ik} \in G$ with $k \neq j$ and $k \neq i$, then $g_{ik} \notin G_{\alpha_e} \cap G_\alpha$ by equations (3.25) and (3.26). We can then conclude that the number of connected components of G is greater the cardinality of I, i.e. the number of $\mathcal{N}(\mathcal{T})$'s factors. ☐

As a particular case of this result we obtain the vice-versa of Theorem 3.2.2.

Corollary 3.2.1. *$\mathcal{N}(\mathcal{T})$ is a factor if and only if G is connected.*

In Theorem 3.2.3 we simply stated that the number of connected components of G has to be greater than the number of factors of $\mathcal{N}(\mathcal{T})$. This should not be surprising since by Theorem 3.2.3 we now know that each connected component represent only one of the many isomorphisms needed to achieve the results of Proposition 3.2.3, and therefore a number of connected components at least equal to the number of factors is needed. In the following Theorem we are able to remove the vagueness left by Theorem 3.2.3 about the connection between card A and card I showing that there is actually a strong connection among the two.

Theorem 3.2.4. *Assume $\mathcal{T}$ to be a uniformly continuous QMS covariant w.r.t. an irreducible representation π of G on $\mathcal{H}$. Given $i, j \in I$, let k_{ij} be the number of different connected components G_α such that $\pi(g)p_i = p_j\pi(g)$ with $g \in G_\alpha$. Then:*

1. *k_{ij} is independent by i, j, i.e. $k_{ij} = k$ for some $k \geq 1$ and for all $i, j \in I$;*
2. *if n is the number of connected components of G, and d the cardinality of I, then $n = kd$.*

Proof. In the proof of Proposition 3.2.3, we observed that

$$\lambda\mathbb{1}_{\mathcal{H}} = \int_G \pi(g)\, p_i\pi(g)^*\mathrm{d}g = \int_G p_{\sigma_g(i)}\, \mathrm{d}g \quad \forall\, l \in I$$

with $\sigma_g(i)$ such that $p_i = \pi(g)^*p_{\sigma_g(i)}\pi(g)$. In particular

$$\lambda\sum_{i=1}^{d} \mathbb{1}_{\mathcal{H}} = \sum_{i=1}^{d} \int_G \pi(g)\, p_i\pi(g)^*\mathrm{d}g = \mathbb{1}_{\mathcal{H}},$$

so $\lambda = 1/d$, where d is number of type I factors of $\mathcal{N}(\mathcal{T})$. Thus, by Theorem 3.2.3, we have that

$$\frac{1}{d} = w(\{g \in G : p_{\sigma_g(i)} = p_j\}) = \sum_{\alpha \in A_{ij}} w(G_\alpha), \tag{3.27}$$

where $A_{ij} = \{\alpha \in A : \pi(g)p_i = p_j\pi(g) \;\; \forall\, g \in G_\alpha\}$ e w is the Haar measure. Since every $G_\alpha = gG_{\alpha_e}$ for some $g \in G_\alpha$, $w(G_\alpha) = w(gG_{\alpha_e}) = w(G_{\alpha_e})$. This implies that

$$\frac{1}{d} = k_{ij}\, w(G_{\alpha_e}),$$

where k_{ij} is the cardinality of A_{ij}. As a consequence $k_{ij} = k$ independent with respect to i and j. Now, since the union of all n connected components is a partition of G, we deduce that

$$n\, w(G_{\alpha_e}) = w(\cup_\alpha G_\alpha) = 1$$

i.e. $n = kd$ and this concludes the proof. $\square$

With this result we have concluded the characterization of a QMS $\mathcal{T}$ covariant w.r.t. an irreducible representation π. In the next section we will build upon the knowledge we gained about the action of irreducible representation to obtain results about more general representation.

3.3 Arbitrary unitary representations

The next step in the study covariant QMSs is to consider a generic representation π instead of a irreducible one. To achieve this we are going to leverage the Peter-Weyl Theorem which states that any representation π on an Hilbert space $\mathcal{H}$ can be decomposed as a direct sum of irreducible representations, and that also $\mathcal{H}$ decomposes accordingly (see Appendix A for the details). The idea is thus to exploit the results we obtained in the previous section about irreducible representations and applying them to every irreducible sub-representation. Before directing our attention to the implications of the Peter-Weyl Theorem, given covariant QMS $\mathcal{T}$ w.r.t. a representation π, we want to study what happens when we restrict the representation to a closed subspace. In this regard we recall that given a closed subspace V of $\mathcal{H}$ it is possible to consider the restriction π_V of π to V, $\pi_V : G \to \mathcal{B}(V)$ by setting

$$\pi_V(g)v = \pi(g)v \quad \forall\, v \in V,\; g \in G. \tag{3.28}$$

obtaining in this way a unitary sub-representation of π. First of all we show that, if π_V is irreducible, the reduced semigroup $\mathcal{T}^p$ associated to the orthogonal projection p onto V is a QMS on $\mathcal{B}(V)$.

Proposition 3.3.1. *Let V be a finite-dimensional π-invariant subspace of $\mathcal{H}$ such that the restriction π_V of π to V is irreducible, and q the orthogonal projection onto V. Let $\mathcal{T}$ be uniformly continuous QMS covariant w.r.t. π. Then the following facts hold:*

1. *$\pi(g)q = q\,\pi(g)$ for all $g \in G$, so that the representation $\pi_V : G \to \mathcal{B}(V)$ is given by*
$$\pi_V(g) = q\pi(g)q = \pi(g)q = q\pi(g) \quad \forall\, g \in G, \tag{3.29}$$

2. *q is $\mathcal{T}$-subharmonic (see Section 2.3 for the relevant results),*

3. *the reduced semigroup $\mathcal{T}^q$ on $\mathcal{B}(V)$,*

$$\mathcal{T}_t^q(x) = q\mathcal{T}_t(x)q \qquad \forall\, x \in \mathcal{B}(V),\ t \geq 0,$$

is a π_V-covariant QMS.

Proof. **1.** Since $\pi(g)q(\mathcal{H}) = \pi(g)V \subseteq q(\mathcal{H})$, we have $\pi(g)q = q\pi(g)q$ for all $g \in G$. Taking the adjoint we get

$$q\pi(g)^* = (\pi(g)q)^* = (q\pi(g)q)^* = q\pi(g^{-1})q = \pi(g)^*q.$$

As a consequence $\pi_V(g) = \pi(g)q = q\pi(g)q = q\pi(g)q$, i.e Equation (3.29) holds.
2. The commutation between $\pi(g)$ and q, and the covariance of $\mathcal{T}$ imply

$$\mathcal{T}_t(q) = \mathcal{T}_t(\pi(g)^*\pi(g)q) = \mathcal{T}_t(\pi(g)^*q\,\pi(g)) = \pi(g)^*\mathcal{T}_t(q)\pi(g)$$

for all $g \in G$ and $t \geq 0$. Therefore, multiplying $\mathcal{T}_t(q)$ by q on both sides, we get

$$q\mathcal{T}_t(q)q = \pi(g)^*q\mathcal{T}_t(q)q\,\pi(g) = \pi_V(g)^*\left(q\mathcal{T}_t(q)q\right)\pi_V(g) \tag{3.30}$$

for all $g \in G$ and $t \geq 0$. Since $q\mathcal{T}_t(q)q$ belongs to $\mathcal{B}(V) = \mathcal{B}(q(\mathcal{H}))$, it intertwines π_V, so that $q\mathcal{T}_t(q)q \in \mathbb{C}\mathbb{1}$ by Schur's Lemma (being π_V irreducible). That is, for each $t \geq 0$ there exists $\lambda(t) \in \mathbb{C}$ such that $q\mathcal{T}_t(q)q = \lambda(t)q$. In particular, by the inequalities

$$0 \leq q\mathcal{T}_t(q)q \leq q \quad \forall\, t \geq 0$$

and the continuity of the map $t \mapsto q\mathcal{T}_t(q)q = \lambda(t)q$, we obtain a continuous function $\lambda\colon [0, +\infty) \to [0, 1]$ such that $\lambda(0) = 1$. We want to show that $\lambda(t) = 1$ also for all $t > 0$, so that $q\mathcal{T}_t(q)q = q$ for all $t \geq 0$, i.e. q is subharmonic. If we assume this not to be true, then we can find $t_0 > 0$ such that $0 < \lambda(t_0) < 1$. Indeed, in the opposite case, for all $t > 0$ we either have $\lambda(t) = 0$ or $\lambda(t) = 1$, i.e. the function λ takes value in $\{0, 1\}$. But the continuity of λ forces to either have $\lambda(t) = 0$ for all $t > 0$ or $\lambda(t) = 1$ for all $t > 0$. The second case contradicts the assumption done on λ, whether the first one gives $1 = \lambda(0) = \lim_{t\to 0}\lambda(t) = 0$ which is impossible. Therefore, there exists $t_0 > 0$ such that $0 < \lambda(t_0) < 1$. Now, taken $v \in V = q(\mathcal{H})$ with $\|v\| = 1$, the vector $w \coloneqq q\mathcal{T}_{t_0}(q)qv = \lambda(t_0)qv = \lambda(t_0)v$ is then a non-zero element in V, so that the set $\{\pi(g)w \,:\, g \in G\}$ is dense in V, being π_V an irreducible representation on V. Therefore, if we define $\epsilon \coloneqq 1 - \lambda(t_0) > 0$, there exists $g \in G$ satisfying

$$\|\pi(g)\lambda(t_0)v - v\| = \|\pi(g)w - v\| < 1 - \lambda(t_0).$$

As a consequence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
(1 - \lambda(t_0))^2 &> \|\pi(g)\lambda(t_0)v - v\|^2 \\
&= \lambda(t_0)^2\|v\|^2 + \|v\|^2 - 2\lambda(t_0)\operatorname{Re}\langle \pi(g)v, v\rangle \\
&\geq \lambda(t_0)^2 + 1 - 2\lambda(t_0)\|\pi(g)v\|\|v\| \\
&= \lambda(t_0)^2 + 1 - 2\lambda(t_0) \\
&= (1 - \lambda(t_0))^2,
\end{aligned}$$

which is a contradiction. Therefore $\lambda(t) = 1$ for all $t \geq 0$, giving q subharmonic.
3. Let $g \in G$, $t \geq 0$ and $x = qxq \in \mathcal{B}(q(\mathcal{H})$. The π-covariance of $\mathcal{T}$ and item 1 give

$$\mathcal{T}^q(\pi(g)^* x\pi(g)) = q\mathcal{T}_t(\pi(g)^* x\pi(g))q = \pi(g)^* q\mathcal{T}_t(x)q\,\pi(g) = \pi_V(g)^*\mathcal{T}_t^q(x)\pi_V(g),$$

which is the thesis. □

We can now analyze the general case. Indeed, by applying the Peter-Weyl Theorem, we know that there exists a family of closed, finite-dimensional, mutually orthogonal subspaces $(V_j)_{j\in J}$ of $\mathcal{H}$ such that $\mathcal{H} = \oplus_{j\in J} V_j$. Moreover, each V_j is π-invariant and such that the restriction π_j of π to V_j is irreducible. By Equation (3.28), the representation $\pi_j\colon G \to \mathcal{B}(V_j)$ is given by

$$\pi_j(g) = q_j\pi(g)q_j = \pi(g)q_j = q_j\pi(g) \quad \forall\, g \in G,$$

where every q_j denotes the orthogonal projection onto V_j. Clearly each of the subspaces V_j falls in the context of the previous Proposition and we can use those results.

Theorem 3.3.1. *Let $\mathcal{H} = \oplus_{j\in J} V_j$ be the decomposition obtained through the Peter-Weyl Theorem applied to a representation π of G, and $(q_j)_{j\in J}$ be the family of projections onto each V_j. Also, let $\mathcal{T}$ be uniformly continuous QMS covariant with respect to π. Then the family $(q_j)_{j\in J}$ is a collection of finite-dimensional mutually orthogonal non-zero projections such that $\sum_{j\in J} q_j = \mathbb{1}$. Moreover, if there exists a normal faithful invariant state, then every q_j is a fixed point of $\mathcal{T}$.*

As a consequence of Theorem 3.3.1, since every q_j is a fixed point, we immediately obtain the equality $\mathcal{N}(\mathcal{T}^j) = q_j\mathcal{N}(\mathcal{T})q_j$, where the reduced semigroup $\mathcal{T}^j$ associated with the projection q_j is simply the restriction of $\mathcal{T}$ to $\mathcal{B}(q_j(\mathcal{H})) = \mathcal{B}(V_j)$. Moreover $\mathcal{T}^j$ is covariant with respect to the irreducible representation π_j. We want then to analyze the relations between the atomic decomposition of every $\mathcal{N}(\mathcal{T}^j)$, and that one of $\mathcal{N}(\mathcal{T})$. Let's start with the simplest case, i.e. when $\mathcal{N}(\mathcal{T})$ is a type I factor.

Proposition 3.3.2. *Let $\mathcal{T}$ be a uniformly continuous QMS with a faithful invariant state such that $\mathcal{N}(\mathcal{T}) = \mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{M}}$ is a type I factor for some Hilbert spaces $\mathcal{K}$, $\mathcal{M}$ satisfying $\mathcal{H} = \mathcal{K} \otimes \mathcal{M}$, and let $(q_j)_{j\in J}$ be a family of harmonic mutually orthogonal projections summing up to the identity. Then:*

1. *$q_j(\mathcal{H}) = \mathcal{K}'_j \otimes \mathcal{M}$ for some $\mathcal{K}'_j$ closed subspace of $\mathcal{K}$, $j \in J$,*
2. *$\oplus_{j\in J}\mathcal{K}'_j = \mathcal{K}$,*
3. *$q_j\mathcal{N}(\mathcal{T})\,q_j = \mathcal{B}(\mathcal{K}'_j) \otimes \mathbb{1}_{\mathcal{M}}$,*

Proof. Let $j \in J$. Since q_j belongs to $\mathcal{N}(\mathcal{T})$ (it is a fixed point), we have $q_j = q'_j \otimes \mathbb{1}_{\mathcal{M}}$ for some q'_j projection in $\mathcal{B}(\mathcal{K})$, i.e. $q_j(\mathcal{H}) = \mathcal{K}'_j \otimes \mathcal{M}$ with $\mathcal{K}'_j = q'_j(\mathcal{H})$ a closed subspace of $\mathcal{K}$. Therefore the equalities

$$\mathcal{K} \otimes \mathcal{M} = \mathcal{H} = \oplus_{j\in J}\left(\mathcal{K}'_j \otimes \mathcal{M}\right) = \left(\oplus_{j\in J}\mathcal{K}'_j\right) \otimes \mathcal{M}$$

give $\mathcal{K} = \oplus_{j\in J}\mathcal{K}'_j$. Moreover $q_j\mathcal{N}(\mathcal{T})\,q_j = q_j\,(\mathcal{B}(\mathcal{K})\otimes \mathbb{1}_{\mathcal{M}})\,q_j = \mathcal{B}(\mathcal{K}'_j)\otimes \mathbb{1}_{\mathcal{M}}$, and so

$$\oplus_{j\in J}q_j\mathcal{N}(\mathcal{T})\,q_j = \oplus_{j\in J}\left(\mathcal{B}(\mathcal{K}'_j)\otimes \mathbb{1}_{\mathcal{M}}\right) = \left(\oplus_{j\in J}\mathcal{B}(\mathcal{K}'_j)\right)\otimes \mathbb{1}_{\mathcal{M}}.$$

This means in particular that we can have $\mathcal{N}(\mathcal{T}) = \oplus_{j\in J}q_j\mathcal{N}(\mathcal{T})\,q_j$ if and only if $\mathcal{B}(\mathcal{K}) = \oplus_{j\in J}\mathcal{B}(\mathcal{K}'_j)$, i.e. $J=\{j_0\}$ and $q'_{j_0} = \mathbb{1}_{\mathcal{K}}$. Clearly this is equivalent to have $q_{j_0} = \mathbb{1}$. □

We can then extend the previous result to the case $\mathcal{N}(\mathcal{T})$ atomic.

Theorem 3.3.2. *Let π be a unitary representation of a compact group G on $\mathcal{B}(\mathcal{H})$ such that $\mathcal{H} = \oplus_{j\in J}V_j$ with $(V_j)_{j\in J}$ a family of invariant subspaces on which the restriction of π is irreducible. Denote by q_j the orthogonal projection onto V_j. Let $\mathcal{T}$ be a π-covariant uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$ having a faithful invariant state, and assume $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I}\left(\mathcal{B}(\mathcal{K}_i)\otimes \mathbb{1}_{\mathcal{M}_i}\right)$ for some collection $(\mathcal{K})_i$ and $(\mathcal{M}_i)_i$ of separable Hilbert spaces such that $\mathcal{H} = \oplus_{i\in I}\left(\mathcal{K}_i\otimes\mathcal{M}_i\right)$. Then for all $j\in J$ we have*

$$V_j = q_j(\mathcal{H}) = \oplus_{i\in I_j}\left(\mathcal{K}_i^j\otimes\mathcal{M}_i\right) = \left(\mathcal{K}_{i_0}^j\otimes\mathcal{M}_{i_0}\right)^{d_j} \tag{3.31}$$

for some family $(\mathcal{K}_i^j)_{i\in I_j}$ of non zero closed subspaces of $\mathcal{K}_i$, $i_0\in I_j$ given by $I_j := \{i\in I \,:\, q_jp_i\neq 0\}$ and d_j the cardinality of this set. Moreover, $\oplus_{j\in J}\mathcal{K}_i^j = \mathcal{K}_i$ for all $i\in I$, and

$$q_j\mathcal{N}(\mathcal{T})\,q_j = \left(\mathcal{B}(\mathcal{K}_{i_0}^j)\otimes \mathbb{1}_{\mathcal{M}_{i_0}}\right)^{d_j}$$

for all $j\in J$.

Proof. Define p_i the orthogonal projection onto $\mathcal{K}_i\otimes\mathcal{M}_i$, $i\in I$. Since p_i and q_j commute for all $j\in J$ and $i\in I$ (p_i is in the center of $\mathcal{N}(\mathcal{T})$ and q_j belongs to it), $q_jp_i = p_iq_j$ is a projection satisfying $\mathcal{T}_t(q_jp_i) = q_jp_i$ for all $t\geq 0$ (see Theorem 3.3.1). Therefore, fixed $i\in I_j$ to have q_jp_i non zero, $(q_jp_i)_{j\in J}$ is a family of harmonic mutually orthogonal projections on $\mathcal{K}_i\otimes\mathcal{M}_i$ such that

$$\sum_{j\in J} q_jp_i = p_i = \mathbb{1}_{\mathcal{K}_i\otimes\mathcal{M}_i}.$$

Moreover, called $\mathcal{T}^i$ the restriction of $\mathcal{T}$ to $\mathcal{B}(p_i(\mathcal{H}))$, the equalities $\mathcal{N}(\mathcal{T}^i) = p_i\mathcal{N}(\mathcal{T})\,p_i = \mathcal{B}(\mathcal{K}_i)\otimes\mathbb{1}_{\mathcal{M}_i}$ hold. We can then apply Proposition 3.3.2 getting

$$q_jp_i(\mathcal{H}) = q_j(\mathcal{K}_i\otimes\mathcal{M}_i) = \mathcal{K}_i^j\otimes\mathcal{M}_i \quad \text{and} \quad q_j\mathcal{N}(\mathcal{T}^i)\,q_j = \mathcal{B}(\mathcal{K}_i^j)\otimes\mathbb{1}_{\mathcal{M}_i}$$

for some non zero closed subspace $\mathcal{K}_i^j$ of $\mathcal{K}_i$ such that $\oplus_{j\in J}\mathcal{K}_i^j = \mathcal{K}_i$. Therefore

$$q_j(\mathcal{H}) = \oplus_{i\in I_j}q_jp_i(\mathcal{H}) = \oplus_{i\in I_j}\left(\mathcal{K}_i^j\otimes\mathcal{M}_i\right) \quad \forall\, j\in J$$

and

$$q_j\mathcal{N}(\mathcal{T})\,q_j = \oplus_{i\in I_j}\left(q_j\mathcal{N}(\mathcal{T}^i)\,q_j\right) = \oplus_{i\in I_j}\left(\mathcal{B}(\mathcal{K}_i^j)\otimes\mathbb{1}_{\mathcal{M}_i}\right), \tag{3.32}$$

being $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I} p_i \mathcal{N}(\mathcal{T})\, p_i = \oplus_{i\in I}\mathcal{N}(\mathcal{T}^i)$ and $q_j p_i = 0$ for $i \in I_j$. Denoting by $\mathcal{T}^{q_j}$ the restriction of $\mathcal{T}$ to $\mathcal{B}(q_j(\mathcal{H}))$, Equation (3.32) gives exactly the atomic decomposition of $\mathcal{N}(\mathcal{T}^{q_j}) = q_j\mathcal{N}(\mathcal{T})\, q_j$. Finally, since the restriction of the representation π to $q_j(\mathcal{H})$ is irreducible, Proposition 3.2.3 and Theorem 3.2.1 imply

$$q_j(\mathcal{H}) = \left(\mathcal{K}^j_{i_0} \otimes \mathcal{M}_{i_0}\right)^{d_j} \quad \text{and} \quad q_j\mathcal{N}(\mathcal{T})\, q_j = \left(\mathcal{B}(\mathcal{K}^j_{i_0}) \otimes \mathbb{1}_{\mathcal{M}_{i_0}}\right)^{d_j}$$

with d_j the cardinality of I_j. □

3.4 Examples

We conclude this section on the analysis of covariant QMSs with some examples on non irreducible representations in order to give a clearer exposition on the results proven so far. Note that in general, if the representation π is not irreducible, the decoherence-free subalgebra of a π-covariant QMS could be not atomic.

Example 3.4.1. Let be consider $G = U(2)$, the compact group of unitary 2×2 matrices, and define a unitary representation π of G on $\mathcal{H} = \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ by setting

$$\pi(g)(u \oplus v) = gu \oplus gv \qquad \forall\, g \in G \;\; u, v \in \mathbb{C}^2.$$

In other words, π is the direct sum of irreducible representations π_1 and π_2 of G on $V_1 = V_2 = \mathbb{C}^2$, with $\pi_i(g) = gz$ for all $g \in G$ and $z \in \mathbb{C}^2$. If we consider on $\mathcal{B}$ the trivial QMS $\mathcal{T}$, defined by $\mathcal{T}_t(x) = x$ for each $x \in \mathcal{B}$ and $t \geq 0$, this is obviously covariant with respect to π and $\mathcal{N}(\mathcal{T}) = \mathcal{B}$. In according with the notations of the previous theorem, q_j is the orthogonal projection into V_j, for $j = 1, 2$, and $q_j\mathcal{N}(\mathcal{T})q_j = \mathcal{B}(\mathbb{C}^2)$. We can observe that in this case $\mathcal{N}(\mathcal{T}) \neq \oplus_{j=1,2}\mathcal{B}(\mathbb{C}^2)$.

Example 3.4.2. In this example we are going to provide a concrete realization of a covariant QMS where its decoherence free algebra is *not* type I but of type II_1. In order to do so, we first recall a few simple objects from the theory of (discrete) groups. Let G be a discrete group with unit element e and consider the Hilbert space of square integrable functions defined on the group, that is $\mathcal{H} = \ell^2(G)$. Such Hilbert space has a natural orthonormal basis that we denote as $\{\, 1_g \,\}_{g\in G}$, and defined as

$$1_g(h) = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}$$

Given two elements $u, v \in \ell^2(G)$ there is a natural definition of convolution, denote by $u * v$, which gives an element in $\ell^\infty(G)$ as

$$(u * v)(g) = \sum_{h\in G} u(gh^{-1})v(h) \qquad \forall g \in G$$

Thanks to the convolution we can define two linear maps from $\ell^2(G)$ to $\ell^\infty(G)$ for each $u \in \ell^2(G)$ as

$$L_u(v) = u * v \qquad R_u(v) = v * u$$

and from these maps we define the following associated algebras

$$\mathcal{L}_G = \left\{ \, L_u \;\middle|\; u \in \ell^2(G),\; L_u \in \mathcal{B}(\ell^2(G)) \, \right\}$$
$$\mathcal{R}_G = \left\{ \, R_u \;\middle|\; u \in \ell^2(G),\; R_u \in \mathcal{B}(\ell^2(G)) \, \right\}$$

For readability sake, we recall also a few key properties satisfied by the objects just defined:

1. $\left(L_{1_g} u\right)(h) = u(g^{-1}h)$ and $\left(R_{1_g} u\right)(h) = u(hg^{-1})$ for all $g, h \in G$ and all $u \in \ell^2(G)$;

2. $L_{1_g} + L_{1_h} = L_{1_g + 1_h}$ and $\alpha L_{1_g} = L_{\alpha 1_g}$ for all $g, h \in G$ and $\alpha \in \mathbb{C}$;

3. $L_{1_g} L_{1_h} = L_{1_g * 1_h} = L_{1_{g^{-1}}}$ and $L^*_{1_g} = L_{1_{g^{-1}}}$ for all $g, h \in G$;

4. $L_{1_e} = \mathbb{1}$

where analogous properties to item 2, 3, and 4 can be proven for R_{1_g}. Moreover, the maps L_{1_g} and R_{1_g} belong to $\mathcal{B}(\ell^2(G))$ and are unitary operators for all $g \in G$, they generate the two von Neumann algebras $\mathcal{L}_G$ and $\mathcal{R}_G$ respectively, which allows to prove that $\mathcal{L}'_G = \mathcal{R}_G$. For a complete proof of the properties just listed see [20, Theorem 6.7.2]. The following Theorem, which descends directly from [20, Proposition 6.7.4, Theorem 6.7.5], allows us to classify the algebras $\mathcal{L}_G$ and $\mathcal{R}_G$.

Theorem 3.4.1. *The von Neumann algebras $\mathcal{L}_G$ and $\mathcal{R}_G$ are finite. Moreover, if $G \neq \{ \, e \, \}$ and the conjugacy class of every $g \neq e$ is infinite, then $\mathcal{L}_G$ and $\mathcal{R}_G$ are type II_1 factors.*

A group satisfying the hypothesis of the Theorem just stated is the free group $\mathcal{F}_n$ of $n \neq 2$ generators, that is the group of words of arbitrary length generated from n letters (see [20, Example 6.7.6.]). Finally, we have introduced all the required notions needed to define and study a covariant QMS with a type II_1 decoherence free subalgebra. Consider the group $\mathcal{F}_n$ for some $n > 2$ and the Hilbert space $\ell^2(\mathcal{F}_n)$ exactly as before. Then, for every $x \in \mathcal{B}(\ell^2(\mathcal{F}_n))$ we define the bounded operator

$$\mathcal{L}(x) = \sum_{g \in G} R^*_{1_g} x R_{1_g} - x$$

where the sum in countable since $\mathcal{F}_n$ is a discrete group. Since $\mathcal{L}$ is expressed in a GKSL form it generates a uniformly continuous QMS $\mathcal{T}$ on $\mathcal{B}(\ell^2(\mathcal{F}_n))$ such that

$$\mathcal{N}(\mathcal{T}) = \left\{ \, R_{1_g} \;\middle|\; g \in G \, \right\}' = \mathcal{R}'_G = \mathcal{L}''_G = \mathcal{L}_G$$

since the Hamiltonian is a multiple of the identity by construction. This is then an example of a QMS with a decoherence free algebra that is also a type II_1 factor. To conclude this example, we are therefore left to show that such QMS is also covariant with respect to some unitary representation of some symmetry group. Luckily, we already carried out most of the work. Indeed, it is enough to note that the map $\pi \colon g \mapsto L_{1_g}$ is a unitary representation, which is simple to prove since we have that

$$\begin{aligned}\left(L_{1_g}L_{1_h}u\right)(g_0) &= \left(L_{1_h}u)(g^{-1}g_0\right)\\ &= u(h^{-1}g^{-1}g_0)\\ &= u((gh)^{-1}g_0)\\ &= \left(L_{1_{gh}}u\right)(g_0)\end{aligned}$$

for all $g,h\in\mathcal{F}_n$ and $u\in\ell^2(\mathcal{F}_n)$ and each L_{1_g} is unitary as already stated previously. The commutativity property for the QMS generator $\mathcal{L}$ is also trivial since we already stated that $\mathcal{L}'_G=\mathcal{R}_G$ and therefore

$$\begin{aligned}L^*_{1_h}\mathcal{L}(x)L_{1_h} &= \sum_{g\in G} L^*_{1_h}R^*_{1_g}xR_{1_g}L_{1_h} - L^*_{1_h}xL_{1_h}\\ &= \sum_{g\in G} R^*_{1_g}L^*_{1_h}xL_{1_h}R_{1_g} - L^*_{1_h}xL_{1_h}\\ &= \mathcal{L}(L^*_{1_h}xL_{1_h})\end{aligned}$$

for all $h,\in\mathcal{B}(\ell^2(\mathcal{F}_n))$ and $x\in\mathcal{F}_n$

Example 3.4.3. For this example we go back to the case of circulant QMSs introduced in Subsection 3.1.1. Recall that the generator for such QMSs for any $x\in M_p(\mathbb{C})$ is defined as

$$\mathcal{L}(x)=\sum_{k=1}^{n-1}\gamma(p-k)J^{*k}xJ^k-x$$

where all the operations between indices are considered modulo p for some $p\geq 2$, γ is a vector in $\mathbb{C}^{p-1}$ such that its components satisfy $\gamma(k)>0$ for all $k=1,\dots,p-1$ and $\sum_{k=1}^{p-1}\gamma(k)=1$, and finally $J=\sum_{k\in\mathbb{Z}_p}|e_k\rangle\langle e_{k+1}|$ given the canonical basis $\{e_k\}_{k\in\mathbb{Z}_p}$ of $\mathbb{C}^p$. Previously we noted that this generator is covariant with respect to the representation $\pi\colon\mathbb{Z}_p\to M_p(\mathbb{C})$ defined as $\pi(k)=J^k$ and we concluded the example stressing the fact that this representation is not irreducible. We will now proceed to give the decomposition in irreducible representations of π and then show how this induces a decomposition of the DF subalgebra of the QMS according to our general results. In order to do so we need to give some preliminary results about the relationship between projections and the matrix J. Recall that we proved that

$$\mathcal{N}(\mathcal{T})=\mathcal{F}(\mathcal{T})=\{\ J\ \}'$$

which implies that any invariant projection p, i.e. $\mathcal{T}_t(p)=p$ for all $t\geq 0$, will also satisfy $J^{*k}pJ^k=p$. From this we immediately have the following Lemma.

Lemma 3.4.1. *Every projection $p=\sum_{i,j\in\mathbb{Z}_p}p_{i,j}\,|e_i\rangle\langle e_j|$ invariant with respect to a circulant QMS $\mathcal{T}$ satisfies $p_{i+1,j+1}=p_{i,j}$.*

Proof. We now that p commutes with J^k for every $k\in\mathbb{Z}_p$ since it is invariant, in particular taking $k=1$ we have $Jp=pJ$. Computing both sides of the equality we obtain

$$\begin{aligned} Jp &= \sum_{k\in\mathbb{Z}_p} |e_k\rangle\langle e_{k+1}| \sum_{i,j\in\mathbb{Z}_p} p_{i,j}\, |e_i\rangle\langle e_j| \\ &= \sum_{i,j,k\in\mathbb{Z}_p} \delta_{k+1,i}\, p_{i,j}\, |e_k\rangle\langle e_j| \\ &= \sum_{ij\in\mathbb{Z}_p} p_{i+1,j}\, |e_i\rangle\langle e_j| \\ &= \sum_{ij\in\mathbb{Z}_p} p_{i+1,j+1}\, |e_i\rangle\langle e_{j+1}| \end{aligned}$$

where the last equality has been obtained by renaming j in $j+1$. Analogously

$$\begin{aligned} pJ &= \sum_{i,j\in\mathbb{Z}_p} p_{i,j}\, |e_i\rangle\langle e_j| \sum_{k\in\mathbb{Z}_p} |e_k\rangle\langle e_{k+1}| \\ &= \sum_{i,j,k\in\mathbb{Z}_p} \delta_{j,k}\, p_{i,j}\, |e_i\rangle\langle e_{k+1}| \\ &= \sum_{ij\in\mathbb{Z}_p} p_{i,j}\, |e_i\rangle\langle e_{j+1}| \end{aligned}$$

and by direct inspection we have the proof. □

By Peter-Weyl Theorem (see Theorem A.2.3) we know that the representation π can be decomposed into irreducible sub-representations $\{\pi_i\}_{i\in I}$ acting on pairwise orthogonal subspaces $\{V_i\}_{i\in I}$. Moreover, by Proposition 3.3.1 and Theorem 3.3.1 we know that the family of projections $\{q_i\}_{i\in I}$ onto each V_i is minimal in $\mathcal{F}(\mathcal{T})$ and each restriction $\mathcal{T}^i$ to V_i is covariant with respect to π_i. In order to give an explicit expression for these projections we need to introduce the *discrete Fourier transform* F which is defined as

$$F = \frac{1}{\sqrt{p}} \sum_{i,j\in\mathbb{Z}_p} \omega^{ij}\, |e_i\rangle\langle e_j|$$

where ω is a primitive p-root of the unity. With this definition we can give the following Lemma characterizing the relation between J and F.

Lemma 3.4.2. *The primary permutation matrix J is diagonalized by the discrete Fourier transform, i.e.*

$$FJF^* = \sum_{i\in\mathbb{Z}_p} \bar{\omega}^i\, |e_i\rangle\langle e_i|$$

Proof. The proof is a straightforward computation

$$\begin{aligned}
FJF^* &= \frac{1}{p}\sum_{i,j\in\mathbb{Z}_p}\omega^{ij}\,|e_i\rangle\langle e_j|\sum_{k\in\mathbb{Z}_p}|e_k\rangle\langle e_{k+1}|\sum_{l,m\in\mathbb{Z}_p}\bar{\omega}^{lm}\,|e_m\rangle\langle e_l| \\
&= \frac{1}{p}\sum_{i,j,k,l,m\in\mathbb{Z}_p}\omega^{ij}\bar{\omega}^{lm}\delta_{j,k}\delta_{k+1,m}\,|e_i\rangle\langle e_l| \\
&= \frac{1}{p}\sum_{i,k,l\in\mathbb{Z}_p}\omega^{ik}\bar{\omega}^{lk}\bar{\omega}^{l}\,|e_i\rangle\langle e_l| \\
&= \sum_{i,l\in\mathbb{Z}_p}\delta_{i,l}\bar{\omega}^{l}\,|e_i\rangle\langle e_l| \\
&= \sum_{i\in\mathbb{Z}_p}\bar{\omega}^{i}\,|e_i\rangle\langle e_i|
\end{aligned}$$

where we have used the orthogonality relation $\sum_{k\in\mathbb{Z}_p}\omega^{ik}\bar{\omega}^{lk} = p\delta_{i,l}$. □

From this Lemma we immediately see that the vectors $\{Fe_i\}_{i\in\mathbb{Z}_p}$ are an orthonormal basis for $\mathbb{C}^p$. If we define the projection q_i as

$$q_i = |Fe_i\rangle\langle Fe_i| = \frac{1}{p}\sum_{j,k\in\mathbb{Z}_p}\omega^{(j-k)i}\,|e_j\rangle\langle e_k|$$

it is easy to see that we have obtained a family of mutually orthogonal projections such that $\sum_{i\in\mathbb{Z}_p} q_i = \mathbb{1}$. Moreover, they are also invariant with respect to $\mathcal{T}$ since $(q_i)_{jk} = \omega^{(j-k)i}$ and therefore they satisfy Lemma 3.4.1. Since each q_i commutes with J^k for every $k \in \mathbb{Z}_p$ by construction, it also commutes with $\pi(k)$ for every $k \in \mathbb{Z}_p$ and therefore its range $V_i = q_i\mathcal{H}$ is π-invariant and as a result the restriction $\pi_i = q_i\pi q_i$ to V_i is a sub-representation of π for all $i \in \mathbb{Z}_p$. Finally, since $\dim q_i\mathcal{H} = 1$ for all $i \in \mathbb{Z}_p$ we can conclude that we have decomposed π is the direct sum of irreducible representations. We conclude this example showing how these projections decompose the DF subalgebra.

Lemma 3.4.3. *Let $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ be the center of $\mathcal{N}(\mathcal{T})$, then $q_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ for all $i \in \mathbb{Z}_p$.*

Proof. Let p be a projection in $\mathcal{N}(\mathcal{T})$. Since it commutes with J by construction, it must also commutes with its spectral projection, and in particular with each $q_i = |Fe_i\rangle\langle Fe_i|$. Moreover, since $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ is generated by its projections, we can conclude that $q_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$. □

Thank to this result we can give the decomposition of $\mathcal{N}(\mathcal{T})$.

Lemma 3.4.4. *Let $\mathcal{T}$ be a circulant QMS, $\{e_i\}_{i\in\mathbb{Z}_p}$ the canonical basis in $\mathbb{C}^p$ and F be the discrete Fourier transform. Defining the projections $q_i = |Fe_i\rangle\langle Fe_i|$ for all $i \in \mathbb{Z}_p$ we have*

$$\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) = \bigoplus_{i\in\mathbb{Z}_p}\mathbb{C}q_i$$

Proof. By Lemma 3.4.3 we know that $q_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ and therefore

$$\mathcal{N}(\mathcal{T}) = \bigoplus_{i\in\mathbb{Z}_p} q_i\mathcal{N}(\mathcal{T})q_i$$

but since the range of each q_i is one dimensional we have $q_i \mathcal{N}(\mathcal{T}) q_i = \mathbb{C} q_i$ which concludes the proof. $\square$

This decomposition of $\mathcal{F}(\mathcal{T})$ can be found also in [21, Proposition 3.1]. Indeed, the authors characterize $\mathcal{F}(\mathcal{T})$ as the span of the projections $|Fe_{i+j}\rangle\langle Fe_i|$ for all $i, j \in \mathbb{Z}_p$ such that $\mathcal{L}(|Fe_{i+j}\rangle\langle Fe_i|) = 0$. To reconcile such decomposition with ours it suffices to note that an operator is in the kernel of $\mathcal{L}$ if it commutes with J, since $\mathcal{F}(\mathcal{T}) = \{ J \}'$, and that a projection p commutes with J if its matrix elements satisfy $p_{i+1,j+1} = p_{i,j}$ according to Lemma 3.4.1. Therefore, since $(|Fe_{i+j}\rangle\langle Fe_i|)_{lm} = \omega^{(l-m)i}\omega^{lj}$, we see that $|Fe_{i+j}\rangle\langle Fe_i|$ commutes with J if $j = 0$, that is if its equal to q_i, which tells us that the two decompositions of $\mathcal{F}(\mathcal{T})$ are equivalent.

Chapter 4

Direct Integrals

In this Chapter we introduce the concept of *direct integrals* that will allow us to obtain a decomposition of both Hilbert spaces and von Neumann algebras that are more general than direct sums. After recalling the basic definitions about direct integral we will try to use these results to the study of the structure of QMS in an effort to expand the existing theory presented so far.

4.1 Basic notions

Definition 4.1.1. Let Γ be a σ-compact, locally compact space and Borel measurable space, let μ be the completion of a Borel measure on Γ, and finally let $\{\mathcal{H}(\gamma)\}$ a family of separable Hilbert spaces indexed by the points $\gamma \in \Gamma$. Then we say that a separable Hilbert space $\mathcal{H}$ is the *direct integral* of $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ over (Γ, μ) and we write

$$\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$$

when for each $u \in \mathcal{H}$ there exists a function $\gamma \mapsto u(\gamma)$ on Γ such that $u(\gamma) \in \mathcal{H}(\gamma)$ for all $\gamma \in \Gamma$. Moreover, the map $\gamma \mapsto \langle u(\gamma), v(\gamma)\rangle$ has to be μ-integrable for all $u, v \in \mathcal{H}$ and

$$\langle u, v\rangle = \int_\Gamma \langle u(\gamma), v(\gamma)\rangle\,\mathrm{d}\mu(\gamma)$$

Finally, if $u_\gamma \in \mathcal{H}(\gamma)$ for each $\gamma \in \Gamma$ and if the map $\gamma \mapsto \langle u_\gamma, v(\gamma)\rangle$ is integrable for all $v \in \mathcal{H}$, then there exists a function $u \in \mathcal{H}$ such that $u(\gamma) = u_\gamma$ for almost every γ. In such case we say that $\int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$ and $\gamma \mapsto u(\gamma)$ are the *direct integral decomposition* of $\mathcal{H}$ and u respectively.

Note that the separability hypothesis for the Hilbert space $\mathcal{H}$ will allow us to consider denumerable sequences that span the entire space, for this reason we will always consider $\mathcal{H}$ to be separable for the rest of the Chapter. The previous definition directly implies a couple of simple results that we are going to clarify in the following.

Remark 4.1.1. Definition 4.1.1 directly implies that for any $u, v \in \mathcal{H}$ there exists an element $w \in \mathcal{H}$ such that $\alpha u(\gamma) + v(\gamma) = w(\gamma)$ for all $\alpha \in \mathbb{C}$ and for almost every γ. Indeed, we have that

$$\langle \alpha u + v - w, a \rangle = \int_\Gamma \langle \alpha u(\gamma) + v(\gamma) - w(\gamma), a(\gamma) \rangle \, \mathrm{d}\mu(\gamma) = 0$$

for all $a \in \mathcal{H}$, which implies that $w = \alpha u + v$. From this we see that if $u(\gamma) = v(\gamma)$ almost everywhere, then $u = v$ because the hypothesis implies that $(u - v)(\gamma) = 0$ almost everywhere and therefore $\|u - v\|^2 = 0$.

From Definition 4.1.1 is easy to see that the span of $\{ \, u(\gamma) \mid u \in \mathcal{H} \, \}$ is $\mathcal{H}(\gamma)$ for almost every $\gamma \in \Gamma$, but in the following Lemma we are going to show that it is actually possible to prove a more general property than this.

Lemma 4.1.1. *Let $\{u_j\}_j$ be denumerable total set in $\mathcal{H}$, and let $\mathcal{H}^0(\gamma) = \overline{\operatorname{span}\{u_j(\gamma)\}}$ Then $\mathcal{H}^0(\gamma) = \mathcal{H}(\gamma)$ for almost every $\gamma \in \Gamma$.*

Proof. Let $\Gamma_0 = \left\{ \, \gamma \;\middle|\; \gamma \in \Gamma, \, \mathcal{H}^0(\gamma) \neq \mathcal{H}(\gamma) \, \right\}$, and let u_γ be a unit vector in $\mathcal{H}(\gamma) \ominus \mathcal{H}^0(\gamma) = \left\{ \, u \in \mathcal{H}(\gamma) \;\middle|\; \langle u, v \rangle = 0 \; \forall v \in \mathcal{H}^0(\gamma) \, \right\}$ if $\gamma \in \Gamma_0$, or 0 if $\gamma \notin \Gamma_0$. Then $\langle u_\gamma, u_j(\gamma) \rangle = 0$ for all γ. Given $v \in \mathcal{H}$, by assumption there exists a sequence $\{v_n\} \subseteq \operatorname{span}\{u_j\}$ such that $\|v - v_n\| \to 0$. Since $v_j = b_1 u_{a_1} + \cdots + b_n u_{a_n}$ then $v_j(\gamma) = b_1 u_{a_1}(\gamma) + \cdots + b_n u_{a_n}(\gamma)$ except for γ in a null set N_j, and so we have that $\langle u_\gamma, v_j(\gamma) \rangle = 0$ for $\gamma \in X \setminus N_j$. Moreover, given that

$$\|v - v_n\|^2 = \int_\Gamma \|v(\gamma) - v_n(\gamma)\|^2 \mathrm{d}\mu(\gamma) \to 0,$$

some subsequence $\{ \, \|v(\gamma) - v_{n_k}(\gamma)\| \, \}_k$ tends to zero except for γ in a null set N_0. Then, if γ is not in the null set $\bigcup_{j=0}^{\infty} N_j$ we have that $\langle u_\gamma, v(\gamma) \rangle = 0$. In particular, we know that the map $\gamma \mapsto \langle u_\gamma, v(\gamma) \rangle$ is integrable for each $v \in \mathcal{H}$, and by Definition 4.1.1 we also know that there exists a $u \in \mathcal{H}$ such that $u_\gamma = u(\gamma)$ almost everywhere. Therefore we find

$$0 = \langle u_\gamma, u(\gamma) \rangle = \langle u_\gamma, u_\gamma \rangle$$

almost everywhere. Therefore we conclude that since u_γ is a unit vector when γ is in Γ_0 then Γ_0 is a null set. $\square$

As a consequence of this result we obtain the equivalence between the Definition of direct integral just presented, and the one given by Takesaki in [22, Chapter IV, Section 8]. More precisely, denoting by $u(\cdot)$ the map $u(\cdot) \in \prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$ with $u(\gamma) \in \mathcal{H}(\gamma)$ for all $\gamma \in \Gamma$, we can define as $\mathcal{M}$ the subset of $\prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$ given by all the elements $u(\cdot) \in \prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$ satisfying the following conditions

- the map $\gamma \mapsto \langle u(\gamma), v(\gamma) \rangle$ is μ-measurable for all $v(\cdot) \in \prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$,
- there exists $\{u_n(\cdot)\} \in \prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$ such that $\{u_n(\gamma)\}_n$ is total in $\mathcal{H}(\gamma)$ for all $\gamma \in \Gamma$.

Then we can identify $\mathcal{H}$ with the Hilbert space

$$\left\{ \, u(\cdot) \in \mathcal{M} \;\middle|\; \int_\Gamma \|u(\gamma)\|^2 \, \mathrm{d}\mu(\gamma) < +\infty \, \right\}$$

together with the scalar product $\langle u, v \rangle = \int_\Gamma \langle u(\gamma), v(\gamma) \rangle \, \mathrm{d}\mu(\gamma)$.

Example 4.1.1. If we consider a constant field, $\mathcal{H}(\gamma) = \mathcal{H}$ for all $\gamma \in \Gamma$, then the direct integral $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$ is just the space of measurable functions from Γ to $\mathcal{H}$ which are square-integrable with respect to μ, that is $\mathcal{H} = L^2(\Gamma, \mu; \mathcal{H})$.

Example 4.1.2. If Γ is discrete and μ is counting measure on Γ, then

$$\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma) = \oplus_{\gamma\in\Gamma}\mathcal{H}(\gamma).$$

Now that we have presented the basic results about the decomposition of an Hilbert space $\mathcal{H}$ through direct integrals, a natural question that arises is if this kind of decomposition extends to the bounded operators $\mathcal{B}(\mathcal{H})$ defined on it. Naturally, given an Hilbert space $\mathcal{H}$ and its direct integral decomposition as $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$, we can easily construct the family $\{\, \mathcal{B}(\mathcal{H}(\gamma)) \mid \gamma \in \Gamma \,\}$ where each $\mathcal{B}(\mathcal{H}(\gamma))$ acts on $\mathcal{H}(\gamma)$. With the following definition we will see how the concept of direct integral decomposition for Hilbert spaces naturally extends to bounded operators.

Definition 4.1.2. Let $\mathcal{H}$ be the direct integral of the family $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ over (Γ, μ), then an operator $x \in \mathcal{B}(\mathcal{H})$ is said to be *decomposable* when there exists a map $\gamma \mapsto x(\gamma)$ on Γ such that $x(\gamma) \in \mathcal{B}(\mathcal{H}(\gamma))$, and for every $u \in \mathcal{H}$, we have $x(\gamma)u(\gamma) = (xu)(\gamma)$ for almost every γ. Moreover, given a scalar function $f\colon \Gamma \to \mathbb{C}$, if $x(\gamma) = f(\gamma)\mathbb{1}(\gamma)$, where $\mathbb{1}(\gamma)$ is identity operator on $\mathcal{H}(\gamma)$, then we say that x is *diagonalizable*.

In the following remark we show that the decomposition of an operator is unique.

Remark 4.1.2. If $x(\gamma)$ and $x'(\gamma)$ are both decomposition of x, then $x(\gamma) = x'(\gamma)$, i.e. every decomposable operator has a unique decomposition. Indeed, let $\{u_j\}_j$ be a denumerable total set in $\mathcal{H}$, then from Lemma 4.1.1 there exists a null set N_0 such that $\{u_j(\gamma)\}_j$ is total in $\{\mathcal{H}(\gamma)\}$ for $\gamma \in \Gamma_0 \setminus N_0$. At the same time we also know from the same Lemma that

$$x(\gamma)u_j(\gamma) = (xu_j)(\gamma) = x'(\gamma)u_j(\gamma)$$

except for γ in a null set N_j. Therefore it follows that the bounded operators $x(\gamma)$ and $x'(\gamma)$ coincide on $\Gamma \setminus \bigcup_{j=0}^\infty N_j$. Conversely, if x and y are decomposable and $x(\gamma) = y(\gamma)$ almost everywhere, then $x = y$. To prove this, for all $u, v \in \mathcal{H}$ consider

$$\begin{aligned}
\langle xu, v\rangle &= \int_\Gamma \langle (xu)(\gamma), v(\gamma)\rangle\,\mathrm{d}\mu(\gamma)\\
&= \int_\Gamma \langle x(\gamma)u(\gamma), v(\gamma)\rangle\,\mathrm{d}\mu(\gamma)\\
&= \int_\Gamma \langle y(\gamma)u(\gamma), v(\gamma)\rangle\,\mathrm{d}\mu(\gamma)\\
&= \langle yu, v\rangle
\end{aligned}$$

In order to have a complete theory for the decomposition of operators we are now going to describe how they behave under the operation of sum, composition and conjugation.

Proposition 4.1.1. *Let $\mathcal{H}$ be the direct integral of $\{\mathcal{H}(\gamma)\}$ over (Γ, μ), and let x_1, x_2 be decomposable operators in $\mathcal{B}(\mathcal{H})$. Then $\alpha x_1 + x_2$, x_1x_2, x_1^* and $\mathbb{1}$ are all decomposable and the following properties hold true for almost every γ:*

1. $(\alpha x_1 + x_2)(\gamma) = \alpha x_1(\gamma) + x_2(\gamma)$;
2. $(x_1x_2)(\gamma) = x_1(\gamma)x_2(\gamma)$;
3. $x_1^*(\gamma) = x_1(\gamma)^*$;
4. $\mathbb{1}(\gamma) = \mathbb{1}_{\mathcal{H}(\gamma)}$;
5. *if* $x_1(\gamma) \leq x_2(\gamma)$ *almost everywhere, then* $x_1 \leq x_2$.

Proof. **1.** Note that defining $(\alpha x_1 + x_2)(\gamma)$ to be $\alpha x_1(\gamma) + x_2(\gamma)$, then for all $u \in \mathcal{H}$ and almost every γ we have

$$\begin{aligned}(\alpha x_1 + x_2)(\gamma)u(\gamma) &= \alpha x_1(\gamma)x(\gamma) + x_2(\gamma)u(\gamma)\\ &= (\alpha x_1 u)(\gamma) + (x_2 u)(\gamma)\\ &= (\alpha x_1 u + x_2 u)(\gamma)\\ &= ((\alpha x_1 + x_2)u)(\gamma)\end{aligned}$$

directly from Definition 4.1.2 and Remark 4.1.1. Therefore $\alpha x_1 + x_2$ is decomposable with decomposition $\gamma \mapsto \alpha x_1(\gamma) + x_2(\gamma)$.
2. Similarly to before, define $(x_1x_2)(\gamma)$ to be $x_1(\gamma)x_2(\gamma)$, then we have

$$(x_1x_2)(\gamma)u(\gamma) = x_1(\gamma)(x_2(\gamma)u(\gamma)) = x_1(\gamma)((x_2u)(\gamma)) = (x_1x_2u)(\gamma)$$

almost everywhere for all $u \in \mathcal{H}$ and therefore we conclude that x_1x_2 is decomposable with decomposition $\gamma \mapsto x_1(\gamma)x_2(\gamma)$.
3. Once again, define $x^*(\gamma)$ to be $x(\gamma)^*$ then for all $u, v \in \mathcal{H}$ and almost every γ we have

$$\langle x^*(\gamma)u(\gamma), v(\gamma)\rangle = \langle u(\gamma), x(\gamma)v(\gamma)\rangle = \langle u(\gamma), (xv)(\gamma)\rangle$$

and the map $\gamma \mapsto \langle u(\gamma), (xv)(\gamma)\rangle$ is integrable. From Definition 4.1.1 we know there exists a $w \in \mathcal{H}$ such that $x^*(\gamma)u(\gamma) = w(\gamma)$ almost everywhere. Moreover, since

$$\begin{aligned}\langle x^*u - w, v\rangle &= \langle u, xv\rangle - \langle w, v\rangle\\ &= \int_\Gamma \langle u(\gamma)x(\gamma)v(\gamma)\rangle \,\mathrm{d}\mu(\gamma) - \int_\Gamma \langle x^*(\gamma)u(\gamma), v(\gamma)\rangle \,\mathrm{d}\mu(\gamma) = 0\end{aligned}$$

for all $v \in \mathcal{H}$ we have that $x^*u - w = 0$. Thus $(x^*u)(\gamma) = w(\gamma) = x(\gamma)^*u(\gamma)$ almost everywhere and x^* is decomposable with decomposition $\gamma \mapsto x(\gamma)^*$.
4. Also in this case, by analogously defining $\mathbb{1}(\gamma)$ to be $\mathbb{1}_{\mathcal{H}(\gamma)}$ we have

$$\mathbb{1}(\gamma)u(\gamma) = \mathbb{1}_{\mathcal{H}(\gamma)}u(\gamma) = (\mathbb{1}u)(\gamma)$$

which implies that $\mathbb{1}$ is decomposable with decomposition $\gamma \mapsto \mathbb{1}_{\mathcal{H}(\gamma)}$.
5. Finally suppose that $x_1(\gamma) \leq x_2(\gamma)$ almost everywhere, then for all $u \in \mathcal{H}$

we have

$$\begin{aligned}\langle x_1 u, u\rangle &= \int_\Gamma \langle (x_1 u)(\gamma), u(\gamma)\rangle \,\mathrm{d}\mu(\gamma)\\ &= \int_\Gamma \langle x_1(\gamma)u(\gamma), u(\gamma)\rangle \,\mathrm{d}\mu(\gamma)\\ &\leq \int_\Gamma \langle x_2(\gamma)u(\gamma), u(\gamma)\rangle \,\mathrm{d}\mu(\gamma)\\ &= \langle x_2 u, u\rangle\end{aligned}$$

and therefore $x_1 \leq x_2$. □

The following Proposition proves a converse of Item 5 of Definition 4.1.1.

Proposition 4.1.2. *Let $\mathcal{H}$ be the direct integral of $\mathcal{H}(\gamma)$ over (Γ, μ), and let x_1 and x_2 be two decomposable self-adjoint operators on $\mathcal{H}$ such that $x_1 \leq x_2$, then $x_1(\gamma) \leq x_2(\gamma)$ almost everywhere. Moreover, if x is decomposable, then the map $\gamma \mapsto \|x(\gamma)\|$ is essentially bounded, measurable and with essential bound $\|x\|$.*

Proof. From Item 1, Proposition 4.1.1, we know that $x_2 - x_1$ is a positive, decomposable operator with decomposition $x_2(\gamma) - x_1(\gamma)$, therefore it will be sufficient to prove that if $0 \leq H$ and H is decomposable, then $0 \leq H(\gamma)$ almost everywhere. Let $\{u_j\}_j \subseteq \mathcal{H}$ a total set. Then there exists a null set $N \subseteq \Gamma$ such that $\{u_j(\gamma)\}_j$ is total in $\mathcal{H}(\gamma)$ for all $\gamma \notin N$. Suppose then that $0 \leq H$, then $0 \leq \langle Hu_j, u_j\rangle = \int_\Gamma \langle H(\gamma)u_j(\gamma), u_j(\gamma)\rangle \,\mathrm{d}\mu(\gamma)$ for all j, by contradiction there are j and $a < 0$ such that $\langle h(\gamma)u_j(\gamma), u_j(\gamma)\rangle \leq a < 0$ for γ in some subset $\Gamma_0 \subseteq \Gamma$ of finite positive measure. Let f be the characteristic function of Γ_0, then $\gamma \mapsto \langle f(\gamma)u_j(\gamma), v(\gamma)\rangle$ is integrable for each $v \in \mathcal{H}$ so that for some $w \in \mathcal{H}$ we have $w(\gamma) = f(\gamma)u_j(\gamma)$ almost everywhere. With this in mind, we have that

$$\begin{aligned}\langle Hw, w\rangle &= \int_\Gamma \langle H(\gamma)f(\gamma)u_j(\gamma), f(\gamma)u_j(\gamma)\rangle \,\mathrm{d}\mu(\gamma)\\ &= \int_{\Gamma_0} \langle H(\gamma)u_j(\gamma), u_j(\gamma)\rangle \,\mathrm{d}\mu(\gamma) \leq a\mu(\Gamma_0) < 0\end{aligned}$$

contradicting the assumption that $0 \leq H$. Therefore we conclude that for all j we have $0 \leq \langle H(\gamma)u_j(\gamma), u_j(\gamma)\rangle$ except for γ in a null set M_j. Letting then $M = \bigcup_{j=0}^\infty M_j$, if $\gamma \notin M \cup N$ we have that $0 \leq \langle H(\gamma)u_j(\gamma), u_j(\gamma)\rangle$ with $\{u_j(\gamma)\}$ a total set in $\mathcal{H}(\gamma)$. It follows that $0 \leq H(\gamma)$ for $\gamma \notin M \cup N$.

Let now x be a decomposable operator, we know then that x^* and x^*x are both decomposable with decompositions $x^*(\gamma)$ and $x^*(\gamma)x(\gamma)$ respectively. Since $\|x(\gamma)\|^2 = \|x^*(\gamma)x(\gamma)\|$, in order to show that $\gamma \mapsto \|x(\gamma)\|$ is measurable and essentially bounded it will suffice to deal with a positive decomposable operator H on $\mathcal{H}$. Now, since $0 \leq H \leq \|H\|\mathbb{1}$, from the previous part of the proof we have that $0 \leq H(\gamma) \leq \|H\|\mathbb{1}(\gamma)$ almost everywhere. Conversely, from Proposition 4.1.1, if $0 \leq H(\gamma) \leq a\mathbb{1}(\gamma)$ almost everywhere, then $0 \leq H \leq a\mathbb{1}$ and $\|H\| \leq a$. It follows than that the essential bound of $\gamma \mapsto \|H(\gamma)\|$ is $\|H\|$. In order to establish that this map is also measurable, we make use of $\{u_j\}$ and N introduced above. It is enough to prove that $\{\, \gamma \in \Gamma \setminus N \mid a < \|H(\gamma)\| < b \,\}$ is a measurable set for all $0 < a < b$. Now, the relation $a < \|H(\gamma)\| < b$ holds if and only if there exists $r, s \in \mathbb{Q} \cap (a, b)$ such that $r < \|H(\gamma)\| \leq s$ for all $\gamma \notin N$, or, in an equivalent way, $r\mathbb{1}(\gamma) < H(\gamma) \leq s\mathbb{1}(\gamma)$ for all $\gamma \notin N$. Since the set $\{u_j\}_j$ is total in $H(\gamma)$ for all $\gamma \notin N$, we get the equality

$$\begin{aligned}\Gamma_s &\coloneqq \{\gamma \in \Gamma \setminus N \mid H(\gamma) \leq s\mathbb{1}(\gamma)\}\\ &= \bigcap_j \{\gamma \in \Gamma \setminus N \mid \langle u_j(\gamma), H(\gamma)u_j\rangle \leq s\|u_j(\gamma)\|^2\}\end{aligned}$$

and moreover, the set Γ_s is measurable due to the measurability of the maps $\gamma \mapsto \langle u_j(\gamma), H(\gamma)u_j(\gamma)\rangle$ and $\gamma \mapsto s\|u_j(\gamma)\|^2$. We can then conclude that

$$\{\gamma \in \Gamma \setminus N \mid a < \|H(\gamma)\| < b\} = \bigcup_{r,s\in\mathbb{Q}\cap(a,b)} \Gamma_s \setminus \Gamma_r$$

is a measurable set. □

The following is Corollary 8.16 from [22], that we report without proof, provides a simple way of determining whether an operator is decomposable or not.

Corollary 4.1.1. *Let $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$. A bounded operator x on $\mathcal{H}$ is decomposable if and only if it commutes with the algebra of all diagonal operators. More precisely, the set $\mathcal{R}$ of all decomposable operators on $\mathcal{H}$ is a von Neumann algebra whose commutant $\mathcal{R}'$ coincides with the abelian algebra of diagonal operators.*

With this in mind, we can give the general Theorem characterizing the decomposition in direct integral of a generic von Neumann algebra (see Definition 8.17 in [22] for further details).

Definition 4.1.3. Let $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$ and let $\{\mathcal{M}(\gamma)\}_{\gamma\in\Gamma}$ be a family of von Neumann algebras on $\mathcal{H}(\gamma)$. If there exists a countable set $\{x_n\}_{n\geq 1}$ of decomposable operators on $\mathcal{H}$ such that $\mathcal{M}(\gamma)$ is the von Neumann algebra generated by $\{x_n(\gamma)\}_{n\geq 1}$ for almost every $\gamma \in \Gamma$, then the von Neumann algebra $\mathcal{M}$ generated by $\{x_n\}_{n\geq 1}$ and the diagonalizable operators is called the *direct integral* of $\{\mathcal{M}(\gamma)\}$. Moreover, $\mathcal{M}$ is written as $\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}(\gamma)\,\mathrm{d}\mu(\gamma)$

Theorem 4.1.1. *Let $\mathcal{M}$ be a direct integral of von Neumann algebras $\{\mathcal{M}(\gamma)\}_{\gamma\in\Gamma}$. Then $\mathcal{M}$ is uniquely determined by $\{\mathcal{M}(\gamma)\}_\gamma$, i.e. an operator x belongs to $\mathcal{M}$ if and only if x is decomposable with $x(\gamma) \in \mathcal{M}(\gamma)$ for almost every $\gamma \in \Gamma$. This means in particular that the algebra of all decomposable operators x such that $x(\gamma) \in \mathcal{M}(\gamma)$ almost everywhere is the direct integral of $\mathcal{M}(\gamma)$. Then its commutant $\mathcal{M}'$ is given by $\mathcal{M}' = \int_\Gamma^\oplus \mathcal{M}(\gamma)'\,\mathrm{d}\mu(\gamma)$. Finally, the diagonal algebra is contained in the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$.*

Note that $\mathcal{M}$ is uniquely determined by the set $\{\mathcal{M}(\gamma)\}_{\gamma\in\Gamma}$, i.e. an operator x belongs to $\mathcal{M}$ if and only if x is decomposable with $x(\gamma) \in \mathcal{M}(\gamma)$ for almost all $\gamma \in \Gamma$. See Theorem 8.18 and Definition 8.19 in [22].

Corollary 4.1.2. *Let $\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}(\gamma)\,\mathrm{d}\mu(\gamma)$ be a direct integral of von Neumann algebras. Its center $\mathcal{Z}(\mathcal{M})$ is also expressed as a direct integral*

$$\mathcal{Z}(\mathcal{M}) = \int_\Gamma^\oplus \mathcal{Z}(\mathcal{M}(\gamma))\,\mathrm{d}\mu(\gamma) \tag{4.1}$$

In particular, $\mathcal{Z}(\mathcal{M})$ coincides with the diagonal algebra if and only if $\mathcal{M}(\gamma)$ is a factor for almost every $\gamma \in \Gamma$.

The following Theorem explains how to obtain the direct integral decomposition of any von Neumann algebra $\mathcal{M}$.

Theorem 4.1.2. *Let $\mathcal{A}$ be an abelian von Neumann algebra on the separable Hilbert space $\mathcal{H}$, then the following hold*

- *there is a (locally compact, complete, separable, metric) measurable space (Γ, μ) such that $\mathcal{H}$ is (up to unitary equivalence) the direct integral of the Hilbert spaces $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ over (Γ, μ)*
- *$\mathcal{A}$ is (again, up to unitary equivalence) the algebra of diagonalizable operators relative to such decomposition.*

Moreover, if we also suppose that $\mathcal{A}$ is a abelian subalgebra of $\mathcal{Z}(\mathcal{M})$ for a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $\mathcal{H}$ decomposable as $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$ with respect to $\mathcal{A}$, then we have

- *$(\mathcal{Z}(\mathcal{M}))(\gamma)$ is the center of $\mathcal{M}(\gamma)$, that is $\mathcal{Z}(\mathcal{M}))(\gamma) = \mathcal{Z}(\mathcal{M}(\gamma))$, almost everywhere,*
- *$\mathcal{M}(\gamma)$ is a factor almost everywhere if and only if $\mathcal{A} = \mathcal{Z}(\mathcal{M})$.*

Proof. See [20, Theorem 14.2.1] and [20, Theorem 14.2.2]. □

The following Theorem gives a characterization of a decomposition of a von Neumann algebra starting from its commutant. For a proof see Theorem 14.2.4 of [23] (pag. 646).

Theorem 4.1.3. *If $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $\mathcal{H}$ is the direct integral of $\{\mathcal{H}(\gamma)\}$ in a decomposition relative to an abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{R}'$. Then $\mathcal{R}(\gamma) = \mathcal{B}(\mathcal{H}(\gamma))$ almost everywhere if and only if $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}'$.*

Proposition 4.1.3. *Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ admitting a direct integral decomposition with respect to $\mathcal{Z}(\mathcal{M})$. Then there exist a decomposition for $\mathcal{M}$*

$$\mathcal{M} = \mathcal{M}_c \oplus \left(\bigoplus_{i\in I} \mathcal{M}_i \right)$$

such that $\mathcal{M}_c$ is either zero or its center has no minimal projections, and each $\mathcal{M}_i$ is a factor.

Proof. First of all, consider the decomposition of $\mathcal{H}$ with respect to $\mathcal{Z}(\mathcal{M})$, then we note that, without any loss of generality, we can consider the space Γ as the union of the unit interval $[0, 1]$ and an at most countable set of atoms $\{\Gamma_n\}_{n\in\mathbb{N}}$ (see [20, Chapter 14, pag. 998]). Thus, the measure μ can be decomposed as the sum of a "continuous" component μ_c (i.e. a Lebesgue measure on $[0, 1]$) and a discrete measure for each discrete atom. Therefore, recalling Example 4.1.2, the decomposition of $\mathcal{H}$ can be written as

$$\mathcal{H} = \int_{[0,1]}^\oplus \mathcal{H}(\gamma)\,\mathrm{d}\mu_c(\gamma) \oplus \left(\bigoplus_{n\in\mathbb{N}} \mathcal{H}(\gamma_n) \right)$$

Given such decomposition, we naturally obtain a family of orthogonal projections. For sake of brevity, let $\mathcal{H}_d = \oplus_{n\in\mathbb{N}}\mathcal{H}(\gamma_n)$ be the discrete part of $\mathcal{H}$, and denote by $\{e_i^m\}_{i\in I_m}$ an orthonormal basis for $\mathcal{H}(\gamma_m)$. Then, it is clear that the family $\{e_i^m\}_{m\in\mathbb{N},\, i\in I_m}$ is an orthonormal basis for $\mathcal{H}_d$. By denoting $I = \cup_{m\in\mathbb{N}} I_m$ we see that for each pair $m \in \mathbb{N}$ and $i \in I_m$ there exists one $n \in I$. Therefore, by relabeling the orthonormal basis as $f_n = e_i^m$ according to the correspondence above, we can define a family of orthonormal projections as

$$p_n = |f_n\rangle\langle f_n|$$

such that $p_n\mathcal{H} = \mathcal{H}(\gamma_n)$. Analogously for the continuous component, we can define the projection

$$q = \int_{[0,1]}^{\oplus} \mathbb{1}(\gamma)\,\mathrm{d}\mu_c(\gamma)$$

such that $q\mathcal{H} = \int_{[0,1]}^{\oplus} \mathcal{H}(\gamma)\,\mathrm{d}\mu_c(\gamma)$. Putting these projections together, we obtain a countable family $\{q, p_i\}_{i\in I}$ of orthonormal, diagonal projections summing up to the identity, and belonging to $\mathcal{Z}(\mathcal{M})$, with each p_i minimal in $\mathcal{Z}(\mathcal{M})$. Thus, we immediately have the desired decomposition of the von Neumann algebra

$$\mathcal{M} = q\mathcal{M} \oplus \left(\bigoplus_{i\in I} p_i\mathcal{M}\right)$$

Moreover, by Theorem 4.1.2, each $p_i\mathcal{M}$ is a factor, while $\mathcal{Z}(q\mathcal{M})$ is the diagonal algebra with respect to the direct integral decomposition $\int_{[0,1]}^{\oplus} \mathcal{H}(\gamma)\,\mathrm{d}\mu_c(\gamma)$, so that it is isomorphic to the multiplication algebra of $L^2([0,1],\mu_c)$ (see for example [20, Example 14.1.4(a), Example 14.1.11(a)]) allowing us to conclude that it does not contain any minimal projection. □

4.2 $\mathcal{N}(\mathcal{T})$ as direct integral of factors

Recall the definition of $\mathcal{N}(\mathcal{T})$ given in Proposition 3.0.2. We will show that every element of $\mathcal{N}(\mathcal{T})$ is decomposable with respect to a suitable "disintegration" of $\mathcal{H}$ in the direct integral on a measurable space (Γ,μ). Moreover, this structure of the DF algebra induces a decomposition of Lindblad operators $\{H, L_k\}_k$ wrt the same direct integral decomposition. Applying Theorem 4.1.2 to the abelian von Neumann subalgebra $\mathcal{A} = \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ of the center of the von Neumann algebra $\mathcal{M} = \mathcal{Z}(\mathcal{N}(\mathcal{T}))'$, there exist von Neumann algebras $\{\mathcal{R}(\gamma)\}$ acting on Hilbert spaces $\{\mathcal{H}(\gamma)\}$ such that

$$\mathcal{Z}(\mathcal{N}(\mathcal{T}))' = \int_{\Gamma}^{\oplus} \mathcal{R}(\gamma)\,\mathrm{d}\mu(\gamma), \qquad \mathcal{H} = \int_{\Gamma}^{\oplus} \mathcal{H}(\gamma)\,\mathrm{d}\mu(\gamma)$$

and $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is the diagonal algebra. Moreover, since $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is a maximal abelian subalgebra of $(\mathcal{Z}(\mathcal{N}(\mathcal{T}))')'$ (they coincide), Theorem 4.1.3 gives $\mathcal{R}(\gamma) = \mathcal{B}(\mathcal{H}(\gamma))$ almost everywhere. Consequently, since $\mathcal{N}(\mathcal{T})$ is contained in $(\mathcal{Z}(\mathcal{N}(\mathcal{T})))'$, we obtain the following result

Theorem 4.2.1. *$\mathcal{Z}(\mathcal{N}(\mathcal{T}))'$ is the algebra of all decomposable operators and every element of $\mathcal{N}(\mathcal{T})$ is decomposable. Moreover, the center $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ of $\mathcal{N}(\mathcal{T})$ is the diagonal algebra with respect to this decomposition.*

4.3 From $\mathcal{N}(\mathcal{T})$ to decomposition of $\mathcal{T}_t$

Let $\mathcal{L}$ be the generator of $\mathcal{T}$ expressed in a GKSL representation

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\sum_{k\geq 1}(L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k)$$

Lemma 4.3.1. *If $\mathcal{M}$ is a von Neumann algebra, then*

$$\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{Z}(\mathcal{M})')$$

Proof. The equality can be shown by a direct computation. Indeed, by the definition of center we have that $\mathcal{Z}(\mathcal{Z}(\mathcal{M})') = \mathcal{Z}(\mathcal{M})'' \cap \mathcal{Z}(\mathcal{M})'$. But by the double commutant Theorem we also have that $\mathcal{Z}(\mathcal{M})'' = \mathcal{Z}(\mathcal{M})$, therefore $\mathcal{Z}(\mathcal{Z}(\mathcal{M})') = \mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{M})'$. Finally, applying the definition of center again we have that $\mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{M})' = \mathcal{Z}(\mathcal{Z}(\mathcal{M}))$, but since the center of a von Neumann algebra is always abelian it holds that $\mathcal{Z}(\mathcal{Z}(\mathcal{M})) = \mathcal{Z}(\mathcal{M})$ and therefore we have the thesis. $\square$

In order to give a decomposition of a QMS $\mathcal{T}$ and its GKSL representation we need this Lemma characterizing *-automorphisms of von Neumann algebras that leave invariant its center.

Lemma 4.3.2. *Let $\alpha\colon \mathcal{M} \to \mathcal{M}$ be a *-automorphism of type I von Neumann algebras. If α preserves $\mathcal{Z}(\mathcal{M})$ then α is inner, that is there exists a unitary operator $U \in \mathcal{M}$ such that $\alpha(x) = UxU^*$ for all $x \in \mathcal{M}$.*

The following proposition shows that, as was in the case of DF subalgebras, in the general framework every operator in the center of $\mathcal{N}(\mathcal{T})$ is a fixed point for the semigroup.

Proposition 4.3.1. *The restriction of every $\mathcal{T}_t$ to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is a *-automorphism. In particular we have $\mathcal{Z}(\mathcal{N}(\mathcal{T})) \subseteq \mathcal{F}(\mathcal{T})$.*

Proof. Since $\mathcal{T}_t$ acts a *-automorphism onto $\mathcal{N}(\mathcal{T})$, it is enough to show that its restriction to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is bijective. So, let $x \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ and $y \in \mathcal{N}(\mathcal{T})$; then there exists $z_t \in \mathcal{N}(\mathcal{T})$ such that $y = \mathcal{T}_t(z_t)$, and thus

$$\mathcal{T}_t(x)y = \mathcal{T}_t(x)\mathcal{T}_t(z_t) = \mathcal{T}_t(xz_t) = \mathcal{T}_t(z_t x) = y\mathcal{T}_t(x),$$

i.e. $\mathcal{T}_t(x)$ belongs to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$. Vice versa, if $y \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$, in particular there exists $x \in \mathcal{N}(\mathcal{T})$ such that $\mathcal{T}_t(x) = y$, i.e $x = e^{-itH} y e^{itH}$ and so for every $z \in \mathcal{N}(\mathcal{T})$

$$zx = ze^{-itH} y e^{itH} = e^{-itH}\mathcal{T}_t(z) y e^{itH} = e^{-itH} y \mathcal{T}_t(z) e^{itH} = xz.$$

This means $x \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$. In order to conclude the proof we have to show that every x in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is a fixed point. Since the restriction of $\mathcal{T}_t$ to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is a *-automorphism on a type I algebra coinciding with its center (being $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ commutative), Lemma 4.3.2 gives $\mathcal{T}_t(x) = UxU^*$ for all $x \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$, with U a unitary operator in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$. Therefore, the equality $\mathcal{T}_t(x) = x$ holds for all $x \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$. $\square$

Thanks to the result just proven we can show that any GKSL representation of $\mathcal{T}$ admits a decomposition in direct integral in a natural way.

Proposition 4.3.2. *The Lindblad operators $\{H, L_k\}_k$ are decomposable. In particular, there exist $H(\gamma), L_k(\gamma) \in \mathcal{B}(\mathcal{H}(\gamma))$ for almost every $\gamma \in \Gamma$, such that*

$$H = \int_\Gamma^\oplus H(\gamma)\,\mathrm{d}\mu(\gamma)$$
$$L_k = \int_\Gamma^\oplus L_k(\gamma)\,\mathrm{d}\mu(\gamma)$$

Proof. We know that $\mathcal{N}(\mathcal{T})$ is contained in the commutant of L_k and L_k^*, so that L_k and L_k^* belong to the commutant of $\mathcal{N}(\mathcal{T})$, that is decomposable by Theorem 4.1.1 Now we want to prove the thesis for H. By Proposition 4.3.1 we have that the von Neumann algebra $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is contained in the set of fixed points $\mathcal{F}(\mathcal{T})$ (see for example [8, Lemma 2.1]), and so every projection of it commutes with the Lindblad operators L_k and H. Consequently H belongs to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))'$, since $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is generated by its projections. □

We can then define on $\mathcal{B}(\mathcal{H}(\gamma))$ the uniformly continuous QMS $\mathcal{T}^\gamma$ generated by the Lindblad operators associated with $\{H(\gamma), L_k(\gamma)\}_k$, for almost every $\gamma \in \Gamma$.

Corollary 4.3.1. *Let be $\mathcal{N}(\mathcal{T}) = \int_\Gamma^\oplus \mathcal{N}(\mathcal{T})(\gamma)\,\mathrm{d}\mu(\gamma)$ the decomposition of $\mathcal{N}(\mathcal{T})$ in direct integrals, then $\mathcal{N}(\mathcal{T})(\gamma) = \mathcal{N}(\mathcal{T}^\gamma)$ for almost every γ.*

Proof. If $x \in \mathcal{N}(\mathcal{T})$, then $x = \int_\Gamma^\oplus x(\gamma)\,\mathrm{d}\mu(\gamma)$ and $\mathcal{T}_t(x) \in \mathcal{N}(\mathcal{T})$, in particular

$$\mathcal{T}_t(x) = \int_\Gamma^\oplus \mathcal{T}_t(x)(\gamma)\,\mathrm{d}\mu(\gamma) = \int_\Gamma^\oplus \mathcal{T}^\gamma(x(\gamma))\,\mathrm{d}\mu(\gamma).$$

If $x \in \mathcal{N}(\mathcal{T})$, $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x)$ and

$$\int_\Gamma^\oplus \mathcal{T}^\gamma(x^*(\gamma)x(\gamma))\,\mathrm{d}\mu(\gamma) = \int_\Gamma^\oplus \mathcal{T}^\gamma(x^*(\gamma))\mathcal{T}_t(x(\gamma)))\,\mathrm{d}\mu(\gamma)$$

So $\mathcal{T}^\gamma(x^*(\gamma)x(\gamma)) = \mathcal{T}^\gamma(x^*(\gamma))\mathcal{T}_t(x(\gamma))$ for almost every γ, i.e. $x(\gamma) \in \mathcal{N}(\mathcal{T}^\gamma)$ for almost every γ. This prove that $\mathcal{N}(\mathcal{T})(\gamma) \subseteq \mathcal{N}(\mathcal{T}^\gamma)$. Vice versa it is trivial. □

Appendix A

Haar measures and group representation theory

A.1 Haar measures

Definition A.1.1. A group G is *topological* if it admits a topology such that the map $(x, y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous or, equivalently, if both $(x, y) \mapsto xy$ from $G \times G$ to G and $x \mapsto x^{-1}$ from G to G are continuous.

For sake of simplicity we introduce a few notations use throughout the Thesis. Let A, B be two subsets of G then we define their product as

$$AB = \{\, ab \mid a \in A,\, b \in B \,\}$$

Analogously, given any $g \in G$, we will use gA and Ag in place of $\{g\}A$ and $A\{g\}$ respectively. Moreover, we will use the notation A^2 for AA (and so on for higher powers) and also

$$A^{-1} = \left\{\, a^{-1} \;\middle|\; a \in A \,\right\}$$

Definition A.1.2. A topological group G is *locally compact* if its topology is locally compact.

Let G be a locally compact group, we denote with $\sigma(G)$ the σ-algebra of Borel sets of G. Note that gA and Ag are in $\sigma(G)$ for every $A \in \sigma(G)$ and every $g \in G$.

Definition A.1.3. A positive regular Borel measure μ on G is said to be a *left (resp. right) Haar measure* if satisfies $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) for every $A \in \sigma(G)$ and every $g \in G$.

The property of invariance under left (resp. right) translation of Borel sets is also often called *left (resp. right) invariance* and is further clarified in the following proposition.

Proposition A.1.1. *A positive regular Borel measure μ is a left (resp. right) Haar measure on G if and only if in satisfies for every $C_c(G)$ and every $g \in G$*

$$\int_G f(gx)\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x)$$

for left measures, and

$$\int_G f(xg)\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x)$$

for right measure

The following Theorem ensures the existence of Haar measures (for a proof see [24, Theorem 2.10]).

Theorem A.1.1. *Every locally compact group admits a left Haar measure.*

We report a few useful properties of Haar measures

Proposition A.1.2. *Let μ be a Haar measure on G, then $\mu(A) > 0$ for every open subset A of G, and $\mu(G) < +\infty$ if and only if G is compact.*

Proposition A.1.3. *Let μ_1 and μ_2 be two left Haar measures, then there exists a constant $c > 0$ such that $\mu_1 = c\mu_2$.*

By the last two Proposition in clear that whenever G is a compact group there always exists a left Haar measure such that $\mu(G) = 1$. It's important to note that in general left and right Haar measures are different mathematical objects that do not coincide. To further explore this point, consider a locally compact group G and a left Haar measure μ on it. Then, for any $g \in G$, we can define another measure ρ_g as the right translation of μ, that is

$$\rho_g(A) = \mu(Ag) \qquad A \in \sigma(G)$$

which is a left Haar measure. Indeed, for every $h \in G$ and $A \in \sigma(G)$ we have

$$\rho_g(hA) = \mu(hAg) = \mu(Ag) = \rho_g(A)$$

But, by Proposition A.1.3, we know that there must exists a positive constant for every possible $g \in G$, that we will denote as $\Delta(g)$, such that $\rho_g = \Delta(g)\mu$. Moreover, the value of $\Delta(g)$ is independent from the choice of μ. Indeed, by the definition of ρ_g we have

$$\Delta(g) = \frac{\mu(Ag)}{\mu(A)} \qquad \forall A \in \sigma(G)$$

which is clearly independent from μ by Proposition A.1.3.

Definition A.1.4. The function $\Delta \colon G \to \mathbb{R}^+$ just discussed is called the *modular function* of G. A group is said to be *unimodular* if $\Delta(g) = 1$ for every $g \in G$, which is equivalent to say that left Haar measures are also right Haar measures.

Remark A.1.1. Every compact group is unimodular

A.2 Group representation theory

Definition A.2.1. Let G be a locally compact group, and $\mathcal{H}$ a separable Hilbert space. Then an homomorphism $\pi \colon G \to \mathcal{B}(\mathcal{H})$ which is continuous w.r.t. the strong topology of $\mathcal{B}(\mathcal{H})$ is called a *representation of G on $\mathcal{H}$*. A representation π is said to be *unitary* if $\pi(g)$ is an unitary operator for every $g \in G$.

Lemma A.2.1. *Let G be locally compact group, and π a representation of G on a Hilbert space $\mathcal{H}$. Then the application $(g, v) \mapsto \pi(g)v$ from $G \times \mathcal{H}$ to $\mathcal{H}$ is continuous.*

Given two elements u, $v \in \mathcal{H}$, the continuous function $g \mapsto \langle \pi(g)u, v \rangle$ from G to $\mathbb{C}$ is called a *representation coefficient* of π. Whenever $u = v$ it is called a *diagonal coefficient.* If the Hilbert space $\mathcal{H}$ on which the representation π acts is finite dimensional with $\dim \mathcal{H} = d$ then the representation *pi* is said to have *dimension d.*

Definition A.2.2. Let G be locally compact group and π a representation of G on a Hilbert space $\mathcal{H}$, then a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to be *π-invariant* (or just invariant) if $\pi(g)\mathcal{K} \subseteq \mathcal{K}$ for every $g \in G$. The representation π is said to be *irreducible* if the only invariant subspaces are $\mathcal{H}$ and $\{ 0 \}$.

Given any π-invariant subspace $\mathcal{K} \subseteq \mathcal{H}$, it is possible to obtain a representation of G on $\mathcal{K}$ by restricting to $\mathcal{K}$ every operator $\pi(g)$. Such representation is called a *subrepresentation* of the original one.

Lemma A.2.2. *Let π be an unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{K} \subseteq \mathcal{H}$ is a π-invariant subspace, then also the orthogonal space $\mathcal{K}^{\perp}$ is π-invariant.*

Proof. Consider $u \in \mathcal{K}^{\perp}$, then, for every $g \in G$ and $v \in \mathcal{K}$, we have

$$\langle \pi(g)u, v \rangle = \langle u, \pi(g)^{*} v \rangle = \left\langle u, \pi(g)^{-1} v \right\rangle = \langle u, \pi(g^{-1})v \rangle = 0$$

since $\pi(g^{-1})v \in \mathcal{K}$. Therefore, we can conclude that $\pi(g)u \in \mathcal{K}^{\perp}$ □

In order to see that the previous Lemma is not necessarily true for non unitary representation, consider the representation $\pi \colon \mathbb{R} \to \mathcal{B}(\mathbb{C}^2)$ given by

$$\pi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Then it is straightforward to see that the only non trivial invariant subspace is given by $\operatorname{span}\left\{ (1,0)^t \right\}$, contradicting the previous Lemma.

Definition A.2.3. Let π be a representation of G on a Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is said to be *cyclic* if the subspace generated by $\{ \pi(g)v \mid g \in G \}$ is *dense* in $\mathcal{H}$.

Definition A.2.4. Let π_1 and π_2 be two unitary representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then, an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be *intertwining* π_1 and π_2 if the following diagram

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "\pi_1(g)"] \arrow[d, "A"'] & \mathcal{H}_1 \arrow[d, "A"] \\ \mathcal{H}_2 \arrow[r, "\pi_2(g)"] & \mathcal{H}_2 \end{tikzcd}$$

is commutative for every $g \in G$, or equivalently if the identity $A\pi_1(g) = \pi_2(g)A$ holds for every $g \in G$. If there exist an unitary intertwining operator, then the two representations are said to be *unitarily equivalent.*

We will denote by $I(\pi_1, \pi_2)$ the set of all operators intertwining π_1 and π_2. When the two representation are the same, i.e. $\pi_1 = \pi_2 = \pi$, the set $I(\pi, \pi)$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ comprised of all the operator commuting with $\pi(g)$ for every $g \in G$, that is $I(\pi, \pi) = \{ \pi(g) \mid g \in G \}'$. The following Lemma is crucial in the characterization of irreducible representations.

Lemma A.2.3 (Schur's Lemma)**.** *Let π be a unitary representations, then it is irreducible if and only if $I(\pi, \pi) = \mathbb{C}\mathbb{1}$. Moreover, let π_1 and π_2 be two unitary irreducible representations, then*

- *if they are equivalent and $A \in I(\pi_1, \pi_2) \implies I(\pi_1, \pi_2) = \mathbb{C}A$;*
- *if they are not equivalent $\implies I(\pi_1, \pi_2) = \{ 0 \}$.*

Proof. First of all, suppose π is a not irreducible representation, and $\mathcal{K} \subseteq \mathcal{H}$ a proper non trivial invariant subspace. Let p be the projection onto $\mathcal{K}$ and $u \in \mathcal{H}$, then $\pi(g)pu \in \mathcal{K}$ and $\pi(g)(u - pu) \in \mathcal{K}^\perp$ for any $g \in G$ by Lemma A.2.2. By a direct computation we have that for every $g \in G$ and every $u \in \mathcal{H}$

$$p\pi(g)u = p\pi(g)pu + p\pi(g)(u - pu) = p\pi(g)pu = \pi(g)pu$$

and therefore $p \in I(\pi, \pi)$, from which $I(\pi, \pi) \neq \mathbb{C}\mathbb{1}$. Vice versa, suppose there exists an operator $A \in I(\pi, \pi)$ such that $A \notin \mathbb{C}\mathbb{1}$. Since $I(\pi, \pi)$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, it must also contain the self-adjoint operators $A + A^*$ and $i(A - A^*)$. It simple to see that at least one of the two, let's call it B, does in fact not belong to $\mathbb{C}\mathbb{1}$. By the spectral Theorem we have that the spectrum $\sigma(B)$ of B must contain more than one element, otherwise we would have $B = \int_{\sigma(B)} \lambda \, \mathrm{d}E(\lambda) = b\mathbb{1}$ for some $b \in \mathbb{C}$. We can therefore say that there exist at least two disjoint nonempty open sets $\omega_1, \omega_2 \in \sigma(B)$ such that the associated spectral projections $E(\omega_j)$ for $j = 1, 2$ satisfy $E(\omega_j) \neq 0$ for $j = 1, 2$ and $E(\omega_1)E(\omega_2) = 0$. From this we see that $E(\omega_1)$, for example, is a non trivial orthogonal projection that commutes with $\pi(g)$ for every $g \in G$. Indeed, since $A \in I(\pi, \pi)$, then also $B \in I(\pi, \pi)$, and again by the spectral Theorem, $E(\omega_1) \in I(\pi, \pi)$. Letting then $\mathcal{K} = E(\omega_1)\mathcal{H}$ and $u \in \mathcal{K}$ we have

$$\pi(g)u = \pi(g)E(\omega_1)u = E(\omega_1)\pi(g)u$$

for every $g \in G$ which implies that $\pi(g)u \in \mathcal{K}$, proving that π admits a nontrivial invariant subspace and concluding the proof of the first part of the Lemma. Suppose now that π_1 and π_2 are two irreducible non equivalent representations. Let $A \in I(\pi_1, \pi_2)$, then $A^* \in I(\pi_2, \pi_1)$. Indeed, for every $g \in G$ we have that

$$A^*\pi_2(g) = \big(\pi_2(g^{-1})A\big)^* = \big(A\pi_1(g^{-1})\big)^* = \pi_1(g)A$$

This directly implies that $A^*A \in I(\pi_1, \pi_1)$ and $AA^* \in I(\pi_2, \pi_2)$. But, since both π_1 and π_2 are irreducible, by the first part of the Lemma we know that there exist c_1, $c_2 \in \mathbb{C}$ such that $A^*A = c_1\mathbb{1}_{\mathcal{H}_1}$ and $AA^* = c_2\mathbb{1}_{\mathcal{H}_2}$. Moreover, since both A^*A and AA^* are positive semidefinite, and since $\|A^*A\| = \|AA^*\| = \|A\|^2$ (given that both $I(\pi_1, \pi_1)$ and $I(\pi_2, \pi_2)$ are C*-subalgebras), we have that $c_1 = c_2 = c \geq 0$. If it was $c > 0$, than $c^{\frac{1}{2}}A$ would be unitary, but this is impossible since π_1 and π_2 are non equivalent. Therefore we must have $c = 0$ and thus $A = 0$ proving the first item on the list. Finally, suppose that π_1 and

π_2 are two irreducible and equivalent representations. Let $U \in I(\pi_1, \pi_2)$ be a unitary operator. If $A \in I(\pi_1, \pi_2)$ then $U^{-1}A \in I(\pi_1, \pi_1)$. But again, since π_1 is irreducible, by the first part of the Lemma, we have that there exists $c \in \mathbb{C}$ such that $U^{-1}A = c\mathbb{1}$, and therefore $A = cU$ concluding the proof. $\square$

Let G be compact group and μ the normalized Haar measure on G. The following Theorem show how is possible to always consider unitary representations of G.

Theorem A.2.1. *Let π be a representation of G on a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot\,,\cdot\rangle$. Define a map $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ as*

$$\langle\!\langle u, v \rangle\!\rangle = \int_G \langle \pi(g)u, \pi(g)v\rangle \,\mathrm{d}\mu(g)$$

then $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ is a scalar product on $\mathcal{H}$ inducing an equivalent norm to the original one and respect to which π is unitary.

Thanks to this Theorem, from now one we will always consider unitary representations unless specified otherwise. This was just an example of how powerful of a tool integration can be. In the next Lemma we prove a crucial result for the main problem of the Thesis.

Lemma A.2.4. *Let π_1 and π_2 two representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Given the operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we define*

$$\widetilde{T} = \int_G \pi_2(g)^* T \pi_1(g) \,\mathrm{d}\mu(g) \tag{A.1}$$

where the integral converges in the strong topology of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\widetilde{T} \in I(\pi_1, \pi_2)$. Moreover, if T is compact, the $\widetilde{T}$ is compact.

Proof. The convergence of the integral is directly implied by the fact that the map $g \mapsto \pi_2(g)^* T \pi_1(g)$ is composition of continuous maps according to Lemma A.2.1. Given $U \in \mathcal{B}(\mathcal{H}_1)$ and $V \in \mathcal{B}(\mathcal{H}_2)$, again by continuity, we have

$$V\widetilde{T} = \int_G V\pi_2(g)^* T \pi_1(g) \,\mathrm{d}\mu(g) \qquad \widetilde{T}U = \int_G V\pi_2(g)^* T \pi_1(g) U \,\mathrm{d}\mu(g)$$

There we can compute

$$\begin{aligned} \pi_2(h)\widetilde{T} &= \int_G \pi_2(gh^{-1})^* T \pi_1(g) \,\mathrm{d}\mu(g) \\ &= \int_G \pi_2(g)^* T \pi_1(gh) \,\mathrm{d}\mu(g) \\ &= \int_G \pi_2(g)^* T \pi_1(g)\pi_1(h) \,\mathrm{d}\mu(g) \\ &= \widetilde{T}\pi_1(g) \end{aligned}$$

that is $\widetilde{T} \in I(\pi_1, \pi_2)$. Suppose now that T is a compact operator, and let $\{u_n\}$ in $\mathcal{H}_1$ converging weakly to 0, then we have

$$\|\widetilde{T}u_n\| \leq \int_G \|\pi_2(g)^{-1} T \pi_1(g) u_n\| \,\mathrm{d}\mu(g) = \int_G \|T\pi_1(g)u_n\| \,\mathrm{d}\mu(g)$$

Clearly, $\lim_{n\to+\infty}\|T\pi_1(g)u_n\| = 0$ and $\|T\pi_1(g)u_n\| \leq \|T\| \sup_n\|u_n\|$ for every $g \in G$. Therefore, by dominated convergence the sequence $\{Tu_n\}$ converges to 0 in norm. □

Lemma A.2.5. *Every unitary representation of G admits a non trivial invariant subspace of finite dimension.*

Proof. Let π be a unitary representation of G on $\mathcal{H}$, and let $v_0 \in \mathcal{H}$ be such that $\|v_0\| = 1$. Define p as the orthogonal projection of $\mathcal{H}$ onto $\mathbb{C}v_0$, i.e. $pv = \langle v, v_0\rangle\, v_0$. Moreover, let $\tilde{p}$ be the averaged operator associated to p according to Equation (A.1), then $\tilde{p}$ is compact (by Lemma A.2.4) and self-adjoint. Indeed, we have

$$\begin{aligned}
\langle v, \tilde{p}w\rangle &= \int_G \left\langle v, \pi(g)^{-1}p\pi(g)w\right\rangle \mathrm{d}\mu(g) \\
&= \int_G \left\langle \pi(g)^{-1}p\pi(g)v, w\right\rangle \mathrm{d}\mu(g) \\
&= \langle \tilde{p}v, w\rangle
\end{aligned}$$

and also

$$\begin{aligned}
\langle \tilde{p}v_0, v_0\rangle &= \int_G \left\langle \pi(g)^{-1}p\pi(g)v_0, v_0\right\rangle \mathrm{d}\mu(g) \\
&= \int_G \|p\pi(g)v_0\| \,\mathrm{d}\mu(g)
\end{aligned}$$

Since the argument of the integral is continuous and its value is 1 when evaluated at $g = e$ we can say that $\tilde{p} \neq 0$ and therefore that its spectrum is such that $\sigma(\tilde{p}) \neq \{\,0\,\}$. Thus, there exists a compact set $K \subset \sigma(\tilde{p}) \setminus \{\,0\,\}$ such that the orthogonal projection $E(K)$ given by the spectral decomposition is non zero. Now, since $\tilde{p} \in I(\pi,\pi)$ by Lemma A.2.4 then also $E(K) \in I(\pi,\pi)$ and therefore the subspace $\mathcal{K} = E(K)\mathcal{H}$ is invariant. To conclude the proof we are only left to prove that $\dim\mathcal{K} < +\infty$. Suppose that $\mathcal{K}$ admits an infinite dimensional orthonormal basis $\{e_n\}$, then tends weakly to 0 and therefore $\{\tilde{p}e_n\}$ tends strongly to 0. But, at the same time we know that $\tilde{p}$ admits a spectral decomposition $\tilde{p} = \int_{\sigma(\tilde{p})} \lambda \,\mathrm{d}E(\lambda)$, so if we define the measure $\rho_{e_n,e_n} = \langle E(\lambda)e_n, e_n\rangle$, we can compute

$$\begin{aligned}
\|\tilde{p}e_n\|^2 &= \left\langle \tilde{p}^2 e_n, e_n\right\rangle \\
&= \int_{\sigma(\tilde{p})} |\lambda|^2 \,\mathrm{d}\rho_{e_n,e_n}(\lambda) \\
&\geq \min_K |\lambda|^2 \int_K \mathrm{d}\rho_{e_n,e_n}(\lambda) \\
&= \min_K |\lambda|^2 \,\langle E(K)e_n, e_n\rangle \\
&= \min_K |\lambda|^2
\end{aligned}$$

which clearly contradicts the hypothesis. □

From this Lemma immediately follows the next Theorem which is crucial to prove the decomposability of every unitary representation of a compact group.

Theorem A.2.2. *Let π be a unitary representation of a compact group G on an Hilbert space $\mathcal{H}$. If π is irreducible, then $\mathcal{H}$ is finite dimensional.*

We conclude this Appendix with (a part of) the Peter-Weyl Theorem which we heavily exploit to prove the main results of this Thesis.

Theorem A.2.3. *Let π be a unitary representation of a compact group G on a Hilbert space $\mathcal{H}$. Then π can be decomposed as the direct sum of irreducible finite dimensional subrepresentations acting on subspaces of $\mathcal{H}$ that are pairwise orthogonal.*

Proof. Let $\mathcal{C}$ be the class of families $\{\mathcal{K}_i\}_{i\in I}$ of finite dimensional π-invariant subspaces, that are also pairwise orthogonal and such that the subrepresentations $\pi|_{\mathcal{K}_i}$ obtained by restriction are irreducible for every $i \in I$. We first show that $\mathcal{C}$ is a non empty class. By Lemma A.2.5 we know that π admits a non trivial, invariant and finite dimensional subspace $\mathcal{K}$. If the restricted representation $\pi|_{\mathcal{K}}$ is reducible, we can apply Lemma A.2.5 again obtaining a non trivial, invariant and finite dimensional subspace of $\mathcal{K}$. By iterating the procedure a finite number of time we are ensured to obtain a non trivial, invariant and finite dimensional subspace on which the restriction of π acts irreducibly. This clearly implies that $\mathcal{C}$ is non empty. By ordering $\mathcal{C}$ by inclusion, we can apply Zorn's Lemma and say that there exists maximal family $\{\mathcal{K}_i\}_{i\in I}$ whose direct sum is exactly the starting space $\mathcal{H}$. In order to show this, suppose the opposite is true, that is that $\mathcal{K} = \oplus_{i\in I}\mathcal{K}_i$ is properly contained in $\mathcal{H}$. Since $\mathcal{K}$ is invariant by construction, then also $\mathcal{K}^{\perp}$ is invariant by Lemma A.2.2 but this implies the existence of an invariant subspace of $\mathcal{K}^{\perp}$ by Lemma A.2.5. This is a clear contradiction to the maximality of $\{\mathcal{K}_i\}_{i\in I}$ from which we can conclude that $\mathcal{H} = \oplus_{i\in I}\mathcal{K}_i$. ∎

Appendix B

Type theory of von Neumann algebras

In this Chapter we recall some of the basic definition about von Neumann algebras and type theory.

B.1 Type theory

Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, then the *commutant* of $\mathcal{A}$ is defined as

$$\mathcal{A}' = \{ x \in \mathcal{B}(\mathcal{H}) \mid xy = yx \ \forall y \in \mathcal{A} \} .$$

Similarly, the *center* of $\mathcal{A}$ is defined as

$$\mathcal{Z}(\mathcal{A}) = \{ x \in \mathcal{A} \mid xy = yx \ \forall y \in \mathcal{A} \}$$

from which we see that $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$.

Definition B.1.1. A von Neumann algebra $\mathcal{A}$ is said to be *factor* if its center satisfies $\mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$.

In order to define what is a "type" in the context of von Neumann algebras, we have to first classify the projections contained in the algebra itself.

Definition B.1.2. Let $\mathcal{A}$ be a von Neumann algebra, and $p, q \in \mathcal{A}$ two projections. Then, we say that p is *equivalent* to q, denote as $p \sim q$, if there exists a partial isometry $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$.

Definition B.1.3. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $p, q \in \mathcal{A}$ two projections. Then p is said to be:

- **minimal** if $p \neq 0$ and $q \leq p$ implies either $q = 0$ or $q = p$;
- **abelian** if $p\mathcal{A}p$ is an abelian algebra;
- **finite** if $q \leq p$ and $q \sim p$ implies $q = p$;

- **semi-finite** if there exists a family of projections $\{p_i\}_{i\in I}$ such that $p = \sum_{i\in I} p_i$;
- **purely infinite** if $p \neq 0$ and there does not exist ant non-zero finite projection $q \leq p$;
- **properly infinite** if $p \neq 0$ and for all projections $z \in \mathcal{Z}(\mathcal{A})$ the projection zp is not finite.

As is common, we say that an von Neumann algebra $\mathcal{A}$ is either minimal, semi-finite, finite, purely infinite or properly infinite, if the unit operator $\mathbb{1} \in \mathcal{A}$ as the corresponding property. Directly from Definition B.1.3 it is straightforward to see that we have following chains of implications

$$\text{minimal} \implies \text{abelian} \implies \text{finite} \implies \text{semi-finite} \implies \text{not purely infinite}$$

and

$$\text{purely infinite} \implies \text{properly infinite}$$

With this in mind we can give the following definition

Definition B.1.4. A von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is said to be

- **type I** if every non-zero projection has a non-zero abelian subprojection;
- **type II** if it is semi-finite and has no non-zero abelian projection;
- **type III** if it is purely infinite.

This type classification allows to completely characterize von Neumann algebras as denoted in the following Theorem

Theorem B.1.1. *Let $\mathcal{A}$ be a von Neumann algebra action on a Hilbert space $\mathcal{H}$. Then there exist unique, central, and pairwise orthogonal projections p_{I}, p_{II}, p_{III} such that $p_{\mathrm{I}} + p_{\mathrm{II}} + p_{\mathrm{III}} = \mathbb{1}$ and $\mathcal{A}p_i$ is of type i for $i =$ I, II, III.*

Corollary B.1.1. *Let $\mathcal{A}$ be a factor, then it can only be of either type* I*, type* II *or type* III*.*

Theorem B.1.2. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and p a projection with central support $z(p) = \mathbb{1}$. Then $\mathcal{A}$ is of a given type if and only if $p\mathcal{A}p$ is of the same type.*

Theorem B.1.3. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is of a given type if and only if $\mathcal{A}'$ is of the same type.*

Now that we have delineated the general results about type theory we can recall a few finer details about each algebra type.

B.1.1 Type I algebras

Definition B.1.5. Let $\mathcal{A}$ be a type I von Neumann algebra acting on a Hilbert space $\mathcal{H}$. $\mathcal{A}$ is said to be of type $\mathrm{I_n}$ if there exists $n \in \mathbb{N}$ such that the unit operator $\mathbb{1}$ can be written as the sum of n equivalent non-zero abelian projections. We denote the type as I_∞ when the number of required projections is infinite.

Proposition B.1.1. *Let $\mathcal{A}$ be a type* I *von Neumann algebra. Then it can be uniquely decomposed as a direct sum of type* $\mathrm{I_n}$ *von Neumann algebras.*

Corollary B.1.2. *A type* I *factor is of type* $\mathrm{I_n}$ *only for one $n \in \mathbb{N}$.*

Theorem B.1.4. *Let $\mathcal{A}$ be a type* $\mathrm{I_n}$ *von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{A} \simeq \mathcal{Z}(\mathcal{A}) \otimes \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ such that $\dim \mathcal{K} = n$. Moreover, if $\mathcal{A}$ is abelian then is of type* $\mathrm{I_1}$*, meanwhile, if $\mathcal{A}$ is a type* $\mathrm{I_n}$ *factor then $\mathcal{A} \simeq \mathcal{B}(\mathcal{K})$.*

B.1.2 Type II algebras

Definition B.1.6. A type II von Neumann algebra $\mathcal{A}$ is said to be of type $\mathrm{II_1}$ if is finite, and is said to be of type $\mathrm{II_\infty}$ if is properly infinite.

Theorem B.1.5. *Let $\mathcal{A}$ be a type* II *von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then there exist unique central projections $p_{\mathrm{II_1}}, p_{\mathrm{II_\infty}}$ such that $p_{\mathrm{II_1}} + p_{\mathrm{II_\infty}} = \mathbb{1}$, and such that $\mathcal{A}p_{\mathrm{II_1}}$ is of type* $\mathrm{II_1}$ *and $\mathcal{A}p_{\mathrm{II_\infty}}$ is of type* $\mathrm{II_\infty}$.

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