Toward a structural theory of learning algebraic decompositions

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0. Introduction

The concept of expressing complicated objects in terms of simpler basis elements is ubiquitous in mathematics. It is central in linear algebra, and permeates many branches of the sciences. Such an expression often allows to answer questions concerning the original object by considering the better-known basis elements. As just one prominent instance, a fundamental challenge in signal and image processing is to fit the parameters of a model to measured data, e. g., for purposes of interpolation, extrapolation or spectral analysis, see, for instance, the textbook of Marple [43].

It is therefore of great importance to have efficient as well as general techniques at one's disposal to analyze data and obtain (or "learn"), such structured decompositions.

The problem is highly relevant in vector spaces of functions, with many important examples being infinite-dimensional. In this scenario, strategies that do not employ direct searches within the basis B become a necessity. An often-used approach is to first consider the subproblem of determining the support of a vector $f \in V$ w. r. t. B, that is, the (finite) set of basis vectors occurring in the expansion of f with a non-zero coefficient. It is usually much easier to determine the coefficients once the support is known. Of course, this strategy may also be applied in the finite-dimensional case.

The paradigmatic instance is Prony's method [56] to decompose a sum f of exponential functions. It uses evaluations to construct a polynomial whose roots correspond to the summands of f. Generalizations and variations of Prony's method have been intensely studied, with motivations stemming from, e. g., signal processing and biology. Classic applications of Prony's method include for example Sylvester's method for Waring decompositions of binary forms [64, 65] and Padé approximation [69, 7]. Over centuries, these tools have been further developed, with [52, 46, 54, 63] being a small sampling of the recent literature. New applications have been found (see, e. g., [30, Section 2.2] for connections to the Berlekamp-Massey algorithm). For a recent overview see [51].

Recently also advances have been made on multivariate versions. Direct attempts can be found in, e. g., [53, 47, 38, 57, 45], for methods based on projections to univariate exponential sums see, e. g., [17, 18, 14]. A numerical variant can be found in, e. g., [20], and further related results and applications in, e. g., [19, 24, 40, 12, 31, 55, 9, 10, 29, 50, 33, 32, 27, 15].

Among the spaces (V, B) for which instances of the problem have been studied are, in particular, the space of polynomials $V = K[\mathrm{y}]$ (K a field) with the monomial basis B [5] or bases of Chebyshev polynomials [40, 31, 46, 55, 27]. These investigations led to analogous results and "Prony-like" decomposition methods for these spaces. The apparent similarities motivate the search for a common framework in which to study and extend these methods.

In 1991, Dress and Grabmeier [19] proposed a framework for the decomposition of character sums, i. e., linear combinations of monoid homomorphisms mapping a commutative monoid to the multiplicative monoid of a field. This setup encompasses, among others, the above mentioned methods for decomposition into of sums of exponentials and sums of monomials. Refinements have been proposed by Grigoriev, Karpinski, and Singer [24] and in 2013 by Peter and Plonka [46], the latter containing an alternative “analytic” approach to the Chebyshev decomposition problem which does not seem to fit into the earlier frameworks.

In order to facilitate the simultaneous study of these classes of methods, we introduce the axiomatic notion of Prony structures (cf. Definition 1.7). We will see that a large part[1] of the instances that could be treated within previously established frameworks and also Lakshman and Saunders' Chebyshev decomposition method are included in this setup.

In typical Prony situations one has a natural identification of basis elements with points in an affine space. For example, in the classic case of exponential sums the basis function $\exp_b$ is identified with its base point $b \in \mathbb{C}^n$. It is this identification that allows to describe the support of f by polynomial equations. A key idea of Prony is to construct Hankel (or Toeplitz) matrices using evaluations of f to obtain the desired data from their kernels.

In our framework we assume that an identification as above is given as part of the initial data. Then suitable sequences of matrices are computed from evaluations of f which are constructed in a way such that their kernels eventually have to yield systems of polynomial equations to determine the support of f.

The thesis is organized as follows. Chapter 1 begins with a brief presentation of Prony's method for the reconstruction of exponential sums, and two well-known variants for polynomial spaces, emphasizing their similarities. With this motivation, we set out on our task of unifying these methods. After fixing the setup and some notation, we introduce our main definition of a Prony structure. Besides the function space and the basis as key parts of the data it consists of families of linear maps and associated ideals defined by their kernels. These ideals are then used to attack the decomposition problem. We discuss properties of evaluation maps on vector spaces of polynomials and their kernels. As one of the main results in this chapter, we prove in Theorem 1.18 a very useful characterization of Prony structures in terms of factorizations through evaluation maps. It can be seen that given some mild assumptions the ideals of a Prony structure are zero-dimensional and radical, which leads to the natural question to provide sufficient conditions which guarantee that the ideals of kernels of evaluation maps have this property. We study this problem, proving a theorem we learned from H. M. Möller on Gröbner bases of zero-dimensional radical ideals with interesting consequences for Prony structures. At the end of the chapter we introduce the category of Prony structures and prove a useful transfer for Prony structures.

In Chapter 2 we discuss examples and applications of Prony structures. We introduce the notion of t-exponential in order to treat Hankel and Toeplitz variants of Prony's

[1] The main exception are characters of not finitely generated monoids, which, roughly speaking, corresponds to the case of functions in an unbounded number of variables.

method for exponential sums w. r. t. various vector space bases in a unified manner. Versions for polynomials with the corresponding monomial and Chebyshev bases and Gaußian sums then follow by the transfer principle.

In Chapter 3 we show how the framework of Prony structures relates to previously known frameworks for the decomposition of sums of characters [19] or eigenfunctions [24, 46].

A priori knowledge can be that functions are supported for example on a torus or a sphere, see, e. g., [36, 37]. Classic techniques do not take this additional information into account. As a novel approach we extend the notion of Prony structures for functions supported on algebraic sets to a relative version in Chapter 4. A first key result is a characterization of such structures in Theorem 4.8. We discuss how to obtain Prony structures in this relative case. Main examples include relative Prony structures for spaces of spherical harmonics.

This work is based on the article [39] by Kunis, Römer, and von der Ohe. A main addition is the unified treatment using the concept of t-exponentials in Chapter 2. We do not cite individual results from this publication. This work also constitutes a continuation, generalization, and update of the earlier thesis [68] of the author. The articles Kunis, Peter, Römer, and von der Ohe [38], Kunis, Möller, Peter, and von der Ohe [36], and Kunis, Möller, and von der Ohe [37] have been the basis of this earlier thesis and they, as well as [68], and are cited appropriately.

1. Prony structures

In this chapter we define our main notion of Prony structure and deduce relevant properties. We start in Section 1.1 by presenting three classic results that will yield typical examples of Prony structures: Prony's reconstruction method for exponential sums, Ben-Or and Tiwari's method for polynomials, and Lakshman and Saunders' method for Chebyshev polynomials. Each of these methods involve the construction of a set of polynomials whose zero locus corresponds to the support of the input function. We take this as motivation to introduce a common abstraction, Prony structures, in Section 1.2. We discuss fundamental properties of Prony structures in Section 1.3 where the close connection to evaluation maps on spaces of polynomials is established. Their relevant basic properties are recalled and additional properties are proven, leading to a theorem of Möller. In Section 1.4 we define structure preserving maps between Prony structures and prove a useful transfer principle that will be applied in the following chapter.

1.1. Prelude: Three classic results

In the following subsections we briefly discuss three well-known and exemplary "Prony-like" results. All three are formulated analogously and to mesh with later material.

1.1.1. Reconstruction of exponential sums

We start with the classic method due to G. C. F. M. Riche, baron de Prony [56]. Let F be an arbitrary field. For $b \in F$ we call the sequence

$$\exp_b\colon \quad \mathbb{N} \longrightarrow F, \qquad k \longmapsto b^k,$$

exponential. Here it is understood that $b^0 = 1$ for all $b \in F$, in particular for $b = 0$. The element b is called the *base* of $\exp_b$; it is uniquely determined since $b = \exp_b(1)$. An *exponential sum* is defined to be an F-linear combination of exponentials.

To an exponential sum f one associates the sequence $(\Phi_d(f))_{d\in\mathbb{N}}$ of linear maps with

$$\Phi_d(f)\colon \quad F[\mathrm{x}]_{\le d} \longrightarrow F^d, \qquad \sum_{k=0}^{d} p_k \mathrm{x}^k \longmapsto P_d(f) \cdot (p_0, \ldots, p_d)^\top,$$

where $P_d(f)$ is the Hankel matrix

$$P_d(f) := \Big(f(i+j)\Big)_{\substack{i=0,\ldots,d-1\\ j=0,\ldots,d}} = \begin{pmatrix} f(0) & \cdots & f(d-1) & f(d) \\ f(1) & \cdots & f(d) & f(d+1) \\ \vdots & \vdots & \vdots & \vdots \\ f(d-1) & \cdots & f(2d-2) & f(2d-1) \end{pmatrix} \in F^{d\times(d+1)}.$$

For a set $A \subseteq F[\mathrm{x}]$ let the zero locus of A be denoted by

$$\mathrm{Z}(A) = \{b \in F \mid p(b) = 0 \text{ for all } p \in A\}.$$

(In the univariate situation at hand, of course $\mathrm{Z}(A)$ consists precisely of the roots of a generator $p \in F[\mathrm{x}]$ of the principal ideal $\langle A \rangle = \langle p \rangle$ of $F[\mathrm{x}]$.) The following theorem forms the basis for Prony's method.

Theorem 1.1 (Prony, 1795): *Let $f = \sum_{i=1}^{r} f_i \exp_{b_i}$ be an exponential sum with non-zero coefficients $f_i \in F$ and pairwise distinct bases $b_i \in F$. Then the following are equivalent:*

(i) $\mathrm{Z}(\ker \Phi_d(f)) = \{b_1, \dots, b_r\}$*;*

(ii) $d \geq r$*.*

Using Prony's theorem, one can approach the problem of computing the bases b_i of the exponential sum f by computing the roots of a polynomial. Always choosing the least index $d = r$ that is sufficient to perform the computations, the size of the matrices and number of evaluations depends only on the number r of summands of f; we will call this number also the *rank of f*. Methods whose computational complexity is essentially determined by the rank of the input may be called *sparse methods*, and Prony's method for the decomposition of exponential sums is among the prime examples.

1.1.2. Sparse monomial interpolation

Prony's method for exponential sums has since attracted much attention and has in particular inspired variants for other families of functions and generalizations.

Among the classes that Prony's method has been adapted to, and closely related to the case of exponential sums, are polynomial functions over a field F. For decompositions w. r. t. the monomial basis a result can be found explicitly as a special case in Ben-Or and Tiwari [5]. It is quickly obtained as a corollary of Theorem 1.1 as follows. We emphasize that [5] is mostly concerned with the efficient treatment of the multivariate case, which we omit in this introductory discussion.

Let F be an arbitrary field. The objective is to determine the support of a polynomial over F w. r. t. the monomial basis using only evaluations. Of course, this cannot be achieved in general with evaluations only in F and hence we allow evaluating in a field extension of F.

Let b be an element in a suitable field $K \geq F$ such that the sequence $(b^i)_{i \in \mathbb{N}} \in K^{\mathbb{N}}$ is injective. For $f \in F[\mathrm{y}]$ let the sequence $(\Phi_d(f))_{d \in \mathbb{N}}$ of K-linear maps be defined by

$$\Phi_d(f)\colon \quad K[\mathrm{x}]_{\leq d} \longrightarrow K^d, \qquad \sum_{k=0}^{d} p_k \mathrm{x}^k \longmapsto P_d(f) \cdot (p_0, \dots, p_d)^\top,$$

where $P_d(f)$ denotes the matrix

$$P_d(f) := \left(f(b^{i+j})\right)_{\substack{i=0,\dots,d-1\\ j=0,\dots,d}} = \begin{pmatrix} f(b^0) & \cdots & f(b^{d-1}) & f(b^d) \\ f(b^1) & \cdots & f(b^d) & f(b^{d+1}) \\ \vdots & \vdots & \vdots & \vdots \\ f(b^{d-1}) & \cdots & f(b^{2d-2}) & f(b^{2d-1}) \end{pmatrix} \in K^{d\times(d+1)}.$$

For $f = \sum_i f_i \mathrm{y}^i \in F[\mathrm{y}]$ let $\operatorname{supp} f = \{i \in \mathbb{N} \mid f_i \neq 0\}$ denote the support of f. The following is a version of Theorem 1.1 for polynomials instead of exponential sums.

Corollary 1.2 (Ben-Or and Tiwari, 1988): *Let $b \in K \geq F$ be chosen as above and let $f \in F[\mathrm{y}]$. Then the following are equivalent:*

(i) $\mathrm{Z}(\ker \Phi_d(f)) = \{b^i \mid i \in \operatorname{supp} f\}$;

(ii) $d \geq |\operatorname{supp} f|$.

Indeed, Corollary 1.2 follows from Theorem 1.1 by the observation that the function $\mathbb{N} \to K$, $i \mapsto f(b^i)$, is an exponential sum with support $\{\exp_{b^i} \mid i \in \operatorname{supp} f\}$. In order to decompose polynomials in this way, it is essential to have an efficient way of computing logarithms w. r. t. the element $b \in K$. Those logarithms may be precomputed for specific choices of b.

1.1.3. Sparse Chebyshev interpolation

In 1995, Lakshman and Saunders [40] proposed the following as a method to compute Chebyshev decompositions of polynomials. Let F be a field of characteristic zero. Recall that the *Chebyshev polynomials* $\mathrm{T}_i \in \mathbb{Z}[\mathrm{y}] \leq F[\mathrm{y}]$, $i \in \mathbb{N}$, are defined inductively by

$$\mathrm{T}_0 := 1, \quad \mathrm{T}_1 := \mathrm{y}, \quad \text{and} \quad \mathrm{T}_i := 2\mathrm{y}\mathrm{T}_{i-1} - \mathrm{T}_{i-2} \text{ for } i \geq 2.$$

It follows immediately from the definition that $\deg(\mathrm{T}_i) = i$. Hence, $B := \{\mathrm{T}_i \mid i \in \mathbb{N}\}$ is an F-basis of $F[\mathrm{y}]$ and for any $d \in \mathbb{N}$, $\{\mathrm{T}_0, \dots, \mathrm{T}_d\}$ is an F-basis of $F[\mathrm{y}]_{\leq d}$.

Let $b \in F$ be suitably chosen (since $\operatorname{char} F = 0$ there is a unique embedding of $\mathbb{Q}$ into F and here it is sufficient that $b \in \mathbb{Q} \subseteq F$ with $b > 1$ as an element of $\mathbb{Q}$), set $u_i := \mathrm{T}_i(b) \in F$, and let the sequence of linear maps $(\Phi_d(f))_{d\in\mathbb{N}}$ be defined by

$$\Phi_d(f)\colon \quad F[\mathrm{x}]_{\leq d} \longrightarrow F^d, \qquad \sum_{k=0}^{d} p_k \mathrm{T}_k \longmapsto P_d(f) \cdot (p_0, \dots, p_d)^\top,$$

where $P_d(f)$ denotes the matrix

$$P_d(f) := \left(f(u_{i+j}) + f(u_{|j-i|})\right)_{\substack{i=0,\dots,d-1\\ j=0,\dots,d}},$$

i. e., $P_d(f)$ is the sum of the $d \times (d+1)$ Hankel matrix

$$H_d(f) = \begin{pmatrix} f(u_0) & \cdots & f(u_{d-1}) & f(u_d) \\ f(u_1) & \cdots & f(u_d) & f(u_{d+1}) \\ \vdots & \vdots & \vdots & \vdots \\ f(u_{d-1}) & \cdots & f(u_{2d-2}) & f(u_{2d-1}) \end{pmatrix}$$

and the $d \times (d+1)$ Toeplitz matrix

$$T_d(f) = \begin{pmatrix} f(u_0) & f(u_1) & f(u_2) & \cdots & f(u_{d-1}) & f(u_d) \\ f(u_1) & f(u_0) & f(u_1) & \cdots & f(u_{d-2}) & f(u_{d-1}) \\ f(u_2) & f(u_1) & f(u_0) & \cdots & f(u_{d-3}) & f(u_{d-2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f(u_{d-1}) & f(u_{d-2}) & f(u_{d-3}) & \cdots & f(u_0) & f(u_1) \end{pmatrix}.$$

For $f = \sum_i f_i \mathrm{T}_i \in F[\mathrm{y}]$ let $\operatorname{supp} f = \{i \in \mathbb{N} \mid f_i \neq 0\}$.

Theorem 1.3 (Lakshman and Saunders, 1995): *Let $u_i \in F$ be as above and let $f \in F[\mathrm{y}]$. Then the following are equivalent:*

(i) $\mathrm{Z}(\ker \Phi_d(f)) = \{u_i \mid i \in \operatorname{supp} f\}$*;*

(ii) $d \geq |\operatorname{supp} f|$.

With b suitably chosen, the sequence $(u_i)_{i\in\mathbb{N}}$ is injective and hence the support of f w. r. t. the Chebyshev basis B is determined by the set $\{u_i \mid i \in \operatorname{supp} f\}$. To perform the method in practice, analogously to the monomial case one must (pre-)compute the "Chebyshev logarithms" of b, i. e., the inverse of the mapping $i \mapsto u_i = \mathrm{T}_i(b)$ for appropriate u_i. A computational illustration of the method is given in Example 2.18.

The preceding discussion indicates a structural analogy of the results of Prony, Ben-Or and Tiwari, and Lakshman and Saunders, which we will investigate more closely in the following.

1.2. Foundation

Motivated by the methods discussed in Section 1.1 and other recent variations and generalizations of Prony's method, we introduce a framework that allows to treat these variants simultaneously and which can be applied in various contexts.

We first fix some notation regarding evaluation maps for polynomials.

Definition 1.4: Let K be a field, $n \in \mathbb{N}$, $S := K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, and let $U \subseteq S$ be a K-subvector space of S. For $X \subseteq K^n$ we set

$$\mathrm{ev}_U^X\colon \quad U \longrightarrow K^X, \qquad p \longmapsto (p(x))_{x\in X},$$

and call ev_U^X *evaluation map at X on U*. Of course, evaluation maps are K-linear. We set

$$\mathrm{I}_U(X) := \ker \mathrm{ev}_U^X\,,$$

and call $\mathrm{I}_U(X)$ the *vanishing space of X on U*.

The space U is called *X-interpolating* if ev_U^X is surjective.

Observe that for $U = S$, $\mathrm{I}_U(X) = \mathrm{I}(X)$ is the usual *vanishing ideal of X*. In this special situation we also set $\mathrm{ev}^X := \mathrm{ev}_S^X$. Note that in general we have

$$\mathrm{I}_U(X) = \mathrm{I}(X) \cap U.$$

Occasionally we will identify the monomial $\mathrm{x}^\alpha \in S$ with its exponent $\alpha \in \mathbb{N}^n$. Then, for $D \subseteq \mathbb{N}^n$ we set $\mathrm{ev}_D^X := \mathrm{ev}_U^X$ and $\mathrm{I}_D(X) := \mathrm{I}_U(X)$ where U denotes the K-subvector space generated by $\{\mathrm{x}^\alpha \mid \alpha \in D\}$ in S, and we call D *X-interpolating* if U is X-interpolating. In Section 1.3 we will state and prove all results on evaluation maps and their kernels that are relevant here. The *zero locus* of a set $A \subseteq S$ is denoted by $\mathrm{Z}(A) = \{x \in K^n \mid p(x) = 0 \text{ for all } p \in A\}$.

In the following three definitions we introduce the central notion of a Prony structure. The key point is to provide a general formal setting that captures the essence of Prony's method with the aim of laying the foundation for a structural theory. In Definition 1.5 we formulate the crucial properties of a Prony structure without reference to the space of elements that are to be decomposed. This forms the basis for Definition 1.7 where the algebraic set X corresponds to the support of a vector.

As usual, for a sequence $a = (a_k)_{k\in\mathbb{N}}$ and a property P, we say that *a satisfies P eventually* or that *$P(a_k)$ holds for all large k* if there is an $m \in \mathbb{N}$ such that for all $k \geq m$, a_k satisfies P.

Definition 1.5 (Prony structure, abstract version): Let $X \subseteq K^n$ be an algebraic set.

(a) Let U be a K-subvector space of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and let W be an arbitrary K-vector space. We say that a K-linear map $\varphi\colon U \to W$ *has a Prony kernel for X* if the following conditions (P_1) and (P_2) are satisfied:

$$\mathrm{Z}(\ker\varphi) = X \qquad (\mathrm{P}_1)$$

and

$$\mathrm{I}_U(X) \subseteq \ker\varphi. \qquad (\mathrm{P}_2)$$

(b) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. We call a sequence

$$\varphi \in \prod_{d\in\mathbb{N}} \mathrm{Hom}_K(S_d, W_d)$$

a *Prony structure for X* if for all large d, φ_d has a Prony kernel for X.

In this case, the number

$$\operatorname{ind}_{\varphi}(X) := \min\{c \in \mathbb{N} \mid \text{for all } d \geq c,\ \varphi_d \text{ has a Prony kernel for } X\}$$

is called *φ-index of X*.

The typical situation we consider is that the set X corresponds to the support of a vector f w. r. t. a basis B of some vector space V. Given such a correspondence, the kernel of φ consists of equations for the support of f. We formalize this in the following definition.

Definition 1.6: Let F be a field, V be an F-vector space, and B be an F-basis of V. For $f \in V$, $f = \sum_{i=1}^{r} f_i b_i$ with $f_1, \ldots, f_r \in F \setminus \{0\}$ and distinct $b_1, \ldots, b_r \in B$, let

$$\operatorname{supp}_B f := \{b_1, \ldots, b_r\} \quad \text{and} \quad \operatorname{rank}_B f := |\operatorname{supp}_B f| = r$$

denote the *support of f* and *rank of f* (w. r. t. B), respectively. For a field K, $n \in \mathbb{N}$, and an injective map $u\colon B \to K^n$ let

$$\operatorname{supp}_u f := \{u(b_1), \ldots, u(b_r)\}.$$

We call $\operatorname{supp}_u f \subseteq K^n$ the *u-support* and its elements the *support labels of f*.

In many situations we will choose $K = F$, but for reasons of flexibility we allow the choice of possibly different fields. Unless mentioned otherwise, we will assume that F, V, B, K, n, and u are given as in Definition 1.6.

Definition 1.7 (Prony structure): Given the setup of Definition 1.6, we define the following:

(a) Let U be a K-subvector space of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and let W be an arbitrary K-vector space. Let $f \in V$. We say that a K-linear map $\varphi\colon U \to W$ *has a Prony kernel for f w. r. t. u* if φ has a Prony kernel for $X = \operatorname{supp}_u f$.

(b) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. Let $f \in V$. We call a sequence $\varphi \in \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(S_d, W_d)$ a *Prony structure for f w. r. t. u* if φ is a Prony structure for $X = \operatorname{supp}_u f$.

In this case, the number

$$\operatorname{ind}_{\varphi}(f) := \operatorname{ind}_{\varphi}(\operatorname{supp}_u f)$$

is called *φ-index of f*.

(c) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. We call a function

$$\Phi\colon \quad V \longrightarrow \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(S_d, W_d)$$

a *Prony structure on* (V, B, u) if for every $f \in V$ the sequence $\Phi(f)$ is a Prony structure for f w. r. t. u.

For $f \in V$ we call the $\Phi(f)$-index of f simply *Φ-index of f* and denote it by $\operatorname{ind}_\Phi(f)$.

Remark 1.8: A key point of a Prony structure Φ on (V, B, u) is that the idea of Prony's method works, i. e. to compute the support of a given $f \in V$ w. r. t. the basis B through a system of polynomial equations. More precisely, one can perform the following (pseudo-)algorithm:

1. Choose $d \in \mathbb{N}$.
2. Determine the linear map $\Phi_d(f)\colon S_d \to W_d$.
3. Compute $U := \ker \Phi_d(f) \subseteq S_d$.
4. Compute $Z := \mathrm{Z}(U) \subseteq K^n$.
5. Compute $u^{-1}[Z] \subseteq B$.

If d is chosen sufficiently large, then the zero locus Z is the u-support and its preimage $u^{-1}[Z]$ is the support of f (and in particular these sets are finite). Note that for this strategy to work in practice, it is important that the maps $\Phi_d(f)$ can be computed from "standard information" on f (such as evaluations if f is a function), in particular without prior knowledge of its support; see also Remark 1.10. Computations will usually be performed with matrices of the linear maps $\Phi_d(f)$. Often computation of the zero locus as well as a good choice of d turn out to be problematic steps.

In classic situations of Prony's method the non-zero coefficients of f w. r. t. B can be computed in an additional step by solving a system of linear equations involving only standard information; this system is finite since one has already computed the support.

Common options for the sequence $(S_d)_{d\in\mathbb{N}}$ of subvector spaces of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ are $S_d = S_{\mathcal{T}_d}$, $S_d = S_{\mathcal{M}_d}$, or $S_d = S_{\mathcal{C}_d}$, where

$$\begin{aligned}
\mathcal{T}_d &:= \Big\{\alpha \in \mathbb{N}^n \;\Big|\; \sum_{j=1}^n \alpha_j \le d\Big\},\\
\mathcal{M}_d &:= \{\alpha \in \mathbb{N}^n \mid \max\{\alpha_j \mid j = 1, \ldots, n\} \le d\},\\
\text{and}\quad \mathcal{C}_d &:= \Big\{\alpha \in \mathbb{N}^n \;\Big|\; \prod_{j=1}^n (\alpha_j + 1) \le d\Big\}.
\end{aligned}$$

Choosing $S_d = S_{\mathcal{T}_d}$ yields the subvector space of polynomials of *total degree* at most d.

The choice of $S_d = S_{\mathcal{M}_d}$ yields the subvector space of polynomials of *maximal degree* at most d. One motivation for considering these spaces lies in the fact that sometimes it is possible to reduce statements involving polynomial functions in several variables to the univariate case using the fact that $K[\mathrm{x}_1, \ldots, \mathrm{x}_n] \cong K[\mathrm{x}_1] \otimes_K \cdots \otimes_K K[\mathrm{x}_n]$. With this technique, estimates of the (total) degree of univariate solutions naturally lead to estimates of the maximal degree of solutions in the multivariate case. Examples include estimates for the degree of polynomials approximating continuous functions in multivariate versions of the Weierstraß approximation theorem, see, e. g., Trefethen [67, Chapter 6]; also proofs of the absolute convergence of Cauchy products of absolutely convergent sequences can be seen in this light, cf., e. g., Forster [21, § 8, proof of Satz 3].

Choosing $S_d = S_{\mathcal{C}_d}$ gives rise to a space of polynomials that is particularly well-suited for zero-testing and interpolation of polynomials. The earliest use of $\mathcal{C}_d$, the non-negative orthant of the *hyperbolic cross of order d*, in the context of Prony-like methods that we are aware of is in articles by Clausen, Dress, Grabmeier, and Karpinski [8] and by Dress and Grabmeier [19]. For more recent applications see in particular Sauer [58] and the recent preprint of Hubert and Singer [27].

Often one sets $W_d = S_d$, $W_d = S_{d-1}$, or chooses a similar relation between the sequences $(W_d)_{d\in\mathbb{N}}$ and $(S_d)_{d\in\mathbb{N}}$.

For the case of total degree we will also use the notation

$$S_{\leq d} := S_{\mathcal{T}_d} = \langle \mathrm{x}^\alpha \mid \alpha \in \mathcal{T}_d \rangle_K, \quad \mathrm{ev}^X_{\leq d} := \mathrm{ev}^X_{\mathcal{T}_d}, \quad \text{and} \quad \mathrm{I}_{\leq d}(X) := \mathrm{I}_{\mathcal{T}_d}(X).$$

Remark 1.9: A framework for the decomposition of sums of characters of commutative monoids has been proposed by Dress and Grabmeier [19] and derivations for sums of eigenfunctions (or more generally eigenvectors) of linear operators have been developed by Grigoriev, Karpinski, and Singer [24] and by Peter and Plonka [46]. We recast these frameworks in the language of Prony structures in Chapter 3. See Remark 3.8 for an overview.

While there is considerable overlap with the one we propose, the approaches make different compromises between generality and effectivity. We aim at a formalization of the most general situation in which Prony's strategy still works. Our treatment is axiomatic rather than the explicit constructions of [19, 24, 46]. While trading in some directness, this abstraction also allows to stay within the language of linear algebra. When dealing with applications, a detour through character sums can seem unnatural (or, as in the Chebyshev decomposition, impossible) given the concrete situation. In this sense, we also find our framework to be more effectively verifiable.

Remark 1.10: For $f \in V$, $X := \mathrm{supp}_u f$, let $\varPhi_d(f) := \mathrm{ev}^X_{\leq d}$. It follows immediately from the noetherianity of the polynomial ring S that the sequence $(\varPhi_d(f))_{d\in\mathbb{N}}$ is a Prony structure for f, cf. Lemma 1.14. We call $\varPhi(f)$ *trivial Prony structure for f*.

The trivial Prony structure itself is useless for the practical computation of the support of f: If one is able evaluate $\varPhi_d(f)$, then one can compute the u-support of f using $\varPhi_d(f)(\mathrm{x}_j) = (b_j)_{b\in X}$ for $d \geq 1$. While this cannot be viewed as a computational shortcut, trivial Prony structures still provide a possible strategy to construct Prony structures that may be obtained from the available data, see Corollary 1.16.

Note also that the function

$$\varPhi\colon \quad V \longrightarrow \prod_{d\in\mathbb{N}} \mathrm{Hom}_K(S_{\leq d}, K^X), \qquad f \longmapsto (\varPhi_d(f))_{d\in\mathbb{N}},$$

is in general *not* a Prony structure on (V, B, u) since the sequences of spaces (W_d) in Definition 1.7 (c) cannot depend on f. This holds for $W_d = K^X = K^{\operatorname{supp}_u f}$ only in the trivial case that $V = \{0\}$.

EXAMPLE 1.11: We reformulate the methods discussed in Section 1.1 in the language of Prony structures as follows.

(a) Let $E := \{\exp_b \mid b \in F\}$ and $\operatorname{Exp} := \langle E \rangle_F$ be the F-vector space of exponential sums. Set

$$u\colon \quad E \longrightarrow F, \qquad \exp_b \longmapsto b = \exp_b(1).$$

Then u is clearly a bijection. Let $\Phi_d(f)$ be defined as in Section 1.1.1 and

$$\Phi\colon \quad \operatorname{Exp} \longrightarrow \prod_{d \in \mathbb{N}} \operatorname{Hom}_F(F[\mathrm{y}]_{\leq d}, F^d), \qquad f \longmapsto (\Phi_d(f))_{d \in \mathbb{N}}.$$

Then by Theorem 1.1, Φ is a Prony structure on $(\operatorname{Exp}, E, u)$. Moreover, one has $\operatorname{ind}_\Phi(f) = \operatorname{rank}_E(f)$ for every exponential sum $f \in \operatorname{Exp}$.

(b) Let $M := \{\mathrm{y}^i \mid i \in \mathbb{N}\}$ be the F-basis of monomials of $F[\mathrm{y}]$. With $b \in K \geq F$ chosen as in Section 1.1.2, set

$$u\colon \quad M \longrightarrow K, \qquad \mathrm{y}^i \longmapsto b^i.$$

By the choice of b, u is an injection. Let $\Phi_d(f)$ be defined as in Section 1.1.2 and

$$\Phi\colon \quad F[\mathrm{y}] \longrightarrow \prod_{d \in \mathbb{N}} \operatorname{Hom}_F(F[\mathrm{y}]_{\leq d}, F^d), \qquad f \longmapsto (\Phi_d(f))_{d \in \mathbb{N}}.$$

Then, by Corollary 1.2, Φ is a Prony structure on $(F[\mathrm{y}], M, u)$. Moreover, one has $\operatorname{ind}_\Phi(f) = \operatorname{rank}_M(f)$ for every polynomial $f \in F[\mathrm{y}]$.

(c) Let F be a field of characteristic zero and $C := \{\mathrm{T}_i \mid i \in \mathbb{N}\}$ be the Chebyshev basis of $F[\mathrm{y}]$. With $b \in F$ chosen as in Section 1.1.3, set

$$u\colon \quad C \longrightarrow K, \qquad \mathrm{T}_i \longmapsto \mathrm{T}_i(b).$$

By the choice of b, u is an injection. Let $\Phi_d(f)$ be defined as in Section 1.1.3 and

$$\Phi\colon \quad F[\mathrm{y}] \longrightarrow \prod_{d \in \mathbb{N}} \operatorname{Hom}_F(F[\mathrm{y}]_{\leq d}, F^d), \qquad f \longmapsto (\Phi_d(f))_{d \in \mathbb{N}},$$

Then, by Theorem 1.3, Φ is a Prony structure on $(F[\mathrm{y}], C, u)$. Moreover, one has $\operatorname{ind}_\Phi(f) = \operatorname{rank}_C(f)$ for every polynomial $f \in F[\mathrm{y}]$.

In each of these cases, this provides a means to compute the support w. r. t. to the respective basis under the assumption that an upper bound $d = d_f \in \mathbb{N}$ of the rank of f is known. Further generalizations and variants will be discussed in Chapters 2 and 3 (see also Peter and Plonka [46], Kunis, Peter, Römer, and von der Ohe [38], Sauer [57], and Mourrain [45]).

Remark 1.12: One might be tempted to remove the technical condition (P_2) from Definition 1.5 (a). For the sake of discussion we say that $\varphi\colon U \to W$ has a *quasi Prony kernel for* $X \subseteq K^n$ if φ satisfies condition (P_1) of Definition 1.5 (a), and define *quasi Prony structures for* $X \subseteq K^n$ respectively *for* $f \in V$ in the nearby way. We observe the following:

(a) All practically relevant examples of quasi Prony structures that we are aware of are indeed Prony structures.

(b) One of the main reasons why we include condition (P_2) in the definition of Prony structures is that the analogues of several of our statements on Prony structures do not hold or are not known to hold for quasi Prony structures; see, for example, Theorem 1.18 and Corollary 4.14.

(c) An "artificial" example of a quasi Prony structure that is not a Prony structure: Let $U := K[\mathrm{x}]_{\leq 2}$, $W := K[\mathrm{x}]_{\leq 1}$, and $\varphi\colon U \to W$ the linear map with $\varphi(1) = 1$, $\varphi(\mathrm{x}) = \mathrm{x}$, and $\varphi(\mathrm{x}^2) = 0$. Then $\ker\varphi = \langle \mathrm{x}^2 \rangle_K \subseteq K[\mathrm{x}]$, so $\mathrm{Z}(\ker\varphi) = \mathrm{Z}(\mathrm{x}^2) = \{0\}$. Hence we see that φ has a quasi Prony kernel for the exponential $f := \exp_0$. Since $\mathrm{x} \in \mathrm{I}_U(0) \setminus \ker\varphi$, φ does not have a Prony kernel for X.

Remark 1.13: (a) The generalization of Prony's problem to polynomial-exponential sums (i. e., sums of functions $\alpha \mapsto p(\alpha)\exp_b(\alpha)$ with polynomials p), also known as "multiplicity case", can be found in the univariate case in Henrici [25, Theorem 7.2 c]. Further developments such as a characterization of sequences that allow interpolation by polynomial-exponential sums have been obtained by Sidi [62] and a variant based on an associated generalized eigenvalue problem is given in Lee [41], see also Peter and Plonka [46, Theorem 2.4] and Stampfer and Plonka [63]. For generalizations of many of these results to the multivariate setting see Mourrain [45]. It would be interesting to extend the notion of Prony structures to also include these cases. We leave this for future work. See also Remark 3.7.

(b) In general, if $\varphi\colon U \to W$ has a Prony kernel for $X \subseteq K^n$ and K is algebraically closed, then we have $\mathrm{rad}(\langle\ker\varphi\rangle) = \mathrm{I}(\mathrm{Z}(\ker\varphi)) = \mathrm{I}(X)$ by Hilbert's Nullstellensatz. It is an interesting problem whether always or under which conditions the ideal $\langle\ker\varphi\rangle$ is already a radical ideal. We return to this question in Section 1.3 where we provide partial answers also over not necessarily algebraically closed fields.

1.3. Prony structures and the evaluation map

In this section we first recall some well-known properties of evaluation maps on vector spaces of polynomials and their kernels. Since they are the vector spaces of polynomials vanishing on a set $X \subseteq K^n$, these kernels play a crucial role in the theory and application of Prony structures. This will be made precise in Theorem 1.18 and its Corollary 1.19.

After providing these essential facts, we study ideal-theoretic issues related to evaluation maps. Inspired by a theorem we learned from H. M. Möller [44], we introduce a condition that allows a characterization of the surjectivity of evaluation maps, cf. Corollary 1.28. We are very grateful toward H. M. Möller for several inspiring discussions related to these results. Consequences of these results for Prony structures are summarized in Corollary 1.32.

Lemma 1.14: *Let $X \subseteq K^n$ be an arbitrary subset. Then there is a $d \in \mathbb{N}$ with $\langle \mathrm{I}_{\leq d}(X) \rangle = \mathrm{I}(X)$. For finite X this implies $\mathrm{Z}(\mathrm{I}_{\leq d}(X)) = X$.*

Proof: This follows immediately from the fact that $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ is noetherian and thus $\mathrm{I}(X)$ is finitely generated. If X is finite, then it is Zariski closed. q. e. d.

The following is a "quantitative" version of Lemma 1.14. The arguments are well-known, see, e. g., Kunis, Peter, Römer, and von der Ohe [38, proof of Theorem 3.1].

Proposition 1.15: (a) *Let $X \subseteq K^n$ be finite. With $d := |X|$ we have*

$$\langle \mathrm{I}_{\leq d}(X) \rangle = \mathrm{I}(X).$$

(b) *Let K be an infinite field. Then for every $d \in \mathbb{N}$ there is an $X \subseteq K^n$ with $|X| = d + 1$ such that $\langle \mathrm{I}_{\leq d}(X) \rangle \subsetneqq \mathrm{I}(X)$.*

Corollary 1.16: *Let $X \subseteq K^n$ be finite. Then for any K-vector space W and injective K-linear map $i \colon K^X \hookrightarrow W$ one has $\mathrm{I}_{\leq d}(X) = \ker(i \circ \mathrm{ev}^X_{\leq d})$. In particular, $\mathrm{Z}(\ker(i \circ \mathrm{ev}^X_{\leq d})) = X$ for all large d. The following diagram illustrates the situation.*

$$\begin{tikzcd}
S_{\leq d} \arrow[r, "\mathrm{ev}^X_{\leq d}"] \arrow[dr, "i \circ \mathrm{ev}^X_{\leq d}"'] & K^X \arrow[d, hook, "i"] \\
 & W
\end{tikzcd}$$

Proof: The first statement clearly holds. Together with Lemma 1.14 this implies the second one. q. e. d.

The following result on polynomial interpolation is well-known and is, in slightly generalized form, also part of von der Ohe [68, Lemma 2.12].

Lemma 1.17: *Let $X \subseteq K^n$ be finite. If $d \in \mathbb{N}$ and $d \geq |X| - 1$ then $\mathrm{ev}^X_{\leq d}$ is surjective.*

Proof: It is easy to see that given $x \in X$, there is a polynomial $p \in S$ of degree $|X| - 1$ such that $p(x) = 1$ and $p(y) = 0$ for $y \in X \setminus \{x\}$ (see, e. g., the proof of Cox, Little, and O'Shea [11, Chapter 5, § 3, Proposition 7]). By linearity this concludes the proof. q. e. d.

As usual, we call a sequence $(S_d)_{d \in \mathbb{N}}$ of K-subvector spaces of $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ a *filtration on S* if $S_d \subseteq S_{d+1}$ for all $d \in \mathbb{N}$. A filtration $(S_d)_{d \in \mathbb{N}}$ on S is called *exhaustive* if $\bigcup_{d \in \mathbb{N}} S_d = S$.

We obtain the following characterization of Prony structures.

THEOREM 1.18: *Let $X \subseteq K^n$ be an algebraic set. Let $(W_d)_{d\in\mathbb{N}}$ be a sequence of K-vector spaces and $(S_d)_{d\in\mathbb{N}}$ be an exhaustive filtration on S. Let $\varphi \in \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(S_d, W_d)$. Then the following are equivalent:*

(i) *φ is a Prony structure for X;*

(ii) *For all large d there is an injective K-linear map $\eta_d \colon K^X \hookrightarrow W_d$ such that the diagram*

$$\begin{tikzcd} S_d \arrow[r, "\varphi_d"] \arrow[dr, "\mathrm{ev}^X_{S_d}"'] & W_d \\ & K^X \arrow[u, hook, dashed, "\eta_d"'] \end{tikzcd}$$

is commutative;

(iii) *For all large d one has $\ker \varphi_d = \mathrm{I}_{S_d}(X)$.*

Proof: (i) $\Rightarrow$ (ii): By Definition 1.4 and since φ is a Prony structure for X, for all large d we have

$$\ker \mathrm{ev}^X_{S_d} = \mathrm{I}_{S_d}(X) \subseteq \ker \varphi_d.$$

Since $(S_d)_{d\in\mathbb{N}}$ is an exhaustive filtration on S, we have $S_{\leq |X|} \subseteq S_d$ for all large d. Then $\mathrm{ev}^X_{S_d}$ is surjective by Lemma 1.17. Together, these facts imply the existence of K-linear maps η_d such that the required diagrams are commutative.

It remains to show that η_d is injective for all large d. Let $c \in \mathbb{N}$ be such that for all $d \geq c$ we have that

$$\mathrm{Z}(\ker \varphi_d) = X, \quad \mathrm{ev}^X_{S_d} \text{ is surjective,} \quad \text{and } \eta_d \text{ exists.}$$

Let $v \in \ker \eta_d$. By the surjectivity of $\mathrm{ev}^X_{S_d}$ we have $\mathrm{ev}^X_{S_d}(p) = v$ for some $p \in S_d$. Then $\varphi_d(p) = \eta_d(\mathrm{ev}^X_{S_d}(p)) = \eta_d(v) = 0$. Thus, and since X is Zariski closed, we have

$$p \in \ker \varphi_d \subseteq \mathrm{I}(\mathrm{Z}(\ker \varphi_d)) = \mathrm{I}(X) = \ker \mathrm{ev}^X.$$

Hence, $v = \mathrm{ev}^X(p) = 0$. Thus, η_d is injective.

(ii) $\Rightarrow$ (iii): Since η_d exists and is injective (for all large d), we have

$$\ker \varphi_d = \ker(\eta_d \circ \mathrm{ev}^X_{S_d}) = \ker \mathrm{ev}^X_{S_d} = \mathrm{I}_{S_d}(X).$$

(iii) $\Rightarrow$ (i): Since $(S_d)_{d\in\mathbb{N}}$ is exhaustive, by Lemma 1.14 for all large d we have

$$\mathrm{Z}(\ker \varphi_d) = \mathrm{Z}(\mathrm{I}_{S_d}(X)) = \mathrm{Z}(\mathrm{I}(X)) = X.$$

Condition (P_2) in Definition 1.5 (a) is obviously satisfied. q. e. d.

For the decomposition of functions one applies Theorem 1.18 to the zero-dimensional set $X := \operatorname{supp}_u f$ for an $f \in V$. Theorem 1.18 intuitively states that to construct a "computable" Prony structure for a given f is to find K-linear embeddings $\eta_d\colon K^{\operatorname{supp}_u f} \hookrightarrow W_d$ into K-vector spaces W_d such that (matrices of) the compositions

$$\Phi_d(f) := \eta_d \circ \operatorname{ev}_{S_d}^{\operatorname{supp}_u f}\colon \quad S_d \longrightarrow W_d$$

can be computed from standard data of f.

In that setting, the following variation of Theorem 1.18 for generating subsets is useful when it is not a priori clear that B is linearly independent. Then the existence of a structure that satisfies variants of the conditions in Theorem 1.18 implies that B is a basis of V and that Φ is a Prony structure.

Corollary 1.19: *Let B be a generating subset of the F-vector space V and $u\colon B \to K^n$ be injective. Let $(S_d)_{d\in\mathbb{N}}$ and $(W_d)_{d\in\mathbb{N}}$ be as in Theorem 1.18 and*

$$\Phi\colon \quad V \longrightarrow \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(S_d, W_d).$$

Then the following are equivalent:

(i) *B is a basis of V and Φ is a Prony structure on (V, B, u).*

(ii) *For all $f \in V$, if $M \subseteq B$ is a finite subset and $f_b \in F \setminus \{0\}$ with $f = \sum_{b\in M} f_b b$, then for all large d there is an injective K-linear map $\eta_d\colon K^{u[M]} \hookrightarrow W_d$ such that the diagram*

$$\begin{tikzcd} S_d \arrow[r, "\Phi_d(f)"] \arrow[dr, "\operatorname{ev}_{S_d}^{u[M]}"'] & W_d \\ & K^{u[M]} \arrow[u, hook, dashed, "\eta_d"'] \end{tikzcd}$$

is commutative;

(iii) *For all $f \in V$, if $M \subseteq B$ is a finite subset and $f_b \in F \setminus \{0\}$ with $f = \sum_{b\in M} f_b b$, then for all large d one has $\ker \Phi_d(f) = \mathrm{I}_{S_d}(u[M])$.*

Proof: (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): This is part of Theorem 1.18.

(ii) $\Rightarrow$ (iii): As in Theorem 1.18.

(iii) $\Rightarrow$ (i): By Theorem 1.18 it is sufficient to show that B is linearly independent. Let $b_1, \ldots, b_r \in B$ be pairwise distinct and $f_1, \ldots, f_r \in F$ such that $f_1 b_1 + \cdots + f_r b_r = 0$. Suppose that $f_1 \neq 0$. By hypothesis, for all large d we have $\ker \Phi_d(f_1 b_1 + \cdots + f_r b_r) = \ker \Phi_d(0) = \mathrm{I}_{S_d}(\emptyset) = S_d$. On the other hand, since $f_1 \neq 0$, $\ker \Phi_d(f_1 b_1 + \cdots + f_r b_r) \subseteq \mathrm{I}_{S_d}(\{u(b_1)\}) \subsetneq S_d$ for all large d, a contradiction. Thus $f_1 = 0$ and B is linearly independent. q. e. d.

Theorem 1.18 and Corollary 1.19 establish a close link between Prony structures and evaluation maps. Therefore carefully studying these maps can reveal joint properties of

Prony structures. In the rest of the section we study ideal-theoretic issues related to the questions raised in Remark 1.13 (b).

In the following we do not distinguish between $\alpha \in \mathbb{N}^n$ and the monomial $\mathrm{x}^\alpha \in \mathrm{Mon}(S)$. For general facts about initial ideals and Gröbner bases see, e. g., the textbooks of Cox, Little, and O'Shea [11] or Kreuzer and Robbiano [34, 35].

Remark 1.20: Let $X \subseteq K^n$ be finite. A direct consequence of Proposition 1.15 (a) is that for all $d \geq |X|$ the vanishing spaces $\mathrm{I}_{\leq d}(X)$ generate the same radical ideal in S (namely, $\mathrm{I}(X)$).

We record an immediate consequence for Prony structures in the following corollary.

Corollary 1.21: *Given the setup of Definition 1.7, let $(S_d)_{d\in\mathbb{N}}$ be an exhaustive filtration on S and φ be a Prony structure for $f \in V$. Then for all large d*

$$\langle \ker \varphi_d \rangle = \mathrm{I}(\mathrm{supp}_u f).$$

In particular, for all large d, $\langle \ker \varphi_d \rangle$ is a radical ideal in S.

Proof: Let $X := \mathrm{supp}_u f$ and $r := \mathrm{rank}_B f = |X|$. By hypothesis there is a $d \in \mathbb{N}$ with $S_{\leq r} \subseteq S_d$, hence by Theorem 1.18 we have $\ker \varphi_d = \mathrm{I}_{S_d}(X) \supseteq \mathrm{I}_{\leq r}(X)$. Since also $S_d \subseteq S_{\leq e}$ for an $e \in \mathbb{N}$, we have

$$\mathrm{I}(X) = \langle \mathrm{I}_{\leq r}(X) \rangle \subseteq \langle \mathrm{I}_{S_d}(X) \rangle \subseteq \langle \mathrm{I}_{S_{\leq e}}(X) \rangle \subseteq \mathrm{I}(X).$$

This concludes the proof. q. e. d.

Observe that for $D \subseteq \mathbb{N}^n$, the ideal $\langle \mathrm{I}_D(X) \rangle$ is not a radical ideal in general. This is shown already by the example $n = 1$, $X = \{0\}$, $D = \{\mathrm{x}_1^2\}$, where $\langle \mathrm{I}_D(X) \rangle = \langle \mathrm{x}_1^2 \rangle$.

On the other hand, in the total degree setting where $d = |X| - 1$ is sufficient for the evaluation map $\mathrm{ev}^X_{\leq d}$ to be surjective for a given $X \subseteq K^n$, $\mathrm{ev}^X_{\leq d}$ can be surjective also for $d < |X| - 1$. Furthermore, it is also possible that $\mathrm{I}_{\leq d}(X)$ generates a radical ideal for small d. The following simple example exhibits both of these behaviors.

Example 1.22: Let

$$X := \{(0, 0), (1, 0), (0, 1)\} \subseteq K^2.$$

One can see immediately that $\mathrm{ev}^X_{\leq 1}$ is bijective by considering its matrix

$$\mathrm{V}^X_{\leq 1} = (t(x))_{\substack{x\in X\\ t\in\mathcal{T}_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in K^{X\times\mathcal{T}_1}.$$

Therefore, $\mathrm{ev}^X_{\leq 1}$ is surjective and $\mathrm{I}_{\leq 1}(X) = \ker \mathrm{ev}^X_{\leq 1} = \{0\}$. So $\langle \mathrm{I}_{\leq 1}(X) \rangle$ is the zero ideal of S, which is prime and thus radical (and of course not equal to $\mathrm{I}(X)$).

Since $|X| = 3$, $\mathrm{ev}^X_{\leq 2}$ is surjective by Lemma 1.17. Hence $\dim_K \mathrm{I}_{\leq 2}(X) = |\mathcal{T}_2| - 3 = 3$ and it is easy to see that

$$\mathrm{I}_{\leq 2}(X) = \langle \mathrm{x}_1(\mathrm{x}_1 - 1), \mathrm{x}_2(\mathrm{x}_2 - 1), \mathrm{x}_1\mathrm{x}_2 \rangle_K.$$

In fact, a computation shows that the vanishing ideal $\mathrm{I}(X)$ of X is generated by $\mathrm{I}_{\leq 2}(X)$.

We consider special situations and prove results related to Corollary 1.21 and Example 1.22.

For a monomial order $<$ on $\operatorname{Mon}(S)$ and an ideal I of S we denote by

$$\mathrm{N}_<(I) := \operatorname{Mon}(S) \setminus \mathrm{in}_<(I)$$

the *normal set of* I. If there seems to be no danger of confusion, we omit the monomial order from the notation and write, e. g., $\mathrm{in}(I)$ and $\mathrm{N}(I)$ for $\mathrm{in}_<(I)$ and $\mathrm{N}_<(I)$, respectively.

For example, for $I = \mathrm{I}(X)$ with $X \subseteq K^2$ as in Example 1.22, one has

$$\mathrm{in}(I) = \langle \mathrm{x}_1^2, \mathrm{x}_1\mathrm{x}_2, \mathrm{x}_2^2 \rangle \text{ and thus } \mathrm{N}(I) = \{1, \mathrm{x}_1, \mathrm{x}_2\}$$

for the degree reverse lexicographic order $<$.

LEMMA 1.23: *Let* $<$ *be a monomial order on* $\operatorname{Mon}(S)$*,* $X \subseteq K^n$ *be finite and* $I := \mathrm{I}(X)$*. Then the following holds:*

(a) $\mathrm{ev}^X_{\mathrm{N}(I)} \colon S_{\mathrm{N}(I)} \to K^X$ *is bijective. In particular,* $|\mathrm{N}(I)| = |X|$*.*

(b) *Let* $D \subseteq \operatorname{Mon}(S)$ *be such that* $\mathrm{ev}^X_D \colon S_D \to K^X$ *is surjective. Then there is a* $C \subseteq \operatorname{Mon}(S)$ *with the following properties:*
   (1) $C \subseteq D$*.*
   (2) $\mathrm{ev}^X_C \colon S_C \to K^X$ *is bijective. In particular,* $|C| = |X| = |\mathrm{N}(I)|$*.*
   (3) *For all* $t \in D \setminus C$ *we have* $\mathrm{ev}^X_D(t) \in \langle \mathrm{ev}^X_C(s) \mid s \in C \text{ and } s < t\rangle_K$*.*

Proof: (a) It is a standard fact that $S_{\mathrm{N}(I)} \cong S/I \cong K^X$, see, for example, Cox, Little, and O'Shea [11, Chapter 5, § 3, Proposition 4]. Let $p \in \ker(\mathrm{ev}^X_{\mathrm{N}(I)})$ and suppose that $p \neq 0$. Then $\mathrm{in}\, p \in \mathrm{in}(I) \cap \mathrm{N}(I) = \emptyset$, a contradiction. Thus, $\mathrm{ev}^X_{\mathrm{N}(I)}$ is injective and hence an isomorphism.

(b) Note that necessarily $|D| \geq |X|$. We prove the assertion by induction on $k = |D| - |X| \in \mathbb{N}$. If $k = 0$, then $|D| = |X|$. So ev^X_D is bijective and $C = D$ works trivially.

Let $k \geq 1$. Then $|D| > |X|$ and the elements $\mathrm{ev}^X_D(t)$, $t \in D$, are linearly dependent in K^X. Hence there are $\lambda_t \in K$ with $\sum_{t \in D} \lambda_t \,\mathrm{ev}^X_D(t) = 0$ and $\lambda_t \neq 0$ for at least one $t \in D$. Let

$$t_0 := \max\nolimits_<\{t \in D \mid \lambda_t \neq 0\} \text{ and } D_1 := D \setminus \{t_0\}.$$

Clearly, $\mathrm{ev}^X_{D_1} \colon S_{D_1} \to K^X$ is surjective and $|D_1| - |X| = k - 1$. By induction hypothesis there is a $C_1 \subseteq D_1$ such that $\mathrm{ev}^X_{C_1} \colon S_{C_1} \to K^X$ is bijective and

$$\mathrm{ev}^X_{D_1}(t) \in \langle \mathrm{ev}^X_{C_1}(s) \mid s \in C_1,\, s < t\rangle_K \text{ for all } t \in D_1 \setminus C_1.$$

Clearly, $C_1 \subseteq D$. We claim that $C := C_1$ fulfills the assertion also for D. It remains to show statement (3) for $t = t_0$. For this let $U := \langle \operatorname{ev}_C^X(s) \mid s \in C,\, s < t_0 \rangle_K$. From the linear dependency above it follows that

$$\operatorname{ev}_D^X(t_0) = \sum_{s \in D_1} \mu_s \operatorname{ev}_{D_1}^X(s) = \sum_{s \in C} \mu_s \operatorname{ev}_C^X(s) + \sum_{s \in D_1 \setminus C} \mu_s \operatorname{ev}_{D_1}^X(s) \text{ with } \mu_s \in K.$$

Trivially $\sum_{s\in C} \mu_s \operatorname{ev}_C^X(s) \in U$ since by the choice of t_0 we have $s < t_0$ for all $s \in D$ with $\mu_s \neq 0$. Also by the choice of t_0 and the induction hypothesis mentioned above we have $\sum_{s \in D_1 \setminus C} \mu_s \operatorname{ev}_{D_1}^X(s) \in U$. Thus we have $\operatorname{ev}_D^X(t_0) \in U$. This concludes the proof. q. e. d.

Remark 1.24: Let the notation be as in Lemma 1.23 (b) and ev_D^X surjective. There are the following interesting questions:

(Q_1) Under which conditions do we have $\mathrm{N}(I) \subseteq D$?

(Q_2) Under which conditions does $C = \mathrm{N}(I)$ satisfy (1), (2), and (3) in Lemma 1.23 (b)?

Of course, $C = \mathrm{N}(I)$ implies that $\mathrm{N}(I) \subseteq D$. A simple example that shows $\mathrm{N}(I) \subseteq D$ does not hold in general is given by $n = 1$, $X = \{1\} \subseteq K$, $D = \{\mathrm{x}_1\} \subseteq \operatorname{Mon}(S)$.

Definition 1.25: Let $<$ be a monomial order on $\operatorname{Mon}(S)$ and $D \subseteq \operatorname{Mon}(S)$ be an order ideal w. r. t. divisibility. We call D *distinguished* (*w. r. t.* $<$) if for all $t \in D$ and $s \in \operatorname{Mon}(S) \setminus D$ we have $t < s$.

For an arbitrary non-empty order ideal $D \subseteq \operatorname{Mon}(S)$ we define

$$\partial(D) := (\mathrm{x}_1 D \cup \cdots \cup \mathrm{x}_n D) \setminus D.$$

We also set

$$\partial(\emptyset) := \{1\}.$$

Usually, $\partial(D)$ is called the *border of* D.

Example 1.26: Our standard examples of distinguished order ideals and a counterexample are the following.

(a) Let $d \in \mathbb{N}$. Then

$$D := \mathcal{T}_d = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \cdots + \alpha_n \leq d\}$$

is a distinguished order ideal w. r. t. $<_{\text{degrevlex}}$ (or any other degree compatible monomial order).

(b) Choose any $w \in (\mathbb{N} \setminus \{0\})^n$, let

$$\deg_w(\mathrm{x}^\alpha) := \sum_{j=1}^n w_j \alpha_j$$

and define a monomial order $<_w$ by letting $\mathrm{x}^\alpha <_w \mathrm{x}^\beta$ if $\deg_w(\mathrm{x}^\alpha) <_w \deg_w(\mathrm{x}^\beta)$ or $\deg_w(\mathrm{x}^\alpha) = \deg_w(\mathrm{x}^\beta)$ and $\mathrm{x}^\alpha <_{\mathrm{lex}} \mathrm{x}^\beta$. Let $d \in \mathbb{N}$ and

$$D := \{\alpha \in \mathbb{N}^n \mid \deg_w(\mathrm{x}^\alpha) \leq d\}.$$

Then D is a distinguished order ideal w. r. t. $<_w$.

(c) Clearly, for any $n \in \mathbb{N}$,

$$D := \mathcal{M}_d = \{\alpha \in \mathbb{N}^n \mid \max\{\alpha_1, \ldots, \alpha_n\} \leq d\}$$

is an order ideal. For $n \geq 2$ and $d \geq 1$, there is no monomial order $<$ on $\mathrm{Mon}(S)$ such that D is a distinguished order ideal w. r. t. $<$. Indeed, if $\mathrm{x}_2 > \mathrm{x}_1$, then $D \ni \mathrm{x}_2\mathrm{x}_1^d > \mathrm{x}_1\mathrm{x}_1^d = \mathrm{x}_1^{d+1} \notin D$.

Lemma 1.27: *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ be finite and $D \subseteq \mathrm{Mon}(S)$ be a distinguished order ideal w. r. t. $<$ such that ev_D^X is surjective. Let $I := \mathrm{I}(X)$ and $C \subseteq D$ be as in Lemma 1.23 (b). For $t \in \mathrm{Mon}(S)$ let $p_t \in S_C$ be the uniquely determined polynomial such that $\mathrm{ev}_C^X(p_t) = \mathrm{ev}^X(t)$ and set $q_t := t - p_t$. Then the following holds:*

(a) *For $t \in \mathrm{Mon}(S)$ we have $q_t \in I$.*

(b) *For $t \in \mathrm{Mon}(S) \setminus C$ we have $\operatorname{supp} p_t \subseteq \{s \in C \mid s < t\}$.*

(c) *For $t \in \mathrm{Mon}(S) \setminus C$ we have $\operatorname{in} q_t = t$.*

(d) *For $p \in I \setminus \{0\}$ we have $\operatorname{supp} p \not\subseteq C$, i. e. $p \notin S_C$.*

(e) *We have $C = \mathrm{N}(I)$.*

Here, $\operatorname{supp} p$ *denotes the support of p w. r. t. the monomial basis of S.*

Proof: (a) This is an immediate consequence of the definition, since $I = \ker(\mathrm{ev}^X)$.

(b) If $t \in D \setminus C$ then there are $\mu_s \in K$ such that

$$\mathrm{ev}_D^X(t) = \sum_{s \in C, s<t} \mu_s \,\mathrm{ev}_C^X(s) = \mathrm{ev}_C^X\Big(\sum_{s \in C, s<t} \mu_s s\Big).$$

Hence $p_t = \sum_{s \in C, s<t} \mu_s s$, and clearly $\operatorname{supp}(p_t) \subseteq \{s \in C \mid s < t\}$. If $t \in \mathrm{Mon}(S) \setminus D$ then $t > s$ for all $s \in D$ since D is a distinguished order ideal. In particular, we see also in this case that $\operatorname{supp}(p_t) \subseteq C = \{s \in C \mid s < t\}$, finishing the proof of the claim.

(c) This is an immediate consequence of part (b).

(d) Suppose that $\operatorname{supp}(p) \subseteq C$. Then $p \in \mathrm{I}_C(X) = \ker(\mathrm{ev}_C^X) = \{0\}$, a contradiction.

(e) If $t \in \mathrm{Mon}(S) \setminus C$ then $t = \mathrm{in}(q_t) \in \mathrm{in}(I)$ by part (c). Thus $\mathrm{N}(I) \subseteq C$ and since $|\mathrm{N}(I)| = |X| = |C|$, we have $\mathrm{N}(I) = C$. q. e. d.

Corollary 1.28: *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ be finite, $I := \mathrm{I}(X)$, and $D \subseteq \mathrm{Mon}(S)$ be a distinguished order ideal w. r. t. $<$. Then the following are equivalent:*

(i) ev_D^X *is surjective;*

(ii) $\mathrm{N}(I) \subseteq D$.

Proof: (i) $\Rightarrow$ (ii): Let $t \in \mathrm{N}(I)$ and let $C \subseteq D$ be as in Lemma 1.23 (b). Then we have $\mathrm{N}(I) = C$ by Lemma 1.27 (e) and thus $\mathrm{N}(I) \subseteq D$.

(ii) $\Rightarrow$ (i): By Lemma 1.23 (a), $\mathrm{ev}_{\mathrm{N}(I)}^X$ is bijective, and since $\mathrm{N}(I) \subseteq D$, ev_D^X is surjective. q. e. d.

The special case of the next theorem for a degree compatible monomial order $<$ and $D = \mathcal{T}_d$ can already be found in [68, Theorem 2.48].

Theorem 1.29 (Möller): *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ finite, and D a distinguished order ideal w. r. t. $<$ such that ev_D^X is surjective. Then there is a Gröbner basis G of $\mathrm{I}(X)$ such that*

$$G \subseteq S_{D \cup \partial(D)} \text{ and } |G| = |D| + |\partial(D)| - |X|.$$

Proof: Let $I := \mathrm{I}(X)$ and let $C = \mathrm{N}(I) \subseteq D$, $p_t \in S_C$, and $q_t = t - p_t$ be as in Lemma 1.27.

Define

$$G := \{q_s \mid s \in D \cup \partial(D) \setminus C\} \subseteq I.$$

We show that G is a Gröbner basis of I. Set $J := \langle \mathrm{in}(G) \rangle_S$. It suffices to show that $J = \mathrm{in}(I)$. It is clear that $J \subseteq \mathrm{in}(I)$. The reverse inclusion is certainly true if $X = \emptyset$, since then

$$I = \langle 1 \rangle = \mathrm{in}(I),\ C = \emptyset,\ 1 \in D \cup \partial(D), \text{ and } 1 = \mathrm{in}(q_1) \in \mathrm{in}(G).$$

Thus let w. l. o. g. $X \neq \emptyset$. Assume that $\mathrm{in}(I) \not\subseteq J$. Then there is a monomial $s \in \mathrm{in}(I) \setminus J$. Let t be a minimal monomial generator of $\mathrm{in}(I)$ with $t \mid s$. Since $t \in \mathrm{in}(I)$ we have $t \notin \mathrm{N}(I) = C$.

Case 1*:* $t \in D$. Then $q_t \in G$ and $t = \mathrm{in}(q_t) \in \mathrm{in}(G)$, hence $s \in \langle \mathrm{in}(G) \rangle = J$, a contradiction.

Case 2*:* $t \notin D$. Since $X \neq \emptyset$ we have $t \neq 1$, so there is a $j \in \{1, \ldots, n\}$ such that $\mathrm{x}_j \mid t$. Let $\widetilde{t} := t/\mathrm{x}_j$. Since t is a minimal generator of $\mathrm{in}(I)$, we have $\widetilde{t} \notin \mathrm{in}(I)$, so $\widetilde{t} \in C \subseteq D$. Hence, $t = \mathrm{x}_j \widetilde{t} \in (\mathrm{x}_j D) \setminus D \subseteq \partial(D) \subseteq \mathrm{in}(G)$. Thus we obtain that $s \in \langle \mathrm{in}(G) \rangle = J$, again a contradiction.

Thus we have $\mathrm{in}(I) \subseteq J$ and G is a Gröbner basis of I. By Lemma 1.27 it is clear that $|G| = |D \cup \partial(D) \setminus C| = |D| + |\partial(D)| - |X|$. Moreover, for $t \in D \cup \partial(D) \setminus C$ we have $\mathrm{supp}(q_t) = \{t\} \cup \mathrm{supp}(p_t) \subseteq \{t\} \cup \{s \in C \mid s < t\} \subseteq D \cup \partial(D)$, i. e., $q_t \in S_{D \cup \partial(D)}$, which concludes the proof. q. e. d.

Note that in Theorem 1.29, in general G contains a border prebasis induced by $\partial(D)$. In particular, if the distinguished order ideal D equals $\mathrm{N}(I)$, then G is a border basis of I. See, e. g., Kreuzer and Robbiano [35, Section 6.4] for further details related to the theory of border bases.

Example 1.30: Let $w \in (\mathbb{N} \setminus \{0\})^n$ and $\deg_w$ and the monomial order $<_w$ be defined as in Example 1.26 (b). Let $D := \{\alpha \in \mathbb{N}^n \mid \deg_w(\mathrm{x}^\alpha) \leq d\}$. Then for any $X \subseteq K^n$ such that ev_D^X is surjective, by Theorem 1.29 there is a Gröbner basis G (w. r. t. $<_w$) of $\mathrm{I}(X)$ of cardinality $|D| + |\partial(D)| - |X|$ contained in $S_{D\cup\partial(D)}$.

For a simple concrete example over any field K of characteristic $\neq 2$, let

$$\begin{aligned} w &:= (1,2), \\ X &:= \{(0,0),(1,0),(-1,0),(0,1)\} \subseteq K^2 \end{aligned}$$

and

$$D := \{1, \mathrm{x}, \mathrm{y}, \mathrm{x}^2\} = \{\mathrm{x}^\alpha \mid \deg_w(\mathrm{x}^\alpha) \leq 2\}.$$

Then $\mathrm{ev}_D^X \colon S_D \to K^X \cong K^4$ is surjective as can easily be seen from its matrix

$$\mathrm{V}_D^X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in K^{X\times D}.$$

Since by Example 1.26 (b), D is a distinguished order ideal w. r. t. $<_w$ for the weight $w = (1,2)$, Theorem 1.29 implies that there is a Gröbner basis G (w. r. t. $<_w$) of $\mathrm{I}(X)$ with $G \subseteq S_{D\cup\partial(D)}$. Note that

$$D \cup \partial(D) = \{1, \mathrm{x}, \mathrm{y}, \mathrm{x}^2, \mathrm{xy}, \mathrm{x}^3, \mathrm{x}^2\mathrm{y}, \mathrm{y}^2\} \subsetneqq \mathcal{T}_3,$$

so in particular, this statement cannot be obtained by specializing Theorem 1.29 to the total degree situation.

We list two immediate consequences of Theorem 1.29 in the following corollary. We state the special case of the total degree part (b) separately; it is also included in [68, Theorem 2.48].

Corollary 1.31: (a) *With notation and assumptions as in Theorem* 1.29*, the ideal generated by* $\mathrm{I}_{D\cup\partial(D)}(X)$ *is a radical ideal in* S*.*

(b) *If* $\mathrm{ev}_{\leq d}^X$ *is surjective, then the ideal generated by* $\mathrm{I}_{\leq d+1}(X)$ *is a radical ideal in* S*.*

We have the following implications for Prony structures.

Corollary 1.32: *Given the setup of Definition* 1.7*, let* φ *be a Prony structure for* $f \in V$ *w. r. t.* u*. Let* $d \in \mathbb{N}$ *be such that* $\ker \varphi_d = \mathrm{I}_{S_d}(\mathrm{supp}_u f)$*. If there is a distinguished order ideal* D *(w. r. t. some monomial order* $<$ *on* $\mathrm{Mon}(S)$*) such that* $\mathrm{ev}_D^{\mathrm{supp}_u f}$ *is surjective and* $D \cup \partial(D) \subseteq S_d$ *then*

$$\langle \ker \varphi_d \rangle = \mathrm{I}(X).$$

In particular, $\langle \ker \varphi_d \rangle$ *is a radical ideal in* S*.*

1.4. Maps between Prony structures

In this section we define structure preserving maps between Prony structures. This gives rise to a category of Prony structures.

We then show a principle to transfer given Prony structures to other vector spaces. In Chapter 2 we will see that many known constructions of Prony structures arise as applications of the transfer principle.

For reasons of brevity, whenever we say that Φ is a Prony structure, we mean that Φ is a Prony structure on (V, B, u) where V is an F-vector space with F-basis B and $u\colon B \to K^n$ is an injection. Similarly, when $\widetilde{\Phi}$ is a Prony structure, then this means that $\widetilde{\Phi}$ is a Prony structure on $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ for an $\widetilde{F}$-vector space $\widetilde{V}$ with $\widetilde{F}$-basis $\widetilde{B}$ and an injection $\widetilde{u}\colon \widetilde{B} \to (\widetilde{K})^{\widetilde{n}}$.

The following is a natural definition of structure preserving maps between Prony structures.

Definition 1.33: Let Φ and $\widetilde{\Phi}$ be Prony structures and let

- $\iota\colon F \to \widetilde{F}$ be a field homomorphism (turning $\widetilde{V}$ into an F-vector space),
- $\varphi\colon V \to \widetilde{V}$ be an F-vector space homomorphism, and
- $\mu\colon \Phi[V] \to \widetilde{\Phi}[\widetilde{V}]$ be a function, where $\Phi[V] = \{(\Phi_d(f))_{d\in\mathbb{N}} \mid f \in V\}$.

Then $\psi := (\iota, \varphi, \mu)$ is called *map of Prony structures from Φ to $\widetilde{\Phi}$*, abbreviated as *Prony map* in the following, and written $\psi\colon \Phi \to \widetilde{\Phi}$, if the inclusion

$$\varphi[B] \subseteq \bigcup_{\widetilde{b}\in\widetilde{B}} \langle \widetilde{b} \rangle$$

holds and the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & \Phi[V] \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\mu} \\ \widetilde{V} & \xrightarrow{\widetilde{\Phi}} & \widetilde{\Phi}[\widetilde{V}] \end{array}$$

Remark 1.34: (a) The definition of Prony map given here is slightly more general than the one in [39] where the inclusion "$\varphi[B] \subseteq \widetilde{B}$" is part of the definition. The general version follows the philosophy that Prony structures see only the support of a vector and beyond this are oblivious of the coefficients. This allows in particular to see the transfer principle below in terms of Prony maps.

(b) Under a similar name certain moment maps are considered in Batenkov and Yomdin [4]. These are not related to our Prony maps.

(c) One might expect a map between (subsets of) K^n and $(\widetilde{K})^{\widetilde{n}}$ (that is compatible with the other data) to play a role in the definition of Prony map. However, such a function is implicitly defined by a Prony map $\psi = (\iota, \varphi, \mu)\colon \Phi \to \widetilde{\Phi}$. Set $h\colon \bigcup_{\widetilde{b}\in\widetilde{B}} \langle \widetilde{b} \rangle \setminus \{0\} \to \widetilde{B}$, $\lambda\widetilde{b} \mapsto \widetilde{b}$. Since u is injective, there is a function

$$\varrho_\psi\colon \quad u[B] \longrightarrow \widetilde{u}[\widetilde{B}], \qquad \ell \longmapsto (\widetilde{u} \circ h \circ \varphi)(u^{-1}(\ell)),$$

that maps elements of $u[B] \subseteq K^n$ to elements of $\widetilde{u}[\widetilde{B}] \subseteq (\widetilde{K})^{\widetilde{n}}$. In other words, the following diagram is commutative.

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d, hook, two heads, "u"] & \bigcup_{\widetilde{b}\in\widetilde{B}} \langle \widetilde{b} \rangle \setminus \{0\} \arrow[r, "h"] & \widetilde{B} \arrow[d, hook, two heads, "\widetilde{u}"] \\
u[B] \arrow[rr, "\varrho_\psi"] & & \widetilde{u}[\widetilde{B}]
\end{tikzcd}$$

Clearly, ϱ_ψ is injective if and only if φ is injective.

Definition 1.35: Let $\mathcal{P} = (\mathcal{O}, \mathrm{Hom}, \mathrm{id}, \circ)$ be defined as follows.

- $\mathcal{O} := \{\Phi \mid \Phi \text{ Prony structure}\}$ is the class of all Prony structures.
- For $\Phi, \widetilde{\Phi} \in \mathcal{O}$,
$$\mathrm{Hom}(\Phi, \widetilde{\Phi}) := \{\psi \mid \psi\colon \Phi \to \widetilde{\Phi} \text{ Prony map}\}$$
is the set of all Prony maps from Φ to $\widetilde{\Phi}$.
- For $\Phi \in \mathcal{O}$, let
$$\mathrm{id}_\Phi := (\mathrm{id}_F, \mathrm{id}_V, \mathrm{id}_{\Phi[V]}).$$
- For $\Phi, \widetilde{\Phi}, \widetilde{\widetilde{\Phi}} \in \mathcal{O}$, $\psi = (\iota, \varphi, \mu) \in \mathrm{Hom}(\Phi, \widetilde{\Phi})$, and $\widetilde{\psi} = (\widetilde{\iota}, \widetilde{\varphi}, \widetilde{\mu}) \in \mathrm{Hom}(\widetilde{\Phi}, \widetilde{\widetilde{\Phi}})$, let
$$\widetilde{\psi} \circ \psi := ((\widetilde{\iota} \circ \iota), (\widetilde{\varphi} \circ \varphi), (\widetilde{\mu} \circ \mu)).$$

It is straightforward to show that $\mathcal{P}$ is a category (cf., e. g., the classic textbooks by Mac Lane [42] or Adámek, Herrlich, and Strecker [1] for the basic notions). We call $\mathcal{P}$ the *category of Prony structures.*

In the following we single out a basic but useful transfer principle for Prony structures that will be applied repeatedly in Chapter 2.

Lemma 1.36 (Transfer principle): *Let $V, \widetilde{V}$ be vector spaces over F resp. $\widetilde{F}$ with F-resp. $\widetilde{F}$-bases $B, \widetilde{B}$, respectively, and let $u\colon B \to K^n$ and $\widetilde{u}\colon \widetilde{B} \to K^n$ be injective. Let $\varphi\colon V \to \widetilde{V}$ (not necessarily linear) and for every $f \in V$ let*

$$\mathrm{supp}_u f = \mathrm{supp}_{\widetilde{u}}(\varphi(f)).$$

Then every Prony structure $\widetilde{\Phi}$ on $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ induces a Prony structure $\varphi^*(\widetilde{\Phi})$ on (V, B, u) with

$$\varphi^*(\widetilde{\Phi})_d(f) = \widetilde{\Phi}_d(\varphi(f))$$

for $f \in V$ and $d \in \mathbb{N}$. The following commutative diagram illustrates the situation.

$$\begin{tikzcd}
V \arrow[rr, "\varphi"] \arrow[drr, "\varphi^*(\widetilde{\Phi})"'] & & \widetilde{V} \arrow[d, "\widetilde{\Phi}"] \\
& & \prod_{d\in\mathbb{N}} \operatorname{Hom}_{\widetilde{K}}(\widetilde{S}_d, \widetilde{W}_d)
\end{tikzcd}$$

Moreover, if φ is linear, then $\psi = (\iota, \varphi, \mu) \in \operatorname{Hom}(\varphi^*(\widetilde{\Phi}), \widetilde{\Phi})$, where $\mu\colon \varphi^*(\widetilde{\Phi})[V] \to \widetilde{\Phi}[\widetilde{V}]$, $\varphi^*(\widetilde{\Phi})(f) \mapsto \widetilde{\Phi}(\varphi(f))$.

Proof: By the hypotheses, for $f \in V$ and all large d we have

$$\operatorname{supp}_u f = \operatorname{supp}_{\widetilde{u}}(\varphi(f)) = \mathrm{Z}(\ker \widetilde{\Phi}_d(\varphi(f))) = \mathrm{Z}(\ker \varphi^*(\widetilde{\Phi})_d(f))$$

and

$$\mathrm{I}_{\widetilde{S}_d}(\operatorname{supp}_u f) = \mathrm{I}_{\widetilde{S}_d}(\operatorname{supp}_{\widetilde{u}}(\varphi(f))) \subseteq \ker(\widetilde{\Phi}_d(\varphi(f))) = \ker(\varphi^*(\widetilde{\Phi})_d(f)).$$

This shows that Φ is a Prony structure on (V, B, u).

Let φ be F-linear. By the hypothesis, for $b \in B$ we have

$$\operatorname{supp}_{\widetilde{u}}(\varphi(b)) = \operatorname{supp}_u b = \{u(b)\},$$

hence $\operatorname{rank}_{\widetilde{B}}(\varphi(b)) = |\operatorname{supp}_{\widetilde{u}}(\varphi(b))| = 1$, and thus $\varphi(b) \in \langle \widetilde{b} \rangle \setminus \{0\}$ for some $\widetilde{b} \in \widetilde{B}$. The required diagram for μ is commutative by the definition of μ. This concludes the proof. q. e. d.

2. Applications of Prony structures

In this chapter we discuss several examples and applications of Prony structures.

In Section 2.1 we introduce a generalized notion of exponentials that contains the ordinary exponentials and also "Chebyshev exponentials". It admits a natural Prony structure that specializes to the Hankel, Toeplitz, and Hankel-plus-Toeplitz structures for sums of (Chebyshev) exponentials and allows to treat Prony structures for these types of functions in a unified way.

Corresponding Prony structures for sparse polynomial interpolation w. r. t. the monomial and the Chebyshev bases are obtained in Section 2.2 as an application of the results of Section 2.1 and the transfer principle from Chapter 1.

A reconstruction method for multivariate Gaußian sums was recently proposed by Peter, Plonka, and Schaback [47]. In Section 2.3 we recast this result in the context of Prony structures as an application of the transfer principle.

2.1. Prony structures for t-exponentials

Multivariate generalizations are among the main directions of recent research on Prony's method, see Potts and Tasche [53], Kunis, Peter, Römer, and von der Ohe [38], Sauer [57, 58], Diederichs and Iske [17, 18], Cuyt and Lee [14], Mourrain [45], and Hubert and Singer [27], among others. For the basis of exponentials, multivariate versions based on either Hankel-like or Toeplitz-like matrices have been studied previously, also in [38, 68]. Since for the Toeplitz case one needs evaluations also at negative arguments, one must either restrict the discussion to a class of exponentials which allows these evaluations, or one has to distinguish two different variants of exponentials. The first one has only non-negative arguments and no constraints on the bases in K^n. The second variant is defined also for negative (integer) arguments under the constraint that the bases lie on the algebraic torus $(K \setminus \{0\})^n$. Observe that it is not possible to define Toeplitz versions of Prony's method for the first variant. This is a technical difficulty that any unification of these methods must cope with.

Moreover, the framework of t-exponentials we introduce in this section also suggests Prony structures for exponential sums w. r. t. a vector space basis derived from Chebyshev polynomials. This will later be used to recast the Lakshman-Saunders method as a simple application of the transfer principle, cf. Section 2.2.2.

Unless stated otherwise, F will always be a field and K will be a field extension of F.

Definition 2.1: Let A be a set and $t = (t_\alpha)_{\alpha \in A} \in F[\mathrm{y}_1^{\pm 1}, \ldots, \mathrm{y}_n^{\pm 1}]^A$. We call $b \in K^n$ *t-admissible* if $t_\alpha(b) \in K$ is defined for all $\alpha \in A$. For a t-admissible $b \in K^n$ let

$$\operatorname{gxp}_b^t\colon \quad A \longrightarrow K, \qquad \alpha \longmapsto t_\alpha(b).$$

We call the functions gxp_b^t *t-exponentials*.

We call a subset $Y \subseteq K^n$ *t-admissible* if every $b \in Y$ is t-admissible. For a t-admissible subset $Y \subseteq K^n$ let

$$B_Y^t := \{\mathrm{gxp}_b^t \mid b \in Y\}$$

and

$$\mathrm{Gxp}_Y^t(F) := \langle B_Y \rangle_F.$$

We call the elements of $\mathrm{Gxp}_Y^t(F)$ *t-exponential sums* (*with bases in* Y).

Before giving examples, we state and prove the following simple lemma.

LEMMA 2.2: *Let A be a set and $t = (t_\alpha)_{\alpha \in A} \in F[\mathrm{y}_1^{\pm 1}, \ldots, \mathrm{y}_n^{\pm 1}]^A$ be such that for all $j = 1, \ldots, n$ we have that*

$$\mathrm{y}_j \in \langle t_\alpha \mid \alpha \in A \rangle_F \quad \textit{or} \quad 1/\mathrm{y}_j \in \langle t_\alpha \mid \alpha \in A \rangle_F.$$

Let $Y \subseteq K^n$ be t-admissible. Then

$$u_Y^t\colon \quad B_Y^t \longrightarrow K^n, \qquad \mathrm{gxp}_b^t \longmapsto b,$$

is well-defined and injective.

Proof: Let $b, c \in Y$ be such that $\mathrm{gxp}_b^t = \mathrm{gxp}_c^t$. We show that $b = c$. Let $j \in \{1, \ldots, n\}$.

Case 1*:* $\mathrm{y}_j \in \langle t_\alpha \mid \alpha \in A \rangle_F$. Then there are $p_\alpha \in F$, almost all zero, with $\mathrm{y}_j = \sum_{\alpha \in A} p_\alpha t_\alpha$. Hence we have

$$b_j = \sum_{\alpha \in A} p_\alpha t_\alpha(b) = \sum_{\alpha \in A} p_\alpha \, \mathrm{gxp}_b^t(\alpha) = \sum_{\alpha \in A} p_\alpha \, \mathrm{gxp}_c^t(\alpha) = \sum_{\alpha \in A} p_\alpha t_\alpha(c) = c_j.$$

Thus $b = c$.

Case 2*:* $1/\mathrm{y}_j \in \langle t_\alpha \mid \alpha \in A \rangle_F$. Analogously to Case 1 one shows $1/b_j = 1/c_j$.

Hence u_Y^t is a function. It is clear that u_Y^t is injective. q. e. d.

The following Example 2.3 provides the fundamental examples of t-exponentials. Part (a) generalizes the univariate case in Example 1.11 (a). See also [68, p. 12].

In part (b) we introduce the new notion of Chebyshev exponentials.

Part (c) is a variation of part (a) where all bases are restricted to lie on the algebraic torus $(K \setminus \{0\})^n$. See also [68, pp. 38f.]. This allows also for non-negative arguments, i. e., the exponentials are functions on the domain $\mathbb{Z}^n$. As a consequence it is possible to define not only sequences of Hankel-like but also of Toeplitz-like matrices associated to an exponential sum (with domain $\mathbb{Z}^n$).

EXAMPLE 2.3: (a) Let $A := \mathbb{N}^n$ and set $t_\alpha := \mathrm{y}^\alpha = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n}$ for $\alpha \in A$. Then every $b \in K^n$ is t-admissible for $t = (t_\alpha)_{\alpha \in A}$. For $b \in K^n$ we have

$$\mathrm{gxp}_b^t\colon \quad \mathbb{N}^n \longrightarrow K, \qquad \alpha \longmapsto b^\alpha,$$

i. e., gxp_b^t is the usual *n-variate exponential with base b*, which we denote also by $\exp_b$.

As in [68], for $Y \subseteq K^n$ we also write

$$\mathrm{Exp}_Y^n(F) := \mathrm{Gxp}_Y^t(F) = \langle B_Y^t \rangle_F$$

and call the elements of $\mathrm{Exp}_Y^n(F)$ (n-*variate*) *exponential sums* (*with domain* $\mathbb{N}^n$).

(b) Let $A := \mathbb{N}$ and F be a field of characteristic zero. For $i \in A$ let $\mathrm{T}_i \in \mathbb{Z}[\mathrm{y}] \leq F[\mathrm{y}]$ denote the *i-th Chebyshev polynomial*, defined inductively by

$$\mathrm{T}_0 := 1, \quad \mathrm{T}_1 := \mathrm{y}, \quad \text{and} \quad \mathrm{T}_i := 2\mathrm{y}\mathrm{T}_{i-1} - \mathrm{T}_{i-2} \text{ for } i \geq 2.$$

Let $t = (\mathrm{T}_i)_{i \in A}$ denote the sequence of Chebyshev polynomials,

Let K be a field extension of F. Every $b \in K$ is t-admissible. For $b \in K^n$ let

$$\mathrm{txp}_b := \mathrm{gxp}_b^t\colon \quad A \longrightarrow K, \qquad i \longmapsto \mathrm{T}_i(b).$$

We call txp_b *Chebyshev exponential with base b.*

For $Y \subseteq K^n$ we also write

$$\mathrm{Txp}_Y(F) := \mathrm{Gxp}_Y^t(F) = \langle B_Y^t \rangle_F$$

and call the elements of $\mathrm{Txp}_Y(F)$ *Chebyshev exponential sums.*

(c) Let $A := \mathbb{Z}^n$ and $t_\alpha := \mathrm{y}^\alpha = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n}$ for $\alpha \in A$. For $t = (t_\alpha)_{\alpha \in A}$, the set of all t-admissible elements is the algebraic torus $(K \setminus \{0\})^n$. For $b \in (K \setminus \{0\})^n$ we have

$$\mathrm{gxp}_b^t\colon \quad \mathbb{Z}^n \longrightarrow K, \qquad \alpha \longmapsto b^\alpha,$$

i. e., gxp_b^t is the usual *n-variate exponential with base b and domain $\mathbb{Z}^n$*, which we denote also by $\exp_{\mathbb{Z},b}$.

As in [68], for $Y \subseteq (K \setminus \{0\})^n$ we also write

$$\mathrm{Exp}_{\mathbb{Z},Y}^n(F) := \mathrm{Gxp}_Y^t(F) = \langle B_Y \rangle_F$$

and call the elements of $\mathrm{Exp}_{\mathbb{Z},Y}^n(F)$ (n-*variate*) *exponential sums* (*with domain* $\mathbb{Z}^n$).

Remark 2.4: Observe that considered merely as vector spaces, the spaces $\mathrm{Exp}_Y^1(F)$ and $\mathrm{Txp}_Y(F)$ are identical. However, here we consider them equipped with the bases of exponentials and Chebyshev exponentials, respectively, and provide the notation to keep track of this distinction.

For the rest of the section we assume that $(t_\alpha)_{\alpha \in A}$ is a family of linearly independent elements of $F[\mathrm{y}_1^{\pm 1}, \ldots, \mathrm{y}_n^{\pm 1}]$ such that $\mathrm{y}_j \in \langle t_\alpha \mid \alpha \in A \rangle_F$ for all $j = 1, \ldots, n$. In particular, $u_Y\colon B_Y \to K^n$ is a well-defined injection according to Lemma 2.2.

Definition 2.5: Let A be a subset of $\mathbb{Z}^n$. For $\alpha \in A$ let $t_\alpha \in F[\mathrm{y}_1^{\pm 1}, \dots, \mathrm{y}_n^{\pm 1}]$ and let $t_\alpha \in F[\mathrm{y}_1, \dots, \mathrm{y}_n]$ for $\alpha \in A \cap \mathbb{N}^n$. In this context, for $\mathcal{I} \subseteq A \cap \mathbb{N}^n$ we write

$$S_\mathcal{I} = \langle t_\alpha \mid \alpha \in \mathcal{I} \rangle_F \leq F[\mathrm{y}_1, \dots, \mathrm{y}_n].$$

Let $Y \subseteq K^n$ be t-admissible for $t := (t_\alpha)_{\alpha \in A}$. For $b \in Y$ and $\Sigma = (\sigma_1, \dots, \sigma_n) \in A^n$ we write

$$\Sigma b := (t_{\sigma_1}(b), \dots, t_{\sigma_n}(b)) \in K^n$$

and for $M \subseteq Y$

$$M_\Sigma := \{\Sigma b \mid b \in M\} \subseteq K^n.$$

Let $\mathcal{I}, \mathcal{J} \subseteq A \cap \mathbb{N}^n$. For $f = \sum_{b \in M} f_b \operatorname{gxp}_b^t \in \operatorname{Gxp}_Y^t(F)$ with fixed finite $M \subseteq Y$ and $f_b \in F \setminus \{0\}$ we set

$$\Phi_\mathcal{I}^\mathcal{J}(f) := \left(\mathrm{ev}_\mathcal{I}^{M_\Sigma}\right)^\top \circ \varkappa_f \circ \mathrm{ev}_\mathcal{J}^M \colon \quad S_\mathcal{J} \longrightarrow (S_\mathcal{I})^*,$$

where $\varkappa_f \colon K^M \to (K^{M_\Sigma})^*$ denotes the K-linear map that maps u_b to $f_b \mathrm{u}_{\Sigma b}{}^*$ for $b \in M$.

Proposition 2.6: *In the notation of Definition 2.5, the matrix of $\Phi_\mathcal{I}^\mathcal{J}(f)$ w. r. t. the bases $\{t_\beta \mid \beta \in \mathcal{J}\}$ of $S_\mathcal{J}$ and $\{t_\alpha{}^* \mid \alpha \in \mathcal{I}\}$ of $(S_\mathcal{I})^*$ is*

$$\Big(\sum_{b \in M} f_b \operatorname{gxp}_b^t(\beta) \operatorname{gxp}_{\Sigma b}^t(\alpha)\Big)_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}}.$$

Proof: For $\beta \in \mathcal{J}$ we have

$$\begin{aligned}
\Phi_\mathcal{I}^\mathcal{J}(t_\beta) &= \left(\mathrm{ev}_\mathcal{I}^{M_\Sigma}\right)^\top \circ \varkappa_f \circ \mathrm{ev}_\mathcal{J}^M(t_\beta) \\
&= \left(\mathrm{ev}_\mathcal{I}^{M_\Sigma}\right)^\top \circ \varkappa_f \Big(\sum_{b \in M} t_\beta(b) \mathrm{u}_b\Big) \\
&= \sum_{b \in M} t_\beta(b) \cdot \left(\mathrm{ev}_\mathcal{I}^{M_\Sigma}\right)^\top \circ \varkappa_f(\mathrm{u}_b) \\
&= \sum_{b \in M} f_b t_\beta(b) \cdot \left(\mathrm{ev}_\mathcal{I}^{M_\Sigma}\right)^\top (\mathrm{u}_{\Sigma b}{}^*) \\
&= \sum_{b \in M} f_b t_\beta(b) \cdot (\mathrm{u}_{\Sigma b}{}^* \circ \mathrm{ev}_\mathcal{I}^{M_\Sigma}) \\
&= \sum_{b \in M} f_b \operatorname{gxp}_b^t(\beta) \cdot (\mathrm{u}_{\Sigma b}{}^* \circ \mathrm{ev}_\mathcal{I}^{M_\Sigma}).
\end{aligned}$$

It is also easy to see that for $b \in M$ we have

$$\mathrm{u}_{\Sigma b}{}^* \circ \mathrm{ev}_\mathcal{I}^{M_\Sigma} = \sum_{\alpha \in \mathcal{I}} \operatorname{gxp}_{\Sigma b}^t(\alpha) \cdot t_\alpha{}^*,$$

i. e., the coordinate vector of the linear form $\mathrm{u}_{\Sigma b}{}^* \circ \mathrm{ev}_\mathcal{I}^{M_\Sigma} \in (S_\mathcal{I})^*$ w. r. t. $\{t_\alpha{}^* \mid \alpha \in \mathcal{I}\}$ is $(\operatorname{gxp}_{\Sigma b}^t(\alpha))_{\alpha \in \mathcal{I}}$. The claimed formula follows. q. e. d.

REMARK 2.7: Proposition 2.6 allows to compute $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ under the assumption that one has an expression of the products

$$\mathrm{gxp}_b^t(\alpha) \cdot \mathrm{gxp}_{\Sigma b}^t(\beta)$$

for $b \in Y$, $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$, as a linear combination of evaluations of gxp_b^t. For applications see the following example.

EXAMPLE 2.8: (a) As in Example 2.3 (a), let $A := \mathbb{N}^n$ and $t_\alpha = \mathrm{y}^\alpha = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n}$ for $\alpha \in A$. Then for $b \in K^n$ and $\alpha, \beta \in A$ we have

$$\exp_b(\alpha) \cdot \exp_b(\beta) = \exp_b(\alpha + \beta),$$

hence by Proposition 2.6 with $\sigma_j = \mathrm{u}_j \in \mathbb{N}^n$ for $j = 1, \ldots, n$, the matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ is the Hankel-like matrix

$$\mathrm{H}_{\mathcal{I}}^{\mathcal{J}}(f) := (f(\beta + \alpha))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}}.$$

(b) As in Example 2.3 (b), let $\operatorname{char} F = 0$, $A := \mathbb{N}$, and $t_i = \mathrm{T}_i$ for $i \in A$. It is well-known that for $i, j \in A$ one has

$$t_i \cdot t_j = \frac{1}{2}(t_{i+j} + t_{|i-j|})$$

and thus for $b \in K^n$

$$\mathrm{txp}_b(i) \cdot \mathrm{txp}_b(j) = \frac{1}{2}\Big(\mathrm{txp}_b(j+i) + \mathrm{txp}_b(|j-i|)\Big).$$

Hence by Proposition 2.6, the matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ is

$$P_{\mathcal{I}}^{\mathcal{J}}(f) = \frac{1}{2}\Big(f(j+i) + f(|j-i|)\Big)_{\substack{i \in \mathcal{I} \\ j \in \mathcal{J}}}.$$

Of course, the map $2 \cdot \Phi_{\mathcal{I}}^{\mathcal{J}}$ has a Prony kernel if and only if $\Phi_{\mathcal{I}}^{\mathcal{J}}$ does so. Thus for our purposes the factor $1/2$ in the above formula can safely be removed.

(c) As in Example 2.3 (c), let $A := \mathbb{Z}^n$ and $t_\alpha = \mathrm{y}^\alpha = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n}$ for $\alpha \in A$. As in part (a), for $b \in (K \setminus \{0\})^n$ and $\alpha, \beta \in A$ we have

$$\exp_b(\alpha) \cdot \exp_b(\beta) = \exp_b(\alpha + \beta).$$

Hence by Proposition 2.6 with $\sigma_j = -\mathrm{u}_j \in \mathbb{Z}^n$ for $j = 1, \ldots, n$, the matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ is the Toeplitz-like matrix

$$\mathrm{T}_{\mathcal{I}}^{\mathcal{J}}(f) := (f(\beta - \alpha))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}}.$$

We summarize the resulting Prony structures in the following corollary.

Corollary 2.9 (Prony structures for exponential sums): *Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$ such that $(S_{\mathcal{J}_d})_{d\in\mathbb{N}}$ and $(S_{\mathcal{I}_d})_{d\in\mathbb{N}}$ are exhaustive filtrations on S. Then the following holds, with $Y \subseteq K^n$ in part* (a), *$Y \subseteq (K \setminus \{0\})^n$ in parts* (b) *and* (c), *and $Y \subseteq K$ in part* (d)*:*

(a) (*See* [68, Corollary 2.19].) *The map*

$$\mathrm{H}\colon \quad \mathrm{Exp}_Y^n(F) \longrightarrow \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d\times\mathcal{J}_d}, \qquad f \longmapsto (\mathrm{H}_d(f))_{d\in\mathbb{N}} = \Big((f(\beta+\alpha))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \Big)_{d\in\mathbb{N}},$$

induces a Prony structure on $(\mathrm{Exp}_Y^n(F), B_Y, u_Y)$.

(b) (*See* [68, Corollary 2.35].) *The map*

$$\mathrm{T}\colon \quad \mathrm{Exp}_{\mathbb{Z},Y}^n(F) \longrightarrow \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d\times\mathcal{J}_d}, \qquad f \longmapsto (\mathrm{T}_d(f))_{d\in\mathbb{N}} = \Big((f(\beta-\alpha))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \Big)_{d\in\mathbb{N}},$$

induces a Prony structure on $(\mathrm{Exp}_{\mathbb{Z},Y}^n(F), B_Y, u_Y)$.

(c) *The map*

$$\mathrm{H}\colon \quad \mathrm{Exp}_{\mathbb{Z},Y}^n(F) \longrightarrow \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d\times\mathcal{J}_d}, \qquad f \longmapsto (\mathrm{H}_d(f))_{d\in\mathbb{N}} = \Big((f(\beta+\alpha))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \Big)_{d\in\mathbb{N}},$$

induces a Prony structure on $(\mathrm{Exp}_{\mathbb{Z},Y}^n(F), B_Y, u_Y)$.

(d) *Let $n = 1$ and* $\operatorname{char} F = 0$. *The map*

$$\Phi_{\mathcal{I}}^{\mathcal{J}}\colon \quad \mathrm{Exp}_Y^1(F) \longrightarrow \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d\times\mathcal{J}_d}, \qquad f \longmapsto \Big(\big(f(j+i) + f(|j-i|) \big)_{\substack{i\in\mathcal{I}_d\\ j\in\mathcal{J}_d}} \Big)_{d\in\mathbb{N}},$$

induces a Prony structure on $(\mathrm{Exp}_Y^n(F), B_Y, u_Y)$.

Proof: These statements follow immediately from the preceding discussion. q. e. d.

Corollary 2.9 implies in particular that the set B_Y is a basis of $\mathrm{Exp}_Y^n(F)$ (and analogously for $\mathrm{Exp}_{\mathbb{Z},Y}^n(F)$), a fact that had been noted explicitly in von der Ohe [68, Corollary 2.22] and more generally in Mourrain [45, Lemma 2.4]; it also follows from the Dedekind lemma (see also Section 3.1).

Remark 3.3 shows that the linear independence of the Chebyshev exponential sums (which holds by part (d)) does not simply follow from the Dedekind lemma. In particular, in all cases the notation $\operatorname{supp}_{u_Y} f$ is justified.

Multivariate versions of Corollary 2.9 (d) are not easily obtained, because of the appearance of "mixed terms" that seemingly cannot be expressed as evaluations at a single point in A. We refer to Hubert and Singer's preprint [27] for recent developments in the multivariate case.

Remark 2.10: Comparing the Prony structures from Corollary 2.9, as previously mentioned, one advantage of the Hankel Prony structure H over the Toeplitz Prony structure T is that H works with exponential sums with arbitrary bases in the n-dimensional affine space while T requires the bases to lie on the algebraic torus. On the other hand, there are relevant results in this context that are known only for Toeplitz matrices; see, e. g., [38, Theorem 3.7] regarding condition numbers.

In the spirit of Díaz and Kaltofen [16] and Garg and Schost [23], we discuss one additional advantage of the Toeplitz variant regarding the number of used evaluations. Let K be a field extension of F. Let I be a set, $V \leq K^I$ be an F-vector space of functions $I \to K$ and B be a basis of V. Moreover, let $\varphi\colon K \to K$ be an F-automorphism of K such that for $b \in B$ we have $\varphi \circ b \in B$. Further, assume that a subset $I_0 \subseteq I$ is given together with a function $\psi\colon I \to I$ such that $\psi[I_0] \subseteq I_1 := I \setminus I_0$ and for every $f \in V$ the following diagram is commutative.

$$\begin{array}{ccc} I & \xrightarrow{\;f\;} & K \\ \downarrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\varphi} \\ I & \xrightarrow{\;f\;} & K \end{array}$$

(It is of course sufficient to check this diagram for every $f = b \in B$.) Thus, under these assumptions, one can replace the evaluations of f at $\alpha \in I_0$ by evaluations of φ at $f(\psi(\alpha))$. Since $\psi(\alpha) \notin I_0$, one does not need the evaluation of f at any element of I_0.

An application is the case $F = \mathbb{R}$, $K = \mathbb{C}$, and the space $V = \mathrm{Exp}^n_{\mathbb{Z},\mathbb{T}^n}(\mathbb{R})$ of exponential sums with real coefficients supported on the analytic torus

$$\mathbb{T}^n = \{z \in \mathbb{C}^n \mid |z_j| = 1 \text{ for } j = 1, \ldots, n\} \subseteq \mathbb{C}^n.$$

Take $\varphi\colon \mathbb{C} \to \mathbb{C}$ to be the complex conjugation and let $I = \mathbb{Z}^n$, $\psi\colon I \to I$, $\alpha \mapsto -\alpha$, with $I_0 = \{\alpha \in I \mid \alpha_1 < 0\}$. In this case, one can often (depending on $\mathcal{I}$ and $\mathcal{J}$) define the Toeplitz matrix $\mathrm{T}_{\mathcal{I},\mathcal{J},d}(f)$ using fewer evaluations than in the Hankel matrix $\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f)$.

Let $f \in \mathrm{Exp}^n_{\mathbb{Z},(K\setminus\{0\})^n}(F)$ be arbitrary. Then the number $s_{\mathrm{H},\mathcal{I},\mathcal{J},d}$ of evaluations needed to define the Hankel matrix $\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f)$ can be different from the number $s_{\mathrm{T},\mathcal{I},\mathcal{J},d}$ of evaluations needed to define $\mathrm{T}_{\mathcal{I},\mathcal{J},d}(f)$, depending on the choice of $\mathcal{I}$ and $\mathcal{J}$. In general one has

$$s_{\mathrm{H},\mathcal{I},\mathcal{J},d} = |\mathcal{I}_d + \mathcal{J}_d| \quad \text{and} \quad s_{\mathrm{T},\mathcal{I},\mathcal{J},d} = |\mathcal{J}_d - \mathcal{I}_d|.$$

Thus for example, in the bivariate case $n = 2$ one has

$$s_{\mathrm{H},\mathcal{M},\mathcal{M},d} = s_{\mathrm{T},\mathcal{M},\mathcal{M},d} \text{ for all } d$$

and

$$s_{\mathrm{H},\mathcal{T},\mathcal{T},2} = 15 \neq 19 = s_{\mathrm{T},\mathcal{T},\mathcal{T},2}.$$

A more detailed discussion of this fact can be found in Josz, Lasserre, and Mourrain [29, Section 2.3.2].

It would be interesting to compare Prony indices $\mathrm{ind}_{\mathrm{H}_{\mathcal{I},\mathcal{J}}}(f)$ and $\mathrm{ind}_{\mathrm{T}_{\mathcal{I},\mathcal{J}}}(f)$ of $f \in \mathrm{Exp}^n_{\mathbb{Z},Y}(F)$ for various choices of the involved parameters.

2.2. Sparse interpolation of polynomials

We give a unified exposition of sparse polynomial interpolation with regards to the monomial basis and the Chebyshev basis based on the notion of t-exponentials.

DEFINITION 2.11: For $j = 1, \dots, n$ let $0 \neq p_j, q_j \in F[\mathrm{y}_j^{\pm 1}] \leq F[\mathrm{y}_1^{\pm 1}, \dots, \mathrm{y}_n^{\pm 1}]$ and set $p := p_1 \cdots p_n$ and $q := q_1 \cdots q_n$. We call $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ *$*$-commuting* if

$$p(q_1, \dots, q_n) = q(p_1, \dots, p_n)$$

holds in $F(\mathrm{y}_1, \dots, \mathrm{y}_n)$. If it is clear which factorizations are meant, we simply call p and q *$*$-commuting* if $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ are $*$-commuting.

EXAMPLE 2.12: (a) If $p, q \in F[\mathrm{y}_1^{\pm 1}, \dots, \mathrm{y}_n^{\pm 1}]$ are "Laurent monomials" (i. e., y^α with $\alpha \in \mathbb{Z}^n$) then p, q are $*$-commuting. Indeed, if $p = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n}$ and $q = \mathrm{y}_1^{\beta_1} \cdots \mathrm{y}_n^{\beta_n}$ with $\alpha, \beta \in \mathbb{Z}^n$, one has

$$p(\mathrm{y}_1^{\beta_1}, \dots, \mathrm{y}_n^{\beta_n}) = \mathrm{y}_1^{\alpha_1 \beta_1} \cdots \mathrm{y}_n^{\alpha_n \beta_n} = q(\mathrm{y}_1^{\alpha_1}, \dots, \mathrm{y}_n^{\alpha_n}).$$

(b) It is well-known that the Chebyshev polynomials T_i, $i \in \mathbb{N}$, satisfy

$$\mathrm{T}_i(\mathrm{T}_j) = \mathrm{T}_j(\mathrm{T}_i)$$

for all $i, j \in \mathbb{N}$. Hence, T_i and T_j are $*$-commuting, and more generally, also $p = \mathrm{T}_{\alpha_1}(\mathrm{y}_1) \cdots \mathrm{T}_{\alpha_n}(\mathrm{y}_n)$ and $q = \mathrm{T}_{\beta_1}(\mathrm{y}_1) \cdots \mathrm{T}_{\beta_n}(\mathrm{y}_n)$ are $*$-commuting since

$$\begin{aligned} p(\mathrm{T}_{\beta_1}, \dots, \mathrm{T}_{\beta_n}) &= \mathrm{T}_{\alpha_1}(\mathrm{T}_{\beta_1}) \cdots \mathrm{T}_{\alpha_n}(\mathrm{T}_{\beta_n}) \\ &= \mathrm{T}_{\beta_1}(\mathrm{T}_{\alpha_1}) \cdots \mathrm{T}_{\beta_n}(\mathrm{T}_{\alpha_n}) = q(\mathrm{T}_{\alpha_1}, \dots, \mathrm{T}_{\alpha_n}). \end{aligned}$$

The following Proposition 2.13 shows that for suitable sequences t of polynomials, a Prony structure on the space of t-exponential sums induces a Prony structure on the space of polynomials w. r. t. t by the transfer principle. We apply Proposition 2.13 in Sections 2.2.1 and 2.2.2 to the monomial and Chebyshev bases.

PROPOSITION 2.13: *For $j = 1, \dots, n$ let A_j be a set, $t^{(j)} = (t_a^{(j)})_{a \in A_j} \in F[\mathrm{y}_j^{\pm 1}]^{A_j}$, $A := A_1 \times \cdots \times A_n$, and $t_\alpha := t_{\alpha_1}^{(1)} \cdots t_{\alpha_n}^{(n)} \in F[\mathrm{y}_1^{\pm 1}, \dots, \mathrm{y}_n^{\pm 1}]$ for $\alpha \in A$. Assume that the t_α, $\alpha \in A$, are linearly independent, let $B := \{t_\alpha \mid \alpha \in A\}$, $V := \langle B \rangle_F$, and assume that $\mathrm{y}_j \in V$ for all $j = 1, \dots, n$. Let $b \in K^n$ be such that b_j is $t^{(j)}$-admissible and such that*

$$u\colon \quad B \longrightarrow K^n, \qquad t_\alpha \longmapsto (t_{\alpha_1}(b_1), \dots, t_{\alpha_n}(b_n)),$$

is injective and $u[B]$ is t-admissible. Let $\widetilde{B} := \{\mathrm{gxp}_c^t \mid c \in u[B]\}$, $\widetilde{V} := \mathrm{Gxp}_{u[B]}^t(F)$, and $\widetilde{u}\colon \widetilde{B} \to K^n$ be according to Lemma 2.2.

For $p \in V$ let

$$f_p\colon \quad A \longrightarrow K, \qquad \alpha \longmapsto p(u(t_\alpha)).$$

Let t_α and t_β be $*$-commuting for all $\alpha, \beta \in A$. Then the following holds:

(a) *For all* $p \in V$ *we have* $f_p \in \widetilde{V}$.

(b) *For all* $p \in V$ *we have* $\operatorname{supp}_u p = \operatorname{supp}_{\widetilde{u}}(f_p)$.

Proof: (a) Let $p \in V$ and let $\operatorname{supp} p = \{\beta \in A \mid t_\beta \in \operatorname{supp}_B p\}$ and $p_\beta \in F$ with $p = \sum_{\beta \in \operatorname{supp} p} p_\beta t_\beta$. Then for $\alpha \in A$ we have

$$
\begin{aligned}
f_p(\alpha) = p(u(t_\alpha)) &= \sum_{\beta \in \operatorname{supp} p} p_\beta t_\beta(u(t_\alpha)) = \sum_{\beta \in \operatorname{supp} p} p_\beta t_\beta(t_{\alpha_1}(b), \ldots, t_{\alpha_n}(b)) \\
&= \sum_{\beta \in \operatorname{supp} p} p_\beta t_\alpha(t_{\beta_1}(b), \ldots, t_{\beta_n}(b)) = \sum_{\beta \in \operatorname{supp} p} p_\beta t_\alpha(u(t_\beta)) \\
&= \sum_{\beta \in \operatorname{supp} p} p_\beta \operatorname{gxp}^t_{u(t_\beta)}(\alpha).
\end{aligned}
$$

This shows that $f_p \in \widetilde{V}$.

(b) Since u is injective, the computation in the proof of part (a) shows that

$$\operatorname{supp}_{\widetilde{u}}(f_p) = \{u(t_\beta) \mid \beta \in \operatorname{supp} p\} = \{u(t) \mid t \in \operatorname{supp}_B p\} = \operatorname{supp}_u p.$$

This concludes the proof. q. e. d.

2.2.1. Sparse interpolation w. r. t. the monomial basis

The following Corollary 2.14 identifies a well-known sparse interpolation technique for polynomials w. r. t. the monomial basis (see, e. g., Ben-Or and Tiwari [5] or Mourrain [45, Section 5.4]) as a Prony structure. In particular, our framework allows a simultaneous treatment of the Hankel and Toeplitz cases.

Let $A := \mathbb{Z}^n$. For $\alpha \in A$ set

$$t_\alpha := \mathrm{y}^\alpha = \mathrm{y}_1^{\alpha_1} \cdots \mathrm{y}_n^{\alpha_n} \in F[\mathrm{y}_1^{\pm 1}, \ldots, \mathrm{y}_n^{\pm 1}].$$

As in Proposition 2.13 set $B := \{t_\alpha \mid \alpha \in A\}$ and $V := \langle B \rangle_F$. Choose a field extension K of F and let $b \in (K \setminus \{0\})^n$ be such that the function

$$u\colon \quad B \longrightarrow K^n, \qquad \mathrm{y}^\alpha \longmapsto (b_1^{\alpha_1}, \ldots, b_n^{\alpha_n}),$$

is injective.[1] Observe that then necessarily $u[B] \subseteq (K \setminus \{0\})^n$. Set $\widetilde{V} := \operatorname{Exp}^n_{\mathbb{Z}, u[B]}(F)$, $\widetilde{B} := \{\exp_c \mid c \in u[B]\}$, and $\widetilde{u}\colon \widetilde{B} \to K^n$, $\exp_c \mapsto c$. For $p \in V$ let

$$f_p\colon \quad A \longrightarrow K, \qquad \alpha \longmapsto p(u(t_\alpha)),$$

be as in Proposition 2.13. By Example 2.3 (c) we have $f_p \in \widetilde{V}$.

[1]For example, for $K = F = \mathbb{C}$, any $b \in \mathbb{C}^n$ such that $b_j \neq 0$ and b_j is not a root of unity for all $j = 1, \ldots, n$ works. Of course, K cannot be finite, for otherwise $u\colon B \to K^n$ cannot be injective. One may always choose $K := F(\mathrm{w})$ (with w an indeterminate over F) and $b := (\mathrm{w}, \ldots, \mathrm{w}) \in K^n$.

Let

$$\varphi\colon \quad V \longrightarrow \widetilde{V}, \qquad p \longmapsto f_p.$$

By Proposition 2.13 the premises of the transfer principle (Lemma 1.36) are satisfied. Hence, any Prony structure $\widetilde{\Phi}$ on the space of exponential sums $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ induces a Prony structure $\varphi^*(\widetilde{\Phi})$ on the space of polynomials (V, B, u). The following corollary describes the matrices of the Prony structures induced by the Hankel and Toeplitz Prony structures from Corollary 2.9.

Corollary 2.14 (Prony structures for monomial-sparse interpolation):
The following holds:

(a) *Let* $\widetilde{\mathrm{H}}$ *be the Prony structure on* $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ *from Corollary* 2.9 (c) *and* $\mathrm{H} := \varphi^*(\widetilde{\mathrm{H}})$. *Then for* $p \in V$ *and* $d \in \mathbb{N}$, *the matrix of* $\mathrm{H}_d(p)$ *is*

$$(f_p(\beta + \alpha))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}} = (p(b_1^{\beta_1 + \alpha_1}, \ldots, b_n^{\beta_n + \alpha_n}))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}}.$$

(b) *Let* $\widetilde{\mathrm{T}}$ *be the Prony structure on* $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ *from Corollary* 2.9 (b) *and* $\mathrm{T} := \varphi^*(\widetilde{\mathrm{T}})$. *Then for* $p \in V$ *and* $d \in \mathbb{N}$, *the matrix of* $\mathrm{T}_d(p)$ *is*

$$(f_p(\beta - \alpha))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}} = (p(b_1^{\beta_1 - \alpha_1}, \ldots, b_n^{\beta_n - \alpha_n}))_{\substack{\alpha \in \mathcal{I} \\ \beta \in \mathcal{J}}}.$$

Proof: The assertions follow immediately from Proposition 2.13, Lemma 1.36, and Corollary 2.9 (c) and (b), respectively. q. e. d.

Example 2.15: The reconstruction method for $p \in V = F[\mathrm{y}_1, \ldots, \mathrm{y}_n]$ from Corollary 2.14 is efficient if p has small rank, i. e., is a "sparse polynomial". To give an illustration, let $n = 2$, $b \in (K \setminus \{0\})^n$ be chosen appropriately and $p = \mathrm{y}^\beta - \mathrm{y}^\gamma \in V$ be a binomial. Then $\operatorname{rank}(f_p) = 2$, hence the polynomial p can be reconstructed, independently of its degree, from the $|\mathcal{T}_3| = \binom{n+3}{3} = \binom{5}{3} = 10$ evaluations used for the matrix $\mathrm{H}_{\mathcal{T}_1, \mathcal{T}_2}(f_p)$.

The number of evaluations of p can be further reduced if p is known to be of degree at most $d - 1$. In this case, $q := p(\mathrm{z}, \mathrm{z}^d, \ldots, \mathrm{z}^{d^{n-1}}) \in F[\mathrm{z}]$ is a binomial of degree at most $d^n - 1$ in *one* variable. The above binomial can thus be reconstructed from four evaluations.

2.2.2. Sparse interpolation w. r. t. the Chebyshev basis

Decomposing a polynomial $f \in \mathbb{Q}[\mathrm{y}]$ w. r. t. the Chebyshev basis B is in principle possible by first decomposing f in terms of the monomial basis (Corollary 2.14) and then computing the Chebyshev decomposition from that. However, the natural assumption of an upper bound on the rank of f w. r. t. B does not imply an upper bound on the rank of f w. r. t. the monomial basis, so that it may be impossible to check the premises of Corollary 2.14. Even if such a bound were given, efficiency would be a concern.

Lakshman and Saunders [40] proposed a sparse method to compute Chebyshev decompositions directly, which follows at this point effortlessly and as in the monomial case (Corollary 2.14) from the case of Chebyshev exponential sums (Corollary 2.9 (d)) and the transfer principle.

As observed by Lakshman and Saunders [40, p. 390], the crucial properties of the Chebyshev polynomials for their Prony structures are that for all $i, j \in \mathbb{N}$ one has the *linearization relation*

$$\mathrm{T}_i \cdot \mathrm{T}_j = \frac{1}{2}(\mathrm{T}_{i+j} + \mathrm{T}_{|i-j|})$$

and the *commutativity relation*

$$\mathrm{T}_i(\mathrm{T}_j) = \mathrm{T}_j(\mathrm{T}_i).$$

We already exploited the linearization relation to obtain Corollary 2.9 (d) and will not need it further. We use the commutativity relation through Proposition 2.13.

It is now straightforward to derive a well-known sparse interpolation technique for polynomials w. r. t. the Chebyshev basis (see, e. g., Lakshman and Saunders [40]) by transferring the Prony structure for Chebyshev exponential sums from Corollary 2.9 (d) to the space of polynomials using Lemma 1.36. To this end, let F be a field of characteristic zero and consider

$$V := F[\mathrm{y}]$$

as an F-vector space with the Chebyshev basis

$$B := \{\mathrm{T}_i \mid i \in \mathbb{N}\}.$$

Choose a field extension K of F and let $b \in K$ be such that the function

$$u\colon \quad B \longrightarrow K, \qquad \mathrm{T}_i \longmapsto \mathrm{T}_i(b),$$

is injective.[2]

Moreover, set $\widetilde{V} := \mathrm{Txp}_{u[B]}(F)$, $\widetilde{B} := \{\mathrm{txp}_c \mid c \in u[B]\}$, and $\widetilde{u}\colon \widetilde{B} \to K$, $\mathrm{txp}_c \mapsto c$.

For $p \in V$ let f_p be as in Proposition 2.13 and let

$$\varphi\colon \quad V \longrightarrow \widetilde{V}, \qquad p \longmapsto f_p.$$

By Proposition 2.13 the premises of the transfer principle (Lemma 1.36) are satisfied. Hence, any Prony structure $\widetilde{\Phi}$ on the space of Chebyshev exponential sums $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ induces a Prony structure $\varphi^*(\widetilde{\Phi})$ on the space of Chebyshev polynomials (V, B, u). The following corollary describes the matrices of the Prony structure $\varphi^*(\widetilde{\Phi})$ induced by the Hankel-plus-Toeplitz Prony structure $\widetilde{\Phi}$ from Corollary 2.9 (d).

Corollary 2.16 (Prony structures for Chebyshev-sparse interpolation):
Let $\widetilde{\Phi}$ be the Prony structure on $(\widetilde{V}, \widetilde{B}, \widetilde{u})$ from Corollary 2.9 (d) *and set $\Phi := \varphi^*(\widetilde{\Phi})$. Then for $p \in V$ and $d \in \mathbb{N}$ the matrix of $\Phi_d(p)$ is*

$$P_d(f) := (f_p(j+i) + f_p(|j-i|))_{\substack{i \in \mathcal{I}_d \\ j \in \mathcal{J}_d}} = (p(t_{j+i}(b)) + p(t_{|j-i|}))_{\substack{i \in \mathcal{I}_d \\ j \in \mathcal{J}_d}}.$$

[2] A choice that always works is $b \in \mathbb{Q} \subseteq F$ with $b > 1$.

Proof: This follows from Proposition 2.13, Lemma 1.36, and Corollary 2.9 (d). q. e. d.

REMARK 2.17: While versions of Corollary 2.9 hold for any basis of polynomials satisfying a linearization relation with fixed coefficients for products (cf. Remark 2.7 and see Corollary 4.15 for a variant in the relative setting of Chapter 4), it is in general not easily possible to obtain corresponding versions of Corollaries 2.14 and 2.16, i. e. sparse interpolation techniques, since bases satisfying commutativity relations are rather elusive and these conditions are not straightforward to replace. However, there are variants for other kinds of Chebyshev bases, see, e. g., Potts and Tasche [55] and Imamoglu, Kaltofen, and Yang [28].

Peter and Plonka show how to view Chebyshev polynomials of the first kind as eigenfunctions of a suitable endomorphism of the space W of continuous real-valued functions on the interval $[-1, 1]$, see [46, Remark 4.6]. Thus, also the "analytic" reconstruction technique for these functions given by Potts and Tasche [55] is recast in the framework for eigenfunction sums. It is however not clear how this might be translated into a purely algebraic version.

Multivariate variants for Chebyshev polynomials of first and second kind can be found in a very recent preprint of Hubert and Singer [27].

EXAMPLE 2.18: We give a toy example computation to illustrate Corollary 2.16. Let

$$f = \mathrm{y}^3 \in \mathbb{Q}[\mathrm{y}].$$

(The polynomial $f = 1/8 \cdot \mathrm{T}_3 + 1/4 \cdot \mathrm{T}_1$ has Chebyshev rank 2.) We choose $b := 2$. Then the matrix of the linear map $\Phi_2(f) = (\varphi^*(\widetilde{\Phi}))_2(f)$ w. r. t. the bases $\{\mathrm{T}_0, \mathrm{T}_1, \mathrm{T}_2\}$ of $F[\mathrm{x}]_{\leq 2}$ and $\{{\mathrm{T}_0}^*, {\mathrm{T}_1}^*\}$ of ${F[\mathrm{x}]_{\leq 1}}^*$ is

$$\begin{aligned} P_2(f) &= \begin{pmatrix} f(\mathrm{T}_0(b)) & f(\mathrm{T}_1(b)) & f(\mathrm{T}_2(b)) \\ f(\mathrm{T}_1(b)) & f(\mathrm{T}_2(b)) & f(\mathrm{T}_3(b)) \end{pmatrix} + \begin{pmatrix} f(\mathrm{T}_0(b)) & f(\mathrm{T}_1(b)) & f(\mathrm{T}_2(b)) \\ f(\mathrm{T}_1(b)) & f(\mathrm{T}_0(b)) & f(\mathrm{T}_1(b)) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 8 & 343 \\ 8 & 343 & 17576 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 343 \\ 8 & 1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 16 & 686 \\ 16 & 344 & 17584 \end{pmatrix} \sim \begin{pmatrix} 1 & 8 & 343 \\ 0 & 1 & 56 \end{pmatrix}. \end{aligned}$$

Thus, in these coordinates the kernel of $\Phi_2(f)$ is generated by $(105, -56, 1)^\top \in \mathbb{Q}^3$. Hence we have

$$\ker(\Phi_2(f)) = \langle \mathrm{T}_2 - 56\mathrm{T}_1 + 105\mathrm{T}_0 \rangle = \langle (\mathrm{x} - 2)(\mathrm{x} - 26) \rangle,$$

and since $\Phi = \varphi^*(\widetilde{\Phi})$ is a Prony structure, we recover the support of f as

$$\begin{aligned} \operatorname{supp}_B f &= u^{-1}[\operatorname{supp}_u f] = u^{-1}[\mathrm{Z}(\ker \Phi_2(f))] = u^{-1}[\{2, 26\}] = u^{-1}[\{\mathrm{T}_1(b), \mathrm{T}_3(b)\}] \\ &= \{\mathrm{T}_1, \mathrm{T}_3\}. \end{aligned}$$

If desired, the coefficients 1/4 and 1/8 can now easily be computed by solving a 2×2-system of linear equations.

Remark 2.19: Probabilistic results related to sparse interpolation in various bases are known in the literature under the name "early termination", see for example Kaltofen and Lee [30]. In the language of the present work, the quest there is to find probabilistic estimates of the Prony index $\operatorname{ind}_{\Phi} f$ of a polynomial f where the Prony structure Φ is given in similar ways as in Corollary 2.14 or Corollary 2.16. The general idea is to perform the interpolation method repeatedly on increasingly large intervals and estimate the probability of having computed the "true" interpolating polynomial in terms of the number of successive intervals with the same result and a bound for the degree of f. For more details and further refinements we refer to [30].

Early termination strategies can also be combined with sparse interpolation methods for rational functions. For details we refer to, e. g., Kaltofen and Yang [31] and Cuyt and Lee [13]. In a related direction, probabilistic methods tailored to sparse polynomial interpolation over finite fields can be found, e. g., in Arnold, Giesbrecht, and Roche [2].

It would be interesting to look for generalizations of these results in the framework of Prony structures. However, in full generality this is unlikely to be fruitful, since one has to be able to make additional assumptions like degree bounds for which the Prony structures are not well-adapted.

Another potential avenue for further research could be the investigation of the computational complexity of Prony structures w. r. t. an underlying model of computation, such as arithmetic circuits in polynomial identity testing. See Shpilka and Yehudayoff [61] and Saxena [59, 60] for recent surveys of this field.

Arnold and Kaltofen [3] propose an error-correcting interpolation method for Chebyshev-sparse polynomials with a bounded number of erroneous evaluations.

We leave the search for suitable settings for the future.

2.3. Gaußian sums

In this section we recast the main result in Peter, Plonka, and Schaback [47] concerning the reconstruction of multivariate Gaußian sums in the framework of Prony structures. We will see that it can be derived as an application of the transfer principle.

To this end, let $A \in \mathbb{R}^{n \times n}$ be a fixed positive definite symmetric matrix. We call the functions

$$\mathrm{g}_{A,t}\colon \quad \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad x \longmapsto \mathrm{e}^{-(x-t)^\top A(x-t)}, \quad t \in \mathbb{R}^n,$$

Gaußians. Let

$$B := \{\mathrm{g}_{A,t} \mid t \in \mathbb{R}^n\} \quad \text{and} \quad V := \langle B \rangle_{\mathbb{C}}.$$

The elements of V are called *Gaußian sums.*

For $t \in \mathbb{R}^n$ set

$$b_{A,t} := \mathrm{e}^{z} = (\mathrm{e}^{z_1}, \dots, \mathrm{e}^{z_n}) \in (\mathbb{R} \setminus \{0\})^n \quad \text{with} \quad z = 2t^\top A \in \mathbb{R}^{1 \times n}$$

and let

$$u\colon \quad B \longrightarrow \mathbb{R}^n, \qquad \mathrm{g}_{A,t} \longmapsto b_{A,t}.$$

Since A is positive definite, $\mathrm{g}_{A,t}$ obtains its unique maximum in t. This implies that u is well-defined. Also since A is positive definite, $b_{A,t} = b_{A,s}$ for $t, s \in \mathbb{R}^n$ implies that $t = s$, and thus u is injective. For the following theorem we set $\widetilde{V} := \mathrm{Exp}^n_{\mathbb{Z},u[B]}(\mathbb{C})$, $\widetilde{B} := \{\exp_b \mid b \in u[B]\}$, and $\widetilde{u}\colon \widetilde{B} \to K^n$, $\exp_b \mapsto b$.

THEOREM 2.20 (Prony structure for Gaußian sums): *For $g \in V$ let*

$$f_g\colon \quad \mathbb{Z}^n \longrightarrow \mathbb{C}, \qquad \alpha \longmapsto g(\alpha) \cdot \mathrm{e}^{\alpha^\top A \alpha}.$$

Then the following holds:

(a) *For all $g \in V$ we have $f_g \in \widetilde{V}$ and $\varphi\colon V \to \widetilde{V}$, $g \mapsto f_g$, is a $\mathbb{C}$-vector space isomorphism with $\varphi(\mathrm{g}_{A,t}) = \lambda_{A,t} \cdot \exp_{b_{A,t}}$ for some $\lambda_{A,t} \in \mathbb{R} \setminus \{0\}$. In particular, B is a basis of V.*

(b) *For all $g \in V$ we have* $\operatorname{supp}_{\widetilde{u}}(f_g) = \operatorname{supp}_u g$.

Hence, any Prony structure on $\widetilde{V}$ (in particular those in Corollary 2.9 (b) *and* (c)) *induces a Prony structure on V by the transfer principle* (*Lemma* 1.36).

Proof: (a) Note that for all $t \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}^n$ and with $\lambda_{A,t} := \mathrm{e}^{-t^\top A t} \in \mathbb{R} \setminus \{0\}$ we have

$$\begin{aligned} f_{\mathrm{g}_{A,t}}(\alpha) &= \mathrm{g}_{A,t}(\alpha) \cdot \mathrm{e}^{\alpha^\top A \alpha} = \mathrm{e}^{-(\alpha - t)^\top A (\alpha - t)} \cdot \mathrm{e}^{\alpha^\top A \alpha} = \mathrm{e}^{-t^\top A t} \cdot \mathrm{e}^{2t^\top A \alpha} \\ &= \lambda_{A,t} \cdot \exp_{b_{A,t}}(\alpha). \end{aligned}$$

By definition we have $b_{A,t} \in u[B]$, and hence $\varphi(\mathrm{g}_{A,t}) = f_{\mathrm{g}_{A,t}} \in \widetilde{V}$. Since clearly $f_{\lambda g + \mu h} = \lambda f_g + \mu f_h$ for all $\lambda, \mu \in \mathbb{C}$ and $g, h \in V$, we have that $\varphi[V] \subseteq \widetilde{V}$ and φ is $\mathbb{C}$-linear. Since $\widetilde{B}$ is a $\mathbb{C}$-basis of $\widetilde{V}$, there is a unique $\mathbb{C}$-linear map $\psi\colon \widetilde{V} \to V$ with $\psi(\exp_{b_{A,t}}) = 1/\lambda_{A,t} \cdot \mathrm{g}_{A,t}$ for all $t \in \mathbb{R}^n$. Then ψ is the inverse of φ and this concludes the proof of part (a).

(b) Let $g = \sum_{t \in F} \mu_t \, \mathrm{g}_{A,t}$ with finite $F \subseteq \mathbb{R}^n$ and $\mu_t \in \mathbb{C} \setminus \{0\}$. Using part (a) we obtain

$$\operatorname{supp}_{\widetilde{u}}(f_g) = \operatorname{supp}_{\widetilde{u}}\Big(\sum_{t \in F} \mu_t \lambda_{A,t} \exp_{b_{A,t}}\Big) = \{b_{A,t} \mid t \in F\} = \operatorname{supp}_u g,$$

i. e., the assertion. q. e. d.

An alternative approach to the reconstruction problem in Theorem 2.20 which is based on Fourier transforms and takes inexact evaluations into account, is proposed in Peter, Potts, and Tasche [48, Section 4].

3. Relations with other frameworks

In this chapter we discuss and compare the framework of Prony structures with previous ones from the literature, namely for character sums (Dress and Grabmeier [19]), for eigenfunction sums (Grigoriev, Karpinski, and Singer [24]), and for eigenvector sums (Peter and Plonka [46]).

3.1. Character sums

The following theorem recasts the Dress-Grabmeier framework [19] for sparse interpolation of character sums in terms of Prony structures. As before, K is a field and $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$.

Recall that a monoid homomorphism $\chi\colon (M, +) \to (K, \cdot)$ from a commutative monoid $(M, +)$ to the multiplicative monoid of K is called *character of M in K*. The Dedekind independence lemma states that the set of characters of M in K is linearly independent as subset of the K-vector space K^M of functions from M to K.

Theorem 3.1 (Prony structure for character sums): *Let $(M, +)$ be a finitely generated commutative monoid generated by the elements $a_1, \dots, a_n \in M$. Consider a set B of characters of M in K and let*

$$V := \langle B\rangle_K$$

be the K-subvector space of K^M generated by B. Let

$$u\colon \quad B \longrightarrow K^n, \qquad \chi \longmapsto (\chi(a_1), \dots, \chi(a_n)).$$

Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$ such that $(S_{\mathcal{J}_d})_{d\in\mathbb{N}}$ and $(S_{\mathcal{I}_d})_{d\in\mathbb{N}}$ are exhaustive filtrations on S. For $f \in V$ and $d \in \mathbb{N}$ set

$$P_d(f) := \Big(f\Big(\sum_{j=1}^{n}(\alpha_j + \beta_j)a_j\Big)\Big)_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d\times\mathcal{J}_d} \cong \mathrm{Hom}_K(S_{\mathcal{J}_d}, (S_{\mathcal{I}_d})^*).$$

Then u is injective and $P_d(f)$ induces a Prony structure on (V, B, u).

Proof: If $u(\chi_1) = u(\chi_2)$ for characters χ_i then $\chi_1(a_j) = \chi_2(a_j)$ for all $j = 1, \dots, n$. Since M is generated by $\{a_1, \dots, a_n\}$, this implies $\chi_1 = \chi_2$, and thus u is injective. For $f \in V$ write $f = \sum_{x\in\mathrm{supp}_u f} f_x\chi_x$ with $f_x \in K \setminus \{0\}$ and $\chi_x \in B$ with $u(\chi_x) = x$ for $x \in \mathrm{supp}_u f$. Let $\varkappa_f\colon K^{\mathrm{supp}_u f} \to (K^{\mathrm{supp}_u f})^*$ be the K-linear map with $\varkappa_f(\mathrm{u}_x) = f_x{\mathrm{u}_x}^*$.

Clearly, $\varkappa_f$ is an isomorphism. A computation on the corresponding matrices shows that one has the following commutative diagram.

$$\begin{tikzcd}
K^{\mathcal{J}_d} \arrow[rrrr, "P_d(f)"] \arrow[d, "\cong"] & & & & (K^{\mathcal{I}_d})^* \\
S_{\mathcal{J}_d} \arrow[r, "\mathrm{ev}_{\mathcal{J}_d}^{\mathrm{supp}_u f}"] & K^{\mathrm{supp}_u f} \arrow[r, "\varkappa_f, \cong"] & (K^{\mathrm{supp}_u f})^* \arrow[r, "(\mathrm{ev}_{\mathcal{I}_d}^{\mathrm{supp}_u f})^\top"] & (S_{\mathcal{I}_d})^* \arrow[u, "\cong"'] &
\end{tikzcd}$$

Thus P is a Prony structure on (V, B, u) by Lemma 1.17 and Theorem 1.18. q. e. d.

Remark 3.2: (a) Since for $b \in K^n$ one has $\exp_b \in \mathrm{Hom}((\mathbb{N}^n, +), (K, \cdot))$, the Dress-Grabmeier framework contains the Prony structures for exponential sums.

(b) Note that Dress and Grabmeier [19] allow arbitrary monoids whereas in Theorem 3.1 we allow only finitely generated ones. Roughly speaking, in applications to function spaces this corresponds to allowing only a fixed finite number n of variables. This is no restriction in any case we have in mind.

(c) While Theorem 3.1 is formulated in the safe knowledge that according to the Dedekind lemma B is linearly independent, a minor variation of the proof using Corollary 1.19 also proves the Dedekind lemma.

Remark 3.3: There is no operation $\oplus$ on $\mathbb{N}$ such that the Chebyshev exponentials txp_b, $b \in \mathbb{Q}$, are characters of $(\mathbb{N}, \oplus)$ in $\mathbb{Q}$. Indeed, otherwise one would have

$$\mathrm{T}_{1\oplus 1}(b) = \mathrm{txp}_b(1 \oplus 1) = \mathrm{txp}_b(1) \cdot \mathrm{txp}_b(1) = \mathrm{T}_1(b) \cdot \mathrm{T}_1(b) = b^2$$

for all $b \in \mathbb{Q}$. Hence, $\mathrm{T}_{1\oplus 1} = \mathrm{y}^2 \in \mathbb{Z}[\mathrm{y}]$. This is a contradiction, since the only Chebyshev polynomial of degree 2 is $\mathrm{T}_2 = 2\mathrm{y}^2 - 1 \neq \mathrm{y}^2$. The framework of Prony structures, being able to handle the basis of Chebyshev exponentials (Corollary 2.9 (d)) thus is strictly more general than the one for character sums.

3.2. Eigenvector and eigenfunction sums

Related decomposition frameworks have been proposed for related families of vector spaces by Grigoriev, Karpinski, and Singer [24] and by Peter and Plonka [46]. In this section, we discuss these frameworks from the point of view of Prony structures.

We start with a method that was given in the case of a single endomorphism $\varphi \in \mathrm{End}_K(W)$ in Peter and Plonka [46]. Roughly speaking, here the number of endomorphisms corresponds to the dimension of the affine space associated to a Prony structure. See also Mourrain [45] for related discussions in the multivariate case with several endomorphisms and the book of Plonka, Potts, Steidl, and Tasche [49, Section 10.4.2].

The framework proposed by Grigoriev, Karpinski, and Singer [24] then follows by specialization to the case of Δ being a point evaluation functional. We derive the Peter-Plonka framework directly from Theorem 3.1.

As usual, the point spectrum of an endomorphism $\varphi \in \operatorname{End}_K(W)$ of a K-vector space W is denoted by

$$\sigma_{\mathrm{p}}(\varphi) = \{\lambda \in K \mid \ker(\varphi - \lambda \operatorname{id}_W) \neq \{0\}\}$$

and for $\lambda \in \sigma_{\mathrm{p}}(\varphi)$ let

$$W_\lambda^\varphi = \ker(\varphi - \lambda \operatorname{id}_W)$$

be the eigenspace of φ w. r. t. λ. For pairwise commuting operators $\varphi_1, \dots, \varphi_n \in \operatorname{End}_K(W)$ and $\alpha \in \mathbb{N}^n$ we use the notation

$$\varphi^\alpha := \varphi_1^{\alpha_1} \circ \cdots \circ \varphi_n^{\alpha_n} \in \operatorname{End}_K(W).$$

Corollary 3.4 (Prony structure for eigenvector sums): *Let $\varphi_1, \dots, \varphi_n \in \operatorname{End}_K(W)$ be pairwise commuting operators and consider $\Lambda \subseteq \prod_{j=1}^n \sigma_{\mathrm{p}}(\varphi_j)$. Assume that for every $\lambda \in \Lambda$ we have $\bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \neq \{0\}$ and choose*

$$b_\lambda \in \bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \setminus \{0\}.$$

Let

$$B := \{b_\lambda \mid \lambda \in \Lambda\}, \quad V := \langle B \rangle_K, \quad \text{and} \quad u\colon B \to K^n,\ b_\lambda \mapsto \lambda.$$

Let $\Delta \in W^* = \operatorname{Hom}_K(W, K)$ be such that

$$V \cap \ker(\Delta) = \{0\}.$$

Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$ such that $(S_{\mathcal{J}_d})_{d\in\mathbb{N}}$ and $(S_{\mathcal{I}_d})_{d\in\mathbb{N}}$ are exhaustive filtrations on S. For $f \in V$ and $d \in \mathbb{N}$ set

$$P_d(f) := \big(\Delta(\varphi^{\alpha+\beta}(f))\big)_{\substack{\alpha\in\mathcal{I}_d \\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d \times \mathcal{J}_d} \cong \operatorname{Hom}_K(S_{\mathcal{J}_d}, (S_{\mathcal{I}_d})^*).$$

Then $P_d(f)$ induces a Prony structure on (V, B, u).

Proof: We apply Theorem 3.1 similarly as in Grigoriev, Karpinski, and Singer [24, pp. 78f.]. Let M denote the submonoid of $(\operatorname{End}_K(W), \circ)$ generated by $\varphi_1, \dots, \varphi_n$. For $\lambda \in \Lambda$ let

$$\chi_\lambda\colon \quad M \longrightarrow K, \qquad \varphi^\alpha \longmapsto \frac{\Delta(\varphi^\alpha(b_\lambda))}{\Delta(b_\lambda)}.$$

Clearly, χ_λ is well-defined. Since $\chi_\lambda(\varphi^\alpha) = \lambda^\alpha$ for every $\alpha \in \mathbb{N}^n$, χ_λ is a character of M in K. Thus, by Theorem 3.1, $Q_d(f) = (f(\varphi^{\alpha+\beta}))_{\alpha\in\mathcal{I}_d,\beta\in\mathcal{J}_d}$ induces a Prony structure on the vector space $X := \langle \chi_\lambda \mid \lambda \in \Lambda\rangle_K \leq K^M$ with respect to the injection $v\colon X \to K^n$, $\chi_\lambda \mapsto \lambda$. Since $\operatorname{supp}_u(b_\lambda) = \lambda = \operatorname{supp}_v(\chi_\lambda)$, the assertion follows. q. e. d.

Observe that there are interesting situations where the condition that the b_λ, $\lambda \in \Lambda$, can be chosen in the desired way is fulfilled. For example this is the case if W is a finite-dimensional $\mathbb{C}$-vector space see, e. g., Horn and Johnson [26, Lemma 1.3.19].

With a little more effort in a direct proof, one can avoid the commutativity assumption in Corollary 3.4 (but of course one still needs that $\bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \neq \{0\}$ for every $\lambda \in \Lambda$).

The case $n = 1$ identifies the method in [46] as a Prony structure.

Corollary 3.5 (Peter and Plonka [46, Theorem 2.1]): *Let $\varphi \in \mathrm{End}_K(W)$ and consider $\Lambda \subseteq \sigma_{\mathrm{p}}(\varphi)$. For $\lambda \in \Lambda$ choose*

$$b_\lambda \in W_\lambda^\varphi \setminus \{0\}.$$

Let

$$B := \{b_\lambda \mid \lambda \in \Lambda\}, \quad V := \langle B \rangle_K, \quad \text{and} \quad u\colon B \to K,\ b_\lambda \mapsto \lambda.$$

Let $\Delta \in W^*$ be such that

$$V \cap \ker(\Delta) = \{0\}.$$

For $f \in V$ and $d \in \mathbb{N}$ set

$$P_d(f) := (\Delta(\varphi^{\alpha+\beta}(f)))_{\substack{\alpha=0,\dots,d-1\\ \beta=0,\dots,d}} \in K^{d\times(d+1)} \cong \mathrm{Hom}_K(S_{\leq d}, (S_{\leq d-1})^*).$$

Then $P_d(f)$ induces a Prony structure on (V, B, u).

Proof: Take $n = 1$, $\mathcal{I}_d = \mathcal{T}_{d-1}$ and $\mathcal{J}_d = \mathcal{T}_d$ in Corollary 3.4. q. e. d.

Example 3.6: Several applications for various choices of the endomorphism φ and the functional Δ can be found in [46], for example, with $\varphi \in \mathrm{End}(W)$ chosen as a Sturm-Liouville differential operator ($W = \mathrm{C}^\infty(\mathbb{R})$) or as a diagonal matrix with distinct elements on the diagonal ($W = K^n$).

Remark 3.7: Besides Corollary 3.5, Peter and Plonka [46, Theorem 2.4] extended their method, e. g., to include generalized eigenvectors and multiplicities; see also Mourrain [45] and Stampfer and Plonka [63]. At present Prony structures do not cover this variation. Since none of the examples we have in mind use generalized eigenvectors and multiplicities, we omit a detailed discussion here. See also Remark 1.13.

Remark 3.8: Summarizing the preceding discussion on frameworks for character [19] and eigenfunction/eigenvector sums [24, 46] and the algebraic and analytic sparse polynomial interpolation techniques w. r. t. the Chebyshev basis [40, 31, 55], we obtain the following diagram of “inclusions”.

$$
\begin{tikzcd}
& \text{Prony structures} & \\
\text{character sums} \arrow[ur, "\text{Theorem 3.1}"] & & \text{algebraic Chebyshev interpol.} \arrow[ul, "\text{Corollary 2.16}"'] \\
\text{eigenvector sums} \arrow[u, "\text{Corollary 3.4}"'] & & \\
\begin{array}{c}\text{eigenfunction sums}\\ \text{(over fields)}\end{array} \arrow[u, "(\text{arbitrary functionals } \Delta)"'] & & \\
& & \text{analytic Chebyshev interpol.} \arrow[ull, "\text{Remark 2.17}"']
\end{tikzcd}
$$

Lakshman and Saunders remark on the possibility to “reconcile” the frameworks for character or eigenfunction sums with their algorithm for sparse polynomial interpolation w. r. t. the Chebyshev basis [40, p. 388]. As the framework of Prony structures is of a very general nature, we would not propose it as a final answer to this question. However, it can be hoped that it will be helpful in finding more particular reconciliations. See also Remark 2.17.

Remark 3.9: There is a close relationship between Prony’s method and Sylvester’s method [64, 65] for computing Waring decompositions of homogeneous polynomials. Although Sylvester’s method does not fit directly into our framework of Prony structures (since it is not a method to reconstruct the support of a function), one may still view it as an application of the classic Prony structure from Theorem 1.1: Given a homogeneous polynomial

$$p = \sum_{i=0}^{d} p_i \mathrm{x}^i \mathrm{y}^{d-i} \in \mathbb{C}[\mathrm{x}, \mathrm{y}]$$

of Waring rank at most r, the linear map associated to the matrix

$$\mathrm{C}(p) := (c_{i+j})_{\substack{i=0,\dots,r \\ j=0,\dots,d-r}} \in \mathbb{C}^{(r+1)\times(d-r+1)}$$

with $c_i := p_i/\binom{d}{i}$ has a Prony kernel for an exponential sum $f_p \in \mathrm{Exp}^1(\mathbb{C})$. Then this exponential sum f_p and its reconstruction as $f_p = \sum_{k=1}^{r} \mu_k \exp_{b_k}$ can be used to compute a Waring decomposition of p. Sylvester’s method has recently been generalized to the multivariate case, cf., in particular, Brachat, Comon, Mourrain, and Tsigaridas [6].

4. Relative Prony structures

A Prony structure on a vector space (V, B, u) can be seen as a tool to solve the problem: Compute a set P of polynomials that has the u-support $\operatorname{supp}_u f \subseteq K^n$ of a given $f \in V$ as its zero locus.

Suppose that a priori one is given a set of polynomials $I \subseteq S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$ with $\operatorname{supp}_u f \subseteq \mathrm{Z}(I)$. For example, one could have $K = \mathbb{R}$ and know that $\operatorname{supp}_u f \subseteq \mathbb{S}^{n-1}$. It would be desirable to be able to exploit this additional information in order to simplify the solution process.

In particular, instead of searching directly for a solution P as above, it would be reasonable to search for a set Q of polynomials such that $P = I \cup Q$ solves the problem. Such a set Q could have fewer elements or indeed be a strict subset of any solution to the original problem.

Prony structures as discussed so far have no way of taking this into account. In this chapter we modify the concept of Prony structures in order to handle and take advantage of the additional information in the relative situation.

In Section 4.1 we present the general setup for the relative setting.

There are two distinct scenarios:

- One already has a Prony structure Φ on (V, B, u) and wants to construct from Φ a "relative Prony structure" Φ/I that takes the equations in I into account.
- The possible construction of a "relative Prony structure w. r. t. I" without using an ordinary Prony structure Φ on (V, B, u).

In Section 4.2 we show one possible construction for the first case and in Section 4.3 we recast an earlier result from [37] concerning the decomposition of function on the sphere w. r. t. a basis of spherical harmonics in this context.

4.1. Definition of relative Prony structures

We begin by giving appropriate variants of earlier definitions for this context. As before, let K always be a field and $S = K[\mathrm{x}] = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$.

Definition 4.1: For $Y \subseteq K^n$ let

$$K[Y] := S/\mathrm{I}(Y)$$

be the usual *coordinate algebra of* Y. For a K-subvector space U of S let

$$K[Y]_U := \{p + \mathrm{I}(Y) \mid p \in U\} \leq K[Y].$$

We call $K[Y]_U$ the *coordinate space of* Y *w. r. t.* U.

Remark 4.2: Let $Y \subseteq K^n$ and $U \leq S$ a K-subvector space. Then we have

$$U/\mathrm{I}_U(Y) \cong K[Y]_U.$$

Indeed, the K-linear map $\varphi\colon U \to K[Y]_U$, $p \mapsto \overline{p} = p + \mathrm{I}(Y)$, is an epimorphism with $\ker \varphi = \mathrm{I}_U(Y)$. In the following we identify these two K-vector spaces.

Definition 4.3: Let $U \leq S$ be a K-subvector space S. For $X \subseteq Y \subseteq K^n$ we call

$$\mathrm{ev}_{U/Y}^X\colon \quad K[Y]_U \longrightarrow K^X, \qquad p + \mathrm{I}_U(Y) \longmapsto \mathrm{ev}_U^X(p) = (p(x))_{x \in X},$$

the *relative evaluation map at X w. r. t. U modulo Y* and

$$\mathrm{I}_{U/Y}(X) := \ker \mathrm{ev}_{U/Y}^X$$

the *relative vanishing space of X on U/Y*.

Remark 4.4: Let $U \leq S$ be a K-subvector space and $X \subseteq Y \subseteq K^n$, with X finite. Then there is a $U_0 \subseteq U$ such that $B := \{p + \mathrm{I}_U(Y) \mid p \in U_0\}$ is a K-basis of $K[Y]_U$. Without loss of generality, choose U_0 such that $|B| = |U_0|$.

Observe that then the matrix of $\mathrm{ev}_{U/Y}^X$ w. r. t. B and the canonical basis of K^X is the Vandermonde matrix $\mathrm{V}_B^X = (p(x))_{x \in X, p \in U_0}$. Hence the matrices of the relative evaluation map $\mathrm{ev}_{U/Y}^X$ and the "ordinary" evaluation map ev_U^X are identical.

Definition 4.5: For $J \subseteq K[Y]$ we call

$$\mathrm{Z}_Y(J) := \{y \in Y \mid \text{for all } q \in S \text{ with } q + \mathrm{I}(Y) \in J,\ q(y) = 0\}$$

the *relative zero locus of J w. r. t. Y*.

After these general preparations, we define relative Prony structures, which are the topic of this chapter. Recall that $Y \subseteq K^n$ is called *algebraic set* if Y is the zero locus of a set of polynomials, i. e., if $Y = \mathrm{Z}(I)$ for some set of polynomials $I \subseteq S$. By Hilbert's basis theorem, I can always be chosen to be finite.

Definition 4.6 (Relative Prony structure, abstract version): Let $X \subseteq Y \subseteq K^n$ be algebraic sets.

(a) Let U be a K-subvector space of $K[Y]$ and let W be an arbitrary K-vector space. We say that a K-linear map $\varphi\colon U \to W$ *has a Prony kernel for X relative to Y* if the following conditions (R_1) and (R_2) are satisfied:

$$\mathrm{Z}_Y(\ker \varphi) = X \tag{R_1}$$

and

$$\mathrm{I}_{U/Y}(X) \subseteq \ker \varphi. \tag{R_2}$$

(b) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $K[Y]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. We call a sequence

$$\varphi \in \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(S_d, W_d)$$

a *Prony structure for X relative to Y* if for all large d, φ_d has a Prony kernel for X relative to Y.

In this case, the least $c \in \mathbb{N}$ such that for all $d \geq c$, φ_d has a Prony kernel for X relative to Y is called *φ-index of X relative to Y* and denoted by $\operatorname{ind}_{\varphi,Y}(X) := c$.

Remark 4.7: Over an infinite field K, "ordinary" Prony structures are precisely the Prony structures relative to $Y = K^n$. This follows immediately from $K[Y] = K[\mathrm{x}]$.

We obtain a characterization of relative Prony structures analogous to the one for ordinary Prony structures in Theorem 1.18.

Theorem 4.8: *Let $X \subseteq Y \subseteq K^n$ be algebraic sets, $(W_d)_{d\in\mathbb{N}}$ be a sequence of K-vector spaces and let $(U_d)_{d\in\mathbb{N}}$ be an exhaustive filtration on $K[Y]$. Let $\varphi \in \prod_{d\in\mathbb{N}} \operatorname{Hom}_K(U_d, W_d)$. Then the following are equivalent:*

(i) *φ is a Prony structure for X relative to Y;*

(ii) *For all large d there is an injective K-linear map $\eta_d\colon K^X \hookrightarrow W_d$ such that the diagram*

$$\begin{tikzcd} U_d \arrow[r, "\varphi_d"] \arrow[dr, "\mathrm{ev}^X_{U_d/Y}"'] & W_d \\ & K^X \arrow[u, hook, dashed, "\eta_d"'] \end{tikzcd}$$

is commutative;

(iii) *For all large d one has $\ker\varphi_d = \mathrm{I}_{U_d/Y}(X)$.*

Proof: Using Remark 4.4 for the surjectivity $\mathrm{ev}^X_{U_d/Y}$ for all large d, the proof is analogous to the one of Theorem 1.18. We write it out for completeness.

(i) $\Rightarrow$ (ii): By Definition 4.3 and since φ is a Prony structure for X relative to Y, for all large d we have

$$\ker \mathrm{ev}^X_{U_d/Y} = \mathrm{I}_{U_d/Y}(X) \subseteq \ker\varphi_d.$$

Since $(U_d)_{d\in\mathbb{N}}$ is an exhaustive filtration on $K[Y]$, there is an exhaustive filtration $(S_d)_{d\in\mathbb{N}}$ on $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ such that $U_d = S_d/\mathrm{I}_{U_d}(Y)$. Hence we have $S_{\leq|X|} \subseteq S_d$ for all large d and $\mathrm{ev}^X_{S_d}$ is surjective by Lemma 1.17. Therefore, also $\mathrm{ev}^X_{U_d/Y}$ is surjective by Remark 4.4. Together, these facts imply the existence of K-linear maps η_d such that the required diagrams are commutative.

It remains to show that η_d is injective for all large d. Let $c \in \mathbb{N}$ be such that for all $d \geq c$ we have that

$$\mathrm{Z}_Y(\ker\varphi_d) = X, \quad \mathrm{ev}^X_{U_d/Y} \text{ is surjective}, \quad \text{and } \eta_d \text{ exists.}$$

Let $v \in \ker \eta_d$. By the surjectivity of $\mathrm{ev}^X_{U_d/Y}$ we have $\mathrm{ev}^X_{U_d/Y}(p) = v$ for some $p \in U_d$. Then $\varphi_d(p) = \eta_d(\mathrm{ev}^X_{U_d/Y}(p)) = \eta_d(v) = 0$. Thus, and since X is Zariski closed, we have

$$p \in \ker \varphi_d \subseteq \mathrm{I}_{U_d/Y}(\mathrm{Z}_Y(\ker \varphi_d)) = \mathrm{I}_{U_d/Y}(X) = \ker \mathrm{ev}^X_{U_d/Y}.$$

Hence, $v = \mathrm{ev}^X_{U_d/Y}(p) = 0$. Thus, η_d is injective.

(ii) $\Rightarrow$ (iii): Since η_d exists and is injective (for all large d), we have

$$\ker \varphi_d = \ker(\eta_d \circ \mathrm{ev}^X_{U_d/Y}) = \ker \mathrm{ev}^X_{U_d/Y} = \mathrm{I}_{U_d/Y}(X).$$

(iii) $\Rightarrow$ (i): Since $(U_d)_{d\in\mathbb{N}}$ is exhaustive, by Lemma 1.14 for all large d we have

$$\mathrm{Z}_Y(\ker \varphi_d) = \mathrm{Z}_Y(\mathrm{I}_{U_d/Y}(X)) = \mathrm{Z}_Y(\mathrm{I}_{S/Y}(X)) = X.$$

Condition (R_2) in Definition 4.6 (a) is obviously satisfied. q. e. d.

Definition 4.9 (Relative Prony structure): Extending the setup of Definition 1.6 let $Y \subseteq K^n$ be an algebraic set.

(a) Let U be a K-subvector space of $K[Y]$ and let W be an arbitrary K-vector space. Let $f \in V$ with

$$\mathrm{supp}_u f \subseteq Y.$$

We say a K-linear map $\varphi\colon U \to W$ *has a Prony kernel for f relative to Y w. r. t. u* if φ has a Prony kernel for $\mathrm{supp}_u f$ relative to Y.

(b) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $K[Y]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. Let $f \in V$ with

$$\mathrm{supp}_u f \subseteq Y.$$

We call a sequence $\varphi \in \prod_{d\in\mathbb{N}} \mathrm{Hom}_K(S_d, W_d)$ a *Prony structure for f relative to Y w. r. t. u* if φ is a Prony structure for $\mathrm{supp}_u f$ relative to Y.

In this case, the number

$$\mathrm{ind}_{\varphi,Y}(f) := \mathrm{ind}_{\varphi,Y}(\mathrm{supp}_u f)$$

is called *φ-index of f relative to Y*.

(c) Let $(S_d)_{d\in\mathbb{N}}$ be a sequence of K-subvector spaces of $K[Y]$ and let $(W_d)_{d\in\mathbb{N}}$ be a sequence of arbitrary K-vector spaces. We call a function

$$\Phi\colon \quad V \longrightarrow \prod_{d\in\mathbb{N}} \mathrm{Hom}_K(S_d, W_d)$$

a *Prony structure on (V, B, u) relative to Y* if for every $f \in V$ the sequence $\Phi(f)$ is a Prony structure for f relative to Y w. r. t. u.

For $f \in V$ we call the $\Phi(f)$-index of f relative to Y simply *Φ-index of f relative to Y* and denote it by $\mathrm{ind}_{\Phi,Y}(f)$.

4.2. Construction of relative Prony structures

The following proposition provides a method to obtain a relative Prony structure from an "ordinary" one. The relative Prony structure then uses smaller matrices.

Definition 4.10: Let $X \subseteq Y \subseteq K^n$ be algebraic sets, U be a K-subvector space of $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$, and let $\varphi \in \mathrm{Hom}_K(U, W)$ have a Prony kernel for X. Observe that

$$\mathrm{I}_U(Y) \subseteq \mathrm{I}_U(X) \subseteq \ker \varphi.$$

Hence there is a unique K-linear map, which we denote by φ/Y, such that the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\pi_{\mathrm{I}_U(X)}"'] & W \\ K[Y]_U \arrow[ur, dashed, "\varphi/Y"'] & \end{tikzcd}$$

is commutative.

The following simple lemma is a variant of [68, Lemma 2.38].

Lemma 4.11: *Let $Y \subseteq K^n$ be an algebraic set and let U be a K-subvector space of $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$. Let W be an arbitrary K-vector space and $\varphi\colon U \to W$ be K-linear. Then one has*

$$Y \cap \mathrm{Z}(\ker \varphi) \subseteq \mathrm{Z}_Y(\ker(\varphi/Y)) \subseteq \mathrm{Z}(\ker \varphi).$$

Proof: Let $y \in Y \cap \mathrm{Z}(\ker \varphi)$ and $\overline{p} \in \ker(\varphi/Y)$. Then $\overline{p} = q + \mathrm{I}_U(Y)$ for some $q \in \ker \varphi$. Hence $\overline{p}(y) = q(y) = 0$. This proves the first inclusion.

To prove the second inclusion, let $y \in \mathrm{Z}_Y(\ker(\varphi/Y))$ and $p \in \ker \varphi$. Then $(\varphi/Y)(p + \mathrm{I}_U(Y)) = \varphi(p) = 0$ and thus $p + \mathrm{I}_U(Y) \in \ker(\varphi/Y)$. It follows that $p(y) = (p + \mathrm{I}_U(Y))(y) = 0$ and the proof is done. q. e. d.

Corollary 4.12: *Let U be a K-subvector space of $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$, W be an arbitrary K-vector space and $\varphi\colon U \to W$ be K-linear. If $X \subseteq Y \subseteq K^n$ are algebraic sets and φ has a Prony kernel for X, then φ/Y has a Prony kernel for X relative to Y.*

Proof: We check that the conditions (R_1) and (R_2) in Definition 4.6 (a) are satisfied.

(R_1): We have to show that $\mathrm{Z}_Y(\ker(\varphi/Y)) = X$.

"$\subseteq$": By Lemma 4.11 and the hypothesis we have $\mathrm{Z}_Y(\ker(\varphi/Y)) \subseteq \mathrm{Z}(\ker \varphi) = X$.

"$\supseteq$": By Lemma 4.11 and the hypothesis we have $X \subseteq Y \cap \mathrm{Z}(\ker \varphi) \subseteq \mathrm{Z}_Y(\ker(\varphi/Y))$.

(R_2): We have to show that $\mathrm{I}_{U/Y}(X) \subseteq \ker(\varphi/Y)$. Let $p \in U$ be such that $p + \mathrm{I}_U(Y) \in \mathrm{I}_{U/Y}(X) = \ker \mathrm{ev}_{U/Y}^X$. Then $\mathrm{ev}_U^X(p) = \mathrm{ev}_{U/Y}^X(p + \mathrm{I}_U(X)) = 0$, i. e., $p \in \mathrm{I}_U(X)$. Since φ has a Prony kernel for X, we have $p \in \ker \varphi$, and thus $(\varphi/Y)(p + \mathrm{I}_U(X)) = \varphi(p) = 0$. q. e. d.

REMARK 4.13: Given bases B of U and C of W, let $A = (a_{(c,b)})_{c\in C, b\in B} \in K^{C\times B}$ be the matrix of φ w. r. t. B, C. Let A/Y be the matrix of φ/Y w. r. t. the basis $\overline{B}$ of $K[Y]_U$ consisting of the vectors $b + \mathrm{I}_U(X)$ with $b \in B \setminus \mathrm{I}_U(X)$ and the basis C of W. Then

$$A/Y = (a_{(c,b)})_{c\in C, b\in\overline{B}} \in K^{C\times\overline{B}}.$$

Thus, with Corollary 4.12 one has a basic method to reduce the size of the involved matrices when applying a Prony structure on a function supported on the algebraic set Y.

Application of the previous discussion to spaces of t-exponential sums yields the following corollary.

COROLLARY 4.14 (Relative Prony structures for t-exponential sums): *In the setup of Definition 2.5 let $Y \subseteq K^n$, in addition to being t-admissible, be an algebraic set. Let $f = \sum_{b\in M} f_b \operatorname{gxp}_b^t \in \operatorname{Gxp}_Y^t(F)$ with fixed finite $M \subseteq Y$ and $f_b \in F \setminus \{0\}$. Then the following holds:*

(a) *The matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)/Y$ is*

$$\Big(\sum_{b\in M} f_b \operatorname{gxp}_b^t(\beta) \operatorname{gxp}_{\Sigma b}^t(\alpha)\Big)_{\substack{\alpha\in\mathcal{I}\\ \beta\in\mathcal{J}'}} \in K^{\mathcal{I}\times\mathcal{J}'}$$

where $\mathcal{J}' \subseteq \mathcal{J}$ with $|\mathcal{J}'| = \dim_K K[Y]_U$ is chosen such that $\{t_\beta + \mathrm{I}_U(Y) \mid \beta \in \mathcal{J}'\}$ is a K-basis of $K[Y]_U$.

(b) *Let $\mathcal{I}, \mathcal{J} \subseteq A \cap \mathbb{N}^n$ be such that $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ has a Prony kernel for f. Let $\mathcal{J}' \subseteq \mathcal{J}$ be as in part* (a) *and $\mathcal{I}' \subseteq \mathcal{I}$ be such that $\{t_\alpha{}^* \mid \alpha \in \mathcal{I}'\}$ is a K-basis of $W := (K[Y]_{S_{\mathcal{I}}})^*$. Then $\Phi_{\mathcal{I}'}^{\mathcal{J}}(f)/Y \colon K[Y]_U \to W$ has a Prony kernel for f relative to Y and the matrix of $\Phi_{\mathcal{I}'}^{\mathcal{J}}(f)/Y$ w. r. t. these bases is*

$$\Big(\sum_{b\in M} f_b \operatorname{gxp}_b^t(\beta) \operatorname{gxp}_{\Sigma b}^t(\alpha)\Big)_{\substack{\alpha\in\mathcal{I}\\ \beta\in\mathcal{J}'}} \in K^{\mathcal{I}'\times\mathcal{J}'}.$$

Proof: (a) By elementary linear algebra the matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)/Y$ is the submatrix of the matrix of $\Phi_{\mathcal{I}}^{\mathcal{J}}(f)$ obtained by deleting the columns indexed by $\mathcal{J} \setminus \mathcal{J}'$. Thus the assertion follows from Proposition 2.6.

(b) This follows in a similar way by linear algebra arguments. q. e. d.

While Corollary 4.14 (a) provides a general way to reduce the number of columns of the involved matrices, in part (b) also the number of rows gets reduced.

4.3. Applications of relative Prony structures

While Section 4.2 yields a general recipe to construct relative Prony structures from “ordinary” ones, in concrete situations it can be possible to achieve better results. We end the chapter with one such example, recasting the main result of [37] in the context of relative Prony structures. See also [68, Section 3.1] for an alternative approach to the reconstruction problem on the sphere. Let $K = \mathbb{R}$, $S = \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_n]$, and

$$Y := \mathbb{S}^{n-1} = \mathrm{Z}\Big(1 - \sum_{j=1}^{n} \mathrm{x}_j^2\Big) = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\} \subseteq \mathbb{R}^n.$$

Consider the $\mathbb{R}$-vector space

$$\mathrm{SH}_{\leq d} := \mathbb{R}[\mathbb{S}^{n-1}]_{\leq d} = S_{\leq d}/\mathrm{I}_{\leq d}(\mathbb{S}^{n-1}) \cong \{p \restriction \mathbb{S}^{n-1} \mid p \in S_{\leq d}\}.$$

Let $\Delta\colon S \to S$, $p \mapsto \sum_{j=1}^{n} \partial_j^2(p)$, denote the *Laplace operator*. A polynomial $p \in S$ is called *harmonic* if $p \in \ker \Delta$.

Let harmH_k be the $\mathbb{R}$-vector space generated by the restrictions $p \restriction \mathbb{S}^{n-1}$ of harmonic homogeneous polynomials $p \in S_k$ of degree k to the sphere, usually called the space of *spherical harmonics*. Using Gallier and Quaintance [22, Theorem 7.13, discussion after Definition 7.15] it is easy to see that one has the decomposition (as vector spaces)

$$\mathrm{SH}_{\leq d} \cong \bigoplus_{k=0}^{d} \mathrm{harmH}_k.$$

For $k = 0, \dots, d$, let $H_k = (y_k^1, \dots, y_k^{d_k})$ be an $\mathbb{R}$-basis of harmH_k. Hence $H_{\leq d} := \bigcup_{k=0}^{d} H_k$ is a basis of $\mathrm{SH}_{\leq d}$. For $x \in \mathbb{S}^{n-1}$ let

$$h_x\colon \quad \{(k, \ell) \mid k \in \mathbb{N},\, \ell = 1, \dots, d_k\} \longrightarrow \mathbb{R}, \qquad (k, \ell) \longmapsto y_k^\ell(x).$$

For finite $X \subseteq \mathbb{S}^{n-1}$ let $W_{\leq d}^X$ be the matrix of $\mathrm{ev}_{\leq d/\mathbb{S}^{n-1}}^X$ w. r. t. $H_{\leq d}$ and the basis of $\mathbb{R}^X$.

Corollary 4.15 (Relative Prony structure for spherical harmonic sums): *Let*

$$B := \{h_x \mid x \in \mathbb{S}^{n-1}\}, \quad V := \langle B \rangle_{\mathbb{R}}, \quad \textit{and} \quad u\colon B \to \mathbb{R}^n,\, h_x \mapsto x.$$

For $f \in V$, $f = \sum_{x \in \mathrm{supp}_u f} f_x h_x$ *with* $f_x \in \mathbb{R} \setminus \{0\}$, *let* $C_f = (f_x \mathrm{u}_x)_{x \in X}$ *and*

$$\widetilde{\mathrm{H}}_d(f) = \big(W_{\leq d}^{\mathrm{supp}_u f}\big)^\top \cdot C_f \cdot W_{\leq d}^{\mathrm{supp}_u f}.$$

Then the function

$$\widetilde{\mathrm{H}}\colon \quad V \longrightarrow \prod_{d \in \mathbb{N}} \mathbb{R}^{H_{\leq d} \times H_{\leq d}}, \qquad f \longmapsto (\widetilde{\mathrm{H}}_d(f))_{d \in \mathbb{N}},$$

induces a Prony structure on (V, B, u) *relative to* $\mathbb{S}^{n-1}$.

Proof: This follows from Kunis, Möller, and von der Ohe [37, Theorem 3.14]. q. e. d.

Remark 4.16: Observe that by [37, Theorem 3.14], the matrix $\widetilde{\mathrm{H}}_d(f)$ can be computed solely from $\Theta(d^{n-1})$ evaluations of f. One may also use Corollary 4.14 (a) or even Corollary 4.14 (b) to get a Prony structure on $\mathrm{SH}_{\leq d}$ relative to $\mathbb{S}^{n-1}$. The matrices so obtained have the same number of columns or the same size as the ones in Corollary 4.15, respectively. But then the number of used evaluations is not in general in $\Theta(d^{n-1})$.

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