# High-Gain Nonlinear Observer Using System State Augmentation

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*Abstract*— **This paper deals with the problem of state estimation of dynamic systems with Lipschitz nonlinearities using a new high gain observer design. The aim of this new design procedure is to reduce the value of the tuning parameter and the observer gain compared to the standard high gain observer on the one hand without solving a set of LMIs as in the LMI-based observer on the other hand. Towards this end, a novel approach based on system state augmentation that transforms the original system of dimension $n$ into a new system whose dimension is $(n + j_s)$, where the new nonlinear function does not depend on $j_s$ last components of the new state. A numerical example is reported to evaluate the effectiveness of the proposed observer for different values of the Lipschitz constant.**

*Index Terms*— **Observer design, high-gain methodology, Lipschitz systems, LMIs.**

## I. INTRODUCTION

The observer is a mathematical tool acting as a sensor, which allows to collect an estimate of unknown states of a given physical system. There are many approaches in literature for designing observers whether for linear or nonlinear case. The first approaches dealt with time-invariant linear systems, using the popular Kalman filter [1] and the Luenberger observer [2], however, with nonlinear systems, the Kalman filter does not guarantee an asymptotic convergence of the estimation error. Therefore, many research activities have been devoted to the design of observers for nonlinear systems. The method of linearizing the estimation error [3], [4], [5], [6], [7], [8], [9] and [10] was one of the starting points for nonlinear observer design. This method has the particularity of transforming the nonlinear system to an affine form according to the state in which the nonlinearities depend only on the input and output using an appropriate change of coordinate, allowing thereafter the design of a Luenberger observer. However, this method does not deliver a systematic transformation and the search for such a transformation to linearize the estimation error is not necessarily easy. A second approach for nonlinear systems is based on LMI (Linear Matrix Inequality), which aims to transform the observability problem into an LMI resolution problem [11] and [12]. Despite theoretical advances in this field with some enhancements that have been proposed recently [13], [14], [15], the problem still remains open. Another approach is based on observability canonical forms which is derived from the fact that the observability of nonlinear systems depends on input [16]. In particular, a diffeomorphism has been proposed in [17] to transform the system uniformly observable and affine in the control into an observable canonical form where the nonlinear part is triangular according to its new coordinates. In case where the considered system is Lipschitz, it is possible to design a high gain observer whose estimation error always converges exponentially towards zero by tuning a single parameter that should be chosen large enough [18]. The design of high gain observers was essentially motivated by its simplicity to implement; however, it has three limitations that are worth mentioning. The first constraint is related to numerical problems concerning large systems as high values of the observer gain are required. The second constraint is the output sensitivity to noise measurements since the correction term of the high gain observer is equal to the product of the observer's gain and the output estimation error. The last constraint is the phenomenon characterized by large amplitudes of the estimates during transient, namely the "peaking phenomenon". To overcome this restrictions, several solutions have been proposed to reduce the sensitivity of the high gain observer to measurement noise. The main solutions are generally based on a time-varying gain that is appropriately updated by taking into account the stability and convergence requirements [19], [20], [13], [21] and [22]. More recently, a high gain observer with a lower gain, called a high gain observer "low power", has been proposed in [23] for a class of nonlinear systems with one output and dimension $n \geq 3$. The cornerstone of this contribution consists in limiting the power of the observer's gain to 2, thus improving the performance of the observer with respect to the measurement noise on the output. Two characteristics of the proposed observer should then be highlighted. First, the dimension of the observer is equal to $2(n-1)$ where $n$ is the size of the original system. Secondly, the observer provides an estimate of the first and last component of the system's state as well as two estimates for each intermediate component of the state. This particular design was then reconsidered in [24] and [25] where they have incorporated saturation functions in order to limit the phenomenon of the "peaking" mentioned before. Another recent high gain observer with the same dimension as the original system and the observer's gain power is limited to 1 was proposed in [26] for the same class of systems considered in [24]. As in [24] and [25], nested saturation functions have been used to limit the peaking phenomenon. However, even if their gain power is limited, the higher dimension of the observer $(2n-2)$ may increase the size of the tuning parameter but from the sensitivity to measurement noise point of view, this new high gain observer is better than the standard one as

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shown in [23].

In this paper, we will present a new observer structure for triangular systems having Lipschitz nonlinearities. The proposed observer is based on system state augmentation which transforms the original system of dimension $n$ into an augmented system of dimension $n + j_s$ which allows to obtain a new threshold on the observer parameter $\theta$ that guarantees the exponential convergence of the estimation error and reduces the value of the observer gain. The paper is organized as follows. In Section II, important background results on the design of high gain observer are presented. The problem formulation is given in section III which describes the motivation behind this work and describes the design methodology of this new observer. A numerical example with simulation results are reported in SectionIV. Finally, the conclusions are drawn in Section VI.

## II. Problem Formulation and Background Results

### A. System Description

Consider the class of nonlinear systems described by the following set of equations:

$$\left\{\begin{array}{l} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \\ f(x) \end{bmatrix} \\ y = x_1 \end{array}\right. \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the Lipschitz property formulated under the following form:

$$\Big| f(x_1 + \Delta_1, \dots, x_n + \Delta_n) - f(x_1, \dots, x_n) \Big| \leq \gamma_f \sum_{j=1}^{n} |\Delta_j| \ . \tag{2}$$

For the sake of compactness, we write system (1) under the form:

$$\left\{\begin{array}{l} \dot{x} = Ax + Bf(x) \\ y = Cx \end{array}\right. , \tag{3}$$

where

$$B = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \tag{4}$$

and the state matrix $A$ is defined by

$$(A)_{i,j} = \left\{\begin{array}{lll} 1 & \text{if} & j = i+1 \\ 0 & \text{if} & j \neq i+1 \end{array}\right. . \tag{5}$$

Consider the following Luenberger observer:

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + L\Big(y - C\hat{x}\Big). \tag{6}$$

The dynamics of the estimation error $\tilde{x} = x - \hat{x}$ is then given by:

$$\dot{\tilde{x}} = \big(A - LC\big)\tilde{x} + B\big[f(x) - f(\hat{x})\big]. \tag{7}$$

### B. Standard high-Gain Design

Here, we recall the basic high gain observer as in [27]. Basically, in the high-gain methodology, we write the observer gain $L$ under the form:

$$L := \mathrm{T}(\theta)K, \ \theta \geq 1. \tag{8}$$

where

$$\mathrm{T}(\theta) := \mathrm{diag}\Big(\theta, \dots, \theta^n\Big) \text{ and } K \in \mathbb{R}^n.$$

In addition, the high-gain methodology focuses on the transformed estimation error

$$\hat{\tilde{x}} := \mathrm{T}^{-1}(\theta)\tilde{x}, \tag{9}$$

where $\mathrm{T}^{-1}(\theta)$ is the inverse of $\mathrm{T}(\theta)$ given by

$$\mathrm{T}^{-1}(\theta) = \mathrm{diag}\Big(\frac{1}{\theta}, \dots, \frac{1}{\theta^n}\Big).$$

It is well-known that the dynamics of the error $\hat{\tilde{x}}$ is given by

$$\dot{\hat{\tilde{x}}} = \theta\big(A - KC\big)\hat{\tilde{x}} + \frac{1}{\theta^n}B\Delta f, \tag{10}$$

with

$$\Delta f := f(x) - f(x - \mathrm{T}(\theta)\hat{\tilde{x}}).$$

From the Lipschitz condition (2) and the fact that $\theta \geq 1$, we can show as in [28] that there always exists a positive scalar constant $k_f$, independent of $\theta$, so that

$$\|\mathrm{T}^{-1}(\theta)B\Delta f\| \leq k_f \|\hat{\tilde{x}}\|. \tag{11}$$

Consequently, by following the high-gain methodology we obtain the following theorem.

**Theorem 1 ( [27]):** If there exist $P > 0$, $\lambda > 0$, $Y$, and $\theta \geq 1$ such that

$$A^T P + PA - C^T Y - Y^T C + \lambda I < 0, \tag{12}$$

$$\theta > \theta_0 = \frac{2k_f \lambda_{\max}(P)}{\lambda}, \tag{13}$$

then the estimation error $\tilde{x}$ is exponentially stable with

$$K = P^{-1}Y^T,$$

where $\lambda_{\max}(P)$ is the largest eigenvalue of the matrix $P$.

*Proof:* For more details about the proof of this theorem, we refer the reader to [27], [28], [29]. ■

One of the drawbacks of the standard high gain observer is clearly related to the increasing power (up to the order $n$) of the high-gain parameter $\theta$, which makes the practical numerical implementation a hard task when $n$ is very large.

### C. Astolfi/Marconi Observer

In [23] a new high-gain observer structure has been proposed for a class of uniformly observable nonlinear systems which are diffeomorphic to the canonical observability form. Specifically, a high-gain observer structure is presented with a gain growing up only to power 2 (regardless the dimension n of the system), at the price of having the observer state dimension $2n-2$. The structure of the proposed observer has the following form:

$$\begin{aligned} \dot{\xi}_i &= A\xi_i + N\xi_{i+1} + T_2(\theta)K_i e_i \quad i = 1\dots n-2 \\ &\vdots \\ \dot{\xi}_{n-1} &= A\xi_{n-1} + Bf(\hat{x}') + T_2(\theta)K_{n-1}e_{n-1}, \end{aligned} \tag{14}$$

where $(A,B,C)$ is a triplet in prime form of dimension 2, $N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\xi_i \in \mathbb{R}^n$, $K_i = (k_{i1}, k_{i1})^\top$ are coefficient to be chosen according to [23, Lemma 1], $T_2(\theta) = \text{diag}(\theta, \theta^2)$ where $\theta$ is the high gain parameter. The variable $\hat{x}' = L_1\xi$ represents an asymptotic estimate of the state $x$ of (3). It is obtained by "extracting" $n$ components from the state $\xi = col(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{2n-n}$ according to the matrix $L_1$ defined as:

$$L_1 = \text{blkdiag}(\underbrace{C, \dots, C}_{(n-2)times}, I_2)$$

and

$$e_1 = y - C\xi_1 \quad e_i = B^\top \xi_{i-1} - C\xi \quad i = 2, \dots, n-1.$$

The redundancy of the observer can be used to extract from $\xi$ an extra state estimation that is

$$\hat{x}'' = L_2\xi, \qquad L_2 = \text{blkdiag}(I_2, \underbrace{B^\top, \dots, B^\top}_{(n-2)times})$$

According to [23], the following proposition shows that the observer (14) recovers the same asymptotic properties for the two estimates $\hat{x}'$ and $\hat{x}''$ of the "standard" high-gain observer. Let $\hat{X} = \text{col}(\hat{x}', \hat{x}'')$, $X = \text{col}(x, x)$.

**Theorem 2 ( [23]):** Consider system (3) and the observer (14) with the coefficients $(k_{i1}\ k_{i2})$ fixed so that the matrix $M$ defined in [23, Lemma 1] is Hurwitz. Then, there exist $\theta^* \geq 1$ such that for any $\theta \geq \theta_0$ and for any $\xi(0) \in \mathbb{R}^{2n-2}$, the variable $\hat{x}$ converges exponentially to $x$.

*Proof:* For the proof we refer the reader to [23]. ■

It is important to notice that several details are omitted in this section to avoid cumbersome notations. We should keep in mind that Theorems 1 and 2 are introduced for clarifying comparisons between different high-gain design procedures. For more details on these theorems and on the definition of some variables, we refer the reader to [27] and [23].

## III. New Solution Using State Augmentation Approach

This section is devoted to the main result of this paper. The motivation of this work is inspired from the HG/LMI design presented in the previous section. We will show that by augmenting the state of the system, we can reduce the value of the tuning parameter and the power of the observer gain.

### A. Motivation

The motivation of developing the new solution comes from work in [30]. Indeed, as demonstrated in [30], if the nonlinear function $f(.)$ satisfies the condition

$$\frac{\partial f}{\partial x_j}(x) \equiv 0, \forall\ j > n - j_s \tag{15}$$

for a given $j_s \geq 0$, then the Lipschitz inequality (11) becomes

$$\|\mathrm{T}^{-1}(\theta)B\Delta f\| \leq \frac{k_f}{\theta^{j_s}}\|\hat{\tilde{x}}\|. \tag{16}$$

It follows that the high-gain inequality (13) becomes

$$\theta > \left(\frac{2k_f\lambda_{\max}(P)}{\lambda}\right)^{\frac{1}{1+j_s}} = \theta_0^{\frac{1}{1+j_s}}. \tag{17}$$

This new threshold on $\theta$ to guarantee exponential convergence of the estimation error is significantly reduced due to the power $\frac{1}{1+j_s}$. Indeed, instead of $\mathrm{T}(\theta)$ in $L$, we have $\mathrm{T}(\theta)^{\frac{1}{1+j_s}}$.

Hence it is important to exploit condition (15) for systems satisfying it, since it allows decreasing considerably the values of the high-gain observer. A solution is proposed in [30] by using a decomposition of the nonlinearity into two parts. Such a solution improves highly the standard high-gain observer, however, the decomposition of the nonlinearity affects the design of the matrices $P$ and $K$ subject to a set of $2^{j_s}$ LMIs to be solved. The aim of this paper is to provide a design procedure without solving $2^{j_s}$ LMIs depending on the nonlinearity of the system. To this end, we propose in this note, a novel approach based on system state augmentation. The idea is to transform the original system of dimension $n$ into a new one with augmented dimension $n+j_s$, where the new nonlinear function does not depend on $j_s$ last components of the new state. The technique is stated in the next subsection.

### B. System state augmentation

This section is devoted to the main result of this paper. The system state augmentation approach and the design procedure are stated in the following Theorem 3.

**Theorem 3:** Let us consider the uniformly observable system:

$$\begin{cases} \dot{x} = \psi(x, u) \\ y = \phi(x, u) \end{cases} \tag{18}$$

Assume that there exists a state transformation (an imbedding) given as:

$$\begin{aligned} \Psi : \mathbb{R}^n &\to \mathbb{R}^{n+j_s} \\ x \to z &= \Psi(x) \end{aligned} \tag{19}$$

which transforms the system (18) into the following:

$$\begin{cases} \dot{z} = A_\Psi z + B_\Psi f_\Psi(z) \\ y = C_\Psi z \end{cases} \tag{20}$$

where $A_\Psi, B_\Psi$, and $C_\Psi$ have the same structure than $A, B$, and $C$, respectively, but with dimension $n + j_s$.
We also have:

$$f_\Psi(z) \triangleq f_\Psi(z_1, \ldots, z_n) \;\Leftrightarrow\; \frac{\partial f_\Psi}{\partial z_j}(z) \equiv 0, \forall\ j > n. \tag{21}$$

Consider the state observer described by (22).

$$\begin{cases} \dot{\hat{z}} = A_\Psi \hat{x} + B_\Psi f_\Psi(\hat{z}) + L_\Psi \Big( y - C_\Psi \hat{z} \Big) \\ \hat{x} = \Phi(\hat{z}), \end{cases} \tag{22}$$

where $\Phi$ is a continuous left invert of the imbedding $\Psi$ satisfying $x = \Phi(z)$ and $L_\Psi \triangleq \mathrm{T}_\Psi(\theta) K_\Psi$, with $\mathrm{T}_\Psi(\theta) \triangleq \mathrm{diag}(\theta, \ldots, \theta^{n+j_s})$. If there exist $P > 0$, $\lambda > 0$, $Y$, and $\theta \geq 1$ such that:

$$A_\Psi^\top P + P A_\Psi - C_\Psi^\top Y - Y^\top C_\Psi + \lambda I < 0, \tag{23}$$

$$K_\Psi \triangleq P^{-1} Y^\top, \tag{24}$$

$$\theta > \theta_\Psi \triangleq \sqrt[1+j_s]{\frac{2 k_{f_\Psi} \lambda_{\max}(P)}{\lambda}}, \tag{25}$$

then the estimation error $\tilde{x} = x - \hat{x}$ converges exponentially towards zero.

*Proof:* The proof is straightforward. Indeed, from Theorem 1, if the conditions (23)-(25) are satisfied, then the error $\tilde{z} = z - \hat{z}$ converges exponentially to zero. The presence of $(1+j_s)^{\text{th}}$ root in (25), as also mentioned in (17), is due to the fact that $f_\Psi$ does not depend on the $j_s$ last components of $z$, which leads to:

$$\|\mathrm{T}_\Psi^{-1}(\theta) B_\Psi \Delta f_\Psi\| \leq \frac{k_{f_\Psi}}{\theta^{j_s}} \|\hat{\tilde{z}}\|, \tag{26}$$

where $\hat{\tilde{z}} = \mathrm{T}_\Psi^{-1}(\theta)\tilde{z}$. Hence, exponential convergence of $\tilde{x}$ towards zero is then preserved due to the continuity of $\Phi$. ∎

### *C. Particular transformation: Adding a chain of integrators*

This section is devoted to a special case of transforming the system (3) into a higher dimensional system by adding $j_s$ integrators. The aim of this section is to show that there exists a transformation satisfying the properties stated in Theorem 3.

Let us consider the following transformation

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n+j_s} \end{pmatrix} = \Psi(x) \triangleq \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x(t)) \\ \frac{\mathrm{d} f(x(t))}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{(j_s-1)} f(x(t))}{\mathrm{d}t^{(j_s-1)}} \end{pmatrix}. \tag{27}$$

It is obvious to see that

$$\dot{z}_i = z_{i+1}, \text{ for } i = 1, \ldots, n + j_s - 1, \tag{28}$$

$$\dot{z}_{n+j_s} = \frac{\mathrm{d}^{j_s} f(x(t))}{\mathrm{d}t^{j_s}} \triangleq f_\Psi(z_1, \ldots, z_n), \tag{29}$$

$$x = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \overbrace{\begin{bmatrix} \mathbb{I}_n & 0_{\mathbb{R}^{n \times j_s}} \end{bmatrix}}^{\Phi} \times z. \tag{30}$$

Then following the previous section, the corresponding observer is

$$\begin{cases} \dot{\hat{z}} = A_\Psi \hat{x} + B_\Psi f_\Psi(\hat{z}) + L_\Psi \Big( y - C_\Psi \hat{z} \Big) \\ \hat{x} = \begin{bmatrix} \mathbb{I}_n & 0_{\mathbb{R}^{n \times j_s}} \end{bmatrix} \hat{z}. \end{cases} \tag{31}$$

The advantage of the proposed augmentation system is the presence of the $\theta_\Psi \triangleq \sqrt[1+j_s]{\frac{2 k_{f_\Psi} \lambda_{\max}(P)}{\lambda}}$, instead of $\frac{2 k_{f_\Psi} \lambda_{\max}(P)}{\lambda}$ if the standard high-gain observer is applied on the augmented system. We are aware that if the standard high-gain observer is applied directly on the original system (3), the obtained value of $\theta_0$ in (13) will be smaller than $\frac{2 k_{f_\Psi} \lambda_{\max}(P)}{\lambda}$. However, the presence of power $\frac{1}{1+j_s}$ will reduce significantly the values of the observer gains.

## IV. Illustrative Example

We will illustrate the particular case of adding a chain of integrators. Towards this end, we consider the following fifth dimensional triangular system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = f(x) \triangleq k_f \sin(x_5) \\ y = x_1 \end{cases} \tag{32}$$

The Lipschitz constant of $f$ is $k_f$.

By adding an integrator, we will get a sixth dimensional system ($j_s = 1$):

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = z_5 \\ \dot{z}_5 = z_6 \\ \dot{z}_6 = f_\Psi^6(x) \triangleq k_f^2 \sin(z_5)\cos(z_5) = \frac{k_f^2}{2}\sin(2z_5) \\ y = z_1 \end{cases} \tag{33}$$

Also, if we add two integrators instead of one, we get the new system of dimension 7 ($j_s = 2$):

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = z_5 \\ \dot{z}_5 = z_6 \\ \dot{z}_6 = z_7 \\ \dot{z}_7 = f_\Psi^7(x) \triangleq k_f^3 \sin(z_5)\cos(2z_5) \\ y = z_1 \end{cases} \tag{34}$$

| **Standard high gain observer** | | | | | | |
|---|---|---|---|---|---|---|
| $k_f$ | $\theta$ | $K$ | | | | |
| 0.1 | 4.7625 | 3.6448 | 6.5296 | 6.9649 | 4.2985 | 1.2402 |
| 1 | 31.7155 | 5.6568 | 13.1152 | 16.4885 | 11.1147 | 3.4150 |
| 5 | 140.6488 | 9.3106 | 24.7578 | 33.1298 | 22.8594 | 7.1489 |

| **Astolfi/Marconi observer [23]** | | | | | |
|---|---|---|---|---|---|
| $k_f$ | $\theta$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ |
| 0.1 | 27.329 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 |
| 1 | 273.4418 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 |
| 5 | 1366.8 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 | 2.9<br>8.0629 |

| **State Augmentation Approach in Theorem 3** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $j_s$ | $k_{f_\Psi}$ | $\theta_\psi = \theta^{\frac{1}{j_s+1}}$ | $K_\psi$ | | | | | | |
| 1 | 0.1 | 1.7345 | 3.4069 | 5.8042 | 6.2497 | 4.4499 | 1.9770 | 0.4284 | |
| | 1 | 8.5805 | 5.9510 | 15.6107 | 23.8689 | 22.7052 | 12.8155 | 3.4185 | |
| | 5 | 39.1490 | 17.5262 | 57.7707 | 96.2147 | 95.0459 | 54.9930 | 15.0285 | |
| 2 | 0.1 | 1.3979 | 3.4103 | 5.8151 | 6.3716 | 4.8225 | 2.5160 | 0.8391 | 0.1384 |
| | 1 | 6.1043 | 6.8077 | 20.5086 | 36.7942 | 42.9236 | 32.6395 | 15.2357 | 3.4267 |
| | 5 | 28.3107 | 37.0682 | 151.6654 | 310.1128 | 387.8794 | 306.2467 | 146.7035 | 33.8209 |

TABLE I: Comparisons between the high-gain observers for different values of $k_f$.

Standard high-gain observer is applied directly to system (32). Also, the Astolfi/Marconi observer of dimension $2(n-1) = 8$ is applied to estimate the states of system (32). Afterwards, the results of Section III-C are applied on (33) and (34) with $j_s = 1$ and $j_s = 2$, respectively. Table I summarize the comparison between all the above high-gain observer. The values in Table I are obtained with $k_f = 0.1$, $k_f = 0.5$, and $k_f = 1$, respectively. It is quite clear from Table I that the proposed methodology provides smaller gains due to the introduction of additional integrators.

## V. Conclusion

In this paper we presented a new technique in the design of high gain observer which is based on system state augmentation. The remarkable assets related to this new design procedure are the ability to provide lower gain compared to the standard high observer in addition to a reduced sensitivity to noise measurements. These advantages are clearly confirmed by the fifth dimensional triangular system numerically simulated for different values of the Lipschitz constant. For future work we will consider another manner to synthesize the observer parameter $K_\Psi$ and the Lyapunov matrix $P$ while relying on the elegant work proposed by Gauthier [31] based on Riccati equation. We also aim to to provide explicit bounds on the estimation error for different high-gain observers developed in the literature.

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