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Doctor of Philosophy

Neutrino Mass Models:
From Type III See-saw to Non-Commutative Geometry

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Foreword

You won't find any science in these few paragraphs. In this preface, I merely wish to explain the underlying reasons, motivations and dreams that brought me to research what I did, and whose implications permeate this whole work. Thus, if you are uninterested, or should you ever get bored halfway through, you may safely skip to the next Chapter. However, I still invite you to read through these few lines, since what is written in here is important and it will give a much deeper and clearer meaning to the "scientific part" of this work as well.

Since my infancy, I've always been a very curious child. I always wanted to understand the reason behind everything I saw in my life, heard of from some chat, or studied about in school. It even went to the point that I started hypothesizing "scientific theories" behind Harry Potter's magic! This ravenous curiosity made me of course a quick learner, which in turn led me to quite the successful academic career: finding an answer to the question *why?* was my burning obsession.

Thus, after finishing high-school, I enrolled in physics with the pretentious goal of finding the ultimate answer to the question *why?*: the Theory of Everything, or at least the theory of Quantum Gravity (back then, I was reading an educational book on string theory[1], which claims to be both, so in my mind the two concepts were basically synonymous at the time). *All* the courses that I took during my university time were chosen for that very reason: deepening as much as I could my understanding of what I judged the fundamental cornerstones of physics, namely the theory of Relativity (both special and then general) and the Quantum theory, that in turn would lead me further down the path towards the TOE – the Final Answer.

Still, whatever I learnt was not enough. During my last year of university, when the time came for me to choose a topic for my Master thesis, I realized that my understanding of the state-of-the-art physics (namely the Standard Model and General Realtivity) was only superficial. For this reason, I decided to pick a Master thesis that would allow me to really delve into the details that still eluded my grasp. After all, how the heck was I supposed to surpass a theory when I didn't even know what I had to surpass?!

[1] *The Elegant Universe*, by Brian Greene

Hence, in my Master thesis I studied the see-saw models – a very small and natural extension of the Standard Model that includes right-handed neutrinos-like particles and also explains in a very elegant and simple way *why* (of course, that word is always my focus) the neutrino masses are so tiny. Actually, what I studied was a slightly more unnatural and less studied variant of the see-saw – the so-called type III see-saw – and to be completely honest, I never once believed that variant of model to be right. Yet, in retrospection, I can definitely say that it was the right call, for it allowed me to learn the meaning to all those "important words" of Quantum Field Theory (QFT) that were still obscure to me (what is a path integral?, what are instantons?, what is an anomaly?. . .)

In the end, because of some unexpected circumstances, I couldn't finish all the work on the type III see-saw that I planned in time for my graduation, so I dedicated the first part of my PhD to completing the analysis, and further deepening my understanding of QFTs. An introduction to the type III see-saw and the results of my research on the topic can be found in Chapters 1-5. As for the reasons that led me from this to the second subject of my research, they will be explained in the Intermission between the first and the second parts of this work.

But before moving on to physics, I wish to emphasize something very important. As I mentioned before, the goal I had when I enrolled in physics was to find the TOE. Today, *my goal is still unchanged.* Of course, I know it is a really ambitious goal. Of course, I know it is extremely unlikely that I'll succeed. After all, many people that care about me remind me again and again that I will probably never find the TOE, that it's too hard a task, that it would probably take one hundred Einsteins to get there. They tell me so again and again because they care about me, and they want to protect me from the disappointment of failure, and I am grateful for their concern.

However! I am *not* afraid of disappointment! Disappointment is just an emotion, why the heck should I fear my own emotions? Moreover, life behaves just like a pendulum. There are good times, and there are bad times. The more bad times you have, the more the pendulum of life will swing and give you that many more good times. And the higher the pendulum goes into the "bad times" side – the more intense the suffering – the more powerful will be the joy and elation that will follow. Withdrawing from disappointment then means depriving oneself of all the most beautiful and wonderful moments life can offer. It is tantamount to not living at all, just surviving through all the monotonous routines that repeat unchanged day after day after day. Living everyday the same day: that's just like being dead before even dying!

It is the perfect recipe for being mediocre. Somehow, we're all encouraged to being mediocre since we were children: "don't make mistakes!", "don't do that, you're strange!", "stop that, behave yourself!"... Well, I *hate* "mediocre", I reject it on a physiological level! I'd rather either succeed and attain greatness, or fail and be a fool rather that being mediocre, because then I'd at least be a fool that gave it his all to what he truly believed in! After all, did Columbus find America by backing down for fear of disappointment? Did Da Vinci achieve greatness by only doing what other people expected him to? Did Einstein conceive General Relativity by keeping his head down, since he definitely "*wasn't a Newton*"? No! They pulled through because they kept believing, they dastardly and doggedly kept believing in themselves, believing in their own ideas and in their own path; they had the courage to keep believing even when all the odds were against them, like unreasonable, obstinate children that won't budge no matter what.

And to obstinately keep unreasonably believing is what I do, and also what I encourage to do to you, whom I'm so grateful to for having read through these feelings of mine. Now, if I did things right, you should feel quite excited, and maybe remember how you used to think when you were a child and thought you were a wizard, or a pirate, or maybe a secret agent, ready for the next breathtaking adventure. And now that we've set the mood, let's dive in into the wonderful world of particle physics.

Chapter 1

Introduction

Neutrino masses are one of the most intriguing riddles amongst the unsolved mysteries of modern physics. Their puzzle is (at least) twofold, for there are two very simple yet profound questions that are still unanswered, namely: how are neutrino masses generated? And why are they so tiny?

As we mentioned, the two questions, albeit simple in their semantic formulation, are deep indeed, and tightly intertwined. The fact that neutrino masses are so tiny with respect to all the other fermions' masses entails that their generation mechanism probably differ from the Higgs mechanism, meaning that it should be some yet unknown phenomenon. On the other hand, undetected phenomena can only reside past our detection range, at energy scales that our experiments cannot reach yet. In other words, neutrino masses are the door that leads to the new physics that lies beyond the furthest grasp of our knowledge, paving the way to the deepest mysteries of our universe.

In this work, we will try and answer to both questions, in a slightly unwonted fashion. In the first part of the work, we will review one very simple yet very natural and effective mechanism of neutrino mass generation, one that explains spontaneously the tininess of neutrino masses, i.e. the see-saw mechanism; and we will do a phenomenological study of one class of models that implement said mechanism. In the second part of the work, we will explore the possible reasons that may lead to such see-saw models, and once again we will dwell upon one class of models that naturally implement those very same reasons that in turn cause the see-saw mechanism to take place, i.e. twisted non-commutative geometry, of which we will study three possible extensions of the Standard Model.

Chapter 2

The Standard Model

2.1 Introduction

The Standard Model is the theory that to date best describes the phenomena that take place at microscopic level, unifying the description of three of the four known fundamental interactions (electromagnetism, weak and strong interactions) in a single theoretical framework. It is a theory, built on the particles observed so far, within the framework of Quantum Field Theories (QFTs), i.e. theories in which each type of particle corresponds to an object called field, and it falls in particular in the subcategory of gauge theories. We will discuss later the fundamental concepts related to field theories and gauge theories.

The Standard Model (SM) depends on about twenty parameters that have to be determined experimentally. That many can seem to be a lot, but they are in fact very few compared to the large number of processes that they allow to explain, moreover with an incredibly accurate agreement with the experimental results.

However, there are still some phenomena that cannot be explained within the SM as it is: among many, the most prominent ones are the origin of neutrino masses, the unified description of *all* fundamental forces, *including gravity*, within the same theoretical framework, and possibly related to this the origin of Dark Matter (DM) and Dark Energy (DE).

This work is devoted to the study of the first two problems; however, we will still report some plausible ideas on how to approach the DM/DE problem in the last section of this first introductory chapter.

2.2 Quantum Field Theories

Quantum Field Theories were born with the QED (Quantum ElectroDynamics) following the introduction of the second quantization formalism that was needed to describe systems with many particles, or systems for which the number of particles can vary. They can also be extended to combine Quantum Mechanics together with Special Relativity so that they can take into account the production and annihilation of particles that was observed experimentally. QFTs are the quantum version of the simpler classical (i.e. non-quantum) Field Theories.

The fundamental ingredient of a Field Theory is of course the *field*, i.e. a space-time function with values either in the real (or sometimes complex) numbers (in case of a classical Field Theory), or in the operators on the Hilbert space of physical states (in case of a QFT). In turn, these operators have values in the distributions, which implies, as we will see in section 2.3, some technical and conceptual difficulties. Each field corresponds to one kind of particle, and in fact particles are actually the excitations of their corresponding fields.

Alongside the fields, the *action* S plays a central role: it is a functional of the fields and the whole theory descends from it. Indeed, once a particular action is specified, the fields (and therefore the particles) taken into account by the corresponding theory are all and only the fields that appear in the action, while the equations of theory (often called *field equations* or *equations of motion*) are determined through the Principle of Stationary Action.

Field Theories can also be used to describe relativistic theories. Within such theories, the action is required to be Lorentz invariant, which ensures the equations of the theory to be the same in all inertial reference frames. Usually, for reasons related to causality, the further assumption of Locality is also made, according to which the action can be written as an integral over the whole space-time of a function $\mathcal{L}$ called Lagrangian Density or, more briefly, Lagrangian:

$$S = \int \mathrm{d}^4x\, \mathcal{L}. \tag{2.1}$$

Under this further hypothesis, for the action to be a Lorentz scalar, the Lagrangian must also be a Lorentz scalar.

Usually, the Lagrangian can be split into two parts, a so-called *free* or *kinetic term*, and an *interaction term*. The free term simply describes how particles (or more properly, *fields*, which represent a bunch of particles – just like a sea collectively represents a lot of water molecules) propagate when they do not interact with anything. The interesting part is the interaction term, which describes how the various field types interact among themselves when many of them are present (i.e. they are non-zero) in the same region. In case of a classical Field Theory, the Lagrangian summarizes all the possible processes that can happen within the theory. However, this ceases to be the case when we turn on quantum effects.

In QFTs, each field can be seen as a collection of creation and annihilation operators of some kind of particles. For this reason, each interaction term of the Lagrangian (which, in the classical version of the theory, represents one possible process involving the fields) has to be re-interpreted as a process in which the existing particles of the initial state are annihilated, and then all the particles of the final state are created. Now, if this were all, the Lagrangian would still contain all the possible processes. However, nature turns out to be much richer than that, for quantum theories allow for an intrinsically quantum phenomenon known as *quantum fluctuations*. The term "quantum fluctuations" refers to the creation and the consequent immediate annihilation of particles, *with no regard to the energy-momentum conservation law*. This means that quantum processes allow for intermediate (and unstable) states that do *not* conserve neither energy nor momentum, even though the final (and stable) state has exactly the same energy and momentum of the initial state. This means in practice that there can be processes in which some particles of total mass 1 (in some appropriate units) transform into a heavier particle of mass 100, which soon thereafter decays into some other particles of total mass 1. These complicated multi-step processes (which are today represented pictorially through Feynman diagrams) are *not* included in the Lagrangian: there is no such term that corresponds to those kind of processes. This means in turn that the Lagrangian, and therefore also the action, are not sufficient to fully describe quantum theories. For this reason, some more sophisticated objects were defined. These objects (the functional generator, the connected diagrams generator, the quantum action) are defined in terms of Feynman's technology of *Path Integrals*, which are actually even today mathematically not fully understood. For this reason, we will not explore the meaning of these objects in depth, but instead we will simply give their intuitive meaning.

It can be shown that the vacuum expectation value of any (time-ordered) product of operators can be expressed into the form

$$\begin{aligned} &\langle 0|\, \mathrm{T}\, \left[\mathcal{O}_1\left(t_1\right)\mathcal{O}_2\left(t_2\right)\dots\mathcal{O}_n\left(t_n\right)\right]|0\rangle \propto \\ \propto &\int \mathrm{D}\phi_i\, \mathcal{O}_1\left(t_1\right)\mathcal{O}_2\left(t_2\right)\dots\mathcal{O}_n\left(t_n\right) e^{iS\left(\phi,\dot{\phi}\right)} \end{aligned} \tag{2.2}$$

where S is the action and ϕ_i are the fields of the theory. Correlation functions (in which all the physics is encoded) actually correspond to such vacuum expectation values of products of field operators, and it turns out that there is a simple way of encoding *all* possible such correlation functions in one single object: the so-called *generating functional*

$$Z[J] = \mathcal{N} \int \mathrm{D}\phi_i \, e^{iS\left(\phi,\dot{\phi}\right) + i\int \mathrm{d}^4x\, J_j(x)\phi_j(x)}. \tag{2.3}$$

In this formula, the external sources J_i are classical fields, i.e. they are fields with values in the complex numbers instead of in the operators on the Hilbert space, and $\mathcal{N}$ is a normalization constant chosen in such a way that

$$Z[0] = \langle 0 \,|\, 0 \rangle = 1. \tag{2.4}$$

All the correlation functions can be expressed in terms of the derivatives of Z with respect to the external sources J_i. For instance, if we derive Z with respect to J_1 and J_2 we get

$$\frac{\delta^2 Z}{\delta J_1(x_1)\, \delta J_2(x_2)} = (i)^2 \int \mathrm{D}\phi \, \phi_1(x_1)\, \phi_2(x_2)\, e^{iS + i\int \mathrm{d}^4x\, J_j(x)\phi_j(x)} \tag{2.5}$$

hence

$$\langle 0|\, \mathrm{T}\, \phi_1(x_1)\, \phi_2(x_2)\, |0\rangle = - \left.\frac{\delta^2 Z}{\delta J_1(x_1)\, \delta J_2(x_2)}\right|_{J=0}. \tag{2.6}$$

In general, one has

$$\langle 0|\, \mathrm{T}\, \phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n)\, |0\rangle = i^{-n} \left.\frac{\delta^n Z}{\delta J_{i_1}(x_1) \ldots J_{i_n}(x_n)}\right|_{J=0}. \tag{2.7}$$

Using this smart trick, one can obtain all possible correlation functions in terms of derivatives of Z – including of course the unphysical disconnected ones. We can restrict ourselves to the physical connected correlation functions only by defining the *connected diagrams generator* W:

$$W[J] = -i \log Z[J]. \tag{2.8}$$

To see that $W[J]$ is indeed the generating functional of connected Feynman diagrams we can proceed as follows. $Z[J]$ is given by the sum of all vacuum-vacuum diagrams (in presence of the external source J), both connected and disconnected, but counting only once those diagrams that differ only by a permutation of vertices in connected subdiagrams. The contribution to $Z[J]$ of a diagram that consists of N connected components will be the product of the contributions of these components, divided by the number $N!$ of permutations of vertices that simply permute all the vertices in one connected component with all the vertices in another. Therefore, the sum of all diagrams is

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \left(iW[J]\right)^N = e^{iW[J]}, \tag{2.9}$$

which proves that $iW[J]$ is the sum of all *connected* vacuum-vacuum diagrams (again, counting only once those diagrams that differ only by a permutation of vertices). Connected correlation functions are defined similarly to Eq. (2.7):

$$\langle 0|\,\mathrm{T}\,\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\,|0\rangle_c = i^{1-n}\,\frac{\delta^n W}{\delta J_{i_1}(x_1)\dots J_{i_n}(x_n)}\bigg|_{J=0}. \tag{2.10}$$

For example, for a single field ϕ we have

$$\frac{\delta W}{\delta J(x)}\bigg|_{J=0} = (-i)\,\frac{1}{Z}\frac{\delta Z}{\delta J(x)}\bigg|_{J=0} = \langle 0|\,\phi(x)\,|0\rangle \tag{2.11}$$

hence

$$\langle 0|\,\phi(x)\,|0\rangle_c = \langle 0|\,\phi(x)\,|0\rangle\,. \tag{2.12}$$

Taking one further derivative yields

$$\begin{aligned}\frac{\delta^2 W}{\delta J(x)\,\delta J(y)}\bigg|_{J=0} = (-i)\left(\frac{1}{Z}\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} - \frac{1}{Z^2}\frac{\delta Z}{\delta J(x)}\frac{\delta Z}{\delta J(y)}\right)\bigg|_{J=0} = \\ i\left(\langle 0|\,\mathrm{T}\,\phi(x)\,\phi(y)\,|0\rangle - \langle 0|\,\phi(x)\,|0\rangle\,\langle 0|\,\phi(y)\,|0\rangle\right),\end{aligned} \tag{2.13}$$

from which we find the connected time-ordered two-point function

$$\langle 0|\,\mathrm{T}\,\phi(x)\,\phi(y)\,|0\rangle_c = \langle 0|\,\mathrm{T}\,\phi(x)\,\phi(y)\,|0\rangle - \langle 0|\,\phi(x)\,|0\rangle\,\langle 0|\,\phi(y)\,|0\rangle\,, \tag{2.14}$$

from which the disconnected component is manifestly subtracted.

One can proceed further and define a generating functional $\Gamma[\Phi]$ for 1-particle-irreducible (1PI) diagrams only (i.e. diagrams that cannot be divided in two disconnected components by cutting one internal line). This functional is called *quantum effective action*, or more briefly *quantum action*, for reasons that will be made clear in the following. Notice that the source is not J any more, but instead $\Phi(x)$ (dropping from now on the index i for simplicity), defined as

$$\Phi(x) = \frac{\delta W[J]}{\delta J(x)}. \tag{2.15}$$

The quantum action is defined as the Legendre transformation of $W[J]$:

$$\Gamma[\Phi] = W[J] - \int \mathrm{d}^4x\,\Phi(x)\,J(x)\,, \tag{2.16}$$

very much like the Lagrangian is the Legendre transform of the Hamiltonian. In Eq. (2.16) it is understood that we inverted Eq. (2.15), so that $J = J[\Phi]$.

We can justify the name "quantum action" of $\Gamma[\Phi]$ as follows. Notice that

$$\frac{\delta\,\Gamma}{\delta\Phi(x)} = \int \mathrm{d}^4 y \left(\frac{\delta\,W}{\delta J(y)} - \Phi(y) \right) \frac{\delta\,J(y)}{\delta\Phi(x)} - J(x) \tag{2.17}$$

or, using Eq. (2.15),

$$\frac{\delta\,\Gamma}{\delta\Phi(x)} = -J(x)\,. \tag{2.18}$$

When $J = 0$, Eq. (2.18) tells us that the possible values for Φ are given by the stationary points of Γ. This can be compared with the classical field equations, which require the classical action $S(\phi)$ to be stationary. Hence Eq. (2.18) can be regarded as the equation of motion for the external field Φ. Notice that, in general, Γ is *not* a local functional of the fields, in the sense that it cannot be expressed as the integral of an effective Lagrangian: $\Gamma[\Phi] \neq \int \mathrm{d}^4x\, \mathcal{L}_{eff}(\Phi(x))$. This reflects the fact that Γ already takes propagators-mediated non-local interactions and loop corrections into account.

We can show that Γ already takes quantum corrections into account proceeding as follows. Let us define the quantity $W_\Gamma[J, g]$ as the connected diagrams generator of the theory described by taking Γ/g as *classical* action:

$$e^{iW_\Gamma[\phi,g]} = \mathcal{N} \int \mathrm{D}\phi\, e^{\frac{i}{g}\left(\Gamma[\phi] + \int \mathrm{d}^4x\, J(x)\phi(x)\right)} \tag{2.19}$$

with an arbitrary constant g that plays the role of $\hbar$ and is a loop-counting parameter. Propagators are the inverse of the term quadratic in ϕ of $\Gamma[\phi]/g$, so it is proportional to g. On the other hand, vertices are proportional to $1/g$. This means that a diagram with V vertices and I internal lines (propagators) will be proportional to g^{I-V}. For any diagram, the number of loops L is

$$L = I - V + 1, \tag{2.20}$$

so the L-loop term in $W_\Gamma[J, g]$ goes like

$$W_\Gamma^{(L)}[J, g] \propto g^{I-V} = g^{L-1}. \tag{2.21}$$

Then, we have

$$W_\Gamma[J, g] = \sum_{L=0}^{\infty} g^{L-1} W_\Gamma^{(L)}[J]\,. \tag{2.22}$$

Let us consider the tree-level contribution of W_Γ, i.e. $W_\Gamma^{(0)}$. In order to isolate it, we consider the limit $g \to 0$. In this limit, the path integral Eq. (2.19) is dominated by the point of stationary phase:

$$\lim_{g\to 0} e^{iW_\Gamma[J,g]} = e^{\frac{i}{g} W_\Gamma^{(0)}[J]} = e^{\frac{i}{g}\left(\Gamma[\Phi] + \int \mathrm{d}^4x\, J(x)\Phi(x)\right)}, \tag{2.23}$$

where $\Phi(x)$ takes the the exponential to extremes, namely

$$\left.\frac{\delta\Gamma}{\delta\phi(x)}\right|_{\phi=\Phi} + J(x) = 0. \tag{2.24}$$

Hence, we found that

$$W_{\Gamma}^{(0)}[J] = W[J] \tag{2.25}$$

which means that Γ takes into account all quantum corrections. Moreover, since any connected diagram can be seen as a tree diagram whose vertices are 1PI diagrams, we conclude that Γ is the sum of all 1PI diagrams as we claimed.

The quantum action can also be computed directly from the classical action without passing through $W[J]$. In fact:

$$e^{i\Gamma[\Phi]} = \int_{1\mathrm{PI}} \mathrm{D}\phi\, e^{iS(\phi+\Phi)}, \tag{2.26}$$

where the subscript 1PI means that the path integral is evaluated over all diagrams (connected or not) in which each connected component is 1PI.

In our discussion on QFTs we did not specify what kind of fields appear in the theory. It turns out that there are some fields that are often unacceptable, since most theories that contain them are inconsistent at the quantum level. The only fields that one can always use to build self-consistent theories are scalar and spinor fields, which correspond to spin 0 and spin $\frac{1}{2}$ fields respectively. Higher-spin fields are allowed only in certain circumstances, such as if the theory is a gauge-theory whose gauge symmetry implies the presence of those higher-spin fields as gauge fields. The higher-spin fields that correspond to the known possible gauge symmetries are spin 1 (Yang-Mills theories), spin $\frac{3}{2}$ (supersymmetric theories) and spin 2 (gravity), hence fields with spin higher than 2 are often excluded. We will explain gauge symmetries and gauge theories in detail in Section (2.4). The reason why high-spin fields are often to be excluded is that theories that involve those fields are usually non-renormalizable, a concept that we will clarify in the next section.

2.3 Renormalization

In order to fully understand the necessity of renormalizability for physical theories, we must first introduce the so-called S matrix.

When one wants to describe any physical process, what has to be studied is the transition from a certain initial state to a certain final state of the system (initial and final states may in general also coincide). In a quantum framework, physical states are represented by vectors (normalized to 1 for convenience) in a

Hilbert space and the probability of transition from one state to another is given by the squared module of their scalar product.

Physical states are identified by a complete set of physical quantities that can be measured at the same time, such as the momentum and spin of each particle of the system. For the sake of brevity, we will group those characteristics into one single index, therefore calling the initial state $|\psi_\alpha\rangle$ and the final state $|\psi_\beta\rangle$. Then, the probability amplitude to move from $|\psi_\alpha\rangle$ to $|\psi_\beta\rangle$ is:

$$S_{\beta\alpha} \equiv \langle\psi_\beta \,|\, \psi_\alpha\rangle\,. \tag{2.27}$$

The matrix $S_{\beta\alpha}$ is known as the *scattering matrix*, or *S matrix* for short. Its squared module is precisely the probability we were searching for.

The S matrix has some properties that will be determinant in the aftermath to understand the need of having renormalizable theories. In fact, it is a matrix which describes the change of basis from the orthonormal basis $|\psi_\alpha\rangle$ to the orthonormal basis $|\psi_\beta\rangle$ (with α and β varying over all permissible values) and therefore has necessarily to be a unitary matrix: this property is extremely important, since it guarantees the entries $S_{\beta\alpha}$ to be, in module, less than 1 so that their squared modules are a well-defined probability. Now, since the S matrix is the object of desire of any physicist (it is what describes, virtually, any physical process), great efforts have been made to find a simple way to calculate it, and today we know exactly how to do it. What one finds out is that the S matrix can be written as:

$$S = \mathbb{I} + a\delta\left(P_f - P_i\right)\mathcal{M}, \tag{2.28}$$

where $\mathbb{I}$ is the identity matrix (as already noticed, in general the initial and the final state may coincide), a is a numerical factor, P_f and P_i are the total four-momenta of, respectively, the final and initial states (the Dirac delta function ensures energy and momentum conservation), and finally $\mathcal{M}$ is known as the *invariant amplitude* (because it is Lorentz-invariant) and it depends on the particular process studied.

One finds out that, for processes involving a total number N of particles, $\mathcal{M}$ can be calculated from the N-points Green's function, that is usually determined perturbatively. Moreover, the Gell-Man–Low formula allows one to write it explicitly in terms of the elements (fields and physical states) of the free theory, that can be solved analytically and is therefore completely known.

Ultimately, it is natural to write $\mathcal{M}$ as a perturbative series. The problem, however, arises now. In fact, almost all orders above the first one of the perturbation series are divergent. And that is not all: the greater the perturbative order, the greater the order of divergence! In practice, those terms that should be negligible are "ever more infinite". This fact has extremely serious consequences: since the S matrix is proportional to $\mathcal{M}$ and $\mathcal{M}$ is infinite, the entries of S turn out to be infinite as well, with a clear breach of probability conservation. To solve this problem the theory of renormalization was born.

The source of the problem can be traced back to the fact that the objects we are handling are not functions, but distributions: it is therefore a mathematical issue that cannot and must not have, in practice, repercussions on physics. The strategy adopted to solve this issue is based on a very simple idea, that has long been considered unsatisfactory or even unacceptable.

The idea in question is to "move the problem where it can not do harm". Some divergent but non-physical (non-measurable) constants are defined and introduced into the Lagrangian so as to cancel exactly the infinities that arise by calculating the divergent terms of the perturbative expansion, so that observable physical quantities are in fact finite (and possibly in agreement with experiments). Theories that only need a finite number of these divergent constants are called *renormalizable theories*. The reason why non-renormalizable theories are useless is that such theories require the measurement of any possible observable in order to define the appropriate renormalization term that allows that observable to have the measured value: in other words, such theories cannot predict the results of any experiment before measuring them. It is therefore compelling to find criteria to determine whether a particular theory is renormalizable or not.

It was found that a necessary condition is that no terms with coefficients with negative mass dimension appear in the Lagrangian, which means that the various terms, deprived of their constant coefficient, must have a mass dimension that is at most the one of the space-time in which the theory is located. Henceforth, the space-time dimension will be considered 4, so that renormalizable theories are the ones whose Lagrangian terms (deprived of their numerical coefficients) have mass dimension less than or equal to 4.

It was also found that a sufficient condition for theories that involve fields that describe particles with spin greater than $\frac{1}{2}$ to be renormalizable is that those fields are massless and arise from a very particular kind of symmetry called gauge symmetry.

2.4 Gauge Theories

Gauge theories are theories with a particular kind of symmetry, called *gauge symmetry*. Before we deal with gauge symmetries, we will speak of symmetries in general in Lagrangian theories. Roughly speaking, a symmetry is a transformation of the theory that leaves the physical phenomena invariant. This tentative definition has different implications for classical and quantum theories, so we will study symmetries in both cases, starting with classical ones.

A classical symmetry is a transformation (executed, as it is said technically, on-shell, i.e. assuming the field equations to be valid) of the fields that make up the Lagrangian, that leaves the action unchanged. In formulas, if we call $\mathcal{L}$ the Lagrangian and S the action, and let δX denote the variation of the quantity X, then a symmetry is a transformation for which:

$$\delta S = 0. \tag{2.29}$$

We recall that the action is the integral of the Lagrangian, so this condition affects the Lagrangian as well:

$$S = \int \mathrm{d}^4x\, \mathcal{L} \quad \Rightarrow \quad \delta S = \int \mathrm{d}^4x\, \delta\mathcal{L} \quad \Rightarrow \quad \delta S = 0 \Leftrightarrow \delta\mathcal{L} = \partial_\mu K^\mu, \tag{2.30}$$

in other words a symmetry of the theory is a transformation that changes the Lagrangian by a four-divergence. Symmetries are extremely important in physics: indeed, there is the Noether theorem, which ensures that each symmetry corresponds to a conserved current called *Noether current*.

Let us now consider a particular case of symmetry, i.e. a symmetry that depends on a parameter α in the following manner:

$$\phi\left(x\right) \to \phi\left(x\right) + \alpha\delta\phi\left(x\right), \tag{2.31}$$

for some field ϕ and with α a real constant. Since this is a symmetry, and therefore the equations governing the theory remain the same after the transformation, it is evident that such a transformation has no physical significance: the predictions of the theory are the same for any value of the parameter α. This means, on the other hand, that it is possible to make such transformations with an appropriate choice of the parameter α in order to make the calculations easier: this process of choosing a particular value of α takes the name of *gauge fixing*, and the symmetry is called a *global gauge symmetry* or, more simply, *global symmetry*.

However, the class of certainly renormalizable theories is much wider than the simple global gauge theories; in particular, there is no reason to require the parameter α to be constant:

$$\phi\left(x\right) \to \phi\left(x\right) + \alpha\left(x\right)\delta\phi\left(x\right). \tag{2.32}$$

In this case, the symmetry is called a *local* or *gauge symmetry*, and theories in which the symmetry is realized in this way are called *gauge theories*. These theories have an additional complication over theories with global symmetries, and this complication has profound repercussions, such as the very existence of gauge bosons and, ultimately, of the fundamental interactions.

The complication arises when considering the field derivatives. The existence of field derivatives within the Lagrangian is necessary, since in their absence all field equations would be simple algebraic equations rather than differential ones, which would lead to a universe unable to evolve: not very interesting. Now, when a local gauge transformation is performed and the field derivative is considered, it is immediately apparent that additional terms appear from the derivative of the gauge parameter.

Since in the SM there are only (with one exception) fermions and gauge bosons, and since its gauge symmetry group is essentially a product of $SU(N)$ with various N, we will narrow down our dissertation to the case of Yang-Mills theories (which are theories with gauge symmetry group $SU(N)$).

Let us consider a set ψ_i of N fermion fields. Then, the most general Lagrangian that is Lorentz-invariant, renormalizable and invariant for global transformations of $SU(N)$:

$$\begin{cases} \psi_i(x) \to U_{ij}\psi_j(x) \\ \psi_i^\dagger(x) \to \psi_j^\dagger(x) U_{ji}^\dagger \end{cases}, \tag{2.33}$$

where $U \in SU(N)$, is the following:

$$\mathcal{L} = i\overline{\psi_i}\partial\!\!\!/\psi_i - m\overline{\psi_i}\psi_i. \tag{2.34}$$

For convenience, we write U in the following form:

$$U = e^{ig\alpha^a T^a}, \tag{2.35}$$

where the T^a are the generators of the algebra of $SU(N)$ and g is a constant, that later we will call *coupling constant*, while the various α^a are the gauge parameters.

Now let us promote the global gauge symmetry to a local one, making the parameters α^a point-dependent:

$$\alpha^a \to \alpha^a(x). \tag{2.36}$$

Now, it is evident that the first term of the Lagrangian (the one with the derivative, called *propagator*) is not gauge-invariant any more:

$$i\overline{\psi_i}\partial\!\!\!/\psi_i \to i\overline{\psi_j}U_{ji}^\dagger\partial\!\!\!/(U_{ik}\psi_k) = i\overline{\psi_i}\partial\!\!\!/\psi_i + i\overline{\psi_j}U_{ji}^\dagger\left(\partial\!\!\!/U_{ik}\right)\psi_k. \tag{2.37}$$

Therefore we need to define a *covariant derivative*, i.e. a modified derivative D that can be used to build a gauge invariant propagator. In particular, if it were true that:

$$D\!\!\!\!/\,\psi_i(x) \to U_{ij}(x)\, D\!\!\!\!/\,\psi_j(x), \tag{2.38}$$

then the term $i\overline{\psi_i}D\!\!\!\!/\,\psi_i$ would be gauge invariant. Then, let us look for a covariant derivative D, that transforms as just said, with form:

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig(A_\mu)_{ij}(x), \qquad (A_\mu)_{ij}(x) = A_\mu^a(x)\, T_{ij}^a. \tag{2.39}$$

Let us find now which gauge transformation law must the fields $A^a_\mu(x)$ satisfy in order for D to be a covariant derivative. For simplicity, we will adopt the matrix notation; we will also avoid writing explicitly the dependence on the point, remembering here that the only objects that depend on it are the vector fields A_μ and the fermion fields ψ:

$$D_\mu\psi = (\partial_\mu - igA_\mu)\,\psi \rightarrow \left(\partial_\mu - igA'_\mu\right) U\psi = \partial_\mu U\psi + U\partial_\mu\psi - igA'_\mu U\psi. \quad (2.40)$$

On the other hand, it has to be true that:

$$D_\mu\psi \rightarrow UD_\mu\psi = U\partial_\mu\psi - igUA_\mu\psi. \quad (2.41)$$

Hence:

$$\partial_\mu U\psi + \cancel{U\partial_\mu\psi} - igA'_\mu U\psi = \cancel{U\partial_\mu\psi} - igUA_\mu\psi \quad (2.42)$$

$$igA'_\mu U\psi = \partial_\mu U\psi + igUA_\mu\psi. \quad (2.43)$$

The last equation is true for every ψ, therefore:

$$igA'_\mu U = \partial_\mu U + igUA_\mu, \quad (2.44)$$

and multiplying from the right by $U^\dagger\,(ig)^{-1}$ we have:

$$A'_\mu = \frac{1}{ig}\partial_\mu UU^\dagger + UA_\mu U^\dagger. \quad (2.45)$$

We can rewrite this last equation in a more elegant way by observing that $UU^\dagger = \mathbb{I}$, therefore $\partial_\mu\left(UU^\dagger\right) = 0$, so $\partial_\mu UU^\dagger = -U\partial_\mu U^\dagger$, hence:

$$A'_\mu = U\left(A_\mu - \frac{1}{ig}\partial_\mu\right)U^\dagger, \quad (2.46)$$

which is the transformation law for A_μ we were looking for, that is, the one which cancels exactly the additional term that arose from the not any more trivial derivative of U.

In summary, we found that the modified Lagrangian

$$\mathcal{L} = i\overline{\psi_i}\not{D}\psi_i - m\overline{\psi_i}\psi_i, \quad (2.47)$$

in which the covariant derivative substitutes the usual partial derivative, has the gauge symmetry realised by the following transformation:

$$\begin{cases} \psi\,(x) \rightarrow U\psi\,(x) \\ \psi^\dagger\,(x) \rightarrow \psi^\dagger\,(x)\,U^\dagger \\ A_\mu \rightarrow U\left(A_\mu - \frac{1}{ig}\partial_\mu\right)U^\dagger \end{cases}, \quad (2.48)$$

where $U = e^{ig\alpha^a T^a}$. Applying Noether's theorem to this symmetry, it is found that for each generator T^a there is a Noether current J^a_μ of form:

$$J^a_\mu = -g\overline{\psi_i}\gamma_\mu T^a_{ij}\psi_j. \tag{2.49}$$

We notice that the introduction of the fields A_μ (called *connections*) was necessary to restore the gauge symmetry in the local case: but each field corresponds to one kind of particles, so we found out that the existence of vector fields (and ultimately of the fundamental interactions, of which the vector bosons are mediators through the couplings $\overline{\psi_i}\gamma_\mu A^\mu \psi_i$ arisen from the covariant derivative) is a necessary consequence of the symmetries of the theory.

Since we introduced new fields – new particles – into our Lagrangian, we have to introduce their propagator, too. Emulating what already known from the electromagnetic theory (which is a gauge theory based on the abelian group $U(1)$), we define a tensor in the following way:

$$F_{\mu\nu} \coloneqq -\frac{1}{ig}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right], \tag{2.50}$$

whose difference from the electromagnetic field tensor is that this time the A_μ are matrices, therefore they do not commute and they give rise to 3- and 4-body self-interaction terms.

With this definition of $F_{\mu\nu}$, the propagator for the fields A_μ is

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{2.51}$$

so the full Lagrangian of the Yang-Mills theory is

$$\mathcal{L} = i\overline{\psi}\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a. \tag{2.52}$$

So far, we described classical symmetries, i.e. symmetries of the classical action. However, in a quantum environment, symmetries of the classical action do not necessarily translate into symmetries of the quantum theory. Indeed, the fundamental object of quantum theories is not the classical action, but instead the *quantum* action, and even if the two actions are in fact related, they do not necessarily have the same symmetries. This is clear if we check what happens in Eq. (2.26) when we perform a classical symmetry transformation:

$$\phi \to \phi' = \phi + \Delta\phi \qquad \text{with} \qquad S\left(\phi'\right) = S\left(\phi\right), \tag{2.53}$$

then

$$e^{i\Gamma[\Phi]} = \int_{\text{1PI}} \mathrm{D}\phi\, e^{iS(\phi+\Phi)} \to \qquad \text{(definition of quantum action)} \tag{2.54}$$

$$\to \int_{1\mathrm{PI}} \mathrm{D}\phi'\, e^{iS(\phi'+\Phi)} = \qquad \text{(transformation of } \phi) \tag{2.55}$$

$$= \int_{1\mathrm{PI}} \mathrm{D}\phi'\, e^{iS(\phi+\Phi)} = \qquad (\phi \to \phi' \text{ is a classical symmetry}) \tag{2.56}$$

$$= \int_{1\mathrm{PI}} \mathcal{J}\, \mathrm{D}\phi\, e^{iS(\phi+\Phi)}, \qquad \text{(Jacobian of the measure)} \tag{2.57}$$

where $\mathcal{J}$ is the Jacobian of the field transformation:

$$\mathcal{J} = \left| \frac{\mathrm{D}\phi'(x)}{\mathrm{D}\phi(y)} \right|. \tag{2.58}$$

If the Jacobian is equal to 1, then the classical symmetry is also a symmetry of the quantum action. Vice versa, if the Jacobian is different from 1, then the quantum action changes, even if the classical action does not. In this case, the classical symmetry is said to be an *anomalous symmetry* of the quantum theory. This nomenclature must not mislead the reader: even if it is called an "anomalous *symmetry*", in fact it is **not** a symmetry of the quantum theory! The term "anomalous symmetry" simply means that the transformation is a symmetry for the classical action.

In general, reading Eq. (2.57), we can deduce that a symmetry of the quantum theory (or, quantum symmetry, as opposed to classical symmetries) is a transformation of the fields that leaves the path integral invariant, i.e. such that

$$\mathrm{D}\phi'\, e^{iS(\phi')} = \mathrm{D}\phi\, e^{iS(\phi)}. \tag{2.59}$$

This means that, in general, a quantum symmetry need not correspond to a classical symmetry: even if both the classical action and the integration measure change, the transformation may still be a symmetry if the two changes compensate each other. Nevertheless, in practice one usually finds classical symmetries, and then checks whether they are anomalous or instead true quantum symmetries.

If a global symmetry is anomalous, it simply means that it is not a symmetry at the quantum level, but it is still an approximate symmetry of the theory. On the other hand, if local symmetries are anomalous, then the quantum theory is inconsistent. In fact, a gauge theory is a theory that describes its physics in a redundant way: the gauge symmetry is the reflection of this redundancy. In turn, this redundancy implies the existence of many unphysical states in the Hilbert space of the theory. For a gauge theory to be consistent, obviously the unphysical states must be decoupled from the physical ones. However, if the gauge symmetry is anomalous, the anomaly prevents the unphysical states from decoupling from the physical ones, thus making the theory inconsistent.

The SM is of course gauge anomaly-free (otherwise we wouldn't still be dealing with an inconsistent theory!). The cancellation of anomalies occurs within each generation of fermions. This has been historically important after the discovery of the τ lepton: the existence of a third generation of leptons implied in turn the existence of a third generation of quarks, since the cancellation of the gauge anomalies requires the presence of an equal number of quarks and leptons. This lead to the prediction of the top and bottom quarks, which were both found experimentally some time later.

2.5 Weak Interactions

After dealing with the Yang-Mills theories, we can finally dedicate ourselves to the proper Standard Model, and more specifically to the theory of electroweak interactions.

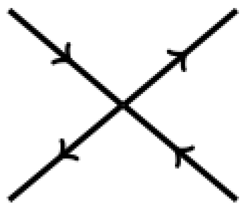
Weak interactions can be described with great approximation by Fermi's theory, described by the following Lagrangian:

$$\mathcal{L}_F = -\frac{G_F}{\sqrt{2}}\overline{p}\gamma_\rho\left(1 - a\gamma_5\right)n\overline{e}\gamma^\rho\left(1-\gamma_5\right)\nu_e - \frac{G_F}{\sqrt{2}}\overline{\nu_\mu}\gamma_\rho\left(1-\gamma_5\right)\mu\overline{e}\gamma^\rho\left(1-\gamma_5\right)\nu_e, \tag{2.60}$$

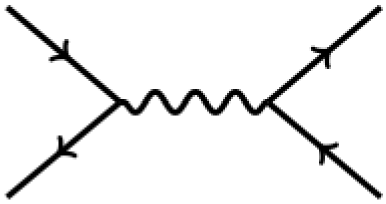
where $G_F \simeq 1.17 \cdot 10^{-5}$ GeV^{-2} is the Fermi constant, p, n, e, μ, ν_e and ν_μ are respectively the proton, neutron, electron, muon, electronic neutrino and muon neutrino fields, and $a \simeq 1.27$ is a numerical coefficient that arises from the fact that protons and neutrons also interact via the strong interaction.

The main problem of this theory is clear: the Fermi constant has mass dimension -2, but a theory is (power-counting) renormalizable if the coefficients of all Lagrangian terms have mass dimension $\alpha \geq 0$. In other words, the theory is not renormalizable, i.e. it cannot calculate the higher order corrections in perturbation theory consistently with probability conservation.

The problem arises from the fact that there are four-fermion interaction terms that correspond to Feynman diagrams of the type:



where each continuous oriented line represents a fermion (or an antifermion). But this is clearly impossible: fermions have mass dimension $3/2$, so no Lagrangian term with more than 2 fermions can be renormalizable (the Lagrangian has mass dimension 4). Again, it is appropriate to seek inspiration from electromagnetism, in which this kind of diagrams appear:



(where the wavy line represents a vector boson, in this case a photon) as order 2 terms, i.e. as terms in which the interaction can be interpreted as if it were in two steps: two fermions interact with the electromagnetic field, emitting a photon; then that same photon interacts in turn with two other fermions, from which it is absorbed.

Generally speaking, once the diagram is converted to mathematical formulae, all the wavy internal lines carry a factor $\frac{1}{k^2-M^2+i\varepsilon}$, where k is the four-momentum of the vector boson, M is its mass and ε is a real infinitesimal quantity, that is introduced in order to make the denominator non-singular. In this case, if M^2 is much bigger than k^2 (that for beta decay is equal to the squared mass of the decaying particle because of the conservation of the four-momentum), the contribution of the vector boson becomes simply the constant $\frac{1}{M^2}$. In the language of the Feynman diagrams, all this means that the wavy line becomes negligible and "narrows" to a single point: the 4-fermion interaction vertex.

This indicates that Fermi's interaction can be thought of as the low energy limit of an order 2 interaction, where the Fermi constant has the dimension of the inverse of a squared mass because it encompasses the contribution $\frac{1}{M^2}$ of the exchanged vector boson. Lagrangian terms of this type, though unsatisfactory at theoretical level, are very useful for practical purposes when it is sufficient to work at tree-level and are called *effective operators*. We will discuss again effective operators in Chapter 3 and see how these are, in a sense, useful tools also at theoretical level.

For now, let us try to build the renormalizable theory that has Fermi's theory as a low energy limit. For simplicity, we will focus on the term describing the muon decay, which has no contributions due to strong interactions. We observe firstly that the Lagrangian can be rewritten in the form:

$$\mathcal{L}_F = -\frac{4G_F}{\sqrt{2}} J_\rho J'^\rho, \tag{2.61}$$

where

$$J^\rho = \overline{\nu_\mu}\gamma^\rho P_L \mu, \tag{2.62}$$

$$J'^{\rho} = \bar{e}\gamma^{\rho} P_L \nu_e \tag{2.63}$$

are currents and the projector $P_L = \frac{1-\gamma_5}{2}$ returns the left-handed chiral components of the fields on which it acts. We would like to rewrite these currents in the form of Noether currents $\overline{\psi_i}\gamma^{\rho} T^a_{ij}\psi_j$, therefore, called τ_i the Pauli's matrices, we define the following objects:

$$\ell^f_L := \begin{pmatrix} P_L \nu^f \\ P_L l^f \end{pmatrix} \equiv \begin{pmatrix} \nu^f_L \\ l^f_L \end{pmatrix} \tag{2.64}$$

$$\tau^+ := \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.65}$$

$$\tau^- := \frac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.66}$$

where f is an index that specifies the lepton flavour, and ν^f and l^f are respectively the neutrino and the charged lepton with flavour f. With these objects we can build, for every flavour, the current $\overline{\ell_L}\gamma^{\rho}\tau^+\ell_L$, which is precisely equal to J^{ρ}:

$$\overline{\ell_L}\gamma^{\rho}\tau^+\ell_L = \begin{pmatrix} \overline{\nu_L} & \overline{l_L} \end{pmatrix} \gamma^{\rho} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ l_L \end{pmatrix} = \begin{pmatrix} \overline{\nu_L} & \overline{l_L} \end{pmatrix} \gamma^{\rho} \begin{pmatrix} l_L \\ 0 \end{pmatrix} = \overline{\nu_L}\gamma^{\rho} l_L = J^{\rho}. \tag{2.67}$$

We can evaluate $J^{\dagger\rho}$:

$$J^{\dagger\rho} = \overline{\ell_L}\gamma^{\rho}\tau^-\ell_L, \tag{2.68}$$

which is still a Noether current. In gauge theories there is one Noether current for each generator of the gauge group algebra, but $[\tau^+, \tau^-]$ is not proportional neither to τ^+ nor to τ^-, therefore the algebra generated by those generators only is not closed. For this reason, if we suppose that the weak interaction can be described by a gauge theory, we expect the existence of at least another current J_3:

$$J^{\rho}_3 = \overline{\ell_L}\gamma^{\rho}\left[\tau^+, \tau^-\right]\ell_L = \overline{\ell_L}\gamma^{\rho}\tau_3\ell_L = \overline{\nu_L}\gamma^{\rho}\nu_L - \overline{l_L}\gamma^{\rho} l_L. \tag{2.69}$$

We observe that J_3 is a neutral current (it couples particles with the same charge), unlike J and $J^{\dagger}$ that are charged currents (they couple particles with different charges). There are no more Noether currents, since $[\tau^{\pm}, \tau_3] \propto \tau^{\pm}$ and hence the algebra is closed.

In summary, we found that the weak interaction can be described by a gauge theory with $SU(2)$ as symmetry group, where the doublets ℓ^f_L transform with the fundamental representation of $SU(2)$, while the right-handed components of the fields (that do not have weak interactions) transform in the trivial representation (i.e. they do not transform at all). This implies the existence of 3 gauge bosons (one for each generator of the group algebra) that permit to build a covariant derivative. These bosons are denoted $W^{1,2,3}$, but usually their linear combinations

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}} \left(W^1_{\mu} \mp i W^2_{\mu} \right) , \tag{2.70}$$

that correspond to particles with definite charge, are used. On the contrary, W_3 already has definite charge, in particular it is a neutral particle. For this reason, it is legitimate to wonder whether it has some kind of relationship with the photon, and indeed this is exactly the case. Obviously, they cannot be the same particle, since the Ws only interact with left-handed particles while electromagnetism is completely indifferent to particle chirality. However, what is found is that W_3 is actually a component of the photon.

2.6 Electroweak Unification

Let us suppose weak and electromagnetic interactions can be described together by the $SU(2)_L \otimes U(1)_Y$ gauge group, where the subscripts L and Y refer to the fact that weak interactions regard only left-handed particles, while $U(1)$ is connected to a new quantum number called hypercharge, which is usually denoted Y. In this case, the gauge transformation takes the form:

$$\begin{cases} \ell_L \to e^{ig\alpha^a \frac{\tau^a}{2}} e^{ig'\beta \frac{Y_L}{2}} \ell_L \\ l_R \to e^{ig'\beta \frac{Y_{l_R}}{2}} l_R \\ \nu_R \to e^{ig'\beta \frac{Y_{\nu_R}}{2}} \nu_R \end{cases} , \tag{2.71}$$

which implies immediately that the hypercharge must be different from the electrical charge. In fact, Y_L must commute with all generators of $SU(2)$, therefore it must be proportional to the identity, which means it has to have the same value for both components of the doublet ℓ_L, that have different electrical charge.

Called B_μ the gauge boson related to $U(1)$, let us write the Lagrangian term related to the interactions between B_μ and W^3_μ with leptons (the so called *neutral current Lagrangian* $\mathcal{L}_{NC}$):

$$\begin{aligned} \mathcal{L}_{NC} = g W^3_\mu \overline{\ell_L} \gamma^\mu \frac{\tau_3}{2} \ell_L + g' B_\mu \left[\overline{\ell_L} \gamma^\mu \frac{Y_L}{2} \ell_L + \overline{l_R} \gamma^\mu \frac{Y_{l_R}}{2} l_R + \overline{\nu_R} \gamma^\mu \frac{Y_{\nu_R}}{2} \nu_R \right] = \\ = \frac{g}{2} W^3_\mu \left[\overline{\nu_L} \gamma^\mu \nu_L - \overline{l_L} \gamma^\mu l_L \right] + \frac{g'}{2} B_\mu \left[Y_L \left(\overline{l_L} \gamma^\mu l_L + \overline{\nu_L} \gamma^\mu \nu_L \right) + \right. \\ \left. + Y_{l_R} \overline{l_R} \gamma^\mu l_R + Y_{\nu_R} \overline{\nu_R} \gamma^\mu \nu_R \right] . \end{aligned} \tag{2.72}$$

We define the following quantities:

$$\Psi := \begin{pmatrix} \nu_L \\ l_L \\ \nu_R \\ l_R \end{pmatrix}, \qquad T^3 := \begin{pmatrix} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \tag{2.73}$$

$$Y := \begin{pmatrix} Y_L & & & \\ & Y_L & & \\ & & Y_{l_R} & \\ & & & Y_{\nu_R} \end{pmatrix}, \qquad Q := \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \tag{2.74}$$

$$\begin{cases} B_\mu =: A_\mu \cos\theta_W - Z_\mu \sin\theta_W \\ W^3_\mu =: A_\mu \sin\theta_W + Z_\mu \cos\theta_W \end{cases}. \tag{2.75}$$

The new fields A_μ and Z_μ (where we would like to identify A_μ with the photon) were defined by means of a rotation so as to leave the kinetic term of the Lagrangian unchanged. With these quantities, we can rewrite $\mathcal{L}_{NC}$ as follows:

$$\begin{aligned} \mathcal{L}_{NC} &= \overline{\Psi}\gamma^\mu \left[g W^3_\mu T^3 + g' B_\mu \frac{Y}{2} \right] \Psi = \\ &= \overline{\Psi}\gamma^\mu \left[g \left(A_\mu \sin\theta_W + Z_\mu \cos\theta_W \right) T^3 + g' \left(A_\mu \cos\theta_W - Z_\mu \sin\theta_W \right) \frac{Y}{2} \right] \Psi = \\ &= \overline{\Psi} \not{A} \left(g \sin\theta_W T^3 + g' \cos\theta_W \frac{Y}{2} \right) \Psi + \overline{\Psi} \not{Z} \left(g \cos\theta_W T^3 - g' \sin\theta_W \frac{Y}{2} \right) \Psi, \end{aligned} \tag{2.76}$$

where we would like to identify $\overline{\Psi}\gamma^\mu \left(g \sin\theta_W T^3 + g' \cos\theta_W \frac{Y}{2} \right) \Psi$ with the electromagnetic current $J^\mu_{EM} = e\overline{\Psi}\gamma^\mu Q \Psi$ (where in this case e is the proton charge). Since g' and Y only appear in the Lagrangian multiplied by each other, we can arbitrarily fix one of the entries of Y, since rescaling Y is equivalent to rescaling g'; therefore, we fix $Y_L = -1$. Then:

$$\begin{aligned} &\overline{\Psi}\gamma^\mu \left(g \sin\theta_W T^3 + g' \cos\theta_W \frac{Y}{2} \right) \Psi = J^\mu_{EM} = e\overline{\Psi}\gamma^\mu Q \Psi \\ &\Rightarrow \left(g \sin\theta_W T^3 + g' \cos\theta_W \frac{Y}{2} \right) = eQ \\ &\Rightarrow \begin{cases} \frac{1}{2} g \sin\theta_W - \frac{1}{2} g' \cos\theta_W = 0 \\ -\frac{1}{2} g \sin\theta_W - \frac{1}{2} g' \cos\theta_W = -e \end{cases} \\ &\Rightarrow g \sin\theta_W = g' \cos\theta_W = e. \end{aligned} \tag{2.77}$$

Therefore, considering again the second equation:

$$\left(g \sin\theta_W T^3 + g' \cos\theta_W \frac{Y}{2} \right) = eQ \qquad \Rightarrow \qquad T^3 + \frac{Y}{2} = Q, \tag{2.78}$$

hence we can find the hypercharge for the right-handed lepton components:

$$\begin{aligned} Y_{l_R} &= 2\left(Q - T^3\right)_{l_R} = -2, \\ Y_{\nu_R} &= 2\left(Q - T^3\right)_{\nu_R} = 0. \end{aligned} \tag{2.79}$$

Similarly, a "Z-charge" can be defined:

$$\begin{aligned} eQ_Z &:= g\cos\theta_W T^3 - g'\sin\theta_W \frac{Y}{2} \\ &= \frac{e}{\sin\theta_W}\cos\theta_W T^3 - \frac{e}{\cos\theta_W}\sin\theta_W\left(Q - T^3\right) \\ eQ_Z &= \frac{e}{\sin\theta_W\cos\theta_W}\left(T^3 - \sin^2\theta_W Q\right). \end{aligned} \tag{2.80}$$

2.7 Weak Interactions for Quarks

So far, we have only talked about leptons: it is now time to include quarks as well in our theory. At quark level, the beta decay and the decay of strange particles are regulated by the current:

$$J^\mu_u = \cos\theta_c \overline{u}\gamma^\mu \frac{1-\gamma_5}{2} d + \sin\theta_c \overline{u}\gamma^\mu \frac{1-\gamma_5}{2} s, \tag{2.81}$$

where $\theta_c \simeq 13°$ is the Cabibbo angle and u, d and s are the quark up, down and strange fields.

At this point, proceeding as in the case of leptons leads to results extremely conflicting with experimental data. Indeed, if we define:

$$q'_L := \frac{1-\gamma_5}{2}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}, \quad T'^+ := \begin{pmatrix} 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \left(T'^-\right)^\dagger \tag{2.82}$$

so that it is:

$$J^\mu = \overline{q'_L}\gamma^\mu T'^+ q'_L, \tag{2.83}$$

we find that the third generator of the algebra induces flavour changing neutral currents (FCNC):

$$T'^3 = \left[T'^+, T'^-\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta_c & -\cos\theta_c\sin\theta_c \\ 0 & -\cos\theta_c\sin\theta_c & -\sin^2\theta_c \end{pmatrix} \tag{2.84}$$

(the off-diagonal terms), that on the contrary are experimentally very suppressed. For example, in the case of the K^+ decay into the two channels

$$K^+ \rightarrow \pi^0 e^+ \nu_e$$

$$K^+ \rightarrow \pi^+ e^+ e^-$$

the ratio of the decay rates would be about

$$r = \left(\frac{\sin\theta_c}{\sin\theta_c \cos\theta_c} \right)^2 \simeq 1.1, \tag{2.85}$$

while the experimental value is

$$r_{exp} \simeq 1.3 \cdot 10^5, \tag{2.86}$$

with an overall error of 5 orders of magnitude.

The problem can be solved by introducing another kind of quark, the charm quark, which is identical to the up quark except for the mass, that contributes to the weak decays through the current:

$$J_c^\mu = -\sin\theta_c \bar{c} \gamma^\mu \frac{1-\gamma_5}{2} d + \cos\theta_c \bar{c} \gamma^\mu \frac{1-\gamma_5}{2} s. \tag{2.87}$$

Let us now define the quantities

$$q_L := \begin{pmatrix} u_L \\ c_L \\ d_L \\ s_L \end{pmatrix}, \qquad T^+ := \begin{pmatrix} 0 & 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & -\sin\theta_c & \cos\theta_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: \left(T^-\right)^\dagger. \tag{2.88}$$

In this way, the complete current can be written as:

$$J^\mu = J_u^\mu + J_c^\mu = \overline{q_L} \gamma^\mu T^+ q_L, \tag{2.89}$$

while the third algebra generator takes the form:

$$T^3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.90}$$

so that the FCNC problem is solved. This mechanism is called *Glashow-Iliopoulos-Maiani* (GIM) *suppression.*

Now we only have to rewrite what we just found in the same forms as the leptons. Then, let us define the quark combinations:

$$\begin{pmatrix} d'_L \\ s'_L \end{pmatrix} = V \begin{pmatrix} d_L \\ s_L \end{pmatrix}, \qquad V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \tag{2.91}$$

that we can group together with the up-type quarks to build $SU(2)$ doublets:

$$q_L^1 = \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \qquad q_L^2 = \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \tag{2.92}$$

in terms of which the current takes the form:

$$J^\mu = \overline{q_L^1}\gamma^\mu T^+ q_L^1 + \overline{q_L^2}\gamma^\mu T^+ q_L^2. \tag{2.93}$$

Now it is easy to assign the quarks their hypercharges:

$$Y_{u_L} = 2\left(Q - T^3\right)_{u_L} = 2\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{1}{3} \tag{2.94}$$

$$Y_{d_L} = 2\left(Q - T^3\right)_{d_L} = 2\left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{3} \tag{2.95}$$

$$Y_{u_R} = 2\left(Q - T^3\right)_{u_R} = 2\left(\frac{2}{3}\right) = \frac{4}{3} \tag{2.96}$$

$$Y_{d_R} = 2\left(Q - T^3\right)_{d_R} = 2\left(-\frac{1}{3}\right) = -\frac{2}{3} \tag{2.97}$$

(and similarly for the other generations' hypercharges).

In fact, today we know that there are more than just two quark families. With n generations, V becomes an $n \times n$ unitary matrix (so that it can guarantee the absence of FCNC). To date we know 3 quark families, so we believe V to be a 3×3 matrix, called *Cabibbo-Kobayashi-Maskawa* (CKM) *matrix*.

Ultimately, the complete weak interactions Lagrangian is:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{CC} + \mathcal{L}_{NC} + \mathcal{L}_{YM}, \tag{2.98}$$

with:

$$\begin{aligned}
\mathcal{L}_0 &= i\overline{\ell_L^f}\not{\partial}\ell_L^f + i\overline{\nu_R^f}\not{\partial}\nu_R^f + i\overline{l_R^f}\not{\partial}l_R^f + \\
&\quad + i\overline{q_L^f}\not{\partial}q_L^f + i\overline{u_R^f}\not{\partial}u_R^f + i\overline{d_R^f}\not{\partial}d_R^f && (2.99)\\
\mathcal{L}_{CC} &= \frac{g}{\sqrt{2}}\left(\sum_{f=1}^{n}\overline{\nu}_L^f\gamma^\mu l_L^f + \sum_{f,g=1}^{n}\overline{u}_L^f\gamma^\mu V_{fg} d_L^g\right) W_\mu^+ + \text{h.c.} && (2.100)\\
\mathcal{L}_{NC} &= e\overline{\Psi}\gamma^\mu Q\Psi A_\mu + e\overline{\Psi}\gamma^\mu Q_Z\Psi Z_\mu && (2.101)\\
\mathcal{L}_{YM} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^+ W_-^{\mu\nu} \\
&\quad + ig\sin\theta_W\left(W_{\mu\nu}^+ W_-^\mu A^\nu - W_{\mu\nu}^- W_+^\mu A^\nu + F_{\mu\nu}W_+^\mu W_-^\nu\right) \\
&\quad + ig\sin\theta_W\left(W_{\mu\nu}^+ W_-^\mu Z^\nu - W_{\mu\nu}^- W_+^\mu Z^\nu + Z_{\mu\nu}W_+^\mu W_-^\nu\right)
\end{aligned}$$

$$+\frac{g^2}{2}\left(2g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)\left[\frac{1}{2}W_\mu^+W_\nu^+W_\rho^-W_\sigma^-\right.$$
$$-W_\mu^+W_\nu^-\left(A_\rho A_\sigma\sin^2\theta_W+Z_\rho Z_\sigma\cos^2\theta_W\right.$$
$$\left.\left.+2A_\rho Z_\sigma\sin\theta_W\cos\theta_W\right)\right], \tag{2.102}$$

where now Ψ also includes left- and right-handed quark fields and where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.103}$$
$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{2.104}$$
$$W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm \tag{2.105}$$

We observe an interesting and, in some ways, worrying fact: not only there are no mass terms for any of the present particles, but also they are prohibited from the symmetries of the theory. In fact, gauge bosons have to be massless (eventual mass terms would violate the gauge symmetry), while fermion masses are prohibited by the fact that the Dirac mass term for a fermion ψ

$$-m\overline{\psi}\psi = -m\left(\overline{\psi_L}\psi_R+\overline{\psi_R}\psi_L\right) \tag{2.106}$$

is not invariant under chiral transformations (i.e. a transformation that acts differently on left-handed and right-handed components) like $SU(2)$ that describes the weak interactions. This seems to invalidate everything we have built so far. In fact, there is a way to solve this incompatibility: it is the *Higgs mechanism*.

2.8 The Higgs Mechanism

Let us suppose that in a certain gauge theory there is a complex scalar field, whose Lagrangian is symmetrical under $U(1)$ global transformations. Then, assuming the renormalizability of the theory, the Lagrangian of this field must necessarily be:

$$\mathcal{L}_\phi = D_\mu\phi^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2, \tag{2.107}$$

where D_μ is the covariant derivative of the theory. According to what we have done so far, we could assume that ϕ transforms under local gauge transformations in the usual way:

$$\phi(x) \to e^{ig\alpha(x)}\phi(x) \tag{2.108}$$

(called *Wigner-Weyl symmetry*) as it happens in Yang-Mills theories. However, since ϕ is a scalar field, this is not the most general possible gauge transformation for ϕ. In fact, since constants are themselves Lorentz scalars, we can assume that ϕ transforms in the following way:

$$\phi(x) \to e^{ig\alpha(x)}\left[\phi(x)+v\right]-v \tag{2.109}$$

(or, equivalently, $(\phi(x) + v) \rightarrow e^{ig\alpha(x)}(\phi(x) + v)$), where v is a constant. In this case, the symmetry is said a *Nambu-Goldstone symmetry*, or it is said to be *spontaneously broken*. In order to understand what this means, we need to study the potential for ϕ:

$$V = m^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2. \tag{2.110}$$

Let us search for the fundamental state for ϕ by finding the minimum of V:

$$\begin{aligned} V'\left(\phi^\dagger_{min}\phi_{min}\right) &= m^2 + 2\lambda\phi^\dagger_{min}\phi_{min} = 0 \\ \Rightarrow \phi^\dagger_{min}\phi_{min} &= -\frac{m^2}{2\lambda}. \end{aligned} \tag{2.111}$$

Now, even if we assume that m^2 can be negative, we find that there are infinite different fundamental states, all of them $\sqrt{-\frac{m^2}{2\lambda}}$ far away from the origin, only differing through an arbitrary phase. When nature chooses, for any reason, one of these fundamental states, experimental observations cease to be $U(1)$ invariant (since one precise choice has been made), however the starting Lagrangian still does remain $U(1)$ invariant. In this sense, the symmetry is said spontaneously broken.

Let us see now how the introduction in the electroweak Lagrangian of a scalar field that transforms in a Nambu-Goldstone way can generate mass terms for every particle. The electroweak symmetry group is not just $U(1)$, but $SU(2) \otimes SU(1)$, therefore we have to assume that ϕ is actually a scalar field doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{2.112}$$

of which we have to determine the hypercharge, that transforms under gauge transformations in the following way:

$$\phi \rightarrow e^{ig\alpha^i(x)\frac{\tau_i}{2}} e^{ig'\beta(x)\frac{Y_\phi}{2}}\left[\phi + \frac{v}{\sqrt{2}}\right] - \frac{v}{\sqrt{2}}, \tag{2.113}$$

where v is obviously a constant doublet:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.114}$$

Moreover, we want the photon to remain massless as experimentally observed, which corresponds to imposing that the electromagnetic $U(1)_{EM}$ symmetry is a Wigner-Weyl symmetry:

$$\phi \rightarrow e^{ieQ\gamma(x)}\phi \qquad \text{under } U(1)_{EM}, \tag{2.115}$$

where $Q = T^3 + \frac{Y}{2}$ as found previously. But this means that:

$$\begin{aligned}
&e^{ieQ\gamma(x)}\left[\phi + \frac{v}{\sqrt{2}}\right] - \frac{v}{\sqrt{2}} = e^{ieQ\gamma(x)}\phi \\
&\Rightarrow e^{ieQ\gamma(x)}\frac{v}{\sqrt{2}} = \frac{v}{\sqrt{2}} \\
&Qv = 0,
\end{aligned} \tag{2.116}$$

which implies:

$$\begin{aligned}
&\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Rightarrow \begin{cases} Q_1 v_1 = 0 \\ Q_2 v_2 = 0 \end{cases} \\
&\Rightarrow \begin{cases} \left(\frac{1}{2} + \frac{Y_\phi}{2}\right) v_1 = 0 \\ \left(-\frac{1}{2} + \frac{Y_\phi}{2}\right) v_2 = 0 \end{cases} \\
&\Rightarrow \begin{cases} Y_\phi = -1 \\ v_2 = 0 \end{cases} \cup \begin{cases} Y_\phi = +1 \\ v_1 = 0 \end{cases} .
\end{aligned} \tag{2.117}$$

Usually, $Y_\phi = +1$ and $v_1 = 0$ is chosen, which means that

$$Q_1 = 1, \tag{2.118}$$
$$Q_2 = 0 \tag{2.119}$$

therefore from here on with a slight abuse of notation we shall write:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{2.120}$$

Let us now see how the covariant derivative applied to ϕ immediately generates mass terms for the vector bosons, with the only exception of the photon. In order to make the calculations easier, we choose a particular gauge (called *unitary gauge*), in which ϕ takes a particularly simple form:

$$\phi = \begin{pmatrix} 0 \\ \frac{H}{\sqrt{2}} \end{pmatrix}, \tag{2.121}$$

where H is a real scalar field. Then we have:

$$D_\mu\left[\phi + \frac{v}{\sqrt{2}}\right] = \partial_\mu\phi - \left(igW_\mu^i\frac{\tau_i}{2} + ig'B_\mu\frac{Y_\phi}{2}\right)\left[\phi + \frac{v}{\sqrt{2}}\right] =$$

$$= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} - \frac{1}{2} ig \left(H + v \right) \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \right.$$
$$\left. - \frac{1}{2} ig' \left(H + v \right) B_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] =$$
$$= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} - \frac{i}{2} \left(H + v \right) \begin{pmatrix} g \left(W_\mu^1 - iW_\mu^2 \right) \\ -gW_\mu^3 + g'B_\mu \end{pmatrix} \right]. \tag{2.122}$$

Now, if we remember that:

$$W_\mu^+ = \frac{1}{\sqrt{2}} \left(W_\mu^1 - iW_\mu^2 \right) \tag{2.123}$$
$$B_\mu = A_\mu \cos\theta_W - Z_\mu \sin\theta_W \tag{2.124}$$
$$W_\mu^3 = A_\mu \sin\theta_W + Z_\mu \cos\theta_W \tag{2.125}$$
$$g \sin\theta_W = g' \cos\theta_W = e \tag{2.126}$$

we find immediately that:

$$D_\mu \left[\phi + \frac{v}{\sqrt{2}} \right] = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} - \frac{i}{2} \left(H + v \right) \begin{pmatrix} g\sqrt{2}W_\mu^+ \\ -\sqrt{g^2 + g'^2} Z_\mu \end{pmatrix} \right], \tag{2.127}$$

which implies that:

$$\left(D_\mu \left[\phi + \frac{v}{\sqrt{2}} \right] \right)^\dagger D^\mu \left[\phi + \frac{v}{\sqrt{2}} \right] =$$
$$= \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{1}{8} \left(H + v \right)^2 \left(2g^2 W_\mu^- W_+^\mu + \left(g^2 + g'^2 \right) Z_\mu Z^\mu \right) =$$
$$= \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{g^2 v^2}{4} W_\mu^- W_+^\mu + \frac{1}{2} \frac{\left(g^2 + g'^2 \right) v^2}{4} Z_\mu Z^\mu +$$
$$+ \frac{1}{4} \left(H^2 + 2Hv \right) \left(g^2 W_\mu^- W_+^\mu + \frac{g^2 + g'^2}{2} Z_\mu Z^\mu \right). \tag{2.128}$$

In this expression, the first term is the propagator of H, while the two following terms are mass terms for the W and Z vector bosons with masses:

$$m_W^2 = \frac{g^2 v^2}{4}, \qquad m_Z^2 = \frac{\left(g^2 + g'^2 \right) v^2}{4}. \tag{2.129}$$

This result takes the name of *Brout-Englert-Higgs mechanism*, or more briefly *Higgs mechanism*. However, the power of the Higgs mechanism does not end here: in fact, this same scalar field through which we conferred mass to vector bosons (called *Higgs field*) allows us to write mass terms also for fermions.

Let us start with the quarks, and let us consider the following Lagrangian:

$$\mathcal{L}_Y^Q = -\sum_{f,g}\left[\overline{q_L'^f}\left(\phi+\frac{v}{\sqrt{2}}\right)Y'_{D\,fg}d_R'^g + \overline{q_L'^f}\left(\phi+\frac{v}{\sqrt{2}}\right)^c Y'_{U\,fg}u_R'^g\right] + \text{h.c.}, \quad (2.130)$$

with

$$q_L' = \begin{pmatrix} u_L' \\ d_L' \end{pmatrix} \quad (2.131)$$

and where the prime on the quark fields means a linear combination of the definite mass fields, the sum on f and g is over all possible flavours, the matrices in the flavour space $Y'_{U,D}$ are called Yukawa matrices, moreover

$$\left(\phi+\frac{v}{\sqrt{2}}\right)^c := \varepsilon\left(\phi+\frac{v}{\sqrt{2}}\right)^*, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.132)$$

is the charge-conjugated field of the Higgs field, of which, as it can be shown, it shares the gauge transformation properties. It is easy to verify through direct calculation that this Lagrangian is gauge and Lorentz invariant and also renormalizable.

We choose again the unitary gauge, we adopt the matrix notation for the flavour space and we evaluate $\mathcal{L}_Y^Q$ explicitly:

$$\begin{aligned}\mathcal{L}_Y^Q &= -\left[\begin{pmatrix} \overline{u_L'} & \overline{d_L'} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{H+v}{\sqrt{2}} \end{pmatrix} Y_D' d_R' + \begin{pmatrix} \overline{u_L'} & \overline{d_L'} \end{pmatrix}\begin{pmatrix} \frac{H+v}{\sqrt{2}} \\ 0 \end{pmatrix} Y_U' u_R'\right] + \text{h.c.} \\ &= -\frac{1}{\sqrt{2}}(H+v)\left(\overline{d_L'}Y_D'd_R' + \overline{u_L'}Y_U'u_R'\right) + \text{h.c.}\ . \end{aligned} \quad (2.133)$$

In general, the $Y'_{U,D}$ matrices will not be diagonal, but they can always be diagonalised via a biunitary transformation:

$$Y'_{U,D} = U^\dagger_{U,D} Y_{U,D} V_{U,D}, \quad (2.134)$$

with U and V unitary and $Y_{U,D}$ diagonal with semidefinite positive eigenvalues. Thus, if we define:

$$d_L := U_D d_L' \quad (2.135)$$

$$d_R := V_D d_R' \quad (2.136)$$

$$u_L := U_U u_L' \quad (2.137)$$

$$u_R := V_U u_R' \quad (2.138)$$

we can write the Lagrangian in the form:

$$\mathcal{L}_Y^Q = -\frac{1}{\sqrt{2}}(H+v)\left(\overline{d_L'}Y_D'd_R' + \overline{u_L'}Y_U'u_R'\right) + \text{h.c.} =$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{2}}(H+v)\left(\overline{d'_L}U_D^\dagger Y_D V_D d'_R + \overline{u'_L}U_U^\dagger Y_U V_U u'_R\right) + \text{h.c.} = \\
&= -\frac{1}{\sqrt{2}}(H+v)\left(\overline{d_L}Y_D d_R + \overline{u_L}Y_U u_R\right) + \text{h.c.} \qquad (2.139)
\end{aligned}
$$

and since $Y_{U,D}$ are diagonal, we can write:

$$
\begin{aligned}
\mathcal{L}_Y^Q &= -\frac{1}{\sqrt{2}}(H+v)\left(Y_D\overline{d_L}d_R + Y_D\overline{d_R}d_L + Y_U\overline{u_L}u_R + Y_U\overline{u_R}u_L\right) = \\
&= -\frac{1}{\sqrt{2}}(H+v)\left(Y_D\overline{d}d + Y_U\overline{u}u\right). \qquad (2.140)
\end{aligned}
$$

Now, if we consider only the Lagrangian terms proportional to v, we see immediately that they are mass terms for the quarks with masses:

$$
m_d^f = \frac{v}{\sqrt{2}}Y_D^{ff}, \qquad m_u^f = \frac{v}{\sqrt{2}}Y_U^{ff}. \qquad (2.141)
$$

The same procedure can be applied to the leptons as well, with the simplification that the term with the charge conjugate Higgs field will not appear. This is because that term would be proportional to the right-handed neutrino field, which has never been observed experimentally and even at theoretical level it would have no interactions and can therefore be safely omitted. In this case, the Yukawa Lagrangian for the leptons takes the form:

$$
\mathcal{L}_Y^L = -\sum_{f,g}\overline{\ell_L'^f}\left(\phi + \frac{v}{\sqrt{2}}\right)Y'_{Lfg}l_R'^g + \text{h.c.}, \qquad (2.142)
$$

with

$$
\ell'_L = \begin{pmatrix} \nu'_L \\ l'_L \end{pmatrix} \qquad (2.143)
$$

and where again the prime means means a linear combination of the fields with definite mass. Retracing the procedure we just followed for the quarks, we arrive to the following form for the Lagrangian:

$$
\mathcal{L}_Y^L = -\frac{1}{\sqrt{2}}(H+v)\,Y_L\bar{l}l, \qquad (2.144)
$$

with Y_L diagonal; the charged leptons masses are:

$$
m_l^f = \frac{v}{\sqrt{2}}Y_L^{ff}. \qquad (2.145)
$$

Indeed, today we know that neutrinos do actually have a small mass, but with the procedure we just followed no neutrino mass term appeared. Of course, this is because we omitted the right-handed neutrinos from our theory, but in fact their very presence involves additional complications, which we will deal with in the next Chapter.

2.9 The Full Standard Model Lagrangian

For completeness, here is the full Lagrangian of the Standard Model in the unitary gauge:

$$\mathcal{L}_{SM} = \mathcal{L}_0^F + \mathcal{L}_0^B + \mathcal{L}_{EM} + \mathcal{L}_{CC} + \mathcal{L}_{NC} + \mathcal{L}_V + \mathcal{L}_H \tag{2.146}$$

where

- $\mathcal{L}_0^F$ is the free Lagrangian for fermions:

$$\mathcal{L}_0^F = \sum_f \left[\overline{\nu^f} i\partial\!\!\!/ \nu^f + \overline{l^f}\left(i\partial\!\!\!/ - m_l\right) l^f + \overline{u^f}\left(i\partial\!\!\!/ - m_u\right) u^f + \overline{d^f}\left(i\partial\!\!\!/ - m_d\right) d^f\right] \tag{2.147}$$

  The index f labels the fermion families.

- $\mathcal{L}_0^B$ is the free Lagrangian for gauge and Higgs bosons:

$$\begin{aligned}\mathcal{L}_0^B &= -\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - \frac{1}{2}W_{\mu\nu}^+ W_-^{\mu\nu} + m_W^2 W_\mu^+ W_-^\mu \\ &\quad -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}m_H^2 H^2\end{aligned} \tag{2.148}$$

  where

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \qquad W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm \tag{2.149}$$

- $\mathcal{L}_{EM}$ is the electromagnetic coupling:

$$\mathcal{L}_{EM} = e\sum_f \left(-\overline{l^f}\gamma^\mu l^f + \frac{2}{3}\overline{u^f}\gamma^\mu u^f - \frac{1}{3}\overline{d^f}\gamma^\mu d^f\right) A_\mu \tag{2.150}$$

- $\mathcal{L}_{CC}$ is the charged-current interaction term:

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}\left[\sum_f \overline{\nu_L^f}\gamma^\mu l_L^f + \sum_{f,g}\overline{u_L^f}\gamma^\mu V_{fg} d_L^g\right] W_\mu^+ + \text{h.c.} \tag{2.151}$$

- $\mathcal{L}_{NC}$ is the neutral-current interaction term:

$$\begin{aligned}\mathcal{L}_{NC} = \frac{e}{4\sin\theta_W\cos\theta_W}\sum_f &\left[\overline{\nu^f}\gamma^\mu\left(1-\gamma_5\right)\nu^f - \overline{l^f}\gamma^\mu\left(1 - 4\sin^2\theta_W - \gamma_5\right) l^f \right.\\ &\left. +\overline{u^f}\gamma^\mu\left(1 - \frac{8}{3}\sin^2\theta_W - \gamma_5\right) u^f - \overline{d^f}\gamma^\mu\left(1 - \frac{4}{3}\sin^2\theta_W - \gamma_5\right) d^f\right]\end{aligned} \tag{2.152}$$

- $\mathcal{L}_V$ contains vector boson interactions among themselves:

$$\begin{aligned}\mathcal{L}_V &= +ig\sin\theta_W\left(W^+_{\mu\nu}W^\mu_-A^\nu - W^-_{\mu\nu}W^\mu_+A^\nu + F_{\mu\nu}W^\mu_+W^\nu_-\right) \\ &\quad +ig\sin\theta_W\left(W^+_{\mu\nu}W^\mu_-Z^\nu - W^-_{\mu\nu}W^\mu_+Z^\nu + Z_{\mu\nu}W^\mu_+W^\nu_-\right) \\ &\quad +\frac{g^2}{2}\left(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\left[\frac{1}{2}W^+_\mu W^+_\nu W^-_\rho W^-_\sigma\right. \\ &\quad -W^+_\mu W^-_\nu\left(A_\rho A_\sigma\sin^2\theta_W + Z_\rho Z_\sigma\cos^2\theta_W\right. \\ &\quad \left.\left.+2A_\rho Z_\sigma\sin\theta_W\cos\theta_W\right)\right] \end{aligned} \tag{2.153}$$

  where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.154}$$

- $\mathcal{L}_H$ is the Higgs interaction Lagrangian:

$$\begin{aligned}\mathcal{L}_H &= \left(\frac{1}{2}m_Z^2 Z_\mu Z^\mu + m_W^2 W^+_\mu W^\mu_-\right)\left(\frac{H^2}{v^2} + \frac{2H}{v}\right) \\ &\quad -\frac{H}{v}\sum_f\left(m_l^f\overline{l^f}l^f + m_u^f\overline{u^f}u^f + m_d^f\overline{d^f}d^f\right) \\ &\quad -\lambda v H^3 - \frac{1}{4}\lambda H^4. \end{aligned} \tag{2.155}$$

2.10 Open Problems

As we already mentioned in the Introduction, there are still many open problems that the SM cannot solve as it is. Some of the most prominent ones are the origin of neutrino masses, the unification of gravity with the other known interactions, and the nature of DM and DE.

As for the first problem, today we know that neutrinos actually do have a tiny mass: the observation of neutrino flavour oscillations is the proof that at least two of them need to be massive, and bounds from cosmological data allow us to put a lower bound on the sum of their masses, so that today we know that neutrino masses are of order $(0.1 \div 1)\, eV$. However, we still do not know for sure where these masses come from. Of course, they could arise from a Yukawa coupling between left- and right-handed neutrinos, just like all the other fermions. But right-handed neutrinos have never been detected so far, so we cannot be sure. There are actually many other possibilities, and in fact we will review some of them in the next Chapter. Moreover, there is another, maybe even more stringent problem: just why are neutrino masses so small? To give an idea of how small they are, the ratio between neutrino masses and the electron mass is more or less the same as the ratio between the electron mass and the top quark mass:

$$\frac{m_\nu}{m_e} \simeq \frac{m_e}{m_t} \sim 10^{-7}, \tag{2.156}$$

with the crucial difference that between the electron and the top quark we can find all the other known particles, but between the electron and the neutrinos there is nothing at all. This huge discrepancy suggests that the mechanism of mass generation is different for neutrinos, and finding out how this mechanism works is the main problem of neutrino masses. We will study in detail one of the most appealing solutions to this problem in the first part of this work, starting from next Chapter.

The second problem is a very, *really* difficult one, so much that today, after more than a century from the birth of Quantum Mechanics and General Relativity (GR), there is still no consolidated theory of quantum gravity, but only many ideas. The most important suggestions are String Theory and Loop Quantum Gravity, but both have very important shortcomings. String Theory claims to be the most predictive theory ever, since it only has *one* free parameter. However, the truth is that it actually suffers from an *enormous* freedom: there are more or less infinitely many possible ways to compactify the additional dimensions that String Theory requires, and to each of those compactifications corresponds a completely different theory. The disappointing fact is that there is *no* known compactifications that leads to a model that resembles the SM. On the other hand, Loop Quantum Gravity seems very promising, but again, there is *no* known way to perform a low-energy limit and check whether it reduces to GR on large scales as it should. In the second part of this work, we will study a very interesting alternative idea, that actually is not as far-reaching as the two mentioned above (it will not describe a quantum theory of gravity), but that somehow allows us to give a unified description of the *quantum* SM together with the *non-quantum* GR, making them both arise from the very same underlying principles. Moreover, this approach can be easily generalised, so it cannot be excluded that the same idea could eventually be used to really describe the full quantum theory of gravity.

This approach we just mentioned actually gives us some interesting hints on how to solve both the DM and the DE problems, even though these hints suggest a quite unpopular way of tackling the problems. These can be reformulated in the following way: we have checked that GR works extremely well till the galactic scale. When we look at larger scales, we find incongruities between GR predictions and experimental data, incongruities which could be solved if we were to introduce additional particles (DM) or additional cosmological energy sources (DE). However, we do not really know if they exist at all, and in fact there are so far no experiments that confirm their existence[1]. Another possibility is that

[1]excluding the controversial results from DAMA [2, 3, 4].

GR has to be modified at scales larger that the galactic scale. It is an unpopular approach, because GR really is a beautiful theory and it somehow feels a shame to modify it, and maybe also because there is a rumour stating that any theory of modified gravity actually generates more problems than it solves and therefore all of them are excluded experimentally. This is actually not completely true, since there are still some theories of modified gravity that can solve both the DM and the DE problems (without introducing any new particle or field content) and that have not been excluded yet. In this work we will actually deal with neither of these problems directly, but we still chose to mention them here because the approach we follow to unify the description of the SM and GR generates some additional lagrangian terms, and those same terms are the ones that appear when one tries to take into account the small quantum fluctuations around GR; moreover, it turns out that those very same terms actually generate the curvature effects that allow many theories of modified gravity to solve both the DM and the DE problems. We will explain this in more detail in section 7.2.6.

Chapter 3

The See-saw Models

3.1 Introduction: The Need of Mass for Neutrinos

Towards the end of the 1960s an apparently inexplicable phenomenon was observed.

The study of solar processes had led to the formulation of the so-called Standard Solar Model, a model that describes the Sun's behaviour. Of course, many experiments were carried out to test its validity, and all of them seemed to show that the model was actually well functional.

All but one.

In the late 1960s, an experiment was attempted to measure the neutrino flux emitted by the Sun: the Homestake experiment [5]. The result was the measurement of a neutrino flux of approximately ⅓ of the expected flow.

Obviously, in the following years several other experiments were carried out to test Homestake's results: in 1985 the Kamiokande experiment [6, 7, 8] started, then, then in 1990 SAGE [9], in 1991 GALLEX [10], then in 1996 Kamiokande was updated to Super-Kamiokande [11, 12, 13], and finally, in 2001, SNO [14, 15] published its first results. All of these experiments confirmed that indeed the electronic neutrino emission from the Sun expected by the Standard Solar Model differed from that measured. But comparing the data of the various experiments, physicists immediately noticed a further problem.

The data collected from the various experiments were in disagreement. For example, Homestake measured a neutrino flux of about ⅓ of the expected one, but Kamiokande and Superkamiokande found a flux equal to about ½ of the expected one. Still, Gallex and SAGE measured a flux of about 0.56 times the expected one, and the situation is analogous for all the other subsequent experiments: in short, all experiments agreed that the predictions of the Standard Solar Model were incorrect, but they did not agree at all on how much they were so.

With one exception.

The Sun, in the various nuclear processes that take place inside it, produces a huge amount of *electronic* neutrinos. For this reason, all the various experiments mentioned above used nuclear reactions involving that particular type of neutrinos. However, SNO also allowed another type of measure: a scattering process of a generic flavour neutrino on a deuton with its consequent disintegration into a proton and a neutron. The solar neutrino flux measurement for this single channel perfectly matched the predictions of the Standard Solar Model.

However, the nuclear reactions that took place in the Sun had already been studied on Earth and it was absolutely certain that the produced neutrinos were necessarily electronic ones. For this reason, it became clear that Pontecorvo's suggestion that neutrinos could *oscillate* between different flavours [16] was probably correct.

The fundamental motivation of why neutrinos oscillate is that they are emitted in states with definite flavour; however, these states are not eigenstates of the free particle Hamiltonian (mass eigenstates), so when they propagate in vacuum they have to be decomposed into the basis of the mass eigenstates and the evolution of these must be considered separately. Let us see how in detail.

We will denote with a Greek index the flavour eigenstates, and with a Roman index the mass eigenstates. Then we have:

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle \,. \tag{3.1}$$

The matrix U is a unitary matrix called Pontecorvo-Maki-Nagakawa-Sakata (PMNS) matrix and it expresses neutrino flavour eigenstates in terms of their mass eigenstates. It depends on 4 parameters: three mixing angles and one complex phase. It is usually parametrised as follows:

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

with

$$c_{ij} = \cos\theta_{ij}, \qquad s_{ij} = \sin\theta_{ij} \tag{3.3}$$

and θ_{ij} the mixing angle between neutrinos of the i-th and j-th generations.

Let us now consider a beam of neutrinos emitted with definite flavour and let us stay in the beam rest frame. The time evolution of each definite mass component will be:

$$|\nu_i(\tau)\rangle = e^{-im_i\tau} |\nu_i\rangle \,, \tag{3.4}$$

with τ the beam's proper time. Then, in a generic inertial frame, the evolution law will be:

$$|\nu_i(t)\rangle = e^{-i(Et-|\vec{p}_i|L)}|\nu_i\rangle\,, \tag{3.5}$$

where L is the distance travelled by the neutrinos, $p_i^\mu = (E, \vec{p}_i)$ is the beam's four-momentum and

$$E = \sqrt{|\vec{p}_i|^2 + m_i^2} \simeq |\vec{p}_i| + \frac{m_i^2}{2|\vec{p}_i|} \simeq |\vec{p}_i| + \frac{m_i^2}{2E} \tag{3.6}$$

is its energy. Since the neutrino masses are extremely small, we can approximate their speed with the speed of light, therefore $L = t$ so that:

$$|\nu_\alpha(t)\rangle = \sum_i U^*_{\alpha i} e^{-i\frac{m_i^2}{2E}t}|\nu_i\rangle\,. \tag{3.7}$$

The probability amplitude for observing a β flavour neutrino after the time t is thus:

$$\langle \nu_\beta\,|\,\nu_\alpha(t)\rangle = \sum_{i,j} U_{\beta j}U^*_{\alpha i}e^{-i\frac{m_i^2}{2E}t}\langle \nu_j\,|\,\nu_i\rangle = \sum_i U_{\beta i}U^*_{\alpha i}e^{-i\frac{m_i^2}{2E}t}. \tag{3.8}$$

Therefore, the corresponding probability is:

$$\begin{aligned} P_{\alpha\to\beta}(t) = |\langle \nu_\beta\,|\,\nu_\alpha(t)\rangle|^2 &= \left|\sum_i U_{\beta i}U^*_{\alpha i}e^{-i\frac{m_i^2}{2E}t}\right|^2 = \\ &= \sum_{i,j}\left(U_{\beta i}U^*_{\alpha i}\right)\left(U_{\beta j}U^*_{\alpha j}\right)^* e^{i\frac{m_j^2-m_i^2}{2E}t}. \end{aligned} \tag{3.9}$$

It is very interesting that neutrino flavour oscillation depend on their squared mass differences. This means that it is necessary to assume that at least one neutrino be massive for oscillation effects to happen at all.

To date, we know almost all the neutrino oscillation parameters. As we can see from Figure 3.1, all absolute values of the squared mass differences are known (with the sign as well for the first two families) as well as all the mixing angles. We still need to determine just two more parameters: the sign of $\Delta m^2_{3\ell}$ and the complex phase δ_{CP}, which may be responsible of another source of CP violation, other than the quark sector.

But what is the origin of neutrino masses? With the SM particles, it is impossible to build a neutrino mass term. Of course, this impossibility arises from the absence of right-handed neutrinos within the SM itself – in contrast with all other fermions. For this reason, it has been studied for a long time how it is possible to extend the SM in a simple way to confer mass on neutrinos.

One of the simplest, but at the same time most effective and full of possibilities ideas is the see-saw mechanism.

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		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.7$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
without SK atmospheric data	$\sin^2\theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$
	$\theta_{12}/^\circ$	$33.44^{+0.78}_{-0.75}$	$31.27 \to 35.86$	$33.45^{+0.78}_{-0.75}$	$31.27 \to 35.87$
	$\sin^2\theta_{23}$	$0.570^{+0.018}_{-0.024}$	$0.407 \to 0.618$	$0.575^{+0.017}_{-0.021}$	$0.411 \to 0.621$
	$\theta_{23}/^\circ$	$49.0^{+1.1}_{-1.4}$	$39.6 \to 51.8$	$49.3^{+1.0}_{-1.2}$	$39.9 \to 52.0$
	$\sin^2\theta_{13}$	$0.02221^{+0.00068}_{-0.00062}$	$0.02034 \to 0.02430$	$0.02240^{+0.00062}_{-0.00062}$	$0.02053 \to 0.02436$
	$\theta_{13}/^\circ$	$8.57^{+0.13}_{-0.12}$	$8.20 \to 8.97$	$8.61^{+0.12}_{-0.12}$	$8.24 \to 8.98$
	$\delta_{CP}/^\circ$	195^{+51}_{-25}	$107 \to 403$	286^{+27}_{-32}	$192 \to 360$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.514^{+0.028}_{-0.027}$	$+2.431 \to +2.598$	$-2.497^{+0.028}_{-0.028}$	$-2.583 \to -2.412$
		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 7.1$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$
	$\theta_{12}/^\circ$	$33.44^{+0.77}_{-0.74}$	$31.27 \to 35.86$	$33.45^{+0.78}_{-0.75}$	$31.27 \to 35.87$
	$\sin^2\theta_{23}$	$0.573^{+0.016}_{-0.020}$	$0.415 \to 0.616$	$0.575^{+0.016}_{-0.019}$	$0.419 \to 0.617$
	$\theta_{23}/^\circ$	$49.2^{+0.9}_{-1.2}$	$40.1 \to 51.7$	$49.3^{+0.9}_{-1.1}$	$40.3 \to 51.8$
	$\sin^2\theta_{13}$	$0.02219^{+0.00062}_{-0.00063}$	$0.02032 \to 0.02410$	$0.02238^{+0.00063}_{-0.00062}$	$0.02052 \to 0.02428$
	$\theta_{13}/^\circ$	$8.57^{+0.12}_{-0.12}$	$8.20 \to 8.93$	$8.60^{+0.12}_{-0.12}$	$8.24 \to 8.96$
	$\delta_{CP}/^\circ$	197^{+27}_{-24}	$120 \to 369$	282^{+26}_{-30}	$193 \to 352$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.517^{+0.026}_{-0.028}$	$+2.435 \to +2.598$	$-2.498^{+0.028}_{-0.028}$	$-2.581 \to -2.414$

Figure 3.1: Values of neutrino squared mass differences and mixing angles. $\Delta m^2_{3\ell} = \Delta m^2_{31} > 0$ in case of normal ordering, while $\Delta m^2_{3\ell} = \Delta m^2_{32} < 0$ in case of inverted ordering. Image from [17].

3.2 The See-saw Mechanism

For simplicity, we will illustrate the idea behind the see-saw models in the simplest case: one single generation of leptons, with the addition of one right-handed neutrino, which transforms as a $SU(2)$ singlet (similarly to all the other right-handed fermions).

This new particle has null charge and hypercharge, so it does not interact and could have been omitted if the neutrino masses were null (as in fact it was done as long as this hypothesis was considered valid).

Then, the following terms have to be added to the SM Lagrangian:

- A kinetic term: $\mathcal{L}_0 = \overline{\nu_R} i \not{D} \nu_R = \overline{\nu_R} i \not{\partial} \nu_R$;
- A Yukawa term: $\mathcal{L}_H = -Y_\nu \left(\overline{\ell_L} \phi \nu_R + \text{h.c.} \right)$. Through the Higgs mechanism, this term generates immediately a Dirac mass term: $\mathcal{L}_{\text{Dirac}} = -m_\nu \left(\overline{\nu_L} \nu_R + \overline{\nu_R} \nu_L \right)$, with $m_\nu = \frac{Y_\nu v}{\sqrt{2}}$.

If the Dirac mass term was the only possible one, the Yukawa coupling Y_ν should be much smaller than the corresponding constant for the charged leptons, so that the neutrino mass is small as observed experimentally:

$$\frac{Y_\nu}{Y_e} = \frac{m_\nu}{m_e} \leq 10^{-6}. \tag{3.10}$$

Now, such a huge difference between orders of magnitude is certainly possible: it is the same between the electron and the top quark. However, between the electron and the top quark masses there are the masses of almost every other known particle. Instead, the gap between the electron and the neutrino masses is completely empty, which seems quite unnatural. Luckily though, there is another mass term compatible with the SM symmetries.

- A Majorana mass term: $\mathcal{L}_{\text{Majorana}} = -\frac{1}{2} M \left(\overline{\nu_R^c} \nu_R + \overline{\nu_R} \nu_R^c \right)$, in which $\nu_R^c = i\gamma_2 \nu_R^*$ is the charge-conjugated right-handed neutrino. The right-handed neutrino is the sole fermion in the SM that can have such a mass term, since it is the only one that has both charge and hypercharge equal to zero. The Majorana mass M is completely unrelated to the Higgs mechanism and to every other known phenomenon, therefore it can assume arbitrarily big values and is to all effects a new mass scale. Moreover, this term violates the conservation of the lepton number, thus M is assumed big enough for the effects of the violation of lepton number (usually suppressed by negative powers of M) to be compatible with observations. It is exactly so as not to introduce this term that the exclusion of right-handed neutrino from the SM was preferred initially.

We can then rewrite the full neutrino mass term in the following way:

$$\mathcal{L}_M = -\frac{1}{2} \begin{pmatrix} \overline{\nu_L^c} & \overline{\nu_R} \end{pmatrix} \begin{pmatrix} 0 & m_\nu \\ m_\nu & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.} \tag{3.11}$$

(since $\overline{\nu_L^c}\nu_R^c = \overline{\nu_R}\nu_L$).

The mass matrix in the previous expression is a real symmetric one and can thus be diagonalised via an orthonormal transformation:

$$\mathcal{L}_M = -\frac{1}{2}\begin{pmatrix}\overline{\nu_1^c} & \overline{\nu_2}\end{pmatrix}\begin{pmatrix}m_1 & 0 \\ 0 & M_2\end{pmatrix}\begin{pmatrix}\nu_1 \\ \nu_2^c\end{pmatrix} + \text{h.c.}, \tag{3.12}$$

with

$$\begin{pmatrix}\nu_1 \\ \nu_2^c\end{pmatrix} = U^T\begin{pmatrix}\nu_L \\ \nu_R^c\end{pmatrix}, \tag{3.13}$$

where U is the orthonormal matrix that diagonalises the mass matrix. The eigenvalues m_1, M_2 are found to be:

$$m_1 = \frac{1}{2}\left(M - \sqrt{M^2 + 4m_\nu^2}\right), \tag{3.14}$$

$$M_2 = \frac{1}{2}\left(M + \sqrt{M^2 + 4m_\nu^2}\right). \tag{3.15}$$

In the limit $M \gg m$ these equations become:

$$m_1 \simeq -\frac{m_\nu^2}{M}, \tag{3.16}$$

$$M_2 \simeq M. \tag{3.17}$$

This mechanism is known as *see-saw mechanism* and it explains in a natural way the observed smallness of neutrino masses: one of the two masses is extremely big and has no observable effect at the electroweak scale, while the other one is suppressed by a factor $\frac{m_\nu}{M}$ with respect to the typical fermion masses. In this way, there is no need to assume unnaturally small values for the Yukawa couplings.

The see-saw mechanism can be easily generalised to n lepton and k right-handed neutrino generations. In this case, m_ν is a $k \times n$ matrix and M is a $k \times k$ matrix. We will study this general case in Chapter 4.

3.3 Neutrino Masses: The Weinberg Operator

Let us consider the following Lagrangian term:

$$\mathcal{L}_W = \frac{1}{2}c_{\alpha\beta}^{d=5}\left(\overline{\ell_{L\alpha}^c}\widetilde{\phi}^*\right)\left(\widetilde{\phi}^\dagger\ell_{L\beta}\right) + \text{h.c.}, \tag{3.18}$$

where ℓ_L are the lepton weak doublets, $\widetilde{\phi} = i\tau_2\phi^*$ with $\phi = (\phi^+, \phi^0)$ the Higgs field, and the Greek indices are flavour indices. $\mathcal{L}_W$ is known as *Weinberg operator* and we can notice that it is non-renormalizable, since its coefficient $c_{\alpha\beta}^{d=5}$ has mass dimension −1: it is thus an effective operator, just like the Fermi Lagrangian we studied in Chapter 2. However, unlike the Lagrangian of the Fermi theory, this operator has a very important feature: it is the only dimension-5 operator compatible with the symmetries and the fields of the SM. It violates the $B - L$ symmetry (with B the baryon number, L the lepton number) and through the spontaneous breaking of the electroweak symmetry it generates a Majorana mass term:

$$\begin{aligned}
&\frac{1}{2} c^{d=5}_{\alpha\beta} \left(\overline{\ell^c_{L\alpha}} \widetilde{\phi}^* \right) \left(\widetilde{\phi}^\dagger \ell_{L\beta} \right) = \\
&= \frac{1}{2} c^{d=5}_{\alpha\beta} \left(\overline{\nu^c_{L\alpha}} \quad \overline{l^c_{L\alpha}} \right) \begin{pmatrix} -\phi^0 \\ \phi^+ \end{pmatrix} \left(-\phi^0 \quad \phi^+ \right) \begin{pmatrix} \nu_{L\beta} \\ l_{L\beta} \end{pmatrix} = \\
&= \frac{1}{2} c^{d=5}_{\alpha\beta} \overline{\nu^c_{L\alpha}} \left(\phi^0 \right)^2 \nu_{L\beta} + \ldots \xrightarrow{\text{Symm. Break.}} \left(\frac{v^2 c^{d=5}_{\alpha\beta}}{2} \right) \overline{\nu^c_{L\alpha}} \nu_{L\beta} + \ldots
\end{aligned} \tag{3.19}$$

which is a Majorana mass term with mass $v^2 c^{d=5}_{\alpha\beta}$.

The interesting thing is that it can be shown that any SM extension that provides masses for the neutrinos at tree-level reduces to this effective operator. Moreover, this operator can be split into two renormalizable operators at tree-level only in three well-precise ways [18]:

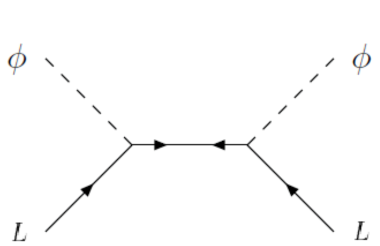
1. Through the introduction of a $SU(2)$ fermion singlet (the right-handed neutrino);

2. Through the introduction of a $SU(2)$ scalar triplet (a Higgs triplet);

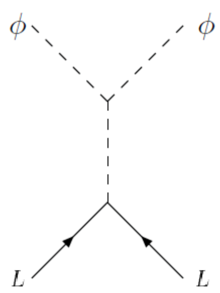
3. Through the introduction of a $SU(2)$ fermion triplet. This is the case studied in detail in this work.

We will try to justify this intuitively, for the detailed demonstration, see [18].

The Feynman diagram corresponding to the Weinberg operator can be stretched in two different ways:



or



Let us consider the charges of the various involved particles: these have obviously to be conserved in the process. The lepton has charges (1, 2, −½) (it is a $SU(3)$ singlet, a $SU(2)$ doublet, and has hypercharge ½), while the Higgs has charges (1, 2, ½). This means that the fermion intermediate particle will be a $SU(3)$ singlet, it will have null hypercharge and its weak isospin will be compatible with the composition of two doublets, thus either a $SU(2)$ singlet or a $SU(2)$ triplet.

It is the same for the scalar intermediary, except that in this case it must be a $SU(2)$ triplet, because otherwise it would not produce a mass term [18].

Depending on the specific particle that is chosen as intermediary, the corresponding models are called respectively *type I see-saw* for the fermion singlet, *type II see-saw* for the scalar triplet, and finally *type III see-saw* for the fermion triplet. Anyhow, the Majorana mass terms specific for each of these models all reduce to the Weinberg operator once the new particles are integrated out (that is, when the limit where the mass of these new particles is much greater than the available energy of any process one wishes to study is considered); for this reason, in order to distinguish the various see-saw models one from the other it is necessary to consider the dimension-6 effective operators as well.

It has been shown that the Weinberg operator is the only dimension-5 one compatible with the SM symmetries; the situation is completely different for the dimension-6 operators: there are about 180 different such operators [19]. If we managed to measure the coefficients of those operators experimentally, comparing their various ratios with what is predicted by the different theories, we could understand which of the many neutrino mass generation mechanisms is the true one. In the next Section, we will describe briefly the fundamental characteristics of the various see-saw models and we will report the effective operators that arise from each model.

3.4 The See-saw Models

3.4.1 Type I See-saw [1]

This is the simplest of the see-saw models, based on the introduction of k right-handed neutrinos (usually denoted by N_R) that transform like $SU(2)$ singlets just like the right-handed components of every other fermion. The lepton Lagrangian is:

$$\mathcal{L}_L = \mathcal{L}_L^0 + \mathcal{L}_L^M, \tag{3.20}$$

where

$$\mathcal{L}_L^0 = i\overline{\ell_L}\not{D}\ell_L + \overline{l_R}\not{D}l_R + i\overline{N_R}\not{D}N_R \tag{3.21}$$

contains the kinetic and gauge interaction terms for the left-handed lepton doublet ℓ_L, the right-handed charged leptons l_R and the right-handed neutrinos N_R, while

$$\mathcal{L}_L^M = -\overline{\ell_L}\phi Y_L l_R - \overline{\ell_L}\widetilde{\phi}Y_N^\dagger N_R - \frac{1}{2}\overline{N_R}M_N N_R^c + \text{h.c.} \tag{3.22}$$

contains the Yukawa interactions with couplings $Y_{L,N}$ with the charged leptons and the right-handed neutrinos, and the Majorana mass term for the right-handed neutrinos, corresponding to the new physical scale M_N. Implicit summation over the flavour indices is to be intended; moreover, we assume to work in the basis that makes M_N diagonal.

In the definite flavour basis, once the right-handed neutrinos are integrated out, the Weinberg operator coefficient is found to be:

$$c^{d=5} = Y_N^T \frac{1}{M_N} Y_N, \tag{3.23}$$

that induces, after spontaneous breaking of the electroweak symmetry, the following mass matrix for left-handed neutrinos:

$$m_\nu = -\frac{v^2}{2}c^{d=5} = -\frac{v^2}{2}Y_N^T\frac{1}{M_N}Y_N \equiv -m_D^T M_N^{-1} m_D \tag{3.24}$$

(where we defined $m_D := \frac{v}{\sqrt{2}}Y_N$), that for a single neutrino generation reduces to the result 3.16 we found in Section 3.2.

It is also found that a unique dimension-6 operator appears:

$$\delta\mathcal{L}^{d=6} = c_{\alpha\beta}^{d=6}\left(\overline{\ell_{L\alpha}}\widetilde{\phi}\right) i\partial\!\!\!/ \left(\widetilde{\phi}^\dagger \ell_{L\beta}\right), \tag{3.25}$$

whose coefficients are

$$c^{d=6} = Y_N^\dagger \frac{1}{M_N^\dagger}\frac{1}{M_N}Y_N. \tag{3.26}$$

When the Higgs doublet acquires a vacuum expectation value (vev), this dimension-6 operator induces corrections to the left-handed neutrino kinetic term:

$$\mathcal{L}_\nu^0 = i\overline{\nu_L}\partial\!\!\!/ \left(1+2\eta_N\right)\nu_L, \qquad \eta_N = \frac{v^2}{4}c^{d=6}. \tag{3.27}$$

Once the neutrino fields are redefined so that the kinetic term is normalised as in the usual way

$$\nu_L \to \nu_L' = \left(1+2\eta_N\right)^{\frac{1}{2}}\nu_L, \tag{3.28}$$

the charged current Lagrangian term for all leptons and the neutral current term for neutrinos only are modified, while the neutral current term for charged leptons remains unchanged. For this reason, there are no FCNC processes for charged leptons in the type I see-saw model. As we will see, this is the main difference between this model and type III see-saw one, in which FCNC processes for charged leptons are already possible at tree-level.

Once the fields are rotated to the basis that makes the mass matrices diagonal, the usual U_{PMNS} matrix that appears in the couplings with the charged currents is replaced by the non-unitary matrix N:

$$N = (1 - \eta_N)\, U_{PMNS}. \tag{3.29}$$

Consequently to the non-unitarity of N, the Fermi constant as measured by experiments cannot be identified with the SM prediction

$$G_F^{SM} = \frac{\sqrt{2} g^2}{8 M_W^2} = \frac{1}{\sqrt{2} v^2}, \tag{3.30}$$

e.g., as we will see in Chapter 4, the Fermi constant as obtained through the decay $\mu^- \to \nu_\mu e^- \overline{\nu_e}$ is related to G_F^{SM} by:

$$G_F^{\mu} = G_F^{SM} \sqrt{\left(N N^\dagger\right)_{ee} \left(N N^\dagger\right)_{\mu\mu}}. \tag{3.31}$$

In turn, this will induce corrections of order η_N in all SM weak processes.

Another, maybe even more remarkable consequence of the non-unitarity of N is the induction of loop-level LFV processes, like e.g. the radiative decay $\mu \to e\gamma$ or $\mu \to e e \overline{e}$. As we will see, all this will be true for the type III see-saw as well, but with the important difference that in the type III see-saw some LFV processes are possible even at tree-level.

3.4.2 Type II See-saw [1]

In this model, one single $SU(2)$ triplet of scalar fields $\vec{\Delta} = (\Delta_1, \Delta_2, \Delta_3)$ with hypercharge 2 is introduced, where

$$\Delta^{++} = \frac{1}{\sqrt{2}} \left(\Delta_1 - i \Delta_2\right), \quad \Delta^{+} = \Delta_3, \quad \Delta^{0} = \frac{1}{\sqrt{2}} \left(\Delta_1 + i \Delta_2\right) \tag{3.32}$$

are its definite charge physical states. The SM symmetries allow the introduction of a Yukawa coupling of the scalar triplet to two lepton doublets $\mathcal{L}_{\Delta,L}$ as well as a coupling of the scalar triplet to the Higgs doublet $\mathcal{L}_{\Delta,\phi}$:

$$\mathcal{L}_{\Delta,L} = \overline{\widetilde{\ell_L}} Y_\Delta \left(\tau \cdot \vec{\Delta}\right) \ell_L + \text{h.c.}\ , \tag{3.33}$$

$$\mathcal{L}_{\Delta,\phi} = \mu_\Delta \widetilde{\phi}^\dagger \left(\tau \cdot \vec{\Delta}\right)^\dagger \phi + \text{h.c.}\ , \tag{3.34}$$

where the τ_i are the Pauli matrices, Y_Δ is a symmetric matrix in the generation space and $\widetilde{\ell_L} = i\tau_2\ell_L^c$. The Lagrangian for the triplet $\vec{\Delta}$ can thus be written as:

$$\mathcal{L}_\Delta = \left(D_\mu\vec{\Delta}\right)^\dagger\left(D^\mu\vec{\Delta}\right) + \left[\overline{\widetilde{\ell_L}}Y_\Delta\left(\tau\cdot\vec{\Delta}\right)\ell_L + \mu_\Delta\widetilde{\phi}^\dagger\left(\tau\cdot\vec{\Delta}\right)^\dagger\phi + \text{h.c.}\right] - \\ - \left[\vec{\Delta}^\dagger M_\Delta^2\vec{\Delta} + \frac{1}{2}\lambda_2\left(\vec{\Delta}^\dagger\vec{\Delta}\right)^2 + \lambda_3\left(\phi^\dagger\phi\right)\left(\vec{\Delta}^\dagger\vec{\Delta}\right) + \frac{\lambda_4}{2}\left(\vec{\Delta}^\dagger T_i\vec{\Delta}\right)^2 + \right. \\ \left. + \lambda_5\left(\vec{\Delta}^\dagger T_i\vec{\Delta}\right)\left(\phi^\dagger\tau_i\phi\right)\right], \quad (3.35)$$

where

$$D_\mu = \partial_\mu - igT_iW_\mu^i - ig'B_\mu\frac{Y}{2}, \quad (3.36)$$

with T_i the generators of the three-dimensional representation of $SU(2)$:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.37)$$

Once the scalar triplets are integrated out, unlike what happens for the fermion see-saw models, in addition to the dimension-5 and 6 operators, a dimension-4 operator arises as well:

$$\delta\mathcal{L}^{d=4} = \frac{|\mu_\Delta|^2}{M_\Delta^2}\left(\widetilde{\phi}^\dagger\vec{\tau}\phi\right)\left(\phi^\dagger\vec{\tau}\widetilde{\phi}\right) = 2\frac{|\mu_\Delta|^2}{M_\Delta^2}\left(\phi^\dagger\phi\right)^2. \quad (3.38)$$

This operator modifies the Higgs 4-fields self-interaction coupling λ:

$$\lambda = \lambda_{SM} + \delta\lambda, \quad (3.39)$$

$$\delta\lambda = -2\frac{|\mu_\Delta|^2}{M_\delta^2}. \quad (3.40)$$

The coefficient of the Weinberg operator this time is given by:

$$c^{d=5} = 4Y_\Delta\frac{\mu_\Delta}{M_\Delta^2}, \quad (3.41)$$

that induces masses for the left-handed neutrinos:

$$m_\nu = -2Y_\Delta v^2\frac{\mu_\Delta}{M_\Delta^2}. \quad (3.42)$$

Finally, the following dimension-6 operators arise:

$$\delta\mathcal{L}_{4\text{ ferm.}} = -\frac{1}{M_\Delta^2}Y_{\Delta ij}Y_{\Delta\alpha\beta}^\dagger\left(\overline{\ell_{L\beta}}\gamma_\mu\ell_{Li}\right)\left(\overline{\ell_{L\alpha}}\gamma^\mu\ell_{Lj}\right), \quad (3.43)$$

$$\delta\mathcal{L}_{6\phi} = -2\left(\lambda_3+\lambda_5\right)\frac{\left|\mu_\Delta\right|^2}{M_\Delta^4}\left(\phi^\dagger\phi\right)^6, \tag{3.44}$$

$$\delta\mathcal{L}_{4\phi} = 4\frac{\left|\mu_\Delta\right|^2}{M_\Delta^4}\left[\left(\phi^\dagger\phi\right)\left(D_\mu\phi\right)^\dagger\left(D^\mu\phi\right)+\left(\phi^\dagger D_\mu\phi\right)^\dagger\left(\phi^\dagger D^\mu\phi\right)\right], \tag{3.45}$$

where the covariant derivative is to be expressed in terms of the Pauli matrices:

$$D_\mu = \partial_\mu - ig\frac{\tau_i}{2}W_\mu^i - ig' B_\mu\frac{Y}{2}. \tag{3.46}$$

Of these, (3.43) induces once again a correction to the Fermi constant

$$G_F = G_F^{SM} + \delta G_F, \tag{3.47}$$

$$\delta G_F = \frac{1}{\sqrt{2}M_\Delta^2}\left|Y_{\Delta_{e\mu}}\right|^2 \tag{3.48}$$

that once again, just like in type I and III see-saw, will induce corrections in all SM weak processes. Then, (3.44) modifies the Higgs potential. Together with the correction to λ (3.39), this becomes

$$V = -m_H^2\left|\phi\right|^2 + \left(\lambda+\delta\lambda\right)\left|\phi\right|^4 + 2\left(\lambda_3+\lambda_5\right)\frac{\left|\mu_\Delta\right|^2}{M_\Delta^4}\left|\phi\right|^6, \tag{3.49}$$

thus inducing a shift in the Higgs vev as well:

$$v^2 = v_{SM}^2 + \delta v^2, \tag{3.50}$$

$$\delta v^2 = -3v^4\frac{\left|\mu_\Delta\right|^4}{M_\Delta^4}\frac{\lambda_3+\lambda_5}{\lambda+\delta\lambda}. \tag{3.51}$$

Finally, (3.45) induces a correction in the measurement of the Z mass:

$$M_Z^2 = \left(M_Z^{SM}\right)^2 + \delta M_Z^2, \tag{3.52}$$

$$\delta M_Z^2 = 2M_Z^2 v^2\frac{\left|\mu_\Delta\right|^2}{M_\Delta^4}. \tag{3.53}$$

3.4.3 Type III See-saw [1]

This is the model we will deal with starting with Chapter 4, we only report here a brief summary of its key points, in analogy to what we did for the type I and II see-saw models. In this case, $SU(2)$ fermion triplets with zero hypercharge $\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ are introduced, where

$$\Sigma^\pm = \frac{\Sigma_1 \mp i\Sigma_2}{\sqrt{2}}, \qquad \Sigma^0 = \Sigma_3 \tag{3.54}$$

are its definite charge physical states. The Lagrangian for $\vec{\Sigma}$ is:

$$\mathcal{L}_\Sigma = i\overline{\vec{\Sigma}_R}\not{D}\vec{\Sigma}_R - \left[\frac{1}{2}\overline{\vec{\Sigma}_R}M_\Sigma\vec{\Sigma}^c_R + \overline{\vec{\Sigma}_R}Y_\Sigma\left(\widetilde{\phi}^\dagger\vec{\tau}\ell_L\right) + \text{h.c.}\right], \tag{3.55}$$

where the covariant derivative is given by

$$D_\mu = \partial_\mu - igT_iW^i_\mu - ig'B_\mu\frac{Y}{2}, \tag{3.56}$$

where, just like in the type II see-saw:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.57}$$

Once the triplets are integrated out, the coefficient of the Weinberg operator is

$$c^{d=5} = Y^T_\Sigma\frac{1}{M_\Sigma}Y_\Sigma, \tag{3.58}$$

that induces the following mass matrix for left-handed neutrinos:

$$m_\nu = -\frac{v^2}{2}c^{d=5} = -\frac{v^2}{2}Y^T_\Sigma\frac{1}{M_\Sigma}Y_\Sigma \equiv -m^T_D M^{-1}_\Sigma m_D \tag{3.59}$$

(where we defined $m_D \coloneqq \frac{v}{\sqrt{2}}Y_\Sigma$), that once again for a single neutrino generation reduces to the result 3.16 we found in Section 3.2.

Once again, a unique dimension-6 operator appears:

$$\delta\mathcal{L}^{d=6} = c^{d=6}_{\alpha\beta}\left(\overline{\ell_{L\alpha}}\vec{\tau}\widetilde{\phi}\right)i\not{D}\left(\widetilde{\phi}^\dagger\vec{\tau}\ell_{L\beta}\right), \tag{3.60}$$

where the coefficients are

$$c^{d=6} = Y^\dagger_\Sigma\frac{1}{M^\dagger_\Sigma}\frac{1}{M_\Sigma}Y_\Sigma. \tag{3.61}$$

Unlike the type I see-saw, however, this time, when the Higgs doublet acquires a vacuum expectation value, this dimension-6 operator induces corrections not only to the left-handed neutrino kinetic term, but also to the charged lepton kinetic term and to the couplings between charged leptons and the W bosons:

$$\begin{aligned}\mathcal{L}_{\text{lept.}} = {} & i\overline{\nu_L}\not{\partial}\left(1+2\eta\right)\nu_L + i\overline{l_L}\not{\partial}\left(1+4\eta\right)l_L + \\ & + \frac{g}{\sqrt{2}}\left[\overline{l_L}\not{W}^-\left(1+4\eta\right)\nu_L + \text{h.c.}\right] - \frac{g}{2}\overline{l_L}\not{W}^3\left(1+8\eta\right)l_L,\end{aligned} \tag{3.62}$$

where

$$\eta = \frac{v^2}{4} c^{d=6}. \tag{3.63}$$

When the neutrino and the charged lepton fields are redefined so that the kinetic term is normalised as in the usual way

$$\nu_L \to \nu'_L = (1+2\eta)^{\frac{1}{2}} \nu_L, \quad l_L \to l'_L = (1+4\eta)^{\frac{1}{2}} l_L \tag{3.64}$$

the charged and neutral current Lagrangian terms for all leptons are modified, this time for the charged leptons as well. For this reason, unlike what happens in the type I see-saw model, this time there are FCNC processes for charged leptons already at tree-level.

Once the fields are rotated to the basis that makes the mass matrices diagonal, once again the usual U_{PMNS} matrix that appears in the couplings with the charged currents is replaced by the non-unitary matrix N:

$$N = (1+\eta)\, U_{PMNS}, \tag{3.65}$$

(notice the opposite sign with respect to Eq. (3.29) of type I see-saw) and once again, consequently to the non-unitarity of N, the Fermi constant as measured by experiments cannot be identified with the SM prediction

$$G_F^{SM} = \frac{\sqrt{2} g^2}{8 M_W^2} = \frac{1}{\sqrt{2} v^2}, \tag{3.66}$$

in particular, as we will see in Chapter 4, the Fermi constant as obtained through the decay $\mu^- \to \nu_\mu e^- \overline{\nu_e}$ is related to G_F^{SM} by:

$$G_F^\mu = G_F^{SM} \sqrt{(NN^\dagger)_{ee} \, (NN^\dagger)_{\mu\mu}}. \tag{3.67}$$

Yet again, as another consequence of the non-unitarity of N, LFV processes are induced, this time already at tree-level (unlike what happens in type I see-saw, in which they appear only at loop-level). For this reason, in type III see-saw processes like e.g. $\mu \to e e \overline{e}$, which happen at tree-level, are more relevant than e.g. $\mu \to e\gamma$, that happens at loop level in type III see-saw as well. As we will see, the appearance of tree-level LFV processes allows one to put very strong constraints on the model parameters, i.e. on the entries of η. All of this will be analyzed in depth in Chapter 4.

3.5 Inverse See-saw

So far, we made only a single assumption on the neutrino Majorana masses, i.e. that these are very large, so that the smallness of the left-handed neutrino masses can be easily justified. However, all the corrections to the SM are proportional to M^{-2}, so if these Majorana masses are *too* large, no effects would be detectable at all. Hence, we wonder whether it is possible to suppose that these Majorana masses are also small enough to have some measurable effects in the experiments in the accelerators we have available today or, at most, in the next generation ones. One possible solution to this problem is the so-called "inverse see-saw".

Let us assume to be in the context of one of the fermion see-saw models: the neutral component of the fermion triplet, Σ^0, behaves to all effects as the fermion singlet N_R, so the following discussion is valid for both cases.

There are several ways in which the two conditions (Majorana masses large enough to have small neutrinos masses but at the same time sufficiently small to produce right-handed neutrinos) are both met. One possibility is that the Yukawa coupling with the neutrinos is small: since the light neutrino mass is

$$m_\nu \sim \frac{v^2}{2} Y^T M^{-1} Y, \tag{3.68}$$

it is sufficient to assume Y to be small (e.g. $Y \sim 10^{-6}$, like for the electron) so that it can be $M \sim 1$ TeV. However, since all corrections to the SM are of order $\eta \sim \frac{v^2}{4} Y^\dagger M^{-2} Y$, with these numbers we could not measure any new effect.

Another possibility arises from the fact that M and Y are actually matrices, not numbers: consequently, in the product $Y^T M^{-1} Y$ there could be cancellations that can make the left-handed neutrino masses small, which in turn would not take place in the product $Y^\dagger M^{-2} Y$, which therefore would not be negligible any more. However, such a hypothesis appears at least unreasonable if it is not supported by any physical motivation. Indeed, it is possible to sustain this hypothesis with a deep reason, that is, by means of an approximate symmetry.

Let us suppose then that the Weinberg operator is made small by a small parameter that violates the lepton number. Surprisingly, it is found that the dimension-6 operators, which a priori should be even more suppressed, automatically preserve the lepton number instead: for this reason, they cannot be proportional to the small parameter that breaks the symmetry and may therefore be sufficiently large to be measured. This is the case of inverse see-saw [20].

Let us see how the inverse see-saw is realised when three fermion singlets or triplets are added. For convenience, we will call $N_{1,2,3}$ the added neutral particles, implying that these may be indifferently $SU(2)$ fermion singlets or the charge-0 components of $SU(2)$ fermion triplets.

Temporarily assuming lepton number conservation, we find that the Dirac mass matrix m_D and the Majorana one M_M must take the form

$$m_D = \frac{v}{\sqrt{2}} \begin{pmatrix} Y_{N_1 e} & Y_{N_1 \mu} & Y_{N_1 \tau} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_M = \begin{pmatrix} 0 & \Lambda & 0 \\ \Lambda & 0 & 0 \\ 0 & 0 & \Lambda' \end{pmatrix}, \tag{3.69}$$

where we assigned lepton numbers so that $L_e = L_\mu = L_\tau = L_{N_1} = -L_{N_2} = 1$ and $L_{N_3} = 0$. In this case, we find $m_\nu = 0$ (left-handed neutrinos are massless), $M_{N_1} = M_{N_2} = \Lambda$ (N_1 and N_2 form a Dirac pair) and finally $M_{N_3} = \Lambda'$ (N_3 is a decoupled Majorana fermion). At the same time, we find that $\eta = \frac{1}{2} m_D^\dagger \frac{1}{M^\dagger} \frac{1}{M} m_D \neq 0$, which shows that corrections to the SM are possible even in the limit of massless neutrinos when the lepton number is conserved.

Let us now introduce some small parameters that break the symmetry:

$$m_D = \frac{v}{\sqrt{2}} \begin{pmatrix} Y_{N_1 e} & Y_{N_1 \mu} & Y_{N_1 \tau} \\ \varepsilon_1 Y_{N_2 e} & \varepsilon_1 Y_{N_2 \mu} & \varepsilon_1 Y_{N_2 \tau} \\ \varepsilon_2 Y_{N_3 e} & \varepsilon_2 Y_{N_3 \mu} & \varepsilon_2 Y_{N_3 \tau} \end{pmatrix}, \qquad M_M = \begin{pmatrix} \mu_1 & \Lambda & \mu_3 \\ \Lambda & \mu_2 & \mu_4 \\ \mu_3 & \mu_4 & \Lambda' \end{pmatrix}. \tag{3.70}$$

In this case it can be shown [1, 21] that $m_\nu \neq 0$ and that it takes qualitatively the form:

$$m_\nu \sim f(Y) \frac{v^2}{2} \frac{\mu}{\Lambda^2}, \tag{3.71}$$

(where μ collectively represents the small, lepton-number violating parameters), while

$$\eta \sim g(Y) \frac{v^2}{4} \frac{1}{\Lambda^2} \tag{3.72}$$

is not proportional to μ and therefore can be big enough to change the model's phenomenology significantly (we will study in detail how the phenomenology is modified by the presence of fermion triplets in Chapter 4) and to be therefore measurable.

3.6 Summary

In this chapter, we have seen how the bizarre results of the solar neutrinos experiments conducted in the second half of the 20th century led to the idea of neutrino flavour oscillation. This oscillation depends directly on the squared mass differences between neutrinos of different type: to date, we know almost all the neutrino oscillation parameters, with the exception of the actual ordering of the various masses (the so-called hierarchy problem) and of the complex phase that could generate CP violation, δ_{CP}.

We have seen that the introduction of a large neutrino Majorana mass to the theory entails in a simple and elegant manner the existence of a very small mass for left-handed neutrinos, through the so-called see-saw mechanism. Therefore, we explored possible ways to make a Majorana mass term possible, studying the Weinberg operator. We argued that, at tree-level, there are only three possibilities to have a neutrino Majorana mass: the three types of see-saw.

We summarized the main aspects of the various see-saw models: the type I, in which right-handed neutrinos are introduced; the type II, in which instead a scalar $SU(2)$ triplet with hypercharge 2 is added; and finally the type III, characterized by the introduction of $SU(2)$ fermion triplets with hypercharge 0, whose neutral components behaves just like the right-handed neutrinos of the type I.

Finally, we have observed that, when the light neutrino masses are kept small by an approximate symmetry (the lepton number), since this symmetry is automatically respected by the effective dimension-6 operators, they can be large enough to have measurable effects in current or next generation accelerators.

It is interesting to observe that it is the *differences* of squared masses that determines neutrino oscillation, not the masses themselves: this means that the situation where only two neutrinos are massive while the third one is massless is compatible with the current experimental observations.

In the next chapter, we will study in detail the phenomenology of the type III see-saw model.

Chapter 4

The Type III See-saw Model

4.1 The Type III See-saw Lagrangian

The type III see-saw model consists in the addition to the Standard Model of n $SU(2)$ (Weyl) fermion triplets with zero hypercharge, $\Sigma_R = (\Sigma_1, \Sigma_2, \Sigma_3)$, where $\Sigma^{\pm} = \frac{1}{\sqrt{2}}(\Sigma_1 \mp i\Sigma_2)$, $\Sigma^0 = \Sigma_3$ are its charge eigenstates. The Lagrangian which describes its interactions can be written as:

$$\mathcal{L}_\Sigma = i\overline{\Sigma}^i_R \not{D} \Sigma^i_R - \frac{1}{2}\overline{\Sigma}^i_R (M_\Sigma)_{ij} \Sigma^{cj}_R - \overline{\Sigma}^i_R (Y_\Sigma)_{i\alpha} \tilde{\phi}^\dagger \tau \ell^\alpha_L + \text{h.c.}\,, \tag{4.1}$$

where $i = 1, \ldots, n$, $\tau = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices, $\tilde{\phi} = i\tau_2\phi^*$ with $\phi = (\phi^+, \phi^0)^t = \left(\phi^+, (v + H + i\xi)/\sqrt{2}\right)^t$, $\ell = (\nu, l)^t$, M_Σ is the Majorana mass matrix for the triplets and Y_Σ the Yukawa couplings between the triplets and the Higgs. M_Σ can be assumed to be real and diagonal without loss of generality; furthermore, in order to study the mixing of the charged components of the triplets with the charged leptons, it is convenient to define the Dirac spinor ψ:

$$\psi = \begin{pmatrix} \Sigma^{+c}_R \\ \Sigma^-_R \end{pmatrix}. \tag{4.2}$$

In terms of ψ and Σ^0, and hereafter omitting the generation index i of the triplets for simplicity, the Lagrangian becomes:

$$\begin{aligned}\mathcal{L}_\Sigma = &\overline{\psi} i\not{\partial}\psi + \Sigma^0_R i\not{\partial}\Sigma^0_R - \overline{\psi} M_\Sigma \psi - \frac{1}{2}\left(\overline{\Sigma^0_R} M_\Sigma \Sigma^{0c}_R + \text{h.c.}\right) \\ &+ g\left(W^+_\mu \overline{\Sigma^0_R}\gamma^\mu P_R \psi + W^+_\mu \overline{\Sigma^{0c}_R}\gamma^\mu P_L \psi + \text{h.c.}\right) - g W^3_\mu \overline{\psi}\gamma^\mu\psi \\ &- \left(\phi^0 \overline{\Sigma^0_R} Y_\Sigma \nu_L + \sqrt{2}\phi^0 \overline{\psi} Y_\Sigma l_L + \phi^+ \overline{\Sigma^0_R} Y_\Sigma l_L - \sqrt{2}\phi^+ \overline{\nu^c_L} Y^t_\Sigma \psi\right),\end{aligned} \tag{4.3}$$

with

$$P_{R,L} = \frac{1 \pm \gamma_5}{2}. \tag{4.4}$$

The lepton mass term in the Lagrangian of the type III see-saw model is therefore:

$$\begin{aligned}\mathcal{L}_M = & -\begin{pmatrix}\overline{l_L} & \overline{\psi_L}\end{pmatrix}\begin{pmatrix}m_l & Y_\Sigma^\dagger v \\ 0 & M_\Sigma\end{pmatrix}\begin{pmatrix}l_R \\ \psi_R\end{pmatrix} \\ & -\frac{1}{2}\begin{pmatrix}\overline{\nu_L^c} & \overline{\Sigma^0}\end{pmatrix}\begin{pmatrix}0 & m_D^t \\ m_D & M_\Sigma\end{pmatrix}\begin{pmatrix}\nu_L \\ \Sigma^{0c}\end{pmatrix} + \text{h.c.}\ ,\end{aligned} \tag{4.5}$$

where $m_D = Y_\Sigma v/\sqrt{2}$. As we can see, both the charged leptons' and the neutrinos' mass matrices are non-diagonal, which means that we have to diagonalise them to get the physical masses. In order to do so, we will also introduce a useful way to parametrize the lepton mixing matrices.

4.2 Parametrisation of the Lepton Mixing Matrices

4.2.1 Neutral Leptons

The neutral lepton mass term in the Lagrangian of the type III see-saw model is:

$$\mathcal{L}_M^{NL} = -\frac{1}{2}\begin{pmatrix}\overline{\nu_L^c} & \overline{\Sigma^0}\end{pmatrix}\begin{pmatrix}0 & m_D^t \\ m_D & M_\Sigma\end{pmatrix}\begin{pmatrix}\nu_L \\ \Sigma^{0c}\end{pmatrix} + \text{h.c.} \tag{4.6}$$

We want to diagonalise the mass matrix. In order to do so, we will follow the procedure used in [22]. In this case, the diagonalisation is easy, since the mass matrix is symmetric and hence can be diagonalised with just one orthogonal matrix:

$$\begin{pmatrix}\overline{\nu_L^c} & \overline{\Sigma^0}\end{pmatrix}\begin{pmatrix}0 & m_D^t \\ m_D & M_\Sigma\end{pmatrix}\begin{pmatrix}\nu_L \\ \Sigma^{0c}\end{pmatrix} \equiv \tag{4.7}$$

$$\equiv \begin{pmatrix}\overline{\nu_L^c} & \overline{\Sigma^0}\end{pmatrix} U \begin{pmatrix}m & 0 \\ 0 & M\end{pmatrix} U^t \begin{pmatrix}\nu_L \\ \Sigma^{0c}\end{pmatrix} \equiv \tag{4.8}$$

$$\equiv \begin{pmatrix}\overline{\nu_L^{c\prime}} & \overline{\Sigma^{0\prime}}\end{pmatrix}\begin{pmatrix}m & 0 \\ 0 & M\end{pmatrix}\begin{pmatrix}\nu_L{}' \\ \Sigma^{0c\prime}\end{pmatrix}, \tag{4.9}$$

with m and M real and diagonal. We choose the following parametrisation:

$$U \equiv A\begin{pmatrix}U_{PMNS} & 0 \\ 0 & \mathbb{I}\end{pmatrix}, \tag{4.10}$$

where A is a matrix which block-diagonalises the full mass matrix:

$$\begin{pmatrix} 0 & m_D^t \\ m_D & M_\Sigma \end{pmatrix} = A \begin{pmatrix} \hat{m} & 0 \\ 0 & M \end{pmatrix} A^t, \tag{4.11}$$

while the second matrix diagonalises the single blocks. Here, we defined $U_{PMNS} = U_{23}(\theta_{23})\, U_{13}(\theta_{13}, \delta)\, U_{12}(\theta_{12})\, \mathrm{diag}\left(e^{-i\alpha_1/2}, e^{-i\alpha_2/2}, 1\right)$ as the usual PMNS matrix multiplied by the diagonal Majorana phases matrix. This means, in particular, that

$$\hat{m} = U_{PMNS}\, m\, U_{PMNS}^t. \tag{4.12}$$

Moreover, we parametrise A in the following way:

$$A = e^{\tilde{\Theta}}, \qquad \tilde{\Theta} = \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger & 0 \end{pmatrix}, \tag{4.13}$$

therefore:

$$U = \begin{pmatrix} c & s \\ -s^\dagger & \hat{c} \end{pmatrix} \begin{pmatrix} U_{PMNS} & 0 \\ 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} cU_{PMNS} & s \\ -s^\dagger U_{PMNS} & \hat{c} \end{pmatrix}, \tag{4.14}$$

where we defined

$$c := \sum_{n=0}^{+\infty} \frac{\left(-\Theta\Theta^\dagger\right)^n}{(2n)!}, \qquad \hat{c} := \sum_{n=0}^{+\infty} \frac{\left(-\Theta^\dagger\Theta\right)^n}{(2n)!}, \qquad s := \sum_{n=0}^{+\infty} \frac{\left(-\Theta\Theta^\dagger\right)^n}{(2n+1)!}\Theta. \tag{4.15}$$

With this parametrisation we can diagonalize the mass matrix. When doing so, we find that, at first order in Θ:

$$\Theta = m_D^\dagger M_\Sigma^{-1} \tag{4.16}$$

$$M = M_\Sigma \tag{4.17}$$

$$\hat{m} = -m_D^t M_\Sigma^{-1} m_D. \tag{4.18}$$

When doing these calculations, one must be careful not to neglect important terms: since $\Theta = m_D^\dagger M_\Sigma^{-1}$, it follows that terms like $\Theta^* M_\Sigma \Theta^\dagger$ are in fact of order $\mathcal{O}(\Theta)$, not $\mathcal{O}(\Theta^2)$ (in this example, $M_\Sigma\Theta^\dagger = m_D$).

Now, remembering that $m_D = \frac{v}{\sqrt{2}} Y_\Sigma$, we can explicitly write all the mixing matrix elements at order $\mathcal{O}\left(\left[\left(vY_\Sigma, m_l\right)/M_\Sigma\right]^2\right)$:

$$U = \begin{pmatrix} (1-\eta)\, U_{PMNS} & \frac{v}{\sqrt{2}} Y_\Sigma^\dagger M_\Sigma^{-1} \\ \left(-\frac{v}{\sqrt{2}} M_\Sigma^{-1} Y_\Sigma\right) U_{PMNS} & 1-\eta' \end{pmatrix}, \tag{4.19}$$

where $\eta = \frac{v^2}{4} Y_\Sigma^\dagger M_\Sigma^{-2} Y_\Sigma$ and $\eta' = \frac{v^2}{4} M_\Sigma^{-1} Y_\Sigma Y_\Sigma^\dagger M_\Sigma^{-1}$.

Notice that, with the definitions given throughout this chapter, we have

$$\hat{m} = -m_D^t M_\Sigma^{-1} m_D = -\frac{v^2}{2} c^{d=5}, \tag{4.20}$$

$$\eta = \frac{v^2}{4} Y_\Sigma^\dagger M_\Sigma^{-2} Y_\Sigma = \frac{v^2}{4} c^{d=6}, \tag{4.21}$$

where we defined $c^{d=5,6}$ in Eq.s (3.58, 3.61).

4.2.2 Charged Leptons

We want now to diagonalise the charged lepton mass term:

$$\mathcal{L}_M^{CL} = -\begin{pmatrix} \overline{l_L} & \overline{\psi_L} \end{pmatrix} \begin{pmatrix} m_l & Y_\Sigma^\dagger v \\ 0 & M_\Sigma \end{pmatrix} \begin{pmatrix} l_R \\ \psi_R \end{pmatrix} + \text{h.c.} \tag{4.22}$$

This time the procedure is somewhat more difficult than that of the neutral case, because the mass matrix is neither symmetric nor hermitian and therefore we will have to use two distinct unitary matrices:

$$\begin{aligned} &\begin{pmatrix} \overline{l_L} & \overline{\psi_L} \end{pmatrix} \begin{pmatrix} m_l & Y_\Sigma^\dagger v \\ 0 & M_\Sigma \end{pmatrix} \begin{pmatrix} l_R \\ \psi_R \end{pmatrix} \equiv \\ &\equiv \begin{pmatrix} \overline{l_L} & \overline{\psi_L} \end{pmatrix} U_L^{TOT} \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix} U_R^{TOT\dagger} \begin{pmatrix} l_R \\ \psi_R \end{pmatrix} \equiv \\ &\equiv \begin{pmatrix} \overline{l'_L} & \overline{\psi'_L} \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} l'_R \\ \psi'_R \end{pmatrix}. \end{aligned} \tag{4.23}$$

In analogy with the neutral case, we choose the following parametrisation for the mixing matrices:

$$U_{LR}^{TOT} \equiv e^{\tilde{\Theta}_{LR}}, \qquad \tilde{\Theta}_{LR} = \begin{pmatrix} 0 & \Theta_{LR} \\ -\Theta_{LR}^\dagger & 0 \end{pmatrix}, \tag{4.24}$$

therefore:

$$U_{LR}^{TOT} = \begin{pmatrix} c_{LR} & s_{LR} \\ -s_{LR}^\dagger & \hat{c}_{LR} \end{pmatrix}, \tag{4.25}$$

where we defined

$$c_{LR} := \sum_{n=0}^{+\infty} \frac{\left(-\Theta_{LR}\Theta_{LR}^\dagger\right)^n}{(2n)!}, \qquad \hat{c}_{LR} := \sum_{n=0}^{+\infty} \frac{\left(-\Theta_{LR}^\dagger\Theta_{LR}\right)^n}{(2n)!},$$
$$s_{LR} := \sum_{n=0}^{+\infty} \frac{\left(-\Theta_{LR}\Theta_{LR}^\dagger\right)^n}{(2n+1)!} \Theta_{LR}. \tag{4.26}$$

With this parametrisation we can diagonalize the mass matrix. When doing so, we find that, at first order in Θ_{LR}:

$$\Theta_L = vY_\Sigma^\dagger M_\Sigma^{-1} \ , \tag{4.27}$$

$$\Theta_R = vm_l Y_\Sigma^\dagger M_\Sigma^{-2} \ . \tag{4.28}$$

Now we can explicitly write all the mixing matrix elements at order $\mathcal{O}\left([(vY_\Sigma, m_l)/M_\Sigma]^2\right)$:

$$U_L^{TOT} = \begin{pmatrix} 1-2\eta & vY_\Sigma^\dagger M_\Sigma^{-1} \\ -vM_\Sigma^{-1}Y_\Sigma & 1-2\eta' \end{pmatrix}, \tag{4.29}$$

$$U_R^{TOT} = \begin{pmatrix} 1 & vm_l Y_\Sigma^\dagger M_\Sigma^{-2} \\ -vM_\Sigma^{-2}Y_\Sigma m_l & 1 \end{pmatrix}, \tag{4.30}$$

where again $\eta = \frac{v^2}{4} Y_\Sigma^\dagger M_\Sigma^{-2} Y_\Sigma$ and $\eta' = \frac{v^2}{4} M_\Sigma^{-1} Y_\Sigma Y_\Sigma^\dagger M_\Sigma^{-1}$.

4.3 Type III See-saw Lagrangian in the Mass Basis

After the rotation of the lepton and Σ fields to the mass basis, the full type III see-saw Lagrangian can be written as:

$$\mathcal{L} = \mathcal{L}_Q + \mathcal{L}_0 + \mathcal{L}_{em} + \mathcal{L}_{CC} + \mathcal{L}_{NC} + \mathcal{L}_H + \mathcal{L}_\xi + \mathcal{L}_{\phi^-}, \tag{4.31}$$

where $\mathcal{L}_Q$ is the quark Lagrangian as in the SM, $\mathcal{L}_0$ is the lepton kinetic term, and:

$$\mathcal{L}_{em} = -e \begin{pmatrix} \bar{l} & \overline{\psi} \end{pmatrix} \gamma^\mu A_\mu \begin{pmatrix} l \\ \psi \end{pmatrix} \tag{4.32}$$

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \begin{pmatrix} \bar{l} & \overline{\psi} \end{pmatrix} \gamma^\mu W_\mu^- \left(P_L g_L^{CC} + P_R g_R^{CC}\right) \begin{pmatrix} \nu \\ \Sigma \end{pmatrix} + \text{h.c.} \tag{4.33}$$

$$\mathcal{L}_{NC} = \frac{g}{c_W} \left[\begin{pmatrix} \bar{l} & \overline{\psi} \end{pmatrix} \gamma^\mu Z_\mu \left(P_L g_L^{NC} + P_R g_R^{NC}\right) \begin{pmatrix} l \\ \psi \end{pmatrix} + \begin{pmatrix} \overline{\nu} & \overline{\Sigma} \end{pmatrix} \gamma^\mu Z_\mu P_L g_\nu^{NC} \begin{pmatrix} \nu \\ \Sigma \end{pmatrix} \right] \tag{4.34}$$

$$\mathcal{L}_H = \frac{g}{2M_W} \begin{pmatrix} \bar{l} & \overline{\psi} \end{pmatrix} H \left(P_L g_L^H + P_R g_R^H\right) \begin{pmatrix} l \\ \psi \end{pmatrix} + \left[\begin{pmatrix} \overline{\nu} & \overline{\Sigma} \end{pmatrix} H P_L g_\nu^H \begin{pmatrix} \nu \\ \Sigma \end{pmatrix} + \text{h.c.} \right] \tag{4.35}$$

$$\mathcal{L}_\xi = \frac{ig}{2M_W} \begin{pmatrix} \bar{l} & \overline{\psi} \end{pmatrix} \xi \left(P_L g_L^\xi + P_R g_R^\xi\right) \begin{pmatrix} l \\ \psi \end{pmatrix} + i \left[\begin{pmatrix} \overline{\nu} & \overline{\Sigma} \end{pmatrix} \xi P_L g_\nu^H \begin{pmatrix} \nu \\ \Sigma \end{pmatrix} - \text{h.c.} \right] \tag{4.36}$$

$$\mathcal{L}_{\phi^-} = -\frac{g}{\sqrt{2}M_W}\begin{pmatrix}\bar{l} & \overline{\psi}\end{pmatrix}\phi^-\left(P_L g_L^{\phi^-} + P_R g_R^{\phi^-}\right)\begin{pmatrix}\nu\\ \Sigma\end{pmatrix} + \text{h.c.} \tag{4.37}$$

with

$$g_L^{CC} = \begin{pmatrix} g_{L_{l\nu}}^{CC} & g_{L_{l\Sigma}}^{CC} \\ g_{L_{\psi\nu}}^{CC} & g_{L_{\psi\Sigma}}^{CC}\end{pmatrix} = \begin{pmatrix}(1+\eta)\,U_{PMNS} & -\frac{v}{\sqrt{2}}Y_\Sigma^\dagger M_\Sigma^{-1} \\ 0 & \sqrt{2}\,(1-\eta')\end{pmatrix} \tag{4.38}$$

$$g_R^{CC} = \begin{pmatrix} g_{R_{l\nu}}^{CC} & g_{R_{l\Sigma}}^{CC} \\ g_{R_{\psi\nu}}^{CC} & g_{R_{\psi\Sigma}}^{CC}\end{pmatrix} = \begin{pmatrix}0 & -\sqrt{2}vm_l Y_\Sigma^\dagger M_\Sigma^{-2} \\ -vM_\Sigma^{-1}Y_\Sigma^* U_{PMNS}^* & \sqrt{2}\,(1-\eta'^*)\end{pmatrix} \tag{4.39}$$

$$g_L^{NC} = \begin{pmatrix} g_{L_{ll}}^{NC} & g_{L_{l\psi}}^{NC} \\ g_{L_{\psi l}}^{NC} & g_{L_{\psi\psi}}^{NC}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} - \cos^2\theta_W - 2\eta & \frac{v}{2}Y_\Sigma^\dagger M_\Sigma^{-1} \\ \frac{v}{2}M_\Sigma^{-1}Y_\Sigma & 2\eta' - \cos^2\theta_W\end{pmatrix} \tag{4.40}$$

$$g_R^{NC} = \begin{pmatrix} g_{R_{ll}}^{NC} & g_{R_{l\psi}}^{NC} \\ g_{R_{\psi}}^{NC} & g_{R_{\psi\psi}}^{NC}\end{pmatrix} = \begin{pmatrix}1 - \cos^2\theta_W & vm_l Y_\Sigma^\dagger M_\Sigma^{-2} \\ vM_\Sigma^{-2}Y_\Sigma m_l & -\cos^2\theta_W\end{pmatrix} \tag{4.41}$$

$$g_\nu^{NC} = \begin{pmatrix} g_{\nu_{\nu\nu}}^{NC} & g_{\nu_{\nu\Sigma}}^{NC} \\ g_{\nu_{\Sigma\nu}}^{NC} & g_{\nu_{\Sigma\Sigma}}^{NC}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}U_{PMNS}^\dagger(1-2\eta)\,U_{PMNS} & \frac{v}{2\sqrt{2}}U_{PMNS}^\dagger Y_\Sigma^\dagger M_\Sigma^{-1} \\ \frac{v}{2\sqrt{2}}M_\Sigma^{-1}Y_\Sigma U_{PMNS} & \eta'\end{pmatrix} \tag{4.42}$$

$$g_L^{H} = \begin{pmatrix} g_{L_{ll}}^{H} & g_{L_{l\psi}}^{H} \\ g_{L_{\psi l}}^{H} & g_{L_{\psi\psi}}^{H}\end{pmatrix} = \begin{pmatrix}m_l\,(6\eta-1) & -vm_l Y_\Sigma^\dagger M_\Sigma^{-1} \\ -vY_\Sigma\,(1-2\eta) - vM_\Sigma^{-2}Y_\Sigma m_l^2 & -v^2 Y_\Sigma Y_\Sigma^\dagger M_\Sigma^{-1}\end{pmatrix} \tag{4.43}$$

$$g_R^{H} = \begin{pmatrix} g_{R_{ll}}^{H} & g_{R_{l\psi}}^{H} \\ g_{R_{\psi l}}^{H} & g_{R_{\psi\psi}}^{H}\end{pmatrix} = \begin{pmatrix}(6\eta-1)\,m_l & -v\,(1-2\eta)\,Y_\Sigma^\dagger - vm_l^2 Y_\Sigma^\dagger M_\Sigma^{-2} \\ -vM_\Sigma^{-1}Y_\Sigma m_l & -v^2 M_\Sigma^{-1}Y_\Sigma Y_\Sigma^\dagger\end{pmatrix} \tag{4.44}$$

$$\begin{aligned} g_\nu^{H} &= \begin{pmatrix} g_{\nu_{\nu\nu}}^{H} & g_{\nu_{\nu\Sigma}}^{H} \\ g_{\nu_{\Sigma\nu}}^{H} & g_{\nu_{\Sigma\Sigma}}^{H}\end{pmatrix} = \\ &= \begin{pmatrix}-\frac{1}{v}m & -\frac{1}{\sqrt{2}}mU_{PMNS}^\dagger Y_\Sigma^\dagger M_\Sigma^{-1} \\ \frac{1}{2\sqrt{2}}\left[(2Y_\Sigma\eta - Y_\Sigma)\,U_{PMNS} - M_\Sigma^{-1}Y_\Sigma^* U_{PMNS}^* m\right] & -\frac{v}{2}Y_\Sigma Y_\Sigma^\dagger M_\Sigma^{-1}\end{pmatrix}\end{aligned} \tag{4.45}$$

$$g_L^{\xi} = \begin{pmatrix} g_{L_{ll}}^{\xi} & g_{L_{l\psi}}^{\xi} \\ g_{L_{\psi l}}^{\xi} & g_{L_{\psi\psi}}^{\xi}\end{pmatrix} = \begin{pmatrix}m_l\,(1+2\eta) & vm_l Y_\Sigma^\dagger M_\Sigma^{-1} \\ -vY_\Sigma\,(1-2\eta) + vM_\Sigma^{-2}Y_\Sigma m_l^2 & -v^2 Y_\Sigma Y_\Sigma^\dagger M_\Sigma^{-1}\end{pmatrix} \tag{4.46}$$

$$g_R^{\xi} = \begin{pmatrix} g_{R_{ll}}^{\xi} & g_{R_{l\psi}}^{\xi} \\ g_{R_{\psi l}}^{\xi} & g_{R_{\psi\psi}}^{\xi}\end{pmatrix} = \begin{pmatrix}-(1+2\eta)\,m_l & v\,(1-2\eta)\,Y_\Sigma^\dagger - vm_l^2 Y_\Sigma^\dagger M_\Sigma^{-2} \\ -vM_\Sigma^{-1}Y_\Sigma m_l & -v^2 M_\Sigma^{-1}Y_\Sigma Y_\Sigma^\dagger\end{pmatrix} \tag{4.47}$$

$$g_L^{\phi^-} = \begin{pmatrix} g_{L_{l\nu}}^{\phi^-} & g_{L_{l\Sigma}}^{\phi^-} \\ g_{L_{\psi\nu}}^{\phi^-} & g_{L_{\psi\Sigma}}^{\phi^-}\end{pmatrix} = \begin{pmatrix}m_l\,(1-\eta)\,U_{PMNS} & \frac{v}{\sqrt{2}}m_l Y_\Sigma^\dagger M_\Sigma^{-1} \\ vM_\Sigma^{-2}Y_\Sigma m_l^2 U_{PMNS} & 0\end{pmatrix} \tag{4.48}$$

$$g_R^{\phi^-} = \begin{pmatrix} g_{R_{l\nu}}^{\phi^-} & g_{R_{l\Sigma}}^{\phi^-} \\ g_{R_{\psi\nu}}^{\phi^-} & g_{R_{\psi\Sigma}}^{\phi^-}\end{pmatrix} =$$

$$= \begin{pmatrix} -U_{PMNS}m & \frac{v}{\sqrt{2}}\left(Y_\Sigma^\dagger - 2\eta Y_\Sigma^\dagger - \frac{3}{2}U_{PMNS}mU_{PMNS}^t Y_\Sigma^t M_\Sigma^{-1}\right) \\ -vY_\Sigma^*\left(1-\eta^*\right)U_{PMNS}^* & 2\sqrt{2}\left(\eta' M_\Sigma - M_\Sigma \eta'^*\right) \end{pmatrix} \tag{4.49}$$

We point out that the coupling matrix for the $\Sigma H \nu$ interaction is exactly the same as in the $\Sigma \xi \nu$ interaction.

As we can see, many couplings that were already present in the Standard Model have been modified and at the same time some new couplings arose. It follows that the phenomenology for this model will be different from the Standard Model one, since many experimental processes that have been measured are different in this model, and processes that are extremely suppressed within the Standard Model (like the FCNC ones) can actually take place at tree-level in this case. For these reasons, we will present the modified rates for some of these processes in the next Section. In particular, we will only evaluate the results for tree-level processes, since this model allows charged lepton flavour violation already at tree-level, therefore the contribution of one-loop corrections is subdominant.

4.4 Main Observables in the Type III See-saw Model

In this section the leading order dependence on $\eta_{\alpha\beta}$ of the most constraining electroweak and flavor observables is presented and discussed. We will use these observables as fit constraints to put bounds on the entries $\eta_{\alpha\beta}$ in Chapter 6.

The SM loop corrections are relevant for these precision observables, and have therefore been taken into account [23] in the results presented in Chapter 6. However, in order to simplify the discussion, they will not be included in the analytic expressions of the observables presented in this section as we are interested in highlighting the corrections stemming from $\eta_{\alpha\beta}$ instead. On the other hand, the 1-loop contributions of the new degrees of freedom are not expected to play an important role on the determination of the bounds on $\eta_{\alpha\beta}$, and therefore they will be neglected [24].

Finally, all the observables will be given in terms of α, M_Z and G_F as measured in μ decay, G_μ [23], making the SM predictions in terms of these three parameters

$$\begin{aligned} \alpha &= (7.2973525698 \pm 0.0000000024) \cdot 10^{-3}\,, \\ M_Z &= (91.1876 \pm 0.0021)\ \text{GeV}\,, \\ G_\mu &= (1.1663787 \pm 0.0000006) \cdot 10^{-5}\ \text{GeV}^{-2}\,. \end{aligned} \tag{4.50}$$

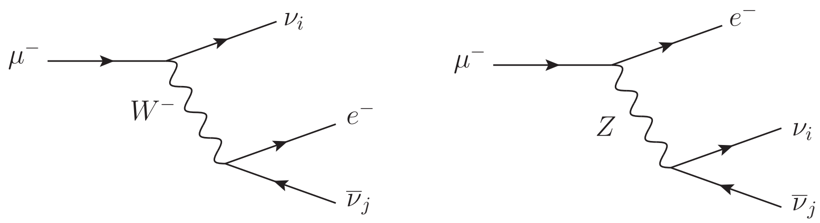


Figure 4.1: Diagrams contributing to the $\mu \to e\nu_i\bar{\nu}_j$ decay rate. At leading order in $\eta_{\alpha\beta}$, the contribution of the LFV decay mediated by the Z boson is subleading and will be neglected.

4.4.1 Constraints from μ-decay: M_W and $\theta_\mathbf{W}$

The presence of new degrees of freedom modifies not only the CC interactions, but also the NC interactions leading to charged Lepton Flavor Violation (LFV) already at tree level. Therefore, the expected decay rate of $\mu \to e\nu_i\bar{\nu}_j$ will receive contributions mediated by both W and Z bosons, as shown in Figure 4.1. However, the contribution mediated by the Z boson and the interference term are proportional to $|\eta_{\mu e}|^2$ and $|\eta_{\mu e}|^3$ respectively, and thus subleading with respect to the linear η_{ee} and $\eta_{\mu\mu}$ corrections present in the W exchange. Then, the decay rate is given by

$$\Gamma_\mu = \frac{m_\mu^5 G_F^2}{192\pi^3}\left(1 + 2\eta_{ee} + 2\eta_{\mu\mu}\right) \equiv \frac{m_\mu^5 G_\mu^2}{192\pi^3}\,. \tag{4.51}$$

Thus, the determination of G_F itself through the muon decay acquires a correction from the $d = 6$ operator $\eta_{\alpha\beta}$ that will affect all other electroweak observables

$$G_F = G_\mu\left(1 - \eta_{ee} - \eta_{\mu\mu}\right)\,. \tag{4.52}$$

In particular, the relation between G_F and M_W allows to constrain the elements η_{ee} and $\eta_{\mu\mu}$ through kinematic measurements of M_W together with M_Z and α, unaffected by $\eta_{\alpha\beta}$

$$G_\mu = \frac{\alpha\pi M_Z^2\left(1 + \eta_{ee} + \eta_{\mu\mu}\right)}{\sqrt{2}M_W^2\left(M_Z^2 - M_W^2\right)}\,. \tag{4.53}$$

Similarly, the weak mixing angle $\sin^2\theta_W$ determined by G_μ, M_Z and α acquires a dependence on η_{ee} and $\eta_{\mu\mu}$

$$\sin^2\theta_W = \frac{1}{2}\left(1 - \sqrt{1 - \frac{2\sqrt{2}\alpha\pi}{G_\mu M_Z^2}\left(1 + \eta_{ee} + \eta_{\mu\mu}\right)}\right)\,. \tag{4.54}$$

Thus, processes containing Z boson couplings to quarks or charged leptons allow to further constrain these parameters.

Observable	SM prediction	Experimental value
$M_W = M_W^{\rm SM}\left(1 - 0.20\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	(80.363 ± 0.006) GeV	(80.379 ± 0.012) GeV

Table 4.1: The W boson mass: in the first column, the leading order corrections on the new physics parameters η; in the second, the SM prediction (including loop corrections) of M_W; in the third, the experimental value.

4.4.2 Constraints from Z decays

The different precision measurements performed by LEP and SLC at the Z peak become a powerful tool to study the extra contributions to the Z couplings under the presence of heavy fermion triplets. Among the possible observables containing Z decays, we found that the invisible decay of the Z, 6 rates of Z decays into different charged fermions, and 6 Z-pole asymmetries are the most constraining. Table 4.2 summarizes the expressions of these 13 observables at leading order in $\eta_{\alpha\beta}$, together with the SM predictions and their experimental values.

Invisible Z decay

The invisible Z decay is corrected directly via the non-diagonal Z coupling to neutrinos and indirectly via Eq. (4.52)

$$\Gamma_{\rm inv} = \frac{G_\mu M_Z^3}{12\sqrt{2}\pi}\Big(3 - 7\left(\eta_{ee} + \eta_{\mu\mu}\right) - 4\eta_{\tau\tau}\Big) \equiv \frac{G_\mu M_Z^3 N_\nu}{12\sqrt{2}\pi}\,, \tag{4.55}$$

where $N_\nu = 2.990 \pm 0.007$ is the number of active neutrinos determined through the invisible Z decay measured by LEP [25].

Z decays into charged fermions

The charged lepton NC interactions are also modified with respect to the SM (see Eq. (3.62)) and thus the decay rate of Z into charged leptons, $\Gamma\left(Z \to l_\alpha \bar{l}_\alpha\right) \equiv \Gamma_l$, is directly sensitive to $\eta_{\alpha\alpha}$

$$\Gamma_l = \frac{G_\mu M_Z^3}{3\sqrt{2}\pi}\left\{\left[\sin^4\theta_W + \left(\sin^2\theta_W - \frac{1}{2}\right)^2\right]\left(1 - \eta_{ee} - \eta_{\mu\mu}\right) + 4\left(\frac{1}{2} - \sin^2\theta_W\right)\eta_{\alpha\alpha}\right\}, \tag{4.56}$$

where the indirect η_{ee} and $\eta_{\mu\mu}$ corrections from the determination of G_F in muon decays have been explicitly added, while $\sin^2\theta_W$ implicitly introduces extra corrections via Eq. (4.54).

Observable	SM prediction	Experimental value
$\Gamma_{\rm inv} = \Gamma_{\rm inv}^{\rm SM}\left(1 - 2.33\left(\eta_{ee} + \eta_{\mu\mu}\right) - 1.33\eta_{\tau\tau}\right)$	(0.50144 ± 0.00004) GeV	(0.4990 ± 0.0015) GeV
$R_e = R_e^{\rm SM}\left(1 - 8.83\eta_{ee} - 0.26\eta_{\mu\mu}\right)$	20.737 ± 0.010	20.804 ± 0.050
$R_\mu = R_\mu^{\rm SM}\left(1 - 0.26\eta_{ee} - 8.83\eta_{\mu\mu}\right)$	20.740 ± 0.010	20.785 ± 0.033
$R_\tau = R_\tau^{\rm SM}\left(1 - 0.26\left(\eta_{ee} + \eta_{\mu\mu}\right) - 8.57\eta_{\tau\tau}\right)$	20.782 ± 0.010	20.764 ± 0.045
$R_c = R_c^{\rm SM}\left(1 - 0.12\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	0.17221 ± 0.00003	0.1721 ± 0.0030
$R_b = R_b^{\rm SM}\left(1 + 0.06\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	0.21582 ± 0.00002	0.21629 ± 0.00066
$\sigma_{\rm had}^0 = \sigma_{\rm had}^{0\,\rm SM}\left(1 + 8.55\eta_{ee} - 0.02\eta_{\mu\mu} - 0.04\eta_{\tau\tau}\right)$	(41.481 ± 0.008) nb	(41.541 ± 0.037) nb
$A_e = A_e^{\rm SM}\left(1 + 30.6\eta_{ee} - 16.5\eta_{\mu\mu}\right)$	0.1469 ± 0.0003	0.1515 ± 0.0019
$A_\mu = A_\mu^{\rm SM}\left(1 - 16.5\eta_{ee} + 30.6\eta_{\mu\mu}\right)$	0.1469 ± 0.0003	0.142 ± 0.015
$A_\tau = A_\tau^{\rm SM}\left(1 - 16.5\eta_{ee} - 16.5\eta_{\mu\mu} + 47.1\eta_{\tau\tau}\right)$	0.1469 ± 0.0003	0.143 ± 0.004
$A_b = A_b^{\rm SM}\left(1 - 0.22\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	0.9347 ± 0.0001	0.923 ± 0.020
$A_c = A_c^{\rm SM}\left(1 - 1.66\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	0.6677 ± 0.0001	0.670 ± 0.027
$A_s = A_s^{\rm SM}\left(1 - 0.22\left(\eta_{ee} + \eta_{\mu\mu}\right)\right)$	0.9356 ± 0.0001	0.90 ± 0.09

Table 4.2: List of the considered flavor conserving observables containing Z: in the first column, the leading order corrections on the parameters η; in the second, the SM predictions (including loop corrections) [23]; in the third, the experimental values [23] used in the fit.

On the other hand, even though the Z boson couplings to quarks remain the same as in the SM, the decay rates of Z into quarks $\Gamma\left(Z \to q\bar{q}\right) \equiv \Gamma_q$ present indirect corrections from G_F and $\sin^2\theta_W$

$$\Gamma_q = \frac{3G_\mu M_Z^3\left(\left(T_q - 2Q_q\sin^2\theta_W\right)^2 + T_q^2\right)}{2\sqrt{2}\pi}\left(1 - \eta_{ee} - \eta_{\mu\mu}\right)\,, \tag{4.57}$$

where Q_q and T_q are the electric charge and the third component of the weak isospin of the given quark q, respectively. Here, η_{ee} and $\eta_{\mu\mu}$ corrections are also implicit in $\sin^2\theta_W$.

These Z decay rates into charged fermions are combined to construct the following observables

$$R_q = \frac{\Gamma_q}{\Gamma_{\rm had}}\,, \quad R_l = \frac{\Gamma_{\rm had}}{\Gamma_l}\,, \quad \text{and} \quad \sigma_{\rm had}^0 = \frac{12\pi\Gamma_e\Gamma_{\rm had}}{M_Z^2\Gamma_Z^2}\,; \tag{4.58}$$

with $\Gamma_{\rm had} \equiv \sum_{q\neq t}\Gamma_q$, and where $\Gamma_Z = \Gamma_e + \Gamma_\mu + \Gamma_\tau + \Gamma_{\rm inv} + \Gamma_{\rm had}$ is the total Z width.

Z asymmetry parameters

Measurements of the Z-pole asymmetries, made by the LEP collaborations and by SLD at SLAC, are additional observables which include the polarization and the forward-backward asymmetry. These observables are ultimately sensitive to the combination (see for instance [23])

$$A_f = \frac{2g_V^f g_A^f}{(g_V^f)^2 + (g_A^f)^2} . \tag{4.59}$$

In particular, including the Type-III Seesaw corrections at leading order, the corresponding expression for charged leptons with flavor α is given by

$$A_\alpha = \frac{1 - 4\sin^2\theta_W}{1 - 4\sin^2\theta_W + 8\sin^4\theta_W} + \frac{64\sin^4\theta_W(1 - 2\sin^2\theta_W)}{(1 - 4\sin^2\theta_W + 8\sin^4\theta_W)^2}\eta_{\alpha\alpha}, \tag{4.60}$$

where $\sin^2\theta_W$ implicitly introduces extra η_{ee} and $\eta_{\mu\mu}$ corrections via Eq. (4.54). In the quark case only these indirect corrections are present

$$A_q = \frac{T_q(T_q - 2Q_q\sin^2\theta_W)}{T_q^2 + 2Q_q^2\sin^4\theta_W - 2Q_qT_q\sin^2\theta_W}. \tag{4.61}$$

4.4.3 Constraints from weak interaction universality tests

Lepton flavor universality of weak interactions can be probed for measuring ratios of decay rates of charged leptons, W, or mesons into charged leptons of different flavors. These decay rates mediated by the W boson acquire corrections proportional to $(1+2\eta_{\alpha\alpha})$, where α is the flavor of the corresponding charged lepton. By doing the ratio between different flavors, the uncertainties of the common variables involved in the two decays cancel out. The weak interaction universality ratios, given by

$$\frac{\Gamma_\alpha^P}{\Gamma_\beta^P} \equiv \frac{\Gamma_\alpha^P|_{\text{SM}}}{\Gamma_\beta^P|_{\text{SM}}}\left(R_{\alpha\beta}^P\right)^2 = \frac{\Gamma_\alpha^P|_{\text{SM}}}{\Gamma_\beta^P|_{\text{SM}}}\left(1 + 2\eta_{\alpha\alpha} - 2\eta_{\beta\beta}\right) , \tag{4.62}$$

become thus powerful observables to indirectly probe for the existence of heavy fermion triplets. In the above equation, the phase space, chirality flip factors and SM loop corrections are encoded in $\Gamma_\alpha^P|_{\text{SM}}$, the SM expectation of the decay width of the parent particle P involving a charged fermion of flavor α. The decay rates containing the SM loop corrections from [26] have been used to derived the experimental constraints on $R_{\alpha\beta}^P$ shown in Table 4.3.

Observable	SM prediction	Experimental value
$R^{\pi}_{\mu e} = (1 + (\eta_{\mu\mu} - \eta_{ee}))$	1	1.0042 ± 0.0022
$R^{\pi}_{\tau\mu} = (1 + (\eta_{\tau\tau} - \eta_{\mu\mu}))$	1	0.9941 ± 0.0059
$R^{W}_{\mu e} = (1 + (\eta_{\mu\mu} - \eta_{ee}))$	1	0.992 ± 0.020
$R^{W}_{\tau\mu} = (1 + (\eta_{\tau\tau} - \eta_{\mu\mu}))$	1	1.071 ± 0.025
$R^{K}_{\mu e} = (1 + (\eta_{\mu\mu} - \eta_{ee}))$	1	0.9956 ± 0.0040
$R^{K}_{\tau\mu} = (1 + (\eta_{\tau\tau} - \eta_{\mu\mu}))$	1	0.978 ± 0.014
$R^{l}_{\mu e} = (1 + (\eta_{\mu\mu} - \eta_{ee}))$	1	1.0040 ± 0.0032
$R^{l}_{\tau\mu} = (1 + (\eta_{\tau\tau} - \eta_{\mu\mu}))$	1	1.0029 ± 0.0029

Table 4.3: List of the most relevant universality ratios: in the first column, the leading order corrections on the η parameters; in the second, the SM prediction; in the third, the experimental values [26] used in the fit.

4.4.4 Unitarity of the CKM matrix

Even though the Unitarity of the CKM quark mixing matrix is not directly affected in the Type-III Seesaw, the elements of the CKM matrix $V_{qq'}$ are measured through processes which are modified by the new degrees of freedom. These modifications will happen also in an indirect way, through the determinations of G_μ and s_W via Eq. (4.52) and Eq. (4.54), as discussed above.

Starting from the Unitarity relation among the three elements $V_{uq'}$ of the CKM matrix, the following relation between $|V_{ud}|$ and $|V_{us}|$ is obtained:

$$|V_{ud}| = \sqrt{1 - |V_{us}|^2}\,, \tag{4.63}$$

where $V_{ub} = (4.13 \pm 0.49) \times 10^{-3}$ [23] has been neglected since $|V_{ub}|^2$ is much smaller than the present accuracy on $|V_{us}|^2$.

In the following, the dependence on the $\eta_{\alpha\beta}$ parameters of the different processes used to constrain the CKM elements $|V_{ud}|$ and $|V_{us}|$ will be discussed. These observables will be incorporated in the global fit as a function of V_{us} via Eq. (4.63). Finally, $|V_{us}|$ will be treated as a nuisance parameter of the global fit, choosing its value in such a way that χ^2 is minimized for each value of the involved $\eta_{\alpha\beta}$ parameters.

Table 4.4 summarizes the dependence on the $\eta_{\alpha\beta}$ parameters of the 9 observables constraining the CKM Unitarity considered and their experimental values.

Determination of $|V_{ud}|$ via Superallowed β decays

The best determination of $|V_{ud}|$ comes from Superallowed $0^+ \to 0^+$ nuclear β decays. It receives both a direct correction with $(1+\eta_{ee})$ from the CC coupling to e, and an indirect one from G_F via Eq. (4.52), resulting in the following expression

$$\left|V_{ud}^{\beta}\right| = (1-\eta_{\mu\mu})\,|V_{ud}|\ . \tag{4.64}$$

Table 4.4 shows the value of $\left|V_{ud}^{\beta}\right|$ based on the 20 different Superallowed β transitions [27].

Observable	SM prediction	Experimental value
$\left\|V_{ud}^{\beta}\right\| = \sqrt{1-\|V_{us}\|^2}(1-\eta_{\mu\mu})$	$\sqrt{1-\|V_{us}\|^2}$	0.97420 ± 0.00021 [27]
$\left\|V_{us}^{\tau\to K\nu}\right\| = \|V_{us}\|\,(1-\eta_{ee}-\eta_{\mu\mu}+\eta_{\tau\tau})$	$\|V_{us}\|$	0.2186 ± 0.0021 [28]
$\left\|V_{us}^{\tau\to K,\pi}\right\| = \|V_{us}\|\,(1-\eta_{\mu\mu})$	$\|V_{us}\|$	0.2236 ± 0.0018 [28]
$\left\|V_{us}^{K_L\to\pi e\overline{\nu}}\right\| = \|V_{us}\|\,(1-\eta_{\mu\mu})$	$\|V_{us}\|$	0.2237 ± 0.0011 [29]
$\left\|V_{us}^{K_L\to\pi\mu\overline{\nu}}\right\| = \|V_{us}\|\,(1-\eta_{ee})$	$\|V_{us}\|$	0.2240 ± 0.0011 [29]
$\left\|V_{us}^{K_S\to\pi e\overline{\nu}}\right\| = \|V_{us}\|\,(1-\eta_{\mu\mu})$	$\|V_{us}\|$	0.2229 ± 0.0016 [29]
$\left\|V_{us}^{K^{\pm}\to\pi e\overline{\nu}}\right\| = \|V_{us}\|\,(1-\eta_{\mu\mu})$	$\|V_{us}\|$	0.2247 ± 0.0012 [29]
$\left\|V_{us}^{K\pm\to\pi\mu\overline{\nu}}\right\| = \|V_{us}\|\,(1-\eta_{ee})$	$\|V_{us}\|$	0.2245 ± 0.0014 [29]
$\left\|V_{us}^{K,\pi\to\mu\nu}\right\| = \|V_{us}\|\,(1-\eta_{\mu\mu})$	$\|V_{us}\|$	0.2315 ± 0.0010 [30]

Table 4.4: List of observables testing the Unitarity of the CKM matrix: in the first column, the leading order corrections on the η parameters; in the second, the SM predictions; in the third, the experimental values [27, 28, 29, 30] used in the fit.

Determination of $|V_{us}|$

$|V_{us}|$ is presently determined via the measurement of τ decays into K or π and semileptonic and leptonic K decays. In this work the values of the form factor $f_+(0)$ and decay constant f_K/f_π involved in these processes have been taken from Ref. [31].

- Via τ decays

The $\tau \to K\nu$ decay rate is proportional to the CKM element $|V_{us}|$ and is directly corrected via the τ CC coupling and indirectly via G_μ

$$|V_{us}^{\tau\to K\nu}| = (1 - \eta_{ee} - \eta_{\mu\mu} + \eta_{\tau\tau})\,|V_{us}|\,. \tag{4.65}$$

Notice that the present experimental value of $\left|V_{us}^{\tau\to K\nu}\right|$ given in Table 4.4 is in tension with other determinations [28].

A complementary way to constrain $|V_{us}|$ is via the ratio $Br\,(\tau \to K\nu)$ / $Br\,(\tau \to \pi\nu)$. In this case, the dependence on the $\eta_{\alpha\beta}$ parameters cancels out. Thus, this observable, sensitive to $|V_{us}|/|V_{ud}|$, remains unaffected by the presence of extra degrees of freedom. When combined with $|V_{ud}^{\beta}|$ from Eq. (4.64), we obtain for $\left|V_{us}^{\tau\to K,\pi}\right|$

$$\left|V_{us}^{\tau\to K,\pi}\right| = (1 - \eta_{\mu\mu})\,|V_{us}|\ . \tag{4.66}$$

- Via K decays

  In the decay rate $K \to \pi l_\alpha \overline{\nu}_\alpha$ (with $\alpha = \mu, e$), the direct $\eta_{\mu\mu}$ (η_{ee}) correction from the W coupling to μ (e) cancels with the indirect $\eta_{\mu\mu}$ (η_{ee}) one introduced by G_μ, leading to the following dependence on the new physics parameters

$$\left|V_{us}^{K\to\pi e\overline{\nu}}\right| = (1 - \eta_{\mu\mu})\,|V_{us}|\ , \tag{4.67}$$

$$\left|V_{us}^{K\to\pi \mu\overline{\nu}}\right| = (1 - \eta_{ee})\,|V_{us}|\ . \tag{4.68}$$

The present determinations of $\left|V_{us}^{K\to\pi e\overline{\nu}}\right|$ and $\left|V_{us}^{K\to\pi\mu\overline{\nu}}\right|$ come from measurements of the different decays of K_L, K_S and $K^\pm$ listed in Table 4.4.

Finally, as in the $\left|V_{us}^{\tau\to K,\pi}\right|$ case discussed above, the ratio $Br\,(K \to \mu\nu)$ / $Br\,(\pi \to \mu\nu)$ is sensitive to $|V_{us}|/|V_{ud}|$ and independent of η. However, when the information of $|V_{ud}^{\beta}|$ from Eq. (4.64) is introduced, an indirect dependence on $\eta_{\mu\mu}$ is induced

$$\left|V_{us}^{K,\pi\to\mu\nu}\right| = (1 - \eta_{\mu\mu})\,|V_{us}|\ . \tag{4.69}$$

4.4.5 LFV observables

In the Type-III Seesaw LFV processes can occur already at tree-level and they are driven by the off-diagonal $|\eta_{\alpha\beta}|$ parameters. The Lepton Flavour Conserving (LFC) processes discussed above constrain, on the other hand, the diagonal parameters $|\eta_{\alpha\alpha}|$. In principle, these are two separate set of bounds since they constrain a priori independent parameters. However, η is a positive-definite matrix, and thus its off-diagonal elements $\eta_{\alpha\beta}$ are bounded by the diagonal ones via the Schwarz inequality:

$$|\eta_{\alpha\beta}| \leq \sqrt{\eta_{\alpha\alpha}\eta_{\beta\beta}}\,. \tag{4.70}$$

Thus, the off-diagonal elements of η are bounded both directly via the LFV processes that will be discussed in this section, and indirectly via LFC processes.

In the following, the most relevant LFV processes are described. Notice that since these observables are already proportional to $|\eta_{\alpha\beta}|^2$, the additional dependence on η from G_μ and $\sin^2\theta_W$ is subleading and, therefore, it will be neglected in the remainder of this section. Table 4.5 summarizes the present experimental bounds and future sensitivities to $|\eta_{\alpha\beta}|$ associated to each LFV process.

$\mu \to e$ conversion in nuclei

The branching ratio for $\mu - e$ conversion in nuclei with atomic number Z and mass number $A \lesssim 100$ normalized to the total nuclear muon capture rate Γ_{capt} is given by [38]

$$R^{\mu\to e}_{{}^A_ZX} = \frac{2\,G_\mu^2\alpha^3 m_\mu^5 Z_{\text{eff}}^4}{\pi^2\Gamma_{\text{capt}}}|F(q)|^2\frac{Q_{\text{W}}^2}{Z}|\eta_{\mu e}|^2\,, \tag{4.71}$$

where Z_{eff} is the effective atomic number due to the screening effect, $F(q)$ is the nuclear form factor as measured from electron scattering, and

$$Q_{\text{W}} = (A+Z)\left(1-\frac{8}{3}s_{\text{W}}^2\right) + (2A-Z)\left(-1+\frac{4}{3}s_{\text{W}}^2\right)\,. \tag{4.72}$$

Γ_{capt} is also measured experimentally with good precision and therefore, information on $\eta_{\mu e}$ can be extracted from Eq. (4.71) in a nuclear-model independent fashion. The strongest experimental bound on this LFV transition is stated by μ to e conversion in ${}^{48}_{22}$Ti, measured by the experiment SINDRUM II [32]. In this case, $Z_{\text{eff}} \simeq 17.6$ and $F\left(q \simeq -m_\mu\right) \simeq 0.54$ [38], $\Gamma_{\text{capt}} \simeq (2.590 \pm 0.012)\cdot 10^6$ s^{-1} [39], and Eq. (4.71) thus reads

$$R^{\mu\to e}_{{}^{48}_{22}\text{Ti}} \simeq 58.88|\eta_{\mu e}|^2\,. \tag{4.73}$$

The corresponding bound on $|\eta_{\mu e}|$ is shown in Table 4.5. This is the strongest present bound on $|\eta_{\mu e}|$. The future PRISM/PRIME [33] sensitivity to this parameter is expected to be improved by three orders of magnitude.

Observable	Present bound on $\|\eta_{\alpha\beta}\|$	Future sensitivity on $\|\eta_{\alpha\beta}\|$
$\boldsymbol{\mu \to e}$ **(Ti)**	$\boldsymbol{\|\eta_{\mu e}\| < 3.0 \cdot 10^{-7}}$ [32]	$\boldsymbol{\|\eta_{\mu e}\| < 1.4 \cdot 10^{-10}}$ [33]
$\mu \to eee$	$\|\eta_{\mu e}\| < 8.7 \cdot 10^{-7}$ [23]	$\|\eta_{\mu e}\| < 1.1 \cdot 10^{-8}$ [34]
$\tau \to eee$	$\|\eta_{\tau e}\| < 3.4 \cdot 10^{-4}$ [23]	$\|\eta_{\tau e}\| < 2.9 \cdot 10^{-5}$ [35]
$\tau \to \mu\mu\mu$	$\|\eta_{\tau\mu}\| < 3.0 \cdot 10^{-4}$ [23]	$\|\eta_{\tau\mu}\| < 2.9 \cdot 10^{-5}$ [35]
$\boldsymbol{\tau \to e\mu\mu}$	$\boldsymbol{\|\eta_{\tau e}\| < 3.0 \cdot 10^{-4}}$ [23]	$\boldsymbol{\|\eta_{\tau e}\| < 2.6 \cdot 10^{-5}}$ [35]
$\boldsymbol{\tau \to \mu ee}$	$\boldsymbol{\|\eta_{\tau\mu}\| < 2.5 \cdot 10^{-4}}$ [23]	$\boldsymbol{\|\eta_{\tau\mu}\| < 2.6 \cdot 10^{-5}}$ [35]
$Z \to \mu e$	$\|\eta_{\mu e}\| < 8.5 \cdot 10^{-4}$ [23]	—
$Z \to \tau e$	$\|\eta_{\tau e}\| < 3.1 \cdot 10^{-3}$ [23]	—
$Z \to \tau\mu$	$\|\eta_{\tau\mu}\| < 3.4 \cdot 10^{-3}$ [23]	—
$h \to \mu e$	$\|\eta_{\mu e}\| < 0.54$ [23]	—
$h \to \tau e$	$\|\eta_{\tau e}\| < 0.14$ [23]	—
$h \to \tau\mu$	$\|\eta_{\tau\mu}\| < 0.20$ [23]	—
$\mu \to e\gamma$	$\|\eta_{\mu e}\| < 1.1 \cdot 10^{-5}$ [23]	$\|\eta_{\mu e}\| < 5.3 \cdot 10^{-6}$ [36]
$\tau \to e\gamma$	$\|\eta_{\tau e}\| < 7.2 \cdot 10^{-3}$ [23]	$\|\eta_{\tau e}\| < 1.2 \cdot 10^{-3}$ [37]
$\tau \to \mu\gamma$	$\|\eta_{\tau\mu}\| < 8.4 \cdot 10^{-3}$ [23]	$\|\eta_{\tau\mu}\| < 1.2 \cdot 10^{-3}$ [37]

Table 4.5: Summary of the present constraints and expected future sensitivities for the most relevant LFV observables considered. The corresponding present and future bounds on $|\eta_{\alpha\beta}|$ have been computed assuming no correlations among the elements of the matrix η and are shown at 2σ. The dominant limits and future sensitivities are highlighted in bold face.

LFV 3-body lepton decays

The LFV decay rate of a lepton with flavor α into three leptons with flavor $\beta \neq \alpha$ is given by

$$\Gamma\left(l_\alpha \to l_\beta l_\beta \bar{l}_\beta\right) = \frac{G_\mu^2 s_{\rm W}^4 m_\alpha^5}{6\pi^3} |\eta_{\alpha\beta}|^2 \,, \tag{4.74}$$

while the τ decay rate into two leptons with flavor $\beta \neq \tau$ and one with flavor $\alpha \neq \beta, \tau$ reads

$$\Gamma\left(\tau \to l_\alpha \bar{l}_\beta l_\beta\right) = \frac{G_\mu^2 m_\tau^5 \left(1 - 4s_{\rm W}^2 + 8s_{\rm W}^4\right)}{48\pi^3} |\eta_{\tau\alpha}|^2. \tag{4.75}$$

The corresponding bounds are listed in Table 4.5. The τ decays into three leptons set the present and near-future most constraining bounds on $|\eta_{\tau e}|$ and $|\eta_{\tau\mu}|$.

LFV Z and h decays

The decay rate of Z and Higgs fields into two leptons of different flavors is given by

$$\Gamma\left(Z \to l_\alpha^\pm l_\beta^\mp\right) = \frac{4M_Z^3 G_\mu}{3\sqrt{2}\pi} |\eta_{\alpha\beta}|^2 \,, \tag{4.76}$$

and

$$\Gamma\left(h \to l_\alpha^\pm l_\beta^\mp\right) = \frac{9G_\mu M_h}{2\sqrt{2}\pi}\left(m_\alpha^2 + m_\beta^2\right)|\eta_{\alpha\beta}|^2 \,, \tag{4.77}$$

respectively. Although these processes are not competitive with respect to LFV lepton decays and $\mu \to e$ conversion in nuclei, we list the bounds on $|\eta_{\alpha\beta}|$ they would lead to in Table 4.5.

Radiative decays

Finally, radiative decays of the type $l_\alpha \to l_\beta \gamma$ would be induced at loop-level [40, 41]

$$\frac{\Gamma\left(l_\alpha \to l_\beta \gamma\right)}{\Gamma\left(l_\alpha \to l_\beta \nu_\alpha \nu_\beta\right)} \simeq \frac{3\alpha}{8\pi}\left(\frac{13}{3} - 6.56\right)^2 |\eta_{\alpha\beta}|^2 \,. \tag{4.78}$$

The present bound on $|\eta_{\mu e}|$ from MEG [42] together with the bounds from $\tau \to e\gamma$ and $\tau \to \mu\gamma$ are listed in Table 4.5. Even though in other extensions of the SM $\mu \to e\gamma$ is the dominant LFV channel, in the Type-III Seesaw other LFV transitions, as μ to e conversion in nuclei, can already occur at tree level and therefore set more stringent bounds than those stemming from radiative decays.

4.4.6 Summary

Summarizing, the following set of 43 observables will be used to derive the most updated global constraints on the Type-III Seesaw parameters:

- the mass of the W boson: M_W;
- the invisible width of the Z: $\Gamma_{\rm inv}$;
- 6 ratios of Z fermionic decays: R_e, R_μ, R_τ, R_c, R_b and $\sigma^0_{\rm had}$;
- 6 Z asymmetry parameters: A_e, A_μ, A_τ, A_b, A_c and A_s;
- 8 ratios of weak decays constraining electroweak (EW) universality: $R^\pi_{\mu e}$, $R^\pi_{\tau\mu}$, $R^W_{\mu e}$, $R^W_{\tau\mu}$, $R^K_{\mu e}$, $R^K_{\tau\mu}$, $R^l_{\mu e}$ and $R^l_{\tau\mu}$;

- 9 weak decays constraining the CKM Unitarity;
- 12 LFV processes: μ to e conversion in Ti, 5 rare lepton decays, 3 radiative decays and 3 Z LFV decays.

The expressions for these observables in the Type-III Seesaw model, the SM predictions and the experimental values are given in Tables 4.1-4.5.

With this list of observables, we will perform a fit of the experimental data in order to find constraints on the entries of η. We will do so in three different scenarios: the General Scenario (G-SS), in which an arbitrary number of fermion triplets is integrated out; the Three Triplets Scenario (3Σ-SS), in which three fermion triplets are added to the SM; and finally the Two Triplets Scenario (2Σ-SS), in which only two fermion triplets are added to the SM. These scenarios will be described in-depth in the next Chapter.

Chapter 5

See-saw Scenarios

In this chapter we will describe the Type-III See-saw scenarios analyzed in the present work and their corresponding parametrizations. As anticipated at the end of last chapter, we will investigate three different scenarios: a General Scenario (G-SS), with an arbitrary number of fermion triplets, and the minimal inverse See-saw scenarios with 3 (3Σ-SS) and 2 fermion triplets (2Σ-SS).

5.1 General scenario (G-SS)

In this general scenario an arbitrary number of fermion triplets heavier than the EW scale is added to the field content without any further assumption. From now on we will refer to this unrestricted case as G-SS (general Seesaw). In this scenario, there are enough independent parameters for the coefficient of the $d = 6$ operator η to be completely independent from that of the $d = 5$ that induces the light neutrino mass matrix $\hat{m}$. Therefore, η cannot be constrained by the light neutrino masses and mixings [43, 44]. We will thus treat all elements of η as independent free parameters in this scenario. Nevertheless, η is a positive-definite matrix and the Schwartz inequality (4.70) holds.

5.2 Three triplets Seesaw scenario (3Σ-SS)

We will also investigate the 3Σ-SS case defined by the following requirements:

- only 3 fermion triplets are added to the SM;
- the mass scale of the triplets is larger than the EW scale;
- even though the neutrino masses are small, large, potentially observable η is allowed;

- the small neutrino masses are radiatively stable.

It turns out that the only way to simultaneously satisfy these requirements is through an underlying lepton number (LN) symmetry [45, 40, 46, 47, 24]. In such a case we have

$$m_D = \frac{v_{EW}}{\sqrt{2}} \begin{pmatrix} Y_{\Sigma_e} & Y_{\Sigma_\mu} & Y_{\Sigma_\tau} \\ \epsilon_1 Y'_{\Sigma_e} & \epsilon_1 Y'_{\Sigma_\mu} & \epsilon_1 Y'_{\Sigma_\tau} \\ \epsilon_2 Y''_{\Sigma_e} & \epsilon_2 Y''_{\Sigma_\mu} & \epsilon_2 Y''_{\Sigma_\tau} \end{pmatrix} \quad \text{and} \quad M_\Sigma = \begin{pmatrix} \mu_1 & \Lambda & \mu_3 \\ \Lambda & \mu_2 & \mu_4 \\ \mu_3 & \mu_4 & \Lambda' \end{pmatrix}, \tag{5.1}$$

where ϵ_i and μ_i are small lepton number violating parameters. By setting all $\epsilon_i = 0$ and $\mu_i = 0$, LN symmetry is recovered if the 3 fermion triplets are assigned LNs 1, -1 and 0 respectively. In this case, we find $m_i = 0$ (3 massless neutrinos), $M_1 = M_2 = \sqrt{\Lambda^2 + \sum_\alpha (|Y_{\Sigma_\alpha}|^2 v^2/2)^2}$ (a heavy Dirac field) and $M_3 = \Lambda'$ (a heavy decoupled Majorana fermion), where m_i and M_i are the mass eigenvalues of the full 6×6 mass matrix including m_D and M_Σ. Substituting Eq. (5.1) in Eq. (3.61), in the LN-conserving limit we find

$$\eta = \frac{1}{2} \begin{pmatrix} |\theta_e|^2 & \theta_e\theta_\mu^* & \theta_e\theta_\tau^* \\ \theta_\mu\theta_e^* & |\theta_\mu|^2 & \theta_\mu\theta_\tau^* \\ \theta_\tau\theta_e^* & \theta_\tau\theta_\mu^* & |\theta_\tau|^2 \end{pmatrix} \quad \text{with} \quad \theta_\alpha \equiv \frac{Y_{\Sigma_\alpha} v_{EW}}{\sqrt{2}\Lambda}. \tag{5.2}$$

The parameters θ_α represent the mixing of the active neutrino ν_α with the neutral component of the heavy fermion triplet that is integrated out to obtain the $d = 5$ and $d = 6$ operators. As it can be seen, given the underlying LN symmetry, this mixing, and hence the $d = 6$ operator η, can be arbitrarily large while the $d = 5$ operator is exactly zero and neutrinos are massless. In other words, the $d = 5$ operator is protected by the LN symmetry while the $d = 6$ is not and hence an approximate LN symmetry can alter the naive expectation of the $d = 6$ operator being subdominant with respect to the $d = 5$. When the small LN-violating parameters are not neglected so as to reproduce the correct pattern of masses and mixings, the following relation has to be satisfied [24]

$$\begin{aligned} \theta_\tau = & \frac{1}{\hat{m}_{e\mu}^2 - \hat{m}_{ee}\hat{m}_{\mu\mu}} \Big(\theta_e \left(\hat{m}_{e\mu}\hat{m}_{\mu\tau} - \hat{m}_{e\tau}\hat{m}_{\mu\mu} \right) \\ & + \theta_\mu \left(\hat{m}_{e\mu}\hat{m}_{e\tau} - \hat{m}_{ee}\hat{m}_{\mu\tau} \right) \pm \sqrt{\theta_e^2 \hat{m}_{\mu\mu} - 2\theta_e\theta_\mu\hat{m}_{e\mu} + \theta_\mu^2\hat{m}_{ee}} \\ & \times \sqrt{\hat{m}_{e\tau}^2\hat{m}_{\mu\mu} - 2\hat{m}_{e\mu}\hat{m}_{e\tau}\hat{m}_{\mu\tau} + \hat{m}_{ee}\hat{m}_{\mu\tau}^2 + \hat{m}_{e\mu}^2\hat{m}_{\tau\tau} - \hat{m}_{ee}\hat{m}_{\mu\mu}\hat{m}_{\tau\tau}} \Big), \end{aligned} \tag{5.3}$$

where only leading order terms in the Seesaw expansion have been considered and $\hat{m}$ contains the information on light neutrino masses and mixings through

Eq. (4.12). This extra constraint leads to correlations among the η matrix elements in Eq. (5.2) not present in the unrestricted scenario G-SS described above[1]. The value of the complex mixing θ_τ is fixed by θ_e and θ_μ through Eq. (5.3), and by the SM neutrino masses and mixings, that are encoded in the $d = 5$ operator $\hat{m}$. Therefore, we will scan the allowed parameter space of the model using as free parameters θ_e and θ_μ, as well as the unknown parameters that characterize $\hat{m}$, i.e. the Dirac phase δ, the Majorana phases α_1 and α_2, the smallest neutrino mass and the mass hierarchy (which can be normal or inverted). We will also consider in the fit the constraint on the sum of the light neutrino masses (from Planck) $\Sigma m_i < 0.12$ eV at 95% CL [48], while we use the best fit values of the remaining oscillation parameters from Ref. [49] which are summarized in Table 5.1.

	Best fit $\pm 1\sigma$
$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$
$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$
$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$
$\Delta m^2_{\rm sol}$	$7.39^{+0.21}_{-0.20} \cdot 10^{-5}$ eV^2
$\lvert\Delta m^2_{\rm atm}\rvert$	$2.528^{+0.029}_{-0.031} \cdot 10^{-3}$ eV^2

Table 5.1: Present best fit values of the light neutrino mixing angles and two squared mass differences from [49].

5.3 Two triplets Seesaw scenario (2Σ-SS)

The 2Σ-SS scenario is a particular case of the 3Σ-SS defined by the same conditions but with the addition of only two fermion triplets instead of three. Notice that this is the most economic realization of the Type-III Seesaw model able to account for the two distinct mass splittings observed in neutrino oscillation experiments.

Analogously to the 3Σ-SS scenario, for this case we have

$$m_D = \frac{v_{EW}}{\sqrt{2}} \begin{pmatrix} Y_{\Sigma_e} & Y_{\Sigma_\mu} & Y_{\Sigma_\tau} \\ \epsilon_1 Y'_{\Sigma_e} & \epsilon_1 Y'_{\Sigma_\mu} & \epsilon_1 Y'_{\Sigma_\tau} \end{pmatrix} \qquad \text{and} \qquad M_\Sigma = \begin{pmatrix} \mu_1 & \Lambda \\ \Lambda & \mu_2 \end{pmatrix}, \tag{5.4}$$

[1]Notice that the η matrix also contains contributions driven by the small LN-violating parameters, ϵ_i and μ_i, which are however subleading and thus neglected in our analysis.

where again ϵ_i and μ_i are small lepton number violating parameters which, once set to zero, imply LN conservation if the two triplets have LNs 1 and -1 respectively. In this limit the mass eigenvalues of the full 6×6 mass matrix are $M_1 = M_2 = \sqrt{\Lambda^2 + \sum_\alpha (|Y_{\Sigma_\alpha}|^2 v_{EW}^2/2)^2}$ (which combine into a heavy Dirac pair) and the light neutrino masses vanish. Eq. (5.2) still holds, showing that large η entries are possible even in the LN-conserving limit with massless neutrinos. Analogously to the 3Σ-SS scenario, upon switching on the LN-violating parameters in Eq. (5.4), neutrino masses and mixings $\hat{m}$ are generated. Given the reduced number of parameters in the Lagrangian, Eq. (4.12) now implies additional correlations

$$\begin{aligned} \theta_\mu &= \frac{\theta_e}{\hat{m}_{ee}}\Big(\hat{m}_{e\mu} \pm i\sqrt{m_2 m_3}\big(U_{13}^* U_{22}^* - U_{12}^* U_{23}^*\big)\Big) \quad \text{for normal hierarchy (NH)}\,, \\ \theta_\mu &= \frac{\theta_e}{\hat{m}_{ee}}\Big(\hat{m}_{e\mu} \pm i\sqrt{m_1 m_2}\big(U_{12}^* U_{21}^* - U_{11}^* U_{22}^*\big)\Big) \quad \text{for inverted hierarchy (IH)}\,, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \theta_\tau &= \frac{\theta_e}{\hat{m}_{ee}}\Big(\hat{m}_{e\tau} \pm i\sqrt{m_2 m_3}\big(U_{13}^* U_{32}^* - U_{12}^* U_{33}^*\big)\Big) \quad \text{for} \quad \text{NH}\,, \\ \theta_\tau &= \frac{\theta_e}{\hat{m}_{ee}}\Big(\hat{m}_{e\tau} \pm i\sqrt{m_1 m_2}\big(U_{12}^* U_{31}^* - U_{11}^* U_{32}^*\big)\Big) \quad \text{for} \quad \text{IH}\,, \end{aligned} \tag{5.6}$$

(with $U_{ij}^* = (U_{PMNS})_{ij}^*$) where both options for the sign in front of the square root are possible but the same choice has to be taken for θ_μ and θ_τ.

Therefore, θ_μ and θ_τ are both proportional to θ_e. In other words, once the known oscillation parameters are fixed to their best fit values, and the remaining unknown parameters[2] characterizing $\hat{m}$ (the Dirac phase δ, the Majorana phase α_2 and the mass hierarchy) are specified, the $d = 6$ operator is fixed up to an overall factor that we parametrize through θ_e.[3] This same conclusion via a different parametrization was first derived in the context of the Type-I Seesaw in [50], and applied to the Type-III Seesaw case in [46, 47].

The parameters characterizing the low energy new physics effects and correlations among them in each of the three cases described in this section are summarized in Table 5.2.

[2]In this minimal scenario one of the light neutrino masses is zero and one of the Majorana phases is nonphysical (α_1 can be set to zero).

[3]θ_e can thus be considered a real parameter since its associated phase becomes a global phase.

	η_{ee}	$\eta_{\mu\mu}$	$\eta_{\tau\tau}$	$\eta_{e\mu}$	$\eta_{e\tau}$	$\eta_{\mu\tau}$
G-SS	$\eta_{ee} > 0$ free	$\eta_{\mu\mu} > 0$ free	$\eta_{\tau\tau} > 0$ free	$\lvert\eta_{e\mu}\rvert \leq \sqrt{\eta_{ee}\eta_{\mu\mu}}$ free	$\lvert\eta_{e\tau}\rvert \leq \sqrt{\eta_{ee}\eta_{\tau\tau}}$ free	$\lvert\eta_{\mu\tau}\rvert \leq \sqrt{\eta_{\mu\mu}\eta_{\tau\tau}}$ free
3Σ-SS	$\eta_{ee} = \frac{\lvert\theta_e\rvert^2}{2}$ free	$\eta_{\mu\mu} = \frac{\lvert\theta_\mu\rvert^2}{2}$ free	$\eta_{\tau\tau} = \frac{\lvert\theta_\tau\rvert^2}{2}$ fixed by Eq. (5.3)	$\eta_{e\mu} = \frac{\theta_e\theta_\mu^*}{2}$ fixed by θ_e, θ_μ	$\eta_{e\tau} = \frac{\theta_e\theta_\tau^*}{2}$ fixed by θ_e, θ_τ	$\eta_{\mu\tau} = \frac{\theta_\mu\theta_\tau^*}{2}$ fixed by θ_μ, θ_τ
2Σ-SS	$\eta_{ee} = \frac{\theta_e^2}{2}$ free	$\eta_{\mu\mu} = \frac{\lvert\theta_\mu\rvert^2}{2}$ fixed by Eq. (5.5)	$\eta_{\tau\tau} = \frac{\lvert\theta_\tau\rvert^2}{2}$ fixed by Eq. (5.6)	$\eta_{e\mu} = \frac{\theta_e\theta_\mu^*}{2}$ fixed by θ_e, θ_μ	$\eta_{e\tau} = \frac{\theta_e\theta_\tau^*}{2}$ fixed by θ_e, θ_τ	$\eta_{\mu\tau} = \frac{\theta_\mu\theta_\tau^*}{2}$ fixed by θ_μ, θ_τ

Table 5.2: Summary of the parameters that characterize the low energy new physics effects of a totally general Type-III Seesaw model (G-SS), and the realizations with 3 and 2 additional heavy triplets (3Σ-SS and 2Σ-SS respectively). η is the coefficient of the $d = 6$ operator while θ_α corresponds to the mixing between ν_α and the neutral component of the heavy fermion triplets. In the 3Σ-SS case, θ_τ is calculated via Eq. (5.3) as a function of θ_e, θ_μ, δ, α_1, α_2, the lightest neutrino mass and the mass hierarchy. In the 2Σ-SS, θ_μ and θ_τ are computed via Eq. (5.5) and Eq. (5.6) respectively as functions of θ_e, δ, α_2 and the mass hierarchy. The remaining oscillation parameters are fixed to their best fit values shown in Table 5.1.

Using the relations we found in this Chapter together with the observables introduced in section 4.4, we performed a fit of the experimental data with the goal of finding constraints on the entries of η for the various scenarios we explored. The results we found are presented in the next Chapter.

Chapter 6

Results and discussion

With all the observables introduced in section 4.4, a χ^2 function has been built under the assumption of gaussianity so as to test the bounds that the experimental constraints can set globally on the Type-III Seesaw parameters. In order to achieve an efficient exploration of the parameter space of the three scenarios introduced in the previous Section, particularly for the G-SS and 3Σ-SS scenarios which are characterized by a relatively large number of parameters, a Markov chain Monte Carlo (MCMC) method has been employed.

For each scenario, $\mathcal{O}(10^7)$ distinct samples have been generated through 20 chains running simultaneously, achieving a convergence for all the free parameters better than $R - 1 < 0.0008$ [51]. These scans are sufficiently well-sampled to allow a frequentist analysis by profiling the χ^2 function of the different subsets of parameters, obtaining the 1D and 2D contours for the preferred regions. The processing of the chains has been performed with the MonteCUBES [52] user interface.

As discussed in the previous Section, the θ_α parameters are not independent in the 3Σ-SS and 2Σ-SS, and the information from neutrino oscillations needs to be considered so as to reproduce the correct neutrino masses and mixings. Thus, θ_{12}, θ_{13}, θ_{23}, Δm^2_{sol}, and Δm^2_{atm}, have been fixed to their best fit values listed in Table 5.1, while the Dirac phase δ, the Majorana phases α_1 and α_2, and the lightest neutrino mass are free parameters in the scan[1], with a constraint on the sum of the light neutrino masses (from Planck) $\Sigma m_i < 0.12$ eV at 95% CL [48].

Even though present oscillation data show a preference for some values of δ, at the 2σ level about half of the parameter space is still allowed and there is some tension between the values favoured by T2K and NOνA analyses. Thus, we allow δ to vary freely in the fit. Moreover, in the NH case, both for the 3Σ-SS and 2Σ-SS scenarios, we have verified that if instead of fixing the mass splittings and

[1]In the 2Σ-SS, the lightest neutrino is exactly massless and therefore α_1 is unphysical.

mixing angles to their best fit values we introduce them as free parameters with their corresponding priors from Ref. [49], the change in the results is negligible. Finally, even though neutrino oscillation data presently disfavor IH at more than 2σ, in order to illustrate the impact of the mass hierarchy in our analysis, we present our results for both IH and NH.

We will show our individual constraints on the $d = 6$ effective operator coefficients $\eta_{\alpha\beta}$, but also the 2D allowed regions projected in the $\sqrt{2\eta_{\alpha\alpha}} - \sqrt{2\eta_{\beta\beta}}$ planes for the three cases under study. In the 3Σ-SS and 2Σ-SS scenarios $\sqrt{2\eta_{\alpha\alpha}} = \theta_\alpha$ is the mixing of the active neutrino ν_α with the neutral component of the heavy fermion triplet (see Eq. (5.2)). Therefore, our results in the G-SS projected on the $\sqrt{2\eta_{\alpha\alpha}} - \sqrt{2\eta_{\beta\beta}}$ can be easily compared with the corresponding bounds on the mixing for the the 3Σ-SS and 2Σ-SS scenarios. For completeness, the individual bounds on $\sqrt{2|\eta_{\alpha\beta}|}$ (the mixing $|\theta_\alpha|$ in the 3Σ-SS and 2Σ-SS cases) are also reported in Appendix A.

6.1 General scenario (G-SS)

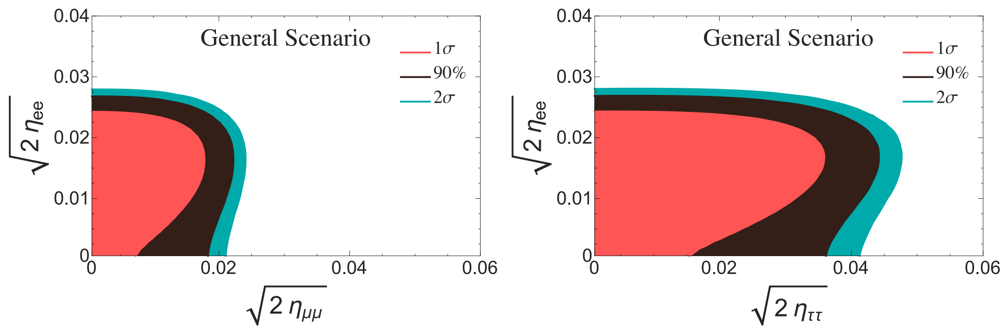


Figure 6.1: Frequentist confidence intervals at 1σ, 90% and 2σ on the parameter space of the G-SS.

In this scenario all the elements of the $d = 6$ operator $\eta_{\alpha\beta}$ are independent free parameters in the fit. In Fig. 6.1 we present the 2 degrees of freedom frequentist allowed regions of the parameter space at 1σ, 90%, and 2σ in red, black and cyan respectively, projected in the $\sqrt{2\eta_{ee}} - \sqrt{2\eta_{\mu\mu}}$ (left panel) and $\sqrt{2\eta_{ee}} - \sqrt{2\eta_{\tau\tau}}$ (right panel) planes. The individual constraints on each $|\eta_{\alpha\beta}|$ at 2σ after profiling over all other parameters are summarized in Table 6.1. With the information on the diagonal elements, and due to the fact that η is a positive-definite Hermitian matrix, bounds on the off-diagonal elements can be derived via Eq. (4.70). These values, collected in the first column of Table 6.1, are derived from data sets independent from the LFV processes discussed in Section 4.4.5 and thus we will refer to them as LFC. In the second column (LFV) of Table 6.1 we show the present constraints on the off-diagonal parameters directly derived from the set of LFV observables considered in Section 4.4.5. Regarding the bounds on the diagonal parameters, both $|\eta_{ee}|$ and $|\eta_{\mu\mu}|$ are $\mathcal{O}(10^{-4})$ while the constraint on $|\eta_{\tau\tau}|$ is $\sim 3-4$ times weaker. The constraints from the LFC and LFV independent sets of data are remarkably similar in magnitude for the $\eta_{\tau e}$ and $\eta_{\tau\mu}$ elements, $\mathcal{O}(10^{-4})$, being the LFV ones slightly more constraining. For the $\eta_{\mu e}$ element, however, the extremely stringent constraint from μ to e conversion in nuclei allows to set an $\mathcal{O}(10^{-7})$ upper bound, three orders of magnitude stronger than the one derived from the LFC data set.

	G-SS		3Σ-SS		2Σ-SS	
	LFC	LFV	NH	IH	NH	IH
η_{ee}	$\mathbf{< 3.2 \cdot 10^{-4}}$	—	$\mathbf{< 3.1 \cdot 10^{-4}}$	$< 3.2 \cdot 10^{-4}$	$\mathbf{< 2.3 \cdot 10^{-7}}$	$< 1.4 \cdot 10^{-5}$
$\eta_{\mu\mu}$	$\mathbf{< 2.1 \cdot 10^{-4}}$	—	$\mathbf{< 1.4 \cdot 10^{-4}}$	$< 1.1 \cdot 10^{-4}$	$\mathbf{< 3.8 \cdot 10^{-6}}$	$< 1.1 \cdot 10^{-6}$
$\eta_{\tau\tau}$	$\mathbf{< 8.5 \cdot 10^{-4}}$	—	$\mathbf{< 6.5 \cdot 10^{-4}}$	$< 3.9 \cdot 10^{-4}$	$\mathbf{< 6.1 \cdot 10^{-6}}$	$< 1.4 \cdot 10^{-6}$
$\eta_{\mu e}$	$< 2.0 \cdot 10^{-4}$	$\mathbf{< 3.0 \cdot 10^{-7}}$	$\mathbf{< 3.0 \cdot 10^{-7}}$	$< 3.0 \cdot 10^{-7}$	$\mathbf{< 3.0 \cdot 10^{-7}}$	$< 3.0 \cdot 10^{-7}$
$\eta_{\tau e}$	$< 4.1 \cdot 10^{-4}$	$\mathbf{< 2.7 \cdot 10^{-4}}$	$\mathbf{< 2.5 \cdot 10^{-4}}$	$< 2.3 \cdot 10^{-4}$	$\mathbf{< 5.4 \cdot 10^{-7}}$	$< 3.6 \cdot 10^{-6}$
$\eta_{\tau\mu}$	$< 2.8 \cdot 10^{-4}$	$\mathbf{< 2.2 \cdot 10^{-4}}$	$\mathbf{< 1.4 \cdot 10^{-4}}$	$< 1.3 \cdot 10^{-4}$	$\mathbf{< 3.6 \cdot 10^{-6}}$	$< 1.2 \cdot 10^{-6}$

Table 6.1: The 2σ constraints on the coefficient of the $d = 6$ operator η are shown. For the G-SS, the off-diagonal entries are bounded in two independent ways: indirectly from LFC observables via Eq. (4.70) and directly from LFV processes. The bold face highlights the most constraining G-SS bounds. For the 3Σ-SS and 2Σ-SS, the constraints are shown separately for normal hierarchy (NH) and inverted hierarchy (IH). The constraints for the NH scenarios are highlighted in bold face since NH provides a better fit to present neutrino oscillation data, while IH is disfavored at more than 2σ [49]. The corresponding bounds on the mixing θ_α between ν_α and the heavy triplets are shown in Appendix A.

6.2 Three and Two Triplets Scenarios (3Σ-SS and 2Σ-SS)

When the number of fermion triplets is ≤ 3, Eq. (4.70) is saturated to an equality $|\eta_{\alpha\beta}| = \sqrt{\eta_{\alpha\alpha}\eta_{\beta\beta}}$, and thus the LFV processes, which a priori constrain only the off-diagonal elements of η, will also contribute to the bounds on the diagonal elements.

In Figure 6.2 (Figure 6.3) we present the 2 dof frequentist contours on the mixings $|\theta_\alpha|$ at 1σ, 90% CL, and 2σ in red, black and cyan, respectively for the 3Σ-SS (2Σ-SS) scenario. The left panels show the allowed regions in the plane $|\theta_e| - |\theta_\mu|$ while the right panels show the allowed regions in the plane $|\theta_e| - |\theta_\tau|$, for normal (top panels) and inverted (bottom panels) neutrino mass hierarchy.

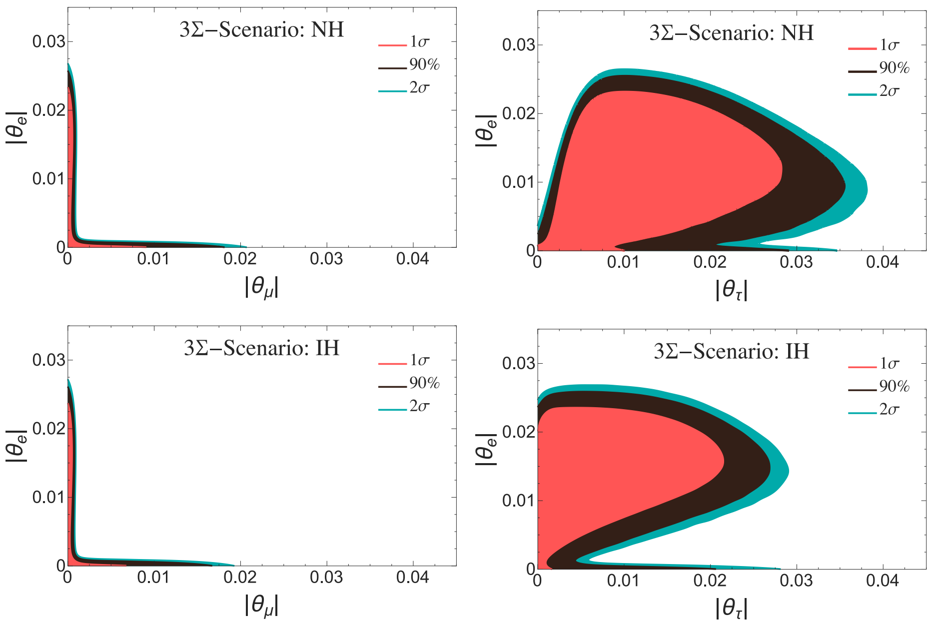


Figure 6.2: Frequentist confidence intervals at 1σ, 90% CL and 2σ on the parameter space of the 3Σ-SS for normal hierarchy (upper panels) and inverted hierarchy (lower panels).

The pronounced hyperbolic shape of the contours on the $|\theta_e| - |\theta_\mu|$ plane of Figure 6.2 are driven by the fact that in this scenario the product of both mixings is directly bounded by μ to e conversion in Ti nuclei. The allowed parameter space is thus dramatically reduced with respect to the G-SS scenario, even if the bounds on the individual parameters are similar. On the other hand the correlation shown in the right panels of the same Figure is determined by the constraints due to the generation of the light neutrino masses and mixing reflected in Eq. (5.3). To be precise, in the 3Σ-SS scenario, θ_τ is determined by θ_e, θ_μ and the light neutrino free parameters.

As can be observed in Figure 6.3, these features are even more pronounced in the minimal scenario with 2 fermion triplets 2Σ-SS since the three mixings are directly proportional to θ_e with a proportionality constant which depends only on the light neutrino free parameters (see Eqs. (5.5) and (5.6)). The overall scale of the three mixings is thus controlled by the most constraining observable, namely

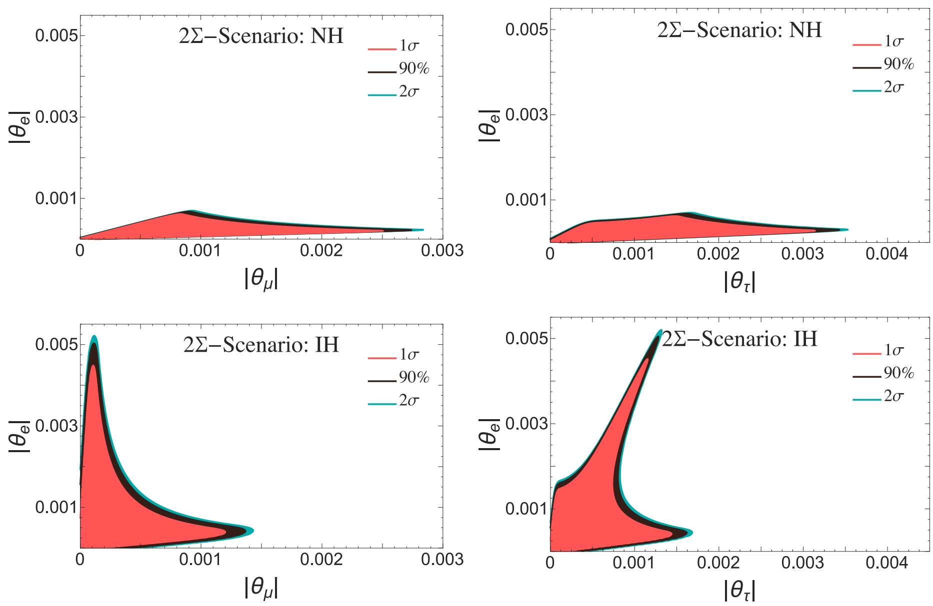


Figure 6.3: Frequentist confidence intervals at 1σ, 90% CL and 2σ on the parameter space of the 2Σ-SS for normal hierarchy (upper panels) and inverted hierarchy (lower panels).

μ to e conversion in nuclei, which sets quite stringent individual bounds on all θ_α. The particular correlations observed arise because in this minimal model the flavour structure is completely determined by the light neutrino parameters.

The constraints on the corresponding $\eta_{\alpha\beta}$ elements in both scenarios, and for both hierarchies, are summarized in Table 6.1. The individual bounds on the 3Σ-SS case are pretty similar to the G-SS scenario, however they are considerably stronger in the 2Σ-SS, ranging from $\mathcal{O}(10^{-5})$ to $\mathcal{O}(10^{-7})$. Finally, notice that in the 3Σ-SS case the results for both light neutrino hierarchies are similar. However, in the 2Σ-SS scenario, the hierarchy has a strong impact in the results since it is a very relevant input regarding the constrained flavor structure of this minimal model (see Eqs. (5.5) and (5.6)).

Intermission

Once again, just like in the Foreword, you will not find any science in this Intermission: here, I will explain the reasons behind my transition from neutrino phenomenology to non-commutative geometry, as well as why this transition is actually very natural. So, if you are uninterested, you may skip to the next Chapter.

So far, we have studied one particular way of explaining the tininess of the neutrino masses, namely the Type III See-saw. In the model, the fact that we added new particles (in this case, the fermion triplets) is actually not so important *per se*; the crucial point of *any* See-saw model is the assumption that neutrinos have a very big Majorana mass. Now, by analogy with what happens for the masses of all other fermions (namely, they are induced by a scalar field, the Higgs field), we could expect this Majorana mass to be induced by some new scalar field as well. Now, in the see-saw models, you do not really care about this extra scalar field, the only thing that matters is the fact that the Majorana mass be there. Nevertheless, one might wonder (and I sure did) "just *why* should that extra scalar field be there?" As far as the Higgs is concerned, it is existence is much more motivated: it has to be there in order for fermionic and bosonic masses to be possible at all, thus making gauge theories compatible with reality. This extra scalar is not so fundamental: the Majorana mass it induces would just explain the *tininess* of neutrino masses, and these are possible (and in principle, could be tiny as well) just by means of the Higgs field. Why should one expect it to be there?

As I think I might perhaps have mentioned in the Foreword (of course, I'm being ironic here), I tend to consider the question "*why*" as quite important, so I started looking for theories or models that would give a reason to the existence of that extra "Majorana-mass" scalar field. Then, I found out that there were people working on a mathematical model that gave a geometrical meaning to scalar fields, thereby providing them with a very fundamental reason for their presence – and they were stationed in the math department, just some hundred metres away from my office! Imagine my surprise!

The "mathematical model" I'm talking about is of course Non-Commutative

Geometry, which will be the subject of the second part of this work. Its scope is actually much wider than just explaining why there are scalars, to the point of leaving me flabbergasted. Under the simple assumption of "the world can be described by a non-commutative space", one automatically gets an *a priori* explanation for:

- why there should be a right-handed neutrino (there should be exactly 16 fermions per generation, and the SM only has 15);
- why the Higgs field should be there (it is a connection, just like the gauge bosons);
- why the action of the SM is what it is (it can be computed from more fundamental objects);
- why *any interaction* should be there: they are all expressions of gravity;
- why one should expect nature to be described by a gauge theory, i.e. why gauge symmetries should be there (it is a reflection of the Morita equivalence of any space to itself);

and much more. In addition, when one assumes nature to be described by a *twisted* non-commutative space, one also gets an extra scalar field that gives Majorana mass to the neutrinos, and on top of that, as an additional bonus, one also gets for free the reason why there should be one time-dimension, as well as why physics should be Lorentz-invariant. Not bad at all, if you ask me!

In short, twisted NCG seemed to be the perfect playground for see-saw models to take place in. Of course, when I found out about all this, I immediately decided that I would use all the experience I had accumulated until then in order to study the twisted non-commutative geometry formulation of the SM, and then study its phenomenology. And so I did.

Now, without further ado, let us dive into the magical world of Non-Commutative Geometry!

Chapter 7

Non-Commutative Geometry

7.1 Why Non-Commutative Geometry?

In this second part of this work, we will talk about Non-Commutative Geometry (hereafter, NCG). But before going into the details of the subject, we will first need to address one very important matter, namely: *why NCG*? Could not we make do with ordinary geometry?

Well, there are many reasons for studying NCG, both mathematical and physical. Let us start with the latter.

There are actually many hints that suggest that ordinary geometry is insufficient to describe physics. The first one that comes to mind is probably Heisenberg's uncertainty principle, that suggests that the very concept of point in space and time might be, in some sense, ill-defined[1]. Let us assume, for instance, that one wants to measure the position of one particle with a better precision that Planck's length. Because of the uncertainty principle, such a measurement requires the uncertainty in momentum (thus also in the corresponding energy) be very large. If the measurement precision is smaller that Planck's length, a huge energy uncertainty is localized in a very small region and a black hole is formed, thus preventing any information from escaping the system (for a more detailed dissertation, see [53]).

But in truth, one does not need to drag quantum physics into the matter, the issue is already present in any gauge theory, even including Maxwell's theory of electromagnetism. General Relativity can be seen as a gauge theory whose gauge transformations are the diffeomorphisms – i.e. coordinate transformations. Changing the coordinates means, roughly speaking, changing the observer. There

[1]More properly, according to the uncertainty principle, the concept of point in the *phase space* is ill-defined. However, this implies problems also for the space-time alone, as the following example makes clear.

is a very clear physical meaning to those transformations. On the other hand, traditional gauge transformations are just a change in the conventions one uses to perform calculations, and have no apparent physical meaning whatsoever. Now, it is best to make this point very clear: we're definitely **not** stating that gauge *symmetries* have no physical meaning. What we're claiming is that performing a gauge *transformation* has no manifest physical significance. Just what could their physical meaning be?

Related to this question is the problem of the geometrization of the theory of elementary particles, which could be classified both as a physical and a mathematical issue. Of course, in principle one might not need to geometrize physics at all, nevertheless the geometrical interpretation of General Relativity (GR) is suggestive enough to encourage the search for a geometrical interpretation of particle physics as well. The complete geometrization of the SM coupled to GR means turning the whole coupled theory into pure gravity on a suitable space. Now, this does not seem possible at first: the gauge invariance group of the action of GR on a manifold $\mathcal{M}$ is the group $\mathrm{Diff}\,(\mathcal{M})$ of diffeomorphisms on $\mathcal{M}$, and the gauge invariance of the action is simply the manifestation of its geometrical nature. When one couples GR to the SM, the gauge invariance group becomes larger, because of the invariance of the matter action under the gauge transformations of the group $\mathcal{G}_{SM}$, which is by construction a group of maps from $\mathcal{M}$ to the small gauge group $G_{SM} = \mathrm{U}\,(1) \times \mathrm{SU}\,(2) \times \mathrm{SU}\,(3)$. $\mathrm{Diff}\,(\mathcal{M})$ acts on $\mathcal{G}_{SM}$ by transformations of the base. This means that the whole gauge symmetry group $\mathcal{U}$ of the full action has the structure of a semidirect product

$$\mathcal{U} = \mathcal{G}_{SM} \rtimes \mathrm{Diff}\,(\mathcal{M})\,. \tag{7.1}$$

Traditionally, in order for $\mathcal{U}$ to have this structure, one postulates that there is a bundle structure over the space-time $\mathcal{M}$. However, this requires an a priori distinction of certain directions in the total space as the fiber directions, with the distinction between base and fiber preserved by the symmetries. On the other hand, it would seem more natural to find some space $\mathcal{X}$ whose group of diffeomorphisms is directly of the form (7.1). Nevertheless, it turns out that $\mathcal{X}$ cannot be an ordinary manifold. In fact, it was shown by W. Thurston, D. Epstein and J. Mather [54] that the component connected to the identity in $\mathrm{Diff}\,(\mathcal{N})$ of any manifold $\mathcal{N}$ is always a simple group, which excludes the possibility of a semidirect product structure like (7.1). Then, one needs to expand the horizon of the possible spaces amongst which one hopes to find $\mathcal{X}$, and NCG happens to be the right direction.

Finally, one purely mathematical reason to study NCG is to generalize the Gelfand-Naimark theorem [55]. The theorem states that there is a duality between topological spaces and commutative algebras. But then, what are *non-*

commutative algebras dual to? In order to answer this question, one needs to deal with NCG.

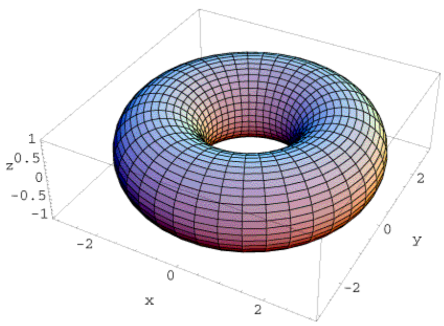
7.2 Introduction to NCG

In this section we will give an intuitive overview of what NCG is. In order to be as easy to understand as possible, we will sometimes be not completely rigorous, so as to allow also the most inexperienced readers to have an idea of the important concepts of NCG. The reader that is not really interested in the mathematical details of NCG should read this section and safely skip the rest of the chapter, in which we will repeat all the topics of this section in a mathematically rigorous way.

7.2.1 Basic notions

So, what is a Non-Commutative Space (NCS)? Roughly speaking, NCSs are a generalization of the usual manifold.

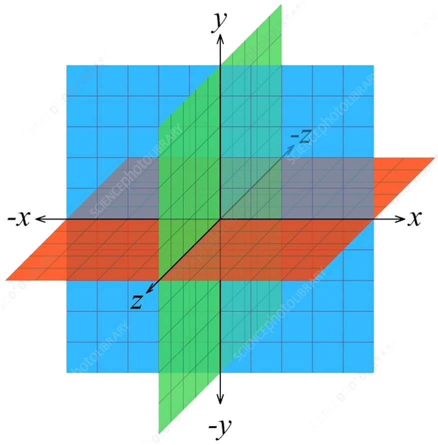
A manifold is basically a smooth collection of points aggregated in some sort of shape, like a ring[2]



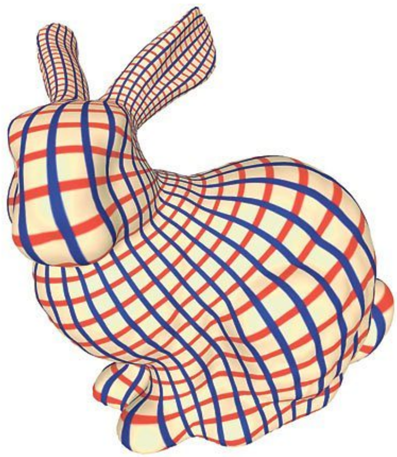
or Euclidean space[3]

[2]Image from https://qph.fs.quoracdn.net/main-qimg-4c7490bb0f4e44bcb2beaa58d1e9117c

[3]Image from https://media.sciencephoto.com/image/c0074231/800wm



or even a rabbit[4].



Each one of those points that make up the manifold is, by definition, an object with no internal structure whatsoever. On the other hand, a NCS is (in some sense) a smooth collection of new objects, that we could call non-commutative points, that conversely to the manifold case *do* have a nontrivial internal structure, like a sphere or even more simply, a pair of "classical" points[5]. So, basically, NCSs are spaces whose points have an internal structure. As we will see, this internal structure will be the reason why we call them "non-commutative".

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[4]Image from https://groups.oist.jp/sites/default/files/styles/group_image/public/eventimg/1711/math.190.2.2.45.jpg?itok=f0oDNTC4

[5]In which case, the Higgs field naturally arises as the connection of the "internal dimension" between the two points.

7.2.2 The Gelfand-Naimark Theorem

Now, let us get just a little bit more technical: how can we describe NCSs in practice?

The main idea is based on a mathematical theorem, called the Gelfand-Naimark Theorem [55]. Its content is quite simple: given an ordinary (topological) space, one can easily construct the set of smooth functions on that space, e.g. by considering the set of polynomials on the space, and then considering its completion. The Gelfand-Naimark Theorem states that the converse is also true: given only the set of smooth functions on some space, but *not given the space*, it is always possible to reconstruct that space just from the set of smooth functions.

The theorem actually also says something more. The set of smooth functions on any one space is by construction a commutative algebra (which means, roughly speaking, that the product of functions is commutative). The theorem then also states that, given any commutative algebra, it is always possible to construct a new space whose set of smooth functions coincides with the commutative algebra one started with.

Here the keyword is "commutative". The theorem does not work if the algebra is not commutative, and the reason is very simple: the product of functions is commutative, hence, if our algebra is to become the set of functions of the new space we're building, it has to be commutative itself!

It should be clear now what the way is to generalize ordinary spaces to NCS. The first step should be to generalize the Gelfand-Naimark construction to manifolds (instead of topological space), in order to encode the metric structure as well. Then, one should further generalize the construction to non-commutative algebras as well, and the result will be a NCS. This was done by Alain Connes – one of the fathers of NCG – who prepared for us all the mathematical machinery needed for the task: the Spectral Triple.

7.2.3 The Spectral Triple

The Spectral Triple [56] is the most general way known so far to describe NCGs. It consists of course of three elements: an algebra of operators, which can be either commutative or non-commutative; a Hilbert space on which the operators act, and finally a differential operator called the Dirac operator. The algebra takes the role of the set of "functions" (now operators) on the space – just like classical observables become operators in Quantum Mechanics. The Dirac operator is called this way because, when one considers a commutative algebra (and then of course the spectral triple simply describes a usual commutative space), it takes the form $\gamma^\mu \partial_\mu$. It has the same function of the metric tensor of the manifold, i.e. measuring distances. Finally, the Hilbert space. This one encodes the fermionic content of the NCS and is something that one actually does not need to specify for commutative spaces. The reason is that the Hilbert space of a spectral triple describing a manifold is somehow forced to be an infinite-dimensional separable Hilbert space. But it can be shown easily that all such Hilbert spaces are all isometrically isomorphic – i.e. they are all morally the very same Hilbert space. Indeed, by definition, a Hilbert space is separable provided it contains a dense countable subset. Zorn's lemma [57] implies that a Hilbert space is separable if and only ifit admits a countable orthonormal basis. All separable Hilbert spaces are therefore isometrically isomorphic to the space of square-summable sequences of complex numbers ℓ^2. Hence there is only one unique allowed Hilbert space for such a spectral triple, therefore there is no need to specify it.

The Spectral Triple is also required to satisfy 8 mathematical axioms, which will be reported in the next section.

Figure 7.1 summarizes the key elements of a Spectral Triple.

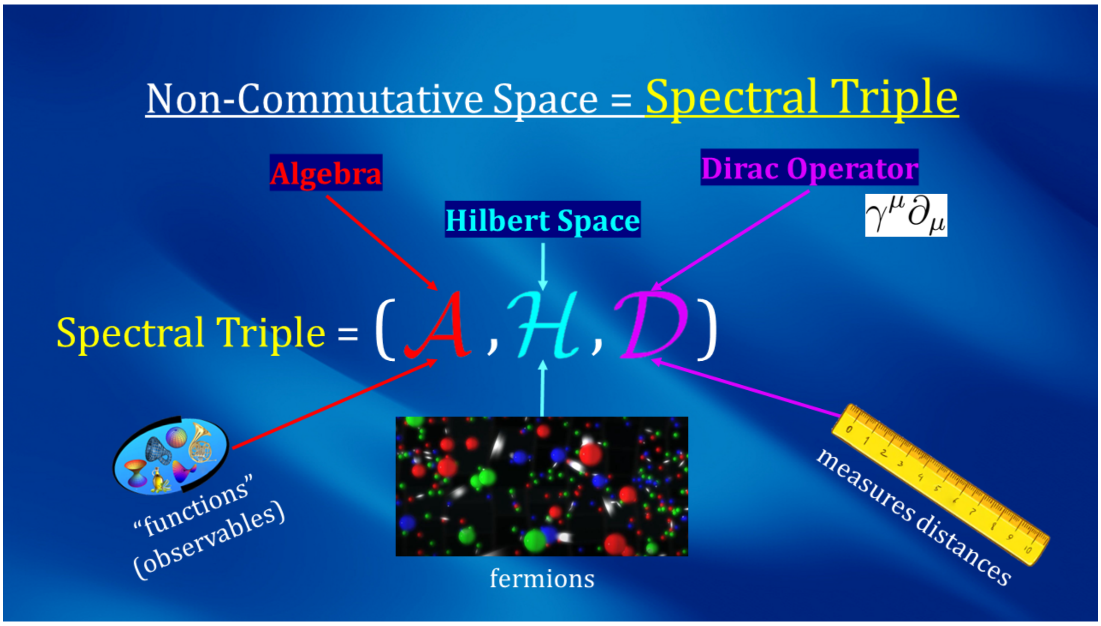

Figure 7.1: Summary of a Spectral Triple.

Amongst the three elements of the Spectral Triple, the Dirac operator has a privileged role, in the sense that, practically, it is used more often than the other two for extracting the physical content of the Spectral Triple. Now, we will show how to do so.

7.2.4 The Action

In standard QFT, one is free to choose which fields to introduce as well as their representations, and then they can build whatever action pleases him, so long as the desired symmetry requirements are satisfied. In the framework of NCG the situation is quite different. One is still free to introduce whatever matter fields pleases them – this is what the Hilbert space of the Spectral Triple is there for – but everything else – *including* the action! – descends from the Spectral Triple [56]:

$$S_f = \langle J\Psi, \mathcal{D}\Psi \rangle \,, \tag{7.2}$$

$$S_b = \lim_{\Lambda\to\infty} \operatorname{Tr} f\left(\frac{\mathcal{D}^2}{\Lambda^2}\right) . \tag{7.3}$$

The first of these two formulas is called in NCG jargon "fermionic action" and it contains all the action terms that involve fermions, both their kinetic terms and their interaction with both bosons and other fermions. The second formula is called "bosonic action" (or "spectral action") and it contains all the action terms involving *only* bosons – again, both their kinetic terms and their interactions with other bosons. The full action is then simply the sum $S_f + S_b$. It is important to stress that the separation of the action into the fermionic and bosonic part is purely computational and has no physical meaning whatsoever: in order to obtain the physics encoded in a NCS one needs to use the full action $S_f + S_b$, since the two have individually no physical meaning.

In (7.2-7.3), Ψ is something like a big multiplet containing all fermionic fields

$$\Psi \sim (\nu,\ e,\ u,\ d,\ \ldots)^t \,,$$

J is the charge conjugation operator, f is a smooth approximation of the characteristic function of the interval [0,1]

$$f\left(x\right) \sim \begin{cases} 1 & \Leftrightarrow\ x \in [0,\ 1] \\ 0 & \text{otherwise} \end{cases} \,,$$

Λ is a cutoff parameter representing the energy up to which one assumes the theory to hold up to, and the brackets denote the scalar product. Anyway, the details of these formulas aren't important right now (they will be in the next section) – the important fact is that they are there at all! This means that there is no freedom regarding the action: everything descends from the Spectral Triple. Once the Spectral Triple is defined, the action is also defined, and so is the full theory.

7.2.5 The Fluctuated Dirac Operator and the Bosons

In the action of a gauge theory, the usual derivative operator should be replaced with the covariant derivative operator. It is the same in NCG: in order to make the action manifestly gauge invariant, the Dirac operator in Eq.s (7.2, 7.3), which has the role of the usual derivative, should be replaced by a "covariant Dirac operator" – or, by using the proper NCG terminology, by the so-called *fluctuated Dirac operator* $\mathcal{D}_A$ [56]:

$$\mathcal{D}_A = \mathcal{D} + A + JAJ^{-1}. \tag{7.4}$$

Just like Ψ in Eq. (7.2) was a multiplet of fermionic fields, here A is itself a big multiplet of bosons (and J is once again the charge conjugation operator). As a technical aside, the reason for adding both A and JAJ^{-1} is in order to ensure that $\mathcal{D}_A$ still be self-adjoint, but this in principle is the same as replacing $\not{\partial}$ with $\not{\partial} + \not{A}$ in the Dirac action.

Now, here comes the key point. In usual QFT, in the covariant derivative appear all the gauge bosons – and of course, this is the same also in the fluctuated Dirac operator. The big difference is that, in QFT, only vector bosons are considered gauge bosons. On the contrary, in NCG *all* boson are considered gauge bosons, both vectors *and* scalars:

$$A \sim \left(A_\mu,\ W^\pm_\mu,\ Z_\mu,\ \ldots,\ H,\ \ldots\right)^t \tag{7.5}$$

In other words, NCG unifies the description of all bosonic fields, both vectors and scalars. In addition, since all bosons are gauge bosons in NCG, there is once again no freedom whatsoever regarding the bosons once might want to add to the action: if one wants to modify the bosonic content of the theory, e.g. by adding an extra scalar, one has to change the Spectral Triple first. As we will see, this fact is quite crucial in the context of the NCG formulation of the SM – i.e. the Connes model [56].

7.2.6 The Connes Model

Alain Connes, himself being one of the fathers of NCG, of course tried to apply all this machinery he came up with to the SM. The model he built carries his name as the Connes Model [56], and its results were quite interesting.

The model predicted 16 fermions per generation, therefore accomodating for the 15 known fermions (left-handed neutrino, electron with 2 chiralities, up and down quarks with two chiralities and 3 colors each) plus for the yet undetected right-handed neutrino; all the fermions had the correct quantum numbers (including the 16$^{\text{th}}$ extra fermion representing the right-handed neutrino). The gauge symmetries and the corresponding gauge vectors were the correct ones; moreover, a scalar with the same quantum numbers as the theorized Higgs boson (at the time, it hadn't been discovered yet) was also predicted.

Moreover, there was a big surprise. The Higgs mass was *not* a free parameter: it was actually a function of other, already-measured parameters! In other words, the Connes Model contained one of the first predictions of the Higgs mass.

And things do not end here. We wrote in the first section of this chapter that NCG in some sense aims to describe all interactions as pure gravity on a suitable space. Then, it shouldn't be a big surprise that, by applying Eq.s (7.2, 7.3) to the Spectral Triple of the SM, the lagrangian one ends up with is the SM lagrangian minimally coupled with GR (i.e. we get GR for free):

$$\mathcal{L} = \mathcal{L}_{GR}\left(g_{\mu\nu}\right) + \mathcal{L}_{SM}\left(\text{SM};\ g_{\mu\nu}\right). \tag{7.6}$$

Now, you might think everything seems too good to be true, especially since you probably never even heard of the Connes model: if all was well, it should have become quite famous, after all. And in fact, if you thought so, you were right: there are actually a few flaws in these results, which have been cleverly concealed so far in order to make clear the potentialities of NCG; but now it is the right time to deal with them.

First of all, the predicted Higgs mass turned out to be wrong – not completely wrong, the order of magnitude is the correct one, but the precise value just is not right:

$$m_{\text{NCG}}^{Higgs} \sim 170\,\text{GeV} \qquad\qquad \leftrightarrow \qquad\qquad m_{\text{true}}^{Higgs} = 125\,\text{GeV}.$$

But there is also another problem, maybe even more important than the last one. All the construction of NCG only works on pure spaces, not in space-*times*. The metric has to be Riemannian, it just cannot be Lorentzian. This has to do with the fact that the Lorentzian scalar product is not positive definite, from which follows that many operators that are bounded in the Riemannian case aren't bounded anymore in the Lorentzian case (from which many mathematical problems follow, which prevent NCG to be self-consistent on space-times).

In order to solve these problems without giving up on NCG, one has to somehow modify its axioms. Twisted NCG was born this way, with the goal to solve the issue of the wrong Higgs mass. Most surprisingly, it turned out to be seemingly able to solve the "no-time issue" as well, as we will see in chapter 8.

Actually, there is also another incongruity between the lagrangian of the Connes model and the one of GR. In fact, the Connes model predicts there be a Weyl term $\Omega_{\mu\nu\rho\sigma}\Omega^{\mu\nu\rho\sigma}$ [56], where

$$\Omega_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{n-2}\left(R_{\mu\sigma}g_{\nu\rho} - R_{\mu\rho}g_{\nu\sigma} + R_{\nu\rho}g_{\mu\sigma} - R_{\nu\sigma}g_{\mu\rho}\right) +$$

$$+\frac{1}{(n-1)(n-2)} R\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right), \quad (7.7)$$

with $R_{\mu\nu\rho\sigma}$ the Riemann tensor, $R_{\mu\nu}$ the Ricci tensor, R the Ricci scalar, $g_{\mu\nu}$ the metric and n the metric dimension of the non-commutative space. Such terms are quite interesting, for they often appear in modified gravity theories that aim to explain DM and DE effects without introducing new fields (see e.g. [58, 59]). In this work we will not explore the impact of such terms, but it is still a topic worthy to be examined in some future work.

7.3 Spectral Triples

In the rest of this chapter, we will reintroduce in full detail all the objects and structures of NCG we mentioned in the last introductory section (following [60, 56]); the uninterested reader may skip directly to Chapter 8. We will be using quite a number of terms that are used very often in mathematics, but not as much in physics; in order not to break the flow of the discussion, we will remind of their meaning in the footnotes. The reader that is already familiar with the terms may skip the footnotes and continue reading.

A Non-Commutative Geometry is given by a *Spectral Triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in the following sense.

Definition 7.3.1. *A* Spectral Triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a set of three objects of the following types:*

- $\mathcal{A}$ *is a unital involutive algebra;*
- $\mathcal{H}$ *is a Hilbert space on which* $\mathcal{A}$ *acts faithfully;*
- $\mathcal{D}$ *is called the* Dirac operator *and is a self-adjoint operator with compact*[6] *resolvent*[7] *such that for each* $a \in \mathcal{A}$*, the commutator* $[\mathcal{D}, a]$ *is bounded*[8]*.*

In addition, one defines the following notions:

Definition 7.3.2. *A Spectral Triple is* graded *or* even *if the Hilbert space* $\mathcal{H}$ *is endowed with a unitary self-adjoint operator* Γ *such that*

[6] An operator $T : X \to Y$ is compact if it it takes bounded sets in X to precompact sets in Y (i.e. to sets whose closure is compact).

[7] Given an operator $\mathcal{O}$, its resolvent is defined as $R(z; \mathcal{O}) = (\mathcal{O} - z\mathbb{I})^{-1}$. $\mathcal{O}$ is said to have compact resolvent if $R(z; \mathcal{O})$ is compact whenever $z \notin \sigma(\mathcal{O})$, where $\sigma(\mathcal{O})$ is the spectrum of $\mathcal{O}$, i.e. the set of λ for which $\mathcal{O} - \lambda\mathbb{I}$ is not invertible.

[8] In order to keep the notations simple, we will omit the representation symbol, so that for $a \in \mathcal{A}$ and $\psi \in \mathcal{H}$ we will write $a\psi \equiv \pi(a)\psi$.

- $a\Gamma = \Gamma a$ *for each* $a \in \mathcal{A}$*;*
- $\mathcal{D}\Gamma = -\Gamma\mathcal{D}$*.*

The operator Γ *is called the* $\mathbb{Z}_2$-grading*, or more simply the* grading*.*

Remark 7.3.1. If Γ is present, then the Hilbert space $\mathcal{H}$ can be split into the sum of two subspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm$ is the (± 1)-eigenspace of Γ.

Definition 7.3.3. *A Spectral Triple is* real *if* $\mathcal{H}$ *is endowed with an anti-linear isometry of* $\mathcal{H}$ *onto itself such that*

- $J^2 = \pm 1$*;*
- $J\mathcal{D} = \pm \mathcal{D}J$*.*

Further, if the Spectral Triple is even, J *should also satisfy*

- $J\Gamma = \pm \Gamma J$*.*

The operator J *is called a* real structure*.*

Given any algebra $\mathcal{A}$, one can define the opposite algebra $\mathcal{A}^\circ = \{a^\circ : a \in \mathcal{A}\}$ with product $a^\circ b^\circ = (ba)^\circ$. Using J one can represent the opposite algebra $\mathcal{A}^\circ$ on the Hilbert space $\mathcal{H}$ by defining $\pi^\circ(b^\circ) = J\pi(b^*) J^{-1}$, which we will abbreviate as

$$b^\circ = Jb^* J^{-1}. \tag{7.8}$$

The opposite algebra can be used to define the right action of the algebra on the elements of $\mathcal{H}$:

$$\psi b := b^\circ \psi. \tag{7.9}$$

In order for the left- and the right actions of $\mathcal{A}$ to be mutually independent, we require the so-called *zeroth-order condition*:

$$[a, b^\circ] = 0. \tag{7.10}$$

If $\mathcal{A}$ is commutative, we *define* $b^\circ = b$, so that the zeroth-order condition is automatically satisfied. Notice that this definition is not consistent with (7.8), but this is not a problem since the algebra is commutative, so that $a^* b^* = (ab)^*$.

In order for the Spectral Triple to actually represent a NCS, it is required to satisfy 7 axioms. Axioms 3-6 fall outside of the scope of this work, but we will report them nevertheless for completeness following [60].

7.3.1 Axiom 1: The Metric Dimension

Axiom 1 (Dimension)**.** *There is an integer* n*, the* metric dimension *of the spectral triple, such that the length element* $ds := |\mathcal{D}|^{-1}$ *is an infinitesimal of order* $1/n$*.*

Here, by "infinitesimal" we mean a compact operator on $\mathcal{H}$. The reason is the following. Conceptually, an infinitesimal is a non-zero quantity T smaller than any positive ε. Since we work with infinite-dimensional Hilbert spaces, we may allow the violation of the requirement $T < \varepsilon$ on a finite-dimensional subspace. T must then be an operator with discrete spectrum, with any non-zero λ in $\sigma(T)$ having finite multiplicity: in other words, the operator T must be compact.

If we denote

$$|T| := \left(T^\dagger T\right)^{\frac{1}{2}}, \tag{7.11}$$

we call *singular values* μ_k of T the eigenvalues of $|T|$. If we arrange the singular values in decreasing order $\mu_0 \geq \mu_1 \geq \mu_2 \geq \ldots$, we say that T is an *infinitesimal of order* α if

$$\mu_k = O\left(k^{-\alpha}\right) \qquad \text{as } k \to \infty. \tag{7.12}$$

Notice that infinitesimals of first order have singular values that form a logarithmically divergent series:

$$\mu_k = O\left(\frac{1}{k}\right) \quad \Rightarrow \quad \sigma_N(T) := \sum_{k<N} \mu_k = O(\log N) \qquad \text{as } N \to \infty. \tag{7.13}$$

The dimension axiom then entails that there is a positive integer n for which the singular values of $\mathcal{D}^{-n}$ form a logarithmically divergent series. The coefficient of logarithmic divergence will be denoted by $⨍ |\mathcal{D}|^{-n}$, where $⨍$ denotes the *non-commutative integral*, that will be defined in section 7.5.

There are two cases for which the metric dimension is 0:

- if the singular values of $|\mathcal{D}|^{-1}$ go to zero exponentially fast (see e.g. [61, 62] for examples); or
- both $\mathcal{A}$ and $\mathcal{H}$ are finite-dimensional, so that $\mathcal{D}$ is just a hermitian matrix. These *finite spectral triple* are very important to construct NCSs corresponding to gauge theories, as we will see in Chapter 9.

7.3.2 Axiom 2: The First-Order Condition

Axiom 2 (First-Order Condition)**.** *For all* $a, b \in \mathcal{A}$ *the following commutation relation holds:*

$$[[\mathcal{D}, a], b^\circ] = 0. \tag{7.14}$$

In the commutative case, the first-order condition expresses the fact that the Dirac operator is a first-order differential operator. Intuitively, if $\psi \in \mathcal{H}$, one has

$$[\partial, a]\,\psi = \partial\,(a\psi) - a\partial\psi = (\partial a)\,\psi + a\partial\psi - a\partial\psi = (\partial a)\,\psi, \tag{7.15}$$

where we have used the Leibniz rule. Then, this shows that the commutator $[\partial, a]$ is just the multiplication by (∂a). Then, remembering that in the commutative case we defined $b^\circ = b$, we find

$$[[\partial, a]\,, b^\circ] = [[\partial, a]\,, b] = [(\partial a)\,, b] = 0. \tag{7.16}$$

On the other hand, if we were to consider the commutator of a with ∂^2 (opposed to ∂), using the Leibniz rule twice one would find

$$\left[\left[\partial^2, a\right], b\right] = 2\,(\partial a)\,(\partial b) \neq 0. \tag{7.17}$$

We can interpret the zeroth- and the first-order conditions as saying that the operators b° commute with the subalgebra of operators on $\mathcal{H}$ generated by all a and $[\mathcal{D}, a]$. This gives rise to a linear representation of the tensor product of several copies of $\mathcal{A}$:

$$a \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n \coloneqq a\,[\mathcal{D}, a_1]\,[\mathcal{D}, a_2] \ldots [\mathcal{D}, a_n]\,. \tag{7.18}$$

In view of the order one condition, we could even replace the first entry $a \in \mathcal{A}$ by $a \otimes b^\circ \in \mathcal{A} \otimes \mathcal{A}^\circ$, writing

$$(a \otimes b^\circ) \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n \coloneqq ab^\circ\,[\mathcal{D}, a_1]\,[\mathcal{D}, a_2] \ldots [\mathcal{D}, a_n]\,. \tag{7.19}$$

Now, $C_n\,(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ) \coloneqq (\mathcal{A} \otimes \mathcal{A}^\circ) \otimes \mathcal{A}^{\otimes n}$ is a bimodule[9] over the algebra $\mathcal{A}$, and this prescription represents it by operators on $\mathcal{H}$. Its elements are called *Hochschild n-chains* with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathcal{A}^\circ$.

7.3.3 Axiom 3: Smoothness of the Algebra

Axioms 3 to 6 fall outside of the scope of this work, but we will report them for sake of completeness, following [60].

Axiom 3 (Regularity). *For any $a \in \mathcal{A}$, $[\mathcal{D}, a]$ is a bounded operator on $\mathcal{H}$, and both a and $[\mathcal{D}, a]$ belong to the domain of smoothness $\cap_{k=1}^{\infty} \operatorname{Dom}\left(\delta^k\right)$ of the derivation δ on $\mathcal{L}\,(\mathcal{H})$[10] given by $\delta\,(T) \coloneqq [|\mathcal{D}|, T]$.*

[9] A bimodule is an abelian group that is both a left and a right module. A left (right) module is an abelian group over the sum operation that is equipped with an external product from the left (right) with the elements of a ring. A ring is once again an abelian group over the sum operation that is equipped with a product operation that is both associative and distributive with respect to the sum.

[10] $\mathcal{L}\,(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$.

In the commutative case, one has $[\mathcal{D}, a] \sim [\partial, a] = (\partial a)$. Then, this axiom amounts to saying that a has derivatives of all orders, i.e. that $\mathcal{A} \subseteq C^\infty(\mathcal{M})$. This can be proved with pseudodifferential calculus: since the principal symbol[11] of $|\mathcal{D}|$ operator is $\sigma_{|\mathcal{D}|}(x, \xi) = 1|\xi|$, one finds that all multiplication operators in $\cap_{k=1}^{\infty} \mathrm{Dom}\left(\delta^k\right)$ are multiplications by smooth functions.

Assuming regularity, using the seminorms[12] $a \mapsto \left\|\delta^k(a)\right\|$ and $a \mapsto \left\|\delta^k([\mathcal{D}, a])\right\|$, one can confer a locally convex topology[13] on $\mathcal{A}$. The completion of $\mathcal{A}$ in this topology is a Fréchet[14] pre-C^*-algebra[15], and Axiom 3 still holds when $\mathcal{A}$ is replaced by its completion [63, Lemma 16]. Also, this topology coincides with the usual one on $C^\infty(\mathcal{M})$ in the commutative case [63, Prop. 20]. Therefore, we may assume that the algebra of a regular spectral triple be a pre-C^*-algebra.

7.3.4 Axiom 4: Orientability

Axioms 3 to 6 fall outside of the scope of this work, but we will report them for sake of completeness, following [60].

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[11] Let P be a differential operator of order n in the derivative D: $P = p(x, D) = \sum_{|k|\leq n} c_k(x) D^k$. Then, the *total symbol* of P is the polynomial p: $p(x, \xi) = \sum_{|k|\leq n} c_k(x) \xi^k$. The *principal symbol* of P is the highest-degree component of p: $\sigma_P(x, \xi) = \sum_{|k|=n} c_k(x) \xi^k$.

[12] A seminorm is a norm that allows $\|x\| = 0$ also for nonzero x.

[13] Let V be a vector space over a field $K \subseteq \mathbb{C}$. Then a subset $C \subseteq V$ is said

- *Convex* if for all $x, y \in C$ and $0 \leq t \leq 1$, $tx + (1-t)y \in C$. In other words, C contains all line segments between points in C.
- *Balanced* if for all $x \in C$, $\lambda x \in C$ if $|\lambda| \leq 1$. This means that for any $x \in C$, C contains the disk with x on its boundary, centred in the origin.
- *Absolutely convex* if it is both balanced and convex.
- *Absorbent* if for all $y \in V$, there exists $r > 0$ such that $y \in tC$ for all $t \in K$ such that $|t| \geq r$. This means that the set C can be scaled out by any "large" value to absorb every point in the space.

Then, a topological vector space is *locally convex* if the origin has a local base of absolutely convex absorbent sets. It can be shown that any vector space endowed with a family of seminorms is always a locally convex space, and vice versa (see e.g. https://en.wikipedia.org/wiki/Locally_convex_topological_vector_space).

[14] A *Fréchet space* is a locally convex complete topological vector space with respect to a topology induced by a translation-invariant metric. Equivalently, it can be defined as a Hausdorff space with topology induced by a countable family of seminorms, that is complete with respect to the family of seminorms.

[15] A *C^*-algebra* $\mathcal{A}$ is an associative involutive algebra over $\mathbb{C}$, complete with respect to a norm, as regards which the product is continuous $\|AB\| \leq \|A\| \|B\|$ and that satisfies the C^*-condition $\|A^*A\| = \|A\|^2$. A *pre-C^*-algebra* is almost a C^* algebra, but its norm need not be complete.

Axiom 4 (Orientability). *There is a Hochschild cycle $\boldsymbol{c} \in C_n\left(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ\right)$ whose representative on $\mathcal{H}$ is*

$$\boldsymbol{c} = \begin{cases} \Gamma, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \tag{7.20}$$

Here $\boldsymbol{c}$ is a Hochschild n-chain as defined in the end of section 7.3.2. We say $\boldsymbol{c}$ is a *cycle* if its boundary is zero, where the Hochschild boundary $b\left(x\right)$ of the n-chain $x = m \otimes a_1 \otimes \ldots \otimes a_n$, with $m \in \mathcal{A} \otimes \mathcal{A}^\circ$, is

$$\begin{aligned} b\left(x\right) \coloneqq\ & m\, a_1 \otimes a_2 \otimes \ldots \otimes a_n - m \otimes a_1\, a_2 \otimes \ldots \otimes a_n + \ldots \\ & + \left(-1\right)^{n-1} m \otimes a_1 \otimes \ldots \otimes a_{n-1}\, a_n + \left(-1\right)^n a_n\, m \otimes a_1 \otimes \ldots \otimes a_{n-1}. \end{aligned} \tag{7.21}$$

This satisfies $b^2 = 0$ and thus makes $C_\bullet\left(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ\right)$ a chain complex, whose homology is the Hochschild homology[16] $H_\bullet\left(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ\right)$. This Hochschild cycle $\boldsymbol{c}$ is the algebraic equivalent of a *volume form* on a non-commutative space. To see that, let us look briefly at the commutative case, where we may replace $\mathcal{A} \otimes \mathcal{A}^\circ$ simply by $\mathcal{A}$. A differential form in $\mathcal{A}^k\left(\mathcal{M}\right)$ is a sum of terms $a_0\, da_1 \wedge \ldots \wedge da_k$, but in the non-commutative case the antisymmetry of the wedge product is lost, so we replace such a form with

$$\boldsymbol{c}' \coloneqq \sum_\sigma \left(-1\right)^\sigma a_0 \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)} \tag{7.22}$$

(sum over n-permutations) in $\mathcal{A}^{\otimes(n+1)} = C_n\left(\mathcal{A}, \mathcal{A}\right)$. Then $b\left(\boldsymbol{c}'\right) = 0$ by cancellation, since $\mathcal{A}$ is commutative, for instance:

$$\begin{aligned} b\left(a \otimes a' \otimes a'' - a \otimes a'' \otimes a'\right) = \left(aa' - a'a\right) \otimes a'' - a \otimes \left(a'a'' - a''a'\right) + \\ + \left(a''a - aa''\right) \otimes a'. \end{aligned} \tag{7.23}$$

[16] An *n-simplex* is the n-dimensional analogue of a triangle, e.g. a point ($n = 0$), a segment ($n = 1$), a triangle ($n = 2$), a tetrahedron ($n = 3$) and so on. An n-simplex is always delimited by $n+1$ points; when one decides upon an order of those points, the simplex becomes an *oriented simplex*. Two oriented n-simplexes delimited by the same points are considered the same oriented simplex if the sequences of delimiting points of the two differ by an even permutation, otherwise, the two oriented simplexes are considered *opposite* to each other (in the intuitive sense of going from point p_1 to p_n in the first one, and going from p_n to p_1 in the opposite one). In this sense, oriented n-simplexes form an abelian group. A set of oriented simplexes is called a *simplicial complex*. If K is a simplicial complex, its *r-chain group* $C_r\left(K\right)$ is the abelian group generated by the r-simplexes of K; the elements of $C_r\left(K\right)$ are called r-chains. The boundary of an r-chain is an $r-1$-chain defined as the antisymmetric linear combinations of all the $r-1$ chains delimited by the same points of the initial r-chain. The boundary of a chain has zero boundary. The sequence of all $C_n\left(K\right)$ is called the *chain complex* of K and is denoted by $C_\bullet\left(K\right)$. An r-chain with zero boundary is called an *r-cycle*. The set of all r-cycles forms the *r-cycle group*, denoted by $Z_r\left(K\right)$. Conversely, the set of boundaries of $r+1$-chains form the *r-boundary group* $B_r\left(K\right)$. The r^{th} *homology group* is defined by $H_r\left(K\right) \coloneqq Z_r\left(K\right) / B_r\left(K\right)$.

In the case $\mathcal{A} = C^\infty(M)$, with M a spin-manifold, chains are represented by Clifford products: $a_0 \otimes a_1 \otimes \ldots \otimes a_n = (-i)^n a_0 \gamma(da_1)\ldots\gamma(da_n)$. The Riemannian volume form on M can be written as $\Omega = \theta^1 \wedge \ldots \wedge \theta^n$ where $\{\theta_k\}$ is an oriented orthonormal basis of 1-forms. Then, the cycle $\boldsymbol{c}$ corresponding to $i^{\lfloor (n+1)/2 \rfloor}\Omega$ is $\boldsymbol{c} = (-i)^{\lfloor n/2 \rfloor}\gamma(\theta^1)\ldots\gamma(\theta^n) = (-i)^{\lfloor n/2 \rfloor}\prod_k \gamma^k = \gamma_5$, i.e. the chirality element, which corresponds to the grading operator on spinors if n is even, or to the identity if n is odd.

7.3.5 Axiom 5: Finiteness of the K-cycle

Axioms 3 to 6 fall outside of the scope of this work, but we will report them for sake of completeness, following [60].

Axiom 5 (Finiteness)**.** *The space of smooth vectors* $\mathcal{H}^\infty := \cap_{k=1}^\infty \operatorname{Dom}(\mathcal{D}^k)$ *is a finitely generated projective left* $\mathcal{A}$*-module*[17] *with a Hermitian pairing*[18] $(\cdot\,|\,\cdot)$ *implicitly defined by*

$$⨍ (\xi\,|\,\eta)\, ds^n := \langle \eta\,|\,\xi \rangle\,. \tag{7.24}$$

Here, the symbol $⨍$ denotes the non-commutative integral, which will be defined in section 7.5. The representation of the algebra as linear operators on $\mathcal{H}$ and the regularity axiom already make $\mathcal{H}^\infty$ a left $\mathcal{A}$-module. Notice that the pairing $(\cdot\,|\,\cdot)$ is linear in the *first* argument, while the inner product $\langle\cdot\,|\,\cdot\rangle$ is linear in the *second* one, so that

$$⨍ a\,(\xi\,|\,\eta)\, ds^n = \langle \eta\,|\,a\xi \rangle\,. \tag{7.25}$$

To see how this defines a Hermitian pairing implicitly, notice that if $a \in \mathcal{A}$, then $a\,ds^n = a\,|\mathcal{D}|^{-n}$ is an infinitesimal of first order because of the dimension axiom, so that the left hand side is defined provided $(\xi\,|\,\eta) \in \mathcal{A}$. As a finitely generated projective left A-module, $\mathcal{H}^\infty \simeq \mathcal{A}^m p$ with $p = p^2 = p^*$ a projection in some $M_m(\mathcal{A})$. In the commutative case, Connes's trace theorem [60, Sec. 5.4] shows

[17] A *free module* is a module that has a basis. A module P is *projective* if it is locally free.

[18] A *pairing* on a finitely generated projective left $\mathcal{A}$-module $(\cdot|\cdot) : \mathcal{H}\times\mathcal{H} \to \mathcal{A}$ is a positive definite sesquilinear form such that

$$\begin{aligned}(\psi + \xi\,|\,\eta) &= (\psi\,|\,\eta) + (\xi\,|\,\eta)\\ (a\xi\,|\,\eta) &= a\,(\xi\,|\,\eta)\\ (\xi\,|\,\eta) &= (\eta\,|\,\xi)^*\\ (\xi\,|\,\xi) &> 0 \quad \text{for } \xi \neq 0\end{aligned}$$

for $\psi, \xi, \eta \in \mathcal{H}$ and $a \in \mathcal{A}$.

that $(\xi\,|\,\eta)$ is just the hermitian product of spinors given by the metric on the spinor bundle.

A point to notice is that

$$⨍ a\,(\xi\,|\,\eta)\,ds^n = \langle \eta\,|\,a\xi\rangle = \langle a^*\eta\,|\,\xi\rangle = ⨍ (\xi\,|\,a^*\eta)\,ds^n = ⨍ (\xi\,|\,\eta)\,a\,ds^n, \quad (7.26)$$

so this axiom implies that $⨍ (\cdot)\,|\mathcal{D}|^{-n}$ defines a *trace* on the algebra $\mathcal{A}$.

7.3.6 Axiom 6: Poincaré duality[19] and K-theory

Axioms 3 to 6 fall outside of the scope of this work, but we will report them for sake of completeness, following [60].

Axiom 6 (Poincaré duality)**.** *The Fredholm index[20] of the operator $\mathcal{D}$ yields a non-degenerate intersection form[21] on the K-theory[22] ring of the algebra $\mathcal{A}\otimes\mathcal{A}^\circ$.*

On a compact oriented n-dimensional manifold $\mathcal{M}$, Poincaré duality is usually formulated [64] as an isomorphism of cohomology (in degree k) with homology

[19] Let $\mathcal{M}$ be a manifold. Then we denote $C_r(\mathcal{M})$ the r-chain group, $Z_r(\mathcal{M})$ the r-cycle group, $B_r(\mathcal{M})$ the r-boundary group, and $H_r(\mathcal{M}) = Z_r(\mathcal{M})/B_r(\mathcal{M})$ the r^{th}homology group (see footnote 16 for more details). One can define their dual spaces: the r^{th}-cochain group $C^r(\mathcal{M})$ is the set of r-forms, the r^{th}-cocycle group $Z^r(\mathcal{M})$ is the set of closed r-forms, the r^{th}-coboundary group $B^r(\mathcal{M})$ is the set of exact r-forms, and the r^{th}-cohomology group is $H^r(\mathcal{M}) \coloneqq Z^r(\mathcal{M})/B^r(\mathcal{M})$. The *Poincaré duality* theorem states that if $\mathcal{M}$ is an n-dimensional oriented closed manifold (compact and without boundary), then $H_r(\mathcal{M}) \simeq H^{n-r}(\mathcal{M})$, $\forall r\in\mathbb{N}$.

[20] A Banach space is a complete normed vector space, i.e. a Hilbert space whose norm is not necessarily associated with an inner product. A Fredholm operator is a bounded linear operator $T : X \to Y$ between Banach spaces with finite-dimensional kernel $\ker T$, finite-dimensional cokernel $\operatorname{coker} T = Y/\operatorname{ran} T$ and with closed range $\operatorname{ran} T$. Intuitively, Fredholm operators are those operators that are invertible "if finite-dimensional effects are ignored". The Fredholm index of a Fredholm operator is $\operatorname{ind} T = \dim\ker T - \dim\operatorname{coker} T$.

[21] A standard n-simplex is given by $\Delta^n = \left\{(t_0,\ldots,t_n)\in\mathbb{R}^{n+1}\middle|\sum_i t_i = 1 \text{ and } t_i\geq 0\right\}$. A *singlular n-simplex* in a topological space X is any continuous function $\sigma : \Delta^n \to X$. Given a p-cochain c_p and a q-cochain d_q, one defines the *cup product* $\smile$ over singular $(p+q)$-simplexes σ as $(c_p \smile d_q)(\sigma) \coloneqq c_p(\sigma|_{\text{faces } 0\to p})\cdot d_q(\sigma|_{\text{faces } p\to p+q})$. The cup product naturally extends to cocycles, coboundaries and to the cohomology groups. If $\mathcal{M}'$ is an n-dimensional connected orientable closed manifold, the *fundamental class* of $\mathcal{M}'$ is the homology class $[\mathcal{M}']\in H_n(\mathcal{M}')$ which corresponds to the generator of $H_n(\mathcal{M}')$; it can be thought of as the orientation of the top-dimensional simplices of the manifold; it represents integration over $\mathcal{M}'$, in the sense that, given an n-form ω, one has $\langle\omega,[\mathcal{M}']\rangle = \int_{\mathcal{M}'}\omega$. The *intersection form* $\lambda_{\mathcal{M}} : H^n(\mathcal{M})\times H^n(\mathcal{M}) \to \mathbb{Z}$ on a $2n$-dimensional connected oriented manifold $\mathcal{M}$ is given by $\lambda_{\mathcal{M}}(a,b) \coloneqq \langle a\smile b,[\mathcal{M}]\rangle$.

[22] *K-theory* is, roughly speaking, the study of a ring generated by vector bundles (i.e. a smooth collection of vector spaces) over a topological space.

(in degree $n-k$; see footnote 19), or equivalently as a nondegenerate bilinear pairing on the cohomology ring[23] $H^{\bullet}(\mathcal{M})$. If $\alpha \in Z^{k}(\mathcal{M})$ and $\beta \in Z^{n-k}(\mathcal{M})$ are closed forms, integration over $\mathcal{M}$ pairs them by

$$(\alpha, \beta) \mapsto \int_{\mathcal{M}} \alpha \wedge \beta. \tag{7.27}$$

Since the right hand side depends only on the cohomology classes of α and β (it vanishes if either α or β is exact), it gives a bilinear map $H^{k}(\mathcal{M}) \times H^{n-k}(\mathcal{M}) \to \mathbb{C}$. Now, each $Z^{k}(\mathcal{M})$ carries a scalar product $(\cdot\,|\,\cdot)$ induced by the metric and orientation on $\mathcal{M}$, given by

$$\alpha \wedge \star\beta =: \varepsilon_{k}\,(\alpha\,|\,\beta)\,\Omega, \tag{7.28}$$

where $\varepsilon_{k} \in \{\pm 1, \pm i\}$, Ω is the volume form on $\mathcal{M}$, and $\star$ denotes the Hodge dual[24]. This pairing is nondegenerate since

$$\int_{\mathcal{M}} \alpha \wedge \left(\varepsilon_{k}^{-1} \star \alpha\right) = \int_{\mathcal{M}} (\alpha\,|\,\alpha)\,\Omega > 0 \text{ for } \alpha \neq 0. \tag{7.29}$$

Now, one could hope to reformulate all this construction that exists on manifolds onto the K-theory ring, as a canonical pairing. This can be done if $\mathcal{M}$ is a spin manifold; the role of the orientation $[\Omega]$ in cohomology is replaced by the K-orientation, so that the corresponding pairing of K-rings is mediated by the Dirac operator [65]. We leave aside the translation from K-theory to cohomology (which is quite the long story) and explain briefly how the intersection form may be computed in the K-context.

K-theory of algebras

There are two abelian groups, $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, associated to a Fréchet pre-C^*-algebra $\mathcal{A}$ [66, 67, 68]. The group $K_0(\mathcal{A})$ gives a rough classification of finitely generated projective modules over $\mathcal{A}$. Let us denote $M_\infty(\mathbb{C})$ the algebra of compact operators of finite rank, and let us define $M_\infty(\mathcal{A}) = \mathcal{A} \otimes M_\infty(\mathbb{C})$. Two projectors in $M_\infty(\mathcal{A})$ have a direct sum $p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$. Two such projectors p and q are equivalent if $p = v^*v$ and $q = vv^*$ for some $v \in M_\infty(\mathcal{A})$. Adding the

[23] The cohomology ring $H^{\bullet}(\mathcal{M})$ is the sequence of all $H^{r}(\mathcal{M})$.

[24] Let $\mathcal{M}$ be an n-dimensional oriented manifold. The Hodge dual $\star\zeta$ of a k-form ζ is the unique $(n-k)$-form such that $\eta \wedge \star\zeta = \langle \eta\,,\,\zeta \rangle\,\Omega$ for each k-form η, where $\langle \cdot\,,\,\cdot \rangle$ is the inner product between $k-$forms and Ω is the volume form on $\mathcal{M}$. Roughly speaking, the Hodge dual of a form ζ is the form whose components are complementary to those of ζ, in the sense that, in two dimensions, one has $\star 1 = dx \wedge dy$, $\star dx = dy$, $\star dy = -dx$ and $\star(dx \wedge dy) = 1$.

equivalence classes by $[p]+[q] =: [p \oplus q]$, we get a semigroup[25], and one can always construct a corresponding group of formal differences $[p]-[q]$: this is $K_0(\mathcal{A})$.

The other group $K_1(\mathcal{A})$ is generated by classes of unitary matrices over $\mathcal{A}$. We nest the unitary groups of various sizes by identifying $u \in U_m(\mathcal{A})$ with $u \oplus \mathbb{I}_k \in U_{m+k}(\mathcal{A})$, and call u, v equivalent if there is a continuous path from u to v in $U_\infty(\mathcal{A}) := \cup_{m\geq 1} U_m(\mathcal{A})$. The group $K_1(\mathcal{A})$ consists of the equivalence classes of $U_\infty(\mathcal{A}^+)$ under this relation[26].

When $\mathcal{A}$ is represented on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, any odd selfadjoint Fredholm[27] operator $\mathcal{D}$ on $\mathcal{H}$ defines an index map $\phi_\mathcal{D} : K_0(\mathcal{A}) \to \mathbb{Z}$ as follows. Denote by $a \mapsto a^+ \oplus a^-$ the representation of $M_m(\mathcal{A})$ on $\mathcal{H}_m^+ \oplus \mathcal{H}_m^- = \mathcal{H}_m = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (m times). Write $\mathcal{D}_m = \mathcal{D} \oplus \ldots \oplus \mathcal{D}$, acting on $\mathcal{H}_m$. Then $p^- \mathcal{D}_m p^+$ is a Fredholm operator from $\mathcal{H}_m^+$ to $\mathcal{H}_m^-$, whose index depends only on the class $[p]$ in $K_0(\mathcal{A})$. We define

$$\phi_\mathcal{D}([p]) := \operatorname{ind}\left(p^- \mathcal{D}_m p^-\right). \tag{7.30}$$

The intersection form

Coming back now to the spectral triple under discussion, we define a pairing on $K_\bullet(\mathcal{A}) := K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$ as follows. The commuting representations of $\mathcal{A}$ and of $\mathcal{A}^\circ$ on the Hilbert space determine a representation of the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$ on $\mathcal{H}$ by

$$a \otimes b^\circ \mapsto aJb^*J^{-1} = Jb^*J^{-1}a. \tag{7.31}$$

If $[p], [q] \in K_0(\mathcal{A})$, then $[p \otimes q^\circ] \in K_0(\mathcal{A} \otimes \mathcal{A}^\circ)$. The *intersection form* for $\mathcal{D}$ is

$$\langle [p]\,,\, [q] \rangle := \phi_\mathcal{D}([p \otimes q^\circ]). \tag{7.32}$$

Poincaré duality is the assertion that this pairing on $K_\bullet(\mathcal{A})$ is nondegenerate.

7.3.7 Axiom 7: The Real Structure

Axiom 7 (Reality)**.** *There is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that the representations of $a \in \mathcal{A}$ and $b^\circ = Jb^*J^{-1} \in \mathcal{A}^\circ$ commute and that satisfies*

$$J^2 = \varepsilon, \qquad J\mathcal{D} = \varepsilon' \mathcal{D} J, \qquad J\Gamma = \varepsilon'' \Gamma J, \tag{7.33}$$

where the numbers $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ define the KO-dimension $\mathfrak{n}$ which is defined mod 8 *by*

[25] A semigroup is a group without inverse elements.
[26] $\mathcal{A}^+$ is the set of positive elements of $\mathcal{A}$.
[27] See footnote 20

$\mathfrak{n}$	0	1	2	3	4	5	6	7
ε	+	+	−	−	−	−	+	+
ε'	+	−	+	+	+	−	+	+
ε''	+		−		+		−	

These tables, with their periodicity in steps of 8, arise from the structure of real Clifford algebra representations. From a physical point of view, the operator J represents the charge-conjugation operator for fermions.

With this last axiom, we finally have all the ingredients we needed to properly define a non-commutative space.

7.3.8 Non-commutative spin geometry

Definition 7.3.4 (Non-commutative spin geometry)**.** *A non-commutative spin geometry is a real graded spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D}; \Gamma, J)$ *that satisfies Axioms 1–7 set out above.*

7.4 Equivalence of Geometries and Gauge Symmetries

So far, we have defined what NCSs are; now, we would like to find a way to compare two of them, and possibly to check whether the two are in fact the very same NCS in two different disguises. Once again, we will follow [60, 56].

7.4.1 Unitary equivalence of Non-Commutative Spaces

In order to compare two NCSs $(\mathcal{A},\ \mathcal{H},\ \mathcal{D};\ \Gamma,\ J)$ and $(\mathcal{A}',\ \mathcal{H}',\ \mathcal{D}';\ \Gamma',\ J')$, we focus first of all on the algebras $\mathcal{A}$ and $\mathcal{A}'$. It is natural to require that these be isomorphic; moreover, since these algebras define spin geometries only through their representations on the Hilbert spaces, we lose nothing by assuming that they are the same algebra $\mathcal{A}$. We can also assume that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are the same, so that $\mathcal{A}$ acts on $\mathcal{H}$ with two possibly different faithful representations. One must then match the operator pairs $\mathcal{D}$ and $\mathcal{D}'$, etc., on the Hilbert space $\mathcal{H}$. We are thus led to the notion of unitary equivalence of spin geometries.

Definition 7.4.1. *Two NCSs* $(\mathcal{A},\ \mathcal{H},\ \mathcal{D};\ \Gamma,\ J)$ *and* $(\mathcal{A},\ \mathcal{H},\ \mathcal{D}';\ \Gamma',\ J')$ *with the* *same algebra and Hilbert space are unitarily equivalent if there is a unitary operator* $U : \mathcal{H} \to \mathcal{H}$ *such that*

- $U\mathcal{D} = \mathcal{D}'U$, $U\Gamma = \Gamma'U$ *and* $UJ = J'U$*;*
- $UaU^{-1} = \sigma(a)$ *for an automorphism* σ *of* $\mathcal{A}$*.*

If in addition $J' = J$, one can easily check by direct computation that $Ub^\circ U^{-1} = (\sigma(b))^\circ$.

It is quite the tedious task to check that, given a NCS $(\mathcal{A}, \mathcal{H}, \mathcal{D};\ \Gamma, J)$ and any unitary operator U on $\mathcal{H}$ such that $U\mathcal{A}U^{-1} = \mathcal{A}$, then $(\mathcal{A}, \mathcal{H}, U\mathcal{D}U^{-1};\ U\Gamma U^{-1}, UJU^{-1})$ is also a NCS, so we will omit the details. The interested reader should refer to [60]. In what follows, we will focus on a particular kind of U.

On any NCS $(\mathcal{A}, \mathcal{H}, \mathcal{D};\ \Gamma, J)$, there is an action by inner automorphisms of the algebra $\mathcal{A}$. If u is a unitary element of the algebra $\mathcal{A}$ (i.e. such that $uu^* = u^*u = 1$), consider the unitary operator on $\mathcal{H}$ given by

$$U \coloneqq u\big(u^{-1}\big)^\circ = uJuJ^{-1} = JuJ^{-1}u. \tag{7.34}$$

If $a \in \mathcal{A}$, then $UaU^{-1} = uau^{-1}$ since JuJ^{-1} commutes with a, so U implements the inner automorphism $\sigma_u(a) \coloneqq uau^{-1}$. Next, since $J^2 = \varepsilon$, one has

$$UJ = uJu = \varepsilon uJ^{-1}u = J^2uJ^{-1}u = JU. \tag{7.35}$$

Similarly, one can show that Γ commutes with U as well: $U\Gamma = \Gamma U$. Thus, the given geometry is unitarily equivalent to $(\mathcal{A}, \mathcal{H}, {}^u\mathcal{D};\ \Gamma, J)$, where

$$\begin{aligned} {}^u\mathcal{D} \coloneqq U\mathcal{D}U^* &= JuJ^{-1}u\mathcal{D}u^*Ju^*J^{-1} = JuJ^{-1}\left(\mathcal{D} + u\left[\mathcal{D}, u^*\right]\right)Ju^*J^{-1} = \\ &= JuJ^{-1}\mathcal{D}Ju^*J^{-1} + u\left[\mathcal{D}, u^*\right] = \mathcal{D} + JuJ^{-1}\left[\mathcal{D}, Ju^*J^{-1}\right] + u\left[\mathcal{D}, u^*\right] = \\ &= \mathcal{D} + u\left[\mathcal{D}, u^*\right] + \varepsilon' Ju\left[\mathcal{D}, u^*\right]J^{-1}, \end{aligned} \tag{7.36}$$

where we used the first-order condition as well as $J\mathcal{D} = \varepsilon'\mathcal{D}J$. Notice that the operator $u\left[\mathcal{D}, u^*\right]$ is bounded and selfadjoint in $\mathcal{L}(\mathcal{H})$.

7.4.2 Morita equivalence and connections

The unitary equivalence of spin geometries helps to eliminate obvious redundancies, but it is by no means the only way to compare geometries. We need a looser notion of equivalence between spin geometries that allows to vary not just the operator data but also the algebra and the Hilbert space. Here the Morita equivalence of algebras [69] gives us a clue as to how to proceed, since Morita-equivalent algebras have equivalent representation theories [70, 71, 69].

Two algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita-equivalent if there exists a $\mathcal{B}$-$\mathcal{A}$ equivalence Hilbert bimodule $\mathcal{E}$, namely a module $\mathcal{E}$ which is at the same time a right Hilbert module[28] over $\mathcal{A}$ with $\mathcal{A}$-valued hermitian structure $\langle\cdot\,,\,\cdot\rangle_{\mathcal{A}}$ and a left Hilbert module over $\mathcal{B}$ with hermitian structure $\langle\cdot\,,\,\cdot\rangle_{\mathcal{B}}$ such that

- The module $\mathcal{E}$ is full[29] both as a right and as a left Hilbert module;
- The hermitian structures are compatible, namely

$$\langle\eta\,,\,\xi\rangle_{\mathcal{B}}\zeta = \eta\langle\xi\,,\,\zeta\rangle_{\mathcal{A}} \qquad \forall\eta,\xi,\zeta\in\mathcal{E}; \tag{7.37}$$

- The left representation of $\mathcal{B}$ on $\mathcal{E}$ is a continuous $*$-representation by operators which are bounded for $\langle\cdot\,,\,\cdot\rangle_{\mathcal{A}}$, namely $\langle b\eta\,,\,b\eta\rangle_{\mathcal{A}} \leq \|b\|^2\langle\eta\,,\,\eta\rangle_{\mathcal{A}}$ for $b\in\mathcal{B}$ and $\eta\in\mathcal{E}$. Similarly, the right representation of $\mathcal{A}$ on $\mathcal{E}$ is a continuous $*$-representation by operators which are bounded for $\langle\cdot\,,\,\cdot\rangle_{\mathcal{B}}$, i.e. $\langle\eta a\,,\,\eta a\rangle \leq \|a\|^2\langle\eta\,,\,\eta\rangle_{\mathcal{B}}$ for $a\in\mathcal{A}$ and $\eta\in\mathcal{E}$.

Given any $\mathcal{B}$-$\mathcal{A}$ equivalence Hilbert bimodule $\mathcal{E}$ one can exchange the role of $\mathcal{A}$ and $\mathcal{B}$ by constructing the associated complex conjugate $\mathcal{A}$-$\mathcal{B}$ equivalence Hilbert bimodule $\overline{\mathcal{E}}$ with a *right* action of $\mathcal{A}$ and a *left* action of $\mathcal{B}$. As an additive group, $\overline{\mathcal{E}}$ is identified with $\mathcal{E}$ and any element of it will be denoted by $\overline{\eta}$, with $\eta\in\mathcal{E}$. Then one gives a conjugate action of $\mathcal{A}$, $\mathcal{B}$ (and complex numbers) with corresponding hermitian structures. The left action by $\mathcal{A}$ and the right action by $\mathcal{B}$ are defined by

$$a\overline{\eta} := \overline{\eta a^*} \qquad \forall a\in\mathcal{A},\overline{\eta}\in\overline{\mathcal{E}} \tag{7.38}$$

$$\overline{\eta}b := \overline{b^*\eta} \qquad \forall b\in\mathcal{B},\overline{\eta}\in\overline{\mathcal{E}}. \tag{7.39}$$

As for the hermitian structures, they are given by

$$\left\langle\overline{\eta}\,,\,\overline{\xi}\right\rangle_{\mathcal{A}} := \langle\eta\,,\,\xi\rangle_{\mathcal{A}}\,, \tag{7.40}$$

$$\left\langle\overline{\eta}\,,\,\overline{\xi}\right\rangle_{\mathcal{B}} := \langle\eta\,,\,\xi\rangle_{\mathcal{B}} \qquad \forall\eta,\xi\in\mathcal{E}. \tag{7.41}$$

There is a theorem that states that two Morita-equivalent algebras have equivalent representation theory (see e.g. [70, 71, 69]). As we mentioned earlier, the only impact of an algebra on its corresponding NCS is through its representation on the Hilbert space. Then, it should be clear that spectral triples involving Morita-equivalent algebras describe actually the very same NCS.

[28]A Hilbert module is basically a Hilbert space whose scalars are elements of a generic algebra $\mathcal{A}$ instead of being elements of $\mathbb{C}$.

[29]Let $\mathcal{E}$ be a Hilbert module over $\mathcal{A}$. Then, the closure of the linear span of $\{\langle\eta\,,\,\xi\rangle_{\mathcal{A}}\,,\ \eta,\xi\in\mathcal{E}\}$ is an ideal of $\mathcal{A}$. If this ideal is the whole $\mathcal{A}$, the module $\mathcal{E}$ is called a full Hilbert module.

If one starts from a specific spectral triple with algebra $\mathcal{A}$, there is a standard procedure to build the spectral triple for the Morita-equivalent algebra $\mathcal{B}$ [60]. We start with any spin geometry $(\mathcal{A},\, \mathcal{H},\, \mathcal{D};\; \Gamma,\, J)$ and a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$, with pairing $(\cdot\,|\,\cdot)$. Using the representation $\mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the opposite representation $b \mapsto b^\circ$, we can regard the space $\mathcal{H}$ as an $\mathcal{A}$-bimodule. This allows us to introduce the vector space

$$\widetilde{\mathcal{H}} \coloneqq \mathcal{E} \otimes \mathcal{A} \otimes \overline{\mathcal{E}}. \tag{7.42}$$

If $p \in M_m(\mathcal{A})$ is the projector ($p = p^* = p^2$) such that $\mathcal{E} = p\mathcal{A}^m$, then $\overline{\mathcal{E}} = \overline{\mathcal{A}}^m p$ and $\widetilde{\mathcal{H}} = pp^\circ\,[\mathcal{H} \otimes M_m(\mathbb{C})]$, so that $\widetilde{\mathcal{H}}$ becomes a Hilbert space under the scalar product

$$\left\langle r \otimes \eta \otimes \overline{q} \,\middle|\, s \otimes \xi \otimes \overline{t} \right\rangle \coloneqq \langle \eta \,|\, (r\,|\,s)\,(t\,|\,q)^\circ \xi \rangle\,. \tag{7.43}$$

If $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is $\mathbb{Z}_2$-graded, there is a corresponding $\mathbb{Z}_2$-grading of $\widetilde{\mathcal{H}}$. The antilinear correspondence $s \mapsto \overline{s}$ between $\mathcal{E}$ and $\overline{\mathcal{E}}$ also gives an obvious way to extend J to $\widetilde{\mathcal{H}}$:

$$\widetilde{J}\left(s \otimes \xi \otimes \overline{t}\right) \coloneqq t \otimes J\xi \otimes \overline{s}. \tag{7.44}$$

Let $\mathcal{B} = \mathcal{E} \otimes \overline{\mathcal{E}}$, then $\mathcal{E}$ is a left-$\mathcal{B}$-module, and the action of $\mathcal{B}$ on $\mathcal{E}$ commutes with the right action of $\mathcal{A}$. Then

$$\mathsf{b} : s \otimes \xi \otimes \overline{t} \mapsto \mathsf{b}s \otimes \xi \otimes \overline{t}, \qquad \mathsf{b} \in \mathcal{B} \tag{7.45}$$

yields a representation of $\mathcal{B}$ on $\widetilde{\mathcal{H}}$, and an opposite representation

$$\mathsf{b}^\circ \coloneqq \widetilde{J}\mathsf{b}^*\widetilde{J}^{-1} : s \otimes \xi \otimes \overline{t} \mapsto s \otimes \xi \otimes \overline{t}\mathsf{b}, \tag{7.46}$$

where $\overline{t}\mathsf{b} \coloneqq \overline{\mathsf{b}^* t}$. b and b° obviously commute.

The nontrivial part of the construction of the new spin geometries $\left(\mathcal{B},\, \widetilde{\mathcal{H}},\, \widetilde{\mathcal{D}};\, \widetilde{\Gamma},\, \widetilde{J}\right)$ is the determination of an appropriate operator $\widetilde{\mathcal{D}}$ on $\widetilde{\mathcal{H}}$. Guided by the differential properties of Dirac operators, the most suitable procedure is to postulate a Leibniz rule:

$$\widetilde{\mathcal{D}}\left(s \otimes \xi \otimes \overline{t}\right) \coloneqq (\nabla s)\,\xi \otimes \overline{t} + s \otimes \mathcal{D}\xi \otimes \overline{t} + s \otimes \xi\overline{(\nabla t)}, \tag{7.47}$$

where ∇s, ∇t belong to some space whose elements can be represented on $\mathcal{H}$ by suitable extensions of the representations of $\mathcal{A}$ and $\mathcal{A}^\circ$. Of course, in order for this last equation to be consistent, ∇ itself should comply with a Leibniz rule. Since $sa \otimes \xi \otimes \overline{t} = s \otimes a\xi \otimes \overline{t}$ for $a \in \mathcal{A}$, this entails

$$(\nabla sa)\,\xi \otimes \overline{t} + s \otimes a\mathcal{D}\xi \otimes \overline{t} = (\nabla s)\,a\xi \otimes \overline{t} + s \otimes \mathcal{D}a\xi \otimes \overline{t}, \tag{7.48}$$

so we infer that

$$\nabla(sa) = (\nabla s)\, a + s \otimes [\mathcal{D}, a]\,. \tag{7.49}$$

To satisfy these requirements, we introduce the space of bounded operators

$$\Omega^1_{\mathcal{D}} := \operatorname{span}\{a\,[\mathcal{D}, b] : a, b \in \mathcal{A}\} \subseteq \mathcal{L}(\mathcal{H})\,, \tag{7.50}$$

which is an $\mathcal{A}$-bimodule under the actions $c \triangleright a\,[\mathcal{D}, b] := ca\,[\mathcal{D}, bc]$ and $a\,[\mathcal{D}, b] \triangleleft c := a\,[\mathcal{D}, bc] - ab\,[\mathcal{D}, c]$. The notation $\Omega^1_{\mathcal{D}}$ is chosen to make clear that the objects $a\,[\mathcal{D}, b]$ are the non-commutative version of 1-forms.

Now, we can form the right $\mathcal{A}$-module $\mathcal{E} \otimes \Omega^1_{\mathcal{D}}$.

Definition 7.4.2. *A connection on $\mathcal{E}$ is a linear mapping*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathcal{D}} \tag{7.51}$$

that satisfies the Leibniz rule (7.49).

Now we just need to ensure that $\widetilde{\mathcal{D}}$ itself be selfadjoint on $\widetilde{\mathcal{H}}$. If $\xi, \eta \in \operatorname{Dom}(\mathcal{D})$, we get

$$\left\langle r \otimes \eta \otimes \overline{q} \,\middle|\, \widetilde{\mathcal{D}}\left(s \otimes \xi \otimes \overline{t}\right)\right\rangle = \langle \eta \,|\, (r \,|\, \nabla s)\,(t \,|\, q)^\circ \xi\rangle + \langle \eta \,|\, (r \,|\, s)\,(t \,|\, q)^\circ \mathcal{D}\xi\rangle + \\ + \langle \eta \,|\, (r \,|\, s)\,(\nabla t \,|\, q)^\circ \xi\rangle\,, \tag{7.52}$$

$$\left\langle \widetilde{\mathcal{D}}\left(r \otimes \eta \otimes \overline{q}\right) \,\middle|\, s \otimes \xi \otimes \overline{t}\right\rangle = \langle \eta \,|\, (\nabla r \,|\, s)\,(t \,|\, q)^\circ \xi\rangle + \langle \mathcal{D}\eta \,|\, (r \,|\, s)\,(t \,|\, q)^\circ \xi\rangle + \\ + \langle \eta \,|\, (r \,|\, s)\,(t \,|\, \nabla q)^\circ \xi\rangle\,. \tag{7.53}$$

This reduces to the condition that

$$(r \,|\, \nabla s) - (\nabla r \,|\, s) = [\mathcal{D}, (r \,|\, s)] \qquad \text{for all } r, s \in \mathcal{E}. \tag{7.54}$$

If this last equation holds, we say that the connection ∇ is *hermitian* with respect to $\mathcal{D}$.

To summarize what we said so far: two NCSs $(\mathcal{A}, \mathcal{H}, \mathcal{D}; \Gamma, J)$ and $\left(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}; \widetilde{\Gamma}, \widetilde{J}\right)$ are *Morita-equivalent* if there exist a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ and an $\Omega^1_{\mathcal{D}}$-valued connection ∇ on $\mathcal{E}$ such that $\mathcal{B} = \mathcal{E} \otimes \overline{\mathcal{E}}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\Gamma}$ are given by Eq. (7.42), $\widetilde{J}$ is given by Eq. (7.44) and $\widetilde{\mathcal{D}}$ by Eq. (7.47).

7.4.3 Gauge Potentials

Any equivalence relation is by definition reflexive, so any algebra $\mathcal{A}$ is Morita-equivalent to itself, in which case the equivalence bimodule is $\mathcal{A}$ itself [72]. We may ask what Morita equivalence entails when the algebra $\mathcal{A}$ is unchanged. Regarded as a right $\mathcal{A}$-module, $\mathcal{A}$ carries a standard hermitian connection with respect to $\mathcal{D}$, namely

$$\mathrm{Ad}_{\mathcal{D}} : \mathcal{A} \to \Omega^1_{\mathcal{D}}, \quad b \mapsto [\mathcal{D}, b] \,. \tag{7.55}$$

By the Leibniz rule (7.49), any connection differs from $\mathrm{Ad}_{\mathcal{D}}$ by an element of $\Omega^1_{\mathcal{D}}$:

$$\nabla b \coloneqq [\mathcal{D}, b] + \mathbb{A} b, \tag{7.56}$$

where

$$\mathbb{A} \coloneqq \sum_i a_i \, [\mathcal{D}, b_i] \tag{7.57}$$

lies in $\Omega^1_{\mathcal{D}}$. We call it a *gauge potential* if it is selfadjoint: $\mathbb{A} = \mathbb{A}^\dagger$. Hermiticity of the connection for the pairing $(a \,|\, b) \coloneqq a^* b$ demands that $a^* \nabla b - (\nabla a)^* \, b = [\mathcal{D}, a^* b]$, i.e. $a^* \left(\mathbb{A} - \mathbb{A}^*\right) b = 0$ for all $a, b \in \mathcal{A}$, so a hermitian connection on $\mathcal{A}$ is indeed given by a gauge potential $\mathbb{A}$.

On substituting the connection (7.56) into Eq. (7.47) for an extended Dirac operator $\widetilde{\mathcal{D}}$, one obtains

$$\begin{aligned} \widetilde{\mathcal{D}} \, (b\xi) &= ([\mathcal{D}, b] + \mathbb{A} b) \, \xi + b \mathcal{D} \xi + \varepsilon' b J \, (\nabla 1) \, J^{-1} \xi = \\ &= \left(\mathcal{D} + \mathbb{A} + \varepsilon' J \mathbb{A} J^{-1}\right) (b\xi) \,. \end{aligned} \tag{7.58}$$

Therefore, the *gauge transformation*

$$\mathcal{D} \mapsto \mathcal{D} + \mathbb{A} + \varepsilon' J \mathbb{A} J^{-1} \tag{7.59}$$

yields a NCS that is Morita-equivalent to the original. Notice that the inner automorphisms of (7.36) yield a special case of (7.58), with $\mathbb{A} = u \, [\mathcal{D}, u^*]$.

7.5 The Non-Commutative Integral

In this chapter we will define the non-commutative integral and relate it to conventional integration on manifolds. Again, we will follow [60, 56].

In the course of the initial development of non-commutative geometry, integration came first, beginning with Segal's early work with traces on operator algebras [73] and continuing with Connes' work on foliations [74]. The introduction of universal graded differential algebras [75] shifted the emphasis to differential calculus based on derivations, which formed the backdrop for the first applications to particle physics [76, 77]. The pendulum later swung back to integral methods, due to the realization [78, 79, 80, 81] that the Yang-Mills functionals could be obtained in this way. In fact, as we will see, the non-commutative integral is the key to understanding the origin of Connes' formula for the spectral action Eq. (7.3).

Early attempts at non-commutative integration [73] used the ordinary trace of Hilbert space operators as an ersatz integral, where traceclass operators play the role of integrable functions. However, for non-commutative geometry one needs an integral that suppresses infinitesimals of order higher than 1, but the ordinary trace diverges for positive first-order infinitesimals, since

$$\mathrm{Tr}\,|T| = \sum_{k=0}^{\infty} \mu_k(T) = \lim_{n\to\infty} \sigma_n(T) = \infty \qquad \text{if } \sigma_n(T) = O(\log n)\,. \tag{7.60}$$

Dixmier [82] found other tracial functionals on compact operators that are more suitable for our purposes: they are finite on first order infinitesimals and vanish on those of higher order. To find them, we must look more closely at the fine structure of infinitesimal operators.

The algebra $\mathcal{K}$ of compact operators on a separable, infinite-dimensional Hilbert space contains the ideal[30] $\mathcal{L}^1$ of traceclass[31] operators, on which $\|T\|_1 := \mathrm{Tr}\,|T|$ is a norm, which is larger than the operator norm $\|T\| = \mu_0(T)$. Each partial sum of singular values σ_n is a norm on $\mathcal{K}$. In fact, one can show [83] that

$$\sigma_n(T) = \sup\{\|TP_n\|_1 : P_n \text{ is a projector of rank } n\}\,. \tag{7.61}$$

Notice that $\sigma_n(T) \le n\mu_0(T) = n\|T\|$. One can also show [60] that the two norms are related by

$$\sigma_n(T) = \inf\{\|R\|_1 + n\|S\| : R, S \in \mathcal{K};\ R + S = T\}\,. \tag{7.62}$$

In order to make clear a few concepts that will be very important later, let us check this equality. Clearly, if $T = R + S$, then $\sigma_n(T) \le \sigma_n(R) + \sigma_n(S) \le \|R\|_1 + n\|S\|$. To show that the infimum is attained, one can assume without loss of generality T be a positive operator, since both sides of Eq. (7.62) are

[30] An *ideal* I of a ring R is an additive subgroup of the ring such that for each $r \in R$ and $x \in I$, one has that rx (or xr) $\in I$. Ideals generalize structures like "even numbers": addition and subtraction of even numbers preserves evenness, and multiplying an even number by any other integer results in another even number. These closure and absorption properties are the defining properties of an ideal.

[31] A *traceclass operator* is a compact operator for which a trace may be defined, such that the trace is finite and independent of the choice of basis.

unchanged if R, S, T are multiplied by a unitary operator V such that $VT = |T|$. Now, let P_n be the projector of rank n whose range is spanned by the eigenvectors of T corresponding to the eigenvalues $\mu_0, \ldots, \mu_{n-1}$. Then $R := (T - \mu_n) P_n$ and $S := \mu_n P_n + T(1 - P_N)$ satisfy $\|R\|_1 = \sum_{k<n} (\mu_k - \mu_n) = \sigma_n(T) - n\mu_n$ and $\|S\| = \mu_n$.

We can think of $\sigma_n(T)$ as the trace of $|T|$ with a cutoff at the scale n. This scale need not be an integer: for any scale $\lambda > 0$, we can define

$$\sigma_\lambda(T) := \inf\left\{\|R\|_1 + \lambda \|S\| : R, S \in \mathcal{K};\ R + S = T\right\}. \tag{7.63}$$

If $0 < \lambda \leq 1$ then $\sigma_\lambda(T) = \lambda \|T\|$. On the other hand, if $\lambda = n + t$ with $0 \leq t < 1$, one can check that

$$\sigma_\lambda(T) = (1 - t)\,\sigma_n(T) + t\sigma_{n+1}(T), \tag{7.64}$$

so that $\lambda \mapsto \sigma_\lambda(T)$ is a piecewise linear, increasing, concave function on $(0, \infty)$. With this definition, each $\sigma_\lambda(T)$ is a norm, hence it satisfies the triangle inequality. For positive compact operators, one can prove a triangle inequality in the opposite direction:

$$\sigma_\lambda(A) + \sigma_\mu(B) \leq \sigma_{\lambda+\mu}(A + B) \qquad \text{if } A, B > 0. \tag{7.65}$$

Hence, we get a sandwich of norms:

$$\sigma_\lambda(A + B) \leq \sigma_\lambda(A) + \sigma_\lambda(B) \leq \sigma_{2\lambda}(A + B) \qquad \text{if } A, B > 0. \tag{7.66}$$

Now, we can precisely define the *first-order infinitesimals* as the following normed ideal of compact operators:

$$\mathcal{L}^{1+} := \left\{T \in \mathcal{K} : \|T\|_{1+} := \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty\right\} \tag{7.67}$$

that obviously includes the traceclass operators $\mathcal{L}^1$.

If we fix $a > e$, if $T \in \mathcal{L}^{1+}$, the function $\lambda \mapsto \sigma_\lambda(T) / \log \lambda$ is continuous and bounded on the interval $[a, \infty)$, i.e. it lies in the C^*-algebra $C_b[a, \infty)$. We then define the *Cesàro mean* of this function:

$$\tau_\lambda(T) := \frac{1}{\log \lambda} \int_a^\lambda \frac{\sigma_u(T)}{\log u} \frac{du}{u}. \tag{7.68}$$

Then, $\lambda \mapsto \tau_\lambda(T)$ lies in $C_b[a, \infty)$ as well, with upper bound $\|T\|_{1+}$. From Eq. (7.66) we can also derive that

$$0 \leq \tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A + B) \leq \left(\|A\|_{1+} + \|B\|_{1+}\right) \log 2 \frac{2 + \log\log \lambda}{\log \lambda}, \tag{7.69}$$

so that $\tau_\lambda(T)$ is 'asymptotically additive' on positive elements of $\mathcal{L}^{1+}$.

We get a true additive functional in two more steps. Firstly, let $\dot{\tau}(A)$ be the class of $\lambda \mapsto \tau_\lambda(A)$ in the quotient C^*-algebra $\mathcal{B} \coloneqq C_b[a,\infty)/C_0[a,\infty)$. Then $\dot{\tau}$ is an additive, positive-homogeneous map from the positive cone of $\mathcal{L}^{1+}$ into $\mathcal{B}$, and $\dot{\tau}(UAU^{-1}) = \dot{\tau}(A)$ for any unitary U, therefore it extends to a linear map $\dot{\tau} : \mathcal{L}^{1+} \to \mathcal{B}$ such that $\dot{\tau}(ST) = \dot{\tau}(TS)$ for $T \in \mathcal{L}^{1+}$ and any bounded S.

Secondly, we compose $\dot{\tau}$ with any state (i.e. normalized positive linear form) $\omega : \mathcal{B} \to \mathbb{C}$. The composition is called a *Dixmier trace*:

$$\mathrm{Tr}_\omega(T) \coloneqq \omega(\dot{\tau}(T)). \tag{7.70}$$

Actually, $\mathcal{B}$ is not separable, hence there is no way to exhibit any particular state. More precisely, a function $f \in C_b[a,\infty)$ has a limit $\lim_{\lambda\to\infty} f(\lambda) = c$ if and only if $\omega(f) = c$ does not depend on ω. Then, we say that $T \in \mathcal{L}^{1+}$ is a *measurable operator* if the function $\lambda \to \tau_\lambda(T)$ converges as $\lambda \to \infty$, in which case any $\mathrm{Tr}_\omega(T)$ equals its limit. We denote by $⨍ T$ the common value of the Dixmier traces, and we call this value *non-commutative integral* of T:

$$⨍ T \coloneqq \lim_{\lambda\to\infty} \tau_\lambda(T). \tag{7.71}$$

7.6 Action Functionals

On a differential manifold, one may use many Riemannian metrics; on a spin manifold with a given Riemannian metric, there may be many distinct (i.e. unitarily inequivalent) spin geometries. An important task, already in the commutative case, is to select, if possible, a particular geometry by some general criterion; in physics we often do so by minimization of an action functional. In the non-commutative case, the minimizing geometries are often not unique, leading to the phenomenon of spontaneous symmetry breaking.

In the following dissertation, we will follow [60, 56].

7.6.1 Algebra automorphisms and the metric

In the rest of this chapter, we fix the data $(\mathcal{A}, \mathcal{H}, \Gamma, J)$ of the spectral triple and consider how the Dirac operator $\mathcal{D}$ may be modified by automorphisms of the algebra $\mathcal{A}$.

The point at issue is that the *automorphism group* of the algebra is just the noncommutative version of the *group of diffeomorphisms* of a manifold. For instance, if $\mathcal{A} = C^\infty(\mathcal{M})$ for a compact smooth manifold $\mathcal{M}$, and if $\alpha \in \mathrm{Aut}(\mathcal{A})$, then each character $\hat{x}$ of $\mathcal{A}$ is the image under α of a unique character[32] $\hat{y}$ (i.e. $\alpha^{-1}(\hat{x})$ is also a character, so it equals $\hat{y}$ for some $y \in \mathcal{M}$). Write $\phi(x) := y$ (with x and y the same as before); then ϕ is a continuous bijection on $\mathcal{M}$ satisfying $\alpha(f)(x) = f(\phi^{-1}(x))$, and the chain rule for derivatives shows that ϕ is itself smooth and hence is a diffeomorphism of $\mathcal{M}$. Then, $\alpha \leftrightarrow \phi$ is a group isomorphism from $\mathrm{Aut}(C^\infty(\mathcal{M}))$ onto $\mathrm{Diff}(\mathcal{M})$.

On a non-commutative *-algebra, there are many *inner automorphisms*

$$\sigma_u(a) := uau^{-1}, \tag{7.72}$$

where u lies in the unitary group $U(\mathcal{A})$; these are of course trivial when A is commutative. We adopt the point of view that these inner automorphisms are henceforth to be regarded as 'internal diffeomorphisms' of our algebra $\mathcal{A}$.

Already in the commutative case, diffeomorphisms change the metric on a manifold. To select a particular metric, one uses often the variational principle. In General Relativity, one works with the Einstein-Hilbert action

$$S_{EH} =\propto \int_{\mathcal{M}} R\sqrt{|g|}\, d^n x = \int_{\mathcal{M}} R\,\Omega, \tag{7.73}$$

where R is the scalar curvature (i.e. the Ricci scalar) of the metric $g_{\mu\nu}$, in order to select a metric minimizing this action. In Yang-Mills theories of particle physics, the bosonic action functional is of the form

$$S_{YM} \propto \int F_{\mu\nu} F^{\mu\nu}, \tag{7.74}$$

where $F_{\mu\nu}$ is the curvature 2-form associated with the corresponding gauge potential.

The question then arises as to what is the general prescription for appropriate action functionals in noncommutative geometry.

Inner automorphisms and gauge potentials

Recall how inner automorphisms act on NCSs: if u is a unitary element of $\mathcal{A}$, the operator $U := uJuJ^{-1}$ implements a unitary equivalence (7.36) between the NCSs determined by $\mathcal{D}$ and by

$${}^u\mathcal{D} = \mathcal{D} + u[\mathcal{D}, u^*] + \varepsilon' Ju[\mathcal{D}, u^*]J^{-1}. \tag{7.75}$$

[32] A character of an algebra $\mathcal{A}$ is a non-zero homomorphism $\mu : \mathcal{A} \to \mathbb{C}$. In this context, $\mathcal{A}$ is the set of smooth functions over a manifold $\mathcal{M}$, and a character is the evaluation of one such functions in a point $x \in \mathcal{M}$; we denote such a character $\hat{x}$.

More generally, any selfadjoint $\mathbb{A} \in \Omega^1_{\mathcal{D}}$ gives rise to a Morita equivalence (7.58) between the geometries determined by $\mathcal{D}$ and by $\mathcal{D} + \mathbb{A} + \varepsilon' J \mathbb{A} J^{-1}$.

It is important to observe that these gauge transformations are trivial when the geometry is commutative. Recall that this means that $\mathcal{A}$ is commutative and that $b^\circ = b$: the charge conjugation J on spinors intertwines multiplication by a function with multiplication by its complex conjugate. Therefore we can write $a = J a^* J^{-1}$ in this case. But then, the first-order condition entails

$$\begin{aligned} Ja\,[\mathcal{D}, b]\,J^{-1} &= a^\dagger J\,[\mathcal{D}, b]\,J^{-1} = J\,[\mathcal{D}, b]\,J^{-1} a^\dagger = \\ &= \left[J\mathcal{D}J^{-1}, JbJ^{-1}\right] a^\dagger = \varepsilon' \left[\mathcal{D}, b^\dagger\right] a^\dagger = -\varepsilon' \left(a\,[\mathcal{D}, b]\right)^\dagger \end{aligned} \tag{7.76}$$

since $J\mathcal{D}J^{-1} = \varepsilon'\mathcal{D}$. Hence $J\mathbb{A}J^{-1} = -\varepsilon'\mathbb{A}^\dagger$ for $\mathbb{A} \in \Omega^1_{\mathcal{D}}$, and thus $\mathbb{A} + \varepsilon' J\mathbb{A}J^{-1} = \mathbb{A} - \mathbb{A}^\dagger$, which vanishes for a self-adjoint gauge potential. This means that, within our postulates, a commutative manifold could support gravity but not electromagnetism; in other words, even to get abelian gauge fields we need that the underlying manifold be non-commutative!

7.6.2 The Fermionic Action

In the Standard Model of particle physics, the following prescription defines the fermionic action functional:

$$S_f\,(\psi, \mathbb{A}) := \left\langle J\tilde{\psi} \,\middle|\, \left(\mathcal{D} + \mathbb{A} + \varepsilon' J\mathbb{A}J^{-1}\right) \tilde{\psi} \right\rangle, \tag{7.77}$$

where $\tilde{\psi}$ is a Graßmann vector in the Fock space $\tilde{\mathcal{H}}_+$ of classical fermions corresponding to the positive eigenspace $\mathcal{H}_+$ of the grading Γ, and may be interpreted as a multiplet of spinors representing elementary particles and antiparticles.

The gauge group $U(\mathcal{A})$ acts on potentials in the following way. If $u \in \mathcal{A}$ is unitary and if $\nabla = \mathrm{Ad}_{\mathcal{D}} + \mathbb{A}$ is a hermitian connection (7.56), then so also is

$$u \nabla u^\dagger = u\,\mathrm{Ad}_{\mathcal{D}}\,u^\dagger + u\mathbb{A}u^\dagger = \mathrm{Ad}_{\mathcal{D}} + u\left[\mathcal{D}, u^\dagger\right] + u\mathbb{A}u^\dagger, \tag{7.78}$$

so that ${}^u\mathbb{A} := u\mathbb{A}u^\dagger + u\left[\mathcal{D}, u^\dagger\right]$ is the gauge-transformed potential. With $U = uJuJ^{-1}$, we get $U\mathbb{A}U^\dagger = u\mathbb{A}u^\dagger$ since JuJ^{-1} commutes with $\Omega^1_{\mathcal{D}}$, and so

$$\begin{aligned} &\mathcal{D} +{}^u \mathbb{A} + \varepsilon' J\,{}^u\mathbb{A}J^{-1} = \\ &\quad = \mathcal{D} + u\left[\mathcal{D}, u^\dagger\right] + \varepsilon' Ju\left[\mathcal{D}, u^\dagger\right] J^{-1} + u\mathbb{A}u^\dagger + \varepsilon' Ju\mathbb{A}u^\dagger J^{-1} = \\ &\quad = U\left(\mathcal{D} + \mathbb{A} + \varepsilon' J\mathbb{A}J^{-1}\right) U^\dagger. \end{aligned} \tag{7.79}$$

The gauge invariance of (7.77) under the group $U(\mathcal{A})$ is now established by

$$\begin{aligned} S_f\,(U\psi, {}^u\mathbb{A}) &= \left\langle U\psi \,\middle|\, \left(\mathcal{D} + {}^u\mathbb{A} + \varepsilon' J\,{}^u\mathbb{A}J^{-1}\right) U\psi \right\rangle = \\ &= \left\langle U\psi \,\middle|\, U\left(\mathcal{D} + \mathbb{A} + \varepsilon' J\,\mathbb{A}J^{-1}\right) \psi \right\rangle = S_f\,(\psi, \mathbb{A})\,. \end{aligned} \tag{7.80}$$

7.6.3 The Provisional Bosonic Action

In Yang-Mills models, the fermionic action is supplemented by a *bosonic action* that is a quadratic functional of the gauge fields or curvatures associated to the gauge potential $\mathbb{A}$. One may formulate the curvature of a connection in non-commutative geometry and obtain a Yang-Mills action. To do so, one can formally introduce the curvature as $\mathbb{F} := d\mathbb{A} + \mathbb{A}^2$, where the notation means

$$d\mathbb{A} := \sum_i [\mathcal{D}, a_i]\,[\mathcal{D}, b_i] \qquad \text{whenever } \mathbb{A} = \sum_i a_i\,[\mathcal{D}, b_i]\,. \tag{7.81}$$

Regrettably, this definition is flawed, since the first sum may be nonzero in cases where the second sum vanishes. For instance, in the commutative case, $a\left[\partial\!\!\!/, a\right] - \left[\partial\!\!\!/, \frac{1}{2}a^2\right] = -i\gamma^\mu\left(a\partial_\mu a - \partial_\mu\left(\frac{1}{2}a^2\right)\right) = 0$ but $\left[\partial\!\!\!/, a\right]\left[\partial\!\!\!/, a\right] = -\left(\partial_\mu a\right)\left(\partial^\mu a\right) < 0$ in general. If we push ahead anyway, we can make a formal check that $\mathbb{F}$ transforms under the gauge group $U(\mathcal{A})$ by ${}^u\mathbb{F} = u\mathbb{F}u^\dagger$. Indeed,

$$\begin{aligned} d\left({}^u\mathbb{A}\right) &= [\mathcal{D}, u]\left[\mathcal{D}, u^\dagger\right] + \sum_i [\mathcal{D}, ua_i]\left[\mathcal{D}, b_j u^\dagger\right] - \sum_i [\mathcal{D}, ua_ib_i]\left[\mathcal{D}, u^\dagger\right] = \\ &= [\mathcal{D}, u]\left[\mathcal{D}, u^\dagger\right] + [\mathcal{D}, u]\,\mathbb{A}u^\dagger - u\mathbb{A}\left[\mathcal{D}, u^\dagger\right] + \sum_i u\,[\mathcal{D}, a_i]\,[\mathcal{D}, b_i]\,u^\dagger, \end{aligned} \tag{7.82}$$

while, using the identity $u\,[\mathcal{D}, u^*]\,u = -\,[\mathcal{D}, u]$, we get

$$\begin{aligned} \left({}^u\mathbb{A}\right)^2 &= u\left[\mathcal{D}, u^\dagger\right]u\left[\mathcal{D}, u^\dagger\right] + u\left[\mathcal{D}, u^\dagger\right]u\mathbb{A}u^\dagger + u\mathbb{A}\left[\mathcal{D}, u^\dagger\right] + u\mathbb{A}^2u^\dagger = \\ &= -\,[\mathcal{D}, u]\left[\mathcal{D}, u^\dagger\right] - [\mathcal{D}, u]\,\mathbb{A}u^\dagger + u\mathbb{A}\left[\mathcal{D}, u^\dagger\right] + u\mathbb{A}^2u^\dagger, \end{aligned} \tag{7.83}$$

and consequently

$${}^u\mathbb{F} := d\left({}^u\mathbb{A}\right) + \left({}^u\mathbb{A}\right)^2 = u\left(d\mathbb{A} + \mathbb{A}^2\right)u^\dagger = u\mathbb{F}u^\dagger. \tag{7.84}$$

Provided that the definition (7.81) can be corrected[33], one can then define a gauge-invariant action as the symmetrized Yang-Mills type functional

$$S_b\left(\mathbb{A}\right) := -\!\!\!\!\!\!\int \left(\mathbb{F} + J\mathbb{F}J^{-1}\right)^2 |\mathcal{D}|^{-n}\,, \tag{7.85}$$

because the non-commutative integral is a trace, so that

$$\begin{aligned} -\!\!\!\!\!\!\int \left({}^u\mathbb{F} + J\,{}^u\mathbb{F}J^{-1}\right)^2 \left|{}^u\mathcal{D}\right|^{-n} &= -\!\!\!\!\!\!\int U\left(\mathbb{F} + J\mathbb{F}J^{-1}\right)^2 |\mathcal{D}|^{-n}\,U^{-1} = \\ &= -\!\!\!\!\!\!\int \left(\mathbb{F} + J\mathbb{F}J^{-1}\right)^2 |\mathcal{D}|^{-n}\,. \end{aligned} \tag{7.86}$$

[33]The ambiguity in (7.81) can be removed by introducing the $\mathcal{A}$-bimodule $\left(\Omega^1_{\mathcal{D}}\right)^2/J_2$, where the subbimodule J_2 consists of the so-called 'junk terms' $\sum_i [\mathcal{D}, a_i]\,[\mathcal{D}, b_i]$ for which $\sum_i a_i\,[\mathcal{D}, b_i] = 0$. Then, by redefining $\mathbb{F}$ as the orthogonal projection of $d\mathbb{A} + \mathbb{A}^2$ on the orthogonal complement of J_2 in $\left(\Omega^1_{\mathcal{D}}\right)^2$, one gets a well-defined curvature and the non-commutative integral of its square gives the desired Yang-Mills action.

7.6.4 The Spectral Action

The Yang-Mills action (7.85), evaluated on a suitable spin geometry, achieves the remarkable feat of reproducing the classical Lagrangian of the Standard Model [65, VI]. However, its computation leads to fearsome algebraic manipulations and very delicate handling of the junk terms, leading one to question whether this action really is fundamental. Connes and Chamseddine [84] made an alternative proposal.

The unitary equivalence $\mathcal{D} \mapsto \mathcal{D} + u\left[\mathcal{D}, u^\dagger\right] + \varepsilon' J u\left[\mathcal{D}, u^\dagger\right] J^{-1}$ is a perturbation by internal diffeomorphisms, and one can regard the Morita equivalence $\mathcal{D} \mapsto \mathcal{D} + \mathbb{A} + \varepsilon' J \mathbb{A} J^{-1}$ as an *internal fluctuation* of $\mathcal{D}$. The correct bosonic action functional should not merely be diffeomorphism invariant – where by diffeomorphisms we mean automorphisms of $\mathcal{A}$ – that is to say, 'of purely gravitational nature', but one can go further and ask that it be *spectrally invariant*. As stated unambiguously by Connes and Chamseddine [78] "The physical action only depends upon $\sigma(\mathcal{D})$".

Since quantum corrections must still be provided for [85], the particular action chosen should incorporate a cutoff scale Λ (roughly comparable to inverse Planck length, or Planck mass, where the commutative spacetime geometry must surely break down), and some suitable cutoff function: $f(t) \geq 0$ for $t \geq 0$ with $f(t) = 0$ for $t \gg 1$. Therefore, Connes and Chamseddine proposed a bosonic action of the form

$$S_b(\mathcal{D}) \coloneqq \lim_{\Lambda \to \infty} \operatorname{Tr} f\left(\frac{\mathcal{D}^2}{\Lambda^2}\right). \tag{7.87}$$

This spectral action turns out to include not only the Standard Model bosonic action but also the Einstein-Hilbert action for gravity, plus some higher-order gravitational terms [78, 81], thereby establishing it firmly as an action for an effective field theory at low energies.

Usually, one takes as f a smooth approximation of the characteristic function $\chi_{[0,1]}$ of the interval $[0, 1]$. In this case, the spectral action simply counts the number of singular values of $\mathcal{D}$ smaller than the cutoff scale Λ.

Chapter 8

Twisted Non-Commutative Geometry

In this chapter, we will outline the main differences between standard NCG and its twisted version, and then we will work out the most important peculiarities of twisted NCG.

Our aim is to somehow modify the axioms of NCG in order to generate a new scalar field, σ, that is capable of solving many of the problems of the Connes model, such as the wrong Higgs mass, the metastability of the electroweak vacuum [56], or of course the arising of the neutrino Majorana mass. Such a field does not arise using the axioms of the standard NCG (i.e. in the Connes model), hence one needs to modify those axioms in order to obtain it. There are several ways to do so, such as enlarging the algebra of the spectral triple [86], breaking the first order condition [56], or working with twisted spectral triples, which was tried initially in [87] with a partial twist (that acted on the electroweak part of the algebra only) and which is what this work focuses on.

The following sections are structured as follows. In sections 8.1 to 8.3 we introduce twisted NCG, summarizing what is already present in literature. In section 8.4 begins the original content of this work, with a brief review of minimal twists and then an in-depth study of the twist-by-grading (already introduced in [88]) and of other minimal twists.

In Chapter 9 we apply twisted NCG to the simplest twist of the Connes model, i.e. the twist-by-grading of the Connes model. Then, in chapter 10 we study other two minimal twists of the Connes model.

8.1 Real Twisted Spectral Triples

Twisted spectral triples have been introduced to build non-commutative geometries from type III algebras [89]. Later, it was found out that an extra scalar field is needed in order to lower the Higgs mass predicted by Connes model to the correct value, and twisted spectral triples are one way to give rise to that new field [56].

Definition 8.1.1. *A* twisted spectral triple $(\mathcal{A}, \mathcal{D}, \mathcal{H})_\rho$ *is a set of four objects of the following types:*

- $\mathcal{A}$ *is a unital involutive algebra;*
- $\mathcal{H}$ *is a Hilbert space on which* $\mathcal{A}$ *acts faithfully as bounded operators;*
- $\mathcal{D}$ *is the* Dirac operator *and is a self-adjoint operator with compact resolvent;*
- ρ *is the* twist *and is an automorphism of* $\mathcal{A}$ *such that the* twisted commutator*, defined as*
$$[\mathcal{D}, a]_\rho \coloneqq \mathcal{D}a - \rho(a)\mathcal{D}, \tag{8.1}$$
is bounded for each $a \in \mathcal{A}$*, and such that*
$$\rho\left(a^\dagger\right) = \rho^{-1}(a)^\dagger . \tag{8.2}$$

Remark 8.1.1. If one requires ρ to be a $*$-automorphism, then the regularity condition implies that $\rho^2 = \mathbb{I}$.

Once again, we define the notions of graded and real twisted spectral triple:

Definition 8.1.2. *A twisted spectral triple is* graded *or* even *if the Hilbert space* $\mathcal{H}$ *is endowed with a unitary self-adjoint operator* Γ *such that*

- $a\Gamma = \Gamma a$ *for each* $a \in \mathcal{A}$*;*
- $\mathcal{D}\Gamma = -\Gamma\mathcal{D}$*.*

The operator Γ *is called the* $\mathbb{Z}_2$-grading*, or more simply the* grading.

Definition 8.1.3. *A twisted spectral triple is* real *if* $\mathcal{H}$ *is endowed with an anti-linear isometry of* $\mathcal{H}$ *onto itself such that*

- $J^2 = \pm 1$*;*
- $J\mathcal{D} = \pm\mathcal{D}J$*.*

Further, if the twisted spectral triple is even, J should also satisfy

- $J\Gamma = \pm \Gamma J$.

The operator J is called a real structure.

As in the non-twisted case, J is required to implement an isomorphism between $\mathcal{A}$ and its opposite algebra $\mathcal{A}^\circ$

$$b \mapsto b^\circ = J b^\dagger J^{-1}, \tag{8.3}$$

and once again one requires the action of $\mathcal{A}^\circ$ to commute with the action of $\mathcal{A}$ (the zeroth-order condition):

$$[a, b^\circ] = 0 \tag{8.4}$$

in order to define the right action of $\mathcal{A}$ on $\mathcal{H}$:

$$\psi b := b^\circ \psi. \tag{8.5}$$

The part of the real structure that is modified is the *first order condition*. In the non-twisted case, it reads $[[D, a], b^\circ] = 0 \; \forall a, b \in \mathcal{A}$. Instead, in the twisted case, we require the twisted first-order condition (which is basically the twisted version of Axiom 2):

$$\left[[\mathcal{D}, a]_\rho, b^\circ\right]_{\rho^\circ} := [\mathcal{D}, a]_\rho - \rho^\circ(b^\circ)[\mathcal{D}, a]_\rho = 0 \qquad \textit{(twisted first-order condition)}, \tag{8.6}$$

where ρ° is the automorphism induced by ρ on the opposite algebra:

$$\rho^\circ(b^\circ) := J\rho\left(b^\dagger\right)J^{-1}. \tag{8.7}$$

If the automorphism ρ is also an inner automorphism of $\mathcal{B}(\mathcal{H})$, that is

$$\rho(a) = \mathcal{R} a \mathcal{R}^\dagger, \tag{8.8}$$

with unitary $\mathcal{R} \in \mathcal{B}(\mathcal{H})$, then ρ is said *compatible with the real structure J* if

$$J\mathcal{R} = \varepsilon''' \mathcal{R} J, \qquad \text{for } \varepsilon''' = \pm 1. \tag{8.9}$$

8.2 Twisted Inner Product

Given a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot\,, \cdot \rangle$ and an automorphism ρ of $\mathcal{B}(\mathcal{H})$, one can define a *ρ-product* $\langle \cdot\,, \cdot \rangle_\rho$ as a new inner product satisfying

$$\langle \phi\,, \mathcal{O}\xi \rangle_\rho = \left\langle \rho(\mathcal{O})^\dagger \phi\,, \xi \right\rangle_\rho \qquad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}),\ \phi, \xi \in \mathcal{H}, \tag{8.10}$$

where the † denotes the hermitian adjoint with respect to the inner product $\langle\cdot\,,\,\cdot\rangle$. Then one calls

$$\mathcal{O}^{+} := \rho\left(\mathcal{O}\right)^{\dagger} \tag{8.11}$$

the ρ-adjoint of $\mathcal{O}$. If ρ is an inner automorphism implemented by the unitary operator $\mathcal{R}$ on $\mathcal{H}$ (i.e. if $\rho\left(\mathcal{O}\right) = \mathcal{R}\mathcal{O}\mathcal{R}^{\dagger}$) then there is a canonical ρ-product:

$$\langle\phi\,,\,\xi\rangle_{\rho} := \langle\phi\,,\,\mathcal{R}\xi\rangle\,. \tag{8.12}$$

The ρ-adjoint is not necessarily an involution. If ρ is a $*$-automorphism (which is always the case if ρ is inner), then $^{+}$ is an involution if and only if $\rho^2 = \mathbb{I}$, in fact:

$$\left(\mathcal{O}^{+}\right)^{+} = \rho\left(\mathcal{O}^{+}\right)^{\dagger} = \rho\left(\rho\left(\mathcal{O}\right)^{\dagger}\right)^{\dagger} = \rho\left(\rho\left(\mathcal{O}\right)\right). \tag{8.13}$$

The same condition comes out for a twisted spectral triple even if one defines the ρ-adjoint solely at the algebraic level, i.e. $a^{+} := \rho\left(a\right)^{\dagger}$, without assuming that $\rho \in \mathrm{Aut}\left(\mathcal{A}\right)$ extends to an automorphism of all $\mathcal{B}\left(\mathcal{H}\right)$. In fact, because of the regularity condition (8.2), one gets

$$\left(a^{+}\right)^{+} = \left(\rho\left(a\right)^{\dagger}\right)^{+} = \left(\rho^{-1}\left(a^{\dagger}\right)\right)^{+} = \rho\left(\rho^{-1}\left(a^{\dagger}\right)\right)^{\dagger} = \rho\left(\rho\left(a\right)^{\dagger}\right)^{\dagger} = \rho^{2}\left(a\right). \tag{8.14}$$

A twisted spectral triple $\left(\mathcal{A},\,\mathcal{H},\,\mathcal{D}\right)_{\rho}$ whose twisting-automorphism ρ extends to an automorphism of $\mathcal{B}\left(\mathcal{H}\right)$ induces a natural twisted inner product on $\mathcal{H}$, which is useful to define a gauge invariant fermionic action.

8.3 Twisted Fermionic Action

The fermionic action (7.77) for a real graded spectral triple $\left(\mathcal{A},\,\mathcal{H},\,\mathcal{D};\;\Gamma,\,J\right)$, can be written as

$$S_f\left(\mathcal{D}_{\mathcal{A}}\right) := \mathfrak{A}_{\mathcal{D}_{\mathcal{A}}}\left(\tilde{\psi},\tilde{\psi}\right), \tag{8.15}$$

constructed from the bilinear form

$$\mathfrak{A}\left(\phi,\xi\right) := \langle J\phi\,,\,\mathcal{D}_{\mathcal{A}}\psi\rangle\,,\qquad \phi,\xi\in\mathcal{H} \tag{8.16}$$

defined by the fluctuated Dirac operator

$$\mathcal{D}_{\mathcal{A}} := \mathcal{D} + \varepsilon'\mathcal{A} + J\mathcal{A}J^{-1}, \tag{8.17}$$

where $\mathcal{A}$ is a self-adjoint element of the set of generalised 1-forms $\Omega^1_{\mathcal{D}}$ defined in Eq. (7.50). Here, $\tilde{\psi}$ is a Graßmann vector in the Fock space $\tilde{\mathcal{H}}_{+}$ of classical fermions corresponding to the positive eigenspace $\mathcal{H}_{+}$ of the grading Γ, and may be interpreted as a multiplet of spinors representing elementary particles and antiparticles.

As we have seen in section 7.6.2, the fermionic action is invariant under gauge transformations of the form

$$\psi \to (\mathrm{Ad}\, u)\, \psi = u\psi u = uu^\circ \psi = uJu^\dagger J^{-1}\psi, \tag{8.18}$$

$$\mathcal{D} \to (\mathrm{Ad}\, u)\, \mathcal{D}\, (\mathrm{Ad}\, u)^\dagger\,, \tag{8.19}$$

with $u \in \mathcal{U}(\mathcal{A})$ a unitary element of $\mathcal{A}$.

In twisted non-commutative geometry, one replaces $\mathcal{D}_\mathcal{A}$ with the twisted fluctuated Dirac operator

$$\mathcal{D}_{\mathbb{A}_\rho} := \mathcal{D} + \mathbb{A}_\rho + \varepsilon' J \mathbb{A}_\rho J^{-1}, \tag{8.20}$$

where $\mathbb{A}_\rho$ is an element of the set of twisted 1-forms

$$\Omega^{1\rho}_\mathcal{D} := \mathrm{span}\left\{ a\,[\mathcal{D}, b]_\rho : a, b \in \mathcal{A} \right\} \tag{8.21}$$

such that $\mathcal{D}_{\mathbb{A}_\rho}$ is self-adjoint; and replaces also the inner product with the ρ-product (8.10) (or (8.12) in case the compatibility condition (8.9) is satisfied). Then, in place of (8.16) one defines

$$\mathfrak{A}^\rho_{\mathcal{D}_{\mathbb{A}_\rho}}(\phi, \xi) := \left\langle J\phi\,,\, \mathcal{D}_{\mathbb{A}_\rho}\xi \right\rangle_\rho = \left\langle J\phi\,,\, \mathcal{R}\mathcal{D}_{\mathbb{A}_\rho}\xi \right\rangle, \qquad \phi, \xi \in \mathrm{Dom}\,\mathcal{D}_{\mathbb{A}_\rho}. \tag{8.22}$$

A twisted gauge transformation is given by

$$\psi \to (\mathrm{Ad}\, u)\, \psi = u\psi u = uu^\circ \psi = uJu^\dagger J^{-1}\psi, \tag{8.23}$$

$$\mathcal{D}_{\mathbb{A}_\rho} \to (\mathrm{Ad}\, \rho(u))\, \mathcal{D}_{\mathbb{A}_\rho}\, (\mathrm{Ad}\, u)^\dagger\,. \tag{8.24}$$

If the twist ρ is compatible with the real structure (8.9), this twisted gauge transformation leaves the twisted bilinear form (8.22) invariant. However, the antisymmetry of $\mathfrak{A}^\rho_{\mathcal{D}_{\mathbb{A}_\rho}}$ is in general not preserved, unless one restricts to the positive eigenspace $\mathcal{H}_\mathcal{R}$ of $\mathcal{R}$, that is

$$\mathcal{H}_\mathcal{R} := \left\{ \xi \in \mathrm{Dom}\,\mathcal{D}_{\mathbb{A}_\rho} : \mathcal{R}\xi = \xi \right\}. \tag{8.25}$$

Then, the twisted fermionic action is

$$S^\rho_f\left(\mathcal{D}_{\mathbb{A}_\rho}\right) := \mathfrak{A}^\rho_{\mathcal{D}_{\mathbb{A}_\rho}}\left(\tilde{\psi}, \tilde{\psi}\right), \tag{8.26}$$

where $\tilde{\psi}$ is the Graßmann vector associated with $\psi \in \mathcal{H}_\mathcal{R}$.

This restriction to $\mathcal{H}_{\mathcal{R}}$ has important consequences. In Connes model of the SM, one assumes the spectral triple of a quasi-commutative space, i.e. the direct product (in a sense that will be made clear in the following) of the spectral triple representing a manifold by a spectral triple of a finite-dimensional Hilbert space. This means that the final Hilbert space will be $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_F$, where $L^2(\mathcal{M}, \mathcal{S})$ is the space of square-integrable spinors on the (Riemannian) spin-manifold $(\mathcal{M}, \mathcal{S})$, and $\mathcal{H}_F$ is the finite-dimensional Hilbert space. In Connes model one restricts to $\mathcal{H}_+$: this ensures that the physical states have well-defined chirality, i.e. that their chirality as spinors of $L^2(\mathcal{M}, \mathcal{S})$ coincides with their chirality as elements of $\mathcal{H}_F$. On the other hand, the elements of $\mathcal{H}_{\mathcal{R}}$ do not have a well-defined chirality – that is, on a *Riemannian* space. It turns out that the restriction to $\mathcal{H}_{\mathcal{R}}$ allows one to obtain a physically meaningful fermionic action in *Lorentzian* signature, even though one starts with a Riemannian manifold.

Before showing how this happens, we will prove two useful lemmas regarding $\mathfrak{A}^{\rho}_{\mathcal{D}_{\mathbb{A}_\rho}}$.

Lemma 8.3.1. *In a real graded spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, *(with* $J^2 = \varepsilon$ *and* $J\mathcal{D} = \varepsilon'\mathcal{D}J$*) one has*

$$\langle J\phi\,,\,\mathcal{D}\xi\rangle = \varepsilon\varepsilon'\,\langle J\xi\,,\,\mathcal{D}\phi\rangle \qquad \forall\phi,\xi \in \mathcal{H}. \tag{8.27}$$

Proof. J, being an antilinear isometry, satisfies $\langle J\phi\,,\,J\xi\rangle = \overline{\langle\phi\,,\,\xi\rangle} = \langle\xi\,,\,\phi\rangle$. Thus

$$\langle J\phi\,,\,\mathcal{D}\xi\rangle = \varepsilon\left\langle J\phi\,,\,J^2\mathcal{D}\xi\right\rangle = \varepsilon\,\langle J\mathcal{D}\xi\,,\,\phi\rangle = \varepsilon\varepsilon'\,\langle\mathcal{D}J\xi\,,\,\phi\rangle = \varepsilon\varepsilon'\,\langle J\xi\,,\,\mathcal{D}\phi\rangle\,. \tag{8.28}$$

□

Remark 8.3.1. For the spectral triple of the SM, one has $n = 10$, which corresponds to $\varepsilon\varepsilon' = -1$, hence the bilinear form $\mathfrak{A}_{\mathcal{D}_{\mathcal{A}}}(\psi, \psi)$ vanishes when evaluated on vectors. However, it is non-zero when evaluated on Graßmann vectors.

Lemma 8.3.2. *In a real twisted spectral triple with twist compatible with the real structure (8.9), one has*

$$\mathfrak{A}^{\rho}_{\mathcal{D}}(\phi, \xi) = \varepsilon'''\mathfrak{A}_{\mathcal{D}}(\phi, \xi) \qquad \forall\phi,\xi \in \mathcal{H}_{\mathcal{R}}. \tag{8.29}$$

Proof. Since $\mathcal{R}^\dagger J = \varepsilon''' J\mathcal{R}$ and $\mathcal{R}^\dagger\phi = \phi$, we have

$$\mathfrak{A}^{\rho}_{\mathcal{D}}(\phi, \xi) = \langle J\phi\,,\,\mathcal{R}\mathcal{D}\xi\rangle = \left\langle\mathcal{R}^\dagger J\phi\,,\,\mathcal{D}\xi\right\rangle = \varepsilon'''\left\langle J\mathcal{R}^\dagger\phi\,,\,\mathcal{D}\xi\right\rangle = \varepsilon'''\,\langle J\phi\,,\,\mathcal{D}\xi\rangle\,. \tag{8.30}$$

□

8.4 Minimal Twist of Non-Commutative Geometries

8.4.1 Twist by Grading

Given any usual (non-twisted) spectral triple $(\mathcal{A},\, \mathcal{H},\, \mathcal{D})$, there is a natural way of constructing a twisted spectral triple [88]. The idea is to replace the commutator with the Dirac operator $[\mathcal{D}, \cdot]$ with a twisted one $[\mathcal{D}, \cdot]_\rho$ while keeping the Hilbert space and the Dirac operator intact. However, for most spectral triples, there is no way to make both $[\mathcal{D}, \cdot]$ and $[\mathcal{D}, \cdot]_\rho$ bounded. For this reason, one has to enlarge the algebra.

Definition 8.4.1 (Minimal twist)**.** *A* minimal twist *of a spectral triple* $(\mathcal{A},\, \mathcal{H},\, \mathcal{D})$ *by a unital involutive algebra* $\mathcal{B}$ *is a twisted spectral triple* $(\mathcal{A}\otimes\mathcal{B},\, \mathcal{H},\, \mathcal{D})_\rho$ *where the initial representation*[1] π_0 *of* $\mathcal{A}$ *on* $\mathcal{H}$ *is related to the representation* π *of* $\mathcal{A}\otimes\mathcal{B}$ *on* $\mathcal{H}$ *by*

$$\pi\left(a\otimes\mathbb{I}_{\mathcal{B}}\right) = \pi_0\left(a\right) \qquad \forall a\in\mathcal{A}, \tag{8.31}$$

where $\mathbb{I}_{\mathcal{B}}$ *is the identity of the algebra* $\mathcal{B}$*.*

If the initial spectral triple is graded, one can use the grading Γ to obtain a natural minimal twist, called *twist by grading.* The idea is that the grading commutes with the representation of $\mathcal{A}$, which means that the representation of $\mathcal{A}$ is actually the direct sum of two representations on the eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of Γ. Therefore, there is enough room to represent twice the algebra $\mathcal{A}$. Therefore, by taking $\mathcal{B}=\mathbb{C}^2$, one gets $\mathcal{A}\otimes\mathbb{C}^2\simeq\mathcal{A}\oplus\mathcal{A}\ni(a,a')$ and

$$\pi\left(a,a'\right) \coloneqq P_+\pi_0\left(a\right) + P_-\pi_0\left(a'\right) = \begin{pmatrix}\pi_+\left(a\right) & 0\\ 0 & \pi_-\left(a\right)\end{pmatrix} \tag{8.32}$$

where $P_\pm \coloneqq \frac{1}{2}\left(\mathbb{I}\pm\Gamma\right)$ and $\pi_\pm\left(a\right)\coloneqq\pi_0\left(a\right)|_{\mathcal{H}_\pm}$ are respectively the projections on $\mathcal{H}_\pm$ and the restrictions on $\mathcal{H}_\pm$ of π_0. If $\pi_\pm$ are faithful, then $(\mathcal{A}\otimes\mathbb{C}^2,\, \mathcal{H},\, \mathcal{D})_\rho$ with the flip automorphism

$$\rho\left(a,a'\right) \coloneqq \left(a',a\right) \tag{8.33}$$

is indeed a twisted spectral triple with grading Γ. Furthermore, if the initial spectral triple is real, then so is this minimal twist, with the same real structure.

The twist ρ is a $*$-automorphism that satisfies the regularity condition $\rho^2=\mathbb{I}$ and coincides on $\pi\left(\mathcal{A}\otimes\mathbb{C}^2\right)$ with the inner automorphism of $\mathcal{B}\left(\mathcal{H}\right)$ implemented

[1] In order to avoid confusions, in this section we will explicitly write the representation symbols.

by the unitary

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_{\mathcal{H}_+} \\ \mathbb{I}_{\mathcal{H}_-} & 0 \end{pmatrix} \tag{8.34}$$

with $\mathbb{I}_{\mathcal{H}_\pm}$ the identity on $\mathcal{H}_\pm$.

Notice that $\mathcal{R} \sim \gamma^0 \otimes \mathbb{I}$. For this reason, the canonical ρ-product associated with the minimal twist of a closed *Riemannian* spin manifold of dimension 4 turns out to coincide with the *Lorentzian* Krein product[2] on the space of Lorentzian spinors. This is the main reason why, by twisting a Riemannian spectral triple, a transition to the Lorentzian signature happens in the action.

8.4.2 Minimal Twist of an Almost-Commutative Geometry

Given an (even dimensional) closed Riemannian spin manifold $\mathcal{M}$, the spectral triple associated with it is

$$\left(C^\infty(\mathcal{M}),\, L^2(\mathcal{M}, S),\, \partial\!\!\!/ = -i\gamma^\mu \nabla_\mu\right) \tag{8.35}$$

where the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$ acts by multiplication on the Hilbert space $L^2(M, S)$ of square integrable spinors, and $\partial\!\!\!/$ is the Dirac operator associated with the spin structure, with ∇_μ the covariant derivative over the spinor bundle of the spin manifold $\mathcal{M}$. Notice that the dimension of $\mathcal{M}$ has to be even so that the spectral triple (8.35) has a grading $\gamma_{\mathcal{M}}$ (the product of the Dirac matrices γ^μ).

For such a spectral triple, the twist by grading described in the previous section 8.4.1 is the only possible minimal twist. However, for a genuinely non-commutative geometry, many more twists are allowed. In particular, this will be true for an almost-commutative geometry:

$$\left(\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F,\, \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F,\, \mathcal{D} = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_{\mathcal{M}} \otimes D_F\right) \tag{8.36}$$

i.e. the product of (8.35) with a finite dimensional graded spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ with grading γ_F (in the equation above $\mathbb{I}_F$ is the identity operator on $\mathcal{H}_F$). The representation

$$\pi_0 = \pi_{\mathcal{M}} \otimes \pi_F \tag{8.37}$$

of $\mathcal{A}$ on $\mathcal{H}$ is the product of $\pi_{\mathcal{M}}$ in (8.35) with the representation π_F of $\mathcal{A}_F$ on $\mathcal{H}_F$ given by the finite dimensional spectral triple. Depending on the degeneracy of the representation π_F, there exists minimal twists with $\mathcal{B}$ of Definition 8.4.1

[2] i.e. the usual scalar product in lorentzian signature.

different from $\mathbb{C}^2$ [88]. In case the representation of $\mathcal{A}_F$ is irreducible, then $\mathcal{B}$ is necessarily $\mathbb{C}^2$ but the representation π in (8.32) is not necessarily the one obtained with the twist-by-grading.

In this section, we will investigate which properties of the grading are necessary to build a minimally twisted partner for an almost commutative geometry. We already know that the commutativity with the initial representation of $\mathcal{A}$ is important to get the two independent representations $\pi_\pm$, but we still have to determine to what extend the commutation properties of the grading Γ with $\mathcal{D}$ and the real structure J are relevant.

Given an almost commutative geometry (8.36), we thus consider an operator

$$T = \mathcal{T} \otimes T_F \tag{8.38}$$

in $\mathcal{B}(\mathcal{H})$ which shares all the properties of a grading but the commutation properties with $\mathcal{D}$ and J, namely T is selfadjoint, $T^2 = \mathbb{I}$, the degeneracy of both its eigenvalues ± 1 are non-zero and T commutes with the representation π_0 of $\mathcal{A}$ in (8.37). The latter is thus the direct sum $\pi_+ \oplus \pi_-$ of the two involutive representations of $\mathcal{A}$ on the eigenspaces $\mathcal{H}_\pm$ of T given by

$$\pi_\pm (a) = \left(\frac{\mathbb{I} \pm T}{2} \pi_0 (a) \right)_{\mathcal{H}_\pm} . \tag{8.39}$$

As in (8.32), the operator T allows to define a representation of $\mathcal{A} \otimes \mathbb{C}^2$ on $\mathcal{H}$

$$\pi (a, a') := \pi_+ (a) \oplus \pi_- (a') = \frac{\mathbb{I} + T}{2} \pi_0 (a) + \frac{\mathbb{I} - T}{2} \pi_0 (a') . \tag{8.40}$$

To avoid domain issues, we assume that $T\mathcal{H} \subset \operatorname{Dom} \mathcal{D}$. We call T a *twisting operator*.

In the rest of this section, we will investigate the properties that T, $\mathcal{T}$ and T_F must satisfy to define a proper twisted spectral triple.

Selfadjointness of T

The selfadjointness of T is to guarantee that the representations $\pi_\pm$ are involutive. This does not imply that $\mathcal{T}$ and T_F are selfadjoint. However, one can show that one may always restrict to this case.

Lemma 8.4.1. *Let $T = \mathcal{T} \otimes T_F$ be a selfadjoint operator on $L^2 (\mathcal{M}, S) \otimes \mathcal{H}_F$ that squares to $\mathbb{I}$. Then there exist two selfadjoint operators $\tilde{\mathcal{T}}$ on $L^2 (\mathcal{M}, S)$ and $\tilde{T}_F$ on $\mathcal{H}_F$, squaring to the identity, such that*

$$T = \tilde{\mathcal{T}} \otimes \tilde{T}_F. \tag{8.41}$$

Proof. The matrix $T_F^\dagger T_F$ is non-zero (otherwise T would not square to $\mathbb{I}$) and positive, thus it admits at least one real eigenvalue $\lambda > 0$, with associated eigenvector $\psi \in \mathcal{H}_F$, and all the other non-zero eigenvalues are also strictly positive. For any $\varphi \in L^2(\mathcal{M}, S)$, one has

$$T^\dagger T(\varphi \otimes \psi) = \mathcal{T}^\dagger \mathcal{T}\varphi \otimes T_F^\dagger T_F \psi = \lambda \mathcal{T}^\dagger \mathcal{T}\varphi \otimes \psi. \tag{8.42}$$

However, by hypothesis, $T^\dagger T = \mathbb{I}$, hence $T^\dagger T(\varphi \otimes \psi) = \varphi \otimes \psi$. Therefore

$$\left(\lambda \mathcal{T}^\dagger \mathcal{T} - \mathbb{I}_\mathcal{M}\right)\varphi \otimes \psi = 0 \qquad \forall \varphi \in L^2(\mathcal{M}, S), \tag{8.43}$$

meaning that

$$\mathcal{T}^\dagger \mathcal{T} = \lambda^{-1}\mathbb{I}_\mathcal{M}. \tag{8.44}$$

Repeating the same analysis for another non-zero eigenvalue λ', one concludes that $\mathcal{T}^\dagger \mathcal{T} = \lambda'^{-1}\mathbb{I}_\mathcal{M}$, implying that $\lambda = \lambda'$. This allows to define

$$\tilde{\mathcal{T}} = \lambda^{\frac{1}{2}}\mathcal{T}, \qquad \tilde{T}_F = \lambda^{-\frac{1}{2}}T_F, \tag{8.45}$$

such that $T = \tilde{\mathcal{T}} \otimes \tilde{T}_F$ with

$$\tilde{\mathcal{T}}^\dagger \tilde{\mathcal{T}} = \lambda \mathcal{T}^\dagger \mathcal{T} = \mathbb{I}_\mathcal{M}. \tag{8.46}$$

From $T^\dagger T = TT^\dagger \mathbb{I}$ it follows that $\tilde{T}_F^\dagger \tilde{T}_F = \tilde{T}_F \tilde{T}_F^\dagger = \mathbb{I}_F$, i.e. $\tilde{T}_F$ is unitary.

To show that $\tilde{\mathcal{T}}$ and $\tilde{T}_F$ are selfadjoint, let us apply $T = T^\dagger$ on $\varphi \otimes \Psi$, where Ψ is an eigenvector of $\tilde{T}_F$ with eigenvalue $\tau \in \mathbb{C}$, with $|\tau| = 1$. Using $\tilde{T}_F^\dagger \Psi = \tau^{-1}\Psi$, one obtains

$$\tilde{\mathcal{T}}\varphi \otimes \tau\Psi = \tilde{\mathcal{T}}^\dagger \varphi \otimes \tau^{-1}\Psi \qquad \forall \varphi \in L^2(\mathcal{M}, S), \tag{8.47}$$

meaning that

$$\tau \tilde{\mathcal{T}} = \tau^{-1}\tilde{\mathcal{T}}^\dagger. \tag{8.48}$$

Redefining $\tau \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$, the equation above shows that $\tilde{\mathcal{T}}$ is selfadjoint. Notice that this redefinition does not invalidate the previous analysis for $T^\dagger T$, since $|\tau| = 1$, meaning that $\left(\tau\tilde{\mathcal{T}}\right)^\dagger \left(\tau\tilde{\mathcal{T}}\right) = \tilde{\mathcal{T}}^\dagger \tilde{\mathcal{T}}$. The selfadjointness of T_F then follows from the one of T. ∎

Boundedness of the Twisted Commutator

In this section, we investigate the boundedness of the twisted commutator

$$[\mathcal{D}, \pi(a, a')]_\rho \tag{8.49}$$

with π the representation (8.40) and ρ the flip (8.33) for T as in (8.38) with $\mathcal{T}$ and T_F selfadjoint and squaring to the identity (which is always possible as explained in Lemma 8.4.1).

As we will see, as a necessary condition, $\mathcal{T}$ will need to be a grading of the spectral triple (8.35) of the manifold. To prove this, we start with the following

Lemma 8.4.2. *For an almost commutative geometry, the twisted commutator*

$$\left[\partial\!\!\!/ \otimes \mathbb{I}_F, \pi\left(a, a'\right)\right]_\rho \tag{8.50}$$

is bounded for any $(a, a') \in \mathcal{A} \otimes \mathbb{C}^2$ if and only if $\mathcal{T}$ anticommutes with $\partial\!\!\!/$.

Proof. Using

$$\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right)(\mathbb{I} + T) = (\mathbb{I} - T)\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right) + \left\{\partial\!\!\!/ \otimes \mathbb{I}_F, T\right\}, \tag{8.51}$$

$$\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right)(\mathbb{I} - T) = (\mathbb{I} + T)\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right) - \left\{\partial\!\!\!/ \otimes \mathbb{I}_F, T\right\}, \tag{8.52}$$

one obtains (identifying $a = \pi_0(a)$)

$$\begin{aligned} \left[\partial\!\!\!/ \otimes \mathbb{I}_F, \pi\left(a, a'\right)\right]_\rho &= \left(\partial\!\!\!/ \otimes \mathbb{I}_F\right)\pi\left(a, a'\right) - \pi\left(a', a\right)\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right) = \\ &= \left(\partial\!\!\!/ \otimes \mathbb{I}_F\right)\left(\frac{\mathbb{I} + T}{2}a + \frac{\mathbb{I} - T}{2}a'\right) - \left(\frac{\mathbb{I} + T}{2}a' + \frac{\mathbb{I} - T}{2}a\right)\left(\partial\!\!\!/ \otimes \mathbb{I}_F\right) = \\ &= \frac{\mathbb{I} - T}{2}\left[\partial\!\!\!/ \otimes \mathbb{I}_F, a\right] + \frac{\mathbb{I} + T}{2}\left[\partial\!\!\!/ \otimes \mathbb{I}_F, a'\right] + \left\{\partial\!\!\!/ \otimes \mathbb{I}_F, T\right\}\left(a - a'\right). \end{aligned} \tag{8.53}$$

For $a = f \otimes m$, with $f \in C^\infty(\mathcal{M})$ and $m \in \mathcal{A}_F$, one has that $\left[\partial\!\!\!/ \otimes \mathbb{I}_F, a\right] = \left[\partial\!\!\!/, f\right] \otimes m$ is bounded, being $\left(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/\right)$ a spectral triple. The same is true for an arbitrary a in $\mathcal{A}$, and also for $\left[\partial\!\!\!/ \otimes \mathbb{I}_F, a'\right]$. So the first two terms in (8.53) are bounded.

If T anticommutes with $\partial\!\!\!/ \otimes \mathbb{I}_F$, the last term in (8.53) is zero, so that the twisted commutator (8.50) is bounded.

Conversely, assume (8.50) is bounded for any (a, a') in $\mathcal{A} \otimes \mathbb{C}^2$. This means that the last term in (8.53) is bounded. For $a - a' = 1 \otimes m$ with 1 the constant function $f(x) = 1$ on $\mathcal{M}$, then this last term is

$$\left\{\partial\!\!\!/ \otimes \mathbb{I}_F, T\right\}\left(a - a'\right) = \left\{\partial\!\!\!/, \mathcal{T}\right\} \otimes T_F m. \tag{8.54}$$

This is bounded if and only if $\left\{\partial\!\!\!/, \mathcal{T}\right\}$ is bounded. For $\psi \in \mathcal{H}_+$, one has

$$\left\{\partial\!\!\!/, \mathcal{T}\right\}\psi = \partial\!\!\!/\psi + \mathcal{T}\partial\!\!\!/\psi = (\mathbb{I} + \mathcal{T})\,\partial\!\!\!/\psi. \tag{8.55}$$

meaning that $\left\{\partial\!\!\!/, \mathcal{T}\right\}$ coincides on $\mathcal{H}_+$ with $(\mathbb{I} + \mathcal{T})\,\partial\!\!\!/$, which is unbounded unless it is zero. A similar result holds for the restriction to $\mathcal{H}_-$, hence the result. □

The finite part of an almost commutative geometry only involves bounded operator. Therefore the boundedness of the twisted commutator (8.49) only depends on the property of $\mathcal{T}$.

Using Lemma 8.4.2, we can prove the following

Proposition 8.4.1. *The twisted commutator (8.49) is bounded only if the component $\mathcal{T}$ of the twisting operator is a grading of the spectral triple (8.35) of the manifold $\mathcal{M}$ and if T_F commutes with the representation π_F of $\mathcal{A}_F$ on $\mathcal{H}_F$.*

Proof. The twisted commutator $\left[\gamma_{\mathcal{M}} \otimes D_F, \pi\left(a, a'\right)\right]_\rho$ is bounded regardless of whether T anticommutes with $\gamma_{\mathcal{M}} \otimes D_F$. Hence (8.49) is bounded if and only if $\left[\partial\!\!\!/ \otimes \mathbb{I}_F, \pi\left(a, a'\right)\right]_\rho$ is bounded, i.e. by Lemma 8.4.2 if and only if $\mathcal{T}$ anticommutes with $\partial\!\!\!/$.

Moreover, by hypothesis T commutes with π, thus, in particular, it commutes with $\pi\left(a \otimes \mathbb{I}_2\right) = \pi_0\left(a\right)$ for any $a \in \mathcal{A}$:

$$\left[T, \pi_0\left(a\right)\right] = 0. \tag{8.56}$$

In particular, this condition must hold true for $a = 1 \otimes m$, with 1 the constant function $f\left(x\right) = 1$ on $\mathcal{M}$. This case case yields (omitting the representation symbol)

$$0 = Ta - aT = \mathcal{T} \otimes T_F m - \mathcal{T} \otimes m T_F = \mathcal{T} \otimes \left[T_F, m\right] \qquad \forall m \in \mathcal{A}_F, \tag{8.57}$$

which implies

$$\left[T_F, m\right] = 0 \qquad \forall m \in \mathcal{A}_F. \tag{8.58}$$

Using this result for a generic $a = f \otimes m$, we get

$$0 = \mathcal{T} f \otimes T_F m - f \mathcal{T} \otimes m T_F = \left[\mathcal{T}, f\right] \otimes T_F m \qquad \forall f \in C^\infty\left(\mathcal{M}\right), \forall m \in \mathcal{A}_F, \tag{8.59}$$

which is equivalent to

$$\left[\mathcal{T}, f\right] = 0 \qquad \forall f \in C^\infty\left(\mathcal{M}\right). \tag{8.60}$$

In other terms, $\mathcal{T}$ commutes with the representation $\pi_{\mathcal{M}}$ of $C^\infty\left(\mathcal{M}\right)$, and T_F commutes with the representation π_F of $\mathcal{A}_F$.

Because of Lemma 8.4.1 $\mathcal{T}$ is selfadjoint and squares to the identity, while because of Lemma 8.4.2 it anticommutes with $\partial\!\!\!/$. So $\mathcal{T}$ has all the properties of a grading of the spectral triple 8.35. □

Notice that the last proposition is not an if and only if, for it does not make explicit the conditions on T_F to guarantee that the twisting operator T actually yields a twisted spectral triple. In particular, the operator T_F may not be a grading of the finite dimensional spectral triple, for so far nothing forces its anti-commutation with D_F. This gives some freedom on twisting an almost commutative geometry, that we will make use of to define different twists of the Connes Model, as we will see in Chapter 10.

Order-Zero Condition

A twisted spectral triple must still satisfy the non-twisted order-zero condition (8.4). The twist by grading of a real twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})_\rho$ automatically satisfies the order zero condition. Namely, if (8.4) holds for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, then one has

$$\left[\pi\left(a, a'\right), J\pi\left(b^*, b'^*\right) J^{-1}\right] = 0 \qquad \forall \left(a, a'\right), \left(b, b'\right) \in \mathcal{A} \otimes \mathbb{C}^2. \tag{8.61}$$

for π in (8.32). In the following lemma, we work out under which condition the same holds true for π given by (8.40).

Lemma 8.4.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be real spectral triple with real structure J. The order-zero condition (8.61) holds for the representation π in (8.40) if and only if*

$$\left[T, JTJ^{-1}\right] = 0 \quad \text{and} \quad \left[\pi_0\left(a\right), JTJ^{-1}\right] = 0 \qquad \forall a \in \mathcal{A}. \tag{8.62}$$

Proof. By defining

$$\alpha \coloneqq a + a', \qquad \alpha' \coloneqq a - a' \tag{8.63}$$

one can rewrite

$$\pi\left(a, a'\right) = \frac{\mathbb{I} + T}{2}\pi_0\left(a\right) + \frac{\mathbb{I} - T}{2}\pi_0\left(a'\right) = \frac{1}{2}\left(\pi_0\left(\alpha\right) + T\pi_0\left(\alpha'\right)\right). \tag{8.64}$$

Similarly, we can write $\pi\left(b, b'\right) = \frac{1}{2}\left(\pi_0\left(\beta\right) + Ti\pi_0\left(\beta'\right)\right)$ for $\beta \coloneqq b + b'$ and $\beta' \coloneqq b - b'$. Then, we have

$$\left[\pi\left(a, a'\right), J\pi\left(b, b'\right) J^{-1}\right] = \frac{1}{4}\left[\pi_0\left(\alpha\right) + T\pi_0\left(\alpha'\right), J\left(\pi_0\left(\beta\right) + T\pi_0\left(\beta'\right)\right) J^{-1}\right]. \tag{8.65}$$

The order-zero condition indicates that this is zero for any $\alpha, \alpha', \beta, \beta' \in \mathcal{A}$. This is equivalent to (omitting the representation symbol)

$$\left[\alpha, J\beta J^{-1}\right] = 0, \qquad \left[\alpha, JT\beta J^{-1}\right] = 0, \tag{8.66}$$

$$\left[T\alpha, J\beta J^{-1}\right] = 0, \qquad \left[T\alpha, JT\beta J^{-1}\right] = 0. \qquad \forall \alpha, \beta \in \mathcal{A}. \tag{8.67}$$

The first condition (8.66) is the order-zero condition for the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ so it is true by hypothesis.

The second condition (8.66) writes

$$0 = \left[\alpha, JTJ^{-1}J\beta J^{-1}\right] = JTJ^{-1}\left[\alpha, J\beta J^{-1}\right] + \left[\alpha, JTJ^{-1}\right] J\beta J^{-1}. \tag{8.68}$$

The first term is always zero by the order zero condition for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The second term is zero for any α, β if and only if(consider e.g. the case β is the unit of $\mathcal{A}$)

$$\left[\alpha, JTJ^{-1}\right] = 0 \qquad \forall \alpha \in \mathcal{A}. \tag{8.69}$$

For the same reasons, the first equation (8.67) written as

$$0 = T\left[\alpha, J\beta J^{-1}\right] + \left[T, J\beta J^{-1}\right]\alpha \tag{8.70}$$

is equivalent to $[T, J\beta J^{-1}] = 0$, i.e., by multiplying by J^{-1} on the left and by J on the right,

$$0 = J^{-1}TJ\beta - \beta J^{-1}TJ = \left[J^{-1}TJ, \beta\right] \tag{8.71}$$

and by remembering that $J^{-1} = \epsilon J$, this is equivalent to (8.69).

Finally, the second condition (8.67) is equivalent to

$$JTJ^{-1}\left[T\alpha, J\beta J^{-1}\right] + \left[T\alpha, JTJ^{-1}\right] J\beta J^{-1} = 0. \tag{8.72}$$

The first term is proportional to the first condition (8.67), hence it is zero. The second term, written as $[T, JTJ^{-1}]\,\alpha + T\,[\alpha, JTJ^{-1}] = 0$, is equivalent to

$$\left[T, JTJ^{-1}\right] = 0. \tag{8.73}$$

Hence the result. □

Remark 8.4.1. The condition (8.73) is equivalent to

$$[T, J]\,\{T, J\} = 0. \tag{8.74}$$

Assuming that the twisting operator is of the form (8.38) and that $\mathcal{T}\mathcal{J} = \pm\mathcal{J}\mathcal{T}$ (which is always true, as we will show in the next section), the conditions (8.62) are equivalent to their restrictions to the finite dimensional spectral triple.

Proposition 8.4.2. *A twisting operator (8.38) with $\mathcal{T}$ such that*

$$\mathcal{T}\mathcal{J} = \pm\mathcal{J}\mathcal{T} \tag{8.75}$$

satisfies the order zero condition if and only if

$$\left[T_F, J_F T J_F^{-1}\right] = 0 \quad \text{and} \quad \left[\pi_F(m), J_F T_F J_F^{-1}\right] = 0 \qquad \forall m \in \mathcal{A}_F. \tag{8.76}$$

Proof. One has $JTJ^{-1} = \mathcal{J}\mathcal{T}\mathcal{J}^{-1} \otimes J_F T_F J_F^{-1} = \pm\mathcal{T} \otimes J_F T_F J_F^{-1}$, so that (8.62) becomes

$$\pm\mathbb{I}_{\mathcal{M}} \otimes \left[T_F, J_F T_F J_F^{-1}\right] = 0, \quad \pm\mathcal{T} f \otimes \left[m, J_F T_F J_F^{-1}\right] = 0 \qquad \forall f \otimes m \in \mathcal{A}. \tag{8.77}$$

□

The real structure $\mathcal{J}$ of the manifold

As anticipated just before Proposition 8.4.2, one can show that the real structure $\mathcal{J}$ of any manifold $\mathcal{M}$ is always a product of gammas, regardless of the dimension of $\mathcal{M}$. In fact, one can show that $\mathcal{J}$ always coincides with the charge conjugation operator often used in physics. To show this, we need some simple preliminary results.

First of all, we recall that in d dimensions the gamma matrices γ_a, $a = 0, 1 \ldots, d-1$ are a d-long set of $N \times N$ hermitian matrices (with $N = 2^{\lfloor d/2 \rfloor}$) that satisfy the Clifford algebra anticommutation relation:

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}\mathbb{I}_N. \tag{8.78}$$

If d is even, one can define the chiral gamma matrix

$$\gamma_{\text{chiral}} \coloneqq i^{\frac{d}{2}} \prod_{a=0}^{d-1} \gamma_a \tag{8.79}$$

which anticommutes with all gamma matrices. If d is odd, γ_{chiral} is proportional to the identity.

One can also define the charge conjugation matrices $C_\pm$ such that

$$C_+ \gamma_a C_+^{-1} = +\overline{\gamma_a}, \qquad C_- \gamma_a C_-^{-1} = -\overline{\gamma_a} \tag{8.80}$$

(where the overline $\bar{\cdot}$ denotes the complex conjugation). If d is even, both $C_\pm$ exist; on the other hand, in odd dimensions only one of the two exists, and in particular C_+ exists for $d = 1, 5 \mod 8$ and C_- exists for $d = 3, 7 \mod 8$.

One can show that C_+ is the product of all the even-indexed gammas, while C_- is the product of all the odd-indexed gammas. To do so, it is best to work in a representation that allows to define the gamma matrices recursively, starting from the Pauli matrices. We will use this particular representation only for the sake of showing this result, while we will use a different representation in Chapters 9 and following.

Lemma 8.4.4. *Let $m = 2\lfloor d/2 \rfloor$. Then, the charge conjugation matrices are a product of either even- or odd-indexed gamma matrices, given by*

$$C_+ = i^{\frac{m}{4}} \prod_{k=1}^{m/2} \gamma_{2k-1}, \qquad C_- = (-i)^{\frac{m}{4}} \prod_{k=0}^{m/2-1} \gamma_{2k} \qquad \textit{for even } \frac{m}{2}, \tag{8.81}$$

$$C_+ = (-i)^{\lfloor \frac{m}{4} \rfloor} \prod_{k=0}^{m/2-1} \gamma_{2k}, \qquad C_- = i^{\lfloor \frac{m}{4} \rfloor} \prod_{k=1}^{m/2} \gamma_{2k-1} \qquad \textit{for odd } \frac{m}{2}. \tag{8.82}$$

Moreover, in any even dimension $d = 2k$

$$C_+ C_- = (-1)^{\lfloor \frac{d}{4} \rfloor} (-1)^{\frac{d}{2}} \gamma_{\text{chiral}} \tag{8.83}$$

and

$$C_-^2 = \mathbb{I}. \tag{8.84}$$

Proof. To start, let us define the gamma matrices in any dimension in a particular representation.

For $d = 2$, we define

$$\gamma_0 = \sigma_1, \qquad \gamma_1 = \sigma_2. \tag{8.85}$$

We also define the chiral gamma matrix

$$\gamma_{\text{chiral}} = -i\gamma_0\gamma_1 = \sigma_3, \tag{8.86}$$

as well as the charge conjugation matrices

$$c_+ = \gamma_0, \qquad c_- = \gamma_1 \tag{8.87}$$

that satisfy

$$c_+ \gamma_a c_+^{-1} = +\overline{\gamma_a} \qquad c_- \gamma_a c_-^{-1} = -\overline{\gamma_a}. \tag{8.88}$$

Notice that γ_{chiral} transforms the same way under the effect of both $c_\pm$:

$$c_\pm \gamma_{\text{chiral}} c_\pm^{-1} = -\overline{\gamma_{\text{chiral}}}. \tag{8.89}$$

Now, let us define the gammas in any even-dimensional $d = 2k$ case. Assuming we have already defined the gammas $\gamma_{a'}, a' = 0, \ldots, d-1$ in d dimensions, with chiral gamma matrix γ_{chiral} and charge conjugation matrices $c_\pm$, we can define the gammas $\Gamma_a, a = 0, \ldots, d+1$ in $d+2$ dimensions:

$$\Gamma_{a'} = \gamma_{a'} \otimes \sigma_3, \qquad \Gamma_d = \mathbb{I} \otimes \sigma_1, \qquad \Gamma_{d+1} = \mathbb{I} \otimes \sigma_2. \tag{8.90}$$

The charge conjugation matrices will be

$$C_+ = c_- \otimes \sigma_1, \qquad C_- = c_+ \otimes \sigma_2. \tag{8.91}$$

The chiral gamma will be

$$\Gamma_{\text{chiral}} = \gamma_{\text{chiral}} \otimes \sigma_3. \tag{8.92}$$

Notice that once again Γ_{chiral} transforms the same way under both $C_\pm$ and moreover the sign in front of $\overline{\Gamma_{\text{chiral}}}$ will be the opposite with respect to the sign in front of $\overline{\gamma_{\text{chiral}}}$:

$$c_\pm \gamma_{\text{chiral}} c_\pm^{-1} = \varepsilon \overline{\gamma_{\text{chiral}}}$$

$$\Rightarrow C_\pm \Gamma_{\text{chiral}} C_\pm^{-1} = c_\pm \gamma_{\text{chiral}} c_\pm^{-1} \otimes \sigma_{1,2}\sigma_3\sigma_{1,2}^{-1} = -\varepsilon \overline{\gamma_{\text{chiral}}} \otimes \sigma_3 = -\varepsilon \overline{\Gamma_{\text{chiral}}}, \quad (8.93)$$

since σ_3 has real entries and it anticommutes with both $\sigma_{1,2}$. Finally, we define the integer

$$n = \frac{d}{2} - 1. \quad (8.94)$$

With these definitions, it is easy to show that

$$\Gamma_0 = \sigma_1 \otimes (\sigma_3)^{\otimes n} \quad (8.95)$$

$$\Gamma_1 = \sigma_2 \otimes (\sigma_3)^{\otimes n} \quad (8.96)$$

$$\Gamma_{2k} = (\mathbb{I}_2)^{\otimes k} \otimes \sigma_1 \otimes (\sigma_3)^{\otimes(n-k)} \quad (8.97)$$

$$\Gamma_{2k+1} = (\mathbb{I}_2)^{\otimes k} \otimes \sigma_2 \otimes (\sigma_3)^{\otimes(n-k)} \quad (8.98)$$

$$\Gamma_{\text{chiral}} = (\sigma_3)^{\otimes \frac{d}{2}} \quad (8.99)$$

$$C_+ = \begin{cases} (\sigma_2 \otimes \sigma_1)^{\otimes \frac{d}{4}} & \text{for even } \frac{d}{2} \\ (\sigma_1 \otimes \sigma_2)^{\otimes \left\lfloor \frac{d}{4} \right\rfloor} \otimes \sigma_1 & \text{for odd } \frac{d}{2} \end{cases} \quad (8.100)$$

$$C_- = \begin{cases} (\sigma_1 \otimes \sigma_2)^{\otimes \frac{d}{4}} & \text{for even } \frac{d}{2} \\ (\sigma_2 \otimes \sigma_1)^{\otimes \left\lfloor \frac{d}{4} \right\rfloor} \otimes \sigma_2 & \text{for odd } \frac{d}{2} \end{cases}. \quad (8.101)$$

Thanks to $\Gamma_{2k}, \Gamma_{2k+1}$ having k identity factors, and using

$$(\sigma_3)^2 = \mathbb{I}_2, \qquad \sigma_3\sigma_1 = i\sigma_2, \qquad \sigma_3\sigma_2 = -i\sigma_1, \quad (8.102)$$

it is easy to show that

$$\Gamma_{\text{chiral}} = i^{\frac{d}{2}} \Gamma_0 \Gamma_1 \dots \Gamma_{d-1}, \quad (8.103)$$

$$C_+ = \begin{cases} i^{\frac{d}{4}} \Gamma_1 \Gamma_3 \Gamma_5 \dots \Gamma_{d-1} = i^{\frac{d}{4}} \prod_{k=1}^{d/2} \Gamma_{2k-1} & \text{for even } \frac{d}{2} \\ (-i)^{\left\lfloor \frac{d}{4} \right\rfloor} \Gamma_0 \Gamma_2 \Gamma_4 \dots \Gamma_{d-2} = (-i)^{\left\lfloor \frac{d}{4} \right\rfloor} \prod_{k=0}^{d/2-1} \Gamma_{2k} & \text{for odd } \frac{d}{2} \end{cases}, \quad (8.104)$$

$$C_- = \begin{cases} (-i)^{\frac{d}{4}} \Gamma_0 \Gamma_2 \Gamma_4 \dots \Gamma_{d-2} = (-i)^{\frac{d}{4}} \prod_{k=0}^{d/2-1} \Gamma_{2k} & \text{for even } \frac{d}{2} \\ i^{\left\lfloor \frac{d}{4} \right\rfloor} \Gamma_1 \Gamma_3 \Gamma_5 \dots \Gamma_{d-1} = i^{\left\lfloor \frac{d}{4} \right\rfloor} \prod_{k=1}^{d/2} \Gamma_{2k-1} & \text{for odd } \frac{d}{2} \end{cases}. \quad (8.105)$$

With these formulas, it is a straightforward calculation to show that

$$C_+ C_- = (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} (-1)^{\frac{d}{2}} \gamma_{\text{chiral}} \quad (8.106)$$

and that

$$C_-^2 = \mathbb{I}. \quad (8.107)$$

Now we can deal with the odd-dimensional case $d = 2k+1$. Consider the previous construction for $d-1$ (which is even) and simply take all Γ_a, $a = 0, \dots, d-2$ matrices, to which append $\Gamma_{d-1} \equiv \Gamma_{\text{chiral}}$.

The charge conjugation matrices are the same as in the $d-1$ case, but only one of the two will transform Γ_{d-1} the same way as all the other gammas, so the other one can be ignored. If we denote C the single charge conjugation matrix, we have

$$C = \begin{cases} C_+ & \text{for even } \frac{d-1}{2} \\ C_- & \text{for odd } \frac{d-1}{2} \end{cases}, \tag{8.108}$$

hence the result.

□

The algebra generated by the gamma matrices spans the whole matrix space, as shown in the following

Lemma 8.4.5. *The set Γ of N^2 matrices (with $N = 2^{\lfloor d/2 \rfloor}$) defined as*

$$\Gamma = \left\{ \mathbb{I}_N, \ \gamma_a, \ \gamma_{[a}\gamma_{b]}, \ \gamma_{[a}\gamma_b\gamma_{c]}, \ \dots \right\} \tag{8.109}$$

containing all possible antisymmetrized products of gammas form a basis of $M(N, \mathbb{C})$. For even d, the products are considered up to d factors and the indices $a, b, c \dots$ range from 0 to $d-1$. For odd d, the products are considered up to $d-1$ factors and the indices range from 0 to $d-2$.

Proof. We define $m = 2\lfloor d/2 \rfloor$, so that the elements of Γ will contain up to m gamma factors for both even and odd d. The number of rank r antisymmetric tensors that one can write using m gamma factors is

$$n_r = \binom{m}{r}, \tag{8.110}$$

so that the total number of elements of Γ is $\sum_{r=0}^{m} = 2^m = N^2$, which is exactly the number of linearly independent components in a $N \times N$ matrix.

Any product of more than m gammas can be reduced to a sum of elements of Γ using the Clifford algebra relation (8.78).

So far, everything is true for both even and odd d. Now, we will restrict to the case of even $d = 2k$, and then deal with the odd d case later.

Using again the relation (8.78), it is easy to show that all gammas are traceless:

$$\begin{aligned} 2\delta_{ab} \operatorname{Tr}(\gamma_c) &= \operatorname{Tr}(\{\gamma_a, \gamma_b\}\gamma_c) = \\ &= \operatorname{Tr}(\gamma_a\gamma_b\gamma_c + \gamma_b\gamma_a\gamma_c) = \\ &= \operatorname{Tr}(\gamma_a\gamma_b\gamma_c + \gamma_a\gamma_c\gamma_b) = \end{aligned}$$

$$= \mathrm{Tr}\left(\gamma_a \left\{\gamma_b, \gamma_c\right\}\right) =$$
$$= 2\delta_{bc}\,\mathrm{Tr}\left(\gamma_a\right)$$

and by taking $a = b \neq c$ we conclude that $\mathrm{Tr}\,\gamma_a = 0$.

It is just as easy to show that all the antisymmetrized products of an even number of gammas are traceless. For instance, we have

$$\mathrm{Tr}\left[\gamma_a, \gamma_b\right] = \mathrm{Tr}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right) = \mathrm{Tr}\left(\gamma_a\gamma_b - \gamma_a\gamma_b\right) = 0, \tag{8.111}$$

where we used the cyclic property of the trace. The trick can be generalized to any even number of antisymmetrized factors. Let us consider for instance the rank 4 case: since the product is antisymmetrized, one can always assume that all the indices are different (otherwise, the product will just be identically zero). Then we simply need to show that

$$\mathrm{Tr}\left(\gamma_a\gamma_b\gamma_c\gamma_d\right) = 0 \tag{8.112}$$

for a, b, c, d all different. To do so, one can anticommute the leftmost gamma factor to the right, yielding a minus sign (since γ_a has to hop through an odd number of gammas):

$$\mathrm{Tr}\left(\gamma_a\gamma_b\gamma_c\gamma_d\right) = -\,\mathrm{Tr}\left(\gamma_b\gamma_c\gamma_d\gamma_a\right) \tag{8.113}$$

but then again, this must be equal to its opposite, thanks again to the cyclic property of the trace:

$$\mathrm{Tr}\left(\gamma_a\gamma_b\gamma_c\gamma_d\right) = +\,\mathrm{Tr}\left(\gamma_b\gamma_c\gamma_d\gamma_a\right). \tag{8.114}$$

Hence we conclude that rank 4 antisymmetric gamma tensors are traceless. The same trick works the same way for any even rank antisymmetric gamma tensor.

Now we show that also all odd rank gamma tensors are traceless (they do not even need to be antisymmetric). This time it is crucial that d is even, for in this case γ_{chiral} anticommutes with all gammas. We have

$$\begin{aligned}\mathrm{Tr}\left(\gamma_a\gamma_b\gamma_c\right) &= \mathrm{Tr}\left(\gamma_{\text{chir}}\gamma_{\text{chir}}\gamma_a\gamma_b\gamma_c\right) = \\ &= -\,\mathrm{Tr}\left(\gamma_{\text{chir}}\gamma_a\gamma_b\gamma_c\gamma_{\text{chir}}\right) \qquad \text{(using that } \{\gamma_{\text{chir}}, \gamma_a\} = 0\text{)} \\ &= +\,\mathrm{Tr}\left(\gamma_{\text{chir}}\gamma_a\gamma_b\gamma_c\gamma_{\text{chir}}\right) \quad \text{(using the cyclic property of the trace)},\end{aligned}$$

hence it is zero. Once again, this proof works for any odd rank gamma tensor.

This means that the only element of Γ that is not traceless is the identity. We can use this fact to show that the elements of Γ are linearly independent. Indeed, if we denote the elements of Γ as Γ_i, we will show that

$$\sum_{i=1}^{N^2} a^i\Gamma_i = 0 \qquad \Leftrightarrow \qquad a^i = 0\ \forall i. \tag{8.115}$$

To do so, we multiply the linear combination by each of the Γ_i and then take the trace. Since all the gammas square to the identity and since they all anticommute if they have different indices, it is easy to show that

$$\Gamma_i^2 = \pm \mathbb{I}_N. \tag{8.116}$$

Then we have

$$\Gamma_j \sum_{i=1}^{N^2} a^i \Gamma_i = \sum_{i=1}^{N^2} a^i \left(\Gamma_j \Gamma_i\right) = a^j \Gamma_j^2 + \sum_{i \neq j} a^i \left(\Gamma_j \Gamma_i\right) = 0. \tag{8.117}$$

For $i \neq j$, $\Gamma_j \Gamma_i$ is of course the product of either an even or an odd number of gamma matrices (partially antisymmetrized in the suitable indices), so it is traceless, while Γ_j^2 is proportional to the identity. Then, by taking the trace, we conclude that $a^j = 0$. Repeating the procedure for all j, we get the result.

Now, we still have to deal with the odd $d = 2k+1$ case. Actually, this is quite simple: the gamma matrices have the same dimension for both the $d = 2k+1$ and for the $d = 2k$ case. Since we have already shown that the set Γ built in the $d = 2k$ case is a basis of $M(N, \mathbb{C})$, that very same set will be a base also if $d = 2k+1$. To characterize it from the $d = 2k+1$ point of view, one has to exclude γ_{d-1} from the set of gammas used to build the antisymmetric products, since it coincides with the γ_{chiral} of the $d = 2k$ case. Indeed, since

$$\prod_{a=0}^{d-1} \gamma_a = \mathbb{I}_N \tag{8.118}$$

when d is odd, and since all gammas are their own inverses, one has that

$$\gamma_{d-1} = \pm \prod_{a=0}^{d-2} \gamma_a \tag{8.119}$$

(where the sign depends on the number of factors). This means that the indices in the products will range from 0 to $d-2$, and that they will have at most $d-1$ factors (otherwise, they would be zero after antisymmetrization). Hence the result. □

Using Lemma 8.4.5, we can prove the following

Theorem 8.4.3. *Let $\mathcal{M}$ be a spin-manifold of dimension d with associated Dirac operator $\partial\!\!\!/ = -i\gamma^a \nabla_a$ and real structure $\mathcal{J}$, with $\mathcal{J}\partial\!\!\!/ = \epsilon' \partial\!\!\!/ \mathcal{J}$; if d is even, let $\gamma_{\mathcal{M}}$ be the grading, that satisfies $\gamma_{\mathcal{M}} \partial\!\!\!/ = -\partial\!\!\!/ \gamma_{\mathcal{M}}$. Then, $\mathcal{J}$ and $\gamma_{\mathcal{M}}$ are both a product of gamma matrices, and in particular:*

- *For even d,* $\gamma_{\mathcal{M}} = \pm\gamma_{\text{chiral}}$ *and* $\mathcal{J} = e^{i\varphi} C_{-\epsilon'} K$*;*
- *For odd d,* $\mathcal{J} = e^{i\varphi} C K$*;*

where φ *is real,* $C_{-\epsilon'} = C_{\mp}$ *for* $\epsilon' = \pm 1$ *are the charge conjugation matrices of the even-dimensional case,* C *is the charge conjugation matrix of the odd-dimensional case, and where* K *denotes the complex conjugation operator:*

$$Kz = \overline{z} \qquad \forall z \in \mathbb{C}. \tag{8.120}$$

Proof. The real structure is by definition antilinear, hence it is proportional to K. We denote

$$\mathcal{J} = \gamma K. \tag{8.121}$$

The real structure must either commute or anticommute with the Dirac operator, hence

$$\mathcal{J}\partial\!\!\!/ = \gamma K \left(-i\right) \gamma^a \left(\partial_a + \omega_a\right) = \epsilon' \partial\!\!\!/ \mathcal{J} = -\epsilon' i \gamma^a \left(\partial_a + \omega_a\right) \gamma K. \tag{8.122}$$

Let us consider the terms with the partial derivative. We have

$$\gamma K \left(-i\right) \gamma^a \partial_a = i \gamma \overline{\gamma^a} \partial_a K = \pm i \left(\gamma C_{\pm}\right) \gamma^a C_{\pm}^{-1} \partial_a K = -\epsilon' i \gamma^a \partial_a \gamma K = -\epsilon' i \gamma^a \gamma \partial_a K.$$

Equating the third term with the last term implies that

$$\pm \left(\gamma C_{\pm}\right) \gamma^a C_{\pm}^{-1} = -\epsilon' \gamma^a \gamma \qquad \forall a. \tag{8.123}$$

Suppose the manifold is even-dimensional, so that we can use any of the two $C_{\pm}$ – let us say we use C_-. We get

$$\left(\gamma C_-\right) \gamma^a C_-^{-1} = \epsilon' \gamma^a \gamma. \tag{8.124}$$

This means that (γC_-) either commutes or anticommutes with all gammas, depending on ϵ'. The only matrix that commutes with all gammas is the identity (which can be easily checked, using Lemma 8.4.5, by evaluating the commutators of all the elements of the basis (8.109) with a generic gamma), and the only matrix that anticommutes with all gammas is γ_{chiral} (which can be checked in a similar way). Then, we conclude that

$$\gamma C_- = \lambda \mathbb{I} \qquad \text{if } \epsilon' = +1, \tag{8.125}$$

$$\gamma C_- = \lambda \gamma_{\text{chiral}} \qquad \text{if } \epsilon' = -1 \tag{8.126}$$

for some $\lambda \in \mathbb{C}$. Using Lemma 8.4.4, this means that

$$\gamma = \lambda C_-^{-1} = \lambda C_- \qquad \text{if } \epsilon' = +1, \tag{8.127}$$

$$\gamma = \lambda\gamma_{\text{chiral}}C_-^{-1} = \lambda' C_+ \qquad \text{if } \epsilon' = -1, \tag{8.128}$$

where again $\lambda' \in \mathbb{C}$.

Using $\mathcal{J}^2 = \epsilon\mathbb{I}$ we can show that λ and λ' must be phases. Indeed, for $\epsilon' = +1$ we have

$$\mathcal{J}^2 = \lambda C_- K \lambda C_- K = |\lambda|^2 C_-\overline{C_-} = |\lambda|^2 (-1)^{\frac{d}{2}} (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} \mathbb{I} = \epsilon\mathbb{I}$$
$$\Rightarrow \quad |\lambda|^2 = (-1)^{\frac{d}{2}} (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} \epsilon, \tag{8.129}$$

where we used

$$\overline{C_-} = (-1)^{\frac{d}{2}} (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} C_- \tag{8.130}$$

that can be obtained by using C_- itself to conjugate the gammas that appear in the definition of C_- (there are $d/2$ gamma factors and $\lfloor d/4 \rfloor$ i factors). For $\epsilon' = -1$ we have

$$\mathcal{J}^2 = \lambda' C_+ K \lambda' C_+ K = |\lambda'|^2 C_+\overline{C_+} = |\lambda'|^2 (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} C_+^2 = |\lambda'|^2 e^{i\varphi}\mathbb{I} = \epsilon\mathbb{I}$$
$$\Rightarrow \quad |\lambda'|^2 = e^{i\varphi'}, \tag{8.131}$$

for some $\varphi, \varphi' \in \mathbb{R}$, where we used

$$\overline{C_+} = (-1)^{\left\lfloor \frac{d}{4} \right\rfloor} C_+ \tag{8.132}$$

that can be obtained by using C_+ itself to conjugate the gammas that appear in the definition of C_+ (there are $\lfloor d/4 \rfloor$ i factors), and

$$C_+^2 = e^{i\varphi}\mathbb{I} \tag{8.133}$$

that is obvious when one considers that C_+ is the product of gammas, that anticommute, and of i factors.

We can use a similar (and much simpler) procedure also for the grading:

$$\gamma_{\mathcal{M}}\partial\!\!\!/ = -\partial\!\!\!/\gamma_{\mathcal{M}} \quad \Rightarrow \quad -i\gamma_{\mathcal{M}}\gamma^a(\partial_a + \omega_a) = -(-i)\gamma^a(\partial_a + \omega_a)\gamma_{\mathcal{M}}. \tag{8.134}$$

Again, let us consider the terms with the partial derivative. We have

$$-i\gamma_{\mathcal{M}}\gamma^a\partial_a = -(-i)\gamma^a\partial_a\gamma_{\mathcal{M}}. \tag{8.135}$$

This implies

$$\gamma^a\gamma_{\mathcal{M}} = -\gamma_{\mathcal{M}}\gamma^a \qquad \forall a, \tag{8.136}$$

hence the grading $\gamma_{\mathcal{M}}$ anticommutes with all gammas, therefore it is proportional to γ_{chiral}. Moreover, it is selfadjoint and squares to the identity, so the proportionality factor must be ± 1.

Suppose now the manifold is odd-dimensional, and let us return to Eq. (8.123):

$$\pm (\gamma C_\pm)\, \gamma^a C_\pm^{-1} = -\epsilon' \gamma^a \gamma \qquad \forall a. \tag{8.137}$$

This time, only one of the two $C_\pm$ can be used. If we denote C the one of the two that must be used, we get that the overall sign s in front of the first member (using (8.108)) is $s = -\epsilon'$, so that the equation becomes

$$(\gamma C)\, \gamma^a C^{-1} = \gamma^a \gamma, \tag{8.138}$$

which then implies that (γC) commutes with all gammas and is therefore proportional to the identity:

$$(\gamma C) = \lambda \mathbb{I} \quad \Rightarrow \quad \gamma = \lambda C^{-1}. \tag{8.139}$$

Remembering that $C_-^{-1} = C_-$ and that $C_+^{-1} = e^{i\varphi} C_+$, meaning that $C^{-1} = e^{i\varphi'} C$ (for some suitable $\varphi, \varphi' \in \mathbb{R}$), we can reabsorb the phase into λ and write

$$\gamma = \lambda C. \tag{8.140}$$

Once again, using that $C\overline{C} = e^{i\varphi''}\mathbb{I}$ and imposing that $\mathcal{J}^2 = \epsilon \mathbb{I}$, we can show that λ is a phase, hence the result. □

Corollary 8.4.3.1. *For a twisting operator* (8.38) $T = \mathcal{T} \otimes T_F$*, one has*

$$\mathcal{T} = \pm \gamma_{\mathcal{M}}. \tag{8.141}$$

Moreover, T *satisfies the order zero condition if and only if*

$$\left[T_F, J_F T J_F^{-1}\right] = 0 \quad \text{and} \quad \left[\pi_F(m), J_F T_F J_F^{-1}\right] = 0 \qquad \forall m \in \mathcal{A}_F. \tag{8.142}$$

Proof. $\mathcal{T}$ is a grading because of Proposition 8.4.1, hence it follows from Theorem 8.4.3 that $\mathcal{T} = \pm\gamma_{\mathcal{M}}$. By definition of grading and real structure, $\mathcal{J}\gamma_{\mathcal{M}} = \pm\gamma_{\mathcal{M}}\mathcal{J}$, hence $\mathcal{T}\mathcal{J} = \pm\mathcal{J}\mathcal{T}$. Then, from Proposition 8.4.2 follows the result. □

In light of Corollary 8.4.3.1, from here on we will assume that T is of the form[3]

$$T = \gamma_{\mathcal{M}} \otimes T_F. \tag{8.143}$$

This structure is quite sensible from the physical point of view: the low-energy limit of a quantum theory should be Newton's physic, and by translating this claim into geometrical language, the low-energy limit of an almost-commutative geometry should be a manifold.

In the next chapter, we will briefly review the non-twisted Connes model, and then will apply all this machinery of twisted spectral triples to the Standard Model.

[3] Actually, T is not necessarily of the form (8.38), but could be (the closure of) a sum of such operators. However, such operators already pave the way to interesting physical applications beyond the Standard Model, as we will see in the next Chapter 9.

Chapter 9

Twist by Grading of the Standard Model

In this Chapter, we will apply the twist by grading procedure described in the last Chapter to the Connes model. As we will see, the twist by grading will not be enough to generate the extra scalar field σ we need to cure the problems of the Connes model, but nevertheless it leads to some very interesting results, i.e. the arising of a new axial vector field X_μ that somehow will allow us to write a fermionic action *in lorentzian signature*.

In section 9.1 we will review the technical details of the Connes model (i.e. the spectral triple and the representation of the algebra). In sections 9.2-9.4 we will adapt the algebra, its representation, the grading and the real structure to the twisted case. In sections 9.5-9.7 we evaluate the fluctuation of the Dirac operator and we identify the physical degrees of freedom. In section 9.8 we find the gauge transformations of the bosons (both scalars and vectors), and finally in section 9.9 we evaluate the fermionic action.

9.1 The non-twisted case: Connes model

9.1.1 The spectral triple of the Standard Model

The usual spectral triple of the Standard Model [90] is the product of the canonical triple of a (closed) riemannian spin manifold $\mathcal{M}$ of even dimension m,

$$C^\infty(\mathcal{M}), \quad L^2(\mathcal{M}, S), \quad \partial\!\!\!/ \tag{9.1}$$

with the finite dimensional spectral triple (called *internal*)

$$\mathcal{A}_{\rm SM} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{32\,n}, \quad D_F \tag{9.2}$$

that describes the gauge degrees of freedom of the Standard Model. In (9.1), $C^\infty(\mathcal{M})$ denotes the algebra of smooth functions on $\mathcal{M}$, that acts by multiplication on the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable spinors as

$$(f\psi)(x) = f(x)\psi(x) \quad \forall f \in C^\infty(\mathcal{M}),\, \psi \in L^2(\mathcal{M}, S),\, x \in \mathcal{M}, \tag{9.3}$$

while

$$\partial\!\!\!/ = -i\gamma^\mu \nabla_\mu \quad \text{with} \quad \nabla_\mu = \partial_\mu + \omega_\mu \tag{9.4}$$

is the Dirac operator on $L^2(\mathcal{M}, S)$ associated with the spin connection ω_μ and the γ^μs are the Dirac matrices associated with the riemannian metric g on $\mathcal{M}$:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I} \quad \forall \mu, \nu = 0, m-1 \tag{9.5}$$

($\mathbb{I}$ is the identity operator on $L^2(\mathcal{M}, S)$ and we label the coordinates of $\mathcal{M}$ from 0 to $m-1$).

In (9.2), n is the number of generations of fermions, and D_F is a $32\,n$ square complex matrix whose entries are the Yukawa couplings of fermions and the coefficients of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix of quarks and of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix of neutrinos. Details are given in section 9.1.3, and the representation of $\mathcal{A}_{\text{SM}}$ on $\mathcal{H}_F$ is in section 9.1.2.

The product spectral triple is

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad \mathcal{D} = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_\mathcal{M} \otimes D_F \tag{9.6}$$

where $\gamma_\mathcal{M}$ is the product of the euclidean Dirac matrices (see appendix B) and $\mathbb{I}_F$ the identity operator on $\mathcal{H}_F$.

The spectral triple (9.1) is graded with grading $\gamma_\mathcal{M}$. The internal spectral triple (9.2) is graded, with grading the operator γ_F on $\mathcal{H}_F$ that takes value $+1$ on right particles or left antiparticles, and -1 on left particles or right antiparticles. The product spectral triple (9.6) is graded, with grading

$$\Gamma = \gamma_\mathcal{M} \otimes \gamma_F. \tag{9.7}$$

The spectral triple (9.1) is also real with real structure $\mathcal{J}$ given by the charge conjugation operator. In dimension $m = 4$, it satisfies

$$\mathcal{J}^2 = -\mathbb{I}, \quad \mathcal{J}\partial\!\!\!/ = \partial\!\!\!/\mathcal{J}, \quad \mathcal{J}\gamma_\mathcal{M} = \gamma_\mathcal{M}\mathcal{J}. \tag{9.8}$$

The real structure of the internal spectral triple (9.2) is the anti-linear operator J_F that exchanges particles with antiparticles on $\mathcal{H}_F$. It satisfies

$$J_F^2 = \mathbb{I}, \quad J_F D_F = D_F J_F, \quad J\gamma_F = -\gamma_F J_F. \tag{9.9}$$

The real structure for the product spectral triple (9.6) is

$$J = \mathcal{J} \otimes J_F. \tag{9.10}$$

For a manifold of dimension $m = 4$, it is such that

$$J^2 = -\mathbb{I}, \quad J\mathcal{D} = \mathcal{D}J, \quad J\Gamma = -\Gamma J. \tag{9.11}$$

9.1.2 Representation of the algebra

To describe the action of $\mathcal{A}_{\text{SM}} \otimes C^\infty(\mathcal{M})$ on $\mathcal{H}$ in (9.6), it is convenient to label the $32n$ degrees of freedom of the finite dimensional Hilbert space $\mathcal{H}_F$ by a multi-index $C\, I\, \alpha$ defined as follows.

- $C = 0, 1$ is for particle ($C = 0$) or anti-particle ($C = 1$);
- $I = 0; i$ with $i = 1, 2, 3$ is the lepto-colour index: $I = 0$ means lepton, while $I = 1, 2, 3$ are for the quark, which exists in three colors;
- $\alpha = \dot{1}, \dot{2}; a$ with $a = 1, 2$ is the flavour index:

$$\dot{1} = \nu_R,\ \dot{2} = e_R,\ 1 = \nu_L,\ 2 = e_L \qquad \text{for leptons } (I = 0), \tag{9.12}$$

$$\dot{1} = u_R,\ \dot{2} = d_R,\ 1 = q_L,\ 2 = d_L \qquad \text{for quarks } (I = i). \tag{9.13}$$

We sometimes use the shorthand notation $\ell_L^a = (\nu_L, e_L)$, $q_L^a = (u_L, d_L)$.

There are $2 \times 4 \times 4 = 32$ choices of triplet of indices (C, I, α), which is the number of fermions per generation. One should also take into account an extra index $n = 1, 2, 3$ for the generations, but for now we will work with one generation only and we will omit it. So from now on

$$\mathcal{H}_F = \mathbb{C}^{32}. \tag{9.14}$$

An element $\psi \in \mathcal{H} = C^\infty(\mathcal{M}) \otimes \mathcal{H}_F$ is thus a 32 dimensional column-vector, in which each component $\psi_{CI\alpha}$ is a Dirac spinor in $L^2(\mathcal{M}, S)$.

Regarding the algebra, unless necessary we omit the symbol of the representation and identify an element $a = (c, q, m)$ in $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}$, where

$$c \in C^\infty(\mathcal{M}, \mathbb{C}), \quad q \in C^\infty(\mathcal{M}, \mathbb{H}), \quad m \in C^\infty(\mathcal{M}, M_3(\mathbb{C})), \tag{9.15}$$

with its representation as bounded operator on $\mathcal{H}$, that is a 32 square matrix whose components[1]

$$a_{CI\alpha}^{DJ\beta} \tag{9.16}$$

[1] D, J, β are column indices with the same range as the line indices C, I, α (the position of the indices was slightly different in [87], the one adopted here makes the tensorial computation more tractable).

are smooth functions acting by multiplication on $L^2(\mathcal{M}, S)$ as in (9.3). Explicitly[2]

$$a = \begin{pmatrix} Q & \\ & M \end{pmatrix}_C^D \tag{9.17}$$

where the 16×16 square matrices Q, M have components

$$Q_{I\alpha}^{J\beta} = \delta_I^J Q_\alpha^\beta, \quad M_{I\alpha}^{J\beta} = \delta_\alpha^\beta M_I^J, \tag{9.18}$$

where

$$Q_\alpha^\beta = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & q \end{pmatrix}_\alpha^\beta, \quad M_I^J = \begin{pmatrix} c & \\ & m \end{pmatrix}_I^J. \tag{9.19}$$

Here, the overbar $\bar{\cdot}$ denotes the complex conjugate, m (evaluated at the point x) identifies with its usual representation as a 3×3 complex matrix and the quaternion q (evaluated at x) acts through its representation as a 2×2 matrix:

$$\mathbb{H} \ni q(x) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}. \tag{9.20}$$

9.1.3 Finite dimensional Dirac operator, grading and real structure

With respect to the particle/antiparticle index C, the internal Dirac operator

$$D_F = D_Y + D_M \tag{9.21}$$

decomposes into a diagonal and an off-diagonal part

$$D_Y = \begin{pmatrix} D_0 & \\ & D_0^\dagger \end{pmatrix}_C^D, \quad D_M = \begin{pmatrix} 0 & D_R \\ D_R^\dagger & 0 \end{pmatrix}_C^D \tag{9.22}$$

containing respectively the Yukawa couplings of fermions and the Majorana mass of the neutrino.

More explicitly, the 16×16 matrices D_0 and D_R are block-diagonal with respect to the lepto-colour index I

$$D_0 = \begin{pmatrix} D_0^\ell & & & \\ & D_0^q & & \\ & & D_0^q & \\ & & & D_0^q \end{pmatrix}_I^J, \quad D_R = \begin{pmatrix} D_R^\ell & & & \\ & 0_4 & & \\ & & 0_4 & \\ & & & 0_4 \end{pmatrix}_I^J, \tag{9.23}$$

[2] The indices after the closing parenthesis are here to recall that the block-entries of $\mathcal{A}$ are labelled by the C, D indices, that is $a_1^1 = Q, a_2^2 = M, a_1^2 = a_2^1 = 0$.

where we write ℓ for $I = 0$ and q for $I = 1, 2, 3$. Each D_0^I is a 4×4 matrix (in the flavor index α),

$$D_0^I = \begin{pmatrix} 0 & \overline{\mathsf{k}^I} \\ \mathsf{k}^I & 0 \end{pmatrix}_\alpha^\beta \quad \text{where} \quad \mathsf{k}^I := \begin{pmatrix} k_u^I & 0 \\ 0 & k_d^I \end{pmatrix}_a^b, \tag{9.24}$$

whose entries are the Yukawa couplings of the elementary fermions

$$k_u^I = \begin{pmatrix} k_\nu, & k_u, & k_u, & k_u \end{pmatrix} \qquad k_d^I = \begin{pmatrix} k_e, & k_d, & k_d, & k_d \end{pmatrix} \tag{9.25}$$

(three of them are equal because the Yukawa coupling of quarks does not depend on the colour). Similarly, D_R^ℓ is a 4×4 matrix (in the flavour index),

$$D_R^\ell = \begin{pmatrix} k_R & \\ & 0_3 \end{pmatrix}_\alpha^\beta \tag{9.26}$$

whose only non-zero entry is the Majorana mass of the neutrino.

In tensorial notations, one has

$$D_R = k_R \, \Xi_{I\alpha}^{J\beta} \tag{9.27}$$

where

$$\Xi_\alpha^\beta := \begin{pmatrix} 1 & \\ & 0_3 \end{pmatrix}_\alpha^\beta, \quad \Xi_I^J := \begin{pmatrix} 1 & \\ & 0_3 \end{pmatrix}_I^J \tag{9.28}$$

and $\Xi_{I\alpha}^{J\beta}$ is a shorthand notation for the tensor $\Xi_I^J \Xi_\alpha^\beta$. Similarly, the internal grading is

$$\gamma_F = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix} = \eta_{C\alpha}^{D\beta} \delta_I^J \tag{9.29}$$

where the blocks in the matrix act respectively on right/left particles, then right/left antiparticles, and we define

$$\eta_\alpha^\beta := \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \quad \eta_C^D := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_C^D \tag{9.30}$$

and $\eta_{C\alpha}^{D\beta}$ holds for $\eta_C^D \eta_\alpha^\beta$. The internal real structure is

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_C^D cc = \xi_C^D \, \delta_{I\alpha}^{J\beta} cc \tag{9.31}$$

where cc denotes the complex conjugation and we define

$$\xi_C^D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_C^D. \tag{9.32}$$

9.2 Twist by Grading of the Standard Model

As we mentioned in section 7.4.3, in the noncommutative geometry description of the Standard Model, the bosonic degrees of freedom are obtained by a so-called fluctuation of the metric, that is the substitution of the operator $\mathcal{D}$ with $\mathcal{D} + \mathbb{A} + \varepsilon' J \mathbb{A} J^{-1}$ where

$$\mathbb{A} = \sum_i a_i[\mathcal{D}, b_i] \qquad a_i, b_i \in \mathcal{A} \tag{9.33}$$

is a generalised 1-form (see [84] for details and the justification of the terminology).

As already noticed in [90, 56], the Majorana mass of the neutrino does not contribute to the bosonic content of the model, for D_M commute with algebra:

$$[\gamma^5 \otimes D_M, a] = 0 \quad \forall a \in \mathcal{A}. \tag{9.34}$$

However, in order to generate the σ field proposed in [91] to cure the electroweak vacuum instability and solve the problem of the computation of the Higgs mass, one precisely needs to make D_M contribute to the fluctuation.

To do this, one possibility consists in substituting the commutator $[\mathcal{D}, a]$ with the twisted commutator (8.1) for a fixed automorphism ρ of $\mathcal{A}$. As shown in [88], starting with a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is almost commutative as in (9.6), then the only way to build a twisted spectral triple with the same Hilbert space and Dirac operator (which, from a physics point of view, means that one looks for models with the same fermionic content as the Standard Model) is to double the algebra and make it act independently on the left and right components of spinors (following actually an idea of [92]). As we will see, the twist by grading is not enough to generate the field σ, so we will have to somewhat modify it, which we will do in Chapter 10; in this Chapter, we will study the easier, "canonical" twist by grading in order to introduce the differences with the Connes Model in a simpler context.

9.3 Algebra and Hilbert space

The algebra $\mathcal{A}$ of the twisted spectral triple of the Standard Model is twice the algebra (9.6),

$$\mathcal{A} = (C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}) \otimes \mathbb{C}^2, \tag{9.35}$$

which is isomorphic to

$$(C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}) \oplus (C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}). \tag{9.36}$$

It acts on the same Hilbert space $\mathcal{H}$ as in the non-twisted case, but now the two copies of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}$ act independently on the right and left components of spinors. To write this action, it is convenient to view an element of $\mathcal{H}$ as a column vector with $4 \times 32 = 128$ components (4 being the number of components of a usual spinor in $L^2(\mathcal{M}, S)$ for $m = 4$). To this aim, one introduce two extra indices to label the degrees of freedom of $L^2(\mathcal{M}, S)$:

- $s = r, l$ is the chirality index;
- $\dot{s} = \dot{0}, \dot{1}$ denotes particle ($\dot{0}$) or anti-particle part ($\dot{1}$).

An element a of (9.36) is a pair of elements of (9.6), namely

$$a = (c, c', \, q, q', \, m, m') \tag{9.37}$$

with

$$c, c' \in C^\infty(\mathcal{M}, \mathbb{C}) \quad q, q' \in C^\infty(\mathcal{M}, \mathbb{H}), \quad m, m' \in C^\infty(\mathcal{M}, M_3(\mathbb{C})). \tag{9.38}$$

Following the "twist by grading" procedure described in section 8.4.1, (c, q, m) acts on the $+1$ eigenspace $\mathcal{H}_+$ of the grading Γ, whereas (c', q', m') acts on the -1 eigenspace $\mathcal{H}_-$. The eigenspace $\mathcal{H}_+$ is the subspace of $\mathcal{H}$ corresponding to the indices $r, \alpha = \dot{1}, \dot{2}$ and $l, \alpha = 1, 2$, while $\mathcal{H}_-$ is spanned by $l, \alpha = 1, 2$ and $r, \alpha = \dot{1}, \dot{2}$. In other terms, $a \in \mathcal{A}$ acts as in (9.17), but now the two 64×64 matrices Q, M are tensor fields of components

$$Q_{\dot{s}sI\alpha}^{\dot{t}tJ\beta} = \delta_{\dot{s}I}^{\dot{t}J} \, Q_{s\alpha}^{t\beta}, \qquad M_{\dot{s}sI\alpha}^{\dot{t}tJ\beta} = \delta_{\dot{s}}^{\dot{t}} \, M_{s\alpha I}^{t\beta J} \tag{9.39}$$

where δ_{sI}^{tJ} denotes the product of the two Kronecker symbols δ_s^t, δ_I^J. In other terms, both Q and M act trivially (i.e. as the identity) on the indices $\dot{s}\dot{t}$, but no longer on the chiral indices st. On the latter, the action is given by

$$Q_{s\alpha}^{t\beta} = \begin{pmatrix} (Q_r)_\alpha^\beta & \\ & (Q_l)_\alpha^\beta \end{pmatrix}_s^t, \qquad M_{s\alpha I}^{t\beta J} = \begin{pmatrix} (M_r)_{\alpha I}^{\beta J} & \\ & (M_l)_{\alpha I}^{\beta J} \end{pmatrix}_s^t, \tag{9.40}$$

with

$$Q_r = \begin{pmatrix} \mathsf{c} & \\ & q' \end{pmatrix}_\alpha^\beta, \quad Q_l = \begin{pmatrix} \mathsf{c}' & \\ & q \end{pmatrix}_\alpha^\beta, \tag{9.41}$$

and

$$M_r = \begin{pmatrix} \mathsf{m}' \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m} \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \; M_l = \begin{pmatrix} \mathsf{m} \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}' \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \tag{9.42}$$

where we denote

$$\mathsf{c} := \begin{pmatrix} c & \\ & \bar{c} \end{pmatrix}, \quad \mathsf{m} := \begin{pmatrix} c & \\ & m \end{pmatrix}_I^J, \mathsf{c}' := \begin{pmatrix} c' & \\ & \bar{c}' \end{pmatrix}, \quad \mathsf{m}' := \begin{pmatrix} c' & \\ & m' \end{pmatrix}_I^J, \tag{9.43}$$

Compared to the usual spectral triple of the Standard Model, $M_{r/l}$ are no longer trivial in the flavour index α.

The twist ρ is the automorphism of $\mathcal{A}$ that exchanges the two components of $\mathcal{A}_{\mathrm{SM}}$, namely

$$\rho(c, c', q, q', m, m') = (c', c, q', q, m', m). \tag{9.44}$$

In terms of the representation, one has

$$\rho(a) = \begin{pmatrix} \rho(Q) & \\ & \rho(M) \end{pmatrix}_C^D \tag{9.45}$$

with

$$\rho(Q)_{s\dot{s}I\alpha}^{t\dot{t}J\beta} = \delta_{\dot{s}I}^{\dot{t}J}\, \rho(Q)_{s\alpha}^{t\beta}, \qquad \rho(M)_{s\dot{s}I\alpha}^{t\dot{t}J\beta} = \delta_{\dot{s}}^{\dot{t}}\, \rho(M)_{s\alpha I}^{t\beta J} \tag{9.46}$$

where

$$\rho(Q)_{s\alpha}^{t\beta} = \begin{pmatrix} (Q_l)_\alpha^\beta & \\ & (Q_r)_\alpha^\beta \end{pmatrix}_s^t, \qquad \rho(M)_{s\alpha I}^{t\beta J} = \begin{pmatrix} (M_l)_{\alpha I}^{\beta J} & \\ & (M_r)_{\alpha I}^{\beta J} \end{pmatrix}_s^t. \tag{9.47}$$

In short, the twist amounts to flipping the left/right indices l/r.

9.4 Grading and real structure

The operators Γ in (9.7) and J in (9.10) are the grading and the real structure for the twisted spectral triple. In particular, as in the non twisted case, the real structure implements an action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}$, that commutes with the one of $\mathcal{A}$. This follows from the general construction of the twisting by grading, yet it is useful for the following to check it explicitly. Let us begin by writing down the representation of the opposite algebra.

Proposition 9.4.1. *For $a \in \mathcal{A}$ as in (9.17), one has (for $\mathcal{M}$ of dimension 4)*

$$JaJ^{-1} = -\begin{pmatrix} \bar{M} & 0 \\ 0 & \bar{Q} \end{pmatrix}_C^D. \tag{9.48}$$

Proof. From (9.10) and (9.31) one has

$$J = \begin{pmatrix} 0 & \mathcal{J} \otimes \mathbb{I}_{16} \\ \mathcal{J} \otimes \mathbb{I}_{16} & 0 \end{pmatrix}_C^D. \tag{9.49}$$

Since $J^{-1} = -J$ by (9.11), using the representation (9.17) of a one obtains (omitting $\mathbb{I}_{16}$)

$$JaJ^{-1} = -JaJ = -\begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}_C^E \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_E^F \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}_F^D = -\begin{pmatrix} \mathcal{J}M\mathcal{J} & 0 \\ 0 & \mathcal{J}Q\mathcal{J} \end{pmatrix}_C^D . \tag{9.50}$$

In addition, $\mathcal{J}$ commutes with the grading $\gamma_{\mathcal{M}}$ (see (9.8)), so it is of the form

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_r & 0 \\ 0 & \mathcal{J}_l \end{pmatrix}_s^t K \tag{9.51}$$

where K denotes complex conjugation and $\mathcal{J}_{r/l}$ are 2×2 matrices carrying the $\dot{s}, \dot{t}$ indices, such that $\mathcal{J}_r\bar{\mathcal{J}}_r = \mathcal{J}_l\bar{\mathcal{J}}_l = -\mathbb{I}_2$. From the explicit form (9.39) of Q and M, one gets (still omitting the indices α, I in which J is trivial)

$$\mathcal{J}Q\mathcal{J} = \begin{pmatrix} \mathcal{J}_r(\delta_{\dot{s}}^{\dot{t}}\bar{Q}_r)\bar{\mathcal{J}}_r & 0 \\ 0 & \mathcal{J}_l(\delta_{\dot{s}}^{\dot{t}}\bar{Q}_l)\bar{\mathcal{J}}_l \end{pmatrix}_s^t = \begin{pmatrix} -\delta_{\dot{s}}^{\dot{t}}\bar{Q}_r & 0 \\ 0 & -\delta_{\dot{s}}^{\dot{t}}\bar{Q}_l \end{pmatrix}_s^t = -\bar{Q}, \tag{9.52}$$

$$\mathcal{J}M\mathcal{J} = \begin{pmatrix} \mathcal{J}_r(\delta_{\dot{s}}^{\dot{t}}\bar{M}_r)\bar{\mathcal{J}}_r & 0 \\ 0 & \mathcal{J}_l(\delta_{\dot{s}}^{\dot{t}}\bar{M}_l)\bar{\mathcal{J}}_l \end{pmatrix}_s^t = \begin{pmatrix} -\delta_{\dot{s}}^{\dot{t}}\bar{M}_r & 0 \\ 0 & -\delta_{\dot{s}}^{\dot{t}}\bar{M}_l \end{pmatrix}_s^t = -\bar{M}, \tag{9.53}$$

hence the result. □

To check the order zero condition, we denote

$$b = (d, d', p, p', n, n') \tag{9.54}$$

another element of $\mathcal{A}$ with $d, d' \in C^\infty(\mathcal{M}, \mathbb{C})$, $p, p' \in C^\infty(\mathcal{M}, \mathbb{H})$ and $n, n' \in C^\infty(\mathcal{M}, M_3(\mathbb{C}))$. It acts on $\mathcal{H}$ by (9.55) as

$$b = \begin{pmatrix} R & \\ & N \end{pmatrix}_C^D \tag{9.55}$$

where R, N are defined as Q, M in (9.39), with

$$R_r = \begin{pmatrix} \mathsf{d} & \\ & p' \end{pmatrix}_\alpha^\beta, \qquad N_r = \begin{pmatrix} \mathsf{n}' \otimes \mathbb{I}_2 & \\ & \mathsf{n} \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta,$$
$$R_l = \begin{pmatrix} \mathsf{d}' & \\ & p \end{pmatrix}_\alpha^\beta, \qquad N_l = \begin{pmatrix} \mathsf{n} \otimes \mathbb{I}_2 & \\ & \mathsf{n}' \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta . \tag{9.56}$$

Corollary 9.4.1.1. *The order-zero condition (8.4) holds.*

Proof. By Prop. 9.4.1, the order zero condition $[a, JbJ^{-1}] = 0$ for all $a, b \in \mathcal{A}$ is equivalent to $[R, \bar{M}] = 0$ and $[N, \bar{Q}] = 0$. By (9.39) and (9.40), one gets

$$[R, M] = \begin{pmatrix} [\delta_I^J R_r, M_r] & 0 \\ 0 & [\delta_I^J R_l, M_l]. \end{pmatrix}_s^t . \tag{9.57}$$

By (9.42), one has

$$[\delta_I^J R_r, M_r] = \begin{pmatrix} [\delta_I^J \mathsf{d}, \mathsf{m}' \otimes \mathbb{I}_2] & 0 \\ 0 & [\delta_I^J p', \mathsf{m} \otimes \mathbb{I}_2] \end{pmatrix}_\alpha^\beta , \tag{9.58}$$

which is zero, as can be seen writing $\delta_I^J \mathsf{d} = \mathbb{I}_2 \otimes \mathsf{d}$ and similarly for $[\delta_I^J p', \mathsf{m} \otimes \mathbb{I}_2]$. The same holds true for $[\delta_I^J R_l, M_l]$. □

9.5 Twisted fluctuation

In the twisted context, fluctuations are similar to (9.33), replacing the commutator for a twisted one [93]. So we consider the *twisted-covariant* Dirac operator

$$\mathcal{D}_{\mathbb{A}_\rho} = \mathcal{D} + \mathbb{A}_\rho + J\mathbb{A}_\rho J^{-1} \tag{9.59}$$

where

$$\mathbb{A}_\rho = \sum_i a_i \, [\mathcal{D}, b_i]_\rho \, , \qquad a_i, b_i \in \mathcal{A} \tag{9.60}$$

is a twisted (generalised) 1-form. The latter decomposes as the sum $\mathbb{A}_\rho = \mathbb{A}_F + \not{\mathbb{A}}$ of two pieces: one that we call the *finite part* of the fluctuation because it comes from the finite dimensional spectral triple, namely

$$\mathbb{A}_F = \sum_i a_i \, [\gamma_\mathcal{M} \otimes D_F, b_i]_\rho \quad a_i, b_i \in \mathcal{A}; \tag{9.61}$$

another one coming from the manifold part of the spectral triple

$$\not{\mathbb{A}} = \sum_i a_i \left[\not{\partial} \otimes \mathbb{I}_F, b_i\right]_\rho \quad a_i, b_i \in \mathcal{A} \tag{9.62}$$

that we call *gauge part* in the following (terminology will become clear later).

To guarantee that the twisted covariant operator (9.59) is selfadjoint, one assumes that the 1-form $\mathbb{A}_\rho$ is selfadjoint (actually this is not a necessary condition, but requiring $\mathbb{A}_\rho$ to be selfadjoint makes sense viewing the fluctuation as a two step process: $\mathcal{D} \to \mathcal{D} + \mathbb{A}_\rho$ then $\mathcal{D} + \mathbb{A}_\rho \to \mathcal{D} + \mathbb{A}_\rho + J\mathbb{A}_\rho J^{-1}$). This means that for physical models, we assume that both $\not{\mathbb{A}}$ and $\mathbb{A}_F$ are selfadjoint.

From now on, we fix the dimension of $\mathcal{M}$ to $m = 4$. The grading and the real structure are (identifying a tensor with its components)

$$\gamma_{\mathcal{M}} = \gamma^5 = \gamma_E^0\gamma_E^1\gamma_E^2\gamma_E^3 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}_s^t = \eta_s^t \delta_{\dot{s}}^{\dot{t}} \tag{9.63}$$

and

$$\mathcal{J} = i\gamma_E^0\,\gamma_E^2\,K = i\begin{pmatrix} \tilde{\sigma}^2 & 0_2 \\ 0_2 & \sigma^2 \end{pmatrix}_{st} K = -i\eta_s^t\tau_{\dot{s}}^{\dot{t}}\,cc, \tag{9.64}$$

where K denotes the complex conjugation and we define

$$\tau_{\dot{s}}^{\dot{t}} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}}^{\dot{t}}, \quad \eta_s^t := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_s^t. \tag{9.65}$$

For the internal spectral triple, one has

$$\gamma_F = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix} = \eta_{C\alpha}^{D\beta}\delta_I^J, \qquad J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_C^D K = \xi_C^D\,\delta_{I\alpha}^{J\beta} \tag{9.66}$$

where the matrix γ_F is written in the basis left/right particles then left/right antiparticles, and we define

$$\eta_\alpha^\beta := \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \quad \eta_C^D := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_C^D, \quad \xi_C^D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_C^D \tag{9.67}$$

with $\eta_{C\alpha}^{D\beta}$ holding for $\eta_C^D\eta_\alpha^\beta$. Thus

$$\Gamma = \gamma_{\mathcal{M}} \otimes \gamma_F = \eta_{sC\alpha}^{tD\beta}\delta_{\dot{s}}^{\dot{t}} \quad \text{and} \quad J = J_{\mathcal{M}} \otimes J_F = -i\eta_s^t\,\tau_{\dot{s}}^{\dot{t}}\,\xi_D^C\,\delta_{I\alpha}^{J\beta}\,K. \tag{9.68}$$

9.6 Scalar part of the twisted fluctuation

The scalar sector of the twisted Standard Model is obtained from the finite part (9.61) of the twisted fluctuation, which in turns decomposes into a diagonal part (determined by the Yukawa couplings of fermions)

$$A_Y = \sum_i a_i\left[\gamma^5 \otimes D_Y, b_i\right]_\rho, \tag{9.69}$$

and an off-diagonal part (determined by the Majorana mass of the neutrino)

$$A_M = \sum_i a_i\left[\gamma^5 \otimes D_M, b_i\right]_\rho. \tag{9.70}$$

As shown below, the former produces the Higgs sector, while the latter is zero for the twist by grading.

9.6.1 The Higgs sector

We begin with the diagonal part (9.69). We first notice that the $M_3(\mathbb{C})$ part of the algebra (9.35) twist-commutes with $\gamma^5 \otimes D_Y$.

Lemma 9.6.1. *For any $b \in \mathcal{A}$ as in (9.55), one has*

$$\left[\gamma^5 \otimes D_Y, b\right]_\rho = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}_C^D \tag{9.71}$$

where S has components

$$S^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \delta^{\dot{t}}_{\dot{s}} \left(\eta^u_s (D_0)^{J\gamma}_{I\alpha} R^{t\beta}_{u\gamma} - \rho(R)^{u\gamma}_{s\alpha}\, \eta^t_u (D_0)^{J\beta}_{I\gamma} \right). \tag{9.72}$$

Proof. From the explicit forms (9.22) of D_Y and (9.55) of b, one has

$$\left[\gamma^5 \otimes D_Y, b\right]_\rho = \begin{pmatrix} [\gamma^5 \otimes D_0, R]_\rho & \\ & \left[\gamma^5 \otimes D_0^\dagger, N\right]_\rho \end{pmatrix}_{CD}.$$

In the tensorial notation, $S := [\gamma^5 \otimes D_0, R]_\rho$ has components

$$S^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \eta^u_s \delta^{\dot{u}}_{\dot{s}} (D_0)^{K\gamma}_{I\alpha}\, \delta^{\dot{t}J}_{\dot{u}K} R^{t\beta}_{u\gamma} - \delta^{\dot{u}K}_{\dot{s}I}\, \rho(R)^{u\gamma}_{s\alpha}\, \eta^t_u \delta^{\dot{t}}_{\dot{u}} (D_0)^{J\beta}_{K\gamma} = \tag{9.73}$$

$$= \delta^{\dot{t}}_{\dot{s}} \left(\eta^u_s (D_0)^{J\gamma}_{I\alpha} R^{t\beta}_{u\gamma} - \rho(R)^{u\gamma}_{s\alpha}\, \eta^t_u (D_0)^{J\beta}_{I\gamma} \right), \tag{9.74}$$

which shows (9.72). To show that

$$[\gamma^5 \otimes D_0^\dagger, N]_\rho = 0, \tag{9.75}$$

let us denote T the left-hand side of the equation above. It has components

$$T^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \eta^u_s \delta^{\dot{u}}_{\dot{s}} (D_0^\dagger)^{K\gamma}_{I\alpha}\, \delta^{\dot{t}}_{\dot{u}} N^{t\beta J}_{u\gamma K} - \delta^{\dot{u}}_{\dot{s}}\, \rho(N)^{u\gamma K}_{s\alpha I}\, \eta^t_u \delta^{\dot{t}}_{\dot{u}} (D_0^\dagger)^{J\beta}_{K\gamma} = \tag{9.76}$$

$$= \delta^t_{\dot{s}} \left(\eta^u_s (D_0^\dagger)^{K\gamma}_{I\alpha} N^{t\beta J}_{u\gamma K} - \rho(N)^{u\gamma K}_{s\alpha I}\, \eta^t_u (D_0^\dagger)^{J\beta}_{K\gamma} \right) = \tag{9.77}$$

$$= \delta^t_{\dot{s}} \begin{pmatrix} (D_0^\dagger)^{K\gamma}_{I\alpha} (N_r)^{\beta J}_{\gamma K} - (N_l)^{\gamma K}_{\alpha I} (D_0^\dagger)^{J\gamma}_{K\alpha} & 0 \\ 0 & -(D_0^\dagger)^{K\gamma}_{I\alpha} (N_l)^{\beta J}_{\gamma K} + (N_r)^{\gamma K}_{\alpha I} (D_0^\dagger)^{J\beta}_{K\gamma} \end{pmatrix} \tag{9.78}$$

Since $(D_0^\dagger)^K_I = \delta^K_I (D_0^I)$ and $(D_0^\dagger)^J_K = \delta^K_J (D_0^J)$ (with no summation on I and J), the upper-left term in (9.78) is

$$\left(D_0^I\right)^\delta_\gamma (N_r)^{\beta J}_{\delta I} - (N_l)^{\gamma J}_{\alpha I} \left(D_0^J\right)^\beta_\gamma =$$

$$= \begin{pmatrix} 0 & \bar{\mathsf{k}}^I(\mathsf{n}\otimes\mathbb{I}_2) \\ \mathsf{k}^I(\mathsf{n}'\otimes\mathbb{I}_2) & 0 \end{pmatrix}_\alpha^\beta - \begin{pmatrix} 0 & (\mathsf{n}\otimes\mathbb{I}_2)\bar{\mathsf{k}}^J \\ (\mathsf{n}'\otimes\mathbb{I}_2)\mathsf{k}^J & 0 \end{pmatrix}_\alpha^\beta \tag{9.79}$$

where we omitted the I, J indices on n. One has

$$\mathsf{k}^I(\mathsf{n}\otimes\mathbb{I}_2) = \begin{pmatrix} k_u^I\mathsf{n} & \\ & k_d^I\mathsf{n} \end{pmatrix}, \quad (\mathsf{n}\otimes\mathbb{I}_2)\mathsf{k}^J = \begin{pmatrix} \mathsf{n}k_u^J & \\ & \mathsf{n}k_d^J \end{pmatrix}, \tag{9.80}$$

and similarly for the terms in n'. Restoring the indices, one has

$$k_u^I n_I^J = \begin{pmatrix} k_u^l d & \\ & k_u^q n \end{pmatrix}_I^J, \quad n_I^J k_u^J = \begin{pmatrix} dk_u^l & \\ & nk_u^q \end{pmatrix}_I^J \tag{9.81}$$

where we write $k_u^{I=0} = k_u^l$ for the lepton, and $k_u^{I=1,2,3} = k_u^q$ for the coloured quarks. Again, in the expression above, there is no summation on I and J: $k_u^I n_I^J$ means the matrix n in which the I^{th} line is multiplied by k_u^I, while in $n_I^J k_u^J$ this is the J^{th} column of n which is multiplied by k_u^J. Therefore

$$\mathsf{k}^I(\mathsf{n}\otimes\mathbb{I}_2) - (\mathsf{n}\otimes\mathbb{I}_2)\mathsf{k}^J = 0. \tag{9.82}$$

One shows in a similar way that $\bar{\mathsf{k}}^I(\mathsf{n}'\otimes\mathbb{I}_2) - (\mathsf{n}'\otimes\mathbb{I}_2)\bar{\mathsf{k}}^J = 0$, so that (9.79) vanishes. Thus the upper-right term in (9.78) is zero. The proof that the lower-right term is zero is similar. Hence (9.75) and the result. □

We now compute the 1-forms generated by the Yukawa couplings of the fermions. In order to do so, we extend the action of the automorphism ρ to any polynomial in q, q', p, p', c, c', d, d'. Namely ρ "primes" what is un-primed, and vice-versa. For instance $\rho(qp' - c'd) = q'p - cd'$.

Proposition 9.6.1. *The diagonal part (9.69) of a twisted 1-form is*

$$A_Y = \begin{pmatrix} A & \\ & 0 \end{pmatrix}_C^D \quad \textit{where} \quad A = \delta_{\dot{s}I}^{tJ} \begin{pmatrix} A_r & \\ & A_l \end{pmatrix}_s^t \tag{9.83}$$

with

$$A_r = \begin{pmatrix} & \bar{\mathsf{k}}^I H_1 \\ H_2\mathsf{k}^I & \end{pmatrix}_\alpha^\beta, \quad A_l = -\begin{pmatrix} & \bar{\mathsf{k}}^I H_1' \\ H_2'\mathsf{k}^I & \end{pmatrix}_\alpha^\beta, \tag{9.84}$$

where $H_{i=1,2}$ and $H_{1,2}' = \rho(H_{1,2})$ are quaternionic fields.

Proof. From (9.17) and lemma 9.6.1, one has $A = Q\,S$. In components, this gives (using the explicit forms (9.39) of Q, R and omitting the summation index i):

$$A_{s\dot{s}I\alpha}^{t\dot{t}J\beta} = Q_{s\dot{s}I\alpha}^{u\dot{u}K\gamma}\delta_{\dot{u}}^{\dot{t}} \left[\eta_u^v\,(D_0)_{K\gamma}^{J\delta}\,R_{v\delta}^{t\beta} - \rho(R)_{u\gamma}^{v\delta}\,\eta_v^t\,(D_0)_{K\delta}^{J\beta}\right] = \tag{9.85}$$

$$= \delta^{tJ}_{\dot{s}I} Q^{u\gamma}_{s\alpha} \left[\eta^v_u \, \left(D^I_0\right)^\delta_\gamma \, R^{t\beta}_{v\delta} - \rho(R)^{v\delta}_{u\gamma} \, \eta^t_v \, \left(D^I_0\right)^\beta_\delta \right], \tag{9.86}$$

where we use $\delta^K_I (D_0)^K_J = \delta^J_I (D^I_0)$ (with no summation on I in the last expression). Since Q is diagonal on the chiral indices s, the only non-zero components of A are for $s = t = r$ and $s = t = l$, namely

$$A^{riJ\beta}_{r\dot{s}I\alpha} = \delta^{iJ}_{\dot{s}I} \left(A^I_r\right)^\beta_\alpha \quad \text{with} \quad \left(A^I_r\right)^\beta_\alpha = (Q_r)^\gamma_\alpha \left[\left(D^I_0\right)^\delta_\gamma (R_r)^\beta_\delta - (R_l)^\delta_\gamma \left(D^I_0\right)^\beta_\delta \right], \tag{9.87}$$

$$A^{liJ\beta}_{l\dot{s}I\alpha} = \delta^{iJ}_{\dot{s}I} \left(A^I_l\right)^\beta_\alpha \quad \text{with} \quad \left(A^I_l\right)^\beta_\alpha = (Q_l)^\gamma_\alpha \left[- \left(D^I_0\right)^\delta_\gamma (R_l)^\beta_\delta + (R_r)^\delta_\gamma \left(D^I_0\right)^\beta_\delta \right]. \tag{9.88}$$

From the explicit expression (9.41), (9.56), (9.24) of $Q_{r/l}$, $R_{r/l}$ and D^I_0 one gets

$$Q_r D^I_0 R_r = \begin{pmatrix} & \mathsf{c}\bar{\mathsf{k}}^I p' \\ q'\mathsf{k}^I \mathsf{d} & \end{pmatrix}^\beta_\alpha, \; Q_r R_l D^I_0 = \begin{pmatrix} & \mathsf{c}\mathsf{d}'\bar{\mathsf{k}}^I \\ q' p \mathsf{k}^I & \end{pmatrix}^\beta_\alpha, \tag{9.89}$$

$$Q_l D^I_0 R_l = \begin{pmatrix} & \mathsf{c}'\bar{\mathsf{k}}^I p \\ q\mathsf{k}^I \mathsf{d}' & \end{pmatrix}^\beta_\alpha, \; Q_l R_r D^I_0 = \begin{pmatrix} & \mathsf{c}'\mathsf{d}\bar{\mathsf{k}}^I \\ q p' \mathsf{k}^I & \end{pmatrix}^\beta_\alpha, \tag{9.90}$$

Using that $\mathsf{c}, \mathsf{c}', \mathsf{d}, \mathsf{d}'$ commute with k^I, one has

$$Q_r (D^I_0 R_r - R_l D^I_0) = \begin{pmatrix} & \bar{\mathsf{k}}^I H_1 \\ H_2 \mathsf{k}^I & \end{pmatrix}^\beta_\alpha, \quad Q_l \left(R_r D^I_0 - D^I_0 R_l\right) = - \begin{pmatrix} & \bar{\mathsf{k}}^I H'_1 \\ H'_2 \mathsf{k}^I & \end{pmatrix}^\beta_\alpha \tag{9.91}$$

where

$$H_1 := \mathsf{c}(p' - \mathsf{d}'), \quad H_2 := q'(\mathsf{d} - p), \quad H'_1 := \mathsf{c}'(p - \mathsf{d}), \quad H'_2 := q(\mathsf{d}' - p'). \tag{9.92}$$

This shows the result. □

Imposing now the selfadjointness condition as stressed at the beginning of this section, we get the

Corollary 9.6.1.1. *A selfadjoint diagonal twisted* 1*-form* (9.69) *is parametrized by two independent scalar quaternionic field* H_r, H_l.

Proof. The twisted 1-form (9.83) is selfadjoint if and only if

$$H_2 = H_1^\dagger =: H_r \quad \text{and} \quad H'_2 = H'^\dagger_1 =: H_l. \tag{9.93}$$

They are independent as follows from their definition (9.92). □

Gathering the previous results, one works out the fields induced by the Yukawa coupling of fermions via a twisted fluctuation of the metric.

Proposition 9.6.2. *A selfadjoint diagonal fluctuation is*

$$D_{A_Y} = \gamma^5 \otimes D_Y + A_Y + JA_YJ^{-1} = \begin{pmatrix} \eta^t_s \delta^{\dot t}_{\dot s} D_0 + A & \\ & \eta^t_s \delta^{\dot t}_{\dot s} D_0^\dagger + \bar{A} \end{pmatrix}^D_C \tag{9.94}$$

where $A = \delta^{\dot t J}_{\dot s I} \begin{pmatrix} A_r & \\ & A_l \end{pmatrix}^t_s$ *is parametrized by two quaternionic fields* H_r, H_l *as*

$$A_r = \begin{pmatrix} & \bar{\mathsf{k}}^I H_r^\dagger \\ H_r \mathsf{k}^I & \end{pmatrix}^\beta_\alpha, \quad A_l = \begin{pmatrix} & \bar{\mathsf{k}}^I H_l^\dagger \\ H_l \mathsf{k}^I & \end{pmatrix}^\beta_\alpha. \tag{9.95}$$

Proof. Remembering that $J^{-1} = -J$, proposition 9.6.1 yields

$$JA_YJ^{-1} = \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}^D_C \begin{pmatrix} A & \\ & 0 \end{pmatrix}^D_C \begin{pmatrix} 0 & -\mathcal{J} \\ -\mathcal{J} & 0 \end{pmatrix}^D_C = \begin{pmatrix} 0 & \\ & -\mathcal{J}A\mathcal{J}^{-1} \end{pmatrix}^D_C. \tag{9.96}$$

From the explicit form (9.31) of $\mathcal{J}$ and (9.83) of A, one obtains (omitting the IJ and $\alpha\beta$ indices in which the real structure J is trivial)

$$\mathcal{J}A\mathcal{J}^{-1} = \eta^u_s \tau^{\dot u}_{\dot s} \, \bar{A}^{v\dot v}_{u\dot u} \, \eta^t_v \tau^{\dot t}_{\dot v} = \eta^u_s \tau^{\dot u}_{\dot s} \begin{pmatrix} \bar{A}_r & 0 \\ 0 & \bar{A}_l \end{pmatrix}^t_s \eta^t_v \tau^{\dot t}_{\dot v} = \begin{pmatrix} -\bar{A}_r & 0 \\ 0 & -\bar{A}_l \end{pmatrix}^t_s = -\bar{A}, \tag{9.97}$$

where we used (9.84) to write $\tau^{\dot u}_{\dot s} \delta^{\dot v}_{\dot u} \tau^{\dot t}_{\dot v} = -\delta^{\dot t}_{\dot s}$.

The result follows summing (9.96) with A_Y given in Prop. 9.6.1 and D_Y given in (9.22), then using corollary 9.6.1.1 to rename H_r and H_l. □

In the non-twisted case, the primed and unprimed quantities are equal, so that one obtains only one quaternionic field

$$H := H_r + H_l = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix}, \tag{9.98}$$

whose complex components ϕ_1, ϕ_2 identify – in the action – with the Higgs doublet. In the twisted case, we obtain two scalar doublets

$$\Phi_r := \begin{pmatrix} \phi_1^r \\ \phi_2^r \end{pmatrix}, \; \Phi_l := \begin{pmatrix} \phi_1^l \\ \phi_2^l \end{pmatrix}, \tag{9.99}$$

(where $\phi^r_{1,2}$, $\phi^l_{1,2}$ are the complex components of H_r, H_l), which couple to the right and on the left part of the Dirac spinors respectively. However, as we will see in section 9.9, the two of them have no individual physical meaning on their own. In fact, they only appear in the fermionic action through the linear combination $h = (H_r + H_l)/2$, therefore there is actually only one physical Higgs doublet in the twisted case as well, that couples to both left and right-handed fermions.

9.6.2 The Off-diagonal Fluctuation

The off-diagonal part (9.70) of the twisted fluctuation is zero for the twist by grading. Indeed, one has the following

Proposition 9.6.3. *Given a spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *with grading* Γ*, if a component* D *of the Dirac operator*

$$\mathcal{D} = D + \textit{other terms} \tag{9.100}$$

commutes with the algebra $\mathcal{A}$*, then it also twist-commutes with the algebra* $\mathcal{A} \otimes \mathbb{C}^2$ *of the corresponding twisted-by-grading spectral triple.*

Proof. We have $[D, a] = 0$ by hypothesis, and $D\Gamma = -\Gamma D$ by definition of Dirac operator. Then, we have

$$2\left[D, \pi\left(a, a'\right)\right]_\rho = \left[D, (1+\Gamma)\, a + (1-\Gamma)\, a'\right]_\rho = \tag{9.101}$$

$$= D\left(a + a'\right) + D\Gamma\left(a - a'\right) - \left(a + a'\right) D + \Gamma\left(a - a'\right) D = \tag{9.102}$$

$$= \left[D, a + a'\right] - \Gamma D\left(a - a'\right) + \Gamma\left(a - a'\right) D = \tag{9.103}$$

$$= \left[D, a + a'\right] - \Gamma\left[D, a - a'\right] = 0. \tag{9.104}$$

□

Corollary 9.6.3.1. *In the twist by grading of the Connes model, the off-diagonal part* (9.70) *of the twisted fluctuation is zero.*

Proof. In the Connes model, one has

$$\left[\gamma^5 \otimes D_M, a\right] = 0. \tag{9.105}$$

Then, the result follows from Proposition 9.6.3. □

As we will see in section 10, twists different from the twist by grading can make the off-diagonal part (9.70) of the twisted fluctuation different from zero, thus generating a new scalar field σ.

9.7 Gauge Part of the Twisted Fluctuation

Here, we compute the twisted fluctuation induced by the free part $\not{D} = \not{\partial} \otimes \mathbb{I}_F$ of the Dirac operator (9.6), that is

$$\not{D} + \not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} \tag{9.106}$$

where $\not{\mathbb{A}}$ is the twisted 1-form (9.62) induced by $\not{D}$, that we call in the following a *free* 1-*form*. As will be checked in section 9.8, the components of this form are the gauge fields of the model.

9.7.1 Dirac matrices and twist

We begin by recalling some useful relations between the Dirac matrices and the twist.

Lemma 9.7.1. *If an operator* $\mathcal{O}$ *on* $L^2(\mathcal{M}, S)$ *twist-commutes with the Dirac matrices,*

$$\gamma^\mu \mathcal{O} = \rho(\mathcal{O})\gamma^\mu \quad \forall \mu, \tag{9.107}$$

and commutes the spin connection ω_μ*, then*

$$\left[\partial\!\!\!/, \mathcal{O}\right]_\rho = -i\gamma^\mu \partial_\mu \mathcal{O}. \tag{9.108}$$

Proof. One has

$$[\gamma^\mu \nabla_\mu, \mathcal{O}]_\rho = [\gamma^\mu \partial_\mu, \mathcal{O}]_\rho + [\gamma^\mu \omega_\mu, \mathcal{O}]_\rho. \tag{9.109}$$

On the one side, the Leibniz rule satisfied by the differential operator ∂_μ together with (9.107) yields

$$\begin{aligned}[\gamma^\mu \partial_\mu, \mathcal{O}]_\rho \psi &= \gamma^\mu \partial_\mu \mathcal{O}\psi - \rho(\mathcal{O})\gamma^\mu \partial_\mu \psi = \\ &= \gamma^\mu(\partial_\mu \mathcal{O})\psi + \gamma^\mu \mathcal{O}\partial_\mu \psi - \rho(\mathcal{O})\gamma^\mu \partial_\mu \psi = \gamma^\mu(\partial_\mu \mathcal{O})\psi. \end{aligned} \tag{9.110}$$

On the other side, by (9.107),

$$[\gamma^\mu \omega_\mu, \mathcal{O}]_\rho = \gamma^\mu \omega_\mu \mathcal{O} - \rho(\mathcal{O})\gamma^\mu \omega_\mu = \gamma^\mu[\omega^\mu, \mathcal{O}] \tag{9.111}$$

vanishes by hypothesis. Hence the result. □

This lemma applies in particular to the components Q and M of the representation of the algebra $\mathcal{A}$ in (9.17). The slight difference is that these components do not act on $L^2(\mathcal{M}, S)$, but on $L^2(\mathcal{M}, S) \otimes \mathbb{C}^{32}$. With a slight abuse of notation, we write

$$\gamma^\mu Q := (\gamma^\mu \otimes \mathbb{I}_{16})\, Q\ ,\ \partial_\mu Q := (\partial_\mu \otimes \mathbb{I}_{16}) Q \tag{9.112}$$

and similarly for M.

Corollary 9.7.0.1. *One has*

$$\gamma^\mu Q = \rho(Q)\gamma^\mu, \quad [\partial\!\!\!/, Q]_\rho = -i\gamma^\mu \partial_\mu Q, \tag{9.113}$$

$$\gamma^\mu M = \rho(M)\gamma^\mu, \quad [\partial\!\!\!/, M]_\rho = -i\gamma^\mu \partial_\mu Q. \tag{9.114}$$

Proof. From (9.40) and omitting the internal indices (on which the action of $\gamma^\mu \otimes \mathbb{I}_{16}$ is trivial), one checks from the explicit form (B.2) of the euclidean Dirac matrices that

$$\gamma_E^\mu Q - \rho(Q)\gamma_E^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}_s^t \begin{pmatrix} Q_r & 0 \\ 0 & Q_l \end{pmatrix}_s^t - \begin{pmatrix} Q_l & 0 \\ 0 & Q_r \end{pmatrix}_s^t \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}_s^t = 0. \tag{9.115}$$

The same holds true for the curved Dirac matrices (B.4), by linear combination.

The commutation with the spin connection follows remembering that the latter is

$$\omega_\mu = \Gamma_\mu^{\rho\nu}\gamma_\rho\gamma_\nu = \Gamma_\mu^{\rho\nu}\begin{pmatrix}\sigma_\mu\tilde{\sigma}_\nu & 0 \\ 0 & \tilde{\sigma}_\mu\sigma_\nu\end{pmatrix}_s^t \tag{9.116}$$

and so commutes with Q, which is diagonal in the s, t indices an trivial in the $\dot{s}, \dot{t}$ indices. □

9.7.2 Free 1-form

With the previous results, it is not difficult to compute a free 1-form (9.62).

Lemma 9.7.2. *A free* 1*-form is*

$$\not{\mathbb{A}} = -i\gamma^\mu A_\mu \quad \textit{with} \quad A_\mu = \begin{pmatrix} Q_\mu & 0 \\ 0 & M_\mu \end{pmatrix}_C^D, \tag{9.117}$$

where we use notations similar to (9.112)*, with*

$$Q_\mu := \sum_i \rho(Q_i)\partial_\mu R_i, \quad M_\mu = \sum_i \rho(M_i)\partial_\mu N_i \tag{9.118}$$

for Q_i, M_i and R_i, N_i the components of a_i, b_i as in (9.17, 9.55).

Proof. Omitting the summation index i, one has

$$\begin{aligned}
\not{\mathbb{A}} = a\left[\not{D}, b\right]_\rho &= \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_C^D \begin{pmatrix} [\not{\partial}, R]_\rho & 0 \\ 0 & [\not{\partial}, N]_\rho \end{pmatrix}_C^D = \\
&= -i\begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_C^D \begin{pmatrix} \gamma^\mu\partial_\mu R & 0 \\ 0 & \gamma^\mu\partial_\mu N \end{pmatrix}_C^D = \\
&= -i\gamma^\mu \begin{pmatrix} \rho(Q)\,\partial_\mu R & 0 \\ 0 & \rho(M)\,\partial_\mu N \end{pmatrix}_C^D,
\end{aligned} \tag{9.119}$$

where the last equality follows from corollary 9.7.0.1. Restoring the index i, one gets the result. □

By computing explicitly the components of $\not{\mathbb{A}}$, one finds that a free 1-form is parametrized by two complex fields c_μ^r, c_μ^l, two quaternionic fields q_μ^r, q_μ^l and two $M_3(\mathbb{C})$-valued fields m_μ^r, m_μ^l.

Proposition 9.7.1. *The components* Q_μ, M_μ *of* $\not{\mathbb{A}}$ *in* (9.117) *are*

$$Q_\mu = \delta^{tJ}_{\dot{s}I}\begin{pmatrix} Q^r_\mu & \\ & Q^l_\mu \end{pmatrix}^t_s, \quad M_\mu = \delta^{t}_{\dot{s}}\begin{pmatrix} M^r_\mu & \\ & M^l_\mu \end{pmatrix}^t_s \tag{9.120}$$

where

$$Q^r_\mu = \begin{pmatrix} \mathsf{c}^r_\mu & \\ & q^r_\mu \end{pmatrix}^\beta_\alpha, \quad Q^l_\mu = \begin{pmatrix} \mathsf{c}^l_\mu & \\ & q^l_\mu \end{pmatrix}^\beta_\alpha \tag{9.121}$$

with $\mathsf{c}^r_\mu = \begin{pmatrix} c^r_\mu & \\ & \bar{c}^r_\mu \end{pmatrix}$, $\mathsf{c}^l_\mu = \begin{pmatrix} c^l_\mu & \\ & \bar{c}^l_\mu \end{pmatrix}$ *and*

$$M^r_\mu = \begin{pmatrix} \mathsf{m}^r_\mu \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}^l_\mu \otimes \mathbb{I}_2 \end{pmatrix}^\beta_\alpha, \; M^l_\mu = \begin{pmatrix} \mathsf{m}^l_\mu \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}^r_\mu \otimes \mathbb{I}_2 \end{pmatrix}^\beta_\alpha \tag{9.122}$$

with $\mathsf{m}^r_\mu = \begin{pmatrix} c^r_\mu & \\ & m^r_\mu \end{pmatrix}^J_I$, $\mathsf{m}^l_\mu = \begin{pmatrix} c^l_\mu & \\ & m^l_\mu \end{pmatrix}^J_I$.

Proof. The form (9.120)-(9.121) of the components of $\not{\mathbb{A}}$ follows calculating explicitly (9.118) using (9.40)-(9.43) for Q_i, M_i and (9.56) for R_i, N_i. Omitting the i index, one finds

$$Q^r_\mu = Q_l\partial_\mu R_r, \quad Q^l_\mu = Q_r\partial_\mu R_l, \quad M^r_\mu = M_l\partial_\mu N_r, \quad M^l_\mu = M_r\partial_\mu N_l. \tag{9.123}$$

This yields (9.121) with

$$c^r_\mu = c'\partial_\mu d, \quad c^l_\mu = c\partial_\mu d', \quad q^r_\mu = q\partial_\mu p', \quad q^l_\mu = q'\partial_\mu p, \tag{9.124}$$

$$m^r_\mu = m\partial_\mu n', \quad m^l_\mu = m'\partial_\mu n \tag{9.125}$$

□

Corollary 9.7.1.1. *A free* 1*-form* $\not{\mathbb{A}}$ *is selfadjoint if and only if*

$$c^l_\mu = -\bar{c}^r_\mu, \quad q^l_\mu = -(q^r_\mu)^\dagger, \quad m^l_\mu = -(m^r_\mu)^\dagger. \tag{9.126}$$

Proof. One has

$$\not{\mathbb{A}}^\dagger = i(A^\mu)^\dagger\gamma^\mu = i\gamma^\mu\rho(A^\mu)^\dagger, \tag{9.127}$$

so $\not{\mathbb{A}}$ is selfadjoint if and only if $\gamma^\mu(\rho(A_\mu) + A^\dagger_\mu) = 0$. Since A_μ is diagonal the s, t indices, the sum $\Delta_\mu := \rho(A_\mu) + A^\dagger_\mu$ is also diagonal with components $\Delta^{r/l}_\mu$. Thus

$$\gamma^\mu\Delta_\mu = \begin{pmatrix} 0 & \sigma^\mu\Delta^l_\mu \\ \tilde{\sigma}^\mu\Delta^r_\mu & \end{pmatrix}^t_s. \tag{9.128}$$

If this is zero, then for any γ^ν

$$\gamma^\nu\gamma^\mu\Delta_\mu = \begin{pmatrix} \sigma^\nu\tilde{\sigma}^\mu\Delta^r_\mu & 0 \\ 0 & \tilde{\sigma}^\nu\sigma^\mu\Delta^l_\mu \end{pmatrix} = 0. \tag{9.129}$$

Being A_μ – hence Δ_μ – trivial in $\dot{s}, \dot{t}$, and since $\operatorname{Tr}\tilde{\sigma}^\mu\sigma^\nu = 2\delta_{\mu\nu}$, the partial trace on the $\dot{s}, \dot{t}$ indices of the expression above yields $\Delta^r_\mu = \Delta^l_\mu = 0$. Therefore $\gamma^\mu(\rho(A_\mu) + A^\dagger_\mu) = 0$ implies

$$\rho(A_\mu) = -A^\dagger_\mu. \tag{9.130}$$

The converse is obviously true. Consequently, $\not{A}$ is selfadjoint if and only if (9.130) holds true.

From (9.117), this is equivalent to $Q^\dagger_\mu = -\rho(Q_\mu)$ and $M^\dagger_\mu = -\rho(M_\mu)$ that is, from (9.120),

$$Q^l_\mu = -(Q^r_\mu)^\dagger \quad \text{ and } M^l_\mu = -(M^r_\mu)^\dagger. \tag{9.131}$$

This is equivalent to (9.126). □

9.7.3 Identification of the physical degrees of freedom

To identify the physical fields, one follows the non twisted case [90] and separates the real from the imaginary parts of the various fields. We thus define two real fields $a_\mu = \operatorname{Re} c^r_\mu$ and $B_\mu = -\frac{2}{g_1}\operatorname{Im} c^r_\mu$ (the signs are such to match the notations of [56], see remark 9.7.1), so that

$$c^r_\mu = a_\mu - i\frac{g_1}{2}B_\mu, \quad c^l_\mu = -\bar{c}^r_\mu = -a_\mu - i\frac{g_1}{2}B_\mu, \tag{9.132}$$

where, as we will see in section 9.9, B_μ corresponds to the gauge field of $U(1)_Y$. Moreover, we denote w_μ and $-\frac{g_2}{2}W^k$ for $k = 1, 2, 3$ the real components of the quaternionic field q^r_μ on the basis $\{\mathbb{I}_2, i\sigma_j\}$ of the (real) algebra of quaternions, so that

$$q^r_\mu = w_\mu\,\mathbb{I}_2 - i\frac{g_2}{2}W^k_\mu\sigma_k, \quad q^l_\mu = -(q^r_\mu)^\dagger = -w_\mu\,\mathbb{I}_2 - i\frac{g_2}{2}W^k_\mu\sigma_k, \tag{9.133}$$

where again, as we will see in section 9.9, W^k_μ are the gauge fields of $SU(2)_L$. Finally, we write m^r_μ as the sum of a selfadjoint part $g_\mu = \frac{1}{2}(m^r_\mu + m^{r\,\dagger}_\mu)$ and an antiselfadjoint part $\frac{1}{2}(m^r_\mu - m^{r\,\dagger}_\mu)$. We denote $V^0_\mu, \frac{g_3}{2}V^m_\mu$ the real-field components of the latter on the basis $\{i\mathbb{I}_3, i\lambda_m\}$ of the (real) vector space of antiselfadjoint 3×3 complex matrices (with $\{\lambda_m, m = 1\ldots 8\}$ the Gell-Mann matrices), so that

$$m^r_\mu = g_\mu + iV^0_\mu\,\mathbb{I}_3 + i\frac{g_3}{2}V^m_\mu\lambda_m, \tag{9.134}$$

$$m^l_\mu = -(m^r_\mu)^\dagger = -g_\mu + iV^0_\mu\,\mathbb{I}_3 + i\frac{g_3}{2}V^m_\mu\lambda_m, \tag{9.135}$$

where V_μ^m are the gluons (once again, as we will see in section 9.9).

The cancellation of anomalies is imposed requiring the the unimodularity condition

$$\operatorname{Tr} A_\mu = 0. \tag{9.136}$$

This yields the same condition as in the non-twisted case.

Proposition 9.7.2. *The unimodularity condition for a selfadjoint free 1-form yields*

$$V_\mu^0 = \frac{g_1}{6} B_\mu. \tag{9.137}$$

Proof. One has $\operatorname{Tr} A_\mu = \operatorname{Tr} Q_\mu + \operatorname{Tr} M_\mu$. On the one side

$$\operatorname{Tr} Q_\mu = \operatorname{Tr} Q_\mu^r + \operatorname{Tr} Q_\mu^l = c_\mu^r + \bar{c}_\mu^r + \operatorname{Tr} q_\mu^r + c_\mu^l + \bar{c}_\mu^l + \operatorname{Tr} q_\mu^l \tag{9.138}$$

vanishes by (9.126), when one notices that $\operatorname{Tr} q^\dagger = \operatorname{Tr} q$ for any quaternion q. On the other side

$$\begin{aligned}\operatorname{Tr} M_\mu = \operatorname{Tr} M_\mu^r + \operatorname{Tr} M_\mu^l = 4 \operatorname{Tr} \mathsf{m}_\mu^r + 4 \operatorname{Tr} \mathsf{m}_\mu^l = \\ = 4(c_\mu^r + \operatorname{Tr} m_\mu^r + c_\mu^l + \operatorname{Tr} m_\mu^l) = 4(-ig_1 B_\mu + 6iV_\mu^0)\end{aligned} \tag{9.139}$$

where we use $c_\mu^r + c_\mu^l = -ig_1 B_\mu$ and $m_\mu^r + m_\mu^l = 2iV_\mu^0 \mathbb{I}_3 + 2ig_3 V_\mu^m \lambda_m$, remembering then that the Gell-Mann matrices are traceless. Hence (9.136) is equivalent to (9.137). □

Let us summarise the results of this section in the following

Proposition 9.7.3. *A unimodular selfadjoint free 1-form $\not{A}$ is parametrized by*

- *two real 1-form fields a_μ, w_μ and a selfadjoint $M_3(\mathbb{C})$-valued field g_μ,*
- *a $\mathfrak{u}(1)$-value field iB_μ, a $\mathfrak{su}(2)$-valued field iW_μ and a $\mathfrak{su}(3)$-valued field iV_μ.*

Proof. Collecting the previous results, denoting $W_\mu := W_\mu^k \sigma_k$ and $V_\mu := V_\mu^m \lambda_m$, one has

$$c_\mu^r = a_\mu - i\frac{g_1}{2} B_\mu, \qquad c_\mu^l = -a_\mu - i\frac{g_1}{2} B_\mu, \tag{9.140}$$

$$q_\mu^r = w_\mu \mathbb{I}_2 - i\frac{g_2}{2} W_\mu, \qquad q_\mu^l = -w_\mu \mathbb{I}_2 - i\frac{g_2}{2} W_\mu, \tag{9.141}$$

$$m_\mu^r = g_\mu + i\left(\frac{g_1}{6} B_\mu \mathbb{I}_3 + \frac{g_3}{2} V_\mu\right), \qquad m_\mu^l = -g_\mu + i\left(\frac{g_1}{6} B_\mu \mathbb{I}_3 + \frac{g_3}{2} V_\mu\right). \tag{9.142}$$

On the one side, a_μ, w_μ are in $C^\infty(\mathcal{M}, \mathbb{R})$ and $g_\mu = {g_\mu}^\dagger$ is in $C^\infty(\mathcal{M}, M_3(\mathbb{C}))$. On the other side, since B_μ is real, $iB_\mu \in C^\infty(\mathcal{M}, i\mathbb{R})$ is a $\mathfrak{u}(1)$-value field.

The Pauli matrices span the space of traceless 2×2 selfadjoint matrices, thus the field iW_μ takes value in the set of antiselfadjoint such matrices, that is $\mathfrak{su}(2)$. Finally, the real span of the Gell-Mann matrices is the space of traceless selfadjoint elements of $M_3(\mathbb{C})$, hence V_μ is a $\mathfrak{su}(3)$-value field. □

In the non-twisted case, the primed and unprimed quantities in (9.124)-(9.125) are equal, meaning that the right and left components of the fields (9.140)-(9.142) are equal, hence

$$a_\mu = w_\mu = g_\mu = 0. \tag{9.143}$$

That the twisting produces some extra 1-form fields has already been pointed out for manifolds in [88], and for electrodynamic in [94]. Actually, such a field (improperly called vector field) appeared initially in the twisted version of the Standard Model presented in [87], but its precise structure – a collection of three selfadjoint field a_μ, w_μ, g_μ, each associated with a gauge field of the Standard Model – had not been worked out there.

Remark 9.7.1. In the non-twisted case, the fields B_μ, W_μ and V_μ coincide with those of the spectral triple of the Standard Model. More precisely, within the conditions of (9.143), then

- our $c^r_\mu = c^l_\mu$ coincides with $-i\Lambda_\mu$ of [56, §15.4][3] The selfadjointness condition (9.126) then implies that Λ_μ is real, in agreement with [56]. Then $B_\mu = \frac{2}{g_1}\Lambda_\mu$ as defined in [56, 1.729] coincides with our $B_\mu = -i\frac{2}{g_1}c^r_\mu = -i\frac{2}{g_1}c^l_\mu$ as defined in (9.132).

- our $q^r_\mu = q^l_\mu$ coincides with $-iQ_\mu$ of [56, §15.4]. The selfadjointness condition (9.126) then implies that Q_μ is selfadjoint, in agreement with [56]. Then $W_\mu = \frac{2}{g_2}Q_\mu$ as defined in [56, 1.739] coincides with our $W_\mu = W^k_\mu\sigma_k = i\frac{2}{g_2}q^r_\mu = i\frac{2}{g_2}q^l_\mu$ in (9.133).

- the identification of our V_μ with the one of the non-twisted case is made after proposition 9.7.4.

Remark 9.7.2. If one does not impose the selfadjointness of $\not{A}$, then one obtains two copies of the bosonic contents of the Standard Model, acting independently on the right and left components of Dirac spinors.

[3] Beware that $\not{\partial}_M$ in the formula of Λ is $i\gamma^\mu\partial_\mu$ [56, 1.580], so that $\Lambda = \Lambda_\mu\gamma^\mu$ is the $U(1)$ part of $-\not{A}$, meaning that Λ_μ is the $U(1)$ part of iA_μ.

9.7.4 Twisted fluctuation of the free Dirac operator

We now compute the free part (9.106) of the twisted fluctuation.

Proposition 9.7.4. *A twisted fluctuation of the free Dirac operator* $\not{D}$ *is* $D_Z = \not{D} + Z$ *where*

$$Z = \not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} = -i\gamma^\mu \begin{pmatrix} Z^\mu & 0 \\ 0 & \overline{Z^\mu} \end{pmatrix}_C^D \quad \textit{with} \quad Z_\mu = \gamma^5 \otimes X_\mu + \mathbb{I}_4 \otimes iY_\mu, \tag{9.144}$$

in which X_μ *and* Y_μ *are selfadjoint* $\mathcal{A}_{SM}$*-valued tensor fields on* $\mathcal{M}$ *with components*

$$(X_\mu)_{\dot{1}I}^{\dot{2}J} = (X_\mu)_{\dot{2}I}^{\dot{1}J} = (Y_\mu)_{\dot{1}I}^{\dot{2}J} = (Y_\mu)_{\dot{2}I}^{\dot{1}J} = 0, \tag{9.145}$$

and

$$(X_\mu)_{\dot{1}I}^{\dot{1}J} = (X_\mu)_{\dot{2}I}^{\dot{2}J} = \begin{pmatrix} 2a_\mu & \\ & a_\mu\mathbb{I}_3 + g_\mu \end{pmatrix}_I^J, \tag{9.146}$$

$$(Y_\mu)_{\dot{1}I}^{\dot{1}J} = \begin{pmatrix} 0 & \\ & -\frac{2g_1}{3}B_\mu\mathbb{I}_3 - \frac{g_3}{2}V_\mu \end{pmatrix}, \quad (Y_\mu)_{\dot{2}I}^{\dot{2}J} = \begin{pmatrix} g_1B_\mu & \\ & \frac{g_1}{3}B_\mu\mathbb{I}_3 - \frac{g_3}{2}V_\mu \end{pmatrix} \tag{9.147}$$

$$(X_\mu)_{aI}^{bJ} = \begin{pmatrix} \delta_a^b\,(w_\mu - a_\mu) & \\ & \delta_a^b w_\mu\mathbb{I}_3 - g_\mu \end{pmatrix}_I^J, \tag{9.148}$$

$$(Y_\mu)_{aI}^{bJ} = \begin{pmatrix} \delta_a^b\frac{g_1}{2}B_\mu - \frac{g_2}{2}(W_\mu)_a^b & \\ & -\delta_a^b\left(\frac{g_1}{6}B_\mu\mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) - \frac{g_2}{2}(W_\mu)_a^b\mathbb{I}_3 \end{pmatrix}_I^J. \tag{9.149}$$

Proof. With $J = -J^{-1}$ as defined in (9.49) one has

$$\begin{aligned} J\not{\mathbb{A}}J^{-1} &= -J(-i\gamma^\mu A_\mu)J^{-1} = -iJ\gamma^\mu A_\mu J^{-1} = i\gamma^\mu J A_\mu J^{-1} = \\ &= i\gamma^\mu \begin{pmatrix} \mathcal{J}M_\mu\mathcal{J}^{-1} & 0 \\ 0 & \mathcal{J}Q_\mu\mathcal{J}^{-1} \end{pmatrix}_C^D \end{aligned}$$

where we use that J is antilinear and anticommutes with γ^μ (lemma B.0.1). Noticing that $\mathcal{J}M_\mu\mathcal{J}^{-1} = -\bar{M}_\mu$ and $\mathcal{J}Q_\mu\mathcal{J}^{-1} = -\bar{Q}_\mu$ (this is shown as in (9.53), (9.52)), one obtains

$$Z_\mu = Q_\mu + \bar{M}_\mu. \tag{9.150}$$

Explicitly, $Z_\mu = \begin{pmatrix} Z_\mu^r & \\ & Z_\mu^l \end{pmatrix}$ where, using the explicit forms (9.122) and (9.121) of Q_μ^r and M_μ^r,

$$Z_\mu^r = Q_\mu^r + \overline{M_\mu^r} = \delta_{\dot{s}}^{\dot{t}} \begin{pmatrix} \mathsf{c}_\mu^r\delta_I^J + \delta_{\dot{a}}^{\dot{b}}\,\overline{\mathsf{m}_\mu^r} & \\ & q_\mu^r\delta_I^J + \delta_a^b\overline{\mathsf{m}_\mu^l} \end{pmatrix}_\alpha^\beta \tag{9.151}$$

and a similar expression for Z^l_μ, exchanging r with l. The components of the matrix above are

$$(Z^r_\mu)^{\dot b}_{\dot a} = \mathsf{c}^r_\mu \delta^J_I + \delta^{\dot b}_{\dot a}\overline{\mathsf{m}^r_\mu} = \begin{pmatrix} c^r_\mu \delta^J_I + \overline{\mathsf{m}^r_\mu} & \\ & \overline{c^r_\mu}\delta^J_I + \overline{\mathsf{m}^r_\mu} \end{pmatrix}^{\dot b}_{\dot a} \tag{9.152}$$

with $(Z^r_\mu)^{\dot 2}_{\dot 1} = (Z^r_\mu)^{\dot 1}_{\dot 2} = 0$,

$$(Z^r_\mu)^{\dot 1}_{\dot 1} = c^r_\mu \delta^J_I + \overline{\mathsf{m}^r_\mu} = \begin{pmatrix} 2a_\mu & \\ & (a_\mu - i\frac{g_1}{2}B_\mu)\mathbb{I}_3 + g_\mu - i\left(\frac{g_1}{6}B_\mu\mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \end{pmatrix}^J_I = \\ = (X^r_\mu)^{\dot 1}_{\dot 1} + i(Y^r_\mu)^{\dot 1}_{\dot 1} \tag{9.153}$$

$$(Z^r_\mu)^{\dot 2}_{\dot 2} = \overline{c^r_\mu}\delta^J_I + \overline{\mathsf{m}^r_\mu} = \\ = \begin{pmatrix} 2a_\mu + ig_1B_\mu & \\ & (a_\mu + i\frac{g_1}{2}B_\mu)\mathbb{I}_3 + g_\mu - i\left(\frac{g_1}{6}B_\mu\mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \end{pmatrix}^J_I = \\ = (X^r_\mu)^{\dot 2}_{\dot 2} + i(Y^r_\mu)^{\dot 2}_{\dot 2}; \tag{9.154}$$

and

$$(Z^r_\mu)^b_a = q^r_\mu\, \delta^J_I + \delta^b_a\overline{\mathsf{m}^l_\mu} = \begin{pmatrix} (q^r_\mu)^1_1\, \delta^J_I + \overline{\mathsf{m}^l_\mu} & (q^r_\mu)^2_1\, \delta^J_I \\ (q^r_\mu)^1_2\, \delta^J_I & \overline{q^r_{\mu}}^2_2\, \delta^J_I + \overline{\mathsf{m}^l_\mu} \end{pmatrix}^b_a \tag{9.155}$$

with

$$(Z^r_\mu)^a_a = (q^r_\mu)^a_a\, \delta^J_I + \overline{\mathsf{m}^l_\mu} = \\ = \begin{pmatrix} w_\mu - i\frac{g_2}{2}(W_\mu)^a_a - a_\mu + i\frac{g_1B_\mu}{2} & \\ & \left(w_\mu - i\frac{g_2}{2}(W_\mu)^a_a\right)\mathbb{I}_3 - g_\mu - i\left(\frac{g_1B_\mu}{6}\mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \end{pmatrix}^J_I = \\ = (X^r_\mu)^a_a + i(Y^r_\mu)^a_a, \tag{9.156}$$

$$(Z^r_\mu)^{b\neq a}_a = (q^r_\mu)^b_a\, \delta^J_I = \begin{pmatrix} -i\frac{g_2}{2}(W_\mu)^b_a & \\ & -i\frac{g_2}{2}(W_\mu)^b_a\mathbb{I}_3 \end{pmatrix}^J_I = (X^r_\mu)^b_a + i(Y^r_\mu)^b_a. \tag{9.157}$$

The matrices X^r_μ and Y^r_μ defined by the equations above are selfadjoint and such that $Z^r_\mu = X^r_\mu + iY^r_\mu$. By the selfadjointness condition, one has

$$Z^l_\mu = Q^l_\mu + \overline{M^l_\mu} = -(Z^r_\mu)^\dagger = -X^r_\mu + iY^r_\mu. \tag{9.158}$$

In other terms, $Z^l_\mu = X^l_\mu + iY^l_\mu$ with

$$X^l_\mu = -X^r_\mu, \quad X^l_\mu = -X^r_\mu. \tag{9.159}$$

Redefining $X_\mu := X^r_\mu = -X^l_\mu$, $Y_\mu := Y^r_\mu = Y^l_\mu$, one obtains the result. □

We collect the components of Z in appendix C: There, we also make explicit that iY_μ coincides exactly with the gauge fields of the Standard Model (including the $\mathfrak{su}(3)$ gauge field V_μ, that represents the gluons). Thus the twist does not modify the gauge content of the model. What it does is to add the selfadjoint part X_μ whose action on spinors breaks chirality. As shown in the next section, this field is invariant under a gauge transformation.

9.8 Gauge Transformations

A gauge transformation is implemented by an action of the group $\mathcal{U}(\mathcal{A})$ of unitary elements of $\mathcal{A}$, both on the Hilbert space and on the Dirac operator. On a twisted spectral triple, these actions have been worked out in [95, 93] and consist in a twist of the original formula of Connes [84]. Explicitly, on the Hilbert space, the fermion fields transform under the adjoint action of $\mathcal{U}(\mathcal{A})$ induced by the real structure, namely

$$\psi \to \operatorname{Ad} u\ \psi := u\psi u = uu^\circ\psi = uJu^*J^{-1}\psi, \quad u \in \mathcal{U}. \tag{9.160}$$

On the other hand, the twisted-covariant Dirac operator D_{A_ρ} (9.59) transforms under the twisted conjugate action of $\operatorname{Ad} u$,

$$D_{A_\rho} \to \operatorname{Ad} \rho(u)\, D_{A_\rho} \operatorname{Ad} u^*. \tag{9.161}$$

By the twisted first-order condition, (9.161) is equivalent to a gauge transformation of the sole fluctuation A_ρ, namely one has

$$\operatorname{Ad} \rho(u)\, D_{A_\rho} \operatorname{Ad} u^* = D + A^u_\rho + JA^u_\rho J^{-1} \tag{9.162}$$

where

$$A^u_\rho := \rho\,(u) \left([D, u^*]_\rho + A_\rho u^* \right). \tag{9.163}$$

This is the twisted version of the law of transformation of generalised 1-forms in ordinary spectral triples, which in turn is a non-commutative generalisation of the law of transformation of the gauge potential in ordinary gauge theories.

To write down the transformation $A_\rho \to A^u_\rho$ for the twisted fluctuation (9.60), we need the explicit form of a unitary u of $\mathcal{A}$. The latter is a pair of elements of[4]

$$\mathcal{U}(\mathbb{C}) \times \mathcal{U}(\mathbb{H}) \times \mathcal{U}(M_3(\mathbb{C})) \simeq U(1) \times SU(2) \times U(3). \tag{9.164}$$

[4]Notice that this is not the gauge group of the Standard Model. However, the Standard Model gauge group is immediately recovered after imposing the unimodularity condition, as we did in section 9.7.3. Indeed, by imposing $\operatorname{Tr} u = 0$, the group $U(1) \times SU(2) \times U(3)$ reduces to $U(1) \times SU(2) \times SU(3)$.

Namely

$$u = (e^{i\alpha}, e^{i\alpha'}, \mathsf{q}, \mathsf{q}', \mathsf{m}, \mathsf{m}') \tag{9.165}$$

with

$$\alpha, \alpha' \in C^\infty(\mathcal{M}, \mathbb{R}), \quad \mathsf{q}, \mathsf{q}' \in C^\infty(\mathcal{M}, SU(2)), \quad \mathsf{m}, \mathsf{m}' \in C^\infty(\mathcal{M}, U(3)). \tag{9.166}$$

It acts on $\mathcal{H}$ as

$$u = \begin{pmatrix} \mathfrak{A} & \\ & \mathfrak{B} \end{pmatrix}_C^D \tag{9.167}$$

where, following (9.39)-(9.43), one has $\mathfrak{A}_{\dot{s}sI\alpha}^{itJ\beta} = \delta_{\dot{s}I}^{iJ} \mathfrak{A}_{s\alpha}^{t\beta}$ and $\mathfrak{B}_{\dot{s}sI\alpha}^{itJ\beta} = \delta_{\dot{s}}^{i} \mathfrak{B}_{s\alpha I}^{t\beta J}$ with

$$\mathfrak{A}_{s\alpha}^{t\beta} = \begin{pmatrix} (\mathfrak{A}_r)_\alpha^\beta & \\ & (\mathfrak{A}_l)_\alpha^\beta \end{pmatrix}_s^t, \qquad \mathfrak{B}_{s\alpha I}^{t\beta J} = \begin{pmatrix} (\mathfrak{B}_r)_{\alpha I}^{\beta J} & \\ & (\mathfrak{B}_l)_{\alpha I}^{\beta J} \end{pmatrix}_s^t, \tag{9.168}$$

in which

$$\mathfrak{A}_r = \begin{pmatrix} \boldsymbol{\alpha} & \\ & \mathsf{q}' \end{pmatrix}_\alpha^\beta, \quad \mathfrak{A}_l = \begin{pmatrix} \boldsymbol{\alpha}' & \\ & \mathsf{q} \end{pmatrix}_\alpha^\beta, \tag{9.169}$$

and

$$\mathfrak{B}_r = \begin{pmatrix} \boldsymbol{m} \otimes \mathbb{I}_2 & 0 \\ 0 & \boldsymbol{m}' \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \; \mathfrak{B}_l = \begin{pmatrix} \boldsymbol{m}' \otimes \mathbb{I}_2 & 0 \\ 0 & \boldsymbol{m} \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \tag{9.170}$$

where we denote

$$\boldsymbol{\alpha} := \begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix}, \; \boldsymbol{m} := \begin{pmatrix} e^{i\alpha} & \\ & \mathsf{m} \end{pmatrix}_I^J, \; \boldsymbol{\alpha}' := \begin{pmatrix} e^{i\alpha'} & \\ & e^{-i\alpha'} \end{pmatrix}, \; \boldsymbol{m}' := \begin{pmatrix} e^{i\alpha'} & \\ & \mathsf{m}' \end{pmatrix}_I^J. \tag{9.171}$$

9.8.1 Gauge sector

Before beginning with the transformation of the free 1-form $\not{\mathbb{A}}$ computed in (9.117), let us recall that a twisted gauge transformation (9.161) does not necessarily preserve the selfadjointness of the Dirac operator. Namely A_ρ^u in (9.163) is not necessarily selfadjoint, even though one starts with a selfadjoint A_ρ.

This may seem as a weakness of the twisted case, since in the non-twisted case selfadjointness is necessarily preserved. Actually this possibility to lose selfadjointness allows to implement Lorentz symmetry and yields – at least for electrodynamics [94] – an interesting interpretation of the component X_μ of the free fluctuation Z of proposition 9.7.4 as a four-vector energy-impulsion.

However, regarding the gauge part of the Standard Model which – as shown below – is fully encoded in the component iY_μ of Z, it is rather natural to ask selfadjointness to be preserved. This reduces the choice of unitaries to pair of elements of (9.164) equal up to a constant.

Proposition 9.8.1. *A unitary u whose action (9.162) preserves the selfadjointness of any unimodular selfadjoint free 1-form $\not{\mathbb{A}}$ is given by (9.165) with*

$$\alpha' = \alpha + K, \quad \mathsf{q} = \mathsf{q}', \quad \mathsf{m}' = \mathsf{m}. \tag{9.172}$$

The components (9.117) of $\not{\mathbb{A}}$ then transform as

$$c_\mu^r \longrightarrow c_\mu^r - i\partial_\mu\alpha, \qquad c_\mu^l \longrightarrow c_\mu^l - i\partial_\mu\alpha, \tag{9.173}$$
$$q_\mu^r \longrightarrow \mathsf{q}\, q_\mu^r\, \mathsf{q}^\dagger + \mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right), \qquad q_\mu^l \longrightarrow \mathsf{q}\, q_\mu^l\, \mathsf{q}^\dagger + \mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right), \tag{9.174}$$
$$m_\mu^r \longrightarrow \mathsf{m}\, m_\mu^r \mathsf{m}^\dagger + \mathsf{m}\left(\partial_\mu \mathsf{m}^\dagger\right), \qquad m_\mu^l \longrightarrow \mathsf{m}\, m_\mu^l \mathsf{m}^\dagger + \mathsf{m}\left(\partial_\mu \mathsf{m}^\dagger\right). \tag{9.175}$$

Proof. From corollary 9.7.0.1 one has (with the same abuse of notations (9.112), now with $\mathbb{I}_{32}$)

$$\not{\mathbb{A}}^u = \rho(u)\left([\not{D}, u^*]_\rho + \not{\mathbb{A}} u^*\right) = -i\gamma^\mu \left(u\left(\partial_\mu u^*\right) + u A_\mu u^*\right). \tag{9.176}$$

Using the explicit forms (9.167) of u and (9.117) of A_μ, one finds

$$\not{\mathbb{A}}^u = -i\gamma^\mu \begin{pmatrix} \mathfrak{A}\left(\partial_\mu \mathfrak{A}^\dagger\right) + \mathfrak{A} Q_\mu \mathfrak{A}^\dagger & 0 \\ 0 & \mathfrak{B}\left(\partial_\mu \mathfrak{B}^\dagger\right) + \mathfrak{B} M_\mu\, \mathfrak{B}^\dagger \end{pmatrix}_C^D \tag{9.177}$$

meaning that a gauge transformation is equivalent to the transformation

$$Q_\mu \longrightarrow Q_\mu + \mathfrak{A}\left(\partial_\mu \mathfrak{A}^\dagger\right) + \mathfrak{A} Q_\mu \mathfrak{A}^\dagger, \quad M_\mu \longrightarrow M_\mu + \mathfrak{B}\left(\partial_\mu \mathfrak{B}^\dagger\right) + \mathfrak{B} M_\mu \mathfrak{B}^\dagger \tag{9.178}$$

From (9.121) and (9.122), these equations are equivalent to

$$c_\mu^r \longrightarrow c_\mu^r + e^{i\alpha}\partial_\mu e^{-i\alpha} = c_\mu^r - i\partial_\mu\alpha, \quad c_\mu^l \longrightarrow c_\mu^l - i\partial_\mu\alpha', \tag{9.179}$$
$$q_\mu^r \longrightarrow \mathsf{q}'\, q_\mu^r\, \mathsf{q}'^\dagger + \mathsf{q}'\left(\partial_\mu \mathsf{q}'^\dagger\right), \qquad q_\mu^l \longrightarrow \mathsf{q}\, q_\mu^l\, \mathsf{q}^\dagger + \mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right), \tag{9.180}$$
$$m_\mu^r \longrightarrow \mathsf{m}\, m_\mu^r \mathsf{m}^\dagger + \mathsf{m}\left(\partial_\mu \mathsf{m}^\dagger\right), \qquad m_\mu^l \longrightarrow \mathsf{m}' m_\mu^l \mathsf{m}'^\dagger + \mathsf{m}'\left(\partial_\mu \mathsf{m}'^\dagger\right). \tag{9.181}$$

For any unitary operator q, one has that $\mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right) = \mathsf{q}[\partial_\mu, \mathsf{q}^\dagger]$ is anti-hermitian (being ∂_μ antihermitian as well). Hence, beginning with a selfadjoint $\not{\mathbb{A}}$ as in (9.126), requiring that $\not{\mathbb{A}}^u$ be selfadjoint is equivalent to

$$\partial_\mu \alpha' = \partial_\mu \alpha, \tag{9.182}$$
$$\mathsf{q}\, q_\mu^l \mathsf{q}^\dagger + \mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right) = \mathsf{q}' q_\mu^l \mathsf{q}'^\dagger + \mathsf{q}'\left(\partial_\mu \mathsf{q}'^\dagger\right), \tag{9.183}$$
$$\mathsf{m}'\, m_\mu^r \mathsf{m}'^\dagger + \mathsf{m}'\left(\partial_\mu \mathsf{m}'^\dagger\right) = \mathsf{m}\, m_\mu^l \mathsf{m}^\dagger + \mathsf{m}\left(\partial_\mu \mathsf{m}^\dagger\right). \tag{9.184}$$

In particular, for q^l_μ equal to the identity the second of these equations yields $\mathsf{q}\left(\partial_\mu \mathsf{q}^\dagger\right) = \mathsf{q}'\left(\partial_\mu \mathsf{q}'^\dagger\right)$ for any q, q'. Hence for any q^l_μ one has $\mathsf{q}\, q^l_\mu \mathsf{q}^\dagger = \mathsf{q}' q^l_\mu \mathsf{q}'^\dagger$. This means that $\mathsf{q}'^\dagger \mathsf{q}$ is in the center of $\mathbb{H}$. Being a unitary, $\mathsf{q}'^\dagger \mathsf{q}$ is thus the identity. So $\mathsf{q} = \mathsf{q}'$. Similarly, one gets that $\mathsf{m}'^\dagger \mathsf{m}$ is in the center of $M_3(\mathbb{C})$, that is a multiple of the identity. Being unitary, $\mathsf{m}'^\dagger \mathsf{m}$ can only be the identity, hence $\mathsf{m}' = \mathsf{m}$. Thus (9.179-9.181) yield the result. □

These transformations of the components of the free 1-form induce the following transformations of the physical fields defined in (9.140)-(9.142).

Proposition 9.8.2. *Under a twisted gauge transformation that preserve selfadjointness, the physical fields a_μ and w_μ are invariant, g_μ undergoes an algebraic (i.e. non-differential) transformation*

$$g_\mu \longrightarrow \mathsf{n} g_\mu \mathsf{n}^\dagger \tag{9.185}$$

and the gauge fields transform as in the Standard Model

$$B_\mu \longrightarrow B_\mu + \frac{2}{g_1}\partial_\mu \alpha, \tag{9.186}$$

$$W_\mu \longrightarrow \mathsf{q} W_\mu \mathsf{q}^\dagger + \frac{2i}{g_2}\mathsf{q}\,(\partial_\mu \mathsf{q}^\dagger), \tag{9.187}$$

$$V_\mu \longrightarrow \mathsf{n} V_\mu \mathsf{n}^\dagger - \frac{2i}{g_3}\mathsf{n}\,(\partial_\mu \mathsf{n}^\dagger), \tag{9.188}$$

where $\mathsf{n} = (\det \mathsf{m})^{-\frac{1}{3}}\mathsf{m}$ is the $SU(3)$ part of m.

Proof. Applying the gauge transformations (9.173)-(9.175) to the physical fields defined through (9.140)-(9.142), one obtains

$$\pm\, a_\mu - i\frac{g_1}{2}B_\mu \to \pm a_\mu - i\left(\frac{g_1}{2}B_\mu + \partial_\mu \alpha\right), \tag{9.189}$$

$$\pm\, w_\mu \mathbb{I}_2 - i\frac{g_2}{2}W_\mu \to \pm w_\mu \mathbb{I}_2 - i\left(\frac{g_2}{2}\mathsf{q} W_\mu \mathsf{q}^\dagger + i\mathsf{q}(\partial_\mu \mathsf{q}^\dagger)\right), \tag{9.190}$$

$$\pm\, g_\mu + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \to \pm \mathsf{m} g_\mu \mathsf{m}^\dagger + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}\mathsf{m}\, V_\mu \mathsf{m}^\dagger - i\mathsf{m}(\partial \mathsf{m}^\dagger)\right), \tag{9.191}$$

where the anti-selfadjointness of $\mathsf{q}(\partial \mathsf{q}^\dagger)$ and $\mathsf{m}(\partial \mathsf{m}^\dagger)$ guarantee that the r.h.s. of (9.190) and (9.191) is split into a selfadjoint and anti-selfadjoint part. The first two equations above yield (9.186)-(9.187) Writing $\mathsf{m} = e^{i\theta}\mathsf{n}$ with $e^{i\theta} = (\det m)^{\frac{1}{3}}$ and $\mathsf{n} \in SU(3)$, then the right hand side of (9.191) becomes

$$\pm\, \mathsf{n} g_\mu \mathsf{n}^\dagger + i\left((\frac{g_1}{6}B_\mu - \partial_\mu \theta)\mathbb{I}_3 + \frac{g_3}{2}\mathsf{n} V_\mu \mathsf{n}^\dagger + \mathsf{n}\partial_\mu \mathsf{n}^\dagger\right). \tag{9.192}$$

where we use $\mathsf{m}\partial_\mu \mathsf{m}^\dagger = -i\partial_\mu\theta = \mathsf{n}\partial_\mu \mathsf{n}^\dagger$. Requiring the unimodularity condition to be gauge invariant forces to identify $-\theta$ with $\frac{\alpha}{3}$, thus reducing the gauge group $U(3)$ to $SU(3)$. This yields (9.185) and (9.188). □

9.8.2 Scalar sector

We now study the gauge transformation of the scalar part of the twisted fluctuation $A_Y + A_M$ computed in section 9.6, beginning with the Yukawa part A_Y in (9.69).

Lemma 9.8.1. *Let u be a unitary of $\mathcal{A}$ as in (9.165). One has*

$$\rho(u)\left[\gamma^5 \otimes D_Y, u^\dagger\right]_\rho + \rho(u) A_Y u^\dagger = \begin{pmatrix} A^u & \\ & 0 \end{pmatrix}_C^D \tag{9.193}$$

where

$$A^u = \delta_{\dot{s}I}^{tJ} \begin{pmatrix} (A^u)^r & \\ & (A^u)^l \end{pmatrix}_s^t, \tag{9.194}$$

with

$$(A^u)^r = \begin{pmatrix} 0 & \bar{\mathsf{k}}^{\mathsf{I}}\left(\boldsymbol{\alpha}'(H_1+\mathbb{I})\,\mathsf{q}'^\dagger - \mathbb{I}\right) \\ \left(\mathsf{q}\,(H_2+\mathbb{I})\,\boldsymbol{\alpha}^\dagger - \mathbb{I}\right)\mathsf{k}^{\mathsf{I}} & 0 \end{pmatrix}, \tag{9.195}$$

$$(A^u)^l = -\begin{pmatrix} 0 & \bar{\mathsf{k}}^{\mathsf{I}}\left(\boldsymbol{\alpha}\,(H_1'+\mathbb{I})\,\mathsf{q}^\dagger - \mathbb{I}\right) \\ \left(\mathsf{q}'\,(H_2'+\mathbb{I})\,\boldsymbol{\alpha}'^\dagger - \mathbb{I}\right)\mathsf{k}^{\mathsf{I}} & 0 \end{pmatrix} \tag{9.196}$$

where $H_{1,2}$ are the components of A_Y, and $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$, q, q' those of u.

Proof. From the formula (9.83) of A_Y and (9.167)-(9.168) of u, one gets

$$\rho(u) A_Y u^\dagger = \begin{pmatrix} \rho(\mathfrak{A}) A \mathfrak{A}^\dagger & \\ & 0 \end{pmatrix}_C^D \quad \text{where} \quad \rho(\mathfrak{A}) A \mathfrak{A}^\dagger = \delta_{\dot{s}I}^{tJ} \begin{pmatrix} \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger & \\ & \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger \end{pmatrix}_s^t, \tag{9.197}$$

where, using (9.84) and (9.169),

$$\mathfrak{A}_l A_r \mathfrak{A}_r^\dagger = \begin{pmatrix} \boldsymbol{\alpha}' & \\ & \mathsf{q} \end{pmatrix} \begin{pmatrix} & \bar{\mathsf{k}}^I H_1 \\ H_2 \mathsf{k}^I & \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^\dagger & \\ & \mathsf{q}'^\dagger \end{pmatrix} = \begin{pmatrix} & \bar{\mathsf{k}}^I \boldsymbol{\alpha}' H_1 \mathsf{q}'^\dagger \\ \mathsf{q} H_2 \boldsymbol{\alpha}^\dagger \mathsf{k}^I & \end{pmatrix}_\alpha^\beta, \tag{9.198}$$

$$\mathfrak{A}_r A_l \mathfrak{A}_l^\dagger = -\begin{pmatrix} \boldsymbol{\alpha} & \\ & \mathsf{q}' \end{pmatrix} \begin{pmatrix} & \bar{\mathsf{k}}^I H_1' \\ H_2' \mathsf{k}^I & \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}'^\dagger & \\ & \mathsf{q}^\dagger \end{pmatrix} = -\begin{pmatrix} & \bar{\mathsf{k}}^I \boldsymbol{\alpha} H_1' \mathsf{q}^\dagger \\ \mathsf{q}' H_2' \boldsymbol{\alpha}'^\dagger \mathsf{k}^I & \end{pmatrix}_\alpha^\beta, \tag{9.199}$$

where we used that k^I, $\bar{\mathsf{k}}^I$ commute with $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ and their conjugate.

The computation of the twisted commutator part in (9.193) is similar to that of A_Y in proposition 9.6.1, with $a_i = \rho(u)$ and $b_i = u^\dagger$ for u as in (9.165), that is

$$\rho(u)\left[\gamma^5 \otimes D_Y, u^\dagger\right]_\rho = \begin{pmatrix} \mathfrak{U} & \\ & 0 \end{pmatrix}_C^D \tag{9.200}$$

where

$$\mathfrak{U} = \delta_{\dot{s}I}^{tJ} \begin{pmatrix} \mathfrak{U}_r & \\ & \mathfrak{U}_l \end{pmatrix}_s^t \text{ with } \mathfrak{U}_r = \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}\mathfrak{H}_1 \\ \mathfrak{H}_2\mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta, \quad \mathfrak{U}_l = -\begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}\mathfrak{H}'_1 \\ \mathfrak{H}'_2\mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta, \tag{9.201}$$

in which $\mathfrak{H}_{i=1,2}$ and $\mathfrak{H}'_{1,2} = \rho(\mathfrak{H}_{1,2})$ given by (9.92) with (remembering (9.171))

$$\mathsf{c} = \boldsymbol{\alpha}',\ \mathsf{c}' = \boldsymbol{\alpha},\ q = \mathsf{q}',\ q' = \mathsf{q} \ \text{ and } \ \mathsf{d} = \boldsymbol{\alpha}^\dagger,\ \mathsf{d}' = \boldsymbol{\alpha}'^\dagger,\ p = \mathsf{q}^\dagger,\ p' = \mathsf{q}'^\dagger; \tag{9.202}$$

that is

$$\mathfrak{H}_1 = \boldsymbol{\alpha}'(\mathsf{q}'^\dagger - \boldsymbol{\alpha}'^\dagger),\ \mathfrak{H}_2 = \mathsf{q}(\boldsymbol{\alpha}^\dagger - \mathsf{q}^\dagger) \ \text{ and } \ \mathfrak{H}'_1 = \boldsymbol{\alpha}(\mathsf{q}^\dagger - \boldsymbol{\alpha}^\dagger),\ \mathfrak{H}'_2 = \mathsf{q}'(\boldsymbol{\alpha}'^\dagger - \mathsf{q}'^\dagger). \tag{9.203}$$

Thus one obtains (9.193) with

$$A^u = \mathfrak{U} + \rho(\mathfrak{A})A\mathfrak{A}^\dagger = \delta_{\dot{s}I}^{tJ} \begin{pmatrix} \mathfrak{U}_r + \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger & \\ & \mathfrak{U}_l + \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger \end{pmatrix}_s^t. \tag{9.204}$$

From (9.198) and (9.201) one obtains the explicit forms of $(A^u)^r$ and $(A^u)^l$

$$(A^u)^r := \mathfrak{U}_r + \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger = \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}(\mathfrak{H}_1 + \boldsymbol{\alpha}' H_1 \mathsf{q}'^\dagger) \\ (\mathfrak{H}_2 + \mathsf{q} H_2 \boldsymbol{\alpha}^\dagger)\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}, \tag{9.205}$$

$$(A^u)^l := \mathfrak{U}_l + \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger = -\begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}(\mathfrak{H}'_1 + \boldsymbol{\alpha} H'_1 \mathsf{q}^\dagger) \\ (\mathfrak{H}'_2 + \mathsf{q}' H'_2 \boldsymbol{\alpha}'^\dagger)\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}. \tag{9.206}$$

The final result follow substituting $\mathfrak{H}_{1,2}$ with their explicit formulas (9.203). □

One easily checks that for a selfadjoint diagonal twisted 1-form A_Y (that is, from corollary 9.6.1.1, when $H_1^\dagger = H_2 = H_r$ and $H_1'^\dagger = H_2' = H_l$) then a twist-invariant unitary u (i.e. $\mathsf{q}' = \mathsf{q}$ and $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + K$) not only preserves the selfadjointness of the free 1-form by proposition 9.8.1, but also the selfadjointness of A_Y, provided that $K = 0$ (if $K \neq 0$, then H_1 and H_2 undergo different gauge transformations, thus forbidding the identification $H_1^\dagger = H_2 = H_r$ – and similarly for H_l – and therefore forcing A_Y not to be selfadjoint). For this reason, from now on we will assume $K = 0$. With this caveat, the gauge transformation of lemma 9.8.1 then reads more clearly as a law of transformation of the complex components (9.99) of the quaternionic fields H_r and H_l.

Proposition 9.8.3. *Let A_Y be a selfadjoint diagonal 1-form parametrized by two quaternionic field H_r, H_l. Under a gauge transformation induced by a twist-invariant unitary $u = (\alpha, \alpha, \mathsf{q}, \mathsf{q}, m, m)$, the components $\phi^r_{1,2}$, $\phi^l_{1,2}$ of H_r, H_l transform as*

$$\begin{pmatrix} \phi^r_1 + 1 \\ \phi^r_2 \end{pmatrix} \longrightarrow \mathsf{q} \begin{pmatrix} \phi^r_1 + 1 \\ \phi^r_2 \end{pmatrix} e^{-i\alpha}, \quad \begin{pmatrix} \phi^l_1 + 1 \\ \phi^l_2 \end{pmatrix} \longrightarrow \mathsf{q} \begin{pmatrix} \phi^l_1 + 1 \\ \phi^l_2 \end{pmatrix} e^{-i\alpha}. \tag{9.207}$$

Proof. A_Y being selfadjoint means that (9.93) holds. A twist-invariant unitary satisfies (9.172) with $K = 0$. Under these conditions, comparing the formula (9.84) of A_Y with its gauge transformed counterpart (9.195)-(9.196), one finds that the fields H_r and H_l undergo the same transformation

$$H_r \longrightarrow \mathsf{q}\,(H_r + \mathbb{I})\,\boldsymbol{\alpha}^\dagger - \mathbb{I}, \tag{9.208}$$

$$H_l \longrightarrow \mathsf{q}\,(H_l + \mathbb{I})\,\boldsymbol{\alpha}^\dagger - \mathbb{I}. \tag{9.209}$$

Written in components (9.99), these equations reads

$$\phi^r_1 \longrightarrow q_{11}\,(\phi^r_1 + 1)e^{-i\alpha} + q_{12}\,\phi^r_2\, e^{-i\alpha} - 1, \tag{9.210}$$

$$\phi^r_2 \longrightarrow q_{21}\,\alpha(\phi^r_1 + 1)e^{-i\alpha} + q_{22}\,\phi^r_2\, e^{-i\alpha} - 1, \tag{9.211}$$

where q_{ij} denote the components of the quaternion q. In matricial form, these equations are nothing but 9.207. □

The transformations (9.207) are similar to those of the Higgs doublet in the Standard Model (see e.g. [96, Prop. 11.5]). In the twisted version of the Standard Model, we thus obtain two Higgs fields, independently coupled to the left and right components of the Dirac spinors. However, as we already mentioned, the two have no individual physical meaning on their own, since they only appear in the fermionic action through the linear combination $h = (H_r + H_l)\,/2$. Therefore there is actually only one physical Higgs doublet in the twisted case as well, that couples to both left and right-handed fermions.

9.9 Fermionic Action

In twisted non-commutative geometry, the fermionic action (i.e. the component of the action that containing all terms that involve fermions) is defined by (8.26). Explicitly:

$$S^\rho_f = \left\langle J\tilde{\psi}\,,\, \mathcal{R}\mathcal{D}_{\mathbb{A}_\rho}\tilde{\psi} \right\rangle \tag{9.212}$$

where J is the real structure, $\mathcal{R}$ is the unitary that implements the twist (8.8), $\mathcal{D}_{\mathbb{A}_\rho}$ is the twisted fluctuated Dirac operator (8.20), and $\tilde{\psi}$ is the Graßmann vector associated with the eigenvector ψ of $\mathcal{R}$.

Since $J\mathcal{R} = -\mathcal{R}J$ in the twisted Connes model, thanks to Lemma 8.3.2 we can rewrite S_f^ρ as

$$S_f^\rho = -\left\langle J\tilde{\psi}\,,\,\mathcal{D}_{\mathbb{A}_\rho}\tilde{\psi}\right\rangle. \tag{9.213}$$

In the previous sections, we calculated $\mathcal{D}_{\mathbb{A}_\rho}$. Then, in order to calculate S_f^ρ, we still need to calculate ψ, $J\psi$, $\mathcal{D}_{\mathbb{A}_\rho}\psi$ and finally we will have to compute the scalar product (9.213). These calculations are really long, since they involve very big matrices acting on very long vectors (with 4×32 components each), yet they are quite easy, for it is all just simple algebra. For this reason, we will not report the full calculations here, but we will only give a detailed sketch of how they should be done, and then we will report the results. The interested reader can find the intermediate steps in the Appendix D.

9.9.1 Eigenvectors of $\mathcal{R}$

The explicit form of $\mathcal{R}$ is

$$(\sigma_1)_s^t\,\delta_{\dot{s}}^{\dot{t}} \otimes \mathbb{I}_F, \tag{9.214}$$

hence any eigenvector of $\mathcal{R}$ must be of the form

$$\Phi \sim \phi \otimes v_F, \qquad \text{with} \qquad \phi_{s\dot{s}} = \begin{pmatrix} \varphi_s \\ \varphi_s \end{pmatrix}_{\dot{s}}, \; v_F \in \mathcal{H}_F \tag{9.215}$$

where φ is a Weyl spinor. If we define a basis of $\mathcal{H}_F$ as follows:

$$\begin{aligned}
e_1 &\equiv \nu_R \equiv \delta_C^0\delta_I^0\delta_\alpha^{\dot{1}} & \overline{e_1} &\equiv \overline{\nu_R} \equiv \delta_C^1\delta_I^0\delta_\alpha^{\dot{1}}\\
e_2 &\equiv e_R \equiv \delta_C^0\delta_I^0\delta_\alpha^{\dot{2}} & \overline{e_2} &\equiv \overline{e_R} \equiv \delta_C^1\delta_I^0\delta_\alpha^{\dot{2}}\\
e_3 &\equiv \nu_L \equiv \delta_C^0\delta_I^0\delta_\alpha^{1} & \overline{e_3} &\equiv \overline{\nu_L} \equiv \delta_C^1\delta_I^0\delta_\alpha^{1}\\
e_4 &\equiv e_L \equiv \delta_C^0\delta_I^0\delta_\alpha^{2} & \overline{e_4} &\equiv \overline{e_L} \equiv \delta_C^1\delta_I^0\delta_\alpha^{2}\\
e_5 &\equiv u_R^1 \equiv \delta_C^0\delta_I^1\delta_\alpha^{\dot{1}} & \overline{e_5} &\equiv \overline{u_R^1} \equiv \delta_C^1\delta_I^1\delta_\alpha^{\dot{1}}\\
e_6 &\equiv d_R^1 \equiv \delta_C^0\delta_I^1\delta_\alpha^{\dot{2}} & \overline{e_6} &\equiv \overline{d_R^1} \equiv \delta_C^1\delta_I^1\delta_\alpha^{\dot{2}}\\
e_7 &\equiv u_L^1 \equiv \delta_C^0\delta_I^1\delta_\alpha^{1} & \overline{e_7} &\equiv \overline{u_L^1} \equiv \delta_C^1\delta_I^1\delta_\alpha^{1}\\
e_8 &\equiv d_L^1 \equiv \delta_C^0\delta_I^1\delta_\alpha^{2} & \overline{e_8} &\equiv \overline{d_L^1} \equiv \delta_C^1\delta_I^1\delta_\alpha^{2}\\
e_9 &\equiv u_R^2 \equiv \delta_C^0\delta_I^2\delta_\alpha^{\dot{1}} & \overline{e_9} &\equiv \overline{u_R^2} \equiv \delta_C^1\delta_I^2\delta_\alpha^{\dot{1}}\\
e_{10} &\equiv d_R^2 \equiv \delta_C^0\delta_I^2\delta_\alpha^{\dot{2}} & \overline{e_{10}} &\equiv \overline{d_R^2} \equiv \delta_C^1\delta_I^2\delta_\alpha^{\dot{2}}\\
e_{11} &\equiv u_L^2 \equiv \delta_C^0\delta_I^2\delta_\alpha^{1} & \overline{e_{11}} &\equiv \overline{u_L^2} \equiv \delta_C^1\delta_I^2\delta_\alpha^{1}\\
e_{12} &\equiv d_L^2 \equiv \delta_C^0\delta_I^2\delta_\alpha^{2} & \overline{e_{12}} &\equiv \overline{d_L^2} \equiv \delta_C^1\delta_I^2\delta_\alpha^{2}\\
e_{13} &\equiv u_R^3 \equiv \delta_C^0\delta_I^3\delta_\alpha^{\dot{1}} & \overline{e_{13}} &\equiv \overline{u_R^3} \equiv \delta_C^1\delta_I^3\delta_\alpha^{\dot{1}}
\end{aligned}$$

$$e_{14} \equiv d_R^3 \equiv \delta_C^0 \delta_I^3 \delta_\alpha^{\dot{2}} \qquad \overline{e_{14}} \equiv \overline{d_R^3} \equiv \delta_C^1 \delta_I^3 \delta_\alpha^{\dot{2}}$$
$$e_{15} \equiv u_L^3 \equiv \delta_C^0 \delta_I^3 \delta_\alpha^1 \qquad \overline{e_{15}} \equiv \overline{u_L^3} \equiv \delta_C^1 \delta_I^3 \delta_\alpha^1$$
$$e_{16} \equiv d_L^3 \equiv \delta_C^0 \delta_I^3 \delta_\alpha^2 \qquad \overline{e_{16}} \equiv \overline{d_L^3} \equiv \delta_C^1 \delta_I^3 \delta_\alpha^2$$

(which is such that $J_F e_i = \overline{e_i}$, $J_F \overline{e_i} = e_i$), we can write the most general eigenvector of $\mathcal{R}$ as follows:

$$\Phi = \sum_{i=1}^{16} \phi_i \otimes e_i + \sum_{i=1}^{16} \phi_{i+16} \otimes \overline{e_i}, \tag{9.216}$$

where

$$\phi_i = \begin{pmatrix} \varphi_i \\ \varphi_i \end{pmatrix} \tag{9.217}$$

with φ_i Weyl spinors.

9.9.2 Calculation of $J\Phi$

Remembering that

$$J = \mathcal{J} \otimes J_F, \tag{9.218}$$

by explicit calculation we have:

$$J\Phi = \sum_{i=1}^{16} \mathcal{J}\phi_i \otimes \overline{e_i} + \sum_{i=1}^{16} \mathcal{J}\phi_{i+16} \otimes e_i. \tag{9.219}$$

9.9.3 Calculation of $D^Y_{A_\rho}\Xi$

Let us define $\Xi \in \mathcal{H}_\mathcal{R}$ as

$$\Xi = \sum_{i=1}^{16} \xi_i \otimes e_i + \sum_{i=1}^{16} \xi_{i+16} \otimes \overline{e_i}, \tag{9.220}$$

where

$$\xi_i = \begin{pmatrix} \zeta_i \\ \zeta_i \end{pmatrix} \tag{9.221}$$

with ζ_i Weyl spinors. By definition, Ξ is an eigenvector of $\mathcal{R}$ (since it has the same form (9.216)).

Now it is time to perform the calculations. We can split $\mathcal{D}_{\mathbb{A}_\rho}$ into many pieces:

$$\mathcal{D}_{\mathbb{A}_\rho} = \partial\!\!\!/ \otimes \mathbb{I}_F + X + iY + \gamma^5 \otimes D_Y + M_1 + \gamma^5 \otimes D_M \tag{9.222}$$

with

$$X = -i\gamma^{\mu}\begin{pmatrix}\gamma^5 \otimes X_{\mu} & \\ & \gamma^5 \otimes \overline{X_{\mu}}\end{pmatrix} \tag{9.223}$$

$$iY = -i\gamma^{\mu}\begin{pmatrix}\mathbb{I}_4 \otimes iY_{\mu} & \\ & \mathbb{I}_4 \otimes \overline{iY_{\mu}}\end{pmatrix} \tag{9.224}$$

$$M_1 = \begin{pmatrix}A & \\ & \overline{A}\end{pmatrix} \tag{9.225}$$

where X_{μ} and Y_{μ} are defined in (9.144), and A is defined in (9.83). Therefore, it is easier to calculate the contributions of the single pieces and then put them together. As an example, we will show how to compute the first components of $(\gamma^5 \otimes D_Y)\,\Xi$; the full result can be found in the Appendix D.

We have

$$\begin{aligned}\left(\gamma^5 \otimes D_Y\right)(\xi_1 \otimes e_1) &= \gamma^5\xi_1 \otimes D_Y e_1 = \\ &= \gamma^5\xi_1 \otimes (D_Y)^{DJ\beta}_{CI\alpha}\,\delta^0_D\delta^0_J\delta^{\mathrm{i}}_{\beta} = \\ &= \gamma^5\xi_1 \otimes \delta^0_C\,(D_0)^{J\beta}_{I\alpha}\,\delta^0_J\delta^{\mathrm{i}}_{\beta} = \\ &= \gamma^5\xi_1 \otimes \delta^0_C\delta^0_I\left(D^{\ell}_0\right)^{\beta}_{\alpha}\delta^{\mathrm{i}}_{\beta} = \\ &= \gamma^5\xi_1 \otimes k_{\nu}\delta^0_C\delta^0_I\delta^1_{\alpha} = \\ &= k_{\nu}\gamma^5\xi_1 \otimes e_3.\end{aligned}$$

Following the same procedure, we can compute the other components of $(\gamma^5 \otimes D_Y)\,\Xi$:

$$\begin{aligned}(\gamma_5 \otimes D_Y)(\xi_2 \otimes e_2) &= k_e\gamma_5\xi_2 \otimes e_4 \\ (\gamma_5 \otimes D_Y)(\xi_3 \otimes e_3) &= \overline{k_{\nu}}\gamma_5\xi_3 \otimes e_1 \\ (\gamma_5 \otimes D_Y)(\xi_4 \otimes e_4) &= \overline{k_e}\gamma_5\xi_4 \otimes e_2 \\ (\gamma_5 \otimes D_Y)(\xi_5 \otimes e_5) &= k_u\gamma_5\xi_5 \otimes e_7 \\ &\cdots\end{aligned}$$

The very same procedure can also be applied for all other terms that appear in the sum (9.222); the results can be found in the Appendix D.

9.9.4 Calculation of the Scalar Products

When computing a scalar product of the form

$$\langle J\Phi\,,\,(\mathcal{O}_{\mathcal{M}} \otimes O_F)\,\Xi\rangle\,, \tag{9.226}$$

the finite-dimensional part $\langle J_F e_i\,,\, O_F e_j\rangle$ will simply select which fermions will be coupled; on the other hand, the manifold part $\langle \mathcal{J}\phi_i\,,\, \mathcal{O}_{\mathcal{M}}\xi_j\rangle$ will lead to interesting effects, so we will deal with it more in detail.

As a shorthand notation, let us define

$$\mathfrak{A}_{\mathcal{O}}(\phi,\xi) = \langle \mathcal{J}\phi\,,\, \mathcal{O}_{\mathcal{M}}\xi\rangle \tag{9.227}$$

for any operator $\mathcal{O} = \mathcal{O}_{\mathcal{M}} \otimes O_F$. With these notations, we have the following

Lemma 9.9.1. *One has*

$$\mathfrak{A}_{\not{\partial}}(\phi,\xi) = 2\int_{\mathcal{M}} d\mu\, \left(\overline{\varphi}^{\dagger}\sigma_2\right)(\sigma_j\partial_j)\,\zeta, \tag{9.228}$$

$$\mathfrak{A}_{-i\gamma^{\mu}\gamma^5 f_{\mu}}(\phi,\xi) = -2\int_{\mathcal{M}} d\mu\, \left(\overline{\varphi}^{\dagger}\sigma_2\right)(if_0\mathbb{I}_2)\,\zeta \tag{9.229}$$

$$\mathfrak{A}_{-i\gamma^{\mu}g_{\mu}}(\phi,\xi) = 2\int_{\mathcal{M}} d\mu\, \left(\overline{\varphi}^{\dagger}\sigma_2\right)(g_j\sigma_j)\,\zeta \tag{9.230}$$

$$\mathfrak{A}_{H_r P_R - H_l P_l}(\phi,\xi) = -\int_{\mathcal{M}} d\mu\, \left(\overline{\varphi}^{\dagger}\sigma_2\right)(H_r + H_l)\,\zeta \tag{9.231}$$

where

$$\phi = \begin{pmatrix}\varphi\\ \varphi\end{pmatrix}, \qquad \xi = \begin{pmatrix}\zeta\\ \zeta\end{pmatrix}, \qquad P_{r,l} = \frac{\mathbb{I}\pm\gamma^5}{2}. \tag{9.232}$$

Proof. One has

$$\mathcal{J}\phi = i\gamma^0\gamma^2 K\phi = i\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\varphi\\ \varphi\end{pmatrix} = i\begin{pmatrix}\tilde{\sigma}^2\varphi\\ \sigma^2\varphi\end{pmatrix}, \tag{9.233}$$

$$\not{\partial}\xi = -i\gamma^{\mu}\partial_{\mu}\xi = -i\begin{pmatrix}0 & \sigma^{\mu}\\ \tilde{\sigma}^{\mu} & 0\end{pmatrix}\partial_{\mu}\begin{pmatrix}\zeta\\ \zeta\end{pmatrix} = -i\begin{pmatrix}\sigma^{\mu}\partial_{\mu}\zeta\\ \tilde{\sigma}^{\mu}\partial_{\mu}\zeta\end{pmatrix}, \tag{9.234}$$

$$-i\gamma^{\mu}\gamma^5 f_{\mu}\xi = -i\begin{pmatrix}0 & \sigma^{\mu}\\ \tilde{\sigma}^{\mu} & 0\end{pmatrix}\begin{pmatrix}f_{\mu}\mathbb{I}_2 & 0\\ 0 & -f_{\mu}\mathbb{I}_2\end{pmatrix}\begin{pmatrix}\zeta\\ \zeta\end{pmatrix} = -i\begin{pmatrix}-f_{\mu}\sigma^{\mu}\zeta\\ f_{\mu}\tilde{\sigma}^{\mu}\zeta\end{pmatrix}, \tag{9.235}$$

$$-i\gamma^{\mu}g_{\mu}\xi = -i\begin{pmatrix}0 & \sigma^{\mu}\\ \tilde{\sigma}^{\mu} & 0\end{pmatrix}g_{\mu}\begin{pmatrix}\zeta\\ \zeta\end{pmatrix} = -i\begin{pmatrix}g_{\mu}\sigma^{\mu}\zeta\\ g_{\mu}\tilde{\sigma}^{\mu}\zeta\end{pmatrix}. \tag{9.236}$$

Noticing that $(\tilde{\sigma}^2)^{\dagger} = \sigma^2$ and $(\sigma^2)^{\dagger} = -\sigma^2$, and using

$$\sigma^{\mu} + \tilde{\sigma}^{\mu} = 2\mathbb{I}_2\delta^{\mu}_0, \qquad \sigma^{\mu} - \tilde{\sigma}^{\mu} = -2i\delta^{\mu}_j\sigma^j, \qquad \sigma^2 = -i\sigma_2, \tag{9.237}$$

one gets

$$\mathfrak{A}_{\not{\partial}}(\phi,\xi) = \left\langle \mathcal{J}\phi\,,\, \not{\partial}\xi\right\rangle = -\left(\overline{\varphi}^{\dagger}\tilde{\sigma}^{2\dagger},\ \ \overline{\varphi}^{\dagger}\sigma^{2\dagger}\right)\begin{pmatrix}\sigma^{\mu}\partial_{\mu}\zeta\\ \tilde{\sigma}^{\mu}\partial_{\mu}\zeta\end{pmatrix} =$$

$$= -\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2\sigma^{\mu}\partial_{\mu}\zeta - \overline{\varphi}^{\dagger}\sigma^2\tilde{\sigma}^{\mu}\partial_{\mu}\zeta = -\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2\left(\sigma^{\mu} - \tilde{\sigma}^{\mu}\right)\partial_{\mu}\zeta = \\ = 2\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma_2\sigma_j\partial_j\zeta; \quad (9.238)$$

$$\mathfrak{A}_{-i\gamma^{\mu}\gamma^5 f_{\mu}}(\phi,\xi) = \left\langle \mathcal{J}\phi\,,\, -i\gamma^{\mu}\gamma^5 f_{\mu}\xi\right\rangle = -\left(\overline{\varphi}^{\dagger}\tilde{\sigma}^{2\dagger},\quad \overline{\varphi}^{\dagger}\sigma^{2\dagger}\right)\begin{pmatrix} -f_{\mu}\sigma^{\mu}\zeta \\ f_{\mu}\tilde{\sigma}^{\mu}\zeta \end{pmatrix} = \\ = \int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2 f_{\mu}\sigma^{\mu}\zeta + \overline{\varphi}^{\dagger}\sigma^2 f_{\mu}\tilde{\sigma}^{\mu}\zeta = \int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2 f_{\mu}\left(\sigma^{\mu} + \tilde{\sigma}^{\mu}\right)\zeta = \\ = -2i\int_{\mathcal{M}} d\mu\, f_0\overline{\varphi}^{\dagger}\sigma_2\zeta; \quad (9.239)$$

$$\mathfrak{A}_{-i\gamma^{\mu}g_{\mu}}(\phi,\xi) = \langle \mathcal{J}\phi\,,\, -i\gamma^{\mu}g_{\mu}\xi\rangle = -\left(\overline{\varphi}^{\dagger}\tilde{\sigma}^{2\dagger},\quad \overline{\varphi}^{\dagger}\sigma^{2\dagger}\right)\begin{pmatrix} g_{\mu}\sigma^{\mu}\zeta \\ g_{\mu}\tilde{\sigma}^{\mu}\zeta \end{pmatrix} = \\ = -\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2 g_{\mu}\sigma^{\mu}\zeta - \overline{\varphi}^{\dagger}\sigma^2 g_{\mu}\tilde{\sigma}^{\mu}\zeta = -\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma^2 g_{\mu}\left(\sigma^{\mu} - \tilde{\sigma}^{\mu}\right)\zeta = \\ = -\int_{\mathcal{M}} d\mu\, \overline{\varphi}^{\dagger}\sigma_2 g_j\sigma_j\zeta. \quad (9.240)$$

□

Remark 9.9.1. Lemma 9.9.1 has impoetant consequences. First of all, Eq.s (9.228, 9.229) imply that the kinetic terms of all fermions will be of the form

$$S^{\rho}_{f,kin.} \sim \int_{\mathcal{M}} d\mu\, \overline{\tilde{\psi}}^{\dagger}\sigma_2\left(if_0\mathbb{I}_2 - \sigma_j\partial_j\right)\tilde{\psi}, \quad (9.241)$$

where f_0 will correspond to the components of X. Moreover, g_{μ} appearing in (9.230) corresponding to the components of Y (i.e. to the gauge vectors), all the gauge vectors will be in the temporal gauge. Finally, $H_{r,l}$ in (9.231) are the two Higgs doublets defined in Corollary 9.6.1.1, which shows that the two will indeed combine into a unique Higgs doublet $H_r + H_l$ as we already anticipated.

We will omit the full calculation of the action since it is once again very long and hardly interesting, it being simply a matter of performing the scalar product of (long, yet finite-dimensional) vectors for the finite part, and identifying the right $\mathfrak{A}_{\mathcal{O}}$ from the ones we computed in Lemma 9.9.1 for the manifold part. The interested reader can find the intermediate steps in the Appendix D. Now, we can finally write the full fermionic action for the twisted-by-grading Connes Model.

9.9.5 Fermionic Action: Physical Fermions

To compute the fermionic action, we have to compute the bilinear form (8.26) on the very same Graßmann vector $\tilde{\phi}$ associated with the eigenvector $\phi \in \mathcal{H}_{\mathcal{R}}$ of $\mathcal{R}$. Then, we can identify its components $\tilde{\varphi}_i$ with the physical fermions through the following identification:

$$\left(i\tilde{\overline{\varphi}}_1^\dagger\sigma_2\right) = \overline{\nu_R} \qquad \tilde{\varphi}_{17} = \nu_R \qquad \left(-i\tilde{\overline{\varphi}}_3^\dagger\sigma_2\right) = \overline{\nu_L} \qquad \tilde{\varphi}_{19} = \nu_L \tag{9.242}$$

$$\left(i\tilde{\overline{\varphi}}_2^\dagger\sigma_2\right) = \overline{e_R} \qquad \tilde{\varphi}_{18} = e_R \qquad \left(-i\tilde{\overline{\varphi}}_4^\dagger\sigma_2\right) = \overline{e_L} \qquad \tilde{\varphi}_{20} = e_L \tag{9.243}$$

$$\left(i\tilde{\overline{\varphi}}_5^\dagger\sigma_2\right) = \overline{u_R^1} \qquad \tilde{\varphi}_{21} = u_R^1 \qquad \left(-i\tilde{\overline{\varphi}}_7^\dagger\sigma_2\right) = \overline{u_L^1} \qquad \tilde{\varphi}_{23} = u_L^1 \tag{9.244}$$

$$\left(i\tilde{\overline{\varphi}}_6^\dagger\sigma_2\right) = \overline{d_R^1} \qquad \tilde{\varphi}_{22} = d_R^1 \qquad \left(-i\tilde{\overline{\varphi}}_8^\dagger\sigma_2\right) = \overline{d_L^1} \qquad \tilde{\varphi}_{24} = d_L^1 \tag{9.245}$$

$$\left(i\tilde{\overline{\varphi}}_9^\dagger\sigma_2\right) = \overline{u_R^2} \qquad \tilde{\varphi}_{25} = u_R^2 \qquad \left(-i\tilde{\overline{\varphi}}_{11}^\dagger\sigma_2\right) = \overline{u_L^2} \qquad \tilde{\varphi}_{27} = u_L^2 \tag{9.246}$$

$$\left(i\tilde{\overline{\varphi}}_{10}^\dagger\sigma_2\right) = \overline{d_R^2} \qquad \tilde{\varphi}_{26} = d_R^2 \qquad \left(-i\tilde{\overline{\varphi}}_{12}^\dagger\sigma_2\right) = \overline{d_L^2} \qquad \tilde{\varphi}_{28} = d_L^2 \tag{9.247}$$

$$\left(i\tilde{\overline{\varphi}}_{13}^\dagger\sigma_2\right) = \overline{u_R^3} \qquad \tilde{\varphi}_{29} = u_R^3 \qquad \left(-i\tilde{\overline{\varphi}}_{15}^\dagger\sigma_2\right) = \overline{u_L^3} \qquad \tilde{\varphi}_{31} = u_L^3 \tag{9.248}$$

$$\left(i\tilde{\overline{\varphi}}_{14}^\dagger\sigma_2\right) = \overline{d_R^3} \qquad \tilde{\varphi}_{30} = d_R^3 \qquad \left(-i\tilde{\overline{\varphi}}_{16}^\dagger\sigma_2\right) = \overline{d_L^3} \qquad \tilde{\varphi}_{32} = d_L^3 \tag{9.249}$$

Notice that these are all Weyl spinors; we will translate the full lagrangian into the more familiar Dirac spinors in section 9.9.10.

9.9.6 Fermionic Action: Kinetic Terms

Using the identifications (9.242-9.249), the kinetic action takes the form

$$S_f^{kin.} = 4\int_{\mathcal{M}} d\mu\, \mathcal{L}_{kin.}, \tag{9.250}$$

with

$$\mathcal{L}_{kin.} = \mathcal{L}_{kin.}^{lept} + \mathcal{L}_{kin.}^{quark} \tag{9.251}$$

where

$$\begin{aligned}\mathcal{L}_{kin.}^{lept} = &\, i\overline{\nu_R}\left(-2ia_0 + \sigma_k\partial_k\right)\nu_R + i\overline{e_R}\left(-2ia_0 + \sigma_k\partial_k\right)e_R + \\ &+ i\overline{\nu_L}\left(iw_0 - ia_0 - \sigma_k\partial_k\right)\nu_L + i\overline{e_L}\left(iw_0 - ia_0 - \sigma_k\partial_k\right)e_L\end{aligned} \tag{9.252}$$

and

$$\mathcal{L}_{kin.}^{quark} = i\overline{u_R^i}\left(i\,\mathrm{Re}\,(g_0)_i^j - ia_0\delta_i^j + \sigma_k\partial_k\right)u_R^j + i\overline{d_R^i}\left(i\,\mathrm{Re}\,(g_0)_i^j - ia_0\delta_i^j + \sigma_k\partial_k\right)d_R^j +$$

$$+ i\overline{u_L^i}\left(i\,\mathrm{Re}\,(g_0)_i^j + iw_0\delta_i^j - \sigma_k\partial_k\right)u_L^j + i\overline{d_L^i}\left(i\,\mathrm{Re}\,(g_0)_i^j + iw_0\delta_i^j - \sigma_k\partial_k\right)d_L^j \tag{9.253}$$

At this point, what one does in twisted QED is to identify

$$if_0\psi = \partial_0\psi \tag{9.254}$$

(with f_0 representing any one of a_0, w_0, g_0 and ψ representing any fermion) so as to recover the kinetic term in Lorentzian signature. In this case, however, one may not do so in full generality for there are 16 independent fermions but only 11 independent f_0s.

One possible way out of this problem might be found in the fact that, currently, there are no known ways to quantize a twisted non-commutative geometry, so at this level the lagrangian should be intended to be classical[5]. The purpose of a classical lagrangian is ultimately to extract the equations of motion, so one may still do the identification (9.254) at the level of the equations of motion. At this level, in each equation of motion will appear just one f_0, so it is possible to view them as mutually independent and safely do the identification (9.254). With this in mind, we will write the usual kinetic terms in Lorentzian signature in the full fermionic action in section 9.9.10. Still, a more in-depth study of this problem is needed, which will be carried out in a forthcoming, dedicated work.

9.9.7 Fermionic Action: Gauge Terms

Using the identifications (9.242-9.249), and once again defining

$$S_{gauge} = 4\int_{\mathcal{M}} d\mu\ \mathcal{L}_{gauge}, \tag{9.255}$$

with

$$\mathcal{L}_{gauge} = \mathcal{L}_B + \mathcal{L}_W + \mathcal{L}_V + \text{h.c.}, \tag{9.256}$$

we get

$$\begin{aligned}\mathcal{L}_B = {} & g_1\overline{e_R}\left(\sigma_k B_k\right)e_R - \frac{g_1}{2}\overline{\ell_L}\left(\sigma_k B_k\right)\ell_L - \\ & -\frac{2}{3}g_1\overline{u_R^i}\left(\sigma_k B_k\right)u_R^i + \frac{g_1}{3}\overline{d_R^i}\left(\sigma_k B_k\right)d_R^i + \frac{g_1}{6}\overline{q_L^i}\left(\sigma_k B_k\right)q_L^i,\end{aligned} \tag{9.257}$$

$$\mathcal{L}_W = \frac{g_2}{2}\overline{\ell_L}\left(\sigma_k W_k\right)\ell_L + \frac{g_2}{2}\overline{q_L^i}\left(\sigma_k W_k\right)q_L^i \tag{9.258}$$

[5]Of course, it can still be used to do phenomenology, in which case one assumes that a quantization method exists and therefore assumes that same lagrangian to be valid also at the quantum level.

$$\mathcal{L}_V = -\frac{g_3}{2}\overline{u^i_R}\left(\sigma_k\left(V_k\right)^j_i\right)u^j_R - \frac{g_3}{2}\overline{d^i_R}\left(\sigma_k\left(V_k\right)^j_i\right)d^j_R + \frac{g_3}{2}\overline{q^i_L}\left(\sigma_k\left(V_k\right)^j_i\right)q^j_L \tag{9.259}$$

where

$$\ell_L = \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}, \qquad q^i_L = \begin{pmatrix}u^i_L\\ d^i_L\end{pmatrix}, \tag{9.260}$$

$$W_i = W^k_i\sigma_k \qquad V_i = V^m_i\lambda_m. \tag{9.261}$$

9.9.8 Fermionic Action: Mass Terms

Using the identifications Eq. (9.242-9.249), and once again defining

$$S_{mass} = 4\int_{\mathcal{M}} d\mu\ \mathcal{L}_{mass}, \tag{9.262}$$

we get

$$\begin{aligned}\mathcal{L}_{mass} = &-i\overline{k_\nu}\,\overline{\nu_R}\nu_L - i\overline{k_e}\,\overline{e_R}e_L + ik_\nu\overline{\nu_L}\nu_R + ik_e\overline{e_L}e_R-\\ &-i\overline{k_u}\,\overline{u^i_R}u^i_L - i\overline{k_d}\,\overline{d^i_R}d^i_L + ik_u\overline{u^i_L}u^i_R + ik_d\overline{d^i_L}d^i_R.\end{aligned} \tag{9.263}$$

If we assume $k_x = im_x$ with m_x real, we recover the correct expression for the mass lagrangian:

$$\begin{aligned}\mathcal{L}_{mass} = &-m_\nu\left(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R\right) - m_e\left(\overline{e_R}e_L + \overline{e_L}e_R\right)-\\ &-m_u\left(\overline{u^i_R}u^i_L + \overline{u^i_L}u^i_R\right) - m_d\left(\overline{d^i_R}d^i_L + \overline{d^i_L}d^i_R\right).\end{aligned} \tag{9.264}$$

9.9.9 Fermionic Action: Higgs Terms

Using the identifications Eq. (9.242-9.249), and once again defining

$$S_{Higgs} = 4\int_{\mathcal{M}} d\mu\ \mathcal{L}_{Higgs}, \tag{9.265}$$

we get

$$\begin{aligned}\mathcal{L}_{Higgs} = &\,ik_\nu\phi_2\overline{\nu_R}\nu_L - ik_\nu\phi_1\overline{\nu_R}e_L + ik_e\overline{\phi_1}\,\overline{e_R}\nu_L + ik_e\overline{\phi_2}\,\overline{e_R}e_L-\\ &-i\overline{k_\nu\phi_2}\,\overline{\nu_L}\nu_R - i\overline{k_e}\phi_1\overline{\nu_L}e_R + i\overline{k_\nu\phi_1}\,\overline{e_L}\nu_R - i\overline{k_e}\phi_2\overline{e_L}e_R+\\ &+ik_u\phi_2\overline{u^1_R}u^1_L - ik_u\phi_1\overline{u^1_R}d^1_L + ik_d\overline{\phi_1}\,\overline{d^1_R}u^1_L + ik_d\overline{\phi_2}\,\overline{d^1_R}d^1_L-\\ &-i\overline{k_u\phi_2}\,\overline{u^1_L}u^1_R - i\overline{k_d}\phi_1\overline{u^1_L}d^1_R + i\overline{k_u\phi_1}\,\overline{d^1_L}u^1_R - i\overline{k_d}\phi_2\overline{d^1_L}d^1_R+\\ &+ik_u\phi_2\overline{u^2_R}u^2_L - ik_u\phi_1\overline{u^2_R}d^2_L + ik_d\overline{\phi_1}\,\overline{d^2_R}u^2_L + ik_d\overline{\phi_2}\,\overline{d^2_R}d^2_L-\\ &-i\overline{k_u\phi_2}\,\overline{u^2_L}u^2_R - i\overline{k_d}\phi_1\overline{u^2_L}d^2_R + i\overline{k_u\phi_1}\,\overline{d^2_L}u^2_R - i\overline{k_d}\phi_2\overline{d^2_L}d^2_R+\end{aligned}$$

$$+ ik_u\phi_2\overline{u^3_R}u^3_L - ik_u\phi_1\overline{u^3_R}d^3_L + ik_d\overline{\phi_1}\overline{d^3_R}u^3_L + ik_d\overline{\phi_2}\overline{d^3_R}d^3_L -$$
$$- i\overline{k_u\phi_2}\overline{u^3_L}u^3_R - i\overline{k_d}\phi_1\overline{u^3_L}d^3_R + i\overline{k_u\phi_1}\overline{d^3_L}u^3_R - i\overline{k_d}\phi_2\overline{d^3_L}d^3_R, \tag{9.266}$$

where

$$\phi_i = \frac{1}{2}\left(\phi^r_i + \phi^l_i\right). \tag{9.267}$$

Remembering that $k_x = im_x$, we define

$$m_x =: \frac{v}{\sqrt{2}}Y_x, \qquad \phi_1 =: i\sqrt{2}\frac{\phi^-}{v/\sqrt{2}}, \qquad \phi_2 =: \frac{H + i\phi^0}{v/\sqrt{2}}, \tag{9.268}$$

so the lagrangian becomes

$$\begin{aligned}\mathcal{L}_{Higgs} = &-H\left[Y_\nu\left(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R\right) + Y_e\left(\overline{e_R}e_L + \overline{e_L}e_R\right) + \right.\\ &\left.+Y_u\left(\overline{u^i_R}u^i_L + \overline{u^i_L}u^i_R\right) + Y_d\left(\overline{d^i_R}d^i_L + \overline{d^i_L}d^i_R\right)\right] + \\ &+ i\phi^0\left[Y_\nu\left(\overline{\nu_R}\nu_L - \overline{\nu_L}\nu_R\right) - Y_e\left(\overline{e_R}e_L - \overline{e_L}e_R\right) + \right.\\ &\left.+Y_u\left(\overline{u^i_R}u^i_L - \overline{u^i_L}u^i_R\right) - Y_d\left(\overline{d^i_R}d^i_L - \overline{d^i_L}d^i_R\right)\right] + \\ &+ i\sqrt{2}\phi^-\left[Y_\nu\overline{e_L}\nu_R - Y_e\overline{e_R}\nu_L + Y_u\overline{d^i_L}u^i_R - Y_d\overline{d^i_R}u^i_L\right] - \\ &- i\sqrt{2}\phi^+\left[Y_\nu\overline{\nu_R}e_L - Y_e\overline{\nu_L}e_R + Y_u\overline{u^i_R}d^i_L - Y_d\overline{u^i_L}d^i_R\right].\end{aligned} \tag{9.269}$$

9.9.10 Fermionic Action for many generations

So far, we only considered the case of one single generation of fermions, now we would like to generalize our description to more generations.

If one considers N generations, the full Hilbert space is the direct sum of N Hilbert spaces for one single generation. Because of this, the elements of the algebra get an additional factor δ^κ_λ, where the Greek letters λ, κ represent family indices. The same happens to the free part of the Dirac operator, while the non-zero entries of its finite part, the k_xs, become matrices in the family indices.

This all means that the boson content of the theory remains the same, but we have many more fermions, and they might not be in their mass basis. By diagonalizing their mass matrices and changing the basis to the mass one, we recover the SM lagrangian with the addition of the extra scalar terms.

In order to avoid confusion, let us proceed step by step, beginning by rewriting the full lagrangian in case of N generations of fermions (for now, ignoring the kinetic terms). We introduce the generation indices λ, κ; furthermore, in order to make the notations less cumbersome, we introduce the color triplets

$$U_{L,R} = \begin{pmatrix} u^1_{L,R} \\ u^2_{L,R} \\ u^3_{L,R} \end{pmatrix}, \qquad D_{L,R} = \begin{pmatrix} d^1_{L,R} \\ d^2_{L,R} \\ d^3_{L,R} \end{pmatrix}, \qquad Q_L = \begin{pmatrix} q^1_L \\ q^2_L \\ q^3_L \end{pmatrix}. \tag{9.270}$$

Then, we get:

$$\mathcal{L}_{gauge} = \mathcal{L}_B + \mathcal{L}_W + \mathcal{L}_V + \text{ h.c.} \tag{9.271}$$

$$\mathcal{L}_B = \sum_{\lambda=1}^{N} \Big[g_1 \overline{e_R^\lambda} \left(\sigma_k B_k\right) e_R^\lambda - \frac{g_1}{2} \overline{\ell_L^\lambda} \left(\sigma_k B_k\right) \ell_L^\lambda - -\frac{2}{3} g_1 \overline{U_R^\lambda} \left(\sigma_j B_j\right) U_R^\lambda + \frac{g_1}{3} \overline{D_R^\lambda} \left(\sigma_j B_j\right) D_R^\lambda + \frac{g_1}{6} \overline{Q_L^\lambda} \left(\sigma_j B_j\right) Q_L^\lambda \Big], \tag{9.272}$$

$$\mathcal{L}_W = \sum_{\lambda=1}^{N} \left[\frac{g_2}{2} \overline{\ell_L^\lambda} \left(\sigma_j W_j^k \sigma_k\right) \ell_L^\lambda + \frac{g_2}{2} \overline{Q_L^\lambda} \left(\sigma_j W_j^k \sigma_k\right) Q_L^\lambda \right], \tag{9.273}$$

$$\mathcal{L}_V = \sum_{\lambda=1}^{N} \left[-\frac{g_3}{2} \overline{U_R^\lambda} \left(\sigma_j V_j^m \lambda_m\right) U_R^\lambda - \frac{g_3}{2} \overline{D_R^\lambda} \left(\sigma_j V_j^m \lambda_m\right) D_R^\lambda + \frac{g_3}{2} \overline{Q_L^\lambda} \left(\sigma_j V_j^m \lambda_m\right) Q_L^\lambda \right], \tag{9.274}$$

$$\mathcal{L}_{mass} = -\sum_{\lambda,\kappa=1}^{N} \Big[\overline{\nu_R^\lambda} \left(m_\nu\right)_\lambda^\kappa \nu_L^\kappa + \overline{\nu_L^\lambda} \left(m_\nu\right)_\lambda^\kappa \nu_R^\kappa + \overline{e_R^\lambda} \left(m_e\right)_\lambda^\kappa e_L^\kappa + \overline{e_L^\lambda} \left(m_e\right)_\lambda^\kappa e_R^\kappa + + \overline{U_R^\lambda} \left(m_u\right)_\lambda^\kappa U_L^\kappa + \overline{U_L^\lambda} \left(m_u\right)_\lambda^\kappa U_R^\kappa + \overline{D_R^\lambda} \left(m_d\right)_\lambda^\kappa D_L^\kappa + \overline{D_L^\lambda} \left(m_d\right)_\lambda^\kappa D_R^\kappa \Big], \tag{9.275}$$

$$\begin{aligned}\mathcal{L}_{Higgs} = \sum_{\lambda,\kappa=1}^{N} \Big\{ &-H \Big[\overline{\nu_R^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_L^\kappa + \overline{\nu_L^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_R^\kappa + \overline{e_R^\lambda} \left(Y_e\right)_\lambda^\kappa e_L^\kappa + \overline{e_L^\lambda} \left(Y_e\right)_\lambda^\kappa e_R^\kappa + \\ &+ \overline{U_R^\lambda} \left(Y_u\right)_\lambda^\kappa U_L^\kappa + \overline{U_L^\lambda} \left(Y_u\right)_\lambda^\kappa U_R^\kappa + \overline{D_R^\lambda} \left(Y_d\right)_\lambda^\kappa D_L^\kappa + \overline{D_L^\lambda} \left(Y_d\right)_\lambda^\kappa D_R^\kappa \Big] + \\ &+ i\phi^0 \Big[\overline{\nu_R^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_L^\kappa - \overline{\nu_L^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_R^\kappa - \overline{e_R^\lambda} \left(Y_e\right)_\lambda^\kappa e_L^\kappa + \overline{e_L^\lambda} \left(Y_e\right)_\lambda^\kappa e_R^\kappa + \\ &+ \overline{U_R^\lambda} \left(Y_u\right)_\lambda^\kappa U_L^\kappa - \overline{U_L^\lambda} \left(Y_u\right)_\lambda^\kappa U_R^\kappa - \overline{D_R^\lambda} \left(Y_d\right)_\lambda^\kappa D_L^\kappa + \overline{D_L^\lambda} \left(Y_d\right)_\lambda^\kappa D_R^\kappa \Big] + \\ &+ i\sqrt{2}\phi^- \Big[\overline{e_L^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_R^\kappa - \overline{e_R^\lambda} \left(Y_e\right)_\lambda^\kappa \nu_L^\kappa + \overline{D_L^\lambda} \left(Y_u\right)_\lambda^\kappa U_R^\kappa - \overline{D_R^\lambda} \left(Y_d\right)_\lambda^\kappa U_L^\kappa \Big] - \\ &- i\sqrt{2}\phi^+ \Big[\overline{\nu_R^\lambda} \left(Y_\nu\right)_\lambda^\kappa e_L^\kappa - \overline{\nu_L^\lambda} \left(Y_e\right)_\lambda^\kappa e_R^\kappa + \overline{U_R^\lambda} \left(Y_u\right)_\lambda^\kappa D_L^\kappa - \overline{U_L^\lambda} \left(Y_d\right)_\lambda^\kappa D_R^\kappa \Big] \Big\}.\end{aligned} \tag{9.276}$$

Notice that in case of multiple generations the real parameters m_x, Y_x become $N \times N$ hermitian matrices. Obviously, in general these matrices will not be diagonal. We want to express the lagrangian in terms of the mass basis, i.e. the basis of fields that diagonalize those matrices. This has already been done in section 2.8.

There is still another issue that needs to be taken care of: the vector fields

B_μ, W^i_μ appearing in the lagrangian are not the physical ones. One still needs to rotate B_μ together with W^3_μ to make A_μ and Z_μ appear. Again, this has been already done in sections 2.6 and 2.7.

Summarizing, the full fermionic lagrangian $\mathcal{L}_f^{\text{TbG}}$ of the twisted by grading Connes Model for N generations of *Dirac* fermions is

$$\mathcal{L}_f^{\text{TbG}} = \mathcal{L}_0^F + \mathcal{L}_{EM} + \mathcal{L}_{CC} + \mathcal{L}_{NC} + \mathcal{L}_{gluons} + \mathcal{L}^\nu_{mass} + \mathcal{L}_{Higgs}, \tag{9.277}$$

where $\mathcal{L}_0^F$ is defined as in (2.147), $\mathcal{L}_{EM}$ as in (2.150), $\mathcal{L}_{CC}$ as in (2.151) and $\mathcal{L}_{NC}$ as in (2.152), but in temporal gauge, i.e. with $A_0 = W_0^\pm = Z_0 = 0$, and with

$$\begin{aligned}\mathcal{L}_{gluons} = \sum_{\lambda=1}^N \frac{g_3}{2} \Big[& \overline{U_R^\lambda} \left(\gamma_k V_k^m \lambda_m\right) U_R^\lambda + \overline{D_R^\lambda} \left(\gamma_k V_k^m \lambda_m\right) D_R^\lambda + \\ & + \overline{U_L^\lambda} \left(\gamma_k V_k^m \lambda_m\right) U_L^\lambda + \overline{D_L^\lambda} \left(\gamma_k V_k^m \lambda_m\right) D_L^\lambda \Big],\end{aligned} \tag{9.278}$$

$$\mathcal{L}^\nu_{mass} = -\sum_{\lambda=1}^N \overline{\nu^\lambda} m_\nu^\lambda \nu^\lambda, \tag{9.279}$$

$$\begin{aligned}\mathcal{L}_{Higgs} = -\sum_{\lambda,\kappa=1}^N \Big\{ & -H \left[\overline{\nu^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu^\kappa + Y_e^\lambda \overline{e^\lambda} e^\lambda + Y_u^\lambda \overline{U^\lambda} U^\lambda + Y_d^\lambda \overline{D^\lambda} D^\lambda \right] - \\ & - i\phi^0 \left[\overline{\nu^\lambda} \left(Y_\nu\right)_\lambda^\kappa \gamma_5 \nu^\kappa + Y_e^\lambda \overline{e^\lambda} \gamma_5 e^\lambda + Y_u^\lambda \overline{U^\lambda} \gamma_5 U^\lambda + Y_d^\lambda \overline{D^\lambda} \gamma_5 D^\lambda \right] + \\ & + i\sqrt{2}\phi^- \left[\overline{e_L^\lambda} \left(Y_\nu\right)_\lambda^\kappa \nu_R^\kappa - Y_e^\lambda \overline{e_R^\lambda} \nu_L^\lambda + Y_u^\lambda \overline{D_L^\lambda} U_R^\lambda - Y_d^\lambda \overline{D_R^\lambda} U_L^\lambda \right] - \\ & - i\sqrt{2}\phi^+ \left[\overline{\nu_R^\lambda} \left(Y_\nu\right)_\lambda^\kappa e_L^\kappa - Y_e^\lambda \overline{\nu_L^\lambda} e_R^\lambda + Y_u^\lambda \overline{U_R^\lambda} D_L^\lambda - Y_d^\lambda \overline{U_L^\lambda} D_R^\lambda \right] \Big\}.\end{aligned} \tag{9.280}$$

The fermionic action of this model is the same of the Connes model, with the only difference being in the signature of the kinetic terms. Of course, this is already an important result, however it is quite probable that the Higgs mass will still be wrong (we cannot be sure since the bosonic action in the twisted case is still to be computed – actually, even defined – but there are no reasons to expect the Higgs mass to miraculously become the right one). As shown by Connes himself [56], the Higgs mass problem as well as other issues can be solved completely by somehow introducing a new neutral scalar field σ, whose vev would induces a Majorana mass for the neutrinos. In order to generate this new field, we need to modify the twist. This is done, for two modifications of the twist by grading, in the next Chapter.

Chapter 10

Minimal Twists of the Standard Model

In the previous Chapter we studied the twist by grading of the Connes Model. The resulting model, albeit interesting, is lacking of what we sought when we directed our attention to the twisted non-commutative geometry, i.e. it lacked the new scalar field σ whose vev would give rise to a big Majorana mass for the neutrinos – thereby providing a fundamental motivation for the see-saw mechanism. In this Chapter, we will study two more twists of the Connes Model that, on the contrary, do give rise to σ.

In one of the first attempts of twisting the Connes Model [87] only part of the algebra has been twisted, but this was still sufficient to generate σ, even though fully twisting by grading is not. Indeed, this is not a coincidence. Twisting only part of the algebra means that the twisted first order condition be only valid for part of the algebra – i.e. it is not completely satisfied. In other words, the model studied in [87] actually breaks the twisted first order condition. In fact, also the two twists presented in this Chapter break the twisted first order condition, and they both generate σ.

Breaking the first order condition is not necessarily a problem in the twisted context: in the untwisted case, the first order condition was a non-commutative version of the statement that the Dirac operator be a first order differential operator. On the other hand, even a commutative version of the twisted first order condition has no such meaning, so it should not be a problem to discard it in the twisted case.

10.1 Existence of σ

The reason why σ does not appear with the canonical twist by grading resides in Proposition 9.6.3, i.e. in that

$$\left[\gamma^5 \otimes D_M, a\right]_\rho = 0 \tag{10.1}$$

when ρ is the twist by grading. Now we will explore the conditions that a different twist must satisfy in order for (10.1) to be non-zero.

Proposition 10.1.1. *Given an almost-commutative space (8.36), if a component $\gamma^5 \otimes D_M$ of the Dirac operator*

$$\mathcal{D} = \not{\partial} \otimes \mathbb{I}_F + \gamma^5 \otimes D_M + \textit{other terms} \tag{10.2}$$

commutes with the algebra $\mathcal{A}$ but does not *twist-commute with the algebra $\mathcal{A} \otimes \mathbb{C}^2$ of the corresponding twisted spectral triple with twist induced by the twisting operator (8.143), then it must be*

$$\{T_F, D_M\} \neq 0. \tag{10.3}$$

Proof. One has

$$2\left[\gamma^5 \otimes D_M, \pi(a, a')\right]_\rho = \left[\gamma^5 \otimes D_M, (1+T)\, a + (1-T)\, a'\right]_\rho = \tag{10.4}$$

$$= \gamma^5 \otimes D_M (a + a') + \gamma^5 \otimes D_M T (a - a') - - (a + a')\, \gamma^5 \otimes D_M + T (a - a')\, \gamma^5 \otimes D_M. \tag{10.5}$$

By hypothesis, $\gamma^5 \otimes D_M (a + a') - (a + a')\, \gamma^5 \otimes D_M$ vanishes. Moreover, that same hypothesis implies that $(a - a')$ commutes with $\gamma^5 \otimes D_M$, so we have

$$2\left[\gamma^5 \otimes D_M, \pi(a, a')\right]_\rho = \left\{\gamma^5 \otimes D_M, \gamma^5 \otimes T_F\right\}(a - a') = \mathbb{I}_{\mathcal{M}} \otimes \{D_M, T_F\}(a - a'). \tag{10.6}$$

This vanishes for all $a, a' \in \mathcal{A}$ if and only if $\{D_M, T_F\} = 0$, hence the result. □

Hereafter, we will consider two different such twists: one for $T_F = T_1^F \equiv \delta_I^J \delta_C^D \eta_\alpha^\beta$, and one for

$$T_F = T_2^F \equiv \delta_I^J \begin{pmatrix} \eta_\alpha^\beta & \\ & \delta_\alpha^\beta \end{pmatrix}_C^D . \tag{10.7}$$

It is easy to show that both $T_{1,2}^F$ do not anticommute with D_M defined in (9.22). Indeed, one has the following

Lemma 10.1.1. *Neither $T_{1,2}^F$ as just defined anticommute with D_M as in (9.22).*

Proof. The result follows from a straightforward calculation. T_1^F being trivial in the CD indices means that the anticommutator $\{T_F, D_M\}$ is not null if $\{D_R, \delta_I^J \eta_\alpha^\beta\} \neq 0$. Again, this is true if $\{D_R^\ell, \eta_\alpha^\beta\} \neq 0$. This anticommutator is proportional to $2k_R$, hence it is not zero.

Things are just a little bit more complicated for T_2^F. In this case, we have

$$\{T_F, D_M\} \neq 0 \qquad \Leftrightarrow \qquad D_R \delta_I^J \left(\delta_\alpha^\beta + \eta_\alpha^\beta\right) \neq 0. \tag{10.8}$$

This is true if $D_R^\ell \left(\delta_\alpha^\beta + \eta_\alpha^\beta\right) \neq 0$. Its top-left entry reads

$$2k_R \neq 0, \tag{10.9}$$

so neither T_2^F anticommutes with D_M. □

10.2 Minimal Twist of the Standard Model through T_1^F

10.2.1 The Representation

The algebra, the Hilbert space, the grading and the Dirac operator are exactly the same as in the twist by grading case; the only (slight) difference is in the representation of the algebra. In particular, the representation will be the same as in the twist by grading except that m and m' have to be replaced with each other:

$$M_r = \begin{pmatrix} \mathsf{m} \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}' \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \; M_l = \begin{pmatrix} \mathsf{m}' \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m} \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \tag{10.10}$$

while Eq.s (9.18-9.41) and (9.43) remain valid.

Remark 10.2.1. Notice that, even though we used $\gamma^5 \otimes T_1^F$ as twisting operator, the grading is still $\Gamma = \gamma^5 \otimes \gamma_F$.

10.2.2 Twisted Fluctuation

Given the strong similarity of the representations of the twist through T_1^F and the twist by grading, it is not surprising that the two fluctuations of the Dirac operators are almost identical, the only difference being that this time A_M defined in (9.70) is non-zero.

Proposition 10.2.1. *The twisted fluctuation of the Dirac operator in the twisted by T_1^F Connes Model is $\mathbb{A}_\rho = \not{\mathbb{A}} + A_Y + A_M$ with $\not{\mathbb{A}}$ and A_Y the same as in the twisted by grading Connes model, and*

$$A_M = \begin{pmatrix} & C \\ D & \end{pmatrix}_C^D \tag{10.11}$$

where

$$C = k_R\, \delta_{\dot{s}}^{\dot{t}} \begin{pmatrix} C_r & \\ & C_l \end{pmatrix}_s^t, \quad D = \bar{k}_R\, \delta_{\dot{s}}^{\dot{t}} \begin{pmatrix} D_r & \\ & D_l \end{pmatrix}_s^t \tag{10.12}$$

with

$$C_r = D_r = \Xi_{I\alpha}^{J\beta}\sigma, \quad C_l = D_l = -\Xi_{I\alpha}^{J\beta}\sigma' \tag{10.13}$$

where σ and σ' are complex fields.

Proof. Using the explicit form (9.22) of D_M, for a in (9.17) and b in (9.55) (with $\mathsf{m} \leftrightarrow \mathsf{m}'$ and $\mathsf{n} \leftrightarrow \mathsf{n}'$) one gets

$$a\left[\gamma^5 \otimes D_M, b\right]_\rho = \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix} \left[\begin{pmatrix} 0 & \gamma^5 \otimes D_R \\ \gamma^5 \otimes D_R^\dagger & 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & N \end{pmatrix}\right]_\rho =$$
$$= \begin{pmatrix} & Q\left((\gamma^5 \otimes D_R)\, N - \rho\,(R)\,(\gamma^5 \otimes D_R)\right) \\ M\left((\gamma^5 \otimes D_R^\dagger)R - \rho\,(N)\,(\gamma^5 \otimes D_R^\dagger)\right) & \end{pmatrix}_C^D . \tag{10.14}$$

With D_R given in (9.27), one computes the upper-right component C of the matrix above:

$$C_{s\dot{s}I\alpha}^{t\dot{t}J\beta} = Q_{s\dot{s}I\alpha}^{u\dot{u}K\gamma}\left[k_R\, \eta_u^v \delta_{\dot{u}}^{\dot{v}}\, \Xi_{K\gamma}^{L\delta} N_{v\dot{v}L\delta}^{t\dot{t}J\beta} - k_R\, \rho(R)_{u\dot{u}K\gamma}^{v\dot{v}L\delta}\eta_v^t \delta_{\dot{v}}^{\dot{t}}\, \Xi_{L\delta}^{J\beta}\right]. \tag{10.15}$$

Since Q, N are diagonal in the s index and proportional to $\delta_{\dot{s}}^{\dot{t}}$, the non-zero components of C are

$$(C_r)_{I\alpha}^{J\beta} = k_R\, \delta_{\dot{s}}^{\dot{t}}\, (Q_r)_{I\alpha}^{K\gamma}\left[\Xi_{K\gamma}^{L\delta}(N_r)_{L\delta}^{J\beta} - (R_l)_{K\gamma}^{L\delta}\, \Xi_{L\delta}^{J\beta}\right], \tag{10.16}$$
$$(C_l)_{I\alpha}^{J\beta} = k_R\, \delta_{\dot{s}}^{\dot{t}}\, (Q_l)_{I\alpha}^{K\gamma}\left[-\Xi_{K\gamma}^{L\delta}(N_l)_{L\delta}^{J\beta} + (R_r)_{K\gamma}^{L\delta}\, \Xi_{L\delta}^{J\beta}\right]. \tag{10.17}$$

Explicitly, from the formula (9.41) for $Q_{r/l}$ and (9.56) of $R_{r/l}, N_{r/l}$, one gets

$$Q_r(\Xi N_r - R_l\Xi) =$$
$$= \begin{pmatrix} \mathsf{c}\delta_I^J & \\ & q'\delta_I^j \end{pmatrix}_\alpha^\beta \left(\begin{pmatrix} \Xi_I^J & \\ & 0_3 \end{pmatrix}_\alpha^\beta \begin{pmatrix} \mathsf{n} \otimes \mathbb{I}_2 & \\ & \mathsf{n}' \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta - \begin{pmatrix} \mathsf{d}'\delta_I^J & \\ & p'\delta_I^J \end{pmatrix}_\alpha^\beta \begin{pmatrix} \Xi_I^J & \\ & 0_3 \end{pmatrix}_\alpha^\beta\right) =$$

$$= \begin{pmatrix} \mathsf{c}\delta_I^J & \\ & q'\delta_I^j \end{pmatrix}_\alpha^\beta \begin{pmatrix} \Xi_I^J \mathsf{n} - d'\Xi_I^J & \\ & 0_3 \end{pmatrix}_\alpha^\beta = \begin{pmatrix} c(\Xi_I^J \mathsf{n} - d'\Xi_I^J) & \\ & 0_3 \end{pmatrix}_\alpha^\beta =$$

$$= \begin{pmatrix} c(d-d')\Xi_I^J & \\ & 0_3 \end{pmatrix}_\alpha^\beta = \sigma \Xi_{\alpha I}^{J\beta} \quad (10.18)$$

and similarly

$$Q_l(-\Xi N_l + R_r\Xi) = -\sigma' \Xi_{\alpha I}^{J\beta} \quad (10.19)$$

where we define the scalar fields

$$\sigma := c(d-d'), \quad \sigma' := c'(d'-d). \quad (10.20)$$

Similarly, one computes that the lower left component D of (10.14) has non zero components

$$D_r = \bar{k}^R \delta_{\dot{s}}^{\dot{t}} M_r(\Xi R_r - N_l\Xi) = \bar{k}^R \delta_{\dot{s}}^{\dot{t}} \Xi_{\alpha I}^{\beta J} c(d-d') = \bar{k}^R \delta_{\dot{s}}^{\dot{t}} \Xi_{\alpha I}^{\beta J} \sigma, \quad (10.21)$$

$$D_l = \bar{k}^R \delta_{\dot{s}}^{\dot{t}} M_l(-\Xi R_l + N_r\Xi) = \bar{k}^R \delta_{\dot{s}}^{\dot{t}} \Xi_{\alpha I}^{\beta J} c'(-d'+d) = -\bar{k}^R \delta_{\dot{s}}^{\dot{t}} \Xi_{\alpha I}^{\beta J} \sigma'. \quad (10.22)$$

□

A free 1-form A_M is self-adjoint if and only if $D_r^\dagger = C_r$ and $D_l^\dagger = C_l$, that is

$$\sigma = \bar{\sigma}, \quad \sigma' = \bar{\sigma}'. \quad (10.23)$$

The part of the twisted fluctuation induced by the Majorana mass of the neutrino is then easily obtained.

Proposition 10.2.2. *An off-diagonal fluctuation is parametrised by two independent real scalar fields σ_r, σ_l:*

$$D_{A_M} = \gamma^5 \otimes D_M + A_M + JA_MJ^{-1} = \delta_{\dot{t}}^{\dot{t}} \begin{pmatrix} 0 & \eta_s^t D_0 + k_R\, \Xi_{I\alpha}^{J\beta}\, \bar{\Sigma}_s^t \\ \eta_s^t D_0^\dagger + \bar{k}_R\, \Xi_{I\alpha}^{J\beta}\, \Sigma_s^t & 0 \end{pmatrix}_C^D, \quad (10.24)$$

where

$$\Sigma = \begin{pmatrix} \sigma_r & \\ & \sigma_l \end{pmatrix}_s^t. \quad (10.25)$$

Proof. As in the proof of proposition 9.6.1, one has

$$JA_MJ^{-1} = \begin{pmatrix} 0 & -\mathcal{J}D\mathcal{J} \\ -\mathcal{J}C\mathcal{J} & 0 \end{pmatrix}_C^D \quad (10.26)$$

with

$$\mathcal{J}C\mathcal{J}^{-1} = \eta_s^u \tau_{\dot{s}}^{\dot{u}}\, \bar{C}_{u\dot{u}}^{v\dot{v}}\, \eta_v^t \tau_{\dot{v}}^{\dot{t}} = -\bar{C} \quad (10.27)$$

and similarly for D. Hence

$$A_M + JA_M J^{-1} = \begin{pmatrix} 0 & C+\overline{D} \\ \overline{C}+D & 0 \end{pmatrix}_C^D. \tag{10.28}$$

The explicit form of Σ follows from (10.12)-(10.13), defining $\sigma_r = \bar{\sigma} + \sigma$ and $\sigma_l = -\bar{\sigma}' - \sigma'$. □

Notice that in the proposition above D_{A_M} is selfadjoint regardless of the selfadjointness of A_M. As well, one does not need to assume that A_M is selfadjoint to ensure that the fields σ_r, σ_l are real.

Remark 10.2.2. The field σ is chiral, in the sense it has two independent components σ_r, σ_l. The one initially worked out in [87] was not chiral. This is because in the latter case, one does not double $M_3(\mathbb{C})$ and identifies the complex component of m with the complex component of Q_r. This means that the component d' of N_l identifies with the component d of R_r, so that (10.19) and (10.21) vanish, that is $C_l = D_r = 0$. Similarly, the component c' of M_l becomes c, so that $D_l = C_r$. One thus retrieves the formula (4.32) of [87] (in which the role of c and d have been interchanged).

10.2.3 Gauge Transformation

As a consequence of Proposition 10.2.1, the gauge transformations of all the bosons already present in the twist by grading case remain the same also in the twist by T_1^F case. Indeed, they are all defined the same way as before (modulo the redefinition $\mathsf{m} \leftrightarrow \mathsf{m}'$), so all the Lemmas and Propositions for their gauge transformations remain valid. However, this time there are the extra scalar fields $\sigma_{r,l}$, so we need to work out the gauge transformations of those as well. One can show that they are actually gauge-invariant.

Proposition 10.2.3. *Under a gauge transformation induced by a twist-invariant unitary u, the real fields σ_r, σ_l parameterising a self-adjoint off-diagonal 1-form A_M (proposition 10.2.2) are invariant.*

Proof. The result amounts to show that A_M is invariant under the gauge transformation

$$A_M \longrightarrow \rho(u)\left[\gamma^5 \otimes D_M, u^\dagger\right]_\rho + \rho(u) A_M u^\dagger. \tag{10.29}$$

Since $u = \rho(u)$ by hypothesis, the twisted-commutator in the expression above coincides with the usual commutator $\left[\gamma^5 \otimes D_M, u^\dagger\right]$ which is zero by (9.34). From the explicit forms (10.11) of A_M and (9.167) of u, one has

$$uA_M u^\dagger = \begin{pmatrix} & \mathfrak{A}C\mathfrak{B}^\dagger \\ \mathfrak{B}D\mathfrak{A}^\dagger & \end{pmatrix}. \tag{10.30}$$

From (10.13), one checks that $\mathfrak{A}C\mathfrak{B}^\dagger$ has components

$$\mathfrak{A}_r C_r \mathfrak{B}_r^\dagger = \sigma_r \, \mathfrak{A}_r \, \Xi_{I\alpha}^{J\beta} \, \mathfrak{B}_r^\dagger = \sigma_r \Xi_{I\alpha}^{J\beta}, \tag{10.31}$$

$$\mathfrak{A}_l C_l \mathfrak{B}_l^\dagger = \sigma_l \, \mathfrak{A}_l \Xi_{I\alpha}^{J\beta} \mathfrak{B}_l^\dagger = \sigma_l \Xi_{I\alpha}^{J\beta}, \tag{10.32}$$

where we use the explicit forms (9.169)-(9.171) of $\mathfrak{A}, \mathfrak{B}$ to get $\mathfrak{A}_r \, \Xi_{I\alpha}^{J\beta} \, \mathfrak{B}_r^\dagger = e^{i\alpha} \, \Xi_{I\alpha}^{J\beta} \, e^{-i\alpha} = \Xi_{I\alpha}^{J\beta}$, and similarly for (10.32). Hence $uA_M u^\dagger = A_M$, and the result. □

Remark 10.2.3. From a physical point of view, it makes sense that $\sigma_{r,l}$ be gauge-invariant. Indeed, their linear combination $\sigma_r - \sigma_l$ will appear in the fermionic action coupled to the right-handed neutrinos, in such a way that its vev becomes a Majorana mass for the neutrinos. The right-handed neutrinos, being completely neutral, are gauge-invariant, hence any scalar coupling to them must be itself gauge-invariant in order to yield a gauge-invariant lagrangian term.

10.2.4 Fermionic Action

Once again, the strong similarity of the representations of the twist by grading and the twist by T_1^F cases imply that their fermionic actions will be almost identical as well. In particular, the fermionic lagrangian $\mathcal{L}_f^{T_1^F}$ of the twisted by T_1^F Connes Model will be

$$\mathcal{L}_f^{T_1^F} = \mathcal{L}_f^{\text{TbG}} + \mathcal{L}_{Majorana}, \tag{10.33}$$

where $\mathcal{L}_f^{\text{TbG}}$ is the one of (9.277) and $\mathcal{L}_{Majorana}$ is a Majorana mass term for the right-handed neutrinos. To show this, one needs to follow the very same procedure described in section 9.9. Once again, we will not do the full computation, since it is very long and hardly interesting (it merely being the computation of the scalar product of two vectors obtained by applying very big matrices to other very long vectors). In order to compute the manifold part of the scalar product,we need the following

Lemma 10.2.1. *One has*

$$\mathfrak{A}_{\gamma^5}(\phi, \xi) = -2 \int_{\mathcal{M}} d\mu \, \left(\overline{\varphi}^\dagger \sigma_2\right) \zeta, \tag{10.34}$$

where $\mathfrak{A}_{\mathcal{O}}$, ϕ and ξ are as in (9.227), (9.232).

Proof. Following the same steps as in the proof of Lemma 9.9.1, one has

$$\gamma^5 \xi = \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = \begin{pmatrix} \zeta \\ -\zeta \end{pmatrix}. \tag{10.35}$$

Then, one has

$$\mathfrak{A}_{\gamma^5}(\phi,\xi) = \langle \mathcal{J}\phi\,,\,\gamma^5\xi\rangle = -i\left(\overline{\varphi}^\dagger\tilde{\sigma}^{2\dagger},\quad \overline{\varphi}^\dagger\sigma^{2\dagger}\right)\begin{pmatrix}\zeta\\ -\zeta\end{pmatrix} =$$

$$= -i\int_{\mathcal{M}} d\mu\;\overline{\varphi}^\dagger\sigma^2\zeta + \overline{\varphi}^\dagger\sigma^2\zeta = -2\int_{\mathcal{M}} d\mu\;\overline{\varphi}^\dagger\sigma_2\zeta, \quad (10.36)$$

where we used $\sigma^2 = -i\sigma_2$. □

Following the same procedure adopted in section 9.9, one can calculate the fermionic action for this model. Using the identifications Eq. (9.242-9.249), and once again defining

$$S_{Majorana} = 4\int_{\mathcal{M}} d\mu\;\mathcal{L}_{Majorana}, \quad (10.37)$$

we get

$$\mathcal{L}_{Majorana} = -\frac{1}{2}\left[-i\overline{k_R}\overline{\nu_R}\left(1+\frac{\sigma}{v_\sigma}\right)\tilde{\varphi}_1 + k_R\left(\tilde{\overline{\varphi}}^\dagger_{17}\sigma_2\right)\left(1+\frac{\sigma}{v_\sigma}\right)\nu_R\right], \quad (10.38)$$

where $\sigma/v_\sigma \equiv \sigma_r - \sigma_l$. Now, we define

$$k_R =: -iM, \qquad\qquad M =: Y_M v_\sigma \in \mathbb{R} \quad (10.39)$$

furthermore, we identify

$$\tilde{\varphi}_1 = \nu_R^c, \qquad\qquad \left(-i\tilde{\overline{\varphi}}_{17}\sigma_2\right) = \overline{\nu_R^c}, \quad (10.40)$$

so that the lagrangian becomes

$$\mathcal{L}_{Majorana} = -\frac{Y_M}{2}\left(\sigma + v_\sigma\right)\left[\overline{\nu_R}\nu_R^c + \overline{\nu_R^c}\nu_R\right]. \quad (10.41)$$

This is the result for a single generation of fermions. For more generations, Y_M becomes a self-adjoint matrix.

Then, the full fermionic lagrangian $\mathcal{L}_f^{T_1^F}$ of the twisted by T_1^F Connes Model for N generations of *Dirac* fermions is

$$\mathcal{L}_f^{T_1^F} = \mathcal{L}_f^{\mathrm{TbG}} + \mathcal{L}_{Majorana}, \quad (10.42)$$

where $\mathcal{L}_f^{\mathrm{TbG}}$ is defined as in (9.277) and

$$\mathcal{L}_{Majorana} = -\frac{1}{2}\sum_{\lambda,\kappa=1}^{N}\left(\sigma + v_\sigma\right)\left[\overline{\nu_R^\lambda}\left(Y_M\right)_\lambda^\kappa\nu_R^{\kappa c} + \overline{\nu_R^{\lambda c}}\left(Y_M\right)_\lambda^\kappa\nu_R^\kappa\right]. \quad (10.43)$$

This model is very interesting: first of all, being twisted, it is Lorentz-invariant; moreover, it contains the scalar field σ that is crucial for sorting out many issues (the wrong Higgs mass, the vacuum metastability [56], the see-saw mechanism); and it is in a sense economical, since it does not generate any more terms than needed. Nevertheless, one may wonder how things might change when considering a yet different twist: it was for this reason that we studied the twist by T_2^F as well. The results will be presented in the next section.

10.3 Minimal Twist of the Standard Model through T_2^F

10.3.1 The Representation

Once again, the algebra, the Hilbert space, the grading and the Dirac operator are exactly the same as in the twist by grading case, and the only difference resides in the representation of the algebra. Once again, the difference will be concentrated in $M_{r,l}$: in particular, the representation will be the same as in the twist by grading except that

$$M_r = \mathsf{m}\,\delta_\alpha^\beta, \quad M_l = \mathsf{m}'\,\delta_\alpha^\beta, \tag{10.44}$$

while Eq.s (9.18-9.41) and (9.43) remain valid.

Remark 10.3.1. Again, even though we used $\gamma^5 \otimes T_2^F$ as twisting operator, the grading is still $\Gamma = \gamma^5 \otimes \gamma_F$.

10.3.2 Twisted Fluctuation

The twisted fluctuation of the Dirac operator in the twisted by T_2^F Connes Model contains all the terms that appear in the twisted by T_1^F model, plus some new terms. Indeed, all the Lemmas and Propositions for the fluctuation remain valid (as one can easily check case by case, by simply renaming some entries of a), except for Lemma 9.6.1 and Proposition 9.6.1. Indeed, in (9.71) S will remain what it is, but the bottom-right entry will be non zero.

Lemma 10.3.1. *In the twisted by T_2^F Connes model, one has*

$$\left[\gamma^5 \otimes D_Y, b\right]_\rho = \begin{pmatrix} S & 0 \\ 0 & \tilde{S} \end{pmatrix}_C^D \tag{10.45}$$

where S is the same as in (9.71) and $\tilde{S}$ has components

$$\tilde{S}_{s\dot{s}I\alpha}^{tiJ\beta} = \delta_{\dot{s}}^{i} \left(\eta_s^u \,(D_0)_{I\alpha}^{I\beta}\, N_{uI}^{tJ} - \rho\,(N)_{sI}^{uJ}\, \eta_u^t\,(D_0)_{J\alpha}^{J\beta} \right), \tag{10.46}$$

where

$$N_{sI\alpha}^{tJ\beta} =: \delta_\alpha^\beta N_{sI}^{tJ}. \tag{10.47}$$

Proof. By explicit calculation, one has

$$\tilde{S}_{s\dot{s}I\alpha}^{t\dot{t}J\beta} = \delta_{\dot{s}}^{\dot{t}} \left(\eta_s^u \left(D_0^\dagger \right)_{I\alpha}^{K\gamma} N_{uK\gamma}^{tJ\beta} - \rho\,(N)_{sI\alpha}^{uK\gamma}\, \eta_u^t \left(D_0^\dagger \right)_{K\gamma}^{J\beta} \right) = \tag{10.48}$$

$$= \delta_{\dot{s}}^{\dot{t}} \left(\eta_s^u \,(D_0)_{I\alpha}^{I\beta}\, N_{uI}^{tJ} - \rho\,(N)_{sI}^{uJ}\, \eta_u^t \,(D_0)_{J\alpha}^{J\beta} \right), \tag{10.49}$$

where we used $D_0^\dagger = D_0$ as well as the fact that D_0 is diagonal in the IJ indices. □

Now we can compute the updated 1-form generated by the Yukawa couplings of the fermions.

Proposition 10.3.1. *In the twisted by T_2^F Connes model, one has*

$$A_Y = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_C^D, \tag{10.50}$$

where A is the same of (9.83) and

$$B = \delta_{\dot{s}}^{\dot{t}} \begin{pmatrix} B_r & \\ & B_l \end{pmatrix}_s^t \tag{10.51}$$

with

$$B_r = \begin{pmatrix} \left(D_0^\ell\right)_\alpha^\beta \sigma & \\ & \left(D_0^r\right)_\alpha^\beta \phi_S \end{pmatrix}, \qquad B_l = \begin{pmatrix} \left(D_0^\ell\right)_\alpha^\beta \sigma' & \\ & \left(D_0^r\right)_\alpha^\beta \phi'_S \end{pmatrix}, \tag{10.52}$$

where σ and σ' are the same of Proposition 10.2.1 and ϕ_S, ϕ'_S are 3×3 complex matrix fields.

Proof. One has $B = M\tilde{S}$. Both M and $\tilde{S}$ are trivial in the $\dot{s}\dot{t}$ indices, and both are diagonal in the st indices. This means that the only non-zero components of B are for $s = t = r$ and $s = t = l$. Moreover, D_0 is diagonal in the IJ indices and one has $(D_0)_{I\alpha}^{J\beta} = \left(D_0^I\right)_\alpha^\beta \delta_I^J$. Hence we get

$$B_{r\dot{s}I\alpha}^{r\dot{t}J\beta} = \delta_{\dot{s}}^{\dot{t}} \,(B_r)_{I\alpha}^{J\beta} \;\text{ with }\; (B_r)_{I\alpha}^{J\beta} = \sum_K \mathsf{m}_I^K \left(\left(D_0^K\right)_\alpha^\beta \mathsf{n}_K^J - \mathsf{n'}_K^J \left(D_0^J\right)_\alpha^\beta \right) \tag{10.53}$$

$$B_{l\dot{s}I\alpha}^{l\dot{t}J\beta} = \delta_{\dot{s}}^{\dot{t}} \,(B_l)_{I\alpha}^{J\beta} \;\text{ with }\; (B_l)_{I\alpha}^{J\beta} = \sum_K \mathsf{m'}_I^K \left(- \left(D_0^K\right)_\alpha^\beta \mathsf{n'}_K^J + \mathsf{n}_K^J \left(D_0^J\right)_\alpha^\beta \right). \tag{10.54}$$

From the explicit expressions (9.43) of m, n and (9.23) of D_0, one gets

$$(B_r)^{J\beta}_{I\alpha} = \begin{pmatrix} \left(D_0^\ell\right)^\beta_\alpha c\,(d-d') & \\ & (D_0^q)^\beta_\alpha m\,(n-n') \end{pmatrix}^J_I \equiv \begin{pmatrix} \left(D_0^\ell\right)^\beta_\alpha \sigma & \\ & (D_0^q)^\beta_\alpha \phi_S \end{pmatrix}, \tag{10.55}$$

$$(B_l)^{J\beta}_{I\alpha} = \begin{pmatrix} \left(D_0^\ell\right)^\beta_\alpha c'\,(d-d') & \\ & (D_0^q)^\beta_\alpha m'\,(n-n') \end{pmatrix}^J_I \equiv \begin{pmatrix} \left(D_0^\ell\right)^\beta_\alpha \sigma' & \\ & (D_0^q)^\beta_\alpha \phi'_S \end{pmatrix}. \tag{10.56}$$

□

Imposing now the selfadjointness condition, we get the

Corollary 10.3.1.1. *A selfadjoint diagonal twisted 1-form (9.60) is parametrized by two scalar quaternionic field H_r, H_l, by two real scalar fields σ, σ' and by two hermitian matrix fields ϕ_S, ϕ'_S.*

Proof. The twisted 1-form (10.50) is selfadjoint if and only if

$$H_2 = H_1^\dagger = H_r \qquad H'_2 = H_1'^\dagger = H_l, \tag{10.57}$$

$$\sigma = \overline{\sigma} \qquad \sigma' = \overline{\sigma'}, \tag{10.58}$$

$$\phi_S = \phi_S^\dagger \qquad \phi'_S = \phi_S'^\dagger. \tag{10.59}$$

□

Gathering the previous results, one works out the fields induced by the Yukawa coupling of fermions via a twisted fluctuation of the metric.

Proposition 10.3.2. *A selfadjoint diagonal fluctuation is*

$$D_{A_Y} = \gamma^5 \otimes D_Y + A_Y + J A_Y J^{-1} = \begin{pmatrix} \eta^t_s \delta^{\dot t}_{\dot s} D_0 + A + \overline{B} & \\ & \eta^t_s \delta^{\dot t}_{\dot s} D_0^\dagger + \overline{A} + B \end{pmatrix} \tag{10.60}$$

where A is as in (9.83) and B is as in (10.51).

Proof. The result follows using the same procedure as in the proof of Proposition 9.6.2. □

10.3.3 Tentative[1] Fermionic Action

Yet again, the strong similarity of the representations of the twist by grading and the twist by T_2^F cases imply that their fermionic actions will be almost identical as well. In particular, the fermionic lagrangian $\mathcal{L}_f^{T_2^F}$ of the twisted by T_2^F Connes Model will be

$$\mathcal{L}_f^{T_2^F} = \mathcal{L}_f^{T_1^F} + \mathcal{L}_{scalar}, \tag{10.61}$$

where $\mathcal{L}_f^{T_1^F}$ is the one of (10.42) and $\mathcal{L}_{scalar}$ contains Yukawa couplings between $\sigma_{r,l}$ and the leptons, and between ϕ_S, ϕ'_S and the quarks. To show this, one needs to follow the very same procedure described in section 9.9. Once again, we will not do the full computation, since it is very long and hardly interesting (it merely being the computation of the scalar product of two vectors obtained by applying very big matrices to other very long vectors), but we will give directly the result, while the calculations will be reported in the Appendix D.

Using the identifications Eq. (9.242-9.249), and once again defining

$$S_{scalar} = 4\int_{\mathcal{M}} d\mu\, \mathcal{L}_{scalar}, \tag{10.62}$$

we get

$$\begin{aligned}\mathcal{L}_{scalar} = \frac{1}{2}\Bigg[&\frac{m_\nu}{v_\sigma}\sigma\overline{\nu_R}\nu_L + \frac{m_\nu}{v_\sigma}\sigma\overline{\nu_L}\nu_R + \frac{m_e}{v_\sigma}\sigma\overline{e_R}e_L + \frac{m_e}{v_\sigma}\sigma\overline{e_L}e_R + \\ &+\frac{m_u}{v_\Phi}\overline{u_R^i}\tilde{\Phi}_S u_L^i + \frac{m_u}{v_\Phi}\overline{u_L^i}\tilde{\Phi}_S u_R^i + \frac{m_d}{v_\Phi}\overline{d_R^i}\tilde{\Phi}_S d_L^i + \frac{m_d}{v_\Phi}\overline{d_L^i}\tilde{\Phi}_S d_R^i\Bigg],\end{aligned} \tag{10.63}$$

where $\sigma/v_\sigma = \sigma_r - \sigma_l$ and $\tilde{\Phi}_S/v_\Phi = \phi_S - \phi'_S$. Notice that the couplings are the very same Yukawas of the Higgs: this makes this model phenomenologically very interesting.

10.3.4 Identification of the Physical Degrees of Freedom

The attentive reader may have noticed that there is a very big problem in (10.63): we stressed in Remark 10.2.3 that σ should be gauge invariant since it couples with the right-handed neutrinos which are gauge invariant. And yet, in (10.63) σ appears once again Yukawa-coupled with the leptons: those lagrangian terms seem

[1] In this section, we will follow the same procedure already used for the twist-by-grading and the twist by T_1^F in order to calculate the fermionic action of this model. However, as we will see, the action so obtained will be inconsistent, as outlined in Proposition 10.3.3. The correct fermionic action will be calculated in section 10.3.5.

to break the gauge symmetry. However, by construction, the twisted fermionic action is gauge invariant, as we showed in section 8.3. How can this be possible?

The answer lies in the fact that in non-commutative geometry the gauge transformations for the bosons are *not* defined as field transformations, but as a transformation of the Dirac operator. This means that it might happen (and this is indeed the case) that the same bosonic fields transform differently in different parts of the lagrangian, because they appear in different entries of the Dirac operator but those entries transform differently. Of course, bosons that act this way cannot have any physical meaning whatsoever. We will need then to identify the physical degrees of freedom of the model.

To do so, we have first to understand how the fields transform under a gauge transformation. It will be sufficient to study the gauge behaviour of those fields that appear in the "new" part B of (10.51) of the Dirac operator, i.e. σ and Φ_S, for we have already studied the gauge behaviour of all other bosons in sections 9.8, 10.2.3.

Proposition 10.3.3. *Under a gauge transformation induced by a twist-invariant unitary u of form (9.167), the fields σ and $\tilde{\Phi}_S$ transform as*

$$k_R\sigma \to k_R\sigma \tag{10.64}$$

$$\mathsf{k}^I\sigma \to \mathsf{q}\mathsf{k}^I\overline{\boldsymbol{\alpha}}\sigma \tag{10.65}$$

$$\mathsf{k}^I\tilde{\Phi}_S \to \mathsf{q}\mathsf{k}^I\overline{\boldsymbol{\alpha}}\mathsf{m}\tilde{\Phi}_S\mathsf{m}^\dagger. \tag{10.66}$$

Proof. Since $\sigma \sim \sigma_r - \sigma_l$ and similarly $\tilde{\Phi}_S \sim \phi_S - \phi'_S$, it is sufficient to show that e.g. σ_r and ϕ_S transform as in (10.64-10.66). For this reason, and since everything is trivial in the *st* indices, we will omit them. As a shorthand, we define

$$A_2 = \begin{pmatrix} 0 & \\ & B \end{pmatrix}_C^D, \tag{10.67}$$

with B as in (10.51).

We want to evaluate the gauge transformation of A_2. By definition of gauge transformation, we have

$$A_2 \to uJu^*J^{-1}A_2\left(uJu^*J^{-1}\right)^\dagger. \tag{10.68}$$

Using the explicit expression (9.167) of u, we get

$$uJu^*J^{-1} = \begin{pmatrix} \mathfrak{A} & \\ & \mathfrak{B} \end{pmatrix}\begin{pmatrix} \mathfrak{B} & \\ & \mathfrak{A} \end{pmatrix} = \begin{pmatrix} \mathfrak{A}\mathfrak{B} & \\ & \mathfrak{B}\mathfrak{A} \end{pmatrix} \tag{10.69}$$

so that

$$uJu^*J^{-1}A_2\left(uJu^*J^{-1}\right)^\dagger. = \begin{pmatrix} 0 & \\ & \mathfrak{B}\mathfrak{A}B\mathfrak{A}^\dagger\mathfrak{B}^\dagger \end{pmatrix}. \tag{10.70}$$

The $C = D = 1$ entry reads

$$\mathfrak{B}\mathfrak{A}B\mathfrak{A}^\dagger\mathfrak{B}^\dagger = \begin{pmatrix} \left(\mathfrak{A}D_0^\ell\mathfrak{A}^\dagger\right)_\alpha^\beta e^{i\alpha}\sigma e^{-i\alpha} & \\ & \left(\mathfrak{A}D_0^q\mathfrak{A}^\dagger\right)_\alpha^\beta \mathsf{m}\phi_S\mathsf{m}^\dagger \end{pmatrix}_I^J . \tag{10.71}$$

In the $\alpha\beta$ indices we have

$$\left(\mathfrak{A}D_0^I\mathfrak{A}^\dagger\right) = \begin{pmatrix} \boldsymbol{\alpha} & \\ & \mathsf{q} \end{pmatrix} \begin{pmatrix} 0 & \overline{\mathsf{k}^I} \\ \mathsf{k}^I & 0 \end{pmatrix} \begin{pmatrix} \overline{\boldsymbol{\alpha}} & \\ & \mathsf{q}^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\alpha}\overline{\mathsf{k}^I}q^\dagger \\ \mathsf{q}\mathsf{k}^I\overline{\boldsymbol{\alpha}} & \end{pmatrix}, \tag{10.72}$$

hence the result. □

Remark 10.3.2. Notice that twist-invariant gauge transformations are, them being twist-invariant by definition, always the same, regardless of the twist adopted.

The gauge transformations (10.65, 10.66) we just found give us a hint on how we should proceed: when σ is Yukawa coupled, it transforms exactly like the Higgs field. Then, we can reabsorb those occurrences of σ into the definition of H. However, there is a problem: H couples to all fermions, while on the other hand σ only couples to leptons. This means that, even if we redefine H in order to make the Yukawa couplings of σ with leptons disappear, new Yukawa couplings of σ with quarks will be generated. However, we still have something that we can use to reabsorb these new terms, i.e. $\tilde{\Phi}_S$, and in particular its trace: by redefining the diagonal entries of $\tilde{\Phi}_S$, we can reabsorb the newly generated σ-quarks couplings as well.

With all this in mind, we redefine the scalar fields in the following way:

$$\phi_1 =: i\sqrt{2}\frac{\phi^-}{v/\sqrt{2}}, \tag{10.73}$$

$$\phi_2 =: \frac{H + i\phi^0}{v/\sqrt{2}} + \frac{\sigma}{2v_\sigma}, \tag{10.74}$$

$$\frac{\Phi_S}{v_\Phi} := \begin{pmatrix} \phi_1 - \phi_1' - \frac{\sigma}{v_\sigma} & \phi_{12} - \phi_{12}' & \phi_{13} - \phi_{13}' \\ \phi_{12}^\dagger - \phi_{12}'^\dagger & \phi_2 - \phi_2' - \frac{\sigma}{v_\sigma} & \phi_{23} - \phi_{23}' \\ \phi_{13}^\dagger - \phi_{13}'^\dagger & \phi_{23}^\dagger - \phi_{23}'^\dagger & \phi_3 - \phi_3' - \frac{\sigma}{v_\sigma} \end{pmatrix}. \tag{10.75}$$

10.3.5 Fermionic Action

With the redefinition (10.73-10.75), the full fermionic lagrangian $\mathcal{L}_f^{T_2^F}$ of the twisted by T_2^F Connes Model for N generations of *Dirac* fermions becomes

$$\mathcal{L}_f^{T_2^F} = \mathcal{L}_f^{T_1^F} + \mathcal{L}_f^{\Phi_S}, \tag{10.76}$$

where $\mathcal{L}_f^{T_1^F}$ is the one of (10.42) and

$$\mathcal{L}_f^{\Phi_S} = \frac{1}{2} \sum_{\lambda=1}^{N} \left[\frac{m_u^\lambda}{v_\Phi} \overline{U_R^\lambda} \Phi_S U_L^\lambda + \frac{m_d^\lambda}{v_\Phi} \overline{D_R^\lambda} \Phi_S D_L^\lambda \right] + \text{ h.c. } , \tag{10.77}$$

where U_{LR}, D_{LR} are defined as in (9.270).

This model is once again very interesting. It contains all the features already present in the twisted by T_1^F Connes model, and on top of that it also contains a new scalar, color-octet field Φ_S that couples to the quarks. These couplings are phenomenologically very interesting, since they are exactly the same Yukawa couplings of the Higgs fields, all rescaled by the unknown parameter v_Φ. This parameter probably needs to be very large to allow the model to be compatible with the experiments; were it to agree with the scale v_σ of σ, this would be yet another hint that something important happens at the $10^{15} GeV$ scale.

Chapter 11

Conclusions and Future Prospects

In this work we dealt with the issue of neutrino masses, focusing our attention to both of its main open problems, namely the mechanism neutrino masses originate from, as well as the reason why they are so tiny. In the first part of this work we studied the see-saw mechanism, that naturally provides an explanation to both matters through the introduction of a very big Majorana mass for the neutrinos. In order to allow for its presence, new particles should be added to the Standard Model, and in particular we studied the so-called Type III See-saw Models, i.e. those models that introduce $SU(2)$ fermion triplets $\overrightarrow{\Sigma}$. These triplets have one neutral component, which has the same behaviour as a right-handed neutrino, and two charged components, which combine together to form a charged lepton-like field. We noticed that both the neutral and the charged components induce a mixing between, respectively, the neutrinos and the charged leptons. These mixings alter the Standard Model couplings between leptons and the $W^{\pm}$ and Z^0 bosons, inducing tree-level flavour changing neutral currents as well as lepton flavour violating processes, which we used to find the upper bounds on the parameters of five Type III See-saw models: a general case, in which an arbitrary number of triplets has been integrated out; two models with three triplets, for both normal and inverted hierarchy of neutrino masses; and two models with two triplets, again for both hierarchies. The constraint we found are quite stringent, of order ranging from 10^{-4} for the general case to 10^{-6}-10^{-7} for the two triplets cases.

Whatever the particle that one adds to the Standard Model to mediate the See-saw Mechanism, the assumption of the presence of a very big Majorana mass remains very natural. In standard QFT, it is normal to add to the lagrangian all possible terms that respect the fundamental symmetries of the theory, for even if they are not explicitly present, they would still be generated by quantum effects. A Majorana mass term for the newly introduced particles respects all the gauge symmetries of the Standard Model, and it only violates the lepton number symmetry, which is actually anomalous and hence not a symmetry at all. Moreover, since it cannot be generated by the Higgs mechanism (for such a coupling would break the gauge symmetries) it must be mediated by some new phenomenon, which then must obviously take place at a higher energy scale than the one we can reach in experiments. Then, it is quite natural for the Majorana mass to represent a very big mass scale. Nevertheless, the see-saw mechanism skirts completely the origin of said Majorana mass. In order to explain its presence, we studied models based on twisted non-commutative geometry.

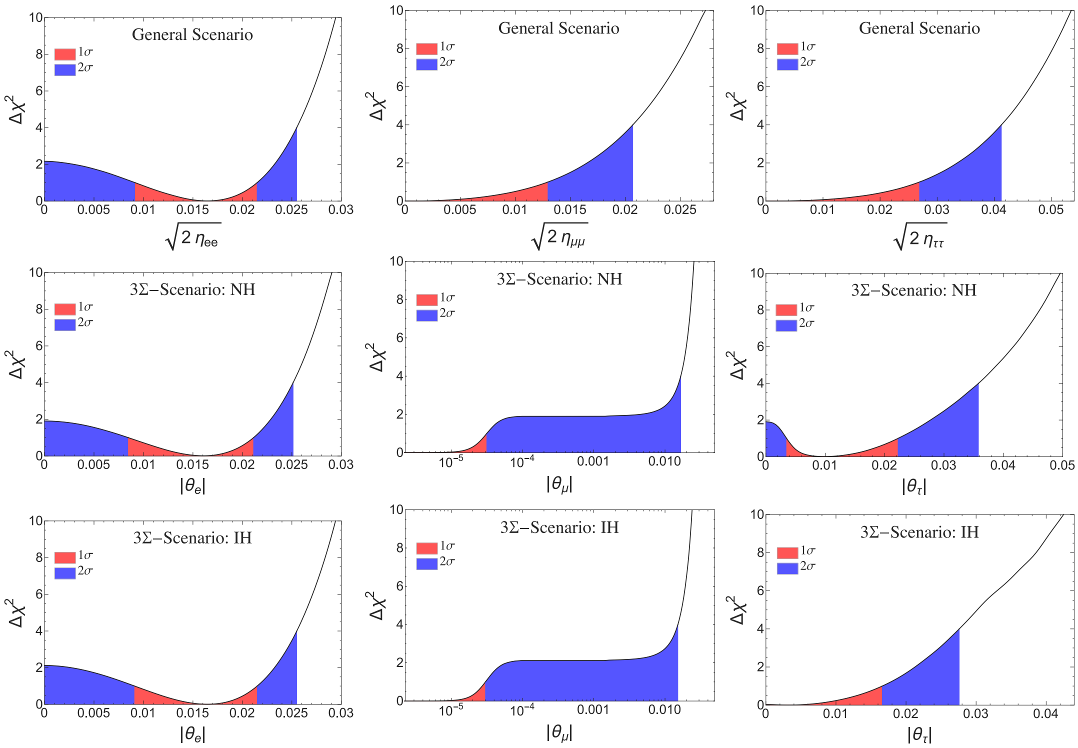
We studied three different twists of the Connes Model (i.e. the non-commutative geometry formulation of the Standard Model), starting with the somehow canonical "twist by grading", and then another two slight modifications of its. For the three twists, we studied the fluctuation of the Dirac operator (that defines the bosonic content of the theory), the gauge transformations of the bosons and finally the fermionic part of the action (i.e. the action terms that involve fermions). The Standard Model particles are always all present, and in addition new bosons appear depending on the twist considered. Common to all three twists, there is a new gauge-invariant axial vector field X_μ that somehow seems to mediate the transition from Euclidean to Lorentzian signature in the action. On top of that, in the two non-canonical twists appears also a new gauge-invariant neutral scalar σ, whose vev generates the neutrino Majorana mass we sought. Finally, in the third twist considered appears a new $SU(3)$ octet scalar field Φ_S, which couples Yukawa-like with the quarks.

The results just presented can be further developed. There is still no known definition of the twisted bosonic part of the action, and this of course is very important in order to have a full action to do a phenomenological analysis with; so the first step would be to properly define the twisted bosonic action and compute it – at least for the three twists we considered. Moreover, non-commutative geometry is an intrinsically non-quantum theory, so a quantization procedure needs to be defined. Furthermore, in non-twisted non-commutative geometry in the bosonic action appear some terms that are typical of modified gravity theories, and the same will probably happen also in the twisted version, once a suitable bosonic action is defined. Assuming this happens, it would be very interesting to check the gravitational implications of twisted non-commutative geometry, in particular by confirming whether it is still compatible with GR experiments, and then checking whether the new action terms are able to explain the observed Dark Matter and/or Dark Energy effects (some work in this direction has already been done in the untwisted case – see e.g. [97]). Finally, we know from the non-twisted case that the natural scale for the new scalar σ is the GUT scale [56], yet there have been few attempts so far to study NCGs with the algebras typical of the GUTs. It would be quite interesting to investigate “Non-Commutative GUTs” and see whether the twisting procedure would still be needed to generate the field σ.

Appendix A

Bounds on the mixing of heavy fermion triplets

For completeness we present here the individual bounds on $\sqrt{2\,|\eta_{\alpha\beta}|}$.



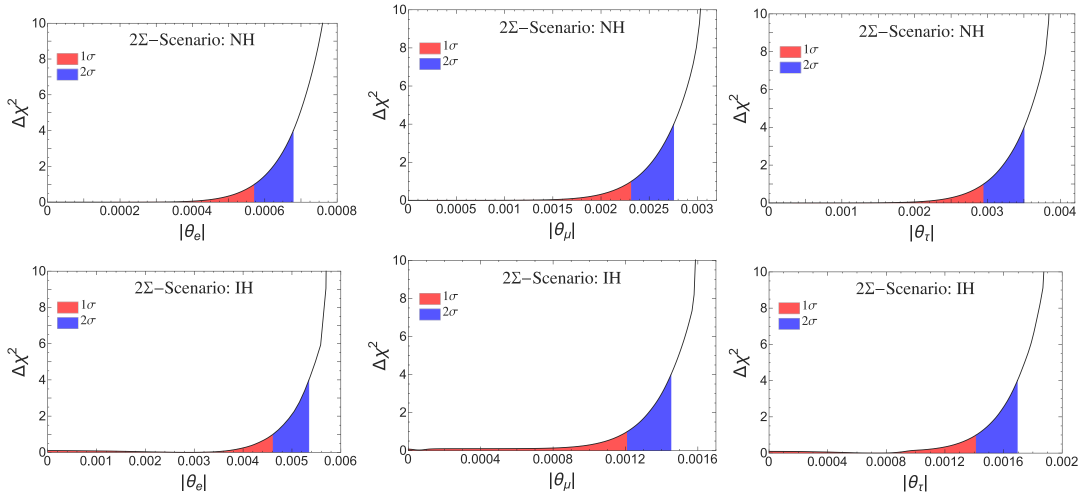
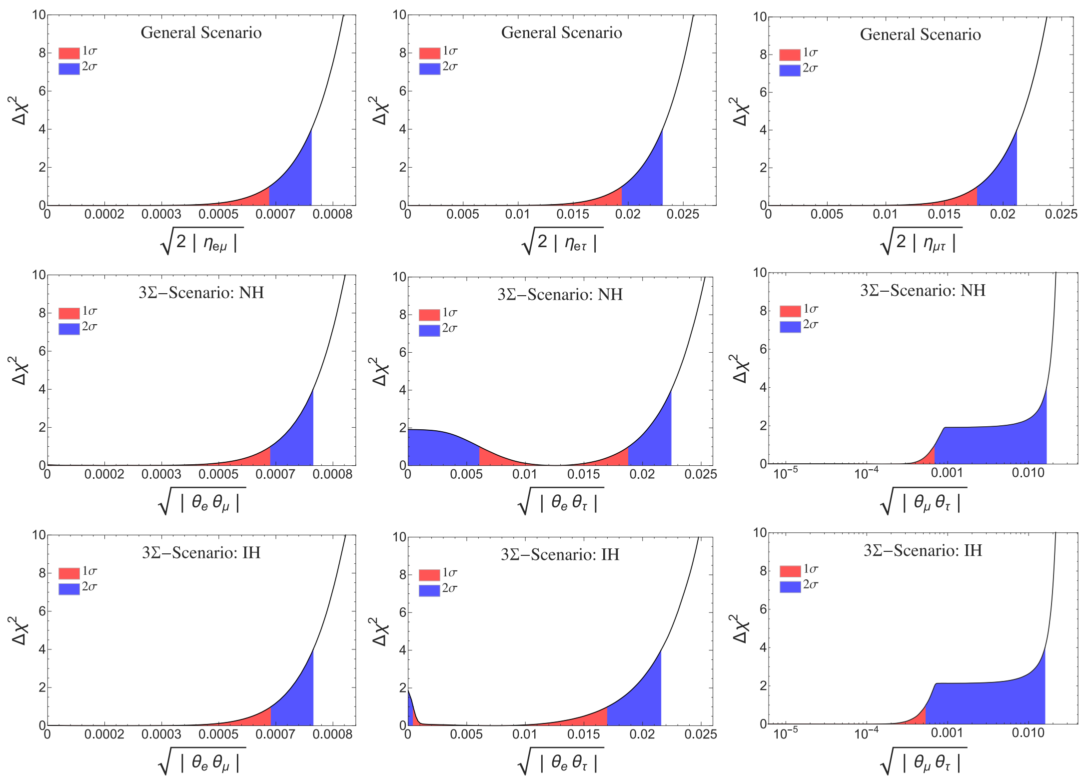


Figure A.1: $\Delta\chi^2$ profile minimized over all fit parameters but one single θ_α (or $\sqrt{2\eta_{\alpha\alpha}}$ for G-SS). In the upper panels the G-SS fit results are plotted, while the middle and lower panels show the results of the 3Σ-SS and the 2Σ-SS, for NH and IH respectively.



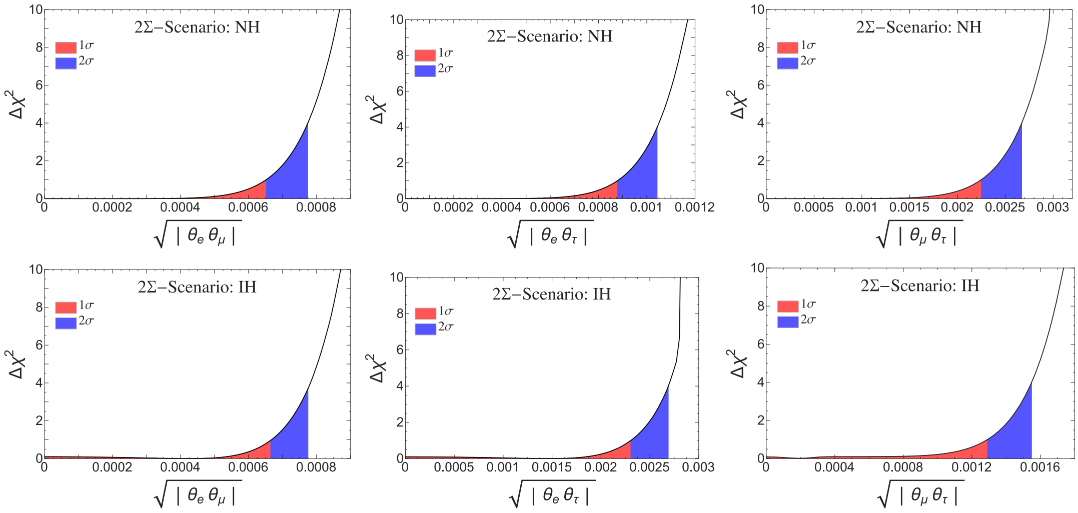


Figure A.2: Constraints on the off-diagonal entries of $\eta_{\alpha\beta}$ (or $|\theta_\alpha\theta_\beta|$ for 3Σ-SS and 2Σ-SS). In the upper panels the G-SS fit results are plotted, while the middle and lower panels show the results of the 3Σ-SS and the 2Σ-SS, for NH and IH respectively.

Appendix B

Dirac matrices and real structure

Let $\sigma_{j=1,2,3}$ be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.1}$$

In four-dimensional euclidean space, the Dirac matrices (in chiral representation) are

$$\gamma_E^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_E^5 := \gamma_E^1 \gamma_E^2 \gamma_E^3 \gamma_E^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{B.2}$$

where, for $\mu = 0, j$, we define

$$\sigma^\mu := \{\mathbb{I}_2, -i\sigma_j\}, \qquad \tilde{\sigma}^\mu := \{\mathbb{I}_2, i\sigma_j\}. \tag{B.3}$$

On a (non-necessarily flat) riemannian spin manifold, the Dirac matrices are linear combinations of the euclidean ones,

$$\gamma^\mu = e_\mu^\alpha \gamma_E^\alpha \tag{B.4}$$

where $\left\{e_\mu^\alpha\right\}$ are the vierbein, which are real fields on $\mathcal{M}$. These Dirac matrices are no longer constant on $\mathcal{M}$. This is a general result of spin geometry that the charge conjugation anticommutes with the Dirac matrices, and commutes with the spin derivative. For sake of completness; we check it explicitly for a four dimensional riemannian manifold;

Lemma B.0.1. *The real structure satisfies*

$$\mathcal{J}\gamma^\mu = -\gamma^\mu \mathcal{J}, \quad \mathcal{J}\omega_\mu^s = \omega_\mu^s \mathcal{J}, \quad \mathcal{J}\nabla_\mu^s = +\nabla_\mu^s \mathcal{J}. \tag{B.5}$$

Proof. Let us first show that $\mathcal{J}$ anticommutes with the euclidean Dirac matrices,

$$\{\mathcal{J}, \gamma_E^\mu\} = 0. \tag{B.6}$$

From the explicit forms (9.64) of $\mathcal{J}$, this is equivalent to

$$\gamma_E^0\,\gamma_E^2\,\bar{\gamma}_E^\mu = -\gamma_E^\mu\gamma_E^0\,\gamma_E^2 \tag{B.7}$$

which is true for $\mu = 0, 2$ since then $\bar{\gamma}_E^\mu = \gamma_E^\mu$ anticommutes with $\gamma_E^0\gamma_E^2$, and is also true for $\mu = 1, 2$ in which case $\bar{\gamma}_E^\mu = -\gamma_E^\mu$ commutes with $\gamma_E^0\gamma_E^2$.

Since the spin connection is a real linear combination of products of two euclidean Dirac matrices, it commutes with $\mathcal{J}$. The latter, having constant components, commutes with ∂_μ, hence also with the spin covariant derivative ∇_μ.

These results hold as well in the curved case, for then one has from (B.4)

$$\{\mathcal{J}, \gamma^\mu\} = e_\mu^\alpha\,\{\mathcal{J}, \gamma_E^\alpha\} = 0. \tag{B.8}$$

□

Appendix C

Components of the gauge sector of the twisted fluctuation

The components of the free twisted fluctuation of proposition 9.7.4 are $Z^r_\mu = \delta^{\dot t}_{\dot s}(Z^r_\mu)^{J\beta}_{I\alpha}$ given by (we invert the order of the lepto-colour and flavour indices in order to make the comparison with the non-twisted case easier)

$$(Z_\mu)^{0\dot 1}_{0\dot 1} = 2a_\mu, \tag{C.1}$$
$$(Z_\mu)^{0\dot 2}_{0\dot 2} = 2a_\mu + ig_1 B_\mu, \tag{C.2}$$
$$(Z_\mu)^{0b}_{0a} = \delta^b_a\left(w^\mu - a_\mu\right) + i\left(\delta^b_a \frac{g_1 B_\mu}{2} - \frac{g_2}{2}(W_\mu)^b_a\right), \tag{C.3}$$
$$(Z_\mu)^{j\dot 1}_{i\dot 1} = \left(a_\mu \delta^j_i + (g_\mu)^j_i\right) - i\left(\frac{2g_1 B_\mu}{3}\delta^j_i + \frac{g_3}{2}(V_\mu)^j_i\right), \tag{C.4}$$
$$(Z_\mu)^{j\dot 2}_{i\dot 2} = \left(a_\mu \delta^j_i + (g_\mu)^j_i\right) + i\left(\frac{g_1 B_\mu}{3}\delta^j_i - \frac{g_3}{2}(V_\mu)^j_i\right), \tag{C.5}$$
$$(Z_\mu)^{jb}_{ia} = \left(\delta^b_a w_\mu \delta^j_i - (g_\mu)^j_i\right) - i\left(\delta^b_a\left(\frac{g_1 B_\mu}{6}\delta^i_i + \frac{g_3}{2}(V_\mu)^j_i\right) + \frac{g_2}{2}(W_\mu)^b_a \delta^j_i\right), \tag{C.6}$$
$$(Z_\mu)^{J\dot 2}_{I\dot 1} = (Z^r_\mu)^{J\dot 1}_{I\dot 2} = 0. \tag{C.7}$$

One then checks that

$$i\begin{pmatrix} (Y_\mu)^{j\dot 1}_{i\dot 1} & & \\ & (Y_\mu)^{j\dot 2}_{i\dot 2} & \\ & & (Y_\mu)^{j\dot b}_{i\dot a}\end{pmatrix}^{\beta}_{\alpha} =$$

$$= \begin{pmatrix} -i\left(\frac{2g_1 B_\mu}{3}\delta_i^j + \frac{g_3}{2}(V_\mu)_i^j\right) & & \\ & i\left(\frac{g_1 B_\mu}{3}\delta_i^j - \frac{g_3}{2}(V_\mu)_i^j\right) & \\ & & -i\left(\delta_a^b\left(\frac{g_1 B_\mu}{6}\delta_i^i + \frac{g_3}{2}(V_\mu)_i^j\right) + \frac{g_2}{2}(W_\mu)_a^b\delta_i^j\right) \end{pmatrix}_\alpha^\beta$$

coincides with the matrix $\mathbb{A}_\mu^q$ of the non-twisted case [56, eq. 1.733], while

$$i\begin{pmatrix} (Y_\mu)_{0\dot{1}}^{0\dot{1}} & & \\ & (Y_\mu)_{0\dot{2}}^{0\dot{2}} & \\ & & (Y_\mu)_{0\dot{a}}^{0\dot{b}} \end{pmatrix}_\alpha^\beta = \begin{pmatrix} 0 & & \\ & ig_1 B_\mu & \\ & & i\left(\delta_a^b\frac{g_1 B_\mu}{2} - \frac{g_2}{2}(W_\mu)_a^b\right) \end{pmatrix}_\alpha^\beta$$

coincides with the matrix $\mathbb{A}_\mu^l$ [56, eq. 1.734].

Appendix D

Calculation of the Fermionic Action

Here we will present the intermediate results needed to calculate the fermionic action of the twisted-by-grading Connes Model (as well as of the other two twists, since this part of the action is the same).

D.1 Calculation of $(\gamma_5 \otimes D_Y)\,\Xi$

We notice that D_Y has exactly one non-zero entry in each row and in each column, therefore $D_Y e_i$ will have exactly one component. Then, by direct calculation we find:

$$(\gamma_5 \otimes D_Y)(\xi_1 \otimes e_1) = k_\nu \gamma_5 \xi_1 \otimes e_3 \tag{D.1}$$
$$(\gamma_5 \otimes D_Y)(\xi_2 \otimes e_2) = k_e \gamma_5 \xi_2 \otimes e_4 \tag{D.2}$$
$$(\gamma_5 \otimes D_Y)(\xi_3 \otimes e_3) = \overline{k_\nu} \gamma_5 \xi_3 \otimes e_1 \tag{D.3}$$
$$(\gamma_5 \otimes D_Y)(\xi_4 \otimes e_4) = \overline{k_e} \gamma_5 \xi_4 \otimes e_2 \tag{D.4}$$
$$(\gamma_5 \otimes D_Y)(\xi_5 \otimes e_5) = k_u \gamma_5 \xi_5 \otimes e_7 \tag{D.5}$$
$$(\gamma_5 \otimes D_Y)(\xi_6 \otimes e_6) = k_d \gamma_5 \xi_6 \otimes e_8 \tag{D.6}$$
$$(\gamma_5 \otimes D_Y)(\xi_7 \otimes e_7) = \overline{k_u} \gamma_5 \xi_7 \otimes e_5 \tag{D.7}$$
$$(\gamma_5 \otimes D_Y)(\xi_8 \otimes e_8) = \overline{k_d} \gamma_5 \xi_8 \otimes e_6 \tag{D.8}$$
$$(\gamma_5 \otimes D_Y)(\xi_9 \otimes e_9) = k_u \gamma_5 \xi_9 \otimes e_{11} \tag{D.9}$$
$$(\gamma_5 \otimes D_Y)(\xi_{10} \otimes e_{10}) = k_d \gamma_5 \xi_{10} \otimes e_{12} \tag{D.10}$$
$$(\gamma_5 \otimes D_Y)(\xi_{11} \otimes e_{11}) = \overline{k_u} \gamma_5 \xi_{11} \otimes e_9 \tag{D.11}$$
$$(\gamma_5 \otimes D_Y)(\xi_{12} \otimes e_{12}) = \overline{k_d} \gamma_5 \xi_{12} \otimes e_{10} \tag{D.12}$$
$$(\gamma_5 \otimes D_Y)(\xi_{13} \otimes e_{13}) = k_u \gamma_5 \xi_{13} \otimes e_{15} \tag{D.13}$$

$$(\gamma_5 \otimes D_Y)(\xi_{14} \otimes e_{14}) = k_d \gamma_5 \xi_{14} \otimes e_{16} \tag{D.14}$$

$$(\gamma_5 \otimes D_Y)(\xi_{15} \otimes e_{15}) = \overline{k_u} \gamma_5 \xi_{15} \otimes e_{13} \tag{D.15}$$

$$(\gamma_5 \otimes D_Y)(\xi_{16} \otimes e_{16}) = \overline{k_d} \gamma_5 \xi_{16} \otimes e_{14} \tag{D.16}$$

$$(\gamma_5 \otimes D_Y)(\xi_{17} \otimes \overline{e_1}) = k_\nu \gamma_5 \xi_{17} \otimes \overline{e_3} \tag{D.17}$$

$$(\gamma_5 \otimes D_Y)(\xi_{18} \otimes \overline{e_2}) = k_e \gamma_5 \xi_{18} \otimes \overline{e_4} \tag{D.18}$$

$$(\gamma_5 \otimes D_Y)(\xi_{19} \otimes \overline{e_3}) = \overline{k_\nu} \gamma_5 \xi_{19} \otimes \overline{e_1} \tag{D.19}$$

$$(\gamma_5 \otimes D_Y)(\xi_{20} \otimes \overline{e_4}) = \overline{k_e} \gamma_5 \xi_{20} \otimes \overline{e_2} \tag{D.20}$$

$$(\gamma_5 \otimes D_Y)(\xi_{21} \otimes \overline{e_5}) = k_u \gamma_5 \xi_{21} \otimes \overline{e_7} \tag{D.21}$$

$$(\gamma_5 \otimes D_Y)(\xi_{22} \otimes \overline{e_6}) = k_d \gamma_5 \xi_{22} \otimes \overline{e_8} \tag{D.22}$$

$$(\gamma_5 \otimes D_Y)(\xi_{23} \otimes \overline{e_7}) = \overline{k_u} \gamma_5 \xi_{23} \otimes \overline{e_5} \tag{D.23}$$

$$(\gamma_5 \otimes D_Y)(\xi_{24} \otimes \overline{e_8}) = \overline{k_d} \gamma_5 \xi_{24} \otimes \overline{e_6} \tag{D.24}$$

$$(\gamma_5 \otimes D_Y)(\xi_{25} \otimes \overline{e_9}) = k_u \gamma_5 \xi_{25} \otimes \overline{e_{11}} \tag{D.25}$$

$$(\gamma_5 \otimes D_Y)(\xi_{26} \otimes \overline{e_{10}}) = k_d \gamma_5 \xi_{26} \otimes \overline{e_{12}} \tag{D.26}$$

$$(\gamma_5 \otimes D_Y)(\xi_{27} \otimes \overline{e_{11}}) = \overline{k_u} \gamma_5 \xi_{27} \otimes \overline{e_9} \tag{D.27}$$

$$(\gamma_5 \otimes D_Y)(\xi_{28} \otimes \overline{e_{12}}) = \overline{k_d} \gamma_5 \xi_{28} \otimes \overline{e_{10}} \tag{D.28}$$

$$(\gamma_5 \otimes D_Y)(\xi_{29} \otimes \overline{e_{13}}) = k_u \gamma_5 \xi_{29} \otimes \overline{e_{15}} \tag{D.29}$$

$$(\gamma_5 \otimes D_Y)(\xi_{30} \otimes \overline{e_{14}}) = k_d \gamma_5 \xi_{30} \otimes \overline{e_{16}} \tag{D.30}$$

$$(\gamma_5 \otimes D_Y)(\xi_{31} \otimes \overline{e_{15}}) = \overline{k_u} \gamma_5 \xi_{31} \otimes \overline{e_{13}} \tag{D.31}$$

$$(\gamma_5 \otimes D_Y)(\xi_{32} \otimes \overline{e_{16}}) = \overline{k_d} \gamma_5 \xi_{32} \otimes \overline{e_{14}} \tag{D.32}$$

D.2 Calculation of $M_1\Xi$

We define

$$M_1 = \begin{pmatrix} A & \\ & \overline{A} \end{pmatrix}_C^D \tag{D.33}$$

where A is defined in (9.83). Before starting with the calculations, let us rewrite M_1 in a more useful form. We have:

$$\begin{aligned} M_1 &= \delta_C^0 \delta_0^D A + \delta_C^1 \delta_1^D \overline{A} = \\ &= \delta_{\dot{s}}^{\dot{t}} \delta_I^J \left[\delta_C^0 \delta_0^D \left(\delta_r^s \delta_t^r \left(A_r^I\right)_\alpha^\beta + \delta_l^s \delta_t^l \left(A_l^I\right)_\alpha^\beta \right) + \delta_C^1 \delta_1^D \left(\delta_r^s \delta_t^r \left(\overline{A_r^I}\right)_\alpha^\beta + \delta_l^s \delta_t^l \left(\overline{A_l^I}\right)_\alpha^\beta \right) \right] = \\ &= \delta_I^J \left[\delta_C^0 \delta_0^D \left(P_r \left(A_r^I\right)_\alpha^\beta + P_l \left(A_l^I\right)_\alpha^\beta \right) + \delta_C^1 \delta_1^D \left(P_r \left(\overline{A_r^I}\right)_\alpha^\beta + P_l \left(\overline{A_l^I}\right)_\alpha^\beta \right) \right] \end{aligned}$$

where

$$P_r \equiv \frac{1+\gamma_5}{2}, \qquad P_l \equiv \frac{1-\gamma_5}{2}. \tag{D.34}$$

Remembering that

$$H_{r,l} = \begin{pmatrix} \phi_1^{r,l} & -\overline{\phi_2^{r,l}} \\ \phi_2^{r,l} & \overline{\phi_1^{r,l}} \end{pmatrix}, \tag{D.35}$$

we find:

$$M_1(\xi_1 \otimes e_1) = k_\nu \left[\left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_1 \otimes e_3 - \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_1 \otimes e_4 \right] \tag{D.36}$$

$$M_1(\xi_2 \otimes e_2) = k_e \left[\left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_2 \otimes e_3 + \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_2 \otimes e_4 \right] \tag{D.37}$$

$$M_1(\xi_3 \otimes e_3) = \overline{k_\nu} \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_3 \otimes e_1 + \overline{k_e} \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_3 \otimes e_2 \tag{D.38}$$

$$M_1(\xi_4 \otimes e_4) = -\overline{k_\nu} \left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_4 \otimes e_1 + \overline{k_e} \left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_4 \otimes e_2 \tag{D.39}$$

$$M_1(\xi_5 \otimes e_5) = k_u \left[\left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_5 \otimes e_7 - \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_5 \otimes e_8 \right] \tag{D.40}$$

$$M_1(\xi_6 \otimes e_6) = k_d \left[\left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_6 \otimes e_7 + \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_6 \otimes e_8 \right] \tag{D.41}$$

$$M_1(\xi_7 \otimes e_7) = \overline{k_u} \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_7 \otimes e_5 + \overline{k_d} \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_7 \otimes e_6 \tag{D.42}$$

$$M_1(\xi_8 \otimes e_8) = -\overline{k_u} \left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_8 \otimes e_5 + \overline{k_d} \left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_8 \otimes e_6 \tag{D.43}$$

$$M_1(\xi_9 \otimes e_9) = k_u \left[\left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_9 \otimes e_{11} - \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_9 \otimes e_{12} \right] \tag{D.44}$$

$$M_1(\xi_{10} \otimes e_{10}) = k_d \left[\left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_{10} \otimes e_{11} + \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_{10} \otimes e_{12} \right] \tag{D.45}$$

$$M_1(\xi_{11} \otimes e_{11}) = \overline{k_u} \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_{11} \otimes e_9 + \overline{k_d} \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_{11} \otimes e_{10} \tag{D.46}$$

$$M_1(\xi_{12} \otimes e_{12}) = -\overline{k_u} \left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_{12} \otimes e_9 + \overline{k_d} \left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_{12} \otimes e_{10} \tag{D.47}$$

$$M_1(\xi_{13} \otimes e_{13}) = k_u \left[\left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_{13} \otimes e_{15} - \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_{13} \otimes e_{16} \right] \tag{D.48}$$

$$M_1(\xi_{14} \otimes e_{14}) = k_d \left[\left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_{14} \otimes e_{15} + \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_{14} \otimes e_{16} \right] \tag{D.49}$$

$$M_1(\xi_{15} \otimes e_{15}) = \overline{k_u} \left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_{15} \otimes e_{13} + \overline{k_d} \left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_{15} \otimes e_{14} \tag{D.50}$$

$$M_1(\xi_{16} \otimes e_{16}) = -\overline{k_u} \left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_{16} \otimes e_{13} + \overline{k_d} \left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_{16} \otimes e_{14} \tag{D.51}$$

$$M_1(\xi_{17} \otimes \overline{e_1}) = \overline{k_\nu} \left[\left(\overline{\phi_2^r} P_r - \overline{\phi_2^l} P_l\right) \xi_{17} \otimes \overline{e_3} - \left(\overline{\phi_1^r} P_r - \overline{\phi_1^l} P_l\right) \xi_{17} \otimes \overline{e_4} \right] \tag{D.52}$$

$$M_1(\xi_{18} \otimes \overline{e_2}) = \overline{k_e} \left[\left(\phi_1^r P_r - \phi_1^l P_l\right) \xi_{18} \otimes \overline{e_3} + \left(\phi_2^r P_r - \phi_2^l P_l\right) \xi_{18} \otimes \overline{e_4} \right] \tag{D.53}$$

$$M_1\left(\xi_{19}\otimes\overline{e_3}\right)=k_\nu\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{19}\otimes\overline{e_1}+k_e\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{19}\otimes\overline{e_2}\tag{D.54}$$

$$M_1\left(\xi_{20}\otimes\overline{e_4}\right)=-k_\nu\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{20}\otimes\overline{e_1}+k_e\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{20}\otimes\overline{e_2}\tag{D.55}$$

$$M_1\left(\xi_{21}\otimes\overline{e_5}\right)=\overline{k_u}\left[\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{21}\otimes\overline{e_7}-\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{21}\otimes\overline{e_8}\right]\tag{D.56}$$

$$M_1\left(\xi_{22}\otimes\overline{e_6}\right)=\overline{k_d}\left[\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{22}\otimes\overline{e_7}+\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{22}\otimes\overline{e_8}\right]\tag{D.57}$$

$$M_1\left(\xi_{23}\otimes\overline{e_7}\right)=k_u\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{23}\otimes\overline{e_5}+k_d\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{23}\otimes\overline{e_6}\tag{D.58}$$

$$M_1\left(\xi_{24}\otimes\overline{e_8}\right)=-k_u\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{24}\otimes\overline{e_5}+k_d\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{24}\otimes\overline{e_6}\tag{D.59}$$

$$M_1\left(\xi_{25}\otimes\overline{e_9}\right)=\overline{k_u}\left[\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{25}\otimes\overline{e_{11}}-\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{25}\otimes\overline{e_{12}}\right]\tag{D.60}$$

$$M_1\left(\xi_{26}\otimes\overline{e_{10}}\right)=\overline{k_d}\left[\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{26}\otimes\overline{e_{11}}+\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{26}\otimes\overline{e_{12}}\right]\tag{D.61}$$

$$M_1\left(\xi_{27}\otimes\overline{e_{11}}\right)=k_u\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{27}\otimes\overline{e_9}+k_d\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{27}\otimes\overline{e_{10}}\tag{D.62}$$

$$M_1\left(\xi_{28}\otimes\overline{e_{12}}\right)=-k_u\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{28}\otimes\overline{e_9}+k_d\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{28}\otimes\overline{e_{10}}\tag{D.63}$$

$$M_1\left(\xi_{29}\otimes\overline{e_{13}}\right)=\overline{k_u}\left[\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{29}\otimes\overline{e_{15}}-\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{29}\otimes\overline{e_{16}}\right]\tag{D.64}$$

$$M_1\left(\xi_{30}\otimes\overline{e_{14}}\right)=\overline{k_d}\left[\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{30}\otimes\overline{e_{15}}+\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{30}\otimes\overline{e_{16}}\right]\tag{D.65}$$

$$M_1\left(\xi_{31}\otimes\overline{e_{15}}\right)=k_u\left(\phi_2^rP_r-\phi_2^lP_l\right)\xi_{31}\otimes\overline{e_{13}}+k_d\left(\overline{\phi_1^r}P_r-\overline{\phi_1^l}P_l\right)\xi_{31}\otimes\overline{e_{14}}\tag{D.66}$$

$$M_1\left(\xi_{32}\otimes\overline{e_{16}}\right)=-k_u\left(\phi_1^rP_r-\phi_1^lP_l\right)\xi_{32}\otimes\overline{e_{13}}+k_d\left(\overline{\phi_2^r}P_r-\overline{\phi_2^l}P_l\right)\xi_{32}\otimes\overline{e_{14}}\tag{D.67}$$

D.3 Calculation of $D_Z\Xi$

Since $D_Z = \not{D} + \not{X} + i\not{Y}$, it is easier to calculate the contribution of the single pieces and then put them together.

D.3.1 Calculation of $\not{D}\Xi$

$\not{D} = \not{\partial} \otimes I_F$, hence

$$\not{D}\Xi=\sum_{i=1}^{16}\not{\partial}\xi_i\otimes e_i+\sum_{i=1}^{16}\not{\partial}\xi_{i+16}\otimes\overline{e_i}.\tag{D.68}$$

D.3.2 Calculation of $\not{X}\Xi$

We have that X_μ is proportional to γ_5. It is useful to write this explicitly, by defining

$$X_\mu =: \gamma_5 \tilde{X}_\mu. \tag{D.69}$$

Then, we have

$$\not{X} = (-i\gamma^\mu) X_\mu = (-i\gamma^\mu) \gamma_5 \tilde{X}_\mu \tag{D.70}$$

Now we can start the computation. We get

$$\not{X}(\xi_1 \otimes e_1) = 2\not{\phi}\gamma_5\xi_1 \otimes e_1 \tag{D.71}$$

$$\not{X}(\xi_2 \otimes e_2) = 2\not{\phi}\gamma_5\xi_2 \otimes e_2 \tag{D.72}$$

$$\not{X}(\xi_3 \otimes e_3) = (\not{\psi} - \not{\phi})\gamma_5\xi_3 \otimes e_3 \tag{D.73}$$

$$\not{X}(\xi_4 \otimes e_4) = (\not{\psi} - \not{\phi})\gamma_5\xi_4 \otimes e_4 \tag{D.74}$$

$$\not{X}(\xi_5 \otimes e_5) = \left(\not{\phi} - (\not{g})^1_1\right)\gamma_5\xi_5 \otimes e_5 - (\not{g})^1_2\gamma_5\xi_5 \otimes e_9 - (\not{g})^1_3\gamma_5\xi_5 \otimes e_{13} \tag{D.75}$$

$$\not{X}(\xi_6 \otimes e_6) = \left(\not{\phi} - (\not{g})^1_1\right)\gamma_5\xi_6 \otimes e_6 - (\not{g})^1_2\gamma_5\xi_6 \otimes e_{10} - (\not{g})^1_3\gamma_5\xi_6 \otimes e_{14} \tag{D.76}$$

$$\not{X}(\xi_7 \otimes e_7) = \left(\not{\psi} + (\not{g})^1_1\right)\gamma_5\xi_7 \otimes e_7 + (\not{g})^1_2\gamma_5\xi_7 \otimes e_{11} + (\not{g})^1_3\gamma_5\xi_7 \otimes e_{15} \tag{D.77}$$

$$\not{X}(\xi_8 \otimes e_8) = \left(\not{\psi} + (\not{g})^1_1\right)\gamma_5\xi_8 \otimes e_8 + (\not{g})^1_2\gamma_5\xi_8 \otimes e_{12} + (\not{g})^1_3\gamma_5\xi_8 \otimes e_{16} \tag{D.78}$$

$$\not{X}(\xi_9 \otimes e_9) = -(\not{g})^2_1\gamma_5\xi_9 \otimes e_5 + \left(\not{\phi} - (\not{g})^2_2\right)\gamma_5\xi_9 \otimes e_9 - (\not{g})^2_3\gamma_5\xi_9 \otimes e_{13} \tag{D.79}$$

$$\not{X}(\xi_{10} \otimes e_{10}) = -(\not{g})^2_1\gamma_5\xi_{10} \otimes e_6 + \left(\not{\phi} - (\not{g})^2_2\right)\gamma_5\xi_{10} \otimes e_{10} - (\not{g})^2_3\gamma_5\xi_{10} \otimes e_{14} \tag{D.80}$$

$$\not{X}(\xi_{11} \otimes e_{11}) = +(\not{g})^2_1\gamma_5\xi_{11} \otimes e_7 + \left(\not{\psi} + (\not{g})^2_2\right)\gamma_5\xi_{11} \otimes e_{11} + (\not{g})^2_3\gamma_5\xi_{11} \otimes e_{15} \tag{D.81}$$

$$\not{X}(\xi_{12} \otimes e_{12}) = +(\not{g})^2_1\gamma_5\xi_{12} \otimes e_8 + \left(\not{\psi} + (\not{g})^2_2\right)\gamma_5\xi_{12} \otimes e_{12} + (\not{g})^2_3\gamma_5\xi_{12} \otimes e_{16} \tag{D.82}$$

$$\not{X}(\xi_{13} \otimes e_{13}) = -(\not{g})^3_1\gamma_5\xi_{13} \otimes e_5 - (\not{g})^3_2\gamma_5\xi_{13} \otimes e_9 + \left(\not{\phi} - (\not{g})^3_3\right)\gamma_5\xi_{13} \otimes e_{13} \tag{D.83}$$

$$\not{X}(\xi_{14} \otimes e_{14}) = -(\not{g})^3_1\gamma_5\xi_{14} \otimes e_6 - (\not{g})^3_2\gamma_5\xi_{14} \otimes e_{10} + \left(\not{\phi} - (\not{g})^3_3\right)\gamma_5\xi_{14} \otimes e_{14} \tag{D.84}$$

$$\not{X}(\xi_{15} \otimes e_{15}) = +(\not{g})^3_1\gamma_5\xi_{15} \otimes e_7 + (\not{g})^3_2\gamma_5\xi_{15} \otimes e_{11} + \left(\not{\psi} + (\not{g})^3_3\right)\gamma_5\xi_{15} \otimes e_{15} \tag{D.85}$$

$$\not{X}(\xi_{16}\otimes e_{16}) = +\left(\not{\theta}\right)^3_1\gamma_5\xi_{16}\otimes e_8 + \left(\not{\theta}\right)^3_2\gamma_5\xi_{16}\otimes e_{12} + \left(\not{\psi} + \left(\not{\theta}\right)^3_3\right)\gamma_5\xi_{16}\otimes e_{16} \tag{D.86}$$

$$\not{X}(\xi_{17}\otimes\overline{e_1}) = 2\not{\phi}\gamma_5\xi_{17}\otimes\overline{e_1} \tag{D.87}$$

$$\not{X}(\xi_{18}\otimes\overline{e_2}) = 2\not{\phi}\gamma_5\xi_{18}\otimes\overline{e_2} \tag{D.88}$$

$$\not{X}(\xi_{19}\otimes\overline{e_3}) = (\not{\psi} - \not{\phi})\gamma_5\xi_{19}\otimes\overline{e_3} \tag{D.89}$$

$$\not{X}(\xi_{20}\otimes\overline{e_4}) = (\not{\psi} - \not{\phi})\gamma_5\xi_{20}\otimes\overline{e_4} \tag{D.90}$$

$$\not{X}(\xi_{21}\otimes\overline{e_5}) = \left(\not{\phi} - (\not{\overline{\theta}})^1_1\right)\gamma_5\xi_{21}\otimes\overline{e_5} - (\not{\overline{\theta}})^1_2\gamma_5\xi_{21}\otimes\overline{e_9} - (\not{\overline{\theta}})^1_3\gamma_5\xi_{21}\otimes\overline{e_{13}} \tag{D.91}$$

$$\not{X}(\xi_{22}\otimes\overline{e_6}) = \left(\not{\phi} - (\not{\overline{\theta}})^1_1\right)\gamma_5\xi_{22}\otimes\overline{e_6} - (\not{\overline{\theta}})^1_2\gamma_5\xi_{22}\otimes\overline{e_{10}} - (\not{\overline{\theta}})^1_3\gamma_5\xi_{22}\otimes\overline{e_{14}} \tag{D.92}$$

$$\not{X}(\xi_{23}\otimes\overline{e_7}) = \left(\not{\psi} + (\not{\overline{\theta}})^1_1\right)\gamma_5\xi_{23}\otimes\overline{e_7} + (\not{\overline{\theta}})^1_2\gamma_5\xi_{23}\otimes\overline{e_{11}} + (\not{\overline{\theta}})^1_3\gamma_5\xi_{23}\otimes\overline{e_{15}} \tag{D.93}$$

$$\not{X}(\xi_{24}\otimes\overline{e_8}) = \left(\not{\psi} + (\not{\overline{\theta}})^1_1\right)\gamma_5\xi_{24}\otimes\overline{e_8} + (\not{\overline{\theta}})^1_2\gamma_5\xi_{24}\otimes\overline{e_{12}} + (\not{\overline{\theta}})^1_3\gamma_5\xi_{24}\otimes\overline{e_{16}} \tag{D.94}$$

$$\not{X}(\xi_{25}\otimes\overline{e_9}) = -(\not{\overline{\theta}})^2_1\gamma_5\xi_{25}\otimes\overline{e_5} + \left(\not{\phi} - (\not{\overline{\theta}})^2_2\right)\gamma_5\xi_{25}\otimes\overline{e_9} - (\not{\overline{\theta}})^2_3\gamma_5\xi_{25}\otimes\overline{e_{13}} \tag{D.95}$$

$$\not{X}(\xi_{26}\otimes\overline{e_{10}}) = -(\not{\overline{\theta}})^2_1\gamma_5\xi_{26}\otimes\overline{e_6} + \left(\not{\phi} - (\not{\overline{\theta}})^2_2\right)\gamma_5\xi_{26}\otimes\overline{e_{10}} - (\not{\overline{\theta}})^2_3\gamma_5\xi_{26}\otimes\overline{e_{14}} \tag{D.96}$$

$$\not{X}(\xi_{27}\otimes\overline{e_{11}}) = +(\not{\overline{\theta}})^2_1\gamma_5\xi_{27}\otimes\overline{e_7} + \left(\not{\psi} + (\not{\overline{\theta}})^2_2\right)\gamma_5\xi_{27}\otimes\overline{e_{11}} + (\not{\overline{\theta}})^2_3\gamma_5\xi_{27}\otimes\overline{e_{15}} \tag{D.97}$$

$$\not{X}(\xi_{28}\otimes\overline{e_{12}}) = +(\not{\overline{\theta}})^2_1\gamma_5\xi_{28}\otimes\overline{e_8} + \left(\not{\psi} + (\not{\overline{\theta}})^2_2\right)\gamma_5\xi_{28}\otimes\overline{e_{12}} + (\not{\overline{\theta}})^2_3\gamma_5\xi_{28}\otimes\overline{e_{16}} \tag{D.98}$$

$$\not{X}(\xi_{29}\otimes\overline{e_{13}}) = -(\not{\overline{\theta}})^3_1\gamma_5\xi_{29}\otimes\overline{e_5} - (\not{\overline{\theta}})^3_2\gamma_5\xi_{29}\otimes\overline{e_9} + \left(\not{\phi} - (\not{\overline{\theta}})^3_3\right)\gamma_5\xi_{29}\otimes\overline{e_{13}} \tag{D.99}$$

$$\not{X}(\xi_{30}\otimes\overline{e_{14}}) = -(\not{\overline{\theta}})^3_1\gamma_5\xi_{30}\otimes\overline{e_6} - (\not{\overline{\theta}})^3_2\gamma_5\xi_{30}\otimes\overline{e_{10}} + \left(\not{\phi} - (\not{\overline{\theta}})^3_3\right)\gamma_5\xi_{30}\otimes\overline{e_{14}} \tag{D.100}$$

$$\not{X}(\xi_{31}\otimes\overline{e_{15}}) = +(\not{\overline{\theta}})^3_1\gamma_5\xi_{31}\otimes\overline{e_7} + (\not{\overline{\theta}})^3_2\gamma_5\xi_{31}\otimes\overline{e_{11}} + \left(\not{\psi} + (\not{\overline{\theta}})^3_3\right)\gamma_5\xi_{31}\otimes\overline{e_{15}} \tag{D.101}$$

$$\not{X}(\xi_{32}\otimes\overline{e_{16}}) = +(\not{\overline{\theta}})^3_1\gamma_5\xi_{32}\otimes\overline{e_8} + (\not{\overline{\theta}})^3_2\gamma_5\xi_{32}\otimes\overline{e_{12}} + \left(\not{\psi} + (\not{\overline{\theta}})^3_3\right)\gamma_5\xi_{32}\otimes\overline{e_{16}} \tag{D.102}$$

D.3.3 Calculation of $i\not{Y}\Xi$

We have that Y_μ is proportional to $\mathbb{I}_4$ in the $s, \dot{s}$ indices. Then, we have

$$\not{Y} = (-i\gamma^\mu)\,Y_\mu =: (-i\gamma^\mu)\,\mathbb{I}_4\tilde{Y}_\mu = (-i\gamma^\mu)\,\tilde{Y}_\mu. \tag{D.103}$$

Now we can start the computation. We get

$$i\not{Y}\left(\xi_1\otimes e_1\right)=0 \tag{D.104}$$

$$i\not{Y}\left(\xi_2\otimes e_2\right)=ig_1\not{B}\xi_2\otimes e_2 \tag{D.105}$$

$$i\not{Y}\left(\xi_3\otimes e_3\right)=i\left(\frac{g_1}{2}\not{B}-\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_1\right)\xi_3\otimes e_3-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_2\xi_3\otimes e_4 \tag{D.106}$$

$$i\not{Y}\left(\xi_4\otimes e_4\right)=-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^2_1\xi_4\otimes e_3+i\left(\frac{g_1}{2}\not{B}-\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^2_2\right)\xi_4\otimes e_4 \tag{D.107}$$

$$\begin{aligned}i\not{Y}\left(\xi_5\otimes e_5\right)=&-i\left(\frac{2}{3}g_1\not{B}+\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_1\right)\xi_5\otimes e_5-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_2\xi_5\otimes e_9-\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_3\xi_5\otimes e_{13}\end{aligned} \tag{D.108}$$

$$\begin{aligned}i\not{Y}\left(\xi_6\otimes e_6\right)=&\,i\left(\frac{g_1}{3}\not{B}-\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_1\right)\xi_6\otimes e_6-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_2\xi_6\otimes e_{10}-\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_3\xi_6\otimes e_{14}\end{aligned} \tag{D.109}$$

$$\begin{aligned}i\not{Y}\left(\xi_7\otimes e_7\right)=&-i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_1+\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_1\right)\xi_7\otimes e_7-\\&-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_2\xi_7\otimes e_8-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_2\xi_7\otimes e_{11}-\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_3\xi_7\otimes e_{15}\end{aligned} \tag{D.110}$$

$$\begin{aligned}i\not{Y}\left(\xi_8\otimes e_8\right)=&-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^2_1\xi_8\otimes e_7-\\&-i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^2_2+\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_1\right)\xi_8\otimes e_8\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_2\xi_8\otimes e_{12}-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^1_3\xi_8\otimes e_{16}\end{aligned} \tag{D.111}$$

$$\begin{aligned}i\not{Y}\left(\xi_9\otimes e_9\right)=&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_1\xi_9\otimes e_5-i\left(\frac{2}{3}g_1\not{B}+\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_2\right)\xi_9\otimes e_9-\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_3\xi_9\otimes e_{13}\end{aligned} \tag{D.112}$$

$$\begin{aligned}i\not{Y}\left(\xi_{10}\otimes e_{10}\right)=&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_1\xi_{10}\otimes e_6+i\left(\frac{g_1}{3}\not{B}-\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_2\right)\xi_{10}\otimes e_{10}-\\&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_3\xi_{10}\otimes e_{14}\end{aligned} \tag{D.113}$$

$$\begin{aligned}i\not{Y}\left(\xi_{11}\otimes e_{11}\right)=&-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_1\xi_{11}\otimes e_7-\\&-i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_1+\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_2\right)\xi_{11}\otimes e_{11}\\&-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^1_2\xi_{11}\otimes e_{12}-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_3\xi_{11}\otimes e_{15}\end{aligned} \tag{D.114}$$

$$i\not{Y}\left(\xi_{12}\otimes e_{12}\right)=-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)^2_1\xi_{12}\otimes e_8-i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)^2_1\xi_{12}\otimes e_{11}$$

$$
\begin{aligned}
&- i\left(\frac{g_1}{6}\not{B} + \frac{g_2}{2}\not{W}^k\left(\sigma_k\right)_2^2 + \frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_2^2\right)\xi_{12}\otimes e_{12} - \\
&- i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_3^2\xi_{12}\otimes e_{16} \qquad \text{(D.115)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{13}\otimes e_{13}\right) = &-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_1^3\xi_{13}\otimes e_5 - i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_2^3\xi_{13}\otimes e_9 \\
&- i\left(\frac{2}{3}g_1\not{B} + \frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_3^3\right)\xi_{13}\otimes e_{13} \qquad \text{(D.116)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{14}\otimes e_{14}\right) = &-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_1^3\xi_{14}\otimes e_6 - i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_2^3\xi_{14}\otimes e_{10} \\
&+ i\left(\frac{g_1}{3}\not{B} - \frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_3^3\right)\xi_{14}\otimes e_{14} \qquad \text{(D.117)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{15}\otimes e_{15}\right) = &-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_1^3\xi_{15}\otimes e_7 - i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_2^3\xi_{15}\otimes e_{11} \\
&- i\left(\frac{g_1}{6}\not{B} + \frac{g_2}{2}\not{W}^k\left(\sigma_k\right)_1^1 + \frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_3^3\right)\xi_{15}\otimes e_{15} - \\
&- i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)_2^1\xi_{15}\otimes e_{16} \qquad \text{(D.118)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{16}\otimes e_{16}\right) = &-i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_1^3\xi_{16}\otimes e_8 - i\frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_2^3\xi_{16}\otimes e_{12} \\
&- i\frac{g_2}{2}\not{W}^k\left(\sigma_k\right)_1^2\xi_{16}\otimes e_{15} - \\
&- i\left(\frac{g_1}{6}\not{B} + \frac{g_2}{2}\not{W}^k\left(\sigma_k\right)_2^2 + \frac{g_3}{2}\not{V}^m\left(\lambda_m\right)_3^3\right)\xi_{16}\otimes e_{16} \qquad \text{(D.119)}
\end{aligned}
$$

$$
i\not{Y}\left(\xi_{17}\otimes\overline{e_1}\right) = 0 \qquad \text{(D.120)}
$$

$$
i\not{Y}\left(\xi_{18}\otimes\overline{e_2}\right) = -ig_1\not{B}\xi_{18}\otimes\overline{e_2} \qquad \text{(D.121)}
$$

$$
i\not{Y}\left(\xi_{19}\otimes\overline{e_3}\right) = -i\left(\frac{g_1}{2}\not{B} - \frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_1^1\right)\xi_{19}\otimes\overline{e_3} + i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_2^1\xi_{19}\otimes\overline{e_4} \qquad \text{(D.122)}
$$

$$
i\not{Y}\left(\xi_{20}\otimes\overline{e_4}\right) = +i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_1^2\xi_{20}\otimes\overline{e_3} - i\left(\frac{g_1}{2}\not{B} - \frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_2^2\right)\xi_{20}\otimes\overline{e_4}, \qquad \text{(D.123)}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{21}\otimes\overline{e_5}\right) = &+i\left(\frac{2}{3}g_1\not{B} + \frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_1^1\right)\xi_{21}\otimes\overline{e_5} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_2^1\xi_{21}\otimes\overline{e_9} + \\
&+ i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_3^1\xi_{21}\otimes\overline{e_{13}} \qquad \text{(D.124)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{22}\otimes\overline{e_6}\right) = &-i\left(\frac{g_1}{3}\not{B} - \frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_1^1\right)\xi_{22}\otimes\overline{e_6} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_2^1\xi_{22}\otimes\overline{e_{10}} + \\
&+ i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_3^1\xi_{22}\otimes\overline{e_{14}} \qquad \text{(D.125)}
\end{aligned}
$$

$$
\begin{aligned}
i\not{Y}\left(\xi_{23}\otimes\overline{e_7}\right) = &+i\left(\frac{g_1}{6}\not{B} + \frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_1^1 + \frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_1^1\right)\xi_{23}\otimes\overline{e_7} + \\
&+ i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)_2^1\xi_{23}\otimes\overline{e_8} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)_2^1\xi_{23}\otimes\overline{e_{11}} +
\end{aligned}
$$

$$+ i\frac{g_3}{2}\not{V}^m \left(\overline{\lambda_m}\right)^1_3 \xi_{23}\otimes\overline{e_{15}} \tag{D.126}$$

$$i\not{Y}\left(\xi_{24}\otimes\overline{e_8}\right) = +i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_1 \xi_{24}\otimes\overline{e_7}+ + i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_2+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^1_1\right)\xi_{24}\otimes\overline{e_8} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^1_2\xi_{24}\otimes\overline{e_{12}} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^1_3\xi_{24}\otimes\overline{e_{16}} \tag{D.127}$$

$$i\not{Y}\left(\xi_{25}\otimes\overline{e_9}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_1\xi_{25}\otimes\overline{e_5} + i\left(\frac{2}{3}g_1\not{B}+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_2\right)\xi_{25}\otimes\overline{e_9}+ + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_3\xi_{25}\otimes\overline{e_{13}} \tag{D.128}$$

$$i\not{Y}\left(\xi_{26}\otimes\overline{e_{10}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_1\xi_{26}\otimes\overline{e_6} - i\left(\frac{g_1}{3}\not{B}-\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_2\right)\xi_{26}\otimes\overline{e_{10}}+ + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_3\xi_{26}\otimes\overline{e_{14}} \tag{D.129}$$

$$i\not{Y}\left(\xi_{27}\otimes\overline{e_{11}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_1\xi_{27}\otimes\overline{e_7}+ + i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^1_1+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_2\right)\xi_{27}\otimes\overline{e_{11}} + i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^1_2\xi_{27}\otimes\overline{e_{12}} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_3\xi_{27}\otimes\overline{e_{15}} \tag{D.130}$$

$$i\not{Y}\left(\xi_{28}\otimes\overline{e_{12}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_1\xi_{28}\otimes\overline{e_8} + i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_1\xi_{28}\otimes\overline{e_{11}} + i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_2+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_2\right)\xi_{28}\otimes\overline{e_{12}}+ + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^2_3\xi_{28}\otimes\overline{e_{16}} \tag{D.131}$$

$$i\not{Y}\left(\xi_{29}\otimes\overline{e_{13}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_1\xi_{29}\otimes\overline{e_5} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_2\xi_{29}\otimes\overline{e_9} + i\left(\frac{2}{3}g_1\not{B}+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_3\right)\xi_{29}\otimes\overline{e_{13}} \tag{D.132}$$

$$i\not{Y}\left(\xi_{30}\otimes\overline{e_{14}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_1\xi_{30}\otimes\overline{e_6} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_2\xi_{30}\otimes\overline{e_{10}} - i\left(\frac{g_1}{3}\not{B}-\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_3\right)\xi_{30}\otimes\overline{e_{14}} \tag{D.133}$$

$$i\not{Y}\left(\xi_{31}\otimes\overline{e_{15}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_1\xi_{31}\otimes\overline{e_7} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_2\xi_{31}\otimes\overline{e_{11}} + i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^1_1+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_3\right)\xi_{31}\otimes\overline{e_{15}}+ + i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^1_2\xi_{31}\otimes\overline{e_{16}} \tag{D.134}$$

$$i\not{Y}\left(\xi_{32}\otimes\overline{e_{16}}\right) = +i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_1\xi_{32}\otimes\overline{e_8} + i\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_2\xi_{32}\otimes\overline{e_{12}}$$

$$+ i\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_1\xi_{32}\otimes\overline{e_{15}}+$$
$$+ i\left(\frac{g_1}{6}\not{B}+\frac{g_2}{2}\not{W}^k\left(\overline{\sigma_k}\right)^2_2+\frac{g_3}{2}\not{V}^m\left(\overline{\lambda_m}\right)^3_3\right)\xi_{32}\otimes\overline{e_{16}} \tag{D.135}$$

D.4 Calculation of the Scalar Products

D.4.1 Definitions

We define the following objects:

$$\mathfrak{A}^{\rho}_{\mathcal{D}}\left(\Phi,\Xi\right) := -\mathfrak{A}_{\mathcal{D}}\left(\Phi,\Xi\right) \tag{D.136}$$
$$\mathfrak{A}_{\mathcal{D}}\left(\Phi,\Xi\right) := \left\langle J\Phi,\mathcal{D}\Xi\right\rangle \tag{D.137}$$
$$\mathfrak{A}_{\mathcal{D}}\left(\phi,\xi\right) := \left\langle \mathcal{J}\phi,\mathcal{D}\xi\right\rangle \tag{D.138}$$
$$\mathfrak{A}_0\left(\phi,\xi\right) := 2\int_{\mathcal{M}} d\mu\left(\overline{\varphi}^{\dagger}\sigma_2\right)\left(\sigma_j\partial_j\right)\zeta \tag{D.139}$$
$$\mathfrak{A}_{\gamma_5}\left(\phi,\xi\right) := -2\int_{\mathcal{M}} d\mu\left(\overline{\varphi}^{\dagger}\sigma_2\zeta\right) \tag{D.140}$$
$$\mathfrak{A}_{\pm}\left(\phi,\xi;H_r,H_l\right) := \mathfrak{A}_{H_rP_r\pm H_lP_l}\left(\phi,\xi\right) \tag{D.141}$$
$$\mathfrak{A}_{H_rP_r\pm H_lP_l}\left(\phi,\xi\right) = -\int_{\mathcal{M}} d\mu\left(\overline{\varphi}^{\dagger}\sigma_2\left(H_r\mp H_l\right)\zeta\right) \tag{D.142}$$
$$\mathfrak{A}_X\left(\phi,\xi;f_\mu\right) := -2i\int_{\mathcal{M}} d\mu f_0\left(\overline{\varphi}^{\dagger}\sigma_2\right)\zeta \tag{D.143}$$
$$\mathfrak{A}_Y\left(\phi,\xi;g_\mu\right) := 2i\int_{\mathcal{M}} d\mu\left(\overline{\varphi}^{\dagger}\sigma_2\right)\left(\sigma_j g_j\right)\zeta \tag{D.144}$$

D.4.2 Calculation of $\mathfrak{A}^{\rho}_{\not{\partial}}\left(\Phi,\Xi\right)$

$$\mathfrak{A}^{\rho}_{\not{\partial}}\left(\Phi,\Xi\right) = -\sum_{i=1}^{16}\mathfrak{A}_0\left(\phi_i,\xi_{i+16}\right) - \sum_{i=1}^{16}\mathfrak{A}_0\left(\phi_{i+16},\xi_i\right) \tag{D.145}$$

D.4.3 Calculation of $\mathfrak{A}^{\rho}_{\not{X}}\left(\Phi,\Xi\right)$

$$\begin{aligned}\mathfrak{A}^{\rho}_{\not{X}}\left(\Phi,\Xi\right) = &-\mathfrak{A}_X\left(\phi_1,\xi_{17};2a_\mu\right) - \mathfrak{A}_X\left(\phi_2,\xi_{18};2a_\mu\right) - \mathfrak{A}_X\left(\phi_3,\xi_{19};w_\mu - a_\mu\right)\\ &- \mathfrak{A}_X\left(\phi_4,\xi_{20};w_\mu - a_\mu\right) - \mathfrak{A}_X\left(\phi_5,\xi_{21};a_\mu - \left(\overline{g_\mu}\right)^1_1\right) + \mathfrak{A}_X\left(\phi_9,\xi_{21};\left(\overline{g_\mu}\right)^1_2\right)\\ &+ \mathfrak{A}_X\left(\phi_{13},\xi_{21};\left(\overline{g_\mu}\right)^1_3\right) - \mathfrak{A}_X\left(\phi_6,\xi_{22};a_\mu - \left(\overline{g_\mu}\right)^1_1\right) + \mathfrak{A}_X\left(\phi_{10},\xi_{22};\left(\overline{g_\mu}\right)^1_2\right)\\ &+ \mathfrak{A}_X\left(\phi_{14},\xi_{22};\left(\overline{g_\mu}\right)^1_3\right) - \mathfrak{A}_X\left(\phi_7,\xi_{23};w_\mu + \left(\overline{g_\mu}\right)^1_1\right) - \mathfrak{A}_X\left(\phi_{11},\xi_{23};\left(\overline{g_\mu}\right)^1_2\right)\end{aligned}$$

$$
\begin{aligned}
&- \mathfrak{A}_X\left(\phi_{15}, \xi_{23}; (\overline{g_\mu})_3^1\right) - \mathfrak{A}_X\left(\phi_8, \xi_{24}; w_\mu + (\overline{g_\mu})_1^1\right) - \mathfrak{A}_X\left(\phi_{12}, \xi_{24}; (\overline{g_\mu})_2^1\right) \\
&- \mathfrak{A}_X\left(\phi_{16}, \xi_{24}; (\overline{g_\mu})_3^1\right) + \mathfrak{A}_X\left(\phi_5, \xi_{25}; (\overline{g_\mu})_1^2\right) - \mathfrak{A}_X\left(\phi_9, \xi_{25}; a_\mu - (\overline{g_\mu})_2^2\right) \\
&+ \mathfrak{A}_X\left(\phi_{13}, \xi_{25}; (\overline{g_\mu})_3^2\right) + \mathfrak{A}_X\left(\phi_6, \xi_{26}; (\overline{g_\mu})_1^2\right) - \mathfrak{A}_X\left(\phi_{10}, \xi_{26}; a_\mu - (\overline{g_\mu})_2^2\right) \\
&+ \mathfrak{A}_X\left(\phi_{14}, \xi_{26}; (\overline{g_\mu})_3^2\right) - \mathfrak{A}_X\left(\phi_7, \xi_{27}; (\overline{g_\mu})_1^2\right) - \mathfrak{A}_X\left(\phi_{11}, \xi_{27}; w_\mu + (\overline{g_\mu})_2^2\right) \\
&- \mathfrak{A}_X\left(\phi_{15}, \xi_{27}; (\overline{g_\mu})_3^2\right) - \mathfrak{A}_X\left(\phi_8, \xi_{28}; (\overline{g_\mu})_1^2\right) - \mathfrak{A}_X\left(\phi_{12}, \xi_{28}; w_\mu + (\overline{g_\mu})_2^2\right) \\
&- \mathfrak{A}_X\left(\phi_{16}, \xi_{28}; (\overline{g_\mu})_3^2\right) + \mathfrak{A}_X\left(\phi_5, \xi_{29}; (\overline{g_\mu})_1^3\right) + \mathfrak{A}_X\left(\phi_9, \xi_{29}; (\overline{g_\mu})_2^3\right) \\
&- \mathfrak{A}_X\left(\phi_{13}, \xi_{29}; a_\mu - (\overline{g_\mu})_3^3\right) + \mathfrak{A}_X\left(\phi_6, \xi_{30}; (\overline{g_\mu})_1^3\right) + \mathfrak{A}_X\left(\phi_{10}, \xi_{30}; (\overline{g_\mu})_2^3\right) \\
&- \mathfrak{A}_X\left(\phi_{14}, \xi_{30}; a_\mu - (\overline{g_\mu})_3^3\right) - \mathfrak{A}_X\left(\phi_7, \xi_{31}; (\overline{g_\mu})_1^3\right) - \mathfrak{A}_X\left(\phi_{11}, \xi_{31}; (\overline{g_\mu})_2^3\right) \\
&- \mathfrak{A}_X\left(\phi_{15}, \xi_{31}; w_\mu + (\overline{g_\mu})_3^3\right) - \mathfrak{A}_X\left(\phi_8, \xi_{32}; (\overline{g_\mu})_1^3\right) - \mathfrak{A}_X\left(\phi_{12}, \xi_{32}; (\overline{g_\mu})_2^3\right) \\
&- \mathfrak{A}_X\left(\phi_{16}, \xi_{32}; w_\mu + (\overline{g_\mu})_3^3\right) - \mathfrak{A}_X\left(\phi_{17}, \xi_1; 2a_\mu\right) - \mathfrak{A}_X\left(\phi_{18}, \xi_2; 2a_\mu\right) \\
&- \mathfrak{A}_X\left(\phi_{19}, \xi_3; w_\mu - a_\mu\right) - \mathfrak{A}_X\left(\phi_{20}, \xi_4; w_\mu - a_\mu\right) - \mathfrak{A}_X\left(\phi_{21}, \xi_5; a_\mu - (g_\mu)_1^1\right) \\
&+ \mathfrak{A}_X\left(\phi_{25}, \xi_5; (g_\mu)_2^1\right) + \mathfrak{A}_X\left(\phi_{29}, \xi_5; (g_\mu)_3^1\right) - \mathfrak{A}_X\left(\phi_{22}, \xi_6; a_\mu - (g_\mu)_1^1\right) \\
&+ \mathfrak{A}_X\left(\phi_{26}, \xi_6; (g_\mu)_2^1\right) + \mathfrak{A}_X\left(\phi_{30}, \xi_6; (g_\mu)_3^1\right) - \mathfrak{A}_X\left(\phi_{23}, \xi_7; w_\mu + (g_\mu)_1^1\right) \\
&- \mathfrak{A}_X\left(\phi_{27}, \xi_7; (g_\mu)_2^1\right) - \mathfrak{A}_X\left(\phi_{31}, \xi_7; (g_\mu)_3^1\right) - \mathfrak{A}_X\left(\phi_{24}, \xi_8; w_\mu + (g_\mu)_1^1\right) \\
&- \mathfrak{A}_X\left(\phi_{28}, \xi_8; (g_\mu)_2^1\right) - \mathfrak{A}_X\left(\phi_{32}, \xi_8; (g_\mu)_3^1\right) + \mathfrak{A}_X\left(\phi_{21}, \xi_9; (g_\mu)_1^2\right) \\
&- \mathfrak{A}_X\left(\phi_{25}, \xi_9; a_\mu - (g_\mu)_2^2\right) + \mathfrak{A}_X\left(\phi_{29}, \xi_9; (g_\mu)_3^2\right) + \mathfrak{A}_X\left(\phi_{22}, \xi_{10}; (g_\mu)_1^2\right) \\
&- \mathfrak{A}_X\left(\phi_{26}, \xi_{10}; a_\mu - (g_\mu)_2^2\right) + \mathfrak{A}_X\left(\phi_{30}, \xi_{10}; (g_\mu)_3^2\right) - \mathfrak{A}_X\left(\phi_{23}, \xi_{11}; (g_\mu)_1^2\right) \\
&- \mathfrak{A}_X\left(\phi_{27}, \xi_{11}; w_\mu + (g_\mu)_2^2\right) - \mathfrak{A}_X\left(\phi_{31}, \xi_{11}; (g_\mu)_3^2\right) - \mathfrak{A}_X\left(\phi_{24}, \xi_{12}; (g_\mu)_1^2\right) \\
&- \mathfrak{A}_X\left(\phi_{28}, \xi_{12}; w_\mu + (g_\mu)_2^2\right) - \mathfrak{A}_X\left(\phi_{32}, \xi_{12}; (g_\mu)_3^2\right) + \mathfrak{A}_X\left(\phi_{21}, \xi_{13}; (g_\mu)_1^3\right) \\
&+ \mathfrak{A}_X\left(\phi_{25}, \xi_{13}; (g_\mu)_2^3\right) - \mathfrak{A}_X\left(\phi_{29}, \xi_{13}; a_\mu - (g_\mu)_3^3\right) + \mathfrak{A}_X\left(\phi_{22}, \xi_{14}; (g_\mu)_1^3\right) \\
&+ \mathfrak{A}_X\left(\phi_{26}, \xi_{14}; (g_\mu)_2^3\right) - \mathfrak{A}_X\left(\phi_{30}, \xi_{14}; a_\mu - (g_\mu)_3^3\right) - \mathfrak{A}_X\left(\phi_{23}, \xi_{15}; (g_\mu)_1^3\right) \\
&- \mathfrak{A}_X\left(\phi_{27}, \xi_{15}; (g_\mu)_2^3\right) - \mathfrak{A}_X\left(\phi_{31}, \xi_{15}; w_\mu + (g_\mu)_3^3\right) - \mathfrak{A}_X\left(\phi_{24}, \xi_{16}; (g_\mu)_1^3\right) \\
&- \mathfrak{A}_X\left(\phi_{28}, \xi_{16}; (g_\mu)_2^3\right) - \mathfrak{A}_X\left(\phi_{32}, \xi_{16}; w_\mu + (g_\mu)_3^3\right) \qquad \text{(D.146)}
\end{aligned}
$$

D.4.4 Calculation of $\mathfrak{A}^{\rho}_{i\not{Y}}(\Phi, \Xi)$

$$
\begin{aligned}
\mathfrak{A}^{\rho}_{i\not{Y}}(\Phi, \Xi) = {} & \mathfrak{A}_Y\left(\phi_2, \xi_{18}; g_1 B_\mu\right) \\
&+ \mathfrak{A}_Y\left(\phi_3, \xi_{19}; \frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^1\right) - \mathfrak{A}_Y\left(\phi_4, \xi_{19}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^1\right) \\
&- \mathfrak{A}_Y\left(\phi_3, \xi_{20}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_4, \xi_{20}; \frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^2\right) \\
&- \mathfrak{A}_Y\left(\phi_5, \xi_{21}; \frac{2}{3} g_1 B_\mu + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^1\right) - \mathfrak{A}_Y\left(\phi_9, \xi_{21}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^1\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \mathfrak{A}_Y\left(\phi_{13}, \xi_{21}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^1\right) \\
&+ \mathfrak{A}_Y\left(\phi_6, \xi_{22}; \frac{g_1}{3} B_\mu - \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^1\right) - \mathfrak{A}_Y\left(\phi_{10}, \xi_{22}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^1\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{14}, \xi_{22}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^1\right) \\
&- \mathfrak{A}_Y\left(\phi_7, \xi_{23}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^1 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^1\right) - \mathfrak{A}_Y\left(\phi_8, \xi_{23}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^1\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{11}, \xi_{23}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^1\right) - \mathfrak{A}_Y\left(\phi_{15}, \xi_{23}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^1\right) \\
&- \mathfrak{A}_Y\left(\phi_7, \xi_{24}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_8, \xi_{24}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^2 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^1\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{11}, \xi_{24}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^1\right) - \mathfrak{A}_Y\left(\phi_{15}, \xi_{24}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^1\right) \\
&- \mathfrak{A}_Y\left(\phi_5, \xi_{25}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_9, \xi_{25}; \frac{2}{3} g_1 B_\mu + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^2\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{13}, \xi_{25}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^2\right) \\
&- \mathfrak{A}_Y\left(\phi_6, \xi_{26}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_{10}, \xi_{26}; \frac{g_1}{3} B_\mu - \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^2\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{14}, \xi_{26}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^2\right) \\
&- \mathfrak{A}_Y\left(\phi_7, \xi_{27}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_{11}, \xi_{27}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^1 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^2\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{12}, \xi_{27}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^1\right) - \mathfrak{A}_Y\left(\phi_{15}, \xi_{27}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^2\right) \\
&- \mathfrak{A}_Y\left(\phi_8, \xi_{28}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_{11}, \xi_{28}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^2\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{12}, \xi_{28}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^2 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^2\right) - \mathfrak{A}_Y\left(\phi_{16}, \xi_{28}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^2\right) \\
&- \mathfrak{A}_Y\left(\phi_5, \xi_{29}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^3\right) - \mathfrak{A}_Y\left(\phi_9, \xi_{29}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^3\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{13}, \xi_{29}; \frac{2}{3} g_1 B_\mu + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^3\right) \\
&- \mathfrak{A}_Y\left(\phi_6, \xi_{30}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^3\right) - \mathfrak{A}_Y\left(\phi_{10}, \xi_{30}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^3\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{14}, \xi_{30}; \frac{g_1}{3} B_\mu - \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^3\right) \\
&- \mathfrak{A}_Y\left(\phi_7, \xi_{31}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^3\right) - \mathfrak{A}_Y\left(\phi_{11}, \xi_{31}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^3\right) \\
&\quad - \mathfrak{A}_Y\left(\phi_{15}, \xi_{31}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^1 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^3\right) - \mathfrak{A}_Y\left(\phi_{16}, \xi_{31}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^1\right) \\
&- \mathfrak{A}_Y\left(\phi_8, \xi_{32}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_1^3\right) - \mathfrak{A}_Y\left(\phi_{12}, \xi_{32}; \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_2^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \mathfrak{A}_Y\left(\phi_{15}, \xi_{32}; \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_1^2\right) - \\
&- \mathfrak{A}_Y\left(\phi_{16}, \xi_{32}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\overline{\sigma_k}\right)_2^2 + \frac{g_3}{2} V_\mu^m \left(\overline{\lambda_m}\right)_3^3\right) \\
&- \mathfrak{A}_Y\left(\phi_{18}, \xi_2; g_1 B_\mu\right) \\
&- \mathfrak{A}_Y\left(\phi_{19}, \xi_3; \frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^1\right) + \mathfrak{A}_Y\left(\phi_{20}, \xi_3; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^1\right) \\
&+ \mathfrak{A}_Y\left(\phi_{19}, \xi_4; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_{20}, \xi_4; \frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^2\right) \\
&+ \mathfrak{A}_Y\left(\phi_{21}, \xi_5; \frac{2}{3} g_1 B_\mu + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^1\right) + \mathfrak{A}_Y\left(\phi_{25}, \xi_5; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^1\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{29}, \xi_5; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^1\right) \\
&- \mathfrak{A}_Y\left(\phi_{22}, \xi_6; \frac{g_1}{3} B_\mu - \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^1\right) + \mathfrak{A}_Y\left(\phi_{26}, \xi_6; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^1\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{30}, \xi_6; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^1\right) \\
&+ \mathfrak{A}_Y\left(\phi_{23}, \xi_7; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^1 + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^1\right) + \mathfrak{A}_Y\left(\phi_{24}, \xi_7; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^1\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{27}, \xi_7; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^1\right) + \mathfrak{A}_Y\left(\phi_{31}, \xi_7; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^1\right) \\
&+ \mathfrak{A}_Y\left(\phi_{23}, \xi_8; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_{24}, \xi_8; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^2 + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^1\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{27}, \xi_8; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^1\right) + \mathfrak{A}_Y\left(\phi_{31}, \xi_8; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^1\right) \\
&+ \mathfrak{A}_Y\left(\phi_{21}, \xi_9; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_{25}, \xi_9; \frac{2}{3} g_1 B_\mu + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^2\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{29}, \xi_9; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^2\right) \\
&+ \mathfrak{A}_Y\left(\phi_{26}, \xi_{10}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^2\right) - \mathfrak{A}_Y\left(\phi_{26}, \xi_{10}; \frac{g_1}{3} B_\mu - \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^2\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{30}, \xi_{10}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^2\right) \\
&+ \mathfrak{A}_Y\left(\phi_{23}, \xi_{11}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_{27}, \xi_{11}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^1 + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^2\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{28}, \xi_{11}; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^1\right) + \mathfrak{A}_Y\left(\phi_{31}, \xi_{11}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^2\right) \\
&+ \mathfrak{A}_Y\left(\phi_{24}, \xi_{12}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_1^2\right) + \mathfrak{A}_Y\left(\phi_{27}, \xi_{12}; \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_1^2\right) \\
&\quad + \mathfrak{A}_Y\left(\phi_{28}, \xi_{12}; \frac{g_1}{6} B_\mu + \frac{g_2}{2} W_\mu^k \left(\sigma_k\right)_2^2 + \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_2^2\right) + \\
&+ \mathfrak{A}_Y\left(\phi_{32}, \xi_{12}; \frac{g_3}{2} V_\mu^m \left(\lambda_m\right)_3^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathfrak{A}_Y\left(\phi_{21},\xi_{13};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_1^3\right)+\mathfrak{A}_Y\left(\phi_{25},\xi_{13};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_2^3\right)\\
&\quad+\mathfrak{A}_Y\left(\phi_{29},\xi_{13};\frac{2}{3}g_1B_\mu+\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_3^3\right)\\
&+\mathfrak{A}_Y\left(\phi_{22},\xi_{14};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_1^3\right)+\mathfrak{A}_Y\left(\phi_{26},\xi_{14};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_2^3\right)\\
&\quad-\mathfrak{A}_Y\left(\phi_{30},\xi_{14};\frac{g_1}{3}B_\mu-\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_3^3\right)\\
&+\mathfrak{A}_Y\left(\phi_{23},\xi_{15};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_1^3\right)+\mathfrak{A}_Y\left(\phi_{27},\xi_{15};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_2^3\right)\\
&\quad+\mathfrak{A}_Y\left(\phi_{31},\xi_{15};\frac{g_1}{6}B_\mu+\frac{g_2}{2}W_\mu^k\left(\sigma_k\right)_1^1+\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_3^3\right)+\mathfrak{A}_Y\left(\phi_{32},\xi_{15};\frac{g_2}{2}W_\mu^k\left(\sigma_k\right)_2^1\right)\\
&+\mathfrak{A}_Y\left(\phi_{24},\xi_{16};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_1^3\right)+\mathfrak{A}_Y\left(\phi_{28},\xi_{16};\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_2^3\right)\\
&\quad+\mathfrak{A}_Y\left(\phi_{31},\xi_{16};\frac{g_2}{2}W_\mu^k\left(\sigma_k\right)_1^2\right)+\mathfrak{A}_Y\left(\phi_{32},\xi_{16};\frac{g_1}{6}B_\mu+\frac{g_2}{2}W_\mu^k\left(\sigma_k\right)_2^2+\frac{g_3}{2}V_\mu^m\left(\lambda_m\right)_3^3\right)
\end{aligned}
$$

D.4.5 Calculation of $\mathfrak{A}^\rho_{\gamma_5\otimes D_Y}(\Phi,\Xi)$

$$
\begin{aligned}
\mathfrak{A}^\rho_{\gamma_5\otimes D_Y}(\Phi,\Xi)=&\,\overline{k_\nu}\mathfrak{A}_{\gamma_5}(\phi_1,\xi_{19})+\overline{k_e}\mathfrak{A}_{\gamma_5}(\phi_2,\xi_{20})+k_\nu\mathfrak{A}_{\gamma_5}(\phi_3,\xi_{17})+k_e\mathfrak{A}_{\gamma_5}(\phi_4,\xi_{18})+\\
&+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_5,\xi_{23})+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_6,\xi_{24})+k_u\mathfrak{A}_{\gamma_5}(\phi_7,\xi_{21})+k_d\mathfrak{A}_{\gamma_5}(\phi_8,\xi_{22})+\\
&+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_9,\xi_{27})+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_{10},\xi_{28})+k_u\mathfrak{A}_{\gamma_5}(\phi_{11},\xi_{25})+k_d\mathfrak{A}_{\gamma_5}(\phi_{12},\xi_{26})+\\
&+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_{13},\xi_{31})+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_{14},\xi_{32})+k_u\mathfrak{A}_{\gamma_5}(\phi_{15},\xi_{29})+k_d\mathfrak{A}_{\gamma_5}(\phi_{16},\xi_{30})+\\
&+k_\nu\mathfrak{A}_{\gamma_5}(\phi_{17},\xi_3)+k_e\mathfrak{A}_{\gamma_5}(\phi_{18},\xi_4)+\overline{k_\nu}\mathfrak{A}_{\gamma_5}(\phi_{19},\xi_1)+\overline{k_e}\mathfrak{A}_{\gamma_5}(\phi_{20},\xi_2)+\\
&+k_u\mathfrak{A}_{\gamma_5}(\phi_{21},\xi_7)+k_d\mathfrak{A}_{\gamma_5}(\phi_{22},\xi_8)+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_{23},\xi_5)+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_{24},\xi_6)+\\
&+k_u\mathfrak{A}_{\gamma_5}(\phi_{25},\xi_{11})+k_d\mathfrak{A}_{\gamma_5}(\phi_{26},\xi_{12})+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_{27},\xi_9)+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_{28},\xi_{10})+\\
&+k_u\mathfrak{A}_{\gamma_5}(\phi_{29},\xi_{15})+k_d\mathfrak{A}_{\gamma_5}(\phi_{30},\xi_{16})+\overline{k_u}\mathfrak{A}_{\gamma_5}(\phi_{31},\xi_{13})+\overline{k_d}\mathfrak{A}_{\gamma_5}(\phi_{32},\xi_{14})
\end{aligned}
\tag{D.147}
$$

D.4.6 Calculation of $\mathfrak{A}^\rho_{M_1}(\Phi,\Xi)$

$$
\begin{aligned}
\mathfrak{A}^\rho_{M_1}(\Phi,\Xi)=&\,k_\nu\mathfrak{A}_-\left(\phi_1,\xi_{19};\phi_2^r,\phi_2^l\right)-k_\nu\mathfrak{A}_-\left(\phi_1,\xi_{20};\phi_1^r,\phi_1^l\right)+\\
&+k_e\mathfrak{A}_-\left(\phi_2,\xi_{19};\overline{\phi_1^r},\overline{\phi_1^l}\right)+k_e\mathfrak{A}_-\left(\phi_2,\xi_{20};\overline{\phi_2^r},\overline{\phi_2^l}\right)+\\
&+\overline{k_\nu}\mathfrak{A}_-\left(\phi_3,\xi_{17};\overline{\phi_2^r},\overline{\phi_2^l}\right)+\overline{k_e}\mathfrak{A}_-\left(\phi_3,\xi_{18};\phi_1^r,\phi_1^l\right)-\\
&-\overline{k_\nu}\mathfrak{A}_-\left(\phi_4,\xi_{17};\overline{\phi_1^r},\overline{\phi_1^l}\right)+\overline{k_e}\mathfrak{A}_-\left(\phi_4,\xi_{18};\phi_2^r,\phi_2^l\right)+\\
&+k_u\mathfrak{A}_-\left(\phi_5,\xi_{23};\phi_2^r,\phi_2^l\right)-k_u\mathfrak{A}_-\left(\phi_5,\xi_{24};\phi_1^r,\phi_1^l\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+ k_d\mathfrak{A}_-\left(\phi_6,\xi_{23};\overline{\phi_1^r},\overline{\phi_1^l}\right) + k_d\mathfrak{A}_-\left(\phi_6,\xi_{24};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_7,\xi_{21};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_7,\xi_{22};\phi_1^r,\phi_1^l\right) - \\
&- \overline{k_u}\mathfrak{A}_-\left(\phi_8,\xi_{21};\overline{\phi_1^r},\overline{\phi_1^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_8,\xi_{22};\phi_2^r,\phi_2^l\right) + \\
&+ k_u\mathfrak{A}_-\left(\phi_9,\xi_{27};\phi_2^r,\phi_2^l\right) - k_u\mathfrak{A}_-\left(\phi_9,\xi_{28};\phi_1^r,\phi_1^l\right) + \\
&+ k_d\mathfrak{A}_-\left(\phi_{10},\xi_{27};\overline{\phi_1^r},\overline{\phi_1^l}\right) + k_d\mathfrak{A}_-\left(\phi_{10},\xi_{28};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_{11},\xi_{25};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{11},\xi_{26};\phi_1^r,\phi_1^l\right) - \\
&- \overline{k_u}\mathfrak{A}_-\left(\phi_{12},\xi_{25};\overline{\phi_1^r},\overline{\phi_1^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{12},\xi_{26};\phi_2^r,\phi_2^l\right) + \\
&+ k_u\mathfrak{A}_-\left(\phi_{13},\xi_{31};\phi_2^r,\phi_2^l\right) - k_u\mathfrak{A}_-\left(\phi_{13},\xi_{32};\phi_1^r,\phi_1^l\right) + \\
&+ k_d\mathfrak{A}_-\left(\phi_{14},\xi_{31};\overline{\phi_1^r},\overline{\phi_1^l}\right) + k_d\mathfrak{A}_-\left(\phi_{14},\xi_{32};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_{15},\xi_{29};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{15},\xi_{30};\phi_1^r,\phi_1^l\right) - \\
&- \overline{k_u}\mathfrak{A}_-\left(\phi_{16},\xi_{29};\overline{\phi_1^r},\overline{\phi_1^l}\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{16},\xi_{30};\phi_2^r,\phi_2^l\right) + \\
&+ \overline{k_\nu}\mathfrak{A}_-\left(\phi_{17},\xi_3;\overline{\phi_2^r},\overline{\phi_2^l}\right) - \overline{k_\nu}\mathfrak{A}_-\left(\phi_{17},\xi_4;\overline{\phi_1^r},\overline{\phi_1^l}\right) + \\
&+ \overline{k_e}\mathfrak{A}_-\left(\phi_{18},\xi_3;\phi_1^r,\phi_1^l\right) + \overline{k_e}\mathfrak{A}_-\left(\phi_{18},\xi_4;\phi_2^r,\phi_2^l\right) + \\
&+ k_\nu\mathfrak{A}_-\left(\phi_{19},\xi_1;\phi_2^r,\phi_2^l\right) + k_e\mathfrak{A}_-\left(\phi_{19},\xi_2;\overline{\phi_1^r},\overline{\phi_1^l}\right) - \\
&- k_\nu\mathfrak{A}_-\left(\phi_{20},\xi_1;\phi_1^r,\phi_1^l\right) + k_e\mathfrak{A}_-\left(\phi_{20},\xi_2;\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_{21},\xi_7;\overline{\phi_2^r},\overline{\phi_2^l}\right) - \overline{k_u}\mathfrak{A}_-\left(\phi_{21},\xi_8;\overline{\phi_1^r},\overline{\phi_1^l}\right) + \\
&+ \overline{k_d}\mathfrak{A}_-\left(\phi_{22},\xi_7;\phi_1^r,\phi_1^l\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{22},\xi_8;\phi_2^r,\phi_2^l\right) + \\
&+ k_u\mathfrak{A}_-\left(\phi_{23},\xi_5;\phi_2^r,\phi_2^l\right) + k_d\mathfrak{A}_-\left(\phi_{23},\xi_6;\overline{\phi_1^r},\overline{\phi_1^l}\right) - \\
&- k_u\mathfrak{A}_-\left(\phi_{24},\xi_5;\phi_1^r,\phi_1^l\right) + k_d\mathfrak{A}_-\left(\phi_{24},\xi_6;\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_{25},\xi_{11};\overline{\phi_2^r},\overline{\phi_2^l}\right) - \overline{k_u}\mathfrak{A}_-\left(\phi_{25},\xi_{12};\overline{\phi_1^r},\overline{\phi_1^l}\right) + \\
&+ \overline{k_d}\mathfrak{A}_-\left(\phi_{26},\xi_{11};\phi_1^r,\phi_1^l\right) + \overline{k_d}\mathfrak{A}_-\left(\phi_{26},\xi_{12};\phi_2^r,\phi_2^l\right) + \\
&+ k_u\mathfrak{A}_-\left(\phi_{27},\xi_9;\phi_2^r,\phi_2^l\right) + k_d\mathfrak{A}_-\left(\phi_{27},\xi_{10};\overline{\phi_1^r},\overline{\phi_1^l}\right) - \\
&- k_u\mathfrak{A}_-\left(\phi_{28},\xi_9;\phi_1^r,\phi_1^l\right) + k_d\mathfrak{A}_-\left(\phi_{28},\xi_{10};\overline{\phi_2^r},\overline{\phi_2^l}\right) + \\
&+ \overline{k_u}\mathfrak{A}_-\left(\phi_{29},\xi_{15};\overline{\phi_2^r},\overline{\phi_2^l}\right) - \overline{k_u}\mathfrak{A}_-\left(\phi_{29},\xi_{16};\overline{\phi_1^r},\overline{\phi_1^l}\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{k_d}\mathfrak{A}_-\left(\phi_{30},\xi_{15};\phi_1^r,\phi_1^l\right)+\overline{k_d}\mathfrak{A}_-\left(\phi_{30},\xi_{16};\phi_2^r,\phi_2^l\right)+\\
&+k_u\mathfrak{A}_-\left(\phi_{31},\xi_{13};\phi_2^r,\phi_2^l\right)+k_d\mathfrak{A}_-\left(\phi_{31},\xi_{14};\overline{\phi_1^r},\overline{\phi_1^l}\right)-\\
&-k_u\mathfrak{A}_-\left(\phi_{32},\xi_{13};\phi_1^r,\phi_1^l\right)+k_d\mathfrak{A}_-\left(\phi_{32},\xi_{14};\overline{\phi_2^r},\overline{\phi_2^l}\right)
\end{aligned}
\tag{D.148}
$$

Appendix E

Calculation of $\mathcal{L}_{Majorana}$

E.1 Calculation of $D^M_{A_\rho}\Xi$

Before starting, let us rewrite $D^M_{A_\rho}$ in a more useful form. We have:

$$\begin{aligned} D^M_{A_\rho} &= \gamma_5 \otimes D_M + \delta^0_C \delta^D_1 \left(C + \overline{D} \right) + \delta^1_C \delta^D_0 \left(\overline{C} + D \right), \\ \gamma_5 \otimes D_M &= \eta^t_s \delta^{\dot{t}}_{\dot{s}} \delta^0_I \delta^J_0 \delta^{\dot{1}}_\alpha \delta^\beta_{\dot{1}} \left(k_R \delta^0_C \delta^D_1 + \overline{k_R} \delta^1_C \delta^D_0 \right), \\ C &= k_R \delta^{\dot{t}}_{\dot{s}} \delta^0_I \delta^J_0 \delta^{\dot{1}}_\alpha \delta^\beta_{\dot{1}} \left(\delta^r_s \delta^t_r \sigma_r + \delta^l_s \delta^t_l \sigma_l \right), \\ D &= \overline{k_R} \delta^{\dot{t}}_{\dot{s}} \delta^0_I \delta^J_0 \delta^{\dot{1}}_\alpha \delta^\beta_{\dot{1}} \left(\delta^r_s \delta^t_r \sigma_r + \delta^l_s \delta^t_l \sigma_l \right). \end{aligned}$$

All this can be recast into:

$$D^M_{A_\rho} = \left(S_M \right)^{t\dot{t}}_{s\dot{s}} \delta^0_I \delta^J_0 \delta^{\dot{1}}_\alpha \delta^\beta_{\dot{1}} \left(k_R \delta^0_C \delta^D_1 + \overline{k_R} \delta^1_C \delta^D_0 \right), \tag{E.1}$$

with

$$\left(S_M \right)^{t\dot{t}}_{s\dot{s}} = \gamma_5 + 2P_r\sigma_r + 2P_l\sigma_l, \tag{E.2}$$

where again

$$P_r \equiv \frac{1+\gamma_5}{2}, \qquad P_l \equiv \frac{1-\gamma_5}{2}. \tag{E.3}$$

Let us evaluate $D^M_{A_\rho}$ on the components of Ξ. Because of the many null entries of $D^M_{A_\rho}$, only two components retain a non-zero value:

$$\begin{aligned} D^M_{A_\rho} \left(\xi_1 \otimes e_1 \right) &= \overline{k_R} S_M \xi_1 \otimes \overline{e_1}, \\ D^M_{A_\rho} \left(\xi_{17} \otimes \overline{e_1} \right) &= k_R S_M \xi_{17} \otimes e_1, \\ D^M_{A_\rho} \left(others \right) &= 0. \end{aligned}$$

E.2 Calculation of $\mathfrak{A}^{\rho}_{D^M_{A_\rho}}(\Phi,\Xi)$

$$\mathfrak{A}^{\rho}_{D^M_{A_\rho}}(\Phi,\Xi) = \overline{k_R}\mathfrak{A}_{S_M}(\phi_1,\xi_1) + k_R\mathfrak{A}_{S_M}(\phi_{17},\xi_{17}). \tag{E.4}$$

Let us evaluate $\mathfrak{A}_{S_M}(\phi,\xi)$:

$$\mathfrak{A}_{S_M}(\phi,\xi) \equiv \langle \mathcal{J}\phi, S_M\xi\rangle \tag{E.5}$$

$$S_M\xi = (\gamma_5 + 2P_r\sigma_r + 2P_l\sigma_l)\,\xi = \begin{pmatrix} I_2(2\sigma_r+1) & 0 \\ 0 & I_2(2\sigma_l-1)\end{pmatrix}\begin{pmatrix}\zeta\\ \zeta\end{pmatrix} = \begin{pmatrix}(2\sigma_r+1)\,\zeta \\ (2\sigma_l-1)\,\zeta\end{pmatrix} \tag{E.6}$$

Then

$$\begin{aligned}(\mathcal{J}\phi)^\dagger(S_M\xi) &= -i\left(-i\overline{\varphi}^\dagger\sigma_2, i\overline{\varphi}\sigma_2\right)\begin{pmatrix}(2\sigma_r+1)\,\zeta\\(2\sigma_l-1)\,\zeta\end{pmatrix} = \\ &= -\overline{\varphi}^\dagger\sigma_2(2\sigma_r+1)\,\zeta + \overline{\varphi}^\dagger\sigma_2(2\sigma_l-1)\,\zeta = -2\overline{\varphi}^\dagger\sigma_2(1+\sigma_r-\sigma_l)\,\zeta\end{aligned} \tag{E.7}$$

therefore

$$\mathfrak{A}_{S_M}(\phi,\xi) = -2\int_{\mathcal{M}} d\mu\left(\overline{\varphi}^\dagger\sigma_2(1+\sigma_r-\sigma_l)\,\zeta\right). \tag{E.8}$$

Appendix F

Calculation of $\mathcal{L}_f^{\Phi_S}$

F.1 Calculation of $M_2\Xi$

We defined

$$M_2 = \begin{pmatrix} \overline{B} & \\ & B \end{pmatrix}_C^D \tag{F.1}$$

where B is defined in (10.51). Before starting with the calculations, let us rewrite M_2 in a more useful form. We have:

$$\begin{aligned} M_2 &= \delta_C^0 \delta_0^D \overline{B} + \delta_C^1 \delta_1^D B = \\ &= \delta_{\dot{s}}^{\dot{t}} \left[\delta_C^0 \delta_0^D \left(\delta_R^s \delta_t^R \left(\overline{B_R}\right)_{I\alpha}^{J\beta} + \delta_L^s \delta_t^L \left(\overline{B_L}\right)_{I\alpha}^{J\beta} \right) + \delta_C^1 \delta_1^D \left(\delta_R^s \delta_t^R \left(B_R\right)_{I\alpha}^{J\beta} + \delta_L^s \delta_t^L \left(B_L\right)_{I\alpha}^{J\beta} \right) \right] = \\ &= \delta_C^0 \delta_0^D \left(P_R \left(\overline{B_R}\right)_{I\alpha}^{J\beta} + P_L \left(\overline{B_L}\right)_{I\alpha}^{J\beta} \right) + \delta_C^1 \delta_1^D \left(P_R \left(B_R\right)_{I\alpha}^{J\beta} + P_L \left(B_L\right)_{I\alpha}^{J\beta} \right) \end{aligned}$$

where again

$$P_R \equiv \frac{1+\gamma_5}{2}, \qquad P_L \equiv \frac{1-\gamma_5}{2}. \tag{F.2}$$

As a shorthand, we define

$$\phi_i = (\phi_S)_{ii}\,, \qquad \phi_{ij} = (\phi_S)_{ij}\,, \tag{F.3}$$

$$\phi'_i = (\phi'_S)_{ii}\,, \qquad \phi'_{ij} = (\phi'_S)_{ij}\,. \tag{F.4}$$

Now we can start with the calculations. We find:

$$M_2 \left(\xi_1 \otimes e_1\right) = \overline{k_\nu} \left(\sigma_r P_R + \sigma_l P_L\right) \xi_1 \otimes e_3 \tag{F.5}$$

$$M_2 \left(\xi_2 \otimes e_2\right) = \overline{k_e} \left(\sigma_r P_R + \sigma_l P_L\right) \xi_2 \otimes e_4 \tag{F.6}$$

$$M_2 \left(\xi_3 \otimes e_3\right) = k_\nu \left(\sigma_r P_R + \sigma_l P_L\right) \xi_3 \otimes e_1 \tag{F.7}$$

$$M_2(\xi_4 \otimes e_4) = k_e(\sigma_r P_R + \sigma_l P_L)\,\xi_4 \otimes e_2 \tag{F.8}$$

$$\begin{aligned} M_2(\xi_5 \otimes e_5) = {} & \overline{k_u}(\phi_1 P_R + \phi_1' P_L)\,\xi_5 \otimes e_7 + \overline{k_u}(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_5 \otimes e_{11} \\ & + \overline{k_u}(\phi_{31} P_R + \phi_{31}' P_L)\,\xi_5 \otimes e_{15} \end{aligned} \tag{F.9}$$

$$\begin{aligned} M_2(\xi_6 \otimes e_6) = {} & \overline{k_d}(\phi_1 P_R + \phi_1' P_L)\,\xi_6 \otimes e_8 + \overline{k_d}(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_6 \otimes e_{12} \\ & + \overline{k_d}(\phi_{31} P_R + \phi_{31}' P_L)\,\xi_6 \otimes e_{16} \end{aligned} \tag{F.10}$$

$$\begin{aligned} M_2(\xi_7 \otimes e_7) = {} & k_u(\phi_1 P_R + \phi_1' P_L)\,\xi_7 \otimes e_5 + k_u(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_7 \otimes e_9 \\ & + k_u(\phi_{31} P_R + \phi_{31}' P_L)\,\xi_7 \otimes e_{13} \end{aligned} \tag{F.11}$$

$$\begin{aligned} M_2(\xi_8 \otimes e_8) = {} & k_d(\phi_1 P_R + \phi_1' P_L)\,\xi_8 \otimes e_6 + k_d(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_6 \otimes e_{10} \\ & + k_d(\phi_{31} P_R + \phi_{31}' P_L)\,\xi_6 \otimes e_{14} \end{aligned} \tag{F.12}$$

$$\begin{aligned} M_2(\xi_9 \otimes e_9) = {} & \overline{k_u}(\phi_{12} P_R + \phi_{12}' P_L)\,\xi_9 \otimes e_7 + \overline{k_u}(\phi_2 P_R + \phi_2' P_L)\,\xi_9 \otimes e_{11} \\ & + \overline{k_u}(\phi_{32} P_R + \phi_{32}' P_L)\,\xi_9 \otimes e_{15} \end{aligned} \tag{F.13}$$

$$\begin{aligned} M_2(\xi_{10} \otimes e_{10}) = {} & \overline{k_d}(\phi_{12} P_R + \phi_{12}' P_L)\,\xi_{10} \otimes e_8 + \overline{k_d}(\phi_2 P_R + \phi_2' P_L)\,\xi_{10} \otimes e_{12} \\ & + \overline{k_d}(\phi_{32} P_R + \phi_{32}' P_L)\,\xi_{10} \otimes e_{16} \end{aligned} \tag{F.14}$$

$$\begin{aligned} M_2(\xi_{11} \otimes e_{11}) = {} & k_u(\phi_{12} P_R + \phi_{12}' P_L)\,\xi_{11} \otimes e_5 + k_u(\phi_2 P_R + \phi_2' P_L)\,\xi_{11} \otimes e_9 \\ & + k_u(\phi_{32} P_R + \phi_{32}' P_L)\,\xi_{11} \otimes e_{13} \end{aligned} \tag{F.15}$$

$$\begin{aligned} M_2(\xi_{12} \otimes e_{12}) = {} & k_d(\phi_{12} P_R + \phi_{12}' P_L)\,\xi_{12} \otimes e_6 + k_d(\phi_2 P_R + \phi_2' P_L)\,\xi_{12} \otimes e_{10} \\ & + k_d(\phi_{32} P_R + \phi_{32}' P_L)\,\xi_{12} \otimes e_{14} \end{aligned} \tag{F.16}$$

$$\begin{aligned} M_2(\xi_{13} \otimes e_{13}) = {} & \overline{k_u}(\phi_{13} P_R + \phi_{13}' P_L)\,\xi_{13} \otimes e_7 + \overline{k_u}(\phi_{23} P_R + \phi_{23}' P_L)\,\xi_{13} \otimes e_{11} \\ & + \overline{k_u}(\phi_3 P_R + \phi_3' P_L)\,\xi_{13} \otimes e_{15} \end{aligned} \tag{F.17}$$

$$\begin{aligned} M_2(\xi_{14} \otimes e_{14}) = {} & \overline{k_d}(\phi_{13} P_R + \phi_{13}' P_L)\,\xi_{14} \otimes e_8 + \overline{k_d}(\phi_{23} P_R + \phi_{23}' P_L)\,\xi_{14} \otimes e_{12} \\ & + \overline{k_d}(\phi_3 P_R + \phi_3' P_L)\,\xi_{14} \otimes e_{16} \end{aligned} \tag{F.18}$$

$$\begin{aligned} M_2(\xi_{15} \otimes e_{15}) = {} & k_u(\phi_{13} P_R + \phi_{13}' P_L)\,\xi_{15} \otimes e_5 + k_u(\phi_{23} P_R + \phi_{23}' P_L)\,\xi_{15} \otimes e_9 \\ & + k_u(\phi_3 P_R + \phi_3' P_L)\,\xi_{15} \otimes e_{13} \end{aligned} \tag{F.19}$$

$$\begin{aligned} M_2(\xi_{16} \otimes e_{16}) = {} & k_d(\phi_{13} P_R + \phi_{13}' P_L)\,\xi_{16} \otimes e_6 + k_d(\phi_{23} P_R + \phi_{23}' P_L)\,\xi_{16} \otimes e_{10} \\ & + k_d(\phi_3 P_R + \phi_3' P_L)\,\xi_{16} \otimes e_{14} \end{aligned} \tag{F.20}$$

$$M_2(\xi_{17} \otimes \overline{e_1}) = k_\nu(\sigma_r P_R + \sigma_l P_L)\,\xi_{17} \otimes \overline{e_3} \tag{F.21}$$

$$M_2(\xi_{18} \otimes \overline{e_2}) = k_e(\sigma_r P_R + \sigma_l P_L)\,\xi_{18} \otimes \overline{e_4} \tag{F.22}$$

$$M_2(\xi_{19} \otimes \overline{e_3}) = \overline{k_\nu}(\sigma_r P_R + \sigma_l P_L)\,\xi_{19} \otimes \overline{e_1} \tag{F.23}$$

$$M_2(\xi_{20} \otimes \overline{e_4}) = \overline{k_e}(\sigma_r P_R + \sigma_l P_L)\,\xi_{20} \otimes \overline{e_2} \tag{F.24}$$

$$\begin{aligned} M_2(\xi_{21} \otimes \overline{e_5}) = {} & k_u(\phi_1 P_R + \phi_1' P_L)\,\xi_{21} \otimes \overline{e_7} + k_u(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_{21} \otimes \overline{e_{11}} \\ & + k_u(\phi_{31} P_R + \phi_{31}' P_L)\,\xi_{21} \otimes \overline{e_{15}} \end{aligned} \tag{F.25}$$

$$M_2(\xi_{22} \otimes \overline{e_6}) = k_d(\phi_1 P_R + \phi_1' P_L)\,\xi_{22} \otimes \overline{e_8} + k_d(\phi_{21} P_R + \phi_{21}' P_L)\,\xi_{22} \otimes \overline{e_{12}}$$

$$\begin{aligned}
&+ k_d\left(\phi_{31}P_R + \phi'_{31}P_L\right)\xi_{22}\otimes\overline{e_{16}} && \text{(F.26)}\\
M_2\left(\xi_{23}\otimes\overline{e_7}\right) &= \overline{k_u}\left(\phi_1 P_R + \phi'_1 P_L\right)\xi_{23}\otimes\overline{e_5} + \overline{k_u}\left(\phi_{21}P_R + \phi'_{21}P_L\right)\xi_{23}\otimes\overline{e_9}\\
&+ \overline{k_u}\left(\phi_{31}P_R + \phi'_{31}P_L\right)\xi_{23}\otimes\overline{e_{13}} && \text{(F.27)}\\
M_2\left(\xi_{24}\otimes\overline{e_8}\right) &= \overline{k_d}\left(\phi_1 P_R + \phi'_1 P_L\right)\xi_{24}\otimes\overline{e_6} + \overline{k_d}\left(\phi_{21}P_R + \phi'_{21}P_L\right)\xi_{24}\otimes\overline{e_{10}}\\
&+ \overline{k_d}\left(\phi_{31}P_R + \phi'_{31}P_L\right)\xi_{24}\otimes\overline{e_{14}} && \text{(F.28)}\\
M_2\left(\xi_{25}\otimes\overline{e_9}\right) &= k_u\left(\phi_{12}P_R + \phi'_{12}P_L\right)\xi_{25}\otimes\overline{e_7} + k_u\left(\phi_2 P_R + \phi'_2 P_L\right)\xi_{25}\otimes\overline{e_{11}}\\
&+ k_u\left(\phi_{32}P_R + \phi'_{32}P_L\right)\xi_{25}\otimes\overline{e_{15}} && \text{(F.29)}\\
M_2\left(\xi_{26}\otimes\overline{e_{10}}\right) &= k_d\left(\phi_{12}P_R + \phi'_{12}P_L\right)\xi_{26}\otimes\overline{e_8} + k_d\left(\phi_2 P_R + \phi'_2 P_L\right)\xi_{26}\otimes\overline{e_{12}}\\
&+ k_d\left(\phi_{32}P_R + \phi'_{32}P_L\right)\xi_{26}\otimes\overline{e_{16}} && \text{(F.30)}\\
M_2\left(\xi_{27}\otimes\overline{e_{11}}\right) &= \overline{k_u}\left(\phi_{12}P_R + \phi'_{12}P_L\right)\xi_{27}\otimes\overline{e_5} + \overline{k_u}\left(\phi_2 P_R + \phi'_2 P_L\right)\xi_{27}\otimes\overline{e_9}\\
&+ \overline{k_u}\left(\phi_{32}P_R + \phi'_{32}P_L\right)\xi_{27}\otimes\overline{e_{13}} && \text{(F.31)}\\
M_2\left(\xi_{28}\otimes\overline{e_{12}}\right) &= \overline{k_d}\left(\phi_{12}P_R + \phi'_{12}P_L\right)\xi_{28}\otimes\overline{e_6} + \overline{k_d}\left(\phi_2 P_R + \phi'_2 P_L\right)\xi_{28}\otimes\overline{e_{10}}\\
&+ \overline{k_d}\left(\phi_{32}P_R + \phi'_{32}P_L\right)\xi_{28}\otimes\overline{e_{14}} && \text{(F.32)}\\
M_2\left(\xi_{29}\otimes\overline{e_{13}}\right) &= k_u\left(\phi_{13}P_R + \phi'_{13}P_L\right)\xi_{29}\otimes\overline{e_7} + k_u\left(\phi_{23}P_R + \phi'_{23}P_L\right)\xi_{29}\otimes\overline{e_{11}}\\
&+ k_u\left(\phi_3 P_R + \phi'_3 P_L\right)\xi_{29}\otimes\overline{e_{15}} && \text{(F.33)}\\
M_2\left(\xi_{30}\otimes\overline{e_{14}}\right) &= k_d\left(\phi_{13}P_R + \phi'_{13}P_L\right)\xi_{30}\otimes\overline{e_8} + k_d\left(\phi_{23}P_R + \phi'_{23}P_L\right)\xi_{30}\otimes\overline{e_{12}}\\
&+ k_d\left(\phi_3 P_R + \phi'_3 P_L\right)\xi_{30}\otimes\overline{e_{16}} && \text{(F.34)}\\
M_2\left(\xi_{31}\otimes\overline{e_{15}}\right) &= \overline{k_u}\left(\phi_{13}P_R + \phi'_{13}P_L\right)\xi_{31}\otimes\overline{e_5} + \overline{k_u}\left(\phi_{23}P_R + \phi'_{23}P_L\right)\xi_{31}\otimes\overline{e_9}\\
&+ \overline{k_u}\left(\phi_3 P_R + \phi'_3 P_L\right)\xi_{31}\otimes\overline{e_{13}} && \text{(F.35)}\\
M_2\left(\xi_{32}\otimes\overline{e_{16}}\right) &= \overline{k_d}\left(\phi_{13}P_R + \phi'_{13}P_L\right)\xi_{32}\otimes\overline{e_6} + \overline{k_d}\left(\phi_{23}P_R + \phi'_{23}P_L\right)\xi_{32}\otimes\overline{e_{10}}\\
&+ \overline{k_d}\left(\phi_3 P_R + \phi'_3 P_L\right)\xi_{32}\otimes\overline{e_{14}} && \text{(F.36)}
\end{aligned}$$

F.2 Calculation of $\mathfrak{A}^{\rho}_{M_2}(\Phi, \Xi)$

$$\begin{aligned}
\mathfrak{A}^{\rho}_{M_2}(\Phi, \Xi) &= k_\nu\mathfrak{A}_+\left(\phi_3, \xi_{17}; \sigma_r, \sigma_l\right) + k_e\mathfrak{A}_+\left(\phi_4, \xi_{18}; \sigma_r, \sigma_l\right) +\\
&+ \overline{k_\nu}\mathfrak{A}_+\left(\phi_1, \xi_{19}; \sigma_r, \sigma_l\right) + \overline{k_e}\mathfrak{A}_+\left(\phi_2, \xi_{20}; \sigma_r, \sigma_l\right) +\\
&+ k_u\mathfrak{A}_+\left(\phi_7, \xi_{21}; \phi_1, \phi'_1\right) + k_u\mathfrak{A}_+\left(\phi_{11}, \xi_{21}; \phi_{21}, \phi'_{21}\right) + k_u\mathfrak{A}_+\left(\phi_{15}, \xi_{21}; \phi_{31}, \phi'_{31}\right) +\\
&+ k_d\mathfrak{A}_+\left(\phi_8, \xi_{22}; \phi_1, \phi'_1\right) + k_d\mathfrak{A}_+\left(\phi_{12}, \xi_{22}; \phi_{21}, \phi'_{21}\right) + k_d\mathfrak{A}_+\left(\phi_{16}, \xi_{22}; \phi_{31}, \phi'_{31}\right) +\\
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_5, \xi_{23}; \phi_1, \phi'_1\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_9, \xi_{23}; \phi_{21}, \phi'_{21}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{13}, \xi_{23}; \phi_{31}, \phi'_{31}\right) +\\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_6, \xi_{24}; \phi_1, \phi'_1\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{10}, \xi_{24}; \phi_{21}, \phi'_{21}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{14}, \xi_{24}; \phi_{31}, \phi'_{31}\right) +\\
&+ k_u\mathfrak{A}_+\left(\phi_7, \xi_{25}; \phi_{12}, \phi'_{12}\right) + k_u\mathfrak{A}_+\left(\phi_{11}, \xi_{25}; \phi_2, \phi'_2\right) + k_u\mathfrak{A}_+\left(\phi_{15}, \xi_{25}; \phi_{32}, \phi'_{32}\right) +\\
&+ k_d\mathfrak{A}_+\left(\phi_8, \xi_{26}; \phi_{12}, \phi'_{12}\right) + k_d\mathfrak{A}_+\left(\phi_{12}, \xi_{26}; \phi_2, \phi'_2\right) + k_d\mathfrak{A}_+\left(\phi_{16}, \xi_{26}; \phi_{32}, \phi'_{32}\right) +
\end{aligned}$$

$$
\begin{aligned}
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_5, \xi_{27}; \phi_{12}, \phi'_{12}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_9, \xi_{27}; \phi_2, \phi'_2\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{13}, \xi_{27}; \phi_{32}, \phi'_{32}\right) + \\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_6, \xi_{28}; \phi_{12}, \phi'_{12}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{10}, \xi_{28}; \phi_2, \phi'_2\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{14}, \xi_{28}; \phi_{32}, \phi'_{32}\right) + \\
&+ k_u\mathfrak{A}_+\left(\phi_7, \xi_{29}; \phi_{13}, \phi'_{13}\right) + k_u\mathfrak{A}_+\left(\phi_{11}, \xi_{29}; \phi_{23}, \phi'_{23}\right) + k_u\mathfrak{A}_+\left(\phi_{15}, \xi_{29}; \phi_3, \phi'_3\right) + \\
&+ k_d\mathfrak{A}_+\left(\phi_8, \xi_{30}; \phi_{13}, \phi'_{13}\right) + k_d\mathfrak{A}_+\left(\phi_{12}, \xi_{30}; \phi_{23}, \phi'_{23}\right) + k_d\mathfrak{A}_+\left(\phi_{16}, \xi_{30}; \phi_3, \phi'_3\right) + \\
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_5, \xi_{31}; \phi_{13}, \phi'_{13}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_9, \xi_{31}; \phi_{23}, \phi'_{23}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{13}, \xi_{31}; \phi_3, \phi'_3\right) + \\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_6, \xi_{32}; \phi_{13}, \phi'_{13}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{10}, \xi_{32}; \phi_{23}, \phi'_{23}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{14}, \xi_{32}; \phi_3, \phi'_3\right) + \\
&+ \overline{k_\nu}\mathfrak{A}_+\left(\phi_{19}, \xi_1; \sigma_r, \sigma_l\right) + \overline{k_e}\mathfrak{A}_+\left(\phi_{20}, \xi_2; \sigma_r, \sigma_l\right) + \\
&+ k_\nu\mathfrak{A}_+\left(\phi_{17}, \xi_3; \sigma_r, \sigma_l\right) + k_e\mathfrak{A}_+\left(\phi_{18}, \xi_4; \sigma_r, \sigma_l\right) + \\
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_{23}, \xi_5; \phi_1, \phi'_1\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{27}, \xi_5; \phi_{21}, \phi'_{21}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{31}, \xi_5; \phi_{31}, \phi'_{31}\right) + \\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_{24}, \xi_6; \phi_1, \phi'_1\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{28}, \xi_6; \phi_{21}, \phi'_{21}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{32}, \xi_6; \phi_{31}, \phi'_{31}\right) + \\
&+ k_u\mathfrak{A}_+\left(\phi_{21}, \xi_7; \phi_1, \phi'_1\right) + k_u\mathfrak{A}_+\left(\phi_{25}, \xi_7; \phi_{21}, \phi'_{21}\right) + k_u\mathfrak{A}_+\left(\phi_{29}, \xi_7; \phi_{31}, \phi'_{31}\right) + \\
&+ k_d\mathfrak{A}_+\left(\phi_{22}, \xi_8; \phi_1, \phi'_1\right) + k_d\mathfrak{A}_+\left(\phi_{26}, \xi_8; \phi_{21}, \phi'_{21}\right) + k_d\mathfrak{A}_+\left(\phi_{31}, \xi_8; \phi_{31}, \phi'_{31}\right) + \\
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_{23}, \xi_9; \phi_{12}, \phi'_{12}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{27}, \xi_9; \phi_2, \phi'_2\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{31}, \xi_9; \phi_{32}, \phi'_{32}\right) + \\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_{24}, \xi_{10}; \phi_{12}, \phi'_{12}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{28}, \xi_{10}; \phi_2, \phi'_2\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{32}, \xi_{10}; \phi_{32}, \phi'_{32}\right) + \\
&+ k_u\mathfrak{A}_+\left(\phi_{21}, \xi_{11}; \phi_{12}, \phi'_{12}\right) + k_u\mathfrak{A}_+\left(\phi_{25}, \xi_{11}; \phi_2, \phi'_2\right) + k_u\mathfrak{A}_+\left(\phi_{29}, \xi_{11}; \phi_{32}, \phi'_{32}\right) + \\
&+ k_d\mathfrak{A}_+\left(\phi_{22}, \xi_{12}; \phi_{12}, \phi'_{12}\right) + k_d\mathfrak{A}_+\left(\phi_{26}, \xi_{12}; \phi_2, \phi'_2\right) + k_d\mathfrak{A}_+\left(\phi_{30}, \xi_{12}; \phi_{32}, \phi'_{32}\right) + \\
&+ \overline{k_u}\mathfrak{A}_+\left(\phi_{23}, \xi_{13}; \phi_{13}, \phi'_{13}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{27}, \xi_{13}; \phi_{23}, \phi'_{23}\right) + \overline{k_u}\mathfrak{A}_+\left(\phi_{31}, \xi_{13}; \phi_3, \phi'_3\right) + \\
&+ \overline{k_d}\mathfrak{A}_+\left(\phi_{24}, \xi_{14}; \phi_{13}, \phi'_{13}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{28}, \xi_{14}; \phi_{23}, \phi'_{23}\right) + \overline{k_d}\mathfrak{A}_+\left(\phi_{32}, \xi_{14}; \phi_3, \phi'_3\right) + \\
&+ k_u\mathfrak{A}_+\left(\phi_{21}, \xi_{15}; \phi_{13}, \phi'_{13}\right) + k_u\mathfrak{A}_+\left(\phi_{25}, \xi_{15}; \phi_{23}, \phi'_{23}\right) + k_u\mathfrak{A}_+\left(\phi_{29}, \xi_{15}; \phi_3, \phi'_3\right) + \\
&+ k_d\mathfrak{A}_+\left(\phi_{22}, \xi_{16}; \phi_{13}, \phi'_{13}\right) + k_d\mathfrak{A}_+\left(\phi_{26}, \xi_{16}; \phi_{23}, \phi'_{23}\right) + k_d\mathfrak{A}_+\left(\phi_{30}, \xi_{16}; \phi_3, \phi'_3\right)
\end{aligned}
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