How long does a generation last? Assessing the relationship between infinite and finite horizon dynamic models

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29 January 2021

How Long does a Generation Last? Assessing the Relationship Between Infinite and Finite Horizon Dynamic Models*

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January 29, 2021

Abstract

This paper aims at assessing the temporal relationship that exists between the time reference of dynamic models with infinite and finite horizon. Specifically, comparing the optimal inter-temporal plans arising from an infinite-horizon model and a 2-period overlapping generations model in their stationary equilibria, I offer a straightforward way to determine the number of time periods of the former that may form a unit of time of the latter. In this way, I show that the theoretical length of a generation is an increasing function of the discount factor of the optimizing agent and I provide an economic rationale for such a relationship grounded on consumption smoothing.

Keywords: Infinite horizon; Overlapping generations; Consumption smoothing; Discount rate.

JEL Classification: C61, C68, E21, E30.

*Comments are welcome.

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1 Introduction

According to a widely accepted view, the length of a generation, i.e. the number of periods between successive young-old relationships in human communities, is something around 25 years.[1] As proof of this, a quarter of a century is also the time interval usually acknowledged for a human generation by demographers and geneticists (e.g. Weiss, 1973; Thomson et al. 2000).

From the very beginning of modern economic science, economists studied a large variety of important relationships that involve human beings in different stages of their life by taking the underlying time perspective in serious consideration (cf. Samuelson, 1958; Diamond, 1965; Galor and Weil, 1996). However, to the best of my knowledge, economic dynamics takes the length of a generation as given without any attempt to provide a possible criterion to measure the actual duration of young-old relationships that are typical of finite-time models. In this paper, I aim at filling this gap by evaluating the theoretical length of a generation from an exquisitely economic point of view.

The starting point of my deepening is the assessment of the temporal relationship that holds between the time reference of commonly used dynamic models with infinite and finite horizon. Specifically, comparing the optimal inter-temporal plans arising from an infinite horizon (IH) model and a companion 2-period overlapping generations (OLG) model in their stationary equilibria, I provide a straightforward way to determine the number of time periods of the former that may form a unit of time of the latter. In other words, analysing the behaviour of a representative household endowed with logarithmic instantaneous preferences that puts forward an optimal intertemporal plan aimed at financing its consumption expenditure by means of its own wealth, I show that the hypothetical length of a generation depends on how heavily the household itself discount future utility streams. To be precise, I show that the number of periods of the IH model that form the first unit of time in the OLG model is an increasing function of the discount factor value. In this way, relying on dynamic models with a sound microfoundation, I am able to give a point evaluation of the theoretical length of a generation as well as to give some insights on how calibrate the rate of intertemporal preference in conventional business cycle models.

The paper is arranged as follows. Section 2 describes the common framework of the analysis. Section 3 develops the IH model. Section 4 sets out the 2-period OLG model. Section 5 makes a comparison between the optimal consumption plans arising from the mentioned dynamic models. Finally, section 6 concludes.

[1]The young (old) are usually the children (parents) of the old (young) born in the previous period.

2 A common framework

Taking time as a discrete phenomenon, I consider an IH and a 2-period OLG model in which a representative household endowed with logarithmic instantaneous preferences puts forward an optimal inter-temporal plan aimed at financing its consumption expenditure by means of its own wealth. In addition, I make the hypothesis that in both cases the household discounts future utility streams with a constant discount rate.

Assuming that the household is called in to choose among $n \geqslant 1$ goods, the maximandum of its inter-temporal problem is the following:

$$\sum_{t=s}^{T}\left(\frac{1}{1+r}\right)^{t-s}\sum_{i=1}^{n} g_i \log\left(c_{i,t}\right) \tag{1}$$

where s is the starting period, T is the final period, $r > 0$ is a measure of real interest rate, $c_{i,t}$ is the real consumption of the i-th good at time t – with $t = s, \ldots, T$ – and g_i – with $i = 1, \ldots, n$ – is weight in terms of instantaneous utility assigned to the i-th good.

For sake of simplicity, I also assume that the instantaneous utility is a log-linearization of a homogenous function of degree one. Therefore, it holds

$$\sum_{i=1}^{n} g_i = 1 \tag{2}$$

Let me now introduce to the structure of the intertemporal budget constraint. In the starting period, the representative household is assumed to be endowed with a real wealth equal to $W_s \geq 0$. This amount of resources can be thought as the sum of the household's human and financial wealth evaluated in the initial period of the optimization problem and it can be alternately consumed or – if saved – invested in the capital market at the prevailing interest rate. Consequently, the intertemporal budget constraint of the household is of the form

$$\sum_{t=s}^{T}\left(\frac{1}{1+r}\right)^{t-s}\sum_{i=1}^{n} c_{i,t} \leqslant W_s \tag{3}$$

In a quite conventional way, the intertemporal budget constraint defined by eq. (3) simply states that the actual value of the consumption expenditure over the n goods carried out from s to T cannot be higher than the value of the initial wealth of the household. Moreover, since the interest rate used to discount future consumption streams is equal to the one used to discount their future instantaneous utilities, such an intertemporal budget constraint will imply the stationarity of the consumption plans in the two models developed below (cf. Ramsey, 1928; Cass, 1965; Koopmans, 1965).

In the remainder of the paper, I will use t to denote the unit of time of the IH model while I will use τ to denote the unit of time of the 2-period OLG model. Consequently, T will be equal to ∞ for the IH model whereas in the OLG model – in which the household is initially young, then after a period it becomes old – T will be equal to $s+1$. Obviously, it seems reasonable to argue that $\tau > t$, i.e. that the time horizon covered by a period of the OLG model is longer than the one covered by a single period of the IH model. Given such a different time perspective, I will also assume that the real interest rate plugged into the IH model – indicated by $r_{\mathrm{IH}} > 0$ – is strictly lower than the one plugged into the OLG model, denoted instead by $r_{\mathrm{OLG}} > 0$. Everything else being equal, this hypothesis means that the representative young household of the OLG model will tend to discount future consumption streams more heavily than the household of the IH model does. However, at the same time, the old household of the OLG model will enjoy a higher return on its financial investment on the capital market. Given this general framework, the main goal of the theoretical analysis that follows is to provide a way to assess the possible magnitude of the ratio τ/t.

3 The IH model

In this section, I develop a simple IH model that draws on Farmer and Plotnikov (2012) and Farmer (2010, Chapter 6). Specifically, the problem of the representative infinitely lived household is assumed to be the following:

$$\max_{\left\{\{c_{i,t}\}_{i=1}^{n}\right\}_{t=s}^{\infty}} \sum_{t=s}^{\infty} \left(\frac{1}{1+r_{\mathrm{IH}}}\right)^{t-s} \sum_{i=1}^{n} g_i \log\left(c_{i,t}\right) \tag{4}$$

s.to

$$\sum_{t=s}^{\infty} \left(\frac{1}{1+r_{\mathrm{IH}}}\right)^{t-s} \sum_{i=1}^{n} c_{i,t} \leqslant W_s \tag{5}$$

The problem above can be solved by writing the implied Lagrangian. Hence,

$$\mathcal{L}(\cdot) \equiv \sum_{t=s}^{\infty} \left(\frac{1}{1+r_{\mathrm{IH}}}\right)^{t-s} \sum_{i=1}^{n} g_i \log\left(c_{i,t}\right) - \lambda \left(\sum_{t=s}^{\infty} \left(\frac{1}{1+r_{\mathrm{IH}}}\right)^{t-s} \sum_{i=1}^{n} c_{i,t} - W_s\right) \tag{6}$$

where λ is the Lagrange multiplier.

The first-order conditions (FOCs) for eq. (6) are given by the following sequences:

$$g_i - \lambda c_{i,t} = 0 \qquad i = 1, \ldots, n \qquad t = s, \ldots, \infty \tag{7}$$

Recalling the result in eq. (2), the aggregation over the n consumption goods reveals that the expressions in eq. (7) can be written as

$$C_t = \frac{1}{\lambda} \qquad t = s, \ldots, \infty \tag{8}$$

where $C_t \equiv \sum_{i=1}^{n} c_{i,t}$ is the aggregate consumption expenditure at time t.

Plugging the result in eq. (8) into the intertemporal budget constraint in eq. (5) allows us to write down the Lagrange multiplier as a function of the real interest rate and the value of initial wealth. Formally speaking, it holds

$$\lambda = \frac{1}{W_s} \frac{1 + r_{\mathrm{IH}}}{r_{\mathrm{IH}}} \tag{9}$$

Substituting eq. (9) into eq. (8) leads to

$$C_t = \frac{r_{\mathrm{IH}}}{1 + r_{\mathrm{IH}}} W_s \tag{10}$$

The expression in eq. (10) reveals that the optimal plan of the infinitely lived household is to consume in each period a fraction of its own wealth equal to $r_{\mathrm{IH}}/(1 + r_{\mathrm{IH}})$. Consequently, the higher the value of the discount rate, i.e. the more impatient the household, the higher the share of the initial wealth allocated to current consumption.

4 The 2-period OLG model

In this section, I develop a simple 2-period OLG model that draws on Guerrazzi (2007, 2010). Specifically, the representative household that lives for 2 periods is assumed to solve the following problem:

$$\max_{\left\{\{c_{i,t}\}_{i=1}^{n}\right\}_{\tau=s}^{s+1}} \sum_{\tau=s}^{s+1} \left(\frac{1}{1 + r_{\mathrm{OLG}}}\right)^{\tau-s} \sum_{i=1}^{n} g_i \log\left(c_{i,\tau}\right) \tag{11}$$

s.to

$$\sum_{i=1}^{n} c_{i,s} + \frac{1}{1 + r_{\mathrm{OLG}}} \sum_{i=1}^{n} c_{i,s+1} \leqslant W_s \tag{12}$$

As before, the problem above can be solved by writing the implied Lagrangian. Hence,

$$\mathcal{L}\left(\cdot\right) \equiv \sum_{\tau=s}^{s+1} \left(\frac{1}{1 + r_{\mathrm{OLG}}}\right)^{\tau-s} \sum_{i=1}^{n} g_i \log\left(c_{i,\tau}\right) - \lambda \left(\sum_{i=1}^{n} c_{i,s} + \frac{1}{1 + r_{\mathrm{OLG}}} \sum_{i=1}^{n} c_{i,s+1} - W_s\right) \tag{13}$$

The FOCs for eq. (13) are given by the following sequences:

$$g_i - \lambda c_{i,\tau} = 0 \qquad i = 1, \ldots, n \qquad \tau = \{s, s+1\} \tag{14}$$

Recalling the result in eq. (2), the aggregation over the n consumption goods reveals the expressions in eq. (14) necessarily imply that

$$C_s = C_{s+1} \tag{15}$$

where $C_s \equiv \sum_{i=1}^{n} c_{i,s}$ is the aggregate consumption expenditure in the starting period when the household is young whereas $C_{s+1} \equiv \sum_{i=1}^{n} c_{i,s+1}$ is the aggregate consumption expenditure in the final period when the household is old.

Substituting the result in eq. (15) into the intertemporal budget constraint in eq. (12) leads to

$$C_\tau = \frac{1 + r_{\text{OLG}}}{2 + r_{\text{OLG}}} W_s \qquad \tau = \{s, s+1\} \tag{16}$$

The expression in eq. (16) reveals that the optimal plan of the household that lives for 2 periods is to consume in each period the fraction $(1 + r_{\text{OLG}})(2 + r_{\text{OLG}})$. Such a fraction is always higher than the share of wealth consumed by the infinitely lived household conveyed by eq. (10) no matter the actual value of the real interest rate. In general, if r is the real interest rate prevailing on the capital market that is also used to discount future utility streams, then it would be possible to show that a household that lives for m periods consumes a fraction $1 + (1 + r)^{-1} + \cdots + (1 + r)^{1-m}$ of its own wealth. Consequently, whenever $m \to \infty$ the expression in eq. (16) collapses to the one in eq. (10).

5 IH versus OLG

In this section, I put forward a comparison between the optimal intertemporal consumption plans of the two household's problems described above by assessing the number of units of time of the IH model may form a unit of time in the 2-period OLG model. A simple way to make such an assessment is to find the number of units of time over which the infinitely lived household consumes the same amount of resources consumed in a unit of time by the household the lives for two periods. Formally speaking this means that the theoretical length of a generation is given by the value of t that solves the following equation:

$$t \frac{r_{\text{IH}}}{1 + r_{\text{IH}}} = \frac{1 + r_{\text{OLG}}}{2 + r_{\text{OLG}}} \tag{17}$$

In order to find a solution to eq. (17) that depends on one configuration only of the rate of interest, it is necessary to make some assumptions about the relationship between r_{IH} and r_{OLG}. In what follows, I will assume that the rate of return prevailing in the OLG model in IH model is achieved only after the theoretical length of a generation. Taking into account the expression in eq. (17), formally speaking this means that

$$(1 + r_{\text{IH}})^t = 1 + r_{\text{OLG}} \tag{18}$$

Plugging eq. (18) into eq. (17) reveals that for each value of r_{HI} the theoretical length of a generation is found by retrieving the value of t that solves the following expression:

$$t\frac{r_{\mathrm{IH}}}{1+r_{\mathrm{IH}}} = \frac{(1+r_{\mathrm{IH}})^t}{1+(1+r_{\mathrm{IH}})^t} \tag{19}$$

On the one hand, given the value of r_{IH}, the expression of the LHS of eq. (19) is equal to zero for $t = 0$ and thereafter it rises linearly with the unit of time of the IH model. On the other hand, the expression on the RHS is equal to 1/2 for $t = 0$ and thereafter it rises at decreasing rates with increases in t. Consequently, as shown in the diagram of Figure 1, there will be only one meaningful solution to eq. (19) – say t_G – and such a solution – according to the resource consumption criterion suggested by eq. (17) – returns the theoretical length of a generation.[2]

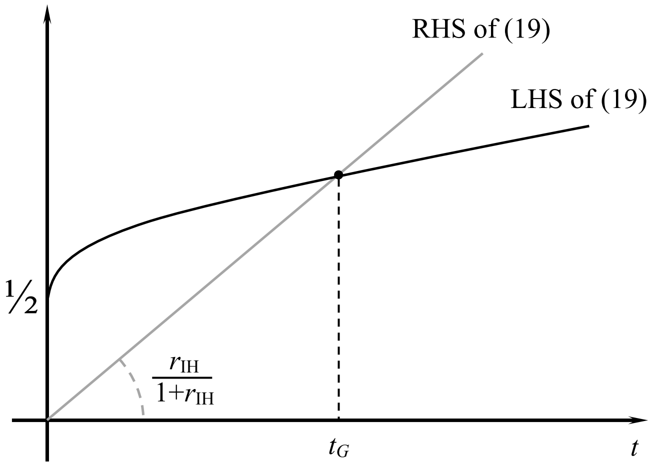

Figure 1: The length of a generation

An intriguing feature of t_G is that it is negatively related to the magnitude of r_{IH}; indeed, for higher (lower) values of the real interest rate the line and the curve depicted in Figure 1 rotate in counter-clockwise (clockwise) direction. However, given the different shapes, the movement of the expression on the RHS of eq. (19) is always more pronounced than the corresponding movement of the one on the LHS. Consequently, higher (lower) values of r_{IH} lead to lower (higher) values of t_G.

An economic rationale for the relationship between the interest rate and the theoretical length of a generation can be given as follows. First, a household that does not care about the future will consume immediately all of its wealth no matter the length of its horizon: indeed, whenever $r_{\mathrm{IH}} \to \infty$, the IH as well as the 2-period OLG model deliver the same intertemporal consumption optimal plan. Consequently, in this case the theoretical length of a generation

[2]Obviously, the point value of r_{IH} can always be tuned in order to have $t_G \in \mathbb{N}$.

coincides with the single period of the IH model. By contrast, when the household starts to care about its future it will smooth its consumption expenditure throughout its relevant time horizon. In general, whenever $r_{\mathrm{IH}} < \infty$, the length of a generation increases with the value of the implied discount factor.

In the real business cycle literature, the discount rate of optimizing consumers is usually calibrated on a quarterly base in order to deliver an implied value of r_{IH} around 1% (e.g. Kydland and Prescott, 1982; Long and Plosser, 1983). Implementing the procedure described above, such a value of the real interest rate implies that time unit of the OLG model would last about 66 periods of the IH model. Therefore, if the time reference of the IH model were a quarter, then a generation would cover about 16 years only. Consequently, if we aim at achieving the conventional figure of 25 years commonly accepted by demographers and geneticists, then the discount factor should be calibrated at highest values by targeting a value of r_{IH} around 0.7%. According to data released by the World Bank for the US economy, such a value of the quarterly real interest rate is close to the estimation of the corresponding annual reference over the last twenty years which amounts to 2.84%.[3] See the time series plotted in Figure 2.

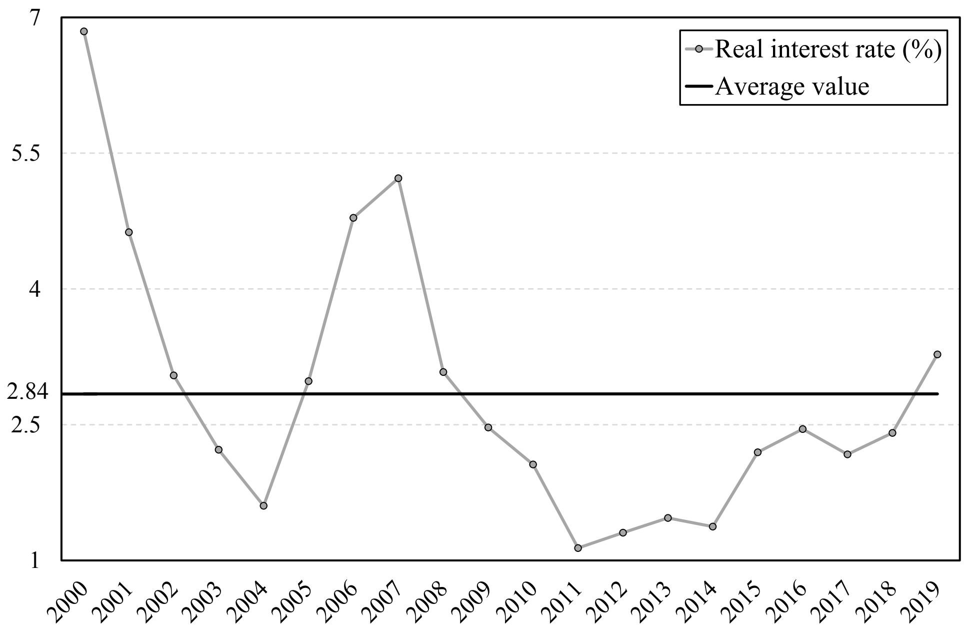

Figure 2: Real interest rates in the US (2000-2019)

6 Concluding remarks

This paper aimed at assessing the relationship between the time reference of infinite and finite horizon dynamic models through the comparison of the consumption plans of the involved opti-

[3]Data on real interest rate can be downloaded from data.worldbank.org.

mizing households. Such a theoretical exploration revealed three remarkable results. First, the length of generation decreases with the value of the real interest rate. Second, the conventional figures exploited to calibrate the discount factor in real business cycles model lead to a theoretical generation that is shorter with respect to the reference of 25 years usually acknowledged by demographers and geneticists. Furthermore, such a reference can be achieved by calibrating the discount rate by targeting the average value of the real interest rates observed in the US over the last two decades.

The analysis carried out in this paper could be developed in different directions. For instance, it could be interesting to see how the results summarized above change when households with different time horizons have also different values of the initial wealth, are endowed with different preferences and/or there is no equality between the discount and the interest rate prevailing on the capital market. Different results could also be achieved by computing in how many periods the present value of the consumption stream from the HI model equals value of consumption of the young household from the OLG model. The implied extensions are left to further developments.

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