

OBSERVER-BASED CONTROL FOR A CLASS OF HYBRID LINEAR AND NONLINEAR SYSTEMS

A. ALESSANDRI¹, F. BEDOUHENE², D. BOUHADJRA^{1,3},
A. ZEMOUCHE³ AND P. BAGNERINI¹

¹ University of Genoa (DIME), Via Opera Pia 15, 16145 Genova, Italy

² Laboratoire de Mathématiques Pures et Appliquées
University Mouloud Mammeri of Tizi-Ouzou
B.P. No. 17 RP, 15000 Tizi-Ouzou, Algeria

³ Centre de Recherche en Automatique de Nancy, University of Lorraine
CNRS UMR 7039, F-54400 Cosnes et Romain, France

ABSTRACT. An approach to output feedback control for hybrid discrete-time systems subject to uncertain mode transitions is proposed. The system dynamics may assume different modes upon the occurrence of a switching that is not directly measurable. Since the current system mode is unknown, a regulation scheme is proposed by combining a Luenberger observer to estimate the continuous state, a mode estimator, and a controller fed with the estimates of both continuous state variables and mode. The closed-loop stability is ensured under suitable conditions given in terms of linear matrix inequalities. Since complexity and conservativeness grow with the increase of the modes, we address the problem of reducing the number of linear matrix inequalities by providing more easily tractable stability conditions. Such conditions are extended to deal with systems having also Lipschitz nonlinearities and affected by disturbances. The effectiveness of the proposed approach is shown by means of simulations.

1. Introduction. Hybrid systems subject to mode transitions have attracted a lot of attention in the last decades owing to their increased modeling capabilities in numerous application fields. However, the control of such systems is still a challenge due to the difficulties to deal with both stability and complexity, thus driving current investigations to develop more efficient solutions. In this paper, we address the control of a class of hybrid systems with switching linear dynamic equations depending on a discrete state, called mode, which is not fully accessible. For such systems, a novel observer-based control scheme is presented that is based a Luenberger observer and a mode predictor to estimate the continuous and discrete states, respectively,

There exist many systems that result from the combination of smooth continuous dynamics and sudden switches (see, among others, [5, 13, 9, 17, 18]). Switching is modelled by means of hybrid automata with a discrete state that usually represents a specific operation mode of the plant. In this context, state estimation aims at reconstructing both the continuous and discrete states by using the measurements

2020 *Mathematics Subject Classification.* Primary: 93C30, 93D05; Secondary: 93C55.

Key words and phrases. Switching system, observer, output feedback, linear matrix inequality.

* Corresponding author: A. Alessandri.

and knowledge of the hybrid model in such a way as to feed the controller. The first results reported in the literature on observer-based control of systems subject to mode transitions require the perfect knowledge of the mode, measured on line [15, 20, 4]. The design of controllers for hybrid systems with constraints on the switching is dealt in [16], where only the switching rule is assumed to be known. In [7] a moving horizon approach is proposed to control such systems by optimizing the attenuation of the external disturbances.

In this paper, we focus on observer-based control for switching discrete-time linear and nonlinear systems by extending the results of [14] since we explicitly account for information on the switching dynamics. Specifically, we address the problem of constructing an observer-based controller by using the Luenberger observer proposed in [2], where sufficient conditions based on observability arguments are presented to ensure the asymptotic stability of the estimation error. Such conditions point out that a delay with respect to the current available information is required to accomplish a correct evaluation of the system mode (see also [1]). Indeed, here we consider the case with no delay permitted in generating the control and propose to perform the estimation of the mode with a mode predictor [3]. Switching-gain observers are analyzed in [19].

To design the proposed output feedback control scheme, a new method is presented that consists in selecting the observer and controller gains by means of a switched quadratic Lyapunov function and a technical result inspired from [12]. In spite of dealing with the unknown system mode as a pure uncertainty likewise in [21], we exploit the information on the switching rule to construct the mode predictor and reduce the number of linear matrix inequalities (LMIs). Contrarily to the approach in [11], where the observer and controller gains are computed by using two dependent sets of LMIs, our LMI conditions can be solved simultaneously. Moreover, the stability of the proposed observer-based scheme is investigated also in the presence of Lipschitz nonlinearities and disturbances by resorting to an H_∞ approach according to [22, 8].

The paper is organized as follows. The proposed observer-based control scheme is presented in Section 2 by referring to a simple example that motivates the approach. Sections 3 and 4 report the main results without and with Lipschitz nonlinearities and disturbances, respectively. In Section 5, numerical results are shown to illustrate the effectiveness of the proposed design methodologies. Conclusions are drawn in Section 6.

Before concluding this section, let us introduce some notations used throughout the paper. Let $(x, y) := [x^\top, y^\top]^\top$, where x and y are column vectors and the symbol $^\top$ means transposition. Given a symmetric matrix S , $S > 0$ (< 0) means that S is positive (negative) definite. For any real column vector z , $|z| := \sqrt{z^\top z}$ is its Euclidean norm. Given a sequence $x := \{x_k\}_{k \in \mathbb{N}_{\geq 0}} \subset \mathbb{R}^s$, $|x|_{l_2^s} := (\sum_{k=0}^{k=\infty} |x_k|^2)^{\frac{1}{2}}$ denotes its l_2 norm and let $l_2^s := \{x = \{x_k\} \subset \mathbb{R}^s, |x|_{l_2^s} < +\infty\}$. Finally, $e_s(i) =$

$(\underbrace{0, \dots, \overset{\text{ith}}{1}, \dots, 0}_{s \text{ components}})^\top \in \mathbb{R}^s$ the unit vector of the canonical basis of \mathbb{R}^s .

2. Observer-based control of switching linear systems. Let us consider a class of switching discrete-time linear systems described by

$$x_{t+1} = A(\lambda_t)x_t + B(\lambda_t)u_t \quad (1a)$$

$$\lambda_{t+1} = F(t, x_{t+1}, u_t, \lambda_t) \quad (1b)$$

$$y_t = C(\lambda_t)x_t \quad (1c)$$

where $t = 0, 1, \dots$ is the time instant, $x_t \in \mathbb{R}^n$ is the continuous state vector and $\lambda_t \in \Lambda = \{1, 2, \dots, q\}$ is the discrete state taking on values only in the finite set Λ ; $y_t \in \mathbb{R}^m$ is the measurement vector; $u_t \in \mathbb{R}^p$ is the input vector; $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$, $\lambda \in \Lambda$, are $n \times n$, $n \times p$, and $m \times n$ real matrices, respectively. The mapping $F : \mathbb{N}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \times \Lambda \rightarrow \Lambda$ represents the dynamics of the discrete state. Such a mapping and the matrices $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ are known but, at each time instant t , both x_t and λ_t are not directly measurable, as the available information is given only by the measures y_t .

We will address the design of output feedback controllers based on the certainty equivalence principle. Such a control scheme needs suitable estimates of both λ_t and x_t . The estimate of x_t is denoted by \hat{x}_t , and is provided by a Luenberger observer, which is adopted to perform estimation by using the measurements y_t , as depicted in Fig. 1. The Luenberger observer is fed by the estimate of λ_t , denoted by $\hat{\lambda}_t$ and given by a mode predictor, which in turn relies on \hat{x}_t . To sum up, the proposed control scheme is composed of three blocks, i.e., Luenberger observer, mode predictor, and controller, which have to be carefully designed in such a way as to ensure closed-loop asymptotic stability. Note that a difficulty arises from the fact that the mode dynamics depends on the full-state vector x_t , which is not at our disposal in general.

As to the Luenberger observer, we refer to the approach presented in [2]. The estimate of the state is thus obtained as follows:

$$\hat{x}_{t+1} = A(\hat{\lambda}_t)\hat{x}_t + B(\hat{\lambda}_t)u_t + L(\hat{\lambda}_t) \left(y_t - C(\hat{\lambda}_t)\hat{x}_t \right) \quad (2)$$

where $\hat{\lambda}_t \in \Lambda$ is an estimate of λ_t and the observer gains $L(\lambda)$, $\lambda \in \Lambda$, are $n \times m$ gain matrices to be chosen. Following [2], a possible strategy to compute $\hat{\lambda}_t$ is that of detecting the discrete state λ_t on the basis of the observations vector $y_{t-\alpha}, y_{t-\alpha+1}, \dots, y_{t+\omega-1}, y_{t+\omega}$ with α and ω nonnegative integers that account for the mode observability issues (see also [1, 10]). Of course, this would cause a delay in generating the control action. Instead, we will exploit the knowledge of the discrete state dynamics by using a mode predictor that estimates the mode one-step ahead based also on the current estimates, i.e.,

$$\hat{\lambda}_t = \hat{F} \left(t, y_t, \hat{x}_t, u_{t-1}, \hat{\lambda}_{t-1} \right) \quad (3)$$

where $\hat{F} : \mathbb{N}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \times \Lambda \rightarrow \Lambda$. Under the state feedback with the certainty equivalence principle, the control action is thus generated as follows:

$$u_t = -K(\hat{\lambda}_t)\hat{x}_t \quad (4)$$

with the $p \times n$ real matrices $K(\lambda)$, $\lambda \in \Lambda$. The mapping \hat{F} may be chosen in such a way as to facilitate the design of the observer-based control, as will be clarified with the next example.

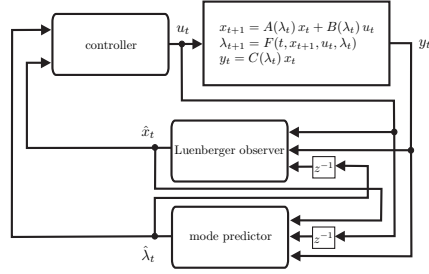


FIGURE 1. Output feedback observer-based control scheme.

Example 1. Consider the following system and switching rule:

$$\begin{aligned}
 x_{t+1} &= A(\lambda_t) x_t + B(\lambda_t) u_t \\
 y_t &= C(\lambda_t) x_t \\
 \lambda_t &= \begin{cases} 1 & \text{if } x_1(t) + \sqrt{3} x_2(t) \geq 0, x_1(t) \geq 0 \\ 2 & \text{if } x_1(t) + \sqrt{3} x_2(t) < 0, x_1(t) - \sqrt{3} x_2(t) \geq 0 \\ 3 & \text{if } x_1(t) - \sqrt{3} x_2(t) < 0, x_1(t) < 0 \end{cases}
 \end{aligned}$$

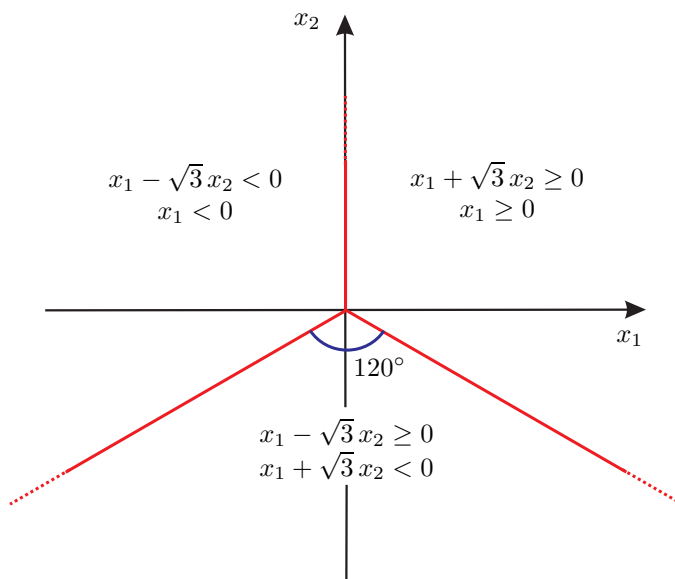


FIGURE 2. State space piecewise regions.

where $x_t \in \mathbb{R}^2$ is the continuous state vector, $u_t \in \mathbb{R}$ is the input vector, $\lambda_t \in \Lambda = \{1, 2, 3\}$ is the discrete state, $y_t \in \mathbb{R}$ is the output; A_i and B_i are given matrices and $C_i = (1 \ 0)$ for $i = 1, 2, 3$. Fig. 2 illustrates the three switching regions. Clearly, the only knowledge of the sign of the current output y_t is not sufficient to discriminate between the discrete state either 1 or 2 if $y_t \geq 0$ and either 2 or 3 if $y_t < 0$. Moreover, an estimate of the second state variable may help to detect the correct mode, as will be clarified in the following.

In the next section, we will consider the problem of determining observer and controller gains.

3. Output feedback control design for switching linear systems. The stability of the closed-loop system under the feedback obtained by combining (2), (3), and (4) is studied by referring to the augmented state vector $z_t := (x_t, e_t)$, where $e_t := x_t - \hat{x}_t$ is the estimation error. Toward this end, let $\Delta A(\lambda_1, \lambda_2) := A(\lambda_1) - A(\lambda_2)$, $\Delta B(\lambda_1, \lambda_2) := B(\lambda_1) - B(\lambda_2)$, and $\Delta C(\lambda_1, \lambda_2) := C(\lambda_1) - C(\lambda_2)$. The dynamics of the augmented state is given by the discrete-time equation:

$$z_{t+1} = \Pi(\lambda_t, \hat{\lambda}_t) z_t \quad (5)$$

where

$$\Pi(\lambda_t, \hat{\lambda}_t) := \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \quad (6)$$

with

$$\begin{aligned} W_{11} &= A(\lambda_t) - B(\lambda_t) K(\hat{\lambda}_t), & W_{12} &= B(\lambda_t) K(\hat{\lambda}_t), \\ W_{21} &= \Delta A(\lambda_t, \hat{\lambda}_t) - \Delta B(\lambda_t, \hat{\lambda}_t) K(\hat{\lambda}_t) - L(\hat{\lambda}_t) \Delta C(\lambda_t, \hat{\lambda}_t), \\ W_{22} &= A(\hat{\lambda}_t) + \Delta B(\lambda_t, \hat{\lambda}_t) K(\hat{\lambda}_t) - L(\hat{\lambda}_t) C(\hat{\lambda}_t). \end{aligned}$$

If we consider the Lyapunov function $V_t(z_t, \hat{\lambda}_t) = z_t^\top P(\hat{\lambda}_t) z_t$, we have

$$V_{t+1} = z_t^\top \Pi(\lambda_t, \hat{\lambda}_t)^\top P(\hat{\lambda}_{t+1}) \Pi(\lambda_t, \hat{\lambda}_t) z_t,$$

and hence stability will be inferred by proving that V_t is strictly decreasing under suitable conditions. Toward this end, the following result is needed (see [12, Lemma 1, p. 1786] for the proof).

Lemma 1. Let X, Y, Z be three matrices of appropriate dimensions with $X = X^\top > 0$ and $Z = Z^\top > 0$. Then, the LMI

$$\begin{bmatrix} -X & Y^\top \\ Y & -Z^{-1} \end{bmatrix} < 0 \quad (7)$$

holds if there exists $\alpha > 0$ so that the following matrix inequality is fulfilled:

$$\begin{bmatrix} -X & \alpha Y^\top & 0 \\ \alpha Y & -2\alpha I & Z \\ 0 & Z & -Z \end{bmatrix} < 0. \quad (8)$$

For the sake of brevity, we denote any matrix M depending on λ_t or $\hat{\lambda}_t$ by M_i and M_j , respectively; similarly, N_{ij} is a matrix that depends on both λ_t and $\hat{\lambda}_t$. For example, $P(\hat{\lambda}_t)$ and $A(\lambda_t) - B(\lambda_t)K(\hat{\lambda}_t)$ will be referred to as P_j and $A_i - B_i K_j$, respectively. Similarly, we will adopt the index k to refer to $\hat{\lambda}_{t+1}$, i.e., for example, by writing P_k for $P(\hat{\lambda}_{t+1})$.

Theorem 1. Assume that there exist matrices $S_i = S_i^\top > 0$, scalars $\alpha_i > 0$ and gain matrices \tilde{K}_i, \tilde{L}_i , $i = 1, \dots, q$, solutions of the LMIs

$$\begin{bmatrix} S_k & \mathbb{T}_{ij} & 0 \\ (*) & 2\alpha_j I & S_j \\ (*) & (*) & S_j \end{bmatrix} > 0, \quad \text{for all } i, j, k \in \Lambda \quad (9)$$

with

$$\begin{aligned} \mathbb{T}_{ij} &= \begin{bmatrix} \alpha_j A_i - B_i \tilde{K}_j & B_i \tilde{K}_j \\ \mathbb{T}_{ij}^{21} & \mathbb{T}_{ij}^{22} \end{bmatrix}, \\ \mathbb{T}_{ij}^{21} &= \alpha_j \Delta A_{ij} - \Delta B_{ij} \tilde{K}_j - \tilde{L}_j \Delta C_{ij}, \\ \mathbb{T}_{ij}^{22} &= \alpha_j A_j + \Delta B_{ij} \tilde{K}_j - \tilde{L}_j C_j. \end{aligned}$$

Then, (5) is asymptotically stable with the gains $K_j = \tilde{K}_j/\alpha_j$ and $L_j = \tilde{L}_j/\alpha_j$, $j \in \Lambda$.

Proof. The stability of (5) is proved by verifying that the Lyapunov function is strictly decreasing out of the origin. Since

$$V_{t+1} - V_t = z_t^\top (\Pi_{ij}^\top P_k \Pi_{ij} - P_j) z_t < 0, \quad \forall i, j, k \in \Lambda,$$

for all $z_t \neq 0$, V_t is decreasing if the following inequalities are satisfied

$$P_j - \Pi_{ij}^\top P_k \Pi_{ij} > 0, \quad \forall i, j, k \in \Lambda \quad (10)$$

where

$$\Pi_{ij} = \begin{bmatrix} A_i - B_i K_j & B_i K_j \\ \Pi_{ij}^{21} & \Pi_{ij}^{22} \end{bmatrix},$$

with

$$\Pi_{ij}^{21} = \Delta A_{ij} - \Delta B_{ij} K_j - L_j \Delta C_{ij}, \quad \Pi_{ij}^{22} = A_j + \Delta B_{ij} K_j - L_j C_j$$

Using the notation $S_i = P_i^{-1}$ for short, the Schur lemma allows establishing the equivalence between (10) and the following:

$$\begin{bmatrix} S_k & \Pi_{ij} \\ (*) & S_j^{-1} \end{bmatrix} > 0, \quad \forall i, j, k \in \Lambda. \quad (11)$$

Notice that (11) is not an LMI because of the presence of both S_k and S_k^{-1} for all $k \in \Lambda$. However, using Lemma 1 (i.e., [12, Lemma 1, p. 1786]), we obtain that (11) holds if there exist positive real scalars α_j such that

$$\begin{bmatrix} S_k & \alpha_j \Pi_{ij} & 0 \\ (*) & 2\alpha_j I & S_j \\ (*) & (*) & S_j \end{bmatrix} > 0, \quad \forall i, j, k \in \Lambda.$$

Using the change of variables $\tilde{K}_j = \alpha_j K_j$ and $\tilde{L}_j = \alpha_j L_j$, we get the LMIs (9), thus concluding the proof. \square

Remark 1. The Schur complement gives rise to the nonlinear inequality (11) from (10) owing to Lemma 1. If we exploited another equivalent Schur form by transforming (10) to

$$\begin{bmatrix} P_j & \Pi_{ij}^\top P_k \\ (*) & P_k \end{bmatrix},$$

we would obtain the product $\Pi_{ij}^\top P_k$, which is more complicated to deal for the following reasons.

- First, to handle the coupling $\Pi_{ij}^\top P_k$, it is convenient to use a diagonal matrix $P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix}$ to simplify and to cancel the off-diagonal bilinear terms. However, even if a diagonal matrix P_k would be used, this would not solve the problem because the matrix Π_{ij} contains decision variables depending on the index j , while P_k depends on the index k . Indeed, this would lead to coupled terms such as $P_k^2 L_j$, where a change of variable is not possible because both P_k^2 and $L_{k,j} := P_k^2 L_j$ are free solutions provided by the LMIs.
- Even in case we replace (artificially only) the term $\Pi_{ij}^\top P_j$ to get the same index j , it is complicated for such a term due to the presence of the observer-based controller gains K_j and L_j in Π_{ij} in both sides. Pre- and post-multiplying the term by positive definite matrices, by using any congruence transformation, will not solve the problem. Such a difficulty is due to the fact that in this paper we deal with a general case where all the system matrices switch ($A(\lambda_t)$, $B(\lambda_t)$, and $C(\lambda_t)$), contrarily to the literature where only the matrix $A(\lambda_t)$ is considered as a switched matrix [11]. In this case, the terms Π_{ij}^{21} and Π_{ij}^{22} are simplified because $\Delta C_{ij} = 0$ and $\Delta B_{ij} = 0$.
- Concerning the product $\Pi_{ij}^\top P_k$, the well-known congruence transformations based on slack variables proposed in [6] cannot be used straightforwardly, unless under strong conditions that lead to very conservative LMIs.

Hence, to avoid such complications, we rely on Lemma 1, which provides a simple solution for the general problem with all the system matrices that may switch.

Let us explain in more detail the motivation of using exactly the Schur transformation (11) instead of

$$\begin{bmatrix} P_j & \Pi_{ij}^\top \\ (*) & P_k^{-1} \end{bmatrix} > 0 \quad (12)$$

which is easier and more direct from (10) without introducing the matrix S_j . Indeed, the use of (11) is more convenient. The reason is simple but hidden in the LMIs. The application of Lemma 1 on the inequality (12) leads to

$$\begin{bmatrix} P_j & \alpha_j \Pi_{ij} & 0 \\ (*) & 2\alpha_j I & P_k \\ (*) & (*) & P_k \end{bmatrix} > 0, \forall i, j, k \in \Lambda \quad (13)$$

and, for the Schur lemma, it follows that

$$P_k < 2\alpha_j I, \forall j, k \in \Lambda \quad (14)$$

because (13) is equivalent to

$$\begin{bmatrix} P_j & \alpha_j \Pi_{ij} \\ (*) & 2\alpha_j I - P_k \end{bmatrix} > 0, \forall i, j, k \in \Lambda.$$

However, inequality (9) in the paper is equivalent to

$$\begin{bmatrix} S_k & \mathbb{T}_{ij} \\ (*) & 2\alpha_j I - S_j \end{bmatrix} > 0, \forall i, j, k \in \Lambda,$$

which leads to

$$S_j < 2\alpha_j I, \forall j \in \Lambda \quad (15)$$

where only the index j is used. Indeed, for a fixed index j , in (14) all the matrices P_k should satisfy the inequality, while in inequality (9) of Theorem 1 only S_j should satisfy (15).

Remark 2. The observer-based controller gains are synthesized together, contrarily to what was proposed for a similar class of systems in [11], where, in order to avoid bilinear matrix inequalities, the authors present an algorithm in two steps. In the first step, a set of LMIs accounts for the gains of the observer, which is designed by following an input-to-state stability formulation. In the second step, the controller gains are deduced by solving other LMI conditions. As compared with [11], our methodology is not affected by the difficulties involved in taking into account additional strong equality constraints as required in [20], which are necessary to avoid bilinear matrix inequalities to synthesize simultaneously the gains.

Remark 3. A reduction of the number of constraints in (9) is crucial to ensure the LMI feasibility. In the worst case, we have q^3 LMIs, but such a number can be reduced in general. If the instantaneous value of the switching mode is known in real time, we have to satisfy (9) with $i = j$, and hence to deal with q^2 LMIs. Moreover, we may account for the knowledge of the switching rule F to set the mapping \hat{F} in such a way to reduce the number of constraints on the overall. For the reader's convenience, consider Example 1 and the mode estimator

$$\hat{\lambda}_t = \begin{cases} 1 & \text{if } y_t + \sqrt{3} \hat{x}_2(t) \geq 0, y_t \geq 0 \\ 2 & \text{if } y_t + \sqrt{3} \hat{x}_2(t) < 0, y_t - \sqrt{3} \hat{x}_2(t) \geq 0 \\ 3 & \text{if } y_t - \sqrt{3} \hat{x}_2(t) < 0, y_t < 0 \end{cases} \quad (16)$$

where $\hat{x}_2(t)$ denotes the second component of \hat{x}_t . Since we instantaneously measure the first state variable, at each time instant t we know that λ_t belongs to either

$\{1, 2\}$ or $\{2, 3\}$. Thus, in the LMIs (9) we have $\hat{\lambda}_t \in \{1, 2\}$ in case $\lambda_t = 1$, i.e., $j = 1, 2$ if $i = 1$ by excluding $j = 3$. Moreover, $\hat{\lambda}_t \in \{1, 2, 3\}$ in case $\lambda_t = 2$ (i.e., $j = 1, 2, 3$ if $i = 2$) and $\hat{\lambda}_t \in \{2, 3\}$ in case $\lambda_t = 3$ (i.e., $j = 2, 3$ if $i = 3$). No reduction can be obtained for the index k , as we have to consider all the cases with $\hat{\lambda}_{t+1} \in \{1, 2, 3\}$. Summing up, we have to solve a design problem with 21 LMIs instead of $3^3 = 27$, since $q = 3$. Fig. 3 provides a pictorial description of the aforesaid.

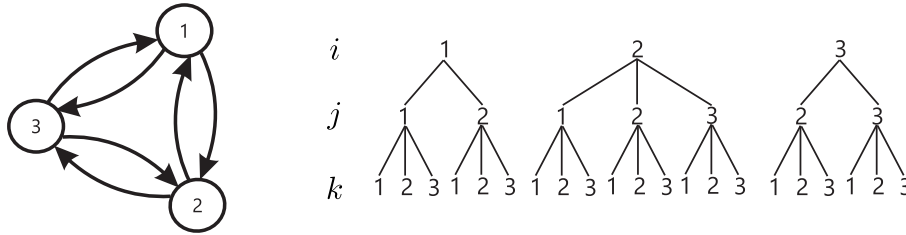


FIGURE 3. Finite state machine of the discrete state dynamics and tree of mode combinations over successive time instants.

4. Output feedback control design for noisy switching Lipschitz nonlinear systems. Consider the system (1) under the effect of additive noises and Lipschitzian nonlinearities:

$$x_{t+1} = A(\lambda_t)x_t + B(\lambda_t)u_t + \phi(\lambda_t, x_t) + D(\lambda_t)\omega_t \quad (17a)$$

$$\lambda_{t+1} = F(t, x_{t+1}, u_t, \lambda_t) \quad (17b)$$

$$y_t = C(\lambda_t)x_t + \psi(\lambda_t, x_t) + E(\lambda_t)\omega_t \quad (17c)$$

where $x_t \in \mathbb{R}^n$ is the continuous state vector, $y_t \in \mathbb{R}^m$ is the measurement vector, $u_t \in \mathbb{R}^p$ is the input vector, $\omega_t \in \mathbb{R}^s$ is the vector of the noises, and $\lambda_t \in \Lambda = \{1, 2, \dots, q\}$ is the discrete state. $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$, and $E(\lambda)$, $\lambda \in \Lambda$, are $n \times n$, $n \times p$, $m \times n$, $n \times s$, and $m \times s$ real matrices, respectively. The functions ϕ and ψ in (17) are globally Lipschitz with respect to the second variable with Lipschitz constants depending on the switching mode λ_t , namely for each $\lambda_t \in \Lambda$ there exist $\gamma_\phi(\lambda_t)$ and $\gamma_\psi(\lambda_t)$ such that:

$$|\phi(\lambda_t, x) - \phi(\lambda_t, z)| \leq \gamma_\phi(\lambda_t)|x - z|, \forall x, z \in \mathbb{R}^n,$$

$$|\psi(\lambda_t, x) - \psi(\lambda_t, z)| \leq \gamma_\psi(\lambda_t)|x - z|, \forall x, z \in \mathbb{R}^n.$$

Moreover, without loss of generality let us assume $\phi(\lambda, 0) = 0$ and $\psi(\lambda, 0) = 0$ for all $\lambda \in \Lambda$.

Based on the certainty equivalence principle, we rely the control scheme presented in Section 2, i.e., we use the observer

$$\hat{x}_{t+1} = A(\hat{\lambda}_t)\hat{x}_t + B(\hat{\lambda}_t)u_t + \phi(\hat{\lambda}_t, \hat{x}_t) + L(\hat{\lambda}_t)\left(y_t - C(\hat{\lambda}_t)\hat{x}_t - \psi(\hat{\lambda}_t, \hat{x}_t)\right) \quad (18)$$

together with (3) and (4) since we suppose to have at disposal a mode prediction.

From (3), (4), (17), and (18) it follows that

$$z_{t+1} = \underbrace{\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}}_{\tilde{A}_{\lambda_t, \hat{\lambda}_t}} z_t + \begin{bmatrix} \phi(\lambda_t, x_t) \\ \Delta\phi(\hat{\lambda}_t, \lambda_t) - L(\hat{\lambda}_t)\Delta\psi(\hat{\lambda}_t, \lambda_t) \end{bmatrix} + \underbrace{\begin{bmatrix} D(\lambda_t) \\ D(\lambda_t) - L(\hat{\lambda}_t)E(\hat{\lambda}_t) \end{bmatrix}}_{G_{\lambda_t, \hat{\lambda}_t}} \omega_t \quad (19)$$

where W_{11} , W_{12} , W_{21} , and W_{22} are defined as in (6); moreover,

$$\Delta\phi(\hat{\lambda}_t, \lambda_t) := \phi(\lambda_t, x_t) - \phi(\hat{\lambda}_t, \hat{x}_t) = \phi(\lambda_t, x_t) - \phi(\lambda_t, 0) - \phi(\hat{\lambda}_t, \hat{x}_t) + \phi(\hat{\lambda}_t, 0) \quad (20a)$$

$$\Delta\psi(\hat{\lambda}_t, \lambda_t) := \psi(\lambda_t, x_t) - \psi(\hat{\lambda}_t, \hat{x}_t) = \psi(\lambda_t, x_t) - \psi(\lambda_t, 0) - \psi(\hat{\lambda}_t, \hat{x}_t) + \psi(\hat{\lambda}_t, 0) \quad (20b)$$

where, for the sake of brevity, we omit the dependence on x_t and \hat{x}_t .

We now need a technical result that allows dealing with the above Lipschitz functions in a more convenient way (see [22, Lemma 6, p. 587] for details).

Lemma 2. Considering a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the two following facts are equivalent:

- φ is γ_φ -Lipschitz with respect to its argument, i.e.,

$$|\varphi(x) - \varphi(z)| \leq \gamma_\varphi |x - z|, \quad \forall x, z \in \mathbb{R}^n,$$

- for all $i, j = 1, \dots, n$ there exist functions $\varphi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and constants $\underline{\gamma}_{\varphi_{ij}}$ and $\bar{\gamma}_{\varphi_{ij}}$ such that, $\forall x, z \in \mathbb{R}^n$,

$$\varphi(x) - \varphi(z) = \sum_{i=1}^n \sum_{j=1}^n \varphi_{ij}(x_1, \dots, x_{j-1}, z_j, \dots, z_n) H_{ij}^n(x - z) \quad (21)$$

and

$$\underline{\gamma}_{\varphi_{ij}} \leq \varphi_{ij}(x_1, \dots, x_{j-1}, z_j, \dots, z_n) \leq \bar{\gamma}_{\varphi_{ij}},$$

where $H_{ij}^n := e_n(i) e_n^\top(j)$.

According to Lemma 2, the Lipschitz property held by $\phi(\lambda_t, \cdot)$ and $\psi(\lambda_t, \cdot)$ leads to the existence of bounded functions $\phi_{ij}(\lambda_t) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi_{ij}(\lambda_t) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ and constants $\underline{\gamma}_{\phi_{ij}}(\lambda_t)$, $\bar{\gamma}_{\phi_{ij}}(\lambda_t)$, for $i, j = 1, \dots, n$, $\bar{\gamma}_{\psi_{ij}}(\lambda_t)$ and $\underline{\gamma}_{\psi_{ij}}(\lambda_t)$, for $i = 1, \dots, m$ and $j = 1, \dots, n$, such that

$$\begin{aligned} \underline{\gamma}_{\phi_{ij}}(\lambda_t) &\leq \phi_{ij}(\lambda_t) \leq \bar{\gamma}_{\phi_{ij}}(\lambda_t), & \underline{\gamma}_{\psi_{ij}}(\lambda_t) &\leq \psi_{ij}(\lambda_t) \leq \bar{\gamma}_{\psi_{ij}}(\lambda_t), \\ \Delta\phi(\hat{\lambda}_t, \lambda_t) &= \left[\sum_{i=1}^n \sum_{j=1}^n \phi_{ij}(\lambda_t) H_{ij}^n \right] x_t - \left[\sum_{i=1}^n \sum_{j=1}^n \phi_{ij}(\hat{\lambda}_t) H_{ij}^n \right] \hat{x}_t, \\ \Delta\psi(\hat{\lambda}_t, \lambda_t) &= \left[\sum_{i=1}^m \sum_{j=1}^n \psi_{ij}(\lambda_t) H_{ij}^{m,n} \right] x_t - \left[\sum_{i=1}^m \sum_{j=1}^n \psi_{ij}(\hat{\lambda}_t) H_{ij}^{m,n} \right] \hat{x}_t, \\ \phi(\lambda_t, x_t) &= \left[\sum_{i=1}^n \sum_{j=1}^n \phi_{ij}(\lambda_t) H_{ij}^n \right] x_t, \end{aligned}$$

where $H_{ij}^{m,n} = e_m(i)e_n^\top(j)$. For the reader's convenience, let us introduce the following notations:

$$\begin{aligned}\Theta_\lambda &:= \sum_{i=1}^n \sum_{j=1}^n \phi_{ij}(\lambda) H_{ij}^n, & \Xi_\lambda &:= \sum_{i=1}^m \sum_{j=1}^n \psi_{ij}(\lambda) H_{ij}^{m,n}, \\ \mathcal{A}(\lambda, \Theta_\lambda) &:= A(\lambda) + \Theta_\lambda, & \mathcal{C}(\lambda, \Xi_\lambda) &:= C(\lambda) + \Xi_\lambda.\end{aligned}$$

Hence, system (19) can be rewritten under the form

$$z_{t+1} = \Pi_{\lambda_t, \hat{\lambda}_t} \begin{pmatrix} \Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t} \end{pmatrix} z_t + G_{\lambda_t, \hat{\lambda}_t} \omega_t \quad (22)$$

where

$$\Pi_{\lambda_t, \hat{\lambda}_t} \begin{pmatrix} \Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t} \end{pmatrix} = \begin{bmatrix} \Pi_{\lambda_t, \hat{\lambda}_t}^{11}(\Theta_{\lambda_t}) & \Pi_{\lambda_t, \hat{\lambda}_t}^{12}(\Theta_{\lambda_t}) \\ \Pi_{\lambda_t, \hat{\lambda}_t}^{21}(\Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) & \Pi_{\lambda_t, \hat{\lambda}_t}^{22}(\Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) \end{bmatrix},$$

with

$$\begin{aligned}\Pi_{\lambda_t, \hat{\lambda}_t}^{11}(\Theta_{\lambda_t}) &:= \mathcal{A}(\lambda_t, \Theta_{\lambda_t}) - B(\lambda_t) K(\hat{\lambda}_t), \\ \Pi_{\lambda_t, \hat{\lambda}_t}^{12}(\Theta_{\lambda_t}) &:= B(\lambda_t) K(\hat{\lambda}_t), \\ \Pi_{\lambda_t, \hat{\lambda}_t}^{21}(\Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) &:= \Delta \mathcal{A}(\lambda_t, \hat{\lambda}_t, \Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}) - L(\hat{\lambda}_t) \Delta \mathcal{C}(\lambda_t, \hat{\lambda}_t, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) \\ &\quad - \Delta B(\lambda_t, \hat{\lambda}_t) K(\hat{\lambda}_t), \\ \Pi_{\lambda_t, \hat{\lambda}_t}^{22}(\Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) &:= \mathcal{A}(\hat{\lambda}_t, \Theta_{\hat{\lambda}_t}) + \Delta B(\lambda_t, \hat{\lambda}_t) K(\hat{\lambda}_t) - L(\hat{\lambda}_t) \mathcal{C}(\hat{\lambda}_t, \Xi_{\hat{\lambda}_t}),\end{aligned}$$

and

$$\begin{aligned}\Delta \mathcal{A}(\lambda_t, \hat{\lambda}_t, \Theta_{\lambda_t}, \Theta_{\hat{\lambda}_t}) &:= \mathcal{A}(\lambda_t, \Theta_{\lambda_t}) - \mathcal{A}(\hat{\lambda}_t, \Theta_{\hat{\lambda}_t}), \\ \Delta \mathcal{C}(\lambda_t, \hat{\lambda}_t, \Xi_{\lambda_t}, \Xi_{\hat{\lambda}_t}) &:= \mathcal{C}(\lambda_t, \Xi_{\lambda_t}) - \mathcal{C}(\hat{\lambda}_t, \Xi_{\hat{\lambda}_t}).\end{aligned}$$

Notice that in view of Lemma 2, for each $\lambda \in \Lambda$, the parameter Θ_λ belongs to bounded convex set $\mathcal{H}_n(\lambda)$ for which the set of vertices is defined as follows:

$$\mathcal{V}_{\mathcal{H}_n}(\lambda) = \left\{ \Phi \in \mathbb{R}^{n \times n}, \Phi_{ij} \in \{\underline{\gamma}_{\phi_{ij}}(\lambda), \bar{\gamma}_{\phi_{ij}}(\lambda)\} \right\}.$$

Similarly, for each $\lambda \in \Lambda$, the parameter Ξ_λ belongs to a bounded convex set $\mathcal{H}_m(\lambda)$ for which the set of vertices is given by:

$$\mathcal{V}_{\mathcal{H}_m}(\lambda) = \left\{ \Psi \in \mathbb{R}^{m \times n}, \Psi_{ij} \in \{\underline{\gamma}_{\psi_{ij}}(\lambda), \bar{\gamma}_{\psi_{ij}}(\lambda)\} \right\}.$$

Our objective consists in finding the gain matrices $K(\lambda)$ and $L(\lambda)$ for all $\lambda \in \Lambda$ such that the closed-loop system (22) with $\omega_t = 0$ is asymptotically stable, and the effect of ω_t on the performance signal

$$Z_t = W z_t \quad (23)$$

is attenuated in the \mathcal{H}_∞ sense, where $W \in \mathbb{R}^{r \times n}$ is a given weight matrix. More precisely, it is required that

$$|Z|_{l_2} \leq \mu |\omega|_{l_2} \quad (24)$$

where $\mu > 0$ is the disturbance attenuation level to be minimized. In the following theorem, we provide new LMI conditions ensuring the robust \mathcal{H}_∞ asymptotic stabilization of the closed loop system (22) in the sense of (24). Likewise in Section

3, the indexes i , j , and k in any matrix correspond to dependence on λ_t , $\hat{\lambda}_t$, and $\hat{\lambda}_{t+1}$, respectively.

Theorem 2. System (22) is \mathcal{H}_∞ asymptotically stable with attenuation level μ if there exist some scalars $\gamma_i > 0$, gain matrices \hat{K}_i and \hat{L}_i , and symmetric matrices S_i for $i = 1, \dots, q$ as a result of the the problem's solution:

$$\begin{aligned} \min \mu \quad \text{s.t.} \quad & \text{LMII}_{i,j,k}(\Theta_i, \Theta_j, \Xi_i, \Xi_j, \gamma_j, \mu) < 0, \forall (\Theta_i, \Theta_j, \Xi_i, \Xi_j) \in \mathcal{V}_{\mathcal{H}n}(i) \\ & \times \mathcal{V}_{\mathcal{H}n}(j) \times \mathcal{V}_{\mathcal{H}m}(i) \times \mathcal{V}_{\mathcal{H}m}(j), \forall i, j, k \in \Lambda \end{aligned} \quad (25)$$

where

$$\text{LMII}_{i,j,k}(\Theta_i, \Theta_j, \Xi_i, \Xi_j, \gamma_j, \mu) := \begin{bmatrix} -S_k & 0 & \tilde{\Pi}_{i,j} & \tilde{G}_{i,j} & 0 & 0 \\ (\star) & -2\gamma_j I & 0 & S_j & 0 & \gamma_j W^\top \\ (\star) & (\star) & -2\gamma_j I & 0 & I & 0 \\ (\star) & (\star) & (\star) & -S_j & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\mu^2 I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I \end{bmatrix},$$

with

$$\tilde{\Pi}_{i,j} := \gamma_j \Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) \text{ and } \tilde{G}_{i,j} = \begin{bmatrix} \gamma_j D_i \\ \gamma_j D_i - \hat{L}_j E_j \end{bmatrix}.$$

The controller and observer gains are given by $K_j = \hat{K}_j/\gamma_j$ and $L_j = \hat{L}_j/\gamma_j$ respectively with $j = 1 \dots q$.

Proof. To ensure (24), we search for a Lyapunov function

$$V_j(t) := V_j(z_t) = z_t^\top P_j z_t,$$

with $P_j = P_j^\top > 0$ such that

$$|Z_t|^2 - \mu^2 |\omega_t|^2 + \Delta V_t < 0 \quad (26)$$

for all $j \in \Lambda$ and t , where $\Delta V_t := V_j(t+1) - V_j(t)$ (see [8] for details). Using (22), ΔV_t becomes

$$\begin{aligned} \Delta V_t &= \left(\Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) z_t + G_{i,j} \omega_t \right)^\top P_k \left(\Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) z_t + G_{i,j} \omega_t \right) \\ &\quad - z_t^\top P_j z_t, \end{aligned}$$

and thus

$$|Z_t|^2 - \mu^2 |\omega_t|^2 + \Delta V_t = \begin{bmatrix} z_t \\ \omega_t \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Sigma_{i,j,k}^{11}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) & \Sigma_{i,j,k}^{12}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) \\ (\star) & \Sigma_{i,j,k}^{22} \end{bmatrix}}_{\Upsilon_{i,j,k}(\Theta_i, \Theta_j, \Xi_i, \Xi_j)} \begin{bmatrix} z_t \\ \omega_t \end{bmatrix} \quad (27)$$

where

$$\begin{aligned}\Upsilon_{i,j}^{11}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) &:= \Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j)^\top P_k \Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) - P_j + W^\top W, \\ \Upsilon_{i,j,k}^{12}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) &:= \Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j)^\top P_k G_{i,j}, \\ \Upsilon_{i,j,k}^{22} &:= G_{i,j}^\top P_k G_{i,j} - \mu^2 I.\end{aligned}$$

Therefore, condition (26) holds if and only if $\Upsilon_{i,j,k}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) < 0$, for all $i, j, k \in \Lambda$, $\Theta_i \in \mathcal{H}_n(i)$, $\Theta_j \in \mathcal{H}_n(j)$, $\Xi(i) \in \mathcal{H}_m(i)$, and $\Xi(j) \in \mathcal{H}_m(j)$, which is equivalent, by the Schur Lemma, to

$$\begin{bmatrix} -S_k & \Pi_{i,j}(\Theta_i, \Theta_j, \Xi_i, \Xi_j) & G_{i,j} \\ (\star) & -S_j^{-1} + W^\top W & 0 \\ (\star) & (\star) & -\mu^2 I \end{bmatrix} < 0 \quad (28)$$

where $S_j^{-1} = P_j$. Using the Schur lemma and Lemma 1, we linearize (28) and deduce that (28) holds if there exist positive real scalars γ_j such that

$$\text{LMI}_{i,j,k}(\Theta_i, \Theta_j, \Xi_i, \Xi_j, \gamma_j, \mu) < 0,$$

which ends the proof. \square

5. Numerical examples. Three case studies are presented. The first and second ones aim at a fair evaluation of the effectiveness of the proposed methodology whenever one may take advantage of the knowledge of the switching rule. Specifically, the second one is based on Example 1, where the knowledge of the switching mode allows reducing the numbers of LMIs to solve. The third case deals with the \mathcal{H}_∞ design in the presence of Lipschitz nonlinearities.

5.1. Case study 1. Let us consider (1) with $\Lambda = \{1, 2, 3\}$,

$$\begin{aligned}A_1 &= \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, & A_2 &= \begin{bmatrix} -0.25 & 1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.1 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0.3 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ B_1 &= [1 \ 1 \ 0]^\top, & B_2 &= [0 \ 1 \ 1]^\top, & B_3 &= [1 \ 1 \ 1]^\top, \\ C_1 &= [1 \ 0 \ 2], & C_2 &= [1.5 \ 0 \ 2], & C_3 &= [2 \ 0 \ 2].\end{aligned}$$

and the three regions with each mode

$$R_1 = \{x_k \in \mathbb{R}^3, x_k^1 < 1\}, \quad R_2 = \{x_k \in \mathbb{R}^3, 1 \leq x_k^1 < 5\}, \quad R_3 = \{x_k \in \mathbb{R}^3, x_k^1 \geq 5\}.$$

that is

$$F(x_t) = \begin{cases} 1, & \text{if } x_t \in R_1 \\ 2, & \text{if } x_t \in R_2 \\ 3, & \text{if } x_t \in R_3. \end{cases}$$

We solved the LMIs (9) and obtained

$$\begin{aligned} K_1 &= [0.0150 \quad 0.0427 \quad 0.2936], & L_1 &= [0.0249 \quad -0.0443 \quad 0.2184]^\top, \\ K_2 &= [0.1189 \quad 0.0048 \quad 0.1865], & L_2 &= [-0.0382 \quad -0.0063 \quad 0.2301]^\top, \\ K_3 &= [0.0956 \quad 0.0518 \quad 0.2241], & L_3 &= [0.0840 \quad -0.0257 \quad 0.1972]^\top. \end{aligned}$$

As to mode estimate, we used the mode predictor $\hat{\lambda}_t = F(\hat{x}_t)$. Based on such a control scheme, the result of a simulation run with initial state $x_0 = [-5 \quad 8 \quad 2.5]^\top \in R_1$ and initial estimated state $\hat{x}_0 = [6.5 \quad 4 \quad 10]^\top \in R_2$ are shown in Fig. 4.

5.2. Case study 2. Consider again Example 1 with the following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, & B_3 &= \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \\ C_1 &= [1 \quad 0], & C_2 &= [1 \quad 0], & C_3 &= [1 \quad 0]. \end{aligned}$$

As pointed out in Remark 3, the knowledge of y_t allows only discriminating that λ_t belongs to either $\{1, 2\}$ or $\{2, 3\}$. Thus, in the LMIs (9), we have $\hat{\lambda}_t \in \{1, 2\}$ in case $\lambda_t = 1$, i.e., $j = 1, 2$ if $i = 1$ by excluding $j = 3$. Moreover, $\hat{\lambda}_t \in \{1, 2, 3\}$ in case $\lambda_t = 2$ (i.e., $j = 1, 2, 3$ if $i = 2$) and $\hat{\lambda}_t \in \{2, 3\}$ in case $\lambda_t = 3$ (i.e., $j = 2, 3$ if $i = 3$). By solving the corresponding set of LMIs (9), we obtained

$$\begin{aligned} K_1 &= [0.5212 \quad 0.04985], & K_2 &= [0.4674 \quad -0.2576], & K_3 &= [0.1023 \quad -0.4376], \\ L_1 &= [0.7743 \quad -0.0218], & L_2 &= [0.5965 \quad -0.1428], & L_3 &= [0.1384 \quad 0.0139]. \end{aligned}$$

It is worth noting that the set of LMIs (9) with all the possible index combinations is infeasible. We chose (16) as a mode predictor. The behaviors of the state and estimated state variables with initial conditions $x_0 = [-6 \quad 10]^\top$ and $\hat{x}_0 = [10 \quad -20]^\top$ are given in Fig. 5.

5.3. Case study 3. In order to validate the approach proposed in Theorem 2, we test the feasibility of the LMI (25) on example taken from [21]. The system is described by the following parameters:

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.036 & 0.126 \\ -0.038 & 0.094 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & C_1 &= \begin{bmatrix} 29 \\ 10 \end{bmatrix}^\top, & D_1 &= \begin{bmatrix} 0.07 \\ 0.018 \end{bmatrix}, & E_1 &= 0.03, \\ A_2 &= \begin{bmatrix} 0.0106 & 0.28 \\ -0.035 & 0.24 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ 2 \end{bmatrix}, & C_2 &= \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}^\top, & D_2 &= \begin{bmatrix} 0.18 \\ 0.01 \end{bmatrix}, & E_2 &= 0.014. \end{aligned}$$

$$\Psi_1(x) = \begin{bmatrix} 0 \\ 0.1(x_1 + \sin x_2) \end{bmatrix}, \quad \Psi_2(x) = \begin{bmatrix} 0 \\ \frac{0.5 \sin x_1}{\sqrt{10}} \end{bmatrix}, \quad W = [0 \quad 0 \quad 0.3 \quad 0.6]$$

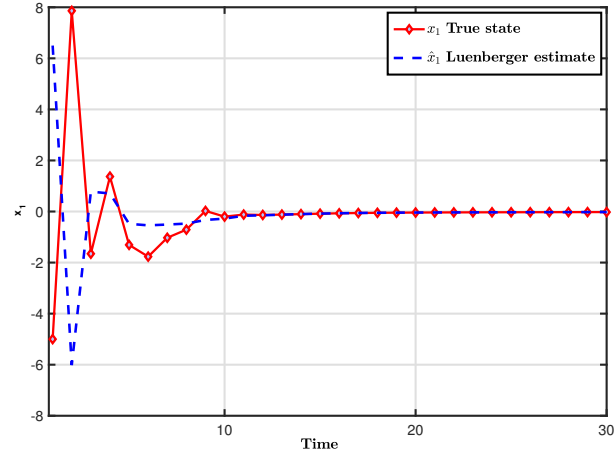
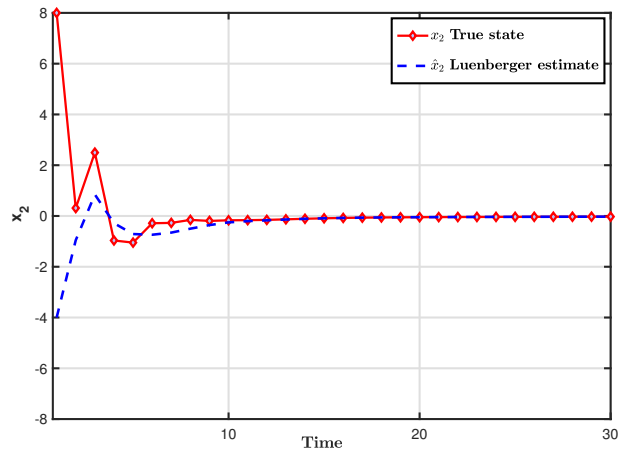
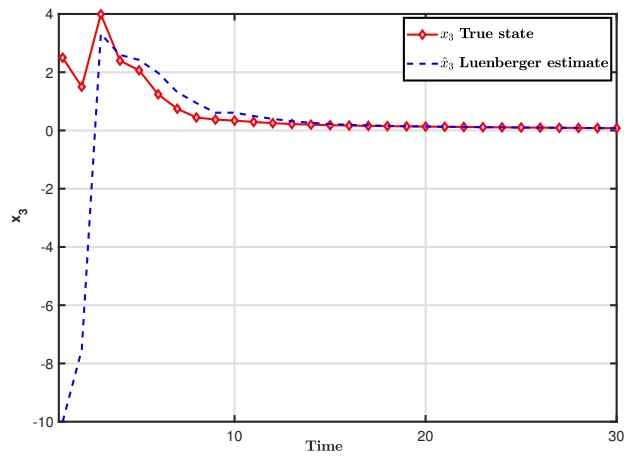
(a) Behavior of x_1 and \hat{x}_1 (b) Behavior of x_2 and \hat{x}_2 (c) Behavior of x_3 and \hat{x}_3

FIGURE 4. Case Study 1: state variables and their estimates.

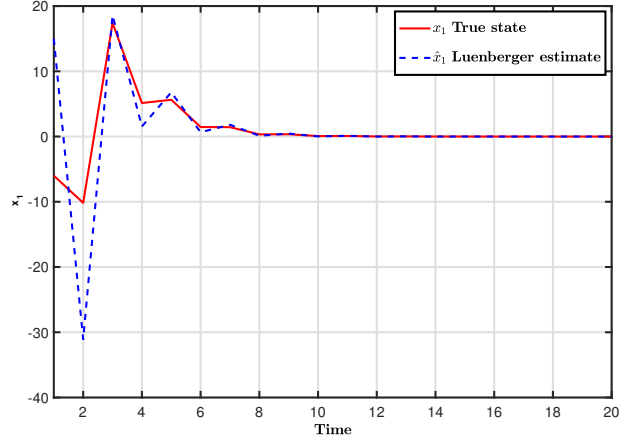
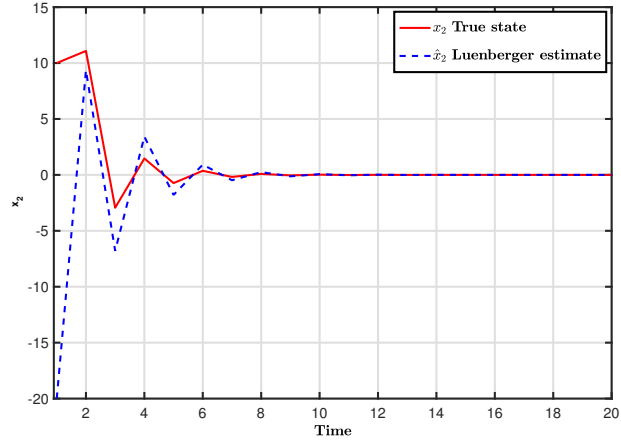
(a) Behavior of x_1 and \hat{x}_1 (b) Behavior of x_2 and \hat{x}_2

FIGURE 5. Case Study 2: state variables and their estimates.

According to Lemma 2, we have:

$$\mathcal{V}_{\mathcal{H}_n}(1) = \left\{ \begin{bmatrix} 0 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0.1 & 0.1 \end{bmatrix} \right\},$$

and

$$\mathcal{V}_{\mathcal{H}_n}(2) = \left\{ \begin{bmatrix} 0 & 0 \\ -\frac{0.5}{\sqrt{10}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{0.5}{\sqrt{10}} & 0 \end{bmatrix} \right\}.$$

The LMI (25) of Theorem 2 was found feasible; we obtained the following observer and the controller gains:

$$L_1 = \begin{bmatrix} -0.0003 \\ 0.0016 \end{bmatrix}, \quad K_1 = [0.0205 \quad 0.0316],$$

$$L_2 = \begin{bmatrix} 0.0167 \\ -0.0031 \end{bmatrix}, \quad K_2 = [-0.0272 \quad -0.0688].$$

These solutions correspond to an optimal value of the disturbance attenuation level equal to 0.1559. The switching signal λ_t was randomly generated with minimum dwell time equal to 3, and its estimate $\hat{\lambda}_t$ was obtained according to the minimum distance criterion, i.e.,

$$\hat{\lambda}_t = \operatorname{argmin}_{i \in \{1,2\}} (|y_t - C_i \hat{x}_t|).$$

The simulations over an horizon of length $T = 100$ with $x_0 = [0.1 \quad 0.6]^\top$, $\hat{x}_0 = [0.2 \quad -0.3]^\top$, and a noise

$$\omega_t = \begin{cases} 1 & \text{if } t \in 20, 21, \dots, 50 \\ 0 & \text{elsewhere} \end{cases} \quad (29)$$

are shown in Fig.s 6-7.

6. Conclusion. We have addressed the problem of observer-based control for hybrid systems with possible nonlinearities and disturbances in the state equations by using a novel approach that explicitly takes into account the switching dynamics. A key advantage of the proposed method is the reduction of the intrinsic conservativeness in the resulting LMIs, which turn to be feasible under more general conditions as compared with those reported in the current literature. This topic may be the subject of future work as well as the investigation on the control of more specific class of hybrid systems such as those that undergo switching with time constraints on the mode transitions. Another direction of research is the control of such hybrid systems by combining moving horizon estimation and model predictive control.

Acknowledgments. F. Bedouhene thanks the Algerian general direction of research (DGRSDT/MESRS-Algeria) for their financial support.

REFERENCES

- [1] A. Alessandri, M. Baglietto and G. Battistelli, [Receding-horizon estimation for switching discrete-time linear systems](#), *IEEE Trans. on Automatic Control*, **50** (2005), 1736–1748.
- [2] A. Alessandri, M. Baglietto and G. Battistelli, [Luenberger observers for switching discrete-time linear systems](#), *International Journal of Control*, **80** (2007), 1931–1943.
- [3] A. Alessandri and F. Boem, [State observers for systems subject to bounded disturbances using quadratic boundedness](#), *IEEE Trans. on Automatic Control*, to appear.
- [4] F. Blanchini, S. Miani and F. Mesquine, [A separation principle for linear switching systems and parametrization of all stabilizing controllers](#), *IEEE Trans. on Automatic Control*, **54** (2009), 279–292.
- [5] S. Court, K. Kunisch and L. Pfeiffer, [Hybrid optimal control problems for a class of semilinear parabolic equations](#), *Discrete and Continuous Dynamical Systems - Series S*, **11** (2018), 1031–1060.
- [6] M. de Oliveira, J. Bernussou and J. Geromel, [A new discrete-time robust stability condition](#), *Systems & Control Letters*, **37** (1999), 261–265.

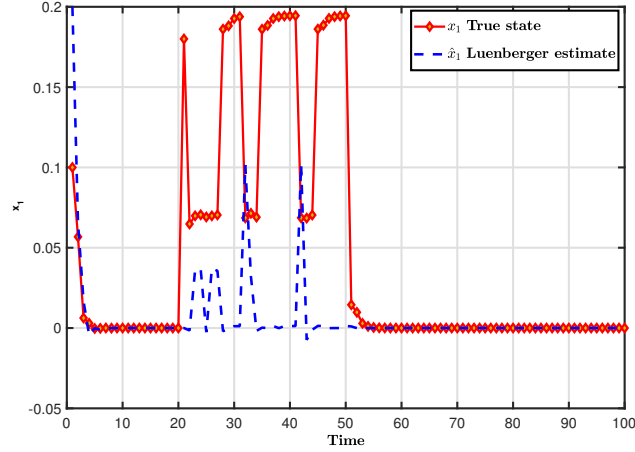
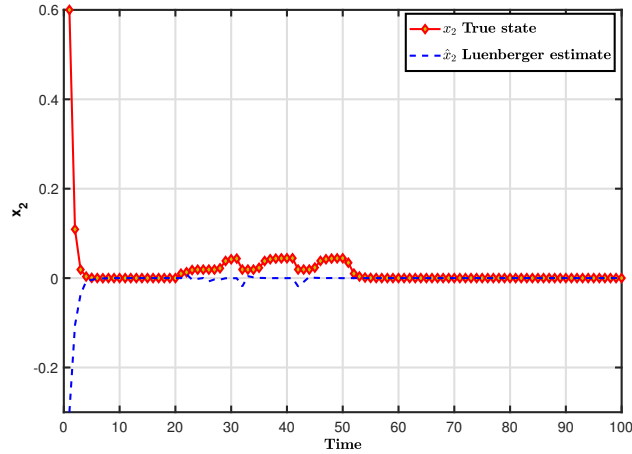
(a) Behavior of x_1 and \hat{x}_1 (b) Behavior of x_2 and \hat{x}_2

FIGURE 6. Case Study 3: state variables and their estimates.

- [7] R. Essick, J.-W. Lee and G. Dullerud, [Control of linear switched systems with receding horizon modal information](#), *IEEE Trans. on Automatic Control*, **59** (2014), 2340–2352.
- [8] B. Grandvallet, A. Zemouche, H. Souley-Ali and M. Boutayeb, [New LMI condition for observer-based \$H_\infty\$ stabilization of a class of nonlinear discrete-time systems](#), *SIAM Journal on Control and Optimization*, **51** (2013), 784–800.
- [9] R. Guo and L. Song, [Optical chaotic secure algorithm based on space laser communication](#), *Discrete and Continuous Dynamical Systems - Series S*, **12** (2019), 1355–1369.
- [10] M. Halimi, G. Millerioux and J. Daafouz, [Model-based modes detection and discernibility for switched affine discrete-time systems](#), *IEEE Trans. on Automatic Control*, **60** (2015), 1501–1514.
- [11] W. Heemels, J. Daafouz and G. Millerioux, [Observer-based control of discrete-time LPV systems with uncertain parameters](#), *IEEE Trans. on Automatic Control*, **55** (2010), 2130–2135.

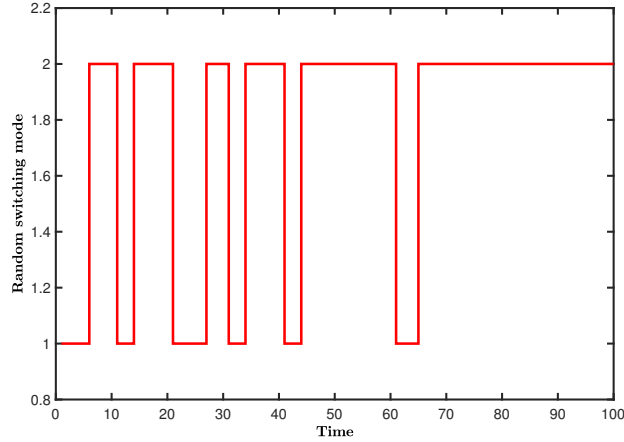
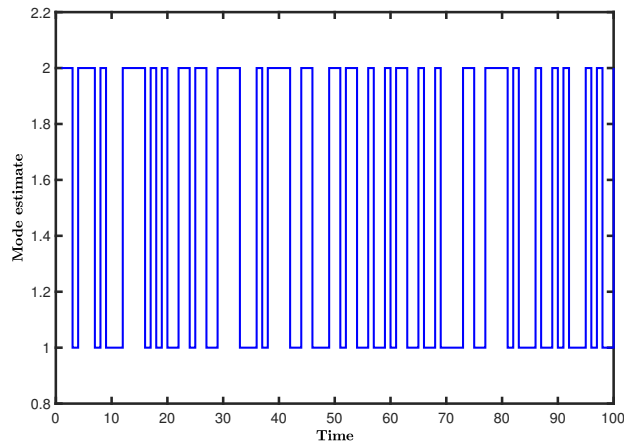
(a) Behavior of the switching mode λ_t (b) Behavior of $\hat{\lambda}_t$

FIGURE 7. Case Study 3: switching signal and its estimate.

- [12] S. Ibrir, [Static output feedback and guaranteed cost control of a class of discrete-time nonlinear systems with partial state measurements](#), *Nonlinear Analysis*, **68** (2008), 1784–1792.
- [13] X.-Q. Jiang and L.-C. Zhang, [Stock price fluctuation prediction method based on time series analysis](#), *Discrete and Continuous Dynamical Systems - Series S*, **12** (2019), 915–927.
- [14] H. Kheloufi, F. Bedouhene, A. Zemouche and A. Alessandri, [Observer-based stabilisation of linear systems with parameter uncertainties by using enhanced LMI conditions](#), *Int. Journal of Control*, **88** (2015), 1189–1200.
- [15] J. Li and Y. Liu, [Stabilization of a class of discrete-time switched systems via observer-based output feedback](#), *Journal of Control Theory and Applications*, **5** (2007), 307–311.
- [16] Z. Li, C. Wen and Y. Soh, [Observer-based stabilization of switching linear systems](#), *Automatica*, **39** (2003), 517–524.
- [17] W. Lv and S. Ji, [Atmospheric environmental quality assessment method based on analytic hierarchy process](#), *Discrete and Continuous Dynamical Systems - Series S*, **12** (2019), 941–955.

- [18] Y. Qiu, W. Chen and Q. Nie, [A hybrid method for stiff reaction-diffusion equations](#), *Discrete and Continuous Dynamical Systems - Series B*, **24** (2019), 6387–6417.
- [19] R. Rajamani, W. Jeon, H. Movahedi and A. Zemouche, [On the need for switched-gain observers for non-monotonic nonlinear systems](#), *Automatica J. IFAC*, **114** (2020), 108814, 12 pp.
- [20] Z. Song and J. Zhao, [Observer-based robust \$H_\infty\$ control for uncertain switched systems](#), *Journal of Control Theory and Applications*, **5** (2007), 278–284.
- [21] W. Xiang, J. Xiao and M. Iqbal, [Robust observer design for nonlinear uncertain switched systems under asynchronous switching](#), *Nonlinear Analysis: Hybrid Systems*, **6** (2012), 754–773.
- [22] A. Zemouche and M. Boutayeb, [On LMI conditions to design observers for Lipschitz nonlinear systems](#), *Automatica*, **49** (2013), 585–591.

Received November 2019; revised February 2020.

E-mail address: alessandri@dime.unige.it

E-mail address: fbedouhene@yahoo.fr

E-mail address: bouhadjra@dime.unige.it

E-mail address: ali.zemouche@univ-lorraine.fr

E-mail address: bagnerini@dime.unige.it