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State observer design method for a class of non-linear systems

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Abstract: In this study, the authors develop a new high-gain observer design method for non-linear systems. This new design provides a lower gain compared to both the high-gain and the enhanced high-gain observer. The idea is to combine the improved high-gain methodology with the linear matrix inequality based observer design technique to build a more general observer that allows one to exploit the benefits of both approaches. A numerical example is given to show the effectiveness of the proposed observer with different values of the Lipschitz constant and of the compromise index.

1 Introduction

Observer design for non-linear systems has been investigated for many decades [1–10]. This is due to its important role in control design systems, diagnosis, health monitoring, and other modern applications like synchronisation of multi-agent systems and cyber-attack detection. There are several methods developed in the literature which can be classified into three categories: extended Kalman filter, Luenberger observer, high-gain (HG) observer methodology, and linear matrix inequality (LMI)-based techniques. However, this paper focuses on HG observers only.

The design of HG observers was essentially motivated by its simplicity to implement due to the use of only one single tuning parameter. However, there are three limitations that make HG observer weak and difficult to be used in sensitive industrial applications. The first limitation is related to numerical issues concerning large systems as high values of the observer gain are required. The second limitation is the sensitivity to measurement noise because high values of the observer gains amplify the noise [11]. The third and last limitation is the peaking phenomenon characterised by large amplitudes of the estimated states in the transient.

To overcome these restrictions, several solutions have been proposed in the literature. The main solutions are generally based on a time-varying gain that is appropriately updated by taking into account the stability and convergence requirements [2, 3, 5, 12, 13]. Recently, a new HG observer has been proposed in [14]. Their contribution consists in limiting the power of the tuning parameter to 2. However, the dimension of the observer is equal to $2(n-1)$ where n is the dimension of the original system, and the power n is only distributed between different additional state variables injected in the observer. Then this power n reappears in the bound of the estimation error when the system is subject to measurement noise. This particular design has been reconsidered in [15, 16] by including saturations to avoid the peaking phenomenon. Another recent HG observer with the same dimension as the original system and where the observer's gain power is limited to 1 was proposed in [17] for the same class of systems considered in [15]. As in [15, 16], nested saturation functions have been used to limit the peaking phenomenon. In [18] a new structure of observers, called HG/LMI observer, has been developed by combining the standard HG methodology with the LMI technique [19]. This new observer has the advantage to provide lower tuning parameter compared to the previous HG observers, without using saturation functions or filtering. In this paper, we develop a new state observer design for systems with multi-non-linearities in triangular form or any system that can be transformed into a triangular structure. The proposed observer has the advantage of allowing more possibilities to choose the design parameters. The idea consists in combining the enhanced HG methodology [6] with the LMI methodology in order to reduce more values of the observer gains. This structure has the advantage to use multiple tuning parameters. Indeed, the observer in [18] becomes a particular case of the observer proposed in this paper by a special choice of the design parameters. It is shown through a simple example that our approach reduces the peaking phenomenon and decreases the sensitivity to high-frequency measurement noise as compared with the HG observer. Note that the system we consider in this paper is time invariant. For time-varying systems with time-varying parameters and stochastic noises, we refer the reader to [11].

2 Preliminaries and problem formulation

2.1 Preliminaries

Before formulating the problem, we introduce some useful preliminaries for the developed approach. We will recall two lemmas, from [6] and [19] respectively which are necessary for the mathematical developments given in the next section.

Lemma 1: Let X and Y be two matrices of adequate dimensions. Then the following inequality holds for any symmetric and non-singular matrix S of appropriate dimension:

$$X^\top Y + Y^\top X \le \frac{1}{2}(X+SY)^\top S^{-1}(X+SY)\,.$$

The next lemma will be used to decompose any Lipschitz non-linear function in a convenient way. Such decompositions play an important role in the observer synthesis and allow enhancing the standard HG observer. However, before stating the lemma, the following definition taken from [19] is needed.

Definition 1: Consider two vectors

$$X \triangleq \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n Z \triangleq \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n\,.$$

For all $i = 0, \dots, n$, we define an auxiliary vector $X^{Z_i} \in \mathbb{R}^n$ corresponding to X and Z as follows:

$$\begin{cases} X^{Z_i} \triangleq \begin{pmatrix} z_1 \\ \vdots \\ z_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} i = 1, \ldots, n, \\ X^{Z_0} \triangleq X. \end{cases} \tag{1}$$

Lemma 2: Consider a function $\Psi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then, the two following claims are equivalent:

- Ψ is γ_Ψ-Lipschitz with respect to its argument, i.e.

$$\left\| \Psi(X) - \Psi(Z) \right\| \le \gamma_\Psi \left\| X - Z \right\|, \forall X, Z \in \mathbb{R}^n; \tag{2}$$

- for all $i, j = 1, \ldots, n$, there exist functions

$$\psi_{ij}: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

and constants $\underline{\gamma}_{\psi_{ij}} \le 0$, $\bar{\gamma}_{\psi_{ij}} \ge 0$, so that $\forall X, Z \in \mathbb{R}^n$,

$$\Psi(X) - \Psi(Z) = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \psi_{ij} H_{ij}\big(X - Z\big), \tag{3}$$

and $-\gamma_\Psi \le \underline{\gamma}_{\psi_{ij}} \le \psi_{ij} \le \bar{\gamma}_{\psi_{ij}} \le \gamma_\Psi$, where

$$\psi_{ij} \triangleq \psi_{ij}\big(X^{Z_{j-1}}, X^{Z_j}\big), H_{ij} = e_n(i) e_n^{\mathrm{T}}(j), \tag{4}$$

and $e_n(i) = \big(0 \cdots \overbrace{01}^{i\text{th component}} 0 \cdots 0\big)^{\mathrm{T}}$.

2.2 Problem formulation

Let us consider the class of non-linear systems which are diffeomorphic to the form of the system studied in [20]:

$$\begin{cases} \dot{x} &= Ax + f(x), \\ y &= Cx, \end{cases} \tag{5}$$

where for $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}$ is the measured output. The matrices $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, and the non-linear function $f: \mathbb{R}^n \to \mathbb{R}^n$ are defined as follows:

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C \triangleq [\, 1\ 0 \cdots 0 \,],$$

$$f(x) \triangleq \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

It should be noticed that, as demonstrated in [21], all uniformly observable systems can be transformed into system (5). Several real-world models can be transformed into the triangular form [21, 22]. The references [20, 23] give more details about this family of systems and its practical importance.

We need to introduce the following notations:

- $\mathbb{R}^n_{\ge 0} = \{(l_i)_{1 \le i \le n} \in \mathbb{R}^n;\ l_i \ge 0,\ \forall i = 1, \ldots, n\}$,
- $\mathbb{R}^n_{> 0} = \{(l_i)_{1 \le i \le n} \in \mathbb{R}^n;\ l_i > 0,\ \forall i = 1, \ldots, n\}$,
- $\mathbb{R}^n_{> 0, \uparrow} = \{(l_i)_{1 \le i \le n} \in \mathbb{R}^n_{> 0};\ l_{k+1} > l_k,\ \forall k = 1, \ldots, n-1\}$.

As usually done for the class of systems (33), we introduce the following Lipschitz assumption on f.

Assumption 1: The function f satisfies the global Lipschitz condition, i.e. there exists a vector $L_f = (L_i^f)_{1 \le i \le n} \in \mathbb{R}^n_{\ge 0}$ such that

$$\begin{vmatrix} f_i(\bar{x}_1 + w_1, \bar{x}_2 + w_2, \ldots + \bar{x}_i + w_i) \\ -f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i) \end{vmatrix} \le \sum_{j=1}^{i} L_i^f \left| w_j \right|,$$

for all $\bar{x} = (\bar{x}_i)_{1 \le i \le n}$, $w = (w_i)_{1 \le i \le n} \in \mathbb{R}^n$ and $i = 1, \ldots, n$.

Instead of the standard HG observer structure, we use in this paper the enhanced structure previously proposed in [6]. Consider the following state observer [6]:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + G(\gamma, K)\,(y - C\hat{x}), \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, for $t \in \mathbb{R}$, and

$$G(\gamma, K) \triangleq \begin{bmatrix} \gamma_1 k_1 & \gamma_2 k_2 & \cdots & \gamma_n k_n \end{bmatrix}^{\mathrm{T}} \triangleq T(\gamma) K,$$

with

$$\begin{aligned} \gamma &\triangleq \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n_{>0}, \\ K &\triangleq \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n, \\ T(\gamma) &\triangleq \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n). \end{aligned}$$

Remark 1: The observer structure (6) is considered with the goal to get multiple tuning parameters contrarily to the standard observer where only one tuning parameter needs to be selected in the design procedure. The use of such a decomposition leads to the involvement of a new matrix of decision variables in the design, namely the matrix Z, which can be tuned to reduce the HG values, as well explained in [6, Section III] by considering a lot of numerical aspects.

As usual in the HG methodology, we consider the transformed error

$$\tilde{x} \triangleq T^{-1}(\gamma)\, e, \tag{7}$$

where $e(t) \triangleq x(t) - \hat{x}(t)$ is the estimation error. After developing the computations, it follows that

$$\begin{aligned} \dot{\tilde{x}} &= \gamma_1\big(A - KC + \Omega(\gamma)\big)\tilde{x} \\ &\quad + T^{-1}(\gamma)\big(f(x) - f(x - T(\gamma)\tilde{x})\big), \end{aligned} \tag{8}$$

where

$$\Omega(\gamma) \triangleq \begin{bmatrix} 0 & z_1 & 0 & \cdots & 0 \\ 0 & 0 & z_2 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & z_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with

$$z_i \triangleq \frac{\gamma_{i+1}}{\gamma_1 \gamma_i} - 1,\ i = 1, 2, \ldots, n-1. \tag{9}$$

The aim is to synthesise the observer parameters γ_i and k_i such that the error $\tilde{x}$ converges exponentially to zero. Usually, this problem is solved by using the standard HG observer methodology [20]. However, in some situations (for instance, larger Lipschitz constants, high dimension of the systems), it leads to extremely high values of the gains, which renders the observer very sensitive to high-frequency measurement noise and causes the picking phenomenon in the transient. To overcome this issue, various improvements have been established in the literature, proposing HG observers with constant or time-varying gains. Limiting our study, in this paper, to observers with constant gains, some recent methods proposed considerable solutions [6, 14, 18]; nevertheless, the problem remains still open for further improvements. In this paper, we will combine between [6] and [18] to propose a new approach. To tackle this problem, a convenient decomposition of the non-linearity and introduction of additional parameters are required. This is the goal of the next section.

3 Observer synthesis methodology

3.1 Preliminary transformations

As stated above, the idea is to exploit the results of [6, 18] to improve the solution of the observer gains. Borrowed from [18, Eq. (54)], the first step consists in decomposing the non-linearity of the system into two parts. From Lemma 2 and after some rearrangements, there exist functions ψ_{ij}, scalars $\underline{\nu}_{\psi_{ik_i(j)}} \leq 0$ and $\bar{\nu}_{\psi_{ik_i(j)}} \geq 0$ such that

$$f(x) - f(x - T(\gamma)\tilde{x}) = \Delta f_1 + \Delta f_2,$$

with

$$\Delta f_1 \triangleq \sum_{i=1}^{n} \sum_{j=1}^{i-j_i} \gamma_j \psi_{ij} e_n(i) \tilde{x}_j,$$

$$\Delta f_2 \triangleq \sum_{i=1}^{n} \sum_{j=1}^{j_i} \gamma_{k_i(j)} \psi_{ik_i(j)} e_n(i) \tilde{x}_{k_i(j)},$$

$$k_i(j) \triangleq i - (j_i - j), 0 \leq j_i \leq i,$$

and

$$\underline{\nu}_{\psi_{ik_i(j)}} \leq \psi_{ij} \leq \bar{\nu}_{\psi_{ik_i(j)}}.$$

By analogy to [18], the first term Δf_1 will be handled by the enhanced HG observer (EHGO)-approach in [6], while the second one, Δf_2, will be associated with the linear part and will be processed by the LMI method [19] as in [18].

Notice that the term $T^{-1}(\gamma)\Delta f_2$ can be rewritten as

$$T^{-1}(\gamma)\Delta f_2 = \sum_{i=1}^{n} \sum_{j=1}^{j_i} \frac{\gamma_{k_i(j)}}{\gamma_i} \psi_{ik_i(j)} e_n(i) \tilde{x}_{k_i(j)}.$$

Now, we will introduce some notations needed to rewrite system (8) under a suitable structure to apply the ideas of [18, 19]. Let us introduce the following matrix function:

$$\mathscr{A}(\Psi^\gamma) \triangleq A + \sum_{i=1}^{n} \sum_{j=1}^{j_i} \psi_{ij}^\gamma e_n(i) e_n^{\mathrm{T}}(k_i(j)), \tag{10}$$

where

$$\Psi^\gamma \triangleq \begin{pmatrix} \psi_{11}^\gamma & \dots & \psi_{ij_i}^\gamma & \dots & \psi_{nj_n}^\gamma \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^d, \tag{11}$$

and

$$\psi_{ij}^\gamma \triangleq \frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \psi_{ik_i(j)}, d \triangleq \sum_{i=1}^{n} j_i.$$

Consequently, system (8) can be expressed as follows:

$$\dot{\tilde{x}} = \gamma_1\big(\mathscr{A}(\Psi^\gamma) - KC + \Omega(\gamma)\big)\tilde{x} + T^{-1}(\gamma)\Delta f_1. \tag{12}$$

3.2 Preliminary results

Before stating the observer design conditions ensuring the exponential convergence of the proposed state observer, we start by introducing some preliminary results, which are necessary for the proposed design procedure. We first define the convex set for any fixed $\gamma \in \mathbb{R}^n_{>0,\uparrow}$:

$$\tilde{\mathscr{H}}^\gamma \triangleq \left\{ \Phi \in \mathbb{R}^d : \frac{\gamma_{k_i(j)} \underline{\nu}_{\psi_{ik_i(j)}}}{\gamma_1 \gamma_i} \leq \Phi_{ij} \leq \frac{\gamma_{k_i(j)} \bar{\nu}_{\psi_{ik_i(j)}}}{\gamma_1 \gamma_i} \right\}. \tag{13}$$

It is obvious that $\tilde{\mathscr{H}}^\gamma$ is a bounded convex. Indeed, from the fact that the function f is Lipschitz, the scalars $\underline{\nu}_{\psi_{ik_i(j)}}$ and $\bar{\nu}_{\psi_{ik_i(j)}}$ are bounded. On the other hand, since $\gamma \in \mathbb{R}^n_{>0,\uparrow}$, we have $\gamma_{k_i(j)}/\gamma_1\gamma_i \leq 1/\gamma_1$. It follows that

$$\frac{\underline{\nu}_{\psi_{ik_i(j)}}}{\gamma_1} \leq \Phi_{ij} \leq \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{\gamma_1},$$

which means that Φ_{ij} is bounded since $\gamma_1 > 0$.

At this stage, the bounded convex set $\tilde{\mathscr{H}}^\gamma$ is not exploitable in an LMI framework because the set of vertices depends on all the parameters $\gamma_i, i = 1, \dots, n$. In other words, it depends on the decision variables z_i, $i = 1, \dots, n-1$. To overcome this obstacle, we need to define a new bounded and convex hyper-rectangle independent of all these observer parameters. Before introducing such a set, we first state the following lemma.

Lemma 3: Let $\gamma \in \mathbb{R}^n_{>0,\uparrow}$ and z_i given by (9). If z_i satisfies $z_i \leq 0$, then there exists $\alpha \in]0, 1]$ such that inequality (14) below holds:

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \leq \frac{1}{(\alpha \gamma_1)^{1+(j_i - j)}}. \tag{14}$$

Proof: From the definition of the variables z_i in Section 2.2 and the assumption $z_i \leq 0$, we get

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1), \quad i = 2, \dots, n,$$

and

$$0 < 1 + z_k \leq 1, \quad i = 1, \dots, n-1.$$

It follows that

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \leq \left[\gamma_1^{1+(j_i-j)} \prod_{k=i-(j_i-j)}^{i} (z_k + 1) \right]^{-1}, \tag{15}$$

and from the Archimedean property, we deduce that there exists $\alpha \in]0, 1]$ so that

$$0 < \alpha \leq z_k + 1 \leq 1. \tag{16}$$

Hence, by exploiting (16) in (15) as $(1/(1+z_k)) \leq 1/\alpha$ inequality (14) is straightforwardly inferred. □

Now, we are ready to introduce a new bounded convex set parameterised by two scalar variables, namely α given as in Lemma 3 and a new tuning parameter $\sigma > 0$ to be included later in the observer design procedure.

Let α and σ be two positive scalars. Define the bounded convex set

$$\mathscr{H}_\alpha^\sigma \triangleq \left\{\Phi \in \mathbb{R}^d: \frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}} \le \Phi_{ij} \le \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}}\right\} \tag{17}$$

for which the set of vertices, $\mathscr{H}_\alpha^\sigma$, is given by

$$\mathscr{V}_{\mathscr{H}_\alpha^\sigma} \triangleq \left\{\Phi \in \mathbb{R}^d: \Phi_{ij} \in \left\{\frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}}, \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}}\right\}\right\}. \tag{18}$$

The next lemma is useful and plays an important role in the design procedure we proposed in this paper.

Lemma 4: Let $\gamma \in \mathbb{R}^n_{>0,\uparrow}$ and $\sigma > 0$ such that $\gamma_1 \ge \sigma$. Let $\alpha \in]0,1]$ be a positive scalar given by (16). Then the following inclusion holds:

$$\tilde{\mathscr{H}}^\gamma \subseteq \mathscr{H}_\alpha^\sigma. \tag{19}$$

Proof: The proof is straightforward by using Lemma 3 and the fact that the quantities $\underline{\nu}_{\psi_{ik_i(j)}}$ and $\bar{\nu}_{\psi_{ik_i(j)}}$ are negative and positive, respectively. The inequality $1/\gamma_1 \le 1/\sigma$ is also used and substituted in (14). □

The next section is devoted to the stability analysis of the estimation error dynamics. By using Lyapunov arguments and the preliminary results provided above, new HG like synthesis conditions will be established.

3.3 Stability analysis

To investigate the stability analysis of the estimation error dynamics, we consider the Lyapunov function

$$V(\tilde{x}) \triangleq \tilde{x}^\top P \tilde{x},$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric and positive-definite matrix. First, let us consider the change of variable $\tilde{K} = PK$. Therefore, after developing the derivative of the function $V(\tilde{x})$ along the trajectories of (12), we obtain

$$\begin{aligned}\dot{V}(\tilde{x}) = \gamma_1 \tilde{x}^\top \Big[&\mathscr{A}(\Psi^\gamma)^\top P + P\mathscr{A}(\Psi^\gamma) - C^\top \tilde{K}^\top - \tilde{K}C \\ &+ \Omega^\top(\gamma)P + P\Omega(\gamma)\Big]\tilde{x} + 2\tilde{x}^\top P T^{-1}(\gamma)\Delta f_1.\end{aligned} \tag{20}$$

Before presenting the stability conditions ensuring the exponential convergence of the estimation error $\tilde{x}$ to zero, we provide some information on the term Δf_1. This term will be handled by using the HG methodology.

Lemma 5: Under the assumptions of Lemma 3, there exists a positive scalar k_{f_1} such that

$$\| T^{-1}(\gamma)\Delta f_1 \| \le \frac{1}{(\alpha\gamma_1)^{j_{\min}}} k_{f_1} \| \tilde{x} \|, \tag{21}$$

where

$$j_{\min} = \min_{\substack{j_i \ne i \\ 1 \le i \le n}} j_i.$$

Proof: We have

$$\Delta f_1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i} \gamma_j \psi_{ij}\tilde{x}_j\right) e_n(i),$$

$$T^{-1}(\gamma)\Delta f_1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i} \frac{\gamma_j}{\gamma_i} \psi_{ij}\tilde{x}_j\right) e_n(i).$$

Therefore

$$\| T^{-1}(\gamma)\Delta f_1 \|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i} \frac{\gamma_j}{\gamma_i} \psi_{ij}\tilde{x}_j\right)^2.$$

Using Assumption 1 and Hölder's inequality, it follows that

$$\begin{aligned}\| T^{-1}(\gamma)\Delta f_1 \|^2 &\le \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i} L_j^f |\tilde{x}_j| \frac{\gamma_j}{\gamma_i}\right)^2 \\ &\le \sum_{i=1}^{n}\left[\left(\max_{1\le j\le i-j_i} L_j^f\right)^2 \left(\frac{\gamma_{i-j_i}}{\gamma_i}\right)^2 \left(\sum_{j=1}^{i-j_i} |\tilde{x}_j|\right)^2\right] \\ &\le \left(\max_{1\le j\le n} L_j^f\right)^2 \sum_{i=1}^{n} (i-j_i)\left(\frac{\gamma_{i-j_i}}{\gamma_i}\right)^2 \| \tilde{x} \|^2 \\ &\le \left[k_{f_1} \max_{1\le i\le n} \frac{\gamma_{i-j_i}}{\gamma_i}\right]^2 \| \tilde{x} \|^2,\end{aligned} \tag{22}$$

where

$$k_{f_1} = \bar{l}_f \sqrt{\left(\frac{n(n+1)}{2} - \sum_{i=1}^{n} j_i\right)}, \bar{l}_f = \max_{1\le j\le n} L_j^f.$$

Since $z_k \le 0, \forall k = 1,\ldots,n-1$, then from Lemma 3, there exists $\alpha \in]0,1]$ such that

$$\frac{\gamma_{i-j_i}}{\gamma_1\gamma_i} \le \frac{1}{(\alpha\gamma_1)^{1+j_i}}.$$

By putting $j_{\min} = \min_{\substack{j_i \ne i \\ 1\le i\le n}} j_i$, we get

$$\max_{1\le i\le n} \frac{\gamma_{i-j_i}}{\gamma_i} \le \frac{1}{(\alpha\gamma_1)^{j_{\min}}}.$$

Then inequality (21) is inferred. This ends the proof. □

Now, we are ready to state the first theorem which provides sufficient design conditions ensuring exponential convergence of the estimation error to zero.

Theorem 1: Assume there exist $P = P^\top > 0$, $\lambda > 0$, $\tilde{K} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^n_{>0,\uparrow}$ and $\sigma > 0$ such that

$$\begin{aligned}&\Big\{\mathscr{A}(\Psi)^\top P + P\mathscr{A}(\Psi) - C^\top \tilde{K}^\top - \tilde{K}C \\ &+ \Omega^\top(\gamma)P + P\Omega(\gamma)\Big\} + \lambda I < 0, \forall \Psi \in \mathscr{V}_{\mathscr{H}_\alpha^\sigma},\end{aligned} \tag{23}$$

and

$$\gamma_1 > \max\left(\sigma, \left[\frac{2k_{f_1}\lambda_{\max}(P)}{\lambda\alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha}\right). \tag{24}$$

Then the estimation error $\tilde{x}(t)$ is exponentially stable.

Proof: From the convexity principle and inclusion (19) for $\gamma_1 \ge \sigma$, if (23) holds, we deduce that

$$\dot{V}(\tilde{x}) \le -\gamma_1\lambda \| \tilde{x} \|^2 + 2\tilde{x}^\top P T^{-1}(\gamma)\Delta f_1. \tag{25}$$

Let $\alpha \in]0,1]$ satisfying (16). Then, from Lemma 5, we have

$$\begin{aligned}2\tilde{x}^\top P T^{-1}(\gamma)\Delta f_1 &\le 2\lambda_{\max}(P)\| \tilde{x} \| \| T^{-1}(\gamma)\Delta f_1 \| \\ &\le \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} k_{f_1} \| \tilde{x} \|^2.\end{aligned} \tag{26}$$

It follows that

$$\dot{V}(\tilde{x}) \le -\left(\gamma_1\lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}}k_{f_1}\right)\|\tilde{x}\|^2 . \quad (27)$$

From the definition of $\dot{V}(\tilde{x})$ and after integrating from 0 to t, we obtain

$$\|\tilde{x}(t)\| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|\tilde{x}_0\| e^{-\left(\gamma_1\lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}}k_{f_1}\right)t}, \quad (28)$$

which means that $\tilde{x}(t)$ converges exponentially to zero as $t \to +\infty$ if

$$\gamma_1 > \lambda - \frac{2k_{f_1}\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} > 0.$$

On the other hand, to guarantee $\gamma \in \mathbb{R}^n_{n,\uparrow}$, we need to have $\gamma_1 \ge \frac{1}{\alpha}$, since γ_1 satisfies (16). These conditions on γ_1 lead to inequality (24). To sum up, the conditions on γ_1 are required for the following reasons:

(1) $\gamma_1 \ge \sigma$ is needed to ensure feasibility of inequality (23). It is justified by inclusion (19);
(2) $\gamma_1 > 1/\alpha$ is needed to guarantee $\gamma \in \mathbb{R}^n_{>,\uparrow}$;
(3) $\gamma_1 > \left[(2k_{f_1}\lambda_{\max}(P))/\lambda\alpha^{j_{\min}}\right]^{1/(1+j_{\min})}$ is required to ensure $\tilde{x}(t)$ converges exponentially to zero, as $t \to \infty$, according to (28).

This ends the proof. □

3.4 LMI formulation

Although Theorem 1 provides sufficient conditions to guarantee the design of the observer parameters K and γ, it still not fully exploitable at this stage because the matrix inequality (23) is not numerically tractable. Indeed, (23) is not LMI and depends on the parameter γ multiplied by the Lyapunov matrix P. To linearise (23) and to render it to be independent of γ, we should separate the coupling $P\Omega(\gamma)$ and use some mathematical tools to make γ vanish from inequality (23). To start the linearisation procedure, we consider the following decomposition of $\Omega(\gamma)$:

$$\Omega(\gamma) \triangleq \Omega(Z) = A_1 Z A_2,$$

where

$$Z = \mathrm{diag}(z_1, \ldots, z_{n-1}) \in \mathbb{R}^{(n-1)\times(n-1)},$$

and

$$A_1 \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times(n-1)},$$

$$A_2 \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n}.$$

Then a simple use of Lemma 1 leads to separate P from $\Omega(Z)$. To satisfy (16), we should include the LMI constraints:

$$Z \le 0, \quad (29a)$$

$$(\alpha - 1)I_{n-1} - Z \le 0. \quad (29b)$$

Hence we are ready to state the following main theorem, which provides LMI-based synthesis conditions ensuring the exponential convergence of the observer.

Theorem 2: If for a fixed $\alpha \in]0,1]$, there exist positive scalars λ, σ, a symmetric positive-definite matrix P, diagonal matrices $S > 0$ and $W \le 0$, and a vector $\tilde{K} \in \mathbb{R}^n$, such that for all $\psi \in \mathscr{V}_{\mathscr{H}^\sigma_\alpha}$ the following conditions are fulfilled:

$$\begin{bmatrix} \mathscr{A}(\psi)^\top P + P\mathscr{A}(\psi) - C^\top\tilde{K}^\top - \tilde{K}C + \lambda I & (*) \\ A_1^\top P + WA_2 & -2S \end{bmatrix} < 0 \quad (30)$$

$$(\alpha - 1)S - W \le 0, \quad (31)$$

then the estimation error $\tilde{x}(t)$ converges exponentially towards zero, as $t \to +\infty$, if the observer parameters are selected as follows:

$$K = P^{-1}\tilde{K},\ Z = S^{-1}W, \quad (32a)$$

$$\gamma_1 > \max\left(\sigma, \left[\frac{2k_{f_1}\lambda_{\max}(P)}{\lambda\alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha}\right), \quad (32b)$$

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1}(z_k + 1), \quad i = 2, \ldots, n. \quad (32c)$$

Proof: First, to get (30), we apply Lemma 1 on inequality (23) of Theorem 1. Indeed, from Lemma 1, we have

$$\Omega^\top P + P\Omega = (A_1P)^\top(ZA_2) + (ZA_2)^\top(A_1P) \le \frac{1}{2}(A_1^\top P + SZA_2)^\top S^{-1}(A_1^\top P + SZA_2).$$

Then, after using the Schur lemma and the change of variables $W = SZ$, $\tilde{K} = PK$, we get (30). Also, condition $W \le 0$ comes from (29a). As for inequality (31), it stems from (29b) after multiplying it by S. This ends the proof. □

3.5 Discussion on the performance of the proposed observer design

The proposed observer design method is more general than those proposed in the literature and related to HG methodology with constant observer gain. Indeed, for particular cases, the design is reduced to some recent methods. Although our idea is taken from [6, 18], this combination is not systematic and has generated major mathematical difficulties (see Lemmas 3–5) necessary for the analysis of the stability of the dynamics of the error. We summarise the particular cases in the following items:

1. If we take $j_i = 0$ and $z_k < 0$, we will get exactly the enhanced HG proposed in [6]. Indeed, in such a case, we have $j_{\min} = 0$, $k_{f_1} = k_f$ and $\mathscr{A}(\psi) \equiv A$.
2. Likewise, if we have $j_i = 0$ (then $j_{\min} = 0$) and $z_k = 0$, we get the standard HG and Theorem 2 will be reduced to the main theorem of the standard HG observer [20].
3. Notice also that the HG/LMI observer proposed in [18] is a particular case of the result in Theorem 2 corresponding to $j_{\min} \ge 1$ and $z_k \equiv 0$.

3.6 Observer design algorithm

The design procedure of the proposed observer can be summarised in the following well-structured design algorithm (see Fig. 1). This algorithm is solved by using Matlab LMI Toolbox and YALMIP.

Remark 2:

if *Check if Assumption 1 holds. If yes,* **then**

Step 1. Set an appropriate value for $\alpha \in]0,1]$, choose a small $\epsilon > 0$ for the gridding, take $\tau = \frac{1}{2}$, $\lambda = 1$, a sufficiently high value $v_{gain} > 0$ and go to **Step 2**.

Step 2. While $\tau + \epsilon < 1$, take $\tau := \tau + \epsilon$ and return to Step 2;

Step 3. Solve the optimization problem (30) and (31) with respect to the decision variables P, $\tilde{K}$, S, W using the gridding method with respect to $\epsilon > 0$.

Step 4. Compute the decision variables P, $\tilde{K}$, Z and W. Take $\sigma = \frac{\tau}{1-\tau}$; $K = P^{-1}\tilde{K}$; $Z = S^{-1}W$, and compute the gains

$$\gamma_1 = \max\left(\sigma, \left[\frac{2k_f \lambda_{\max}(P)}{\lambda \alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha}\right);$$

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1}(z_k+1), \quad i = 2, \ldots, n;$$

$$G = T(\gamma)P^{-1}\tilde{K}.$$

if *If* $v_{gain} > \|G\|$ **then**

put $v_{gain} = \|G\|$ and go to **Step 2**.

else

$G^* = T(\gamma)P^{-1}\tilde{K}$, with G^* as the final result of the procedure.

else

The observer gains cannot be computed.

Fig. 1 *Algorithm 1: EHG/LMI observer design*

Table 1 Gains G^* for different values of $j_{\min}$ and σ

$j_{\min}$	0	1	2
σ	—	2.3557	1.7548
G^*	$\begin{bmatrix} 24.9 \\ 472.9 \\ \boxed{3739.5} \end{bmatrix}$	$\begin{bmatrix} 7.8474 \\ 29.6747 \\ \boxed{48.6805} \end{bmatrix}$	$\begin{bmatrix} 6.1598 \\ 18.4522 \\ \boxed{24.8470} \end{bmatrix}$

1. Algorithm 1 (Fig. 1) is based on the use of the gridding technique with respect to the parameter $\sigma > 1$. Such a technique consists in scaling $\sigma > 1$ by defining $\tau \in]0.5, 1]$, via the change of variable $\tau = \sigma/(1+\sigma)$. Then, we assign a uniform subdivisions of the interval $]0.5, 1]$, and solve (30) and (31) for each subdivision. Note that LMIs (30) and (31) are always feasible for any $\sigma \geq 1$ (or $\tau \in [0.5, 1[$ and $\epsilon > 0$). This issue has been addressed in [18, Eqs. (46)–(47)]. However, the choice of ϵ has an impact on the value of σ, which is explicitly related to the value of the HG parameter γ_1. Therefore, the smaller is ϵ, the smaller is the value of the obtained parameter γ_1. Of course, this could generate significant computational complexity. This is the reason why we introduce the stop test $v_{\text{gain}} \leq \| G \|$.
2. The scalar α is fixed a priori. In the case of our illustrative example, one set $\alpha = 0.95$. Indeed, the smaller is the value of $1 - \alpha > 0$, the smaller is the value of the observer parameter γ_1.

4 Illustrative example

This section is dedicated to a simple numerical example to show the validity and effectiveness of the proposed design technique. We show the benefits of combining the enhanced HG and the LMI observer design method for different values of the compromise index and the Lipschitz constant. We aim also to compare our proposed approach, in particular, with the one developed recently in [18] and the classical HG observer.

Consider the third-order system

$$\begin{cases} \dot{x} &= Ax + f(x), \\ y &= Cx, \end{cases} \tag{33}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0],$$

and

$$f: \mathbb{R}^3 \to \mathbb{R}^3;\ x \mapsto \quad \left(0, 0, \frac{k_f}{\sqrt{3}} \sum_{i=1}^{3} \sin x_i\right).$$

The non-linearity f satisfies Assumption 1 with $L_1 = L_2 = 0$ and $L_3 = k_f$.

Now, to design the observer parameters, we will follow all the steps of Algorithm 1 (Fig. 1). Let us choose $\alpha = 0.95$ and $v_{\text{gain}} = 2e18$. For the gridding, we choose $\epsilon = 10^{-3}$.

According to the values of the index $j_{\min}$, γ_f and σ, we obtain the following gains G^* for different values of $j_{\min}$ and $k_f = 1$ (see Table 1).

Now we will provide some comparisons between the standard HG (HGO), the enhanced HG observer (EHGO), the HG/LMI observer and the proposed technique in Theorem 2.

Table 2 shows the numerical comparisons between the HG/LMI observer proposed in [18] and the classical HG observer. The superiority of the method compared to the authors' previous work

Table 2 Comparisons between the HG observers for different values of k_f and $j_{\min}$

$j_{\min}$	k_f	HGO θ_0	HGO K			EHGO [6] γ	EHGO [6] K		
	0.1	2.34	1.06	0.86	0.29	2.34	1.06	0.86	0.29
0	1	23.45	1.06	0.86	0.29	14.82	7.46	5.95	1.87
	10	234.51	1.06	0.86	0.29	148.24	7.46	5.95	1.86

$j_{\min}$	k_f	$n_{\rm LMI}$	HG/LMI observer [18] σ	HG/LMI observer [18] $\theta_{j_{\min}}^{\frac{1}{1+j_i}}$	HG/LMI observer [18] K			Theorem 2 σ	Theorem 2 $\gamma_1^{\frac{1}{1+j_{\min}}}$	Theorem 2 K		
	0.1		1.004	1.004	2.5021	2.9498	1.4199	1.1834	1.1794	1.62	1.30	0.49
1	1	2	2.3557	2.3557	4.8587	8.7086	5.8140	2.9063	2.8924	2.70	3.5133	1.9832
	10		9.7527	9.7527	13.9080	28.0327	26.5827	10.4943	10.2405	5.6260	11.1557	10.1883
	0.1		1.004	1.0040	2.5930	3.1640	1.6028	1.008	1.0054	1.7455	1.5160	0.6135
2	1	4	1.7548	1.7548	5.4481	10.6211	8.4631	1.9674	1.9565	3.1310	4.7674	3.2630
	10		6.0423	6.0423	16.9046	49.8802	72.2843	6.2993	6.1383	7.7352	20.9695	27.7039

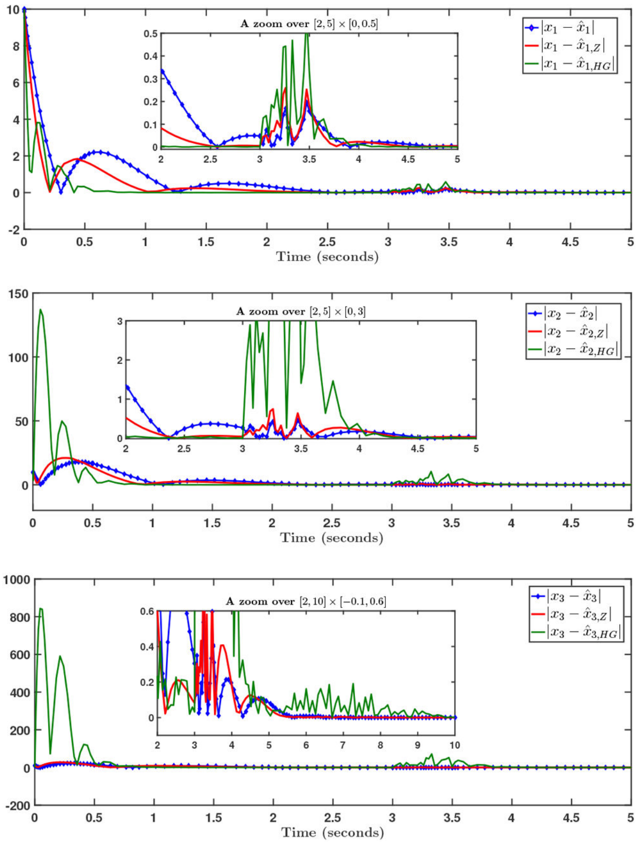

Fig. 2 *Absolute values of the errors for* $k_f = 1$, $j_{\min} = 2$, $\alpha = 0.95$ *and* $\sigma = 1.9674$

and in the papers [6, 18] is clearly illustrated. On the other hand, it was demonstrated that the authors' previous work improves the existing HG design techniques, namely the standard HG observer and the Astolfi/Marconi observer.

The advantage of the compromise index $j_{\min}$ is shown for different values of k_f. We can see clearly that our method provides a smaller gain as compared with the other methods. For example, for fixed $k_f = 10$, for $j_{\min} = 1$, it suffices to solve two LMIs to reduce significantly the gain from $\theta_0 = 234.51$ obtained by classical HG into 10.2405. For $j_{\min} = 2$, the same gain is reduced to 6.2993. This shows the importance of the introduction of increasing values of the index $j_{\min}$.

Fig. 2 shows the results of a simulation run in the presence of an additive uniform random noise on the output when $t \in [0.5, 1] \cup [3, 3.5]$ which is a Gaussian distributed random signal with mean zero and variance 0.01, initial state equal to $[5 \quad 5 \quad 5]^{\rm T}$ and initial estimated state equal to $[-5 \quad -5 \quad -5]^{\rm T}$ for all the discussed methods.

Denote by $\hat{x} = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3]^{\rm T}$, $\hat{x}_Z = [\hat{x}_{1,Z} \quad \hat{x}_{2,Z} \quad \hat{x}_{3,Z}]^{\rm T}$ and $\hat{x}_{\rm HG} = [\hat{x}_{1,{\rm HG}} \quad \hat{x}_{2,{\rm HG}} \quad \hat{x}_{3,{\rm HG}}]^{\rm T}$ the state estimates for system (33) by using the observer design method proposed in [18], in the present paper and the classical HG observer, respectively.

The curves of the absolute values $|x - \hat{x}_i|$, $|x - \hat{x}_{i,Z}|$ and $|x - \hat{x}_{i,{\rm HG}}|, i = 1, \ldots, 3$ are depicted in Fig. 2. We can see that the errors converge towards zeros. Note, however, that the transient performance of the new designed observer is better, although the convergence speeds are almost the same. The proposed new technique is able to avoid the peaking phenomenon. We also see a significant improvement of the sensitivity to high-frequency measurement noise; the estimated states are not so much disturbed as the standard HG.

5 Conclusion

In this paper a general structure of state observer is proposed that comprises many of the methods reported in the literature by making particular choices on the observer design parameters. Especially, we generalised the work presented in [18] with more many possibilities of choosing the design parameters of the gain. The stability of the estimation error is shown using a Lyapunov function after having successfully established new HG-like synthesis conditions. A numerical example was provided in order to demonstrate the performances of the proposed approach.

The application of the proposed design procedure on practical models is the objective of our future work with deep and refined theoretical results.

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