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## On uniform continuity of posterior distributions

Emanuele Dolera <sup>a,\*</sup>, Edoardo Mainini <sup>b</sup><sup>a</sup> Dipartimento di Matematica "F. Casorati", Università di Pavia, via Ferrata 5, I-27100 Pavia, Italy<sup>b</sup> Dipartimento di ingegneria meccanica, energetica, gestionale e dei trasporti, Università di Genova, via all'Opera Pia 15, I-16145 Genova, Italy

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### ABSTRACT

In the setting of dominated statistical models, we provide conditions yielding strong continuity of the posterior distribution with respect to the observed data. We show some applications, with special focus on exponential models.

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## 1. Introduction

We investigate the notion of *well-posedness* of a Bayesian statistical inference. For a given *conditional probability distribution*, we refer to well-posedness as a continuity property with respect to the conditioning variable. Indeed, we aim at quantitative estimates of the discrepancy between two inferences in terms of the distance between the observations. Our problem could be compared with the Bayesian sensitivity analysis by specifying that we are working under fixed prior and statistical model, the imprecision being concerned only with the data.

Few general results are available on this topic, even if it naturally arises – sometimes as a technical tool – in connection with different Bayesian procedures, such as consistency (Diaconis and Freedman, 1986), Ghosal and van der Vaart (2017, Chapters 6–9), mixture approximations (West, 1979; Regazzini and Sazonov, 2001), deconvolution (Efron, 2016), inverse problems (Stuart, 2010) and computability (Ackerman et al., 2017). While the pioneering paper (Zabell, 1979), essentially inspired by foundational questions, dealt with the qualitative definition of continuity for conditional distributions, more recent studies highlight the relevance of modulus of continuity estimates. We refer, for instance, to the well-posedness theory developed in Stuart (2010) and to different results found in Dashti and Stuart (2017), Cotter et al. (2009), Iglesias et al. (2014) and Latz (2019). Our contribution moves in the same direction.

In this work, we confine ourselves to dealing with the *posterior distribution*. We introduce two measurable spaces  $(\mathbb{X}, \mathcal{X})$  and  $(\Theta, \mathcal{T})$ , representing the space of the observations and the parameters, respectively. We further introduce a probability measure  $\pi$  on  $(\Theta, \mathcal{T})$ , the *prior distribution*, and a probability kernel  $\nu(\cdot | \cdot) : \mathcal{X} \times \Theta \rightarrow [0, 1]$ , called *statistical model*. We assume that:

\* Corresponding author.

E-mail addresses: [emanuele.dolera@unipv.it](mailto:emanuele.dolera@unipv.it) (E. Dolera), [mainini@dime.unige.it](mailto:mainini@dime.unige.it) (E. Mainini).

- $\mathbb{X}$  is a metric space with distance  $d_{\mathbb{X}}$  and  $\mathcal{X}$  coincides with the Borel  $\sigma$ -algebra on  $\mathbb{X}$ ;
- $\Theta$  is a Polish space and  $\mathcal{T}$  coincides with its Borel  $\sigma$ -algebra;
- the model is *dominated*:  $\forall \theta \in \Theta, \nu(\cdot | \theta) \ll \lambda$ , for some  $\sigma$ -finite measure  $\lambda$  on  $(\mathbb{X}, \mathcal{X})$ .

For any  $\theta \in \Theta$ , we consider a density  $f(\cdot | \theta)$  of  $\nu(\cdot | \theta)$  w.r.t.  $\lambda$  and we put  $\rho(x) := \int_{\Theta} f(x | \theta) \pi(d\theta)$ , since  $(x, \theta) \mapsto f(x | \theta)$  proves to be  $\mathcal{X} \otimes \mathcal{T}$ -measurable. See [Kallenberg \(2002, Chapter 5\)](#) for details. In this framework, the well-known *Bayes theorem* provides an explicit form of the posterior distribution, namely

$$\pi(B | x) = \frac{\int_B f(x | \theta) \pi(d\theta)}{\rho(x)}$$

for any  $B \in \mathcal{T}$  and  $x$  such that  $\rho(x) > 0$ . Thus, the Bayes mapping  $x \mapsto \pi(\cdot | x)$  can be seen as a measurable function from  $\{x \in \mathbb{X} | \rho(x) > 0\}$  into the space of all probability measures on  $(\Theta, \mathcal{T})$  endowed with the topology of weak convergence. Our main task is to find sufficient conditions on  $\pi$  and  $f(\cdot | \cdot)$  such that  $x \mapsto \pi(\cdot | x)$  satisfies a uniform continuity condition of the following form: given a modulus of continuity  $w : [0, +\infty) \rightarrow [0, +\infty)$  and a set  $K \subseteq \{x | \rho(x) > 0\}$ , there exists a constant  $L_w(K)$  such that

$$d_{TV}(\pi(\cdot | x), \pi(\cdot | y)) \leq L_w(K) w(d_{\mathbb{X}}(x, y)) \quad \forall x, y \in K \quad (1)$$

holds, where  $d_{TV}(\pi_1, \pi_2) := \sup_{B \in \mathcal{T}} |\pi_1(B) - \pi_2(B)|$  denotes the *total variation distance*.

In order to motivate the study of a property like (1), let us briefly discuss some of its applications to Bayesian inference. By itself, uniform continuity is of interest in the theory of regular conditional distributions ([Tjur, 1980, Sections 9.6–9](#)), [Pfanzagl \(1979\)](#). Indeed, a natural approximation of the posterior is

$$\bar{\pi}_x(B) := \int_{B \times U_x} f(y | \theta) \pi(d\theta) \lambda(dy) / \int_{\Theta \times U_x} f(y | \theta) \pi(d\theta) \lambda(dy) \quad (B \in \mathcal{T})$$

where  $U_x$  stands for a suitable neighborhood of  $x$ . Thus, (1) would express the approximation error  $d_{TV}(\pi(\cdot | x), \bar{\pi}_x)$ . Anyway, the main applications are concerned with the theory of  $n$  exchangeable observations, where  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  and the model  $f(\cdot | \theta)$  is in product form, by de Finetti's representation theorem. The main advantage of an estimate like (1) arises when  $d_{\mathbb{X}}(x, y)$  is re-expressed in terms of a sufficient statistic (e.g., the empirical measure), so that the asymptotic behavior of the posterior for large  $n$  could be studied by resorting to the asymptotic behavior of such statistic. We believe that uniform continuity would represent a new technique to prove asymptotic properties of the posterior distribution, like Bayesian consistency. See Sections 3.2 and 3.5. Finally, uniform continuity would represent also a powerful technical tool to solve the problem of approximating the posterior by mixtures, on the basis of a discretization of the sample space. See [Regazzini and Sazonov \(2001\)](#) and, in particular, Proposition 2 therein, where an estimate like (1) would allow to quantitatively determine how fine the discretization should be in order to achieve a desired degree of approximation.

## 2. Continuous dependence on data

In the sequel, we refer to a *modulus of continuity* as a continuous strictly increasing function  $w : [0, +\infty) \rightarrow [0, +\infty)$  such that  $w(0) = 0$ , and we consider the space of  $w$ -continuous functions. In particular, we say that  $g : K \subseteq \mathbb{X} \rightarrow \mathbb{R}$  belongs to  $C^w(K)$  if

$$|g|_{C^w(K)} := \sup_{\substack{x, y \in K \\ x \neq y}} \frac{|g(y) - g(x)|}{w(d_{\mathbb{X}}(x, y))} < +\infty.$$

If  $w(r) = r^\alpha$ ,  $\alpha \in (0, 1]$ , we get the class  $C^{0,\alpha}(K)$  of Hölder continuous functions.

**Theorem 2.1.** *In the same setting of Section 1, suppose that  $R(K) := \inf_K \rho > 0$  is fulfilled for some  $K \subseteq \mathbb{X}$  and that, for a suitable modulus of continuity  $w$ , there holds*

$$A_{w,f,\pi,K} := \int_{\Theta} |f(\cdot | \theta)|_{C^w(K)} \pi(d\theta) < +\infty. \quad (2)$$

Then, (1) is satisfied with  $L_w(K) = \frac{A_{w,f,\pi,K}}{R(K)}$ .

**Proof.** First of all, by assumption (2), and since  $\rho(x) = \int_{\Theta} f(x | \theta) \pi(d\theta)$ , there holds

$$|\rho(y) - \rho(x)| \leq \int_{\Theta} |f(y | \theta) - f(x | \theta)| \pi(d\theta) \leq A_{w,f,\pi,K} w(d_{\mathbb{X}}(y, x)) \quad (3)$$

for any  $x, y \in K$ , so that  $\rho \in C^w(K)$ . The dual formulation of the total variation (see, e.g., [Gibbs and Su \(2002\)](#)) reads

$$d_{TV}(\pi(\cdot | x), \pi(\cdot | y)) = \frac{1}{2} \sup_{|\zeta| \leq 1} \left( \int_{\Theta} \zeta(\theta) \pi(d\theta | x) - \int_{\Theta} \zeta(\theta) \pi(d\theta | y) \right)$$

for any  $x, y \in \mathbb{X}$ , the supremum being taken among all continuous functions  $\zeta : \Theta \rightarrow \mathbb{R}$  such that  $|\zeta(\theta)| \leq 1$  for any  $\theta \in \Theta$ . For any such  $\zeta$ , define  $\Phi_\zeta(x) := \int_\Theta \zeta(\theta) \pi(d\theta | x)$  and note that the Bayes formula entails  $\rho(x)\Phi_\zeta(x) = \int_\Theta \zeta(\theta) f(x | \theta) \pi(d\theta)$ . We shall prove the  $w$ -continuity of the map  $x \mapsto \Phi_\zeta(x)$  on  $K$ . First of all, this map satisfies  $|\Phi_\zeta(x)| \leq 1$  for any  $x \in \mathbb{X}$  since  $|\zeta(\theta)| \leq 1$ . Then, for  $x, y \in K$ , there holds

$$|\rho(y)\Phi_\zeta(y) - \rho(x)\Phi_\zeta(x)| = \left| \int_\Theta \zeta(\theta)(f(y | \theta) - f(x | \theta)) \pi(d\theta) \right| \leq A_{w,f,\pi,K} w(d_{\mathbb{X}}(y, x)), \tag{4}$$

yielding  $\rho\Phi_\zeta \in C^w(K)$ . Since  $\rho \geq R(K) > 0$  on  $K$ , and  $|\Phi_\zeta| \leq 1$ , for any  $x, y \in K$  we get

$$R(K)|\Phi_\zeta(y) - \Phi_\zeta(x)| \leq |\rho(y)\Phi_\zeta(y) - \rho(y)\Phi_\zeta(x)| \leq |\rho(y)\Phi_\zeta(y) - \rho(x)\Phi_\zeta(x)| + |\rho(y) - \rho(x)|. \tag{5}$$

For any  $x, y \in K$  such that  $x \neq y$ , (3)–(5) entail

$$R(K) \frac{|\Phi_\zeta(y) - \Phi_\zeta(x)|}{w(d_{\mathbb{X}}(y, x))} \leq \frac{|\rho(y)\Phi_\zeta(y) - \rho(x)\Phi_\zeta(x)|}{w(d_{\mathbb{X}}(y, x))} + \frac{|\rho(y) - \rho(x)|}{w(d_{\mathbb{X}}(y, x))} \leq 2A_{w,f,\pi,K}.$$

The latter estimate being uniform with respect to  $\zeta$ , we conclude that

$$\frac{d_{TV}(\pi(\cdot | y), \pi(\cdot | x))}{w(d_{\mathbb{X}}(y, x))} = \frac{1}{2} \sup_{|\zeta| \leq 1} \frac{|\Phi_\zeta(y) - \Phi_\zeta(x)|}{w(d_{\mathbb{X}}(y, x))} \leq \frac{A_{w,f,\pi,K}}{R(K)}$$

holds for any  $x, y \in K$  such that  $x \neq y$ , proving the theorem.  $\square$

**Remark 2.2.** If  $\lambda(\mathbb{X}) < +\infty$ , we can take  $K = \mathbb{X}$  in Theorem 2.1. If  $R(\mathbb{X}) > 0$ , we get  $w$ -continuity on the whole  $\mathbb{X}$  for the map  $x \mapsto \pi(\cdot | x)$ , w.r.t.  $d_{TV}$ .

Some examples may also be treated within the following simple

**Corollary 2.3.** *In the same framework of Theorem 2.1, take  $K \subseteq \mathbb{X}$  such that (2) holds. In addition, suppose there exist  $g : \Theta \rightarrow \mathbb{R}$  and  $h : \mathbb{X} \rightarrow \mathbb{R}$  such that  $g > 0$  on  $\Theta$ ,  $\inf_K h > 0$  and  $f(x | \theta) \geq g(\theta)h(x)$  for any  $\theta \in \Theta$  and  $x \in K$ . Then, (1) is satisfied with  $L_w(K) = A_{w,f,\pi,K} \cdot \left[ \inf_{x \in K} h(x) \int_\Theta g(\theta) \pi(d\theta) \right]^{-1}$ .*

Let us now consider the Euclidean case, letting  $\mathbb{X} \subseteq \mathbb{R}^d$  have nonempty interior and  $K \subseteq \mathbb{X}$  be an open set with Lipschitz boundary. Usually, a Sobolev regularity might be simpler to verify, the Hölder regularity following then by Sobolev embedding. For instance, for  $p > d$ , by Morrey inequality there exists a constant  $C_{1,d,p}(K)$  such that  $|g|_{C^{0,\alpha}(K)} \leq C_{1,d,p}(K)|g|_{W^{1,p}(K)}$  holds for any  $g \in W^{1,p}(K)$ , with  $\alpha = 1 - d/p$  and  $|g|_{W^{1,p}(K)} := \|\nabla g\|_{L^p(K)}$ . More generally, if  $1 > s > d/p$ , the fractional Sobolev embedding (see, e.g., Demengel and Demengel (2012)) states that

$$|g|_{C^{0,\alpha}(K)} \leq C_{s,d,p}(K)|g|_{W^{s,p}(K)} \tag{6}$$

holds with a suitable constant  $C_{s,d,p}(K)$ ,  $\alpha = s - d/p$  and

$$|g|_{W^{s,p}(K)} := \left( \int_K \int_K \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} dx dy \right)^{\frac{1}{p}} < +\infty$$

for any  $g \in L^p(K)$ . We readily obtain the following

**Corollary 2.4.** *In the same framework of Theorem 2.1, let  $d/p < s \leq 1$  and let  $K \subseteq \mathbb{X} \subseteq \mathbb{R}^d$  be an open set with Lipschitz boundary. Let  $B_{p,s,f,\pi,K} := \int_\Theta |f(\cdot | \theta)|_{W^{s,p}(K)} \pi(d\theta) < +\infty$  and  $R(K) := \inf_K \rho > 0$ . Then, for  $\alpha = s - d/p$  and  $w(r) = r^\alpha$ , (1) is satisfied with  $L_w(K) = \frac{C_{s,d,p}(K) B_{p,s,f,\pi,K}}{R(K)}$ .*

### 3. Examples and applications

#### 3.1. Exponential models

A remarkably interesting statistical model is the *exponential family*, which includes many popular distributions, such as the Gaussian, the exponential and the gamma. For terminology and basic results about this family, see, e.g., Brown (1986). For the sake of definiteness, we consider a  $\sigma$ -finite reference measure  $\lambda$  on  $(\mathbb{X}, \mathcal{X})$  and a measurable map  $\mathbf{t} : \mathbb{X} \rightarrow \mathbb{R}^k$  such that the interior  $\Delta$  of the convex hull of the support of  $\lambda \circ \mathbf{t}^{-1}$  is nonempty and  $\Lambda := \left\{ \mathbf{y} \in \mathbb{R}^k \mid \int_{\mathbb{X}} e^{\mathbf{y} \cdot \mathbf{t}(x)} \lambda(dx) < +\infty \right\}$  is a nonempty open subset of  $\mathbb{R}^k$ . As for  $f(\cdot | \theta)$ , we resort to the so-called *canonical parametrization*, by which  $\Theta = \Lambda$ ,  $\theta = \mathbf{y}$ ,

$$f(x | \theta) = e^{\theta \cdot \mathbf{t}(x) - M(\theta)}, \quad M(\theta) := \log \left( \int_{\mathbb{X}} e^{\theta \cdot \mathbf{t}(x)} \lambda(dx) \right)$$

and, for any  $\theta \in \Theta$ ,  $f(\cdot | \theta)$  is a probability density function w.r.t.  $\lambda$ . Now, given a prior  $\pi$  on  $(\Theta, \mathcal{F})$  and a set  $K$  compactly contained in the interior of  $\mathbb{X}$ , we observe that

$$R(K) := \inf_{x \in K} \rho(x) \geq \int_{\Theta} \inf_{x \in K} f(x | \theta) \pi(d\theta) = \int_{\Theta} e^{\inf_{x \in K} \theta \cdot \mathbf{t}(x)} e^{-M(\theta)} \pi(d\theta),$$

where the last term is positive if  $\mathbf{t}$  is continuous. On the other hand, we have

$$\int_{\Theta} |f(\cdot | \theta)|_{C^w(K)} \pi(d\theta) \leq |\mathbf{t}|_{C^w(K)} \int_{\Theta} |\theta| e^{\sup_{z \in t(K)} \theta \cdot \mathbf{y}} e^{-M(\theta)} \pi(d\theta).$$

Therefore, if we suppose  $\mathbf{t} \in C^w(K)$  and that the integral  $\int_{\Theta} |\theta| e^{\sup_{z \in t(K)} \theta \cdot \mathbf{y}} e^{-M(\theta)} \pi(d\theta)$  is finite, we can invoke [Theorem 2.1](#) to obtain the  $w$ -continuity on  $K$  of the posterior distribution.

We finally notice that, in connection with an exponential model, it is natural to choose a *conjugate prior*, yielding an explicit form of the posterior ([Diaconis and Ylvisaker, 1979](#)). Thus, the LHS of (1) can be directly computed, claiming a fair comparison with the RHS. Actually, nothing seems lost at the level of the modulus of continuity, though our constant  $L_w(K)$  is usually sub-optimal. To illustrate this phenomenon, we can take  $f(x | \theta) = \theta e^{-\theta x}$ , with  $x \in \mathbb{X} = [0, +\infty)$  and  $\theta \in \Theta = (0, +\infty)$ . Chosen a conjugate prior like  $\pi(d\theta) = e^{-\theta} d\theta$ , we observe that (1) holds with  $K = [0, M]$ , for any  $M > 0$ , and  $w(r) = r$ . But our constant  $L_w(K)$  behaves asymptotically like  $M^2$  for large  $M$ , whilst the optimal one remains bounded as  $M$  grows.

### 3.2. Exponential models for $n$ exchangeable observations

Here, we adapt the result of Section 3.1 to the  $n$ -observations setting, assuming exchangeability. In this case  $x = (x_1, \dots, x_n) \in \mathbb{X}^n$  and the statistical model is of the form  $(\mathbb{X}^n, \Theta) \ni (x, \theta) \mapsto f(x | \theta) = \prod_{i=1}^n \tilde{f}(x_i | \theta)$  for some density  $\tilde{f}(\cdot | \cdot) : \mathbb{X} \times \Theta \rightarrow \mathbb{R}$ . If  $\tilde{f}$  belongs to the exponential family considered in Section 3.1, we have  $\prod_{i=1}^n \tilde{f}(x_i | \theta) = \exp\{\theta \cdot (\sum_{i=1}^n \mathbf{t}(x_i)) - nM(\theta)\}$ . By Neyman’s factorization lemma, we rewrite the model as

$$f(x | \theta) = \exp\{\tau_n(\theta) \cdot \bar{\mathbf{t}}_n(X_n(x)) - \bar{M}_n(\tau_n(\theta))\},$$

for suitable functions  $X_n : \mathbb{X}^n \rightarrow \mathbb{X}^k$ ,  $\tau_n : \Theta \rightarrow \mathbb{R}^k$ ,  $\bar{\mathbf{t}}_n : \mathbb{X}^k \rightarrow \mathbb{R}^k$ ,  $\bar{M}_n : \mathbb{R}^k \rightarrow \mathbb{R}$ , with  $X_n$  symmetric and  $k$  standing for the dimension of  $\Theta$ . We can recast the statistical model by considering  $\tau_n$  as the new parameter, and  $X_n$  as the observable. Indeed, we introduce

$$g(X, \tau) = \exp\{\tau \cdot \bar{\mathbf{t}}_n(X) - \bar{Q}_n(\tau)\} \exp\{\bar{Q}_n(\tau) - \bar{M}_n(\tau)\},$$

where  $e^{\bar{Q}_n(\tau)} := \int_{\mathbb{X}^k} e^{\tau \cdot \bar{\mathbf{t}}_n(z)} \lambda^{\otimes k}(dz)$  and  $\lambda$  is the reference measure on  $\mathbb{X}$ . Letting  $\varphi_n(\tau) := \exp\{\bar{Q}_n(\tau) - \bar{M}_n(\tau)\}$ , we have  $g(X, \tau) = h(X | \tau) \varphi_n(\tau)$ , where  $h(\cdot | \tau)$  is a probability density with respect to the product measure  $\lambda^{\otimes k}$ , parametrized by  $\tau \in \mathbb{R}^k$ . Given a prior  $\pi$  on  $(\Theta, \mathcal{F})$ , the posterior  $\pi_n(d\theta | x)$  reads

$$\frac{\prod_{i=1}^n \tilde{f}(x_i | \theta) \pi(d\theta)}{\int_{\Theta} \prod_{i=1}^n \tilde{f}(x_i | t) \pi(dt)} = \frac{h(X_n(x) | \tau_n(\theta)) \varphi_n(\tau_n(\theta)) \pi(d\theta)}{\int_{\Theta} h(X_n(x) | \tau_n(t)) \varphi_n(\tau_n(t)) \pi(dt)} = \frac{h(X_n(x) | \tau_n(\theta)) \bar{\pi}_n(d\theta)}{\int_{\Theta} h(X_n(x) | \tau_n(t)) \bar{\pi}_n(dt)},$$

where  $\bar{\pi}_n(d\theta) := \frac{\varphi_n(\tau_n(\theta)) \pi(d\theta)}{\int_{\Theta} \varphi_n(\tau_n(t)) \pi(dt)}$ , provided that the denominator is finite. Then, if [Theorem 2.1](#) can be applied in terms of the new model  $\tilde{h}$  and prior  $\bar{\pi}_n$ , the thesis (1) reads

$$d_{TV}(\pi_n(\cdot | x), \pi_n(\cdot | y)) \leq L_w(K) w(d_{\mathbb{X}^k}(X_n(x), X_n(y))) \quad \forall x, y \in K,$$

where  $d_{\mathbb{X}^k}$  denotes the product distance. We stress that a bound in terms of  $d_{\mathbb{X}^k}(X_n(x), X_n(y))$  is statistically more meaningful than a bound in terms of  $d_{\mathbb{X}^n}(x, y)$ , as the former agrees with the symmetry assumption coming from exchangeability.

### 3.3. Global regularity for models with $\mathbb{X} \subset \mathbb{R}^d$

When  $\lambda(\mathbb{X}) < +\infty$ , we can check whether [Theorem 2.1](#) holds with  $K = \mathbb{X}$ , yielding a global uniform continuity. We discuss the case  $\mathbb{X} \subset \mathbb{R}^d$  with  $\lambda = \mathcal{L}^d$ , the  $d$ -dimensional Lebesgue measure, forcing  $\mathbb{X}$  to be bounded. Many popular models do not satisfy the assumptions of [Theorem 2.1](#) with  $K = \mathbb{X}$ , but only with some  $K$  compactly contained in the interior of  $\mathbb{X}$ . This is the case of Beta and Dirichlet models. On the other hand, a model that fits the assumptions of [Corollary 2.3](#) is the Bradford distribution, given by

$$f(x | \theta) := \frac{\theta}{(1 + \theta x) \log(1 + \theta)}$$

with  $x \in \mathbb{X} := [0, 1]$  and  $\theta \in \Theta := (-1, +\infty)$ . Such a model is used for the description of the occurrences of references in a set of documents on the same subject (Leimkuhler, 1967). Choosing

$$g(\theta) := \begin{cases} \frac{\theta}{\log(1 + \theta)} & \text{if } \theta \in (-1, 0] \\ \frac{\theta}{(1 + \theta)\log(1 + \theta)} & \text{if } \theta \in (0, +\infty), \end{cases}$$

Corollary 2.3 entails global Lipschitz-continuity, i.e. (1) with  $L_w(\mathbb{X}) = \frac{\int_{\Theta} C_1(\theta)\pi(d\theta)}{\int_{\Theta} g(\theta)\pi(d\theta)}$  and  $w(r) = r$ , where  $C_1(\theta) := \sup_{x \in \mathbb{X}} |\partial_x f(x | \theta)|$ , provided that  $\pi$  satisfies  $\int_{\Theta} C_1(\theta)\pi(d\theta) < +\infty$ .

### 3.4. Infinite-dimensional models

One of the merits of our approach consists in the fact that we can handle also complex statistical models of non parametric type. Two noteworthy examples in Bayesian analysis are the infinite dimensional exponential family and the infinite mixture models. See Giné and Nickl (2016) for a comprehensive treatment.

As for the first model, keeping in mind the Karhunen–Loève theorem (Kallenberg, 2002, Chapter 13), we confine ourselves to considering densities of the form  $f(x | \theta) = e^{\theta(x)} \left( \int_0^T e^{\theta(y)} dy \right)^{-1}$  with  $\mathbb{X} = [0, T]$ , with a fixed  $T > 0$  and  $\Theta = C([0, T]; \mathbb{R})$ . See also (Lenk, 1991). After fixing a prior  $\pi$ , we show how Theorem 2.1 can be applied. First, we deal with the condition on the infimum of  $\rho$  with  $K = \mathbb{X}$ . In fact, we have  $\inf_{x \in \mathbb{X}} f(x | \theta) \geq \frac{1}{T} e^{-r_T(\theta)}$ , where  $r_T(\theta) := \sup_{x \in \mathbb{X}} \theta(x) - \inf_{x \in \mathbb{X}} \theta(x)$  denotes the range of the (random) trajectory  $\theta$ . Whence,

$$R(\mathbb{X}) \geq \frac{1}{T} \int_{\Theta} e^{-r_T(\theta)} \pi(d\theta) = \frac{1}{T} \int_0^{+\infty} e^{-s} F_{r_T}(s) ds > 0,$$

where  $F_{r_T}$  stands for the density of  $r_T$  with respect to the Lebesgue measure. To check (2), we consider a Hölder condition with exponent  $\gamma \in (0, 1)$ . Thus, we note that  $|f(\cdot | \theta)|_{C^{0,\gamma}(\mathbb{X})} \leq \frac{1}{T} e^{r_T(\theta)} |\theta|_{C^{0,\gamma}(\mathbb{X})}$ . By Hölder’s inequality, for  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we write

$$\int_{\Theta} |f(\cdot | \theta)|_{C^{0,\gamma}(\mathbb{X})} \pi(d\theta) \leq \frac{1}{T} \left( \int_{\Theta} |\theta|_{C^{0,\gamma}([0,T])}^p \pi(d\theta) \right)^{\frac{1}{p}} \left( \int_0^{+\infty} e^{qs} F_{r_T}(s) ds \right)^{\frac{1}{q}}.$$

By the fractional Sobolev inequality (6), for  $s = \frac{\gamma p + 1}{p}$  we have

$$\begin{aligned} \int_{\Theta} |\theta|_{C^{0,\gamma}([0,T])}^p \pi(d\theta) &\leq C_{s,1,p}^p(\mathbb{X}) \int_{\Theta} \int_0^T \int_0^T \frac{|\theta(x) - \theta(y)|^p}{|x - y|^{1+sp}} dx dy \pi(d\theta) \\ &= C_{s,1,p}^p(\mathbb{X}) \int_0^T \int_0^T \frac{1}{|x - y|^{1+sp}} \left( \int_{\mathbb{R}^2} |u - v|^p h_{x,y}(u, v) du dv \right) dx dy, \end{aligned} \tag{7}$$

where  $h_{x,y}$  is the two-times density of  $\theta$ . Typically, the Kolmogorov–Chentsov condition (Kallenberg, 2002, Chapter 3)

$$\int_{\mathbb{R}^2} |u - v|^p h_{x,y}(u, v) du dv \leq Q(p, \lambda) |x - y|^{1+\lambda} \tag{8}$$

holds for some  $\lambda > 0$  and some  $Q(p, \lambda) > 0$ . See, e.g., Kallenberg (2002). If (8) is verified for  $\lambda > sp - 1 = \gamma p$ , then the last term of (7) is finite. Summing up, if there exists  $q > 1$  such that  $Z_q(r_T) := \left( \int_0^{+\infty} e^{qs} F_{r_T}(s) ds \right)^{\frac{1}{q}} < +\infty$ , and if (8) holds for some  $\lambda > 0$  and  $p = q/(q - 1)$ , then, as soon as  $\gamma < \min\{1 - \frac{1}{p}, \frac{\lambda}{p}\}$ , (1) holds with  $w(r) = r^\gamma$  and

$$A_{w,f,\pi,\mathbb{X}} = \frac{1}{T} C_{s,1,p}(\mathbb{X}) Z_q(r_T) \left( Q(p, \lambda) \int_0^T \int_0^T |x - y|^{\lambda - \gamma p - 1} dx dy \right)^{\frac{1}{p}}.$$

The case of  $\pi$  equal to the Wiener measure deserves some attention. Indeed, (8) is satisfied with  $\lambda = p/2 - 1$  and  $p > 2$  (Kallenberg, 2002, Chapter 13). Moreover, we have (see Feller (1951))

$$F_{r_T}(s) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{+\infty} (-1)^k k^2 \exp \left\{ -\frac{k^2 s^2}{2T} \right\},$$

yielding  $Z_q(r_T) < +\infty$  for all  $q \in \mathbb{R}$ . Therefore, we obtain Hölder continuity of the posterior distribution for any exponent  $\gamma < \frac{1}{2}$ .

The second model of interest, namely the so-called infinite dimensional mixture model, is based on a family of densities of the form  $f(x | \theta) = \int_{\mathbb{R}} \kappa(x; t) \theta(dt)$ , with  $\mathbb{X} = \mathbb{R}$  and  $\Theta$  equal to the space of all probability measures  $\mathcal{P}(\mathbb{R})$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

The kernel  $\kappa$  consists of a family of densities (in the  $x$ -variable) parametrized by  $t \in \mathbb{R}$ . A noteworthy case of interest is the Gaussian kernel  $\kappa(x; t) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x-t)^2\}$ . Now, after fixing a prior  $\pi$  of nonparametric type (e.g. the Ferguson–Dirichlet prior), the application of [Theorem 2.1](#) is straightforward. First, for kernels in the form  $\kappa(x; t) = \kappa(x-t)$ , condition [\(2\)](#) holds even independently of  $\pi$ , provided that  $\sup_{x \in \mathbb{R}} |\kappa'(x)| < +\infty$ . For the condition on  $R(K)$ , for some compact  $K \subset \mathbb{R}$ , it is enough to assume that  $\inf_{x \in K} \int_{\mathbb{R}} \kappa(x; t) \bar{\theta}(dt) > 0$ , where  $\bar{\theta}(B) := \int_{\mathcal{P}(\mathbb{R})} \theta(B) \pi(d\theta)$ ,  $B \in \mathcal{B}(\mathbb{R})$ .

### 3.5. Application to Bayesian consistency

We have seen in [Section 3.2](#) that, in presence of exchangeable observations, the posterior can be written as

$$\pi_n(d\theta | x) = \frac{\prod_{i=1}^n \tilde{f}(x_i | \theta)}{\int_{\Theta} \prod_{i=1}^n \tilde{f}(x_i | t) \pi(dt)} \pi(d\theta) = \frac{\exp\{n \int_{\mathbb{X}} \log \tilde{f}(y | \theta) \epsilon_n^x(dy)\}}{\int_{\Theta} \exp\{n \int_{\mathbb{X}} \log \tilde{f}(y | t) \epsilon_n^x(dy)\} \pi(dt)} \pi(d\theta)$$

where  $x = (x_1, \dots, x_n)$  and  $\epsilon_n^x(\cdot) := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}(\cdot)$  denotes the *empirical measure*. In the theory of Bayesian consistency, one fixes  $\theta_0 \in \Theta$  and generates from  $\nu(\cdot | \theta_0)$  a sequence  $\{\xi_i\}_{i \geq 1}$  of i.i.d. random variables. The objective is to prove that the posterior piles up near the true value  $\theta_0$ , i.e.  $d(\pi(\cdot | \xi_1, \dots, \xi_n), \delta_{\theta_0}) \rightarrow 0$  in probability, for some weak distance  $d$  (e.g., Prokhorov or bounded-Lipschitz metric ([Gibbs and Su, 2002](#))) between probability measures on  $(\Theta, \mathcal{F})$ , possibly with an estimation of the convergence rate. To establish a link with our theory, we introduce the probability kernel

$$\pi_n^*(d\theta | \mu) := \frac{\exp\{n \int_{\mathbb{X}} \log \tilde{f}(y | \theta) \mu(dy) - M_n(\theta)\}}{\int_{\Theta} \exp\{n \int_{\mathbb{X}} \log \tilde{f}(y | t) \mu(dy) - M_n(t)\} \pi(dt)} e^{M_n(\theta) - \bar{M}_n} \pi(d\theta)$$

where  $\mu \in \mathcal{M}$ , a subset of probability measures containing in its closure both  $\nu(\cdot | \theta_0)$  and  $\epsilon_n^{(\xi_1, \dots, \xi_n)}(\cdot)$ , with

$$M_n(\theta) := \log \left\{ \int_{\mathcal{M}} \exp\{n \int_{\mathbb{X}} \log \tilde{f}(y | \theta) \mu(dy)\} \eta(d\mu) \right\}$$

for some measure  $\eta$  on  $\mathcal{M}$ , and  $\bar{M}_n := \log \left\{ \int_{\Theta} e^{M_n(\theta)} \pi(d\theta) \right\}$ . In this notation,  $\pi_n(d\theta | x) = \pi_n^*(d\theta | \epsilon_n^x)$ . Whence,

$$d(\pi(\cdot | \xi_1, \dots, \xi_n), \delta_{\theta_0}) \leq d(\pi_n^*(\cdot | \nu(\cdot | \theta_0)), \delta_{\theta_0}) + d(\pi_n^*(\cdot | \epsilon_n^{(\xi_1, \dots, \xi_n)}), \pi_n^*(\cdot | \nu(\cdot | \theta_0))) .$$

As for the first term on the RHS, convergence to zero is well-known with explicit rates, as a consequence of the so-called Kullback–Leibler property ([Ghosal and van der Vaart, 2017](#), Definition 6.15). The second term on the RHS can be studied under expectation, by splitting it as follows:

$$\mathbb{E}_{\theta_0} [d(\pi_n^*(\cdot | \epsilon_n), \pi_n^*(\cdot | \nu(\cdot | \theta_0))) \mathbb{1}_{E_n}] + \mathbb{E}_{\theta_0} [d(\pi_n^*(\cdot | \epsilon_n), \pi_n^*(\cdot | \nu(\cdot | \theta_0))) \mathbb{1}_{E_n^c}] \quad (9)$$

with  $E_n := \{D(\epsilon_n^{(\xi_1, \dots, \xi_n)}, \nu(\cdot | \theta_0)) \leq \epsilon_n\}$ , where  $\{\epsilon_n\}_{n \geq 1}$  is a vanishing sequence of positive numbers and  $D$  a weak distance (e.g., Prokhorov or bounded-Lipschitz metric ([Gibbs and Su, 2002](#))) between probability measures on  $(\mathbb{X}, \mathcal{X})$ . If the distance  $d$  is bounded, the second term in [\(9\)](#) is handled in terms of  $\mathbb{P}_{\theta_0}[D(\epsilon_n, \nu(\cdot | \theta_0)) > \epsilon_n]$ , and hence resorting to well-known large deviations inequalities for empirical processes ([Kallenberg, 2002](#), Chapter 27). Finally, if  $d \leq d_{TV}$  (see [Gibbs and Su \(2002\)](#)), we can study the first term in [\(9\)](#) by applying [Theorem 2.1](#), with  $K = \{\mu \in \mathcal{M} | D(\mu, \nu(\cdot | \theta_0)) \leq \epsilon_n\}$  and  $w(r) = r^\alpha$  for some  $\alpha \in (0, 1]$ . The role of the local Hölder continuity is now functional to reducing the analysis of the first term in [\(9\)](#) to that of  $\mathbb{E}_{\theta_0}[D(\epsilon_n^{(\xi_1, \dots, \xi_n)}, \nu(\cdot | \theta_0))^\alpha]$ , whose rates of contraction are well-known ([Fournier and Guillin, 2015](#)).

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### References

- Ackerman, N.L., Freer, C.E., Roy, D.M., 2017. On computability and disintegration. *Math. Struct. Comp. Sci.* 27, 1287–1314.
- Brown, L.D., 1986. Fundamentals of Statistical Exponential Families with Application in Statistical Decision Theory. In: Institute of Mathematical Statistics Lecture Notes-Monograph Series, vol. 9, Hayward, California.
- Cotter, S.L., Dashti, M., Robinson, J.C., Stuart, A.M., 2009. Bayesian inverse problems for functions and applications to fluid mechanics. *Inverse Problems* 25, 115008.
- Dashti, M., Stuart, A.M., 2017. The Bayesian approach to inverse problems. In: Ghanem, R., Higdon, D., Owhadi, H. (Eds.), *Handbook of Uncertainty Quantification*. Springer, Cham.
- Demengel, F., Demengel, G., 2012. Functional spaces for the theory of elliptic partial differential equations. In: *Universitext*. Springer, London; EDP Sciences, Les Ulis.
- Diaconis, P., Freedman, D., 1986. On consistency of Bayes estimates. *Ann. Statist.* 14, 1–26.

- Diaconis, P., Ylvisaker, D., 1979. Conjugate priors for exponential families. *Ann. Statist.* 7, 269–281.
- Efron, B., 2016. Empirical Bayes deconvolution estimates. *Biometrika* 103, 1–20.
- Feller, W., 1951. The asymptotic distribution of the range of sums of independent random variables. *Ann. Math. Statist.* 22, 427–432.
- Fournier, N., Guillin, A., 2015. On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* 162, 707–738.
- Ghosal, S., van der Vaart, A., 2017. *Fundamentals of Nonparametric Bayesian Inference*. Cambridge University Press, Cambridge.
- Gibbs, A.L., Su, F.E., 2002. On choosing and bounding probability metrics. *Internat. Statist. Rev.* 70, 419–435.
- Giné, E., Nickl, R., 2016. *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge University Press, Cambridge.
- Iglesias, M.A., Lin, K., Stuart, A.M., 2014. Well-posed Bayesian geometric inverse problems arising in subsurface flow. *Inverse Problems* 30, 114001.
- Kallenberg, O., 2002. *Foundations of Modern Probability*, second ed. Springer-Verlag, New York.
- Latz, J., **On the well-posedness of Bayesian inverse problems**. Preprint arXiv:1902.10257.
- Leimkuhler, F.F., 1967. Bradford's distribution. *J. Doc.* 23 (3), 197–207.
- Lenk, P.J., 1991. Towards a practicable Bayesian nonparametric density estimator. *Biometrika* 78, 531–543.
- Pfanzagl, J., 1979. Conditional distributions as derivatives. *Ann. Probab.* 7, 1046–1050.
- Regazzini, E., Sazonov, V.V., 2001. Approximation of laws of random probabilities by mixtures of Dirichlet distributions with applications to nonparametric bayesian inference. *Theory Probab. Appl.* 45, 93–110.
- Stuart, A.M., 2010. Inverse problems: a Bayesian perspective. *Acta Numer.* 19, 451–559.
- Tjur, T., 1980. *Probability Based on Radon Measures*. In: *Wiley Series in Probability and Mathematical Statistics*, Wiley, Chichester.
- West, M., 1979. Approximating posterior distributions by mixtures. *J. R. Stat. Soc. Ser. B Stat. Methodol.* 55, 409–422.
- Zabell, S.L., 1979. Continuous versions of regular conditional distributions. *Ann. Probab.* 7, 159–165.