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On the Irreducibility of Some Quiver Varieties

Claudio BARTOCCI †[1]†[2], *Ugo BRUZZO* †[3]†[4]†[5]†[6]†[7], *Valeriano LANZA* †[8] *and Claudio L.S. RAVA* †[1]

†[1] *Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy*
E-mail: *bartocci@dima.unige.it*, *clsrava@gmail.com*
URL: http://www.dima.unige.it/~bartocci/index.htm

†[2] *Laboratoire SPHERE, CNRS, Université Paris Diderot (Paris 7), 75013 Paris, France*

†[3] *SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy*
E-mail: *bruzzo@sissa.it*
URL: http://www.people.sissa.it/~bruzzo/webpage/

†[4] *Departamento de Matemática, Universidade Federal da Paraíba, Campus I, João Pessoa, PB, Brasil*

†[5] *IGAP (Institute for Geometry and Physics), Trieste, Italy*

†[6] *INFN (Istituto Nazionale di Fisica Nucleare), Sezione di Trieste, Italy*

†[7] *Arnold-Regge Center for Algebra, Geometry and Theoretical Physics, Torino, Italy*

†[8] *Departamento de Análise, IME, Universidade Federal Fluminense, Rua Professor Marcos Waldemar de Freitas Reis, Niterói, RJ, Brazil*
E-mail: *vlanza@id.uff.br*

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Abstract. We prove that certain quiver varieties are irreducible and therefore are isomorphic to Hilbert schemes of points of the total spaces of the bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$ for $n \geq 1$.

Key words: quiver representations; Hilbert schemes of points

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1 Introduction

Nakajima's quiver varieties were introduced by Hiraku Nakajima in [11] to study the moduli spaces of instantons on ALE spaces, and have been extensively studied since then, see, e.g., [8, 10, 12, 13]. They provide a modern and significant example of how algebra and geometry can be sometimes so deeply, yet surprisingly connected: in fact, their main feature is that they allow one to put in relation some moduli spaces of bundles (or torsion-free sheaves) over certain smooth projective varieties with some moduli spaces of representations of suitable algebras (the so-called *path algebras* of a quiver and quotients of them). A major example of this bridge is given by the moduli space of framed sheaves on $\mathbb{P}^2$, which can be identified with the moduli space of semistable representations of the ADHM quiver (see [12] for details).

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The way this relation is usually looked at is the one that inspired Nakajima's first pioneering work: the philosophy is to use the algebraic data we get on one side (usually called *ADHM data*) to parameterize the geometric moduli spaces we have on the other side, i.e., the objects we are actually interested in (see for example [4, 6, 14]). But sometimes it may be useful to switch roles and use the geometric interpretation as a "tool" to prove something interesting per se on the algebraic side. For instance, this is the case when one deals with irreducibility problems: to determine whether a variety of matrices is irreducible is known to be a challenging problem (see [15] and references therein), and in the specific case of Nakajima's quiver varieties the conclusive result by Crawley-Boevey stating that *all of them* are indeed irreducible has been achieved only by using hyperkähler geometry techniques [5].

In [1] we introduced a collection of new quiver varieties $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$, $n \geq 1$ (see below for the notation); for $n \neq 2$ they are not Nakajima's quiver varieties, as the quivers involved are not doubles. We proved that $\mathcal{M}(\Lambda_1, \vec{v}_c, w_c, \vartheta_c)$ is isomorphic to the Hilbert scheme of points of the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$, and, in particular, that it is therefore irreducible (as the Hilbert scheme is so [7]). For $n \geq 2$ we only proved a weaker result, i.e., that only a certain connected component of $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$ can be identified with $\mathrm{Hilb}^c(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$. However, as $\mathcal{M}(\Lambda_2, \vec{v}_c, w_c, \vartheta_c)$ is a Nakajima quiver variety, its irreducibility follows from Crawley-Boevey's result, so that one only has to determine whether the varieties $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$ are irreducible for $n \geq 3$. In this paper we prove this fact, completing the work of [1], actually showing directly that $\mathrm{Hilb}^c(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$ is isomorphic to the whole $\mathcal{M}(\Lambda_n, \vec{v}_c, w_c, \vartheta_c)$. As this technique also works for the case $n = 2$ we include it as well.

2 Some background

The quivers we are going to consider are extracted from the ADHM data for the Hilbert schemes of points of the varieties $\mathrm{Hilb}^c(X_n)$, where X_n is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$, and, in turn, the construction of the ADHM data is based on the description of the moduli spaces of framed sheaves on the Hirzebruch surfaces Σ_n in terms of monads that was given in [2]. We denote by H and E the classes in $\mathrm{Pic}(\Sigma_n)$ of the sections of the natural ruling $\Sigma_n \to \mathbb{P}^1$ that square to n and $-n$, respectively. We fix a curve ℓ_∞ in Σ_n belonging to the class H (the "line at infinity"). A framed sheaf on Σ_n is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a rank r torsion-free sheaf which is trivial along ℓ_∞, and $\theta\colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism. A morphism between framed sheaves $(\mathcal{E}, \theta)$, $(\mathcal{E}', \theta')$ is by definition a morphism $\Lambda\colon \mathcal{E} \longrightarrow \mathcal{E}'$ such that $\theta' \circ \Lambda|_{\ell_\infty} = \theta$. The moduli space parameterizing isomorphism classes of framed sheaves $(\mathcal{E}, \theta)$ on Σ_n with Chern character $\mathrm{ch}(\mathcal{E}) = \left(r, aE, -c - \frac{1}{2}na^2\right)$, where $r, a, c \in \mathbb{Z}$ and $r \geq 1$, will be denoted $\mathcal{M}^n(r, a, c)$. We normalize the framed sheaves so that $0 \leq a \leq r - 1$.

A monad M on a scheme X is a three-term complex of locally free $\mathcal{O}_X$-modules of finite rank, having nontrivial cohomology only at the middle term (cf. [16, Definition II.3.1.1]). It was proved in [2] that a framed sheaf $(\mathcal{E}, \theta)$ on Σ_n with invariants (r, a, c) is the cohomology of a monad

$$M(\alpha, \beta)\colon \quad 0 \longrightarrow \mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}} \longrightarrow 0\,, \tag{2.1}$$

where $\vec{k}$ is the quadruple (n, r, a, c), and

$$\mathcal{U}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(0, -1)^{\oplus k_1}, \qquad \mathcal{V}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1, -1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4}, \qquad \mathcal{W}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1, 0)^{\oplus k_3},$$

with

$$k_1 = c + \frac{1}{2}na(a-1), \qquad k_2 = k_1 + na, \qquad k_3 = k_1 + (n-1)a, \qquad k_4 = k_1 + r - a.$$

The space $L_{\vec{k}}$ of pairs in $\mathrm{Hom}(\mathcal{U}_{\vec{k}},\mathcal{V}_{\vec{k}})\oplus\mathrm{Hom}(\mathcal{V}_{\vec{k}},\mathcal{W}_{\vec{k}})$ fitting into (2.1), such that the cohomology of the complex is torsion-free and trivial at infinity, is a smooth algebraic variety. There is a principal $\mathrm{GL}(r,\mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$ whose fibre at a point (α,β) is the space of framings for the corresponding cohomology of (2.1). The algebraic group

$$G_{\vec{k}}=\mathrm{Aut}(\mathcal{U}_{\vec{k}})\times\mathrm{Aut}(\mathcal{V}_{\vec{k}})\times\mathrm{Aut}(\mathcal{W}_{\vec{k}})$$

acts freely on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^n(r,a,c)$ is the quotient $P_{\vec{k}}/G_{\vec{k}}$ [2, Theorem 3.4]. This is nonempty if and only if $c+\frac{1}{2}na(a-1)\geq 0$, and when nonempty, it is a smooth algebraic variety of dimension $2rc+(r-1)na^2$.

When $r=1$ we can assume $a=0$, and there is an identification

$$\mathcal{M}^n(1,0,c)\simeq\mathrm{Hilb}^c(\Sigma_n\setminus\ell_\infty)=\mathrm{Hilb}^c(X_n).$$

A first step to construct ADHM data for the Hilbert schemes of points of the varieties X_n is to show that the Hilbert schemes can be covered by open subsets that are isomorphic to the Hilbert scheme of $\mathbb{C}^2$, and therefore have an ADHM description, according to Nakajima. Then one proves that these "local ADHM data" can be glued to provide ADHM data for the Hilbert schemes of X_n.

Let $P^n(c)$ be the set of collections $(A_1,A_2;C_1,\ldots,C_n;e)$ in $\mathrm{End}(\mathbb{C}^c)^{\oplus n+2}\oplus\mathrm{Hom}(\mathbb{C}^c,\mathbb{C})$ satisfying the conditions

$$\text{(P1)}\quad\begin{cases}A_1C_1A_2=A_2C_1A_1, & & \text{when } n=1,\\ A_1C_q=A_2C_{q+1}, & \text{for}\quad q=1,\ldots,n-1, & \text{when } n>1;\\ C_qA_1=C_{q+1}A_2 & & \end{cases}$$

(P2) $A_1+\lambda A_2$ is a *regular pencil* of matrices, i.e., there exists $[\nu_1,\nu_2]\in\mathbb{P}^1$ such that $\det(\nu_1A_1+\nu_2A_2)\neq 0$;

(P3) for all values of the parameters $([\lambda_1,\lambda_2],(\mu_1,\mu_2))\in\mathbb{P}^1\times\mathbb{C}^2$ satisfying

$$\lambda_1^n\mu_1+\lambda_2^n\mu_2=0$$

there is no nonzero vector $v\in\mathbb{C}^c$ such that

$$\begin{cases}C_1A_2v=-\mu_1v,\\ C_nA_1v=(-1)^n\mu_2v,\\ v\in\ker e\end{cases}\qquad\text{and}\qquad(\lambda_2A_1+\lambda_1A_2)\,v=0.$$

The group $\mathrm{GL}(c,\mathbb{C})\times\mathrm{GL}(c,\mathbb{C})$ acts on $P^n(c)$ according to

$$(A_i,C_j,e)\mapsto\left(\phi_2A_i\phi_1^{-1},\phi_1C_j\phi_2^{-1},e\phi_1^{-1}\right)$$

for $i=1,2$, $j=1,\ldots,n$, $(\phi_1,\phi_2)\in\mathrm{GL}(c,\mathbb{C})\times\mathrm{GL}(c,\mathbb{C})$.

The following result expresses the fact that the collections $(A_1,A_2;C_1,\ldots,C_n;e)$ satisfying conditions (P1) to (P3) are ADHM data for the varieties $\mathrm{Hilb}^c(X_n)$ (this is Theorem 3.1 in [2]).

Theorem 2.1. *$P^n(c)$ is a principal $\mathrm{GL}(c,\mathbb{C})\times\mathrm{GL}(c,\mathbb{C})$-bundle over $\mathrm{Hilb}^c(X_n)$.*

3 The main result

Now we turn to the purpose of this paper, namely, proving that the Hilbert schemes of points of the varieties X_n are isomorphic to moduli spaces of representations of suitable quivers. For

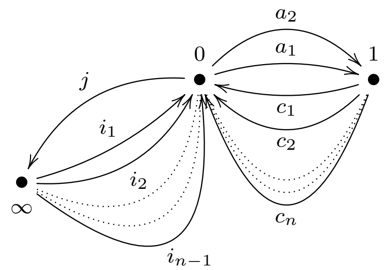

Figure 1. The quivers Q_n.

any $n \geq 2$ let Q_n be the framed quiver in Fig. 1, where ∞ is the framing vertex. Let J_n be the two sided ideal of $\mathbb{C}Q_n$ generated by the relations

$$\begin{cases} a_2 c_{q+1} - a_1 c_q = 0, \\ c_{q+1} a_2 - c_q a_1 - i_q j = 0 \end{cases} \qquad \text{for} \quad q = 1, \dots, n-1. \tag{3.1}$$

Our purpose is to describe the spaces of representations of the quiver Q_n with relations J_n, i.e., the spaces of representations of the quotient algebra $\Lambda_n = \mathbb{C}Q_n/J_n$.

We recall some basic definitions. Given $\vec{v} = (v_0, v_1) \in \mathbb{N}^2$ and $w \in \mathbb{N}$, a $(\vec{v}, w)$-*dimensional representation of* Λ_n is the datum of a triple of $\mathbb{C}$-vector spaces V_0, V_1, W, with $\dim V_i = v_i$, $\dim W = w$, and of an element $(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1})$ in

$$\mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_1, V_0)^{\oplus n} \oplus \mathrm{Hom}_{\mathbb{C}}(V_0, W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_0)^{\oplus n-1}$$

satisfying the relations determined by equations (3.1), namely

$$\begin{cases} A_2 C_{q+1} - A_1 C_q = 0, \\ C_{q+1} A_2 - C_q A_1 - f_q e = 0 \end{cases} \qquad \text{for} \quad q = 1, \dots, n-1. \tag{Q1}$$

The space $\mathrm{Rep}(\Lambda_n, \vec{v}, w)$ of all $(\vec{v}, w)$-dimensional representations of Λ_n is an affine variety, on which the group $G_{\vec{v}} = \mathrm{GL}(v_0, \mathbb{C}) \times \mathrm{GL}(v_1, \mathbb{C})$ acts by basis change. Indeed, we ignore the action of $\mathrm{GL}(w, \mathbb{C})$ on the vector space W attached to the framing vertex. As usual, to get a well behaved quotient space one has to perform a GIT construction by introducing a suitable notion of stability. This was done by A. King [9] and, in a slightly different way, by A. Rudakov [17]. In the case of a quiver with a framing vertex, the following definition can be shown to be equivalent to the King–Rudakov one [3, 5].

Definition 3.1. Fix $\vartheta \in \mathbb{R}^2$. A $(\vec{v}, w)$-dimensional representation (V_0, V_1, W) of Λ_n is said to be ϑ-*semistable* if, for any subrepresentation $S = (S_0, S_1) \subseteq (V_0, V_1)$, one has:

$$\text{if } S_0 \subseteq \ker e, \text{ then } \vartheta \cdot (\dim S_0, \dim S_1) \leq 0; \tag{3.2}$$

$$\text{if } S_0 \supseteq \mathrm{Im}\, f_i \text{ for } i = 1, \dots, n-1, \text{ then } \vartheta \cdot (\dim S_0, \dim S_1) \leq \vartheta \cdot (v_0, v_1). \tag{3.3}$$

A ϑ-semistable representation is ϑ-*stable* if a strict inequality holds in (3.2) whenever $S \neq 0$ and in (3.3) whenever $S \neq (V_0, V_1)$.

Let $\mathrm{Rep}(\Lambda_n, \vec{v}, w)^{\mathrm{ss}}_{\vartheta}$ be the subset of $\mathrm{Rep}(\Lambda_n, \vec{v}, w)$ consisting of ϑ-semistable representations. By [9, Proposition 5.2], the coarse moduli space of $(\vec{v}, w)$-dimensional ϑ-semistable representations of Λ_n is the GIT quotient

$$\mathcal{M}(\Lambda_n, \vec{v}, w, \vartheta) = \mathrm{Rep}(\Lambda_n, \vec{v}, w)^{\mathrm{ss}}_{\vartheta} /\!/ G_{\vec{v}}.$$

It can be proved that the open subset $\mathcal{M}^{\mathrm{s}}(\Lambda_n,\vec{v},w,\vartheta)\subset\mathcal{M}(\Lambda_n,\vec{v},w,\vartheta)$ consisting of stable representations makes up a fine moduli space. Notice that, for quivers without a framing, this holds only when the dimension vector is primitive [9, Proposition 5.3], whilst this requirement is not necessary in the case of framed quivers [5]. Theorem 4.5 of [1] states that the Hilbert scheme of points $\mathrm{Hilb}^c(X_n)$ can be embedded into $\mathcal{M}(\Lambda_n,\vec{v},w,\vartheta)$ for suitable choices of $\vec{v}$, w, and ϑ. Precisely, one has the following result:

Theorem 3.2. *For every $n\geq 2$ and $c\geq 1$ let*

$$\vec{v}_c=(c,c),\qquad w_c=1,\qquad \vartheta_c=(2c,1-2c),$$

and let $\mathcal{H}(n,c)$ be the irreducible component of $\mathcal{M}(\Lambda_n,\vec{v}_c,1,\vartheta_c)$ given by the equations

$$f_1=f_2=\cdots=f_{n-1}=0. \tag{3.4}$$

Then $\mathrm{Hilb}^c(X_n)\simeq\mathcal{H}(n,c)$.

Let $\mathrm{pr}\colon\ \mathrm{Rep}(\Lambda_n,\vec{v}_c,1)^{\mathrm{ss}}_{\vartheta_c}\to\mathcal{M}(\Lambda_n,\vec{v}_c,1,\vartheta_c)$ be the quotient map. The proof of Theorem 3.2 basically consists in proving that the counterimage $\mathrm{pr}^{-1}(\mathcal{H}(n,c))=:Z_n(c)$ coincides with the total space of the principal fibration $P^n(c)$ we introduced in Section 2. As it is quite involved and requires a few intermediate Lemmas and Propositions, we refer the reader to [1] for further details. Here we only note that the starting point is given by the stability conditions in Definition 3.1.

Remark 3.3. The set of $(\vec{v}_c,w_c)$-dimensional representations of Λ_n which are semistable according to Definition 3.1 does not change if we let the stability parameter vary inside the open cone

$$\Gamma_c=\left\{\vartheta=(\vartheta_0,\vartheta_1)\in\mathbb{R}^2\,|\,\vartheta_0>0,\,-\vartheta_0<\vartheta_1<-\frac{c-1}{c}\vartheta_0\right\}.$$

It can be shown that for any stability parameter $\bar{\vartheta}$ on the open rays

$$R_1=\left\{(\vartheta_0,\vartheta_1)\in\mathbb{R}^2\,|\,\vartheta_0>0,\,\vartheta_0+\vartheta_1=0\right\},$$
$$R_2=\left\{(\vartheta_0,\vartheta_1)\in\mathbb{R}^2\,|\,\vartheta_0>0,\,(c-1)\vartheta_0+c\vartheta_1=0\right\}$$

there exist representations which are $\bar{\vartheta}$-semistable, but not ϑ_c-semistable. So, Γ_c is a chamber in the space $\mathbb{R}^2_{(\vartheta_0,\vartheta_1)}$ of stability parameters and the closed rays $\overline{R_1}$, $\overline{R_2}$ are its walls. Furthermore, inside Γ_c semistability and stability are equivalent (cf. [1, Lemma 4.7]): in particular, points in $\mathcal{M}(\Lambda_n,\vec{v}_c,1,\vartheta_c)$ can be thought of as $G_{\vec{v}_c}$-orbits of representations in $\mathrm{Rep}(\Lambda_n,\vec{v}_c,1)$.

A full description of the chamber/wall decomposition of the space $\mathbb{R}^2_{(\vartheta_0,\vartheta_1)}$ will be the object of a future work.

We wish to prove that the component $\mathcal{H}(n,c)$ of $\mathcal{M}(\Lambda_n,\vec{v}_c,1,\vartheta_c)$ introduced in Theorem 3.2 coincides with the whole moduli space $\mathcal{M}(\Lambda_n,\vec{v}_c,1,\vartheta_c)$ (this will be Theorem 3.8). Let us introduce the following notation

$$\mathcal{R}(\Lambda_n,c)=\mathrm{Rep}(\Lambda_n,\vec{v}_c,1);\qquad \mathcal{R}^{\mathrm{ss}}(\Lambda_n,c)=\mathrm{Rep}(\Lambda_n,\vec{v}_c,1)^{\mathrm{ss}}_{\vartheta_c}.$$

Given a representation $(A_1,A_2;C_1,\dots,C_n;e;f_1,\dots,f_{n-1})\in\mathcal{R}(\Lambda_n,c)$, we form the pencil $A_1+\lambda A_2$, with $\lambda\in\mathbb{C}$. We recall that a pencil $A_1+\lambda A_2$ is *regular* if there is a point $[\nu_1,\nu_2]\in\mathbb{P}^1$ such that $\det(\nu_1A_1+\nu_2A_2)\neq 0$.

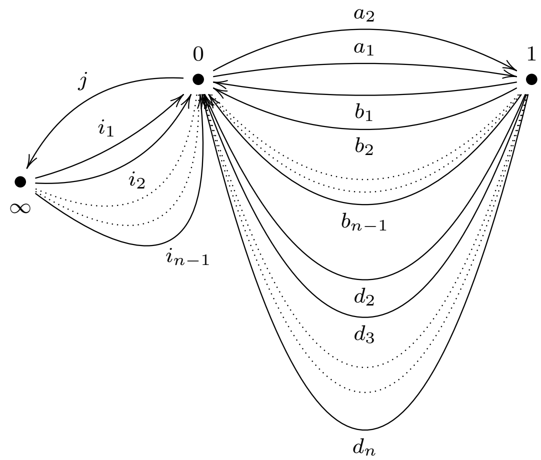

Figure 2. The quivers Q'_n for $n \geq 3$.

To prove Theorem 3.8 it is convenient to introduce "augmented" framed quivers defined as follows: let $Q'_2 = Q_2$, and, for every $n \geq 3$, let Q'_n be the framed quiver in Fig. 2. Let $J'_2 = J_2$ and, for all $n \geq 3$, let J'_n be the two sided ideal of $\mathbb{C}Q'_n$ generated by the relations

$$\begin{cases} a_2 d_{q+1} - a_1 b_q = 0, \\ d_{q+1} a_2 - b_q a_1 - i_q j = 0 \end{cases} \qquad \text{for} \quad q = 1, \dots, n-1. \tag{3.5}$$

We set $\Lambda'_n = \mathbb{C}Q'_n / J'_n$ for all $n \geq 2$. Notice that $\Lambda'_2 = \Lambda_2$; for $n \geq 3$, the algebra Λ_n can be obtained by taking the quotient of Λ'_n by a suitable ideal. Indeed, let K_n be the two sided ideal of Λ'_n generated by the relations

$$\bar{b}_q = \bar{d}_q \qquad \text{for} \quad q = 2, \dots, n-1, \tag{3.6}$$

where $\bar{x}$ is the class in Λ'_n of the element $x \in \mathbb{C}Q'_n$. Let $\tilde{p}_n \colon \mathbb{C}Q'_n \longrightarrow \mathbb{C}Q_n$ be the $\mathbb{C}$-algebra morphism determined by the assignments

$$\tilde{p}_n(a_q) = a_q, \qquad \tilde{p}_n(b_q) = c_q, \qquad \tilde{p}_n(d_q) = c_q, \qquad \tilde{p}_n(j) = j, \qquad \tilde{p}_n(i_q) = i_q. \tag{3.7}$$

It is straightforward that $\tilde{p}_n$ is surjective and that its kernel is the two sided ideal $L_n \subset \mathbb{C}Q'_n$ generated by the relations

$$b_q = d_q \qquad \text{for} \quad q = 2, \dots, n-1. \tag{3.8}$$

It follows directly from equation (3.7) that $\tilde{p}_n$ maps the set of generators of J'_n (see equation (3.5)) onto the set of generators of J_n (see equation (3.1)), so that

$$\tilde{p}_n(J'_n) = J_n.$$

Then it is not hard to check that $\tilde{p}_n$ induces a surjective morphism $p_n \colon \Lambda'_n \to \Lambda_n$, whose kernel, by equations (3.6) and (3.8), is

$$\ker p_n = L_n / (L_n \cap J'_n) = K_n.$$

In conclusion, we have proved the following lemma.

Lemma 3.4. *There is an isomorphism of $\mathbb{C}$-algebras $\Lambda'_n / K_n \simeq \Lambda_n$.*

One of the reasons to introduce the augmented quivers Q'_n is that their path algebras carry an action of the group $\mathrm{SO}(2,\mathbb{C})$ which descends to the quotient algebra Λ'_n. This action will be instrumental in proving the regularity of the pencil $A_1+\lambda A_2$.

Elements of $\mathrm{SO}(2,\mathbb{C})$ will be denoted by $\nu = \left(\begin{smallmatrix} \nu_1 & \nu_2 \\ -\nu_2 & \nu_1 \end{smallmatrix}\right)$. Given arrows

$$(a_1, a_2; b_1, \dots, b_{n-1}; d_2, \dots, d_n; j; i_1, \dots, i_{n-1})$$

as above and $\nu \in \mathrm{SO}(2,\mathbb{C})$, we set

$$\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \nu \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \begin{pmatrix} b'_q \\ d'_{q+1} \end{pmatrix} = \nu^{-1} \begin{pmatrix} b_q \\ d_{q+1} \end{pmatrix} \qquad \text{for} \quad q = 1, \dots, n-1.$$

The assignment

$$\begin{aligned} &(a_1, a_2; b_1, \dots, b_{n-1}; d_2, \dots, d_n; j; i_1, \dots, i_{n-1}) \\ &\qquad \longmapsto (a'_1, a'_2; b'_1, \dots, b'_{n-1}; d'_2, \dots, d'_n; j; i_1, \dots, i_{n-1}), \end{aligned}$$

induces an action

$$\widetilde{\Phi}_n \colon \ \mathrm{SO}(2,\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}Q'_n),$$

which leaves invariant the generators of the ideal J'_n, that is,

$$\widetilde{\Phi}_n(\nu)\big(J'_n\big) = J'_n.$$

So one has an induced action

$$\Phi_n \colon \ \mathrm{SO}(2,\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\Lambda'_n).$$

We wish now to study the space $\mathrm{Rep}(\Lambda'_n, \vec{v}_c, 1) = \mathcal{R}(\Lambda'_n, c)$ of (c,c)-dimensional framed representations of Λ'_n and its open subset $\mathrm{Rep}(\Lambda'_n, \vec{v}_c, 1)^{\mathrm{ss}}_{\vartheta_c} = \mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c)$ of ϑ_c-semistable representations (defined analogously to Definition 3.1). For $n=2$ there is nothing new, since $\mathcal{R}(\Lambda'_2, c) = \mathcal{R}(\Lambda_2, c)$ and $\mathcal{R}^{\mathrm{ss}}(\Lambda'_2, c) = \mathcal{R}^{\mathrm{ss}}(\Lambda_2, c)$. For $n \geq 3$, $\mathcal{R}(\Lambda'_n, c)$ is the affine subvariety of the vector space

$$\mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_1, V_0)^{\oplus 2n-2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_0, W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_0)^{\oplus n-1}$$

whose points $(A_1, A_2; B_1, \dots, B_{n-1}; D_2, \dots, D_n, e; f_1, \dots, f_{n-1})$ satisfy the relations determined by equations (3.5), namely,

$$\begin{cases} A_2 D_{q+1} = A_1 B_q, \\ D_{q+1} A_2 = B_q A_1 + f_q e \end{cases} \qquad \text{for} \quad q = 1, \dots, n-1. \tag{Q1$'$}$$

Lemma 3.5. *$\mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c)$ is the open subset of $\mathcal{R}(\Lambda'_n, c)$ determined by the conditions:*

(Q2$'$) *for all subrepresentations $S = (S_0, S_1)$ such that $S_0 \subseteq \ker e$, one has $\dim S_0 \leq \dim S_1$, and, if $\dim S_0 = \dim S_1$, then $S = 0$;*

(Q3$'$) *for all subrepresentations $S = (S_0, S_1)$ such that $S_0 \supseteq \mathrm{Im}\, f_i$, for $i = 1, \dots, n-1$, one has $\dim S_0 \leq \dim S_1$.*

Proof. Given a subrepresentation (S_0, S_1), we set $s_i = \dim S_i$, $i = 0, 1$. By substituting the definitions of $\vec{v}_c$ and ϑ_c given in Theorem 3.2 into equations (3.2) and (3.3) one gets

$$\text{if } S_0 \subseteq \ker e, \text{ then } s_0 \leq s_1 - \frac{s_1}{2c}; \tag{3.9}$$

$$\text{if } S_0 \supseteq \mathrm{Im}\, f_i \text{ for } i = 1, \dots, n-1, \text{ then } s_0 \leq s_1 + \frac{1}{2} - \frac{s_1}{2c}. \tag{3.10}$$

Whenever $s_1 > 0$, one has $0 < \frac{s_1}{2c} < 1$; hence, equation (3.9) is equivalent to condition (Q2$'$). On the other hand, as $0 \leq \frac{1}{2} - \frac{s_1}{2c} < \frac{1}{2}$, equation (3.10) is equivalent to condition (Q3$'$). ∎

Proposition 3.6. *For each point of $\mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c)$ the associated matrix pencil $A_1 + \lambda A_2$ is regular.*

Proof. Let $(A_1, A_2; B_1, \ldots, B_{n-1}; D_2, \ldots, D_n, e; f_1, \ldots, f_{n-1})$ be a point of $\mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c)$, and assume that $A_1 + \lambda A_2$ is singular. If $c = 1$, then $A_1 + \lambda A_2$ is singular if and only if $A_1 = A_2 = 0$. But this implies the subrepresentation $(V_0, 0)$ does not satisfy condition (Q3′). Hence we can assume $c \geq 2$. The fact that the pencil $A_1 + \lambda A_2$ is singular implies that there is a nontrivial element

$$v(\lambda) = \sum_{\alpha=0}^{\varepsilon} (-\lambda)^\alpha v_\alpha \in V_0 \otimes_{\mathbb{C}} \mathbb{C}[\lambda] \tag{3.11}$$

such that

$$(A_1 + \lambda A_2) v(\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{C}. \tag{3.12}$$

By arguing as in the proof of [1, Lemma 4.11], one can show that the minimal degree polynomial solution $v(\lambda)$ for the pencil $A_1 + \lambda A_2$ has necessarily degree $\varepsilon > 0$. Let us inductively define the vector spaces $\{U_i\}_{i \in \mathbb{N}}$ as follows:

$$\begin{cases} U_0 = \langle v_0, \ldots, v_\varepsilon \rangle, \\ U_{2k+1} = A_1(U_{2k}) + A_2(U_{2k}) & \text{for} \quad k \geq 0, \\ U_{2k} = \displaystyle\sum_{q=1}^{n-1} B_q(U_{2k-1}) + \sum_{q=2}^{n} D_q(U_{2k-1}) & \text{for} \quad k \geq 1. \end{cases}$$

Note that each U_j, with j even, is a subspace of V_0, while each U_j, with j odd, is a subspace of V_1. So, if we introduce the subspaces

$$S_0 = \sum_{k=0}^{\infty} U_{2k} \subset V_0, \qquad S_1 = \sum_{k=0}^{\infty} U_{2k+1} \subset V_1,$$

it follows that (S_0, S_1) is a subrepresentation of (V_0, V_1). We will show that this subrepresentation fails to satisfy either condition (Q2′) or condition (Q3′) of Lemma 3.5, so that one gets a contradiction.

By substituting equation (3.11) into equation (3.12) one finds out that

$$\begin{cases} A_1 v_0 = 0, \\ A_1 v_\alpha = A_2 v_{\alpha-1}, & \alpha = 1, \ldots, \varepsilon, \\ A_2 v_\varepsilon = 0, \end{cases} \tag{3.13}$$

so that

$$U_1 = \langle A_1 v_1, \ldots, A_1 v_\varepsilon \rangle = A_1(U_0). \tag{3.14}$$

There are two possible cases, either i) $U_0 \subseteq \ker e$, or ii) $U_0 \not\subseteq \ker e$.

i) If we suppose that $U_0 \subseteq \ker e$, equation (3.13) and condition (Q1′) imply that

$$U_2 = \sum_{q=2}^{n} D_q A_1(U_0).$$

By letting $w_{q,\alpha} = D_q A_1 v_\alpha$, $\alpha = 1, \ldots, \varepsilon$, for each $q = 2, \ldots, n$ we obtain an element

$$(w_{q,1}, \ldots, w_{q,\varepsilon}) \in U_2^{\oplus \varepsilon}$$

such that $\sum_{\alpha=0}^{\varepsilon-1}(-\lambda)^{\alpha}w_{q,\alpha+1}$ is a polynomial solution for the pencil $A_1+\lambda A_2$ of degree $\varepsilon-1$. Since we have supposed ε to be minimal, one has $(w_{q,1},\dots,w_{q,\varepsilon})=0$. From that it is easy to deduce that $U_2=0$ and that $(S_0,S_1)=(U_0,U_1)$. So, since $\ker A_1\cap U_0\neq 0$ by equation (3.13), then equation (3.14) entails that (S_0,S_1) is a subrepresentation violating condition (Q2′).

ii) Suppose now that U_0 is not contained in $\ker e$. So, there is at least one $\gamma\in\{0,\dots,\varepsilon\}$ such that $e(v_\gamma)\neq 0$. Condition (Q1′) implies that

$$\operatorname{Im} f_q=\langle f_q e(v_\gamma)\rangle\subseteq U_2 \qquad \text{for all} \quad q=1,\dots,n-1. \tag{3.15}$$

To simplify computations, we may assume $\gamma=0$ and $e(v_0)=1$. Actually, one checks that the $\mathrm{SO}(2,\mathbb{C})$ action on Λ'_n induces an action on $\mathcal{R}(\Lambda'_n,c)$, which commutes with the $G_{\vec{v}_c}$ action defined on the same space, and therefore it restricts to an $\mathrm{SO}(2,\mathbb{C})$ action on $\mathcal{R}^{\mathrm{ss}}(\Lambda'_n,c)$. Moreover, this action preserves the regularity of the matrix pencil $A_1+\lambda A_2$. An element $\nu=\left(\begin{smallmatrix}\nu_1 & \nu_2\\ -\nu_2 & \nu_1\end{smallmatrix}\right)\in\mathrm{SO}(2,\mathbb{C})$ produces a change of basis

$$(v_0,\dots,v_\varepsilon)\mapsto \nu\cdot(v_0,\dots,v_\varepsilon)=(v'_0,\dots,v'_\varepsilon),$$

so that

$$e(v'_0)=\sum_{\alpha=0}^{\varepsilon}(-\nu_2)^{\alpha}\nu_1^{\varepsilon-\alpha}e(v_\alpha).$$

Since $(e(v_0),\dots,e(v_\varepsilon))\neq(0,\dots,0)$, there is $\nu\in\mathrm{SO}(2,\mathbb{C})$ so that $e(v'_0)\neq 0$. Moreover, $e(v'_0)$ can be assumed to be 1.

Next, by using condition (Q1′) and equation (3.13), along with the identity $A_2(\operatorname{Im} f_q)=\langle A_2 f_q(1)\rangle=\langle A_2 f_q(e(v_0))\rangle$, it is not hard to show that

$$A_2(\operatorname{Im} f_q)\subseteq A_1(U_2) \qquad \text{for all} \quad q=1,\dots,n-1. \tag{3.16}$$

Now we show that

$$U_{2k+1}\subseteq\sum_{l=1}^{k}A_1(U_{2l}) \tag{3.17}$$

for all $k\geq 1$. Assume $k=1$. By using equations (3.13), (Q1′) and condition $e(v_0)=1$, one gets

$$f_q e(v_\alpha)=e(v_\alpha)D_{q+1}A_1v_1$$

for $q=1,\dots,n-1$. Hence, by using equations (3.13) and (Q1′) again one shows that

$$U_2=\sum_{q=2}^{n}D_qA_1(U_0).$$

Then U_3 is spanned by the sets of vectors

$$\{A_1D_qA_1v_\alpha\}_{\substack{q=2,\dots,n\\ \alpha=1,\dots,\varepsilon}}\subseteq A_1(U_2), \qquad \{A_2D_qA_1v_\alpha\}_{\substack{q=2,\dots,n\\ \alpha=1,\dots,\varepsilon}}\subseteq A_2(U_2)$$

and it follows directly from equations (Q1′) that $A_2D_qA_1v_\alpha\in A_1(U_2)$, for $q=2,\dots,n$. So $U_3\subseteq A_1(U_2)$.

Let us now suppose that equation (3.17) holds true for $1\leq k\leq m$, with $m\geq 1$. This means that U_{2m+1} is spanned by vectors of the form A_1w with $w\in U_{2l}$, $l=1,\dots,m$. By noticing that

U_{2m+2} is spanned by vectors of the form B_pA_1w and D_qA_1w', with $w \in U_{2l}$ and $w' \in U_{2l'}$ for $l, l' = 1, \ldots, m$, and by using equation (Q1′) and the inductive hypothesis one finds out that

$$U_{2m+2} \subseteq \sum_{l=1}^{m}\sum_{q=2}^{n} D_qA_1(U_{2l}) + \sum_{q=1}^{n-1} \operatorname{Im} f_q.$$

From this it follows

$$\begin{aligned} A_1(U_{2m+2}) + A_2(U_{2m+2}) = U_{2m+3} \\ &\subseteq A_1(U_{2(m+1)}) + \sum_{l=1}^{m}\sum_{q=2}^{n} A_2D_qA_1(U_{2l}) + \sum_{q=1}^{n-1} A_2(\operatorname{Im} f_q) \\ &\subseteq A_1(U_{2(m+1)}) + A_1(U_2) + \sum_{l=1}^{m}\sum_{q=2}^{n} A_2D_qA_1(U_{2l}), \end{aligned} \tag{3.18}$$

where in the last step equation (3.16) has been used. For $q = 2, \ldots, n$, equation (Q1′) implies that

$$A_2D_qA_1(U_{2l}) \subseteq A_1(U_{2(l+1)}).$$

Thus, from equation (3.18) we may conclude that

$$U_{2(m+1)+1} \subseteq \sum_{l=1}^{m+1} A_1(U_{2l}),$$

so that the inclusion (3.17) is proved.

This and equation (3.14) imply that

$$S_1 = \sum_{k=0}^{\infty} U_{2k+1} \subseteq A_1(U_0) + \sum_{k=1}^{\infty}\sum_{l=1}^{k} A_1(U_{2l}) = \sum_{k=0}^{\infty} A_1(U_{2k}) = A_1\left(\sum_{k=0}^{\infty} U_{2k}\right) = A_1(S_0).$$

But $A_1(S_0) \subseteq S_1$, so that

$$S_1 = A_1(S_0).$$

By equation (3.13), one has $\ker A_1 \cap S_0 \neq 0$, and therefore $\dim S_1 < \dim S_0$. Finally, equation (3.15) implies that the subrepresentation (S_0, S_1) violates condition (Q3′). ■

When $n \geq 3$, there is a map $\mathcal{R}(\Lambda_n, c) \longrightarrow \mathcal{R}(\Lambda'_n, c)$ given by

$$(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \mapsto (A_1, A_2; C_1, \ldots, C_{n-1}; C_2, \ldots, C_n, e; f_1, \ldots, f_{n-1}).$$

This map provides a $G_{\vec{v}_c}$-equivariant isomorphism of $\mathcal{R}(\Lambda_n, c)$ onto the subvariety of $\mathcal{R}(\Lambda'_n, c)$ cut by the equations

$$B_q = D_q \qquad \text{for} \quad q = 2, \ldots, n-1$$

(cf. equations (3.6)). Through this isomorphism $\mathcal{R}(\Lambda_n, c)$ may be regarded as a closed subvariety of $\mathcal{R}(\Lambda'_n, c)$.

Lemma 3.7. *When $n \geq 3$, one has that*

$$\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c) = \mathcal{R}^{\mathrm{ss}}(\Lambda'_n, c) \cap \mathcal{R}(\Lambda_n, c).$$

Proof. Semistability is a numerical condition which is to be checked on the set of all submodules of a given representation. Hence, it is enough to show that for any left Λ_n-module M, an abelian subgroup $N \subset M$ is a left Λ_n-submodule if and only if it is a left Λ'_n-submodule (notice that M has also a natural structure of left Λ'_n-module, induced by restriction of scalars; cf. Lemma 3.4). However, precisely because the algebra Λ_n is a quotient of Λ'_n, the category Λ_n-**mod** is a full subcategory of Λ'_n-**mod**, and this implies in particular that the set of all subobjects of a given Λ_n-module is the same in the two categories. ■

Theorem 3.8. *The component $\mathcal{H}(n,c)$ of the moduli space $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$ defined by equations* (3.4) *coincides with the whole of $\mathcal{M}(\Lambda_n, \vec{v}_c, 1, \vartheta_c)$.*

Proof. For each representation $(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1}) \in \mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$, equations (3.4) hold if and only if the pencil $A_1 + \lambda A_2$ is regular (condition (P2) in Section 2 and in [1]):

- in the proof of Proposition 4.9 of [1] it has been shown that condition (P2) holds in $Z_n(c) = \mathrm{pr}^{-1}(\mathcal{H}(n,c))$; i.e., equations (3.4) imply the regularity of the pencil;
- further on, in the proof of Theorem 4.5 of [1] it has been shown that $Z_n(c)$ actually coincides with the open subset of $\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$ (denoted $\mathcal{R}_n(c)$ in [1]) where condition (P2) is satisfied; i.e., the regularity of the pencil implies equations (3.4).

So, the orbit of a ϑ_c-semistable representation

$$(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1}) \in \mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)$$

lies in $\mathcal{H}(n,c)$ if and only if the pencil $A_1 + \lambda A_2$ is regular. Then the conclusion follows from Proposition 3.6 and Lemma 3.7. ■

4 A remark involving the 2-Kronecker quiver

We want to rephrase Proposition 3.6 is a slightly different way which involves the Kronecker quiver with two arrows Q_K

$$\begin{tikzcd} \overset{0}{\bullet} \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & \overset{1}{\bullet} \end{tikzcd}$$

The new claim, Proposition 4.2, may be regarded as a statement in relative Geometric Invariant Theory.

The vector space of $\vec{v}_c = (c,c)$-dimensional representations of Q_K is the space $\mathrm{Rep}(Q_K, \vec{v}_c) = \mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2}$. Since Definition 3.1 only applies to framed quivers, we need a slightly different notion of semistability. So we recall from [9, 17] that, given $\vartheta \in \mathbb{R}^2$, a $\vec{v}_c$-dimensional representation of Q_K is said to be *ϑ-semistable* if, for any proper nontrivial subrepresentation supported by $(S_0, S_1) \subseteq (V_0, V_1)$, one has

$$\frac{\vartheta \cdot (\dim S_0, \dim S_1)}{\dim S_0 + \dim S_1} \leq \frac{\vartheta \cdot \vec{v}_c}{2c}. \tag{4.1}$$

A ϑ-semistable representation is *ϑ-stable* if strict inequality holds in (4.1).

As in Section 3, we set $\vartheta_c = (2c, 1-2c)$.

Lemma 4.1. *A point $(A_1, A_2) \in \mathrm{Rep}(Q_K, \vec{v}_c)$ is ϑ_c-semistable if and only if the matrix pencil $A_1 + \lambda A_2$ is regular.*

Proof. Let (A_1, A_2) be a representation of Q_K supported by the pair of vector spaces (V_0, V_1), and consider a proper subrepresentation supported by (S_0, S_1). If the stability parameter is $\vartheta_c = (2c, 1-2c)$, the inequality (4.1) is equivalent to

$$\frac{2c \dim S_0 + (1-2c)\dim S_1}{\dim S_0 + \dim S_1} \leq \frac{1}{2},$$

which is in turn equivalent to

$$\dim S_0 \leq \dim S_1. \tag{4.2}$$

It is not hard to show that (4.2) implies

$$\dim(A_1(S) + A_2(S)) \geq \dim S \quad \text{for all vector subspaces } S \subseteq V_0. \tag{4.3}$$

Conversely, if condition (4.3) is satisfied, then, given any subrepresentation supported by $S = (S_0, S_1)$, one has

$$\dim S_1 \geq \dim(A_1(S_0) + A_2(S_0)) \geq \dim S_0.$$

Finally, by [1, Lemma 4.10] condition (4.3) is equivalent to the fact that the matrix pencil $A_1 + \lambda A_2$ is regular. ■

Recall that $\mathcal{R}(\Lambda_n, c)$ is the affine subvariety of

$$\operatorname{Rep}(Q_n, \vec{v}_c, 1) = \operatorname{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} \oplus \operatorname{Hom}_{\mathbb{C}}(V_1, V_0)^{\oplus n} \oplus \operatorname{Hom}_{\mathbb{C}}(V_0, W) \oplus \operatorname{Hom}_{\mathbb{C}}(W, V_0)^{\oplus n-1}$$

defined by equations (Q1). Let us denote by $\pi_n \colon \mathcal{R}(\Lambda_n, c) \to \operatorname{Rep}(Q_K, \vec{v}_c)$ the restriction of the natural projection $\operatorname{Rep}(Q_n, \vec{v}_c, 1) \to \operatorname{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 2} = \operatorname{Rep}(Q_K, \vec{v}_c)$.

As a straightforward consequence of Lemma 4.1, Proposition 3.6 may be rephrased in the following terms.

Proposition 4.2. *Each $(\vec{v}_c, 1)$-dimensional ϑ_c-semistable representation of Λ_n is mapped by π_n to a $\vec{v}_c$-dimensional ϑ_c-semistable representation of Q_K:*

$$\pi_n\big(\mathcal{R}^{\mathrm{ss}}(\Lambda_n, c)\big) \subseteq \operatorname{Rep}(Q_K, \vec{v}_c)^{\mathrm{ss}}_{\vartheta_c}.$$

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