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Generalized Heegner cycles and p-adic L-functions in a quaternionic setting

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Contents

Introduction

In their recent paper [CH], Castella and Hsieh proved vanishing results for Selmer groups associated with Galois representations attached to newforms twisted by Hecke characters of an imaginary quadratic field. These results are obtained under the so-called *Heegner hypothesis* that the imaginary quadratic field satisfies with respect to the level of the modular form. In particular, Castella and Hsieh prove the rank 0 case of the Bloch–Kato conjecture for L-functions of modular forms in their setting.

The key point of the work of Castella and Hsieh is a remarkable link between certain arithmetic objects called *generalized Heegner cycles* that were introduced by Bertolini, Darmon and Prasanna in [BDP] and suitably defined p-adic L-functions, which are instead objects of p-adic analytic nature, interpolating special values of complex L-series.

In this thesis, we extend several of the results of Castella–Hsieh to a quaternionic setting, that is, the setting that arises when one works under a "relaxed" Heegner hypothesis. In particular, we are interested in an analogous link between the quaternionic generalized Heegner cycles defined by Brooks in [Brooks] and suitable p-adic L-functions.

From a broader point of view, L-functions, be they complex or p-adic, are expected to encode a lot of information on the arithmetic objects they are attached to (for examples, abelian varieties, modular forms, Galois representations). Celebrated examples are the class number formula

$$\operatorname{res}_{s=1} \zeta_K(s) \doteq h_K,$$

which relates the residue at $s = 1$ of the Dedekind zeta function associated with a number field K to the class number of K, and the Birch–Swinnerton-Dyer conjecture

$$\operatorname{ord}_{s=1} L(E/K, s) = \operatorname{rank}_{\mathbb{Z}} E(K),$$

which predicts an equality between the order of vanishing at $s = 1$ of the L-function associated with an elliptic curve E over a number field K (the *analytic rank* of E over K) with the rank of the finitely generated Mordell–Weil group $E(K)$ of K-rational points of E (the *algebraic rank* of E over K). The symbol $\doteq$ in the class number formula means that the equality holds up to an explicit non-zero multiplicative factor that is comparatively less important than the main terms.

Birch–Swinnerton-Dyer conjecture and Kolyvagin's method

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and let K be an imaginary quadratic field satisfying the *Heegner hypothesis* relative to N:

- all the primes dividing N split in K.

In a series of landmark papers culminating in [Ko], Kolyvagin proved that if a certain distinguished K-rational point (called a *Heegner point*) on E is non-torsion then $E(K)$

has rank one. In combination with the Gross–Zagier formula for the central derivative of $L(E/K, s)$ ([GZ]) and other sophisticated results of an analytic nature, Kolyvagin's theorem gives a proof of the Birch–Swinnerton-Dyer conjecture for E when the analytic rank of E over $\mathbb{Q}$ is at most 1.

The significance of Kolyvagin's theorem, whose importance can be hardly overestimated, lies not only in its statement but also in the strategy on which its proof is based. More precisely, the proof builds essentially on two fundamental ingredients: *Selmer groups* and the *Euler system* of Heegner points.

In a nutshell, Kolyvagin proved his theorem by measuring the size of some Selmer groups $\mathrm{Sel}_p(E/K)$, which are suitable Galois cohomology groups associated with E/K and primes p, via a method that is based on the formal Euler system properties of the collection of Heegner points on E. These points give a systematic supply of distinguished points on E that are rational over abelian estension of K and satisfy specific conditions of compatibility with respect to trace maps. It is worth emphasizing that the Heegner hypothesis is crucial for this construction.

Bloch–Kato conjecture

To an elliptic curve $E/\mathbb{Q}$ and a prime p we can attach a p-adic Galois representation

$$\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(T_p(E)) \cong \mathrm{GL}_2(\mathbb{Z}_p) \subseteq \mathrm{GL}_2(\mathbb{Q}_p),$$

where $T_p(E)$ is the p-adic Tate module of E. In fact, all the arguments in Kolyvagin's strategy can be formulated in terms of Galois representations. Remarkably, it turns out that the link between Heegner points, Selmer groups, Galois representations and L-functions is a particular instance of a much more general phenomenon.

Namely, with every Galois representation V and every number field K one can associate a Selmer group $\mathrm{Sel}(K, V)$ and an L-function. The Bloch–Kato conjecture ([BK]) predicts a relation of the form

$$\mathrm{Sel}(K, V) \quad \longleftrightarrow \quad L\text{-function associated with } V.$$

Furthermore, as in the Birch–Swinnerton-Dyer conjecture for elliptic curves (or, more generally, abelian varieties), the L-function should control the size of the Selmer group. In fact, the Birch–Swinnerton-Dyer conjecture can be viewed as a special case of the Bloch–Kato conjecture.

In this thesis, we are interested in the Galois representations attached by Deligne to (higher weight) modular forms ([Del]).

Modular forms and Kolyvagin's method for Heegner cycles

Thanks to sophisticated results due to Shimura, Deligne and Serre, it is possible to attach to a large class of modular forms f a two-dimensional Galois representations of the form

$$V_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(F),$$

where $G_{\mathbb{Q}}$ is the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ and F is a suitable p-adic field (by which we mean, as customary, a finite extension of $\mathbb{Q}_p$). These representations are obtained from étale cohomology groups of Kuga–Sato varieties. More precisely, the two dimensional Galois representation V_f of $G_{\mathbb{Q}}$ associated with a modular form f of weight k and level N is obtained as a subspace of the étale cohomology group $H_{\text{ét}}^{k-1}(W_{k-2}, \mathbb{Q}_p)$ that is stable under the action of a certain Hecke algebra. Here W_{k-2} is the Kuga–Sato

variety defined as a smooth compactification of the $(k-2)$-fold fibre product of the universal elliptic curve over the modular curve $X_1(N)$.

In this more general context, the role of Heegner points is played by *Heegner cycles*, which are distinguished algebraic cycles in the Chow group $\mathrm{CH}^{k/2}(W_{k-2})$ of W_{k-2}.

In the case of modular forms of even weight $k \geq 4$, Nekovář built in [Nek92] an Euler system made up of cohomology classes $\{c_n\}_n$, with $c_n \in H^1(K_n, V_f(k/2))$. Here H^1 stands for the first Galois cohomology group, K is an imaginary quadratic field satisfying the Heegner hypothesis with respect to the level N of f, $V_f(k/2)$ is the self-dual (Tate) twist of the representation V_f and K_n is the ring class field of K of conductor n. These cohomology classes are obtained as the images of particular Heegner cycles via the p-adic étale Abel–Jacobi maps

$$\mathrm{AJ}_{K_n} : \mathrm{CH}^{k/2}(W_{k-2}/K_n)_0 \longrightarrow H^1(K_n, V_f(k/2)).$$

Extending the method of Kolyvagin, Nekovář bounded the size of a Selmer group attached to V_f and K. This led to a proof of the Bloch–Kato conjecture in a rank 1 situation and of the finiteness of a suitably chosen primary part of the relevant Shararevich–Tate group.

Generalized Heegner cycles

Starting from a modular form f, we are interested in studying the representation $V_{f,\chi}$ defined as the twist of $V_f(k/2)$ by a Galois character χ.

In [BDP], Bertolini, Darmon and Prasanna introduced a distinguished collection of algebraic cycles, coming from graphs of isogenies between elliptic curves, lying in the product of the Kuga–Sato variety W_{k-2} with a power of a fixed elliptic curve A. These cycles are called *generalized Heegner cycles* and, roughly speaking, play the role of Heegner cycles for $V_{f,\chi}$.

Later, Castella and Hsieh constructed in [CH] an Euler systems for generalized Heegner cycles; they proved, among other results, a theorem that establishes, under suitable hypotheses, the vanishing of the Selmer group $\mathrm{Sel}(K, V_{f,\chi})$ associated with the representation $V_{f,\chi}$. This proves the Bloch–Kato conjecture in the setting of Castella and Hsieh:

$$\dim_F \mathrm{Sel}(K, V_{f,\chi}) = \mathrm{ord}_{s=k/2} L(f, \chi, s) = 0.$$

Building upon results from [BDP], the proof by Castella–Hsieh is based on a link between this system of generalized Heegner cycles and a certain p-adic L-function attached to f, and on a generalization of Kolyvagin's method. We emphasize that the Heegner hypothesis is essential in [CH].

The quaternionic setting: relaxing the Heegner hypothesis

What happens if we want to weaken the Heegner hypothesis? More explicitly, we would like to generalize the work of Castella and Hsieh to the case of an imaginary quadratic field K that does not satisfy the classical Heegner hypothesis, but instead satisfies the following *generalized Heegner hypothesis* relative to the level N of the modular form f:

- no prime factor of N ramifies in K, if a prime ℓ is inert in K then ℓ^2 does not divide N and the number of prime factors of N that are inert in K is *even*.

In this setting, we cannot work with Kuga–Sato varieties over classical modular curves, as we are not able to construct Heegner cycles on these varieties without the Heegner

hypothesis. The right substitutes for modular curves in this context are Shimura curves, so it is natural to work with Kuga–Sato varieties fibered over Shimura curves.

In [Brooks], Brooks introduced a collection of generalized Heegner cycles on a Kuga–Sato variety over a Shimura curve Sh, coming from graphs of isogenies between abelian surfaces. The curve Sh has the form of a quotient of the complex upper half plane under the action of a group that is determined by an order in an indefinite quaternion algebra over $\mathbb{Q}$. Brooks proved results that generalize (some of) those in [BDP] to this "quaternionic" setting, according to the following picture:

$$
\begin{array}{rcl}
\text{Heegner hypothesis} & \longleftrightarrow & \text{Generalized Heegner hypothesis,} \\
\text{Modular curves} & \longleftrightarrow & \text{Shimura curves,} \\
\text{Elliptic curves} & \longleftrightarrow & \text{Abelian surfaces.}
\end{array}
$$

Building on the work of Brooks, our goal is to generalize to a quaternionic context the key result of [CH] relating their p-adic L-function to the system of generalized Heegner classes. As said before, this is a crucial point for the proof of the vanishing of the Selmer group. We construct a system of generalized Heegner cycles on the Kuga–Sato variety over the Shimura curve Sh and a p-adic L-function defined as a p-adic measure given as a sum of values of a variation of f, as a modular form over our Shimura curve, at certain CM abelian surfaces. With these ingredients at hand, we will prove results on the Selmer group $\mathrm{Sel}(K, V_{f,\chi})$.

It is worth remarking that we expect the results of this thesis to play a key role in the proof of a generalization of Castella's specialization results ([Cas]) for Howard's big Heegner points in Hida families ([How]) to the quaternionic big Heegner points introduced by Longo and Vigni ([LV]). We plan to address this question in a future project.

Outline of the thesis

First of all, we fix some notation. Let $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a newform of weight $k = 2r + 2 \geq 4$ and level N. Fix an odd prime $p \nmid N$ and a field embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let F be a finite extension of $\mathbb{Q}_p$ containing the image of the Fourier coefficients of f under i_p and let K be an imaginary quadratic field of discriminant D_K and ring of integers $\mathcal{O}_K$ in which p splits as $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ splits, with $\mathfrak{p}$ determined by i_p. Let $\chi : \mathrm{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_F^\times$ be a locally algebraic anticyclotomic character of infinity type $(j, -j)$ and conductor $c_0 p^s \mathcal{O}_K$ (see section 3.5). Denote by $V_{f,\chi} := V_f(k/2) \otimes \chi$ the twist of $V_f(k/2)$ by χ seen as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, by $L(f, \chi, s)$ the associated Rankin L-series and by $\mathrm{Sel}(K, V_{f,\chi})$ the Block–Kato Selmer group associated with $V_{f,\chi}$ and K. Assume that:

1. $p \nmid 2N\phi(N^+)$ (where ϕ is Euler's function);
2. c_0 is prime to N, i.e., the conductor of χ is prime to N;
3. either $D_K > 3$ is odd or $8 \mid D_K$;
4. $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K.

Moreover, assume that K satisfies the generalized Heegner hypothesis relative to N as described above, and factor N as $N = N^+ N^-$, where N^+ is a product of primes that split in K and N^- is a (necessarily square-free) product of an *even* number of primes that are inert in K.

Our first theorem on Selmer groups, which corresponds to Theorem 6.4.1, is a vanishing result.

Theorem A. *If f is p-ordinary and $L(f, \chi, k/2) \neq 0$, then*

$$\dim_F \operatorname{Sel}(K, V_{f,\chi}) = 0.$$

Denote by $\epsilon(V_{f,\chi})$ the sign of the functional equation of $L(f, \chi, s)$. Our second theorem on Selmer groups, which corresponds to Theorem 6.4.2, is a one-dimensionality result.

Theorem B. *If $\epsilon(V_{f,\chi}) = -1$ and $z_\chi \neq 0$, then*

$$\operatorname{Sel}(K, V_{f,\chi}) = F \cdot z_\chi.$$

In the statement above, z_χ is a suitable cohomology class in $H^1(K, V_{f,\chi})$ that comes from an Euler system of generalized Heegner classes.

Let us conclude this introduction by briefly sketching the structure of the thesis.

In Chapter I we recall basic facts about quaternion algebras and Shimura curves as moduli spaces of abelian surfaces. We introduce the Shimura curve Sh we will work with and also the notions of modular forms and p-adic modular forms over Shimura curves.

In Chapter II we review Serre–Tate theory and study deformations of abelian surfaces, which we will use to get power series expansions at ordinary CM points for modular forms over Shimura curves (for which q-expansions are not available).

In Chapter III we define our analytic anticyclotomic p-adic L-function $\mathscr{L}_{f,\psi}$ as a measure on $\operatorname{Gal}(K_{p^\infty}/K)$ with values in the ring of Witt vectors $\mathcal{W} = W(\overline{\mathbb{F}}_p)$, where $K_{p^\infty} = \cup_n K_{p^n}$ for K_{p^n} the ring class field of conductor p^n of K, $\overline{\mathbb{F}}_p$ is an algebraic closure of the field $\mathbb{F}_p$ with p elements and ψ is an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$ and conductor $c_0 \mathcal{O}_K$ with $(c_0, pN^+) = 1$. We close the chapter with an interpolation formula for our p-adic L-function, which we will use later to obtain the reciprocity law of Chapter VI, relating the value of $\mathscr{L}_{f,\psi}$ at ϕ to the central critical value $L(f, \chi, k/2)$, where ϕ is an anticyclotomic Hecke character of infinity type $(n, -n)$ with $n \geq 0$ and p-power conductor such that $\chi = \psi\phi$.

In Chapter IV, following Brooks, we introduce a family of generalized Heegner cycles on the generalized Kuga–Sato variety over our Shimura curve Sh. More precisely, these cycles live in a Chow group of the generalized Kuga–Sato variety $\mathcal{X}_r = \mathcal{A}^r \times A^r$, where $\mathcal{A}$ is the universal object of the fine moduli problem associated with Sh and A is a fixed abelian surface with CM by K. Then we apply a p-adic Abel–Jacobi map to obtain cohomology classes from generalized Heegner cycles. In this way, we construct a system of generalized Heegner classes associated with f and χ, and indexed by fractional ideals of K, for which we prove compatibility properties.

In Chapter V we establish a relation between values of $\mathscr{L}_{f,\psi}$ at Galois characters ϕ of infinity type $(-k/2-j, k/2+j)$ and Bloch–Kato logarithms of generalized Heegner cycles associated with χ of infinity type $(j, -j)$, with $-k/2 < j < k/2$, where $\chi = \psi^{-1}\phi^{-1}$. This relation, for which we refer to Theorem 5.5.1, has the form

$$\mathscr{L}_{f,\psi}(\phi) = (\text{something}) \cdot \langle \log_{\mathfrak{p}}(z_\chi), * \rangle,$$

where "something" is an explicit non-zero coefficient that is comparatively less important than the main terms in the formula. The key ingredient to establish this Gross–Zagier type formula is the work of Brooks: we link our p-adic L-function to the differential operator $\theta = t\frac{d}{dt}$ on the Serre–Tate coordinates and then we use Brooks's results to obtain a formula suitably relating θ to our generalized Heegner cycles.

Finally, in Chapter VI we use the previous formula and the interpolation property to establish, under a p-ordinarity assumption on f, a reciprocity law relating the analytic p-adic L-function $\mathscr{L}_{f,\psi}$ to an algebraic p-adic L-function obtained as a sort of image of an Iwasawa cohomology class $\boldsymbol{z}_f \in H^1\big(K_{p^\infty}, V_f(k/2)\big)$, obtained as an inverse limit of generalized Heegner classes, under a big logarithm map. This reciprocity law and the costruction of an anticyclotomic Euler system associated with generalized Heegner classes, combined with an extension of Kolyvagin's method for anticyclotomic Euler systems developed in [CH], lead to the proof of Theorem A. As for Theorem B, its proof rests once again on an extension of Kolyvagin's method applied to another anticyclotomic Euler system associated with generalized Heegner classes.

Notation

If F is a number field or a local field, whose ring of integers will be denoted by $\mathcal{O}_F$, we fix an algebraic closure $\overline{F}$ of F and write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F.

For any prime number p, we fix an immersion $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of the chosen algebraic closure of $\mathbb{Q}_p$.

Unadorned tensor products are always taken over $\mathbb{Z}$.

We denote by $\mathbb{A}_F$ the adele ring of a number field F and by $\hat{F}$ the ring of finite adeles of F.

For an imaginary quadratic field K and an integer $n \geq 1$, we denote by K_n the ring class field of K of conductor n; in particular, K_1 is the Hilbert class field of K.

Finally, for an integer $n \geq 1$, we write ζ_n for the primitive n-th root of unity $e^{\frac{2\pi i}{n}} \in \mathbb{C}^\times$.

Chapter 1

Shimura curves

1.1 Shimura curves

In this section we give a short introduction to Shimura curves attached to quaternion algebras over $\mathbb{Q}$.

1.1.1 Quaternion algebras

We briefly introduce quaternion algebras, a standard reference for which is [Vi] (a nice introduction to the subject can also be found in [Voi]).

Definition 1.1.1. A **quaternion algebra** over a field F with $\operatorname{char}(F) \neq 2$ is an algebra B, that is a 4-dimensional vector space over F with a basis $1, i, j, k$ with the following multiplicative relations:
$$i^2 = a, \ j^2 = b, \ k = ij = -ji,$$
for some $a, b \in F \smallsetminus \{0\}$. The quaternion algebra determined by elements $a, b \in F \smallsetminus \{0\}$ is denoted by $\left(\frac{a,b}{F}\right)$.

Examples:

1. The $\mathbb{R}$-algebra $H = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is the algebra of Hamilton's quaternions.
2. The algebra of 2×2 matrices $M_2(F)$ is a quaternion algebra over F; indeed there is an isomorphism $\left(\frac{1,1}{F}\right) \cong M_2(F)$ induced by $i \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $j \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This the only quaternion algebra over F which is not a division algebra.

   We are interested in rational quaternion algebras, i.e., quaternion algebras over $\mathbb{Q}$. Denote by $\hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$ the profinite completion of $\mathbb{Z}$ and by $\hat{\mathbb{Q}} := \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adeles of $\mathbb{Q}$. Let B be a rational quaternion algebra; for each place v of $\mathbb{Q}$, possibly $v = \infty$, denote by $B_v := B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ and by $\hat{B} := B \otimes_{\mathbb{Q}} \hat{\mathbb{Q}}$ the adelization of B. We say that B is **split** at v if $B_v \cong M_2(\mathbb{Q}_v)$ and it is **ramified** at v otherwise. Because of the Hilbert's reciprocity law, the numbers of ramified places in B is finite and even. Furthermore, the set of primes ramified classifies completely (up to isomorphism) the rational quaternion algebra B; for each finite even set S of primes (possibly containing ∞) there is a unique rational quaternion algebra B ramified at exactly the primes of S. Denote by $D := \prod_{p \text{ finite}}^{\text{ramified}} p$ the discriminant of B, which determines B up to isomorphism. We say that B is **indefinite** if B is split at ∞, i.e., if there exists an isomorphism of $\mathbb{R}$-algebras $\Phi_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, and that B is **definite** otherwise.

   Consider now a quaternion algebra B over a field F of characteristic 0 and denote by $\mathcal{O}_F$ the ring of integers of F.

Definition 1.1.2. An **order** R in B is a $\mathcal{O}_F$-lattice that is also a subring of B. An order is said to be **maximal** if it is not properly contained in another order.

Another interesting type of order is an **Eichler order**, i.e., an order which is the intersection of two maximal orders.

Take a quaternion algebra B over $\mathbb{Q}$ of discriminant D. If ℓ is a prime such that $\ell \mid D$, then B_ℓ has a unique maximal order R_ℓ. If $\ell \nmid D$, we can choose an isomorphism $\Phi_\ell : B_\ell \cong M_2(\mathbb{Q}_\ell)$ and consider $R_\ell := \Phi_\ell^{-1}(M_2(\mathbb{Z}_\ell))$, which is a maximal order in B_ℓ. Then $R := B \cap \prod_\ell R_\ell$ is a maximal order in B. If N^+ is an integer prime to D, we consider the order in B defined by

$$R_{N^+} := \left\{ b \in R \mid \Phi_\ell(b) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) \pmod{\ell} \text{ for each } \ell \mid N^+ \right\},$$

that is the standard Eichler order of B of level N^+ in R (it depends on the choice of the isomorphisms Φ_ℓ).

Each quaternion algebra B over a field F can be endowed with a standard involution.

Definition 1.1.3. An **involution** on B is an F-linear map $(\cdot)^* : B \to B$ such that

- $1^* = 1$;
- $(xy)^* = y^* x^*$ for any $x, y \in B$;
- $(x^*)^* = x$ for any $x \in B$.

An involution is **standard** if $x^* x \in F$ for all $x \in B$.

1. If $B = \left(\frac{a, \, b}{F} \right)$ with $\mathrm{char}(F) \neq 2$, then the map

$$x = u + vi + wj + zk \longmapsto \overline{x} = u - vi - wj - zk$$

   defines a standard involution on B, called *main involution*.
2. If $B = M_2(F)$, the map
$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \longmapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix} \right)$$
   defines a standard involution on $M_2(F)$.

Consider an indefinite rational quaternion algebra B. By the Skolem–Noether theorem, any positive involution on B is given by $b \mapsto t^{-1} \overline{b} t$ with $t^2 \in \mathbb{Q}_{<0}$, with $b \mapsto \overline{b}$ the main involution.

1.1.2 Shimura curves

For a more general and detailed introduction to the theory of Shimura curves, see [Mi90]. Let B be an indefinite rational quaternion algebra, $b \mapsto b^*$ a positive involution on B, and $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R}$ be the disjoint union of the upper and lower complex half planes. Denote by $\mathcal{O}_B$ a maximal order of B and fix an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \cong M_2(\mathbb{R})$. For K a compact open subgroup of $\hat{B}^\times$, consider the double coset space

$$Sh_K(B) := B^\times \backslash \left(\mathcal{H}^\pm \times \hat{B}^\times \right) / K,$$

where the action is given by

$$b \cdot (z, g) \cdot k := (bz, bgk)$$

for $b \in B^\times, z \in \mathcal{H}^\pm, g \in \hat{B}^\times$ and $k \in K$, i.e., K acts naturally on the right on $\hat{B}^\times$ by right multiplication, $B^\times$ acts on the left on $\hat{B}^\times$ through the diagonal embedding

$B \hookrightarrow \hat{B}$ and on $\mathcal{H}^{\pm}$ through the fixed isomorphism $\Phi_{\infty} : B \otimes \mathbb{R} \cong M_2(\mathbb{R})$ by the usual linear fractional transformations

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot z := \frac{az + b}{cz + d}.$$

When K is sufficiently small, $Sh_K(B)$ is a finite disjoint union of quotients $\Gamma \backslash \mathcal{H}$, with Γ arithmetic subgroup of $SL_2(\mathbb{R})$, so it is a finite disjoint union of Riemann surfaces and it has a structure of an algebraic curve over $\mathbb{C}$ (cf. [Mi90, Chapter II, §2]).

Let

$$Sh(B) := \varprojlim_K Sh_K(B).$$

This is a scheme over $\mathbb{C}$ whose complex points are

$$Sh(B)(\mathbb{C}) = B^{\times} \backslash \mathcal{H}^{\pm} \times \hat{B}^{\times}.$$

See [Mi90, Lemma 10.1]. There is a continuous action of $\hat{B}^{\times}$ on $Sh(B)$ defined by

$$[(z, g)] \cdot h := [(z, gh)].$$

If K is normal then the action of $\hat{B}^{\times}$ is well defined also on $Sh_K(B)$. Indeed, the action of $\hat{B}^{\times}$ is defined by the maps

$$Sh_K(B) \longrightarrow Sh_{h^{-1}Kh}(B)$$
$$[(z, g)] \longmapsto [(z, gh)].$$

The scheme $Sh(B)$ together with this continuous action of $\hat{B}^{\times}$ is called the **Shimura curve** associated with the quaternionic algebra B. The operation defined by $g \in \hat{B}^{\times}$ on $Sh(B)$ is called **Hecke operator** associated with g. For more details, see [Mi90, Chapter II].

The Shimura curve $Sh(B)$ is the Shimura variety associated with the Shimura datum (G, X) where G is the algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}) = B^{\times}$ and $X = \mathcal{H}^{\pm}$ seen as the conjugacy class through the action of $G(\mathbb{R})$ of a morphism $h : \mathbb{C}^{\times} \to G(\mathbb{R})$. See again [Mi90, Chapter II, §2].

1.1.3 Shimura curves as moduli spaces

Shimura curves can be seen as moduli spaces classifying abelian surfaces with quaternionic multiplication by $\mathcal{O}_B$ and certain level structures. An abelian surface over a field F is a projective algebraic variety over F of dimension 2 that is also an algebraic group. A standard reference for abelian varieties is [Mum]; another good source are Milne's notes [AV]. A quaternionic multiplication on an abelian surface A over F is an embedding $\mathcal{O}_B \hookrightarrow \text{End}_F(A)$. We will see this in more detail in the case of the Shimura curve we will work with.

For a description of the moduli interpretation, we closely follow [Mi79, §1] and work with an integral form of G, that is to say that we take G as the group scheme over $\mathbb{Z}$ such that $G(R) = (\mathcal{O}_B \otimes R)^{\times}$ for any ring R. Let $V := V(\mathbb{Z})$ be a free $\mathbb{Z}$-module of rank 4 with an action of $\mathcal{O}_B$.

Lemma 1.1.4. *There exists a unique nondegenerate alternating form* $\psi : V(\mathbb{Q}) \times V(\mathbb{Q}) \to \mathbb{Q}$ *on* $V(\mathbb{Q}) := V \otimes_{\mathbb{Z}} \mathbb{Q}$ *such that*

1. $\psi(V, V) \subseteq Z$;

2. $\psi(ut, u) < 0$ for all $u \neq 0, u \in V(\mathbb{R})$;
3. $\psi(bu, v) = \psi(u, b^*v)$ for all $u, v \in V(\mathbb{Q})$;
4. for any B-automorphism α of $V(\mathbb{Q})$ there exists $\mu(\alpha) \in \mathbb{Q}^\times$ such that $\psi(\alpha u, \alpha v) = \psi(u, \alpha v)$ for all $u, v \in V(\mathbb{Q})$. Moreover if ϕ' is another nondegenerate alternating form on $V(\mathbb{Q})$ satisfying (3) then there is $q \in \mathbb{Q}^\times$ such that $\psi(u, cv) = \psi'(u, v)$ for all $u, v \in V(\mathbb{Q})$.

This means that on the complex manifold $\mathbb{C}^2/V$ with the $\mathcal{O}_B$-action there exists a Riemann form for V whose corresponding Rosati involution induces $b \mapsto b*$ on B and any such two forms are equivalent, i.e., there exists a unique polarization compatible with the $\mathcal{O}_B$-action.

Take $V = V(\mathbb{Z}) = \mathcal{O}_B$ with the natural action of $\mathcal{O}_B$. Note that for any ring R, we can identify $(\mathcal{O}_B \otimes R)^\times$ with $\mathrm{Aut}_{\mathcal{O}_B \otimes R}(V(R))$, because any $\mathcal{O}_B \otimes R$ endomorphism of $V(R) = \mathcal{O}_B \otimes R$ is the right multiplication by an element of $\mathcal{O}_B \otimes R$. Taking now $R = \mathbb{R}$, we define a homomorphism $h : \mathbb{C}^\times \to G(\mathbb{R}) = \mathrm{Aut}_{\mathcal{O}_B \otimes R}(V(R))$ such that $h(i)$ is the right multiplication by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, using the isomorphism $\Phi_\infty : G(\mathbb{R}) = (B \otimes \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R})$. The form ψ of the previous lemma is a Riemann form on (V, h), so that (V, h) defines an abelian surface $A = V(\mathbb{R})/V$ over $\mathbb{C}$ with a QM-structure $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}_{\mathbb{C}}(A)$ induced by the $\mathcal{O}_B$-action on V. However, in this case, the QM-structure determines the polarization up to a certain equivalence, so we do not need to include it in the construction.

Denote by $T(A)$ the complete Tate module of A that is the projective limit of the torsion groups $A[n]$, i.e.,

$$T(A) := \varprojlim_n A[n] \cong \prod_\ell T_\ell(A),$$

where $T_\ell(A)$ is the usual ℓ-adic Tate module of A. Then there is a natural isomorphism $T(A) \cong V(\hat{\mathbb{Z}})$ given by

$$T(A) = \varprojlim_n A[n] = \varprojlim_n n^{-1}V/V \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} \otimes V = \hat{\mathbb{Z}} \otimes V.$$

If K is an open compact subgroup of $G(\hat{\mathbb{Z}}) = \hat{\mathcal{O}}_B$ and $\alpha_1, \alpha_2 : T(A) \cong V(\hat{\mathbb{Z}})$ two isomorphisms, they are said to be K-equivalent if $\alpha_1 = (-\cdot k) \circ \alpha_2$ for some $k \in K$. For example, if K is the kernel of the natural map $G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/M\mathbb{Z})$ then give a K-equivalence class of isomorphisms $T(A) \cong V(\hat{\mathbb{Z}})$ is the same as to give an isomorphism $V(\mathbb{Z}/M\mathbb{Z}) \cong A[M](\mathbb{C})$ that is a full level M structure.

Theorem 1.1.5. *There is a bijection between the set of complex points $Sh_K(B)(\mathbb{C}) = B^\times \backslash \mathcal{H}^\pm \times \hat{B}^\times / K$ and the set of isomorphism classes of triples (A, ι, ϕ), where A is an abelian surface over $\mathbb{C}$, ι is a quaternionic multiplication $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ and ϕ is a K- equivalence class of isomorphisms $T(A) \overset{\cong}{\to} V(\hat{\mathbb{Z}})$.*

We want to give explicitly the bijection $x = (z, g) \mapsto (A_x, \iota_x, \phi_x)$ of the previous theorem. If $z = i$, $g = 1$ then $A_x = A = V(\mathbb{R})/V$, $\iota_x = \iota$ and ϕ_x is the class of the natural isomorphism $T(A) \overset{\cong}{\to} V(\hat{\mathbb{Z}})$ seen before. For a general $x = (z, g) \in \mathcal{H}^\pm \times \hat{B}^\times$, put $V_g = V(\hat{\mathbb{Z}})g \cap V(\mathbb{Q})$ and $h_z : \mathbb{C}^\times \to G(\mathbb{R})$ defined by $\tau \mapsto g_\infty h(\tau) g_\infty^{-1}$, where $g_\infty \in G(\mathbb{R})$ such that $g_\infty i = z$. Then A_x is the abelian surface defined by (V_g, h_z), so $A_x = V(\mathbb{R})/V_g$ with ι_x defined by the action of $\mathcal{O}_B$ on V_g and ϕ_x determined by $T(A_x) \cong V_g(\hat{\mathbb{Z}}) = V(\hat{\mathbb{Z}})g \cong V(\hat{\mathbb{Z}})$, where the last $\cong$ is right multiplication by g^{-1}. See [Mi79] for details.

1.2 Shimura curves of level $V_1(N^+)$

We introduce now the Shimura curves we will work with. Let B be an indefinite rational quaternion algebra of discriminant D and let $\mathcal{H}^{\pm} = \mathbb{C} - \mathbb{R}$ be the disjoint union of the upper and lower complex half planes. Fix isomorphisms $\Phi_\ell : B_\ell \cong M_2(\mathbb{Q}_\ell)$ for each prime $\ell \nmid D$ and denote by $\mathcal{O}_B$ a maximal order of B such that each Φ_ℓ induces an isomorphism $\mathcal{O}_B \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)$. Fix also an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \cong M_2(\mathbb{R})$. For N^+ positive integer prime to D, consider the map

$$\pi_{N^+} : \hat{\mathcal{O}}_B^{\times} \twoheadrightarrow \prod_{\ell \mid N^+} (\mathcal{O}_B \otimes \mathbb{Z}_\ell)^{\times} \cong \prod_{\ell \mid N^+} \mathrm{GL}_2(\mathbb{Z}_\ell) \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/N^+\mathbb{Z}).$$

Denote by $\hat{\Gamma}_{1,N^+}$ the open compact subgroup of $\hat{\mathcal{O}}_B^{\times}$ composed of the elements $b \in \hat{\mathcal{O}}_B^{\times}$ such that $\pi_{N^+}(b) \in \{ \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix} \right) \in GL_2(\mathbb{Z}/N^+\mathbb{Z}) \}$. Consider the space of double cosets

$$X_{N^+} := B^{\times} \backslash \mathcal{H}^{\pm} \times \hat{B}^{\times} / \hat{\Gamma}_{1,N^+}.$$

Here, as already seen before, $\hat{\Gamma}_{1,N^+}$ acts naturally on the right on $\hat{B}^{\times}$ by right multiplication, while $B^{\times}$ acts on the left on $\hat{B}^{\times}$ through the diagonal embedding $B \hookrightarrow \hat{B}$ and on $\mathcal{H}^{\pm}$ under the fixed isomorphism Φ_∞ by the usual linear fractional transformations

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

This is the Shimura curve associated with the Shimura datum $G(\mathbb{Q}) = B^{\times}$, $X = \mathcal{H}^{\pm}$ and $K = \hat{\Gamma}_{1,N^+}$.

Because B is indefinite, there is a bijection

$$X_{N^+} \cong \mathcal{H}/\Gamma_{1,N^+},$$

where $\mathcal{H}$ is the classical upper half plane and Γ_{1,N^+} is the subgroup of matrices in $\Phi_\infty((\hat{\Gamma}_{1,N^+} \cap B)^{\times})$ of determinant 1 ([BD, §1.3]). This bijection endows X_{N^+} with a Riemann surface structure and gives, as a consequence, an analytic description of X_{N^+}.

The coset space X_{N^+} admits a model over $\mathbb{Q}$, which is the fine moduli scheme classifying abelian surfaces with quaternionic multiplication by $\mathcal{O}_B$ and certain level structures.

Definition 1.2.1. Let S be a $\mathbb{Z}[1/D]$-scheme. An **abelian surface with quaternionic multiplication by $\mathcal{O}_B$** (abelian surface with QM, for short) **over** S is a pair (A, i) where

1. A is an abelian scheme A/S of relative dimension 2;
2. i is an optimal inclusion $i : \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$ giving an action of $\mathcal{O}_B$ on A.

A morphism of abelian surfaces with QM is a morphism of abelian surfaces that respects the action of $\mathcal{O}_B$.

Abelian surfaces with quaternionic multiplication are often called *false elliptic curves*.

Definition 1.2.2. Let $N^+ > 0$ be an integer prime to D. A **level $V_1(N^+)$-structure**, or an arithmetic level N^+ structure, on a QM abelian surface (A, i) is an inclusion

$$\mu_{N^+} \times \mu_{N^+} \hookrightarrow A[N^+]$$

of group schemes over S, commuting with the action of $\mathcal{O}_B$, where μ_{N^+} denotes the group scheme of N^+th roots of unity. The action of $\mathcal{O}_B$ on the left hand side is via the isomorphism $\mathcal{O}_B \otimes \mathbb{Z}/N^+\mathbb{Z} \cong M_2(\mathbb{Z}/N^+\mathbb{Z})$ induced by the chosen isomorphisms $\Phi_\ell : B_\ell \cong M_2(\mathbb{Q}_\ell)$, through which one has $\mathcal{O}_B \otimes \mathbb{Z}_\ell \cong M_2(\mathbb{Z}_\ell)$ for each $\ell \mid N^+$.

A morphism of QM abelian surfaces with $V_1(N^+)$-level structure is a morphism of QM abelian surfaces that respects the level structures.

If A is an abelian surface over an algebraically closed field k, a $V_1(N^+)$-level structure can be thought of as an orbit of full level N^+ structures, i.e., isomorphisms

$$\mathcal{O}_B \otimes \mathbb{Z}/N^+\mathbb{Z} \cong A[N^+]$$

commuting with the action of $\mathcal{O}_B$, under the natural action of the subgroup $\left\{ \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix} \right) \in GL_2(\mathbb{Z}/N^+\mathbb{Z}) \right\}$ of $GL_2(\mathbb{Z}/N^+\mathbb{Z})$. See [Brooks, §2.2] for details.

The moduli problem of QM abelian surfaces with $V_1(N^+)$-level structure is representable, as asserted by

Theorem 1.2.3. *For $N^+ > 3$, the moduli problem that assigns to a $Z[1/DN^+]$-scheme S the set of isomorphism classes of QM-abelian surfaces over S with $V_1(N^+)$-level structure is representable by a smooth proper $\mathbb{Z}[1/DN^+]$-scheme X.*

For details, see [Brooks, §2.2. and §2.3], [Buz, §2] or [Kas, §2 and §3].

The complex points of X are naturally identified with the compact Riemann surface $X_{N^+} \cong \mathcal{H}/\Gamma_{1,N^+}$.

The $\mathbb{Z}[1/DN^+]$-scheme X from Theorem 1.2.3 is called the **Shimura curve of level** $V_1(N^+)$ associated to the indefinite quaternion algebra B and we will denote it by X_{N^+}, using the same notation for the scheme, the Riemann surface and the double coset space.

Theorem 1.2.3 says that the scheme $X_{N^+}/\mathbb{Z}[1/DN^+]$ represents the moduli functor

$$F : \mathrm{Sch}\,/\mathbb{Z}[1/DN^+] \longrightarrow \mathcal{S}et$$

given by

$$F(S) := \left\{ \begin{array}{c} \text{isomorphism classes of abelian surfaces } (A,\iota,\nu) \\ \text{over S with QM and } V_1(N^+)\text{-level structure} \end{array} \right\}.$$

Therefore there exists a universal object that is the isomorphism class of QM abelian surfaces with $V_1(N^+)$-level structure in $F(X_{N^+})$ corresponding to $\mathrm{id}_{X_{N^+}}$ through the isomorphism

$$F(X_{N^+}) \cong \mathrm{Hom}_{\mathbb{Z}[1/DN^+]}(X_{N^+}, X_{N^+}).$$

We denote by $\pi : \mathcal{A} \to X_{N^+}$ this *universal QM abelian surface* over X_{N^+}. It is called "universal" because every isomorphism class in $F(S)$ comes from $\mathcal{A}$; more precisely, an isomorphism class corresponding to a morphism $f \in X_{N^+}(S) = \mathrm{Hom}_{\mathbb{Z}[1/DN^+]}(S, X_{N^+})$ is the isomorphism class of $(\mathcal{A}, \iota_{\mathcal{A}}, \nu_{\mathcal{A}}) \times_f S$. For each geometric point $x : \mathrm{Spec}(L) \to \mathcal{A}$, the fiber $\mathcal{A}_x := \mathcal{A} \times_x \mathrm{Spec}(L)$ is an abelian surface with QM by $\mathcal{O}_B$ and $V_1(N^+)$-level structure defined over L, representing the isomorphism class that corresponds to the point x.

1.3   Hecke operators

The Shimura curve X_{N^+} comes equipped with a ring of Hecke correspondences, which can be introduced by using the adelic description of X_{N^+}. See, for example, [BD, §1.5]; here the construction is for the Shimura curve relative to level structures "of type Γ_0", but it can be done also in our case. In terms of abelian surfaces the Hecke operators acts in the following way. For a prime ℓ, a QM abelian surface A over a field k of

characteristic prime to ℓ has $\ell + 1$ cyclic $\mathcal{O}_B$-submodules annihilated by ℓ. Denote them by $C_0, \ldots, C_\ell$ and consider the isogenies $\psi_i : A \twoheadrightarrow A/C_i$ of QM abelian surfaces. If ν_A is a $V_1(N^+)$-level structure on A and $\ell \nmid N^+$, then ψ_i induces a $V_1(N^+)$-level structure $\nu_A \circ \psi_i$ on A/C_i. If $\ell \nmid N^+D$, the "good" Hecke operator T_ℓ can be described by

$$T_\ell(A, \iota_A, \nu_A) = \sum_{i=0}^{\ell} (A/C_i, \iota_i, \nu_i).$$

For more details, see §4.1 and [Brooks, §3.6].

1.4 Igusa tower

We are interested in working with p-adic modular forms over our Shimura curve, which are defined analogously to Katz's generalized p-adic modular forms. Therefore we want to work on a cover of the ordinary locus of the Shimura curve.

Fix a prime $p \nmid N^+D$. Since X_{N^+} is a scheme over $\mathbb{Z}[1/N^+D]$, it can be viewed as a scheme over $\mathbb{Z}_{(p)}$. For simplicity, denote by Sh the curve $X_{N^+/\mathbb{Z}_{(p)}}$. Since Sh is a fine moduli scheme for QM abelian surfaces over $\mathbb{Z}_{(p)}$-schemes with level structures, there is a universal abelian surface $\mathcal{A} \to Sh$, which is the one associated with X_{N^+} but tensored with $\mathbb{Z}_{(p)}$ over $\mathbb{Z}[1/DN^+]$.

Recall that a QM abelian surface A over a field k of characteristic p is said to be *ordinary* if $A[p](\bar{k}) \cong (\mathbb{Z}/p\mathbb{Z})^2$, and *supersingular* otherwise. Indeed, a QM abelian surface in characteristic p, is either ordinary or supersingular; equivalently, it is isogenous either to a product of ordinary elliptic curves or to a product of supersingular elliptic curves, respectively. Consider the *ordinary locus* Sh^{ord} of Sh, i.e., the locus on which the Hasse invariant does not vanish, that is the scheme obtained by removing the supersingular points of Sh in the fiber at p, which are those points which correspond in the moduli interpretation to abelian surfaces which have supersingular reduction modulo p. See [Kas] for details about the ordinary locus and the Hasse invariant.

Let $\mathcal{A}^{\mathrm{ord}} \to Sh^{\mathrm{ord}}$ be the universal ordinary QM abelian surface over Sh^{ord}, that is the fiber product $\mathcal{A}^{\mathrm{ord}} = \mathcal{A} \times_{Sh} Sh^{\mathrm{ord}}$. Consider the functor $I_n : \mathrm{Sch}_{/Sh^{\mathrm{ord}}} \to \mathcal{S}et$ that takes an Sh^{ord}-scheme S to the set of closed immersions $\mu_{p^n} \times \mu_{p^n} \hookrightarrow \mathcal{A}^{\mathrm{ord}}[p^n]$ of finite flat group schemes over S respecting the $\mathcal{O}_B$-action. This functor is representable by a scheme $I_{n/Sh^{\mathrm{ord}}}$. Then $I_{n/\mathbb{Z}_{(p)}}$ classifies quadruples $(A, i, \nu_{N^+}, \nu_{p^n})$, where A is an abelian surface, ι a quaternionic multiplication, ν_{N^+} a $V_1(N^+)$-level structure and ν_{p^n} an $\mathcal{O}_B$-immersion $\mu_{p^n} \times \mu_{p^n} \hookrightarrow A[p^n]$. There is a tower

$$\cdots \longrightarrow I_{n+1} \longrightarrow I_n \longrightarrow I_{n-1} \longrightarrow \cdots.$$

Consider the formal scheme $I_{/\mathbb{Z}_{(p)}} := \varprojlim_n I_{n/\mathbb{Z}_{(p)}}$. This formal scheme parametrizes compatible sequences of isomorphism classes of quadruples $(A, i, \nu_{N^+}, \nu_{p^n})$, where A is an ordinary abelian surface, ι a quaternionic multiplication and ν_{N^+}, ν_{p^n} respectively a $V_1(N^+)$ and $V_1(p^n)$ level structures. But a sequence of compatible $V_1(p^n)$-level structures is the same as a $V_1(p^\infty)$-level structure, that is an immersion $\nu_{p^\infty} : \mu_{p^\infty} \times \mu_{p^\infty} \hookrightarrow A[p^\infty]$. Therefore this tower parametrizes isomorphism classes of quadruples $(A, i, \nu_{N^+}, \nu_{p^\infty})$.

There is a bijection

$$I(\mathbb{C}) \cong \varprojlim_n X_{N^+p^n}(\mathbb{C}),$$

between complex points of I and compatible sequences $\{x_n\}_n$ of complex points $x_n = (A, i, \nu_{N^+}, \nu_{p^n}) \in X_{N^+p^n}(\mathbb{C})$.

See [Hi04] for details about Igusa schemes in the case of modular curves (§6.2.12) and more in general for Shimura varieties (Ch. 8). See also [Hi09].

1.5 CM points on Shimura curves

In this section we will construct a collection of CM points in our Shimura curves, i.e., points that correspond to abelian surfaces with complex multiplication, indexed by fractional ideals of orders in an imaginary quadratic field. We denote again by Sh the curve X_{N^+} seen as a scheme over $\mathbb{Z}_{(p)}$. Since also Sh is the fine moduli scheme for QM abelian surfaces over $\mathbb{Z}_{(p)}$-schemes with level structure, it has a universal abelian surface $\mathcal{A} \to Sh$, which is the one of X_{N^+} but tensored by $\mathbb{Z}_{(p)}$ over $\mathbb{Z}[1/DN^+]$.

1.5.1 Abelian surfaces with QM and CM over $\mathbb{C}$

Theorem 1.5.1. *Let (A, i) be an abelian surface with QM by $\mathcal{O}_B$ over $\mathbb{C}$. Then either*

1. *A is simple and $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q} = B$, or*
2. *A is not simple, $A \sim E^2$ is isogenous to the product of an elliptic curve E with CM by an imaginary quadratic field K which embeds in B and $\mathrm{End}^0(A) \cong M_2(K)$.*

In particular, we are interested in the second case of the previous theorem. Abelian surfaces with QM that satisfy that second condition are said to have **complex multiplication** (CM for short) by K. Suppose that (A, i) is an abelian surface over $\mathbb{C}$ with QM by $\mathcal{O}_B$ and $V_1(N^+p^n)$-level structure. Then the ring

$$\mathrm{End}_{\mathcal{O}_B}(A) := \big\{ f \in \mathrm{End}(A) \mid f \circ i(b) = i(b) \circ f \text{ for all } b \in \mathcal{O}_B \big\}$$

is either $\mathbb{Z}$ or an order in an imaginary quadratic field K. If K is an imaginary quadratic field and $\mathrm{End}_{\mathcal{O}_B}(A) = \mathcal{O}_c$, where $\mathcal{O}_c$ is the order of conductor c in $\mathcal{O}_K$, then A is said to have complex multiplication by $\mathcal{O}_c$ and the point $P = [(A, i)] \in X_{N^+p^n}(\mathbb{C})$ is said to be a **CM point of conductor** c.

1.5.2 Generalized Heegner hypothesis

Let K be an imaginary quadratic field of discriminant D_K and consider a positive integer N, which will ultimately be the level of our modular form, such that $(N, D_K) = 1$. Suppose that K satisfies the **generalized Heegner hypothesis** relative to N:

- no prime factor of N ramifies in K, if a prime ℓ is inert in K then ℓ^2 does not divide N and the number of prime factors of N that are inert in K is *even*.

Factor N as a product $N = N^+N^-$ where N^+ is a product of primes that split in K and N^- is a (necessarily square-free) product of (an even number of) primes that are inert in K.

Let B be the indefinite rational quaternion algebra over of discriminant $D = N^-$ and fix a prime $p \nmid N$ that splits in K and B. Then the field K embeds into B, thanks to the following

Theorem 1.5.2. *For a quadratic field K and a rational quaternion algebra B, the following properties are equivalent:*

20

1. *K embeds into B;*
2. *K splits B;*
3. *K is not split at any place where B is ramified.*

Proof. This is a classical result: see, for example, [Voi, Chapter 5]. $\qquad\square$

We will work with the Shimura curves $X_{N^+p^n}$ defined before relatively to the fixed K, B, p. This hypothesis concerning the behavior of primes dividing Np in K and B will allow us to construct CM points on our Shimura curves. Regarding this, see [Dar, Lemma 4.17] or [BD, §2.2] (here the Shimura curves considered are different but the result holds also in our case).

1.5.3   Products of CM elliptic curves

Start with an elliptic curve E over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_K$, take $E := \mathbb{C}/\mathcal{O}_K$. Consider on E a $\Gamma_1(M)^{\mathrm{arit}}$-level structure given by a morphism

$$\mu_M : \mu\!\!\!/_M \longhookrightarrow E[M],$$

where $M > 3$ is an integer prime to D. Consider now the self product $A := E \times E$ that is an abelian surface over $\mathbb{C}$; then its endomorphism ring is $\mathrm{End}(A) \cong M_2(\mathrm{End}(E)) \cong M_2(\mathcal{O}_K)$. Since K splits B, we can embed K in B and choose a basis $\{b_1, b_2\}$ of B over K with $b_1, b_2 \in \mathcal{O}_B$. Then we have an an immersion $B \hookrightarrow M_2(K) = M_2(\mathrm{End}^0(E)) = \mathrm{End}^0(A)$ such that $\mathcal{O}_B \hookrightarrow M_2(\mathcal{O}_K) = M_2(\mathrm{End}(E)) = \mathrm{End}(A)$. See [Mi79, §2]. Hence $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ is a quaternionic multiplication for A.

Consider the isomorphism $i_K : B \otimes_{\mathbb{Q}} K \cong M_2(K)$ induced by ι and put

$$e := i_K^{-1}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) \in B \otimes_{\mathbb{Q}} K,$$

which is an idempotent such that $e^* = e$. Then the decomposition of A is induced by $A = eA \oplus (1-e)A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)A \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)A \cong E \times E$ (multiplication by $\alpha := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ gives an isomorphism $eA \overset{\alpha}{\cong} (1-e)A$). So $A[M] = eA[M] \oplus (1-e)A[M]$. The choice of a level structure on $eA[M] = E[M]$ induces a $V_1(M)$-level structure on $A[M]$, because of the request of compatibility of the level structure with respect to the action of $\mathcal{O}_B$ (and consequently of B). See also the last lines of [Brooks, §2.2]. Hence the fixed level structure μ_M on E induces a $V_1(M)$-level structure

$$\nu_M : \mu\!\!\!/_M \times \mu\!\!\!/_M \longhookrightarrow A[M]$$

on A.

Therefore, starting from a $\Gamma_1(N^+p^n)^{\mathrm{arit}}$-level structure on E, we obtain a quadruple $(A, \iota, \nu_{N^+}, \nu_{p^n})$ which determines a CM point in $Sh_n(\mathbb{C})$. Starting from a $\Gamma_1(N^+p^\infty)^{\mathrm{arit}}$-level structure on E, we obtain a quadruple $(A, \iota, \nu_{N^+}, \nu_{p^\infty})$ which can be seen as a compatible sequence of CM points in the Shimura tower $X_{N^+p^n}$.

Start now with the elliptic curve $E_c = \mathbb{C}/\mathcal{O}_c$ over $\mathbb{C}$ with complex multiplication by an order $\mathcal{O}_c$ of K, with $(c, N) = 1$. The isogeny

$$E = \mathbb{C}/\mathcal{O}_K \longrightarrow E = \mathbb{C}/\mathcal{O}_c$$
$$\overline{z} \longmapsto \overline{cz},$$

induces an isogeny

$$\phi_c : A = E \times E \longrightarrow A_c := E_c \times E_c$$

of complex abelian surfaces. Take on A_c the quaternionic multiplication $\iota_c : \mathcal{O}_B \hookrightarrow \mathrm{End}(A_c)$ determined by compatibility with ϕ_c:

$$\iota_c(b)(\phi_c(a)) = \phi_c(\iota(b)a),$$

for any $b \in B$, $a \in A$. As before, a $\Gamma_1(M)^{\mathrm{arit}}$-level structure $\mu_{c,M}$ on E_c, with M prime to D, induces a $V_1(M)$-level structure $\nu_{c,M}$ on A_c.

Therefore, starting from a $\Gamma_1(N^+ p^n)^{\mathrm{arit}}$-level structure on E_c, we obtain a quadruple $(A_c, \iota_c, \nu_{N^+}, \nu_{p^n})$ which determines a CM point of conductor c in $X_{N^+ p^n}(\mathbb{C})$. Starting from a $\Gamma_1(N^+ p^\infty)^{\mathrm{arit}}$-level structure on E, we obtain a quadruple $(A, i, \nu_{c,N^+}, \nu_{c,p^\infty})$ which can be seen as a compatible sequence of CM points in the Shimura tower $X_{N^+ p^n}$.

Note that the isogeny ϕ_c doesn't necessarily respect the chosen level structures if p divides c.

1.5.4 The action of $\mathrm{Pic}(\mathcal{O}_c)$

Denote by $\mathrm{Pic}(\mathcal{O}_c)$ the Picard group of the order $\mathcal{O}_c$ of conductor c of an imaginary quadratic field K, that is

$$\mathrm{Pic}(\mathcal{O}_c) = K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_c^\times = I_c(\mathcal{O}_c)/P_c(\mathcal{O}_c),$$

where $I_c(\mathcal{O}_c)$ is the group of fractional ideals of $\mathcal{O}_c$ coprime to c and $P_c(\mathcal{O}_c)$ is the subgroup of $I_c(\mathcal{O}_c)$ of principal fractional ideals.

Consider a quadruple $(A, \iota, \nu_{N^+}, \nu_{p^\infty})$ where (A, ι) is a QM abelian surface with CM by $\mathcal{O}_c$. There is an action of $\mathrm{Pic}(\mathcal{O}_c)$ on the isomorphism classes of these quadruples, defined by

$$\mathfrak{a} \star (A, \iota, \nu_{N^+}, \nu_{p^\infty}) := (A_{\mathfrak{a}}, \iota_{\mathfrak{a}}, \nu_{\mathfrak{a},N^+}, \nu_{\mathfrak{a},p^\infty}),$$

where the representative $\mathfrak{a}$ is chosen to be integral and prime to $N^+ pc$. Here $A_{\mathfrak{a}} := A/A[\mathfrak{a}]$, where $A[\mathfrak{a}]$ is the subgroup of the elements of A that are killed by all the endomorphisms in $\mathfrak{a}$. The quaternionic multiplication $\iota_{\mathfrak{a}}$ and the level structures $\nu_{\mathfrak{a},N^+}, \nu_{\mathfrak{a},p^\infty}$ are induced by the ones of A. Denote by $\varphi_{\mathfrak{a}}$ the quotient isogeny $A \twoheadrightarrow A/A[\mathfrak{a}]$, that is an isogeny of degree $N(\mathfrak{a})^2 = (\#\mathcal{O}_c/\mathfrak{a})^2$ and so prime to $N^+ p$; define

$$
\begin{aligned}
\iota_{\mathfrak{a}} : \mathcal{O}_B &\longrightarrow \mathrm{End}(A_{\mathfrak{a}}) \\
b &\longmapsto \left(\varphi_{\mathfrak{a}}(x) \mapsto \varphi_{\mathfrak{a}}\big(\iota(b)(x)\big) \right)
\end{aligned}
$$

and

$$\nu_{\mathfrak{a},N^+} : \mu_{N^+} \times \mu_{N^+} \xrightarrow{\nu_{N^+}} A[N^+] \overset{\varphi_{\mathfrak{a}}}{\cong} A_{\mathfrak{a}}[N^+],$$

$$\nu_{\mathfrak{a},p^\infty} : \mu_{p^\infty} \times \mu_{p^\infty} \xrightarrow{\nu_{p^\infty}} A[p^\infty] \overset{\varphi_{\mathfrak{a}}}{\cong} A_{\mathfrak{a}}[p^\infty].$$

See [Brooks, §2.5].

If $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K_c/K)$ corresponds to $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$ through the classical Artin reciprocity map, then by Shimura's reciprocity law there is an equality

$$(A, \iota, \nu_{N^+}, \nu_{p^\infty})^{\sigma_{\mathfrak{a}}} = \mathfrak{a} \star (A, \iota, \nu_{N^+}, \nu_{p^\infty}).$$

See, again, [Brooks, §2.5].

1.5.5 Construction of CM points

We want to introduce CM points indexed by ideals of orders of K.

Take an element $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$ and choose the representative $\mathfrak{a}$ to be integral and prime to N^+pc. Consider the elliptic curve $E_{\mathfrak{a}} := \mathbb{C}/\mathfrak{a}^{-1}$ with the $\Gamma_1(N^+p^\infty)^{\mathrm{arith}}$-level structure defined in [CH, §2.3 and §2.4]. Put $A_{\mathfrak{a}} := E_{\mathfrak{a}} \times E_{\mathfrak{a}}$, which, by the theory of complex multiplication, is an abelian surface defined over the ring class field K_c of K of conductor c. The abelian surface $A_{\mathfrak{a}}$ has QM by $\mathcal{O}_B$ and we can consider the quadruple

$$x(\mathfrak{a}) := (A_{\mathfrak{a}}, i_{\mathfrak{a}}, \nu_{\mathfrak{a},N^+}, \nu_{\mathfrak{a},p^\infty}),$$

where $\nu_{\mathfrak{a},N^+}, \nu_{\mathfrak{a},p^\infty}$ are the level structures induced by the ones of $E_{\mathfrak{a}}$ (as in §1.5.3). We write $x(c) = (A_c, i_c, \nu_{c,N^+}, \nu_{c,p^\infty})$ when $\mathfrak{a} = \mathcal{O}_c$, and $x(1) = (A, i, \nu_{N^+}, \nu_{p^\infty})$ when $\mathfrak{a} = \mathcal{O}_K$.

We have already used the notation above in the previous section for the action of the Picard group on a quadruple: the reason is that there is an equality

$$\mathfrak{a} \star (A_c, i_c, \nu_{c,N^+}, \nu_{c,p^\infty}) = (A_{\mathfrak{a}}, i_{\mathfrak{a}}, \nu_{\mathfrak{a},N^+}, \nu_{\mathfrak{a},p^\infty}).$$

Indeed, $\mathfrak{a} \star A_c = A_c/A_c[\mathfrak{a}] \cong E_c/E_c[\mathfrak{a}] \times E_c/E_c[\mathfrak{a}]$ because $\mathfrak{a} \subseteq \mathcal{O}_c \hookrightarrow M_2(\mathcal{O}_c)$ acts diagonally and the level structures are induced by the isogeny $\varphi_{\mathfrak{a}} : A_c \twoheadrightarrow A_c/A_c[\mathfrak{a}]$ that is the product of the isogeny $E_c \twoheadrightarrow E_c/E_c[\mathfrak{a}]$.

1.6 Modular forms on Shimura curves

We recall here the definitions and some properties of modular forms and p-adic modular forms on Shimura curves. The references are [Kas], [Brooks], [EdVP], [Hi04].

1.6.1 Geometric modular forms on Shimura curves

We will need integrality conditions only at p, so we define modular forms over algebras R over the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal generated by p. Let $(A \to \mathrm{Spec}(R), \iota)$ be a QM abelian surface over a $\mathbb{Z}_{(p)}$-algebra R. Then $\pi_*\Omega_{A/R}$, where $\Omega_{A/R}$ is the bundle of relative differentials, inherits an action of $\mathcal{O}_B$ which tensored with the scalar action of $\mathbb{Z}_p$ gives an action of $M_2(\mathbb{Z}_p)$ on $\pi_*\Omega_{A/R}$. Write $\underline{\omega}_{A/R}$ for $e\pi_*\Omega_{A/R}$. If $\mathcal{A} \to Sh$ is the universal QM abelian surface, then $\mathcal{A} \otimes R \to Sh \otimes R$ is the universal object for $Sh \otimes R$. In the particular case $\pi : \mathcal{A} \otimes R \to Sh \otimes R$ of the universal QM abelian surface over a $\mathbb{Z}_p$-algebra R, we just write $\underline{\omega}_R$ for $e\pi_*\Omega_{\mathcal{A} \otimes R/Sh \otimes R}$.

In analogy with the case of elliptic modular forms (see, for example, [BDP, §1], in particular equation (1.1.1)), we give a geometric definition à la Katz for modular forms on Sh over a $\mathbb{Z}_{(p)}$-algebra R. For a nice exposition of Katz modular forms in the case of modular curves, see [Go, Chapter 1]. The geometric definition for modular forms on Shimura curves is due to Kassaei, see [Kas, §4.1]. We closely follow [Brooks].

Definition 1.6.1. A modular form with respect to B of weight $k \in \mathbb{Z}$ and level $V_1(N^+)$ over R is a global section of $\underline{\omega}_R^{\otimes k}$, i.e., an element of $H^0(Sh \otimes R, \underline{\omega}_R^{\otimes k})$. We denote by $M_k(Sh, R)$ the space of modular forms with respect to B, of weight $k \in \mathbb{Z}$ and level $V_1(N^+)$ over R.

Alternatively, one can define modular forms in the following ways.

Definition 1.6.2. Let R_0 be an R-algebra. A **test object** is a quadruple $(A/R_0, \iota, \nu, \omega)$ consisting of a QM abelian surface A over R_0, a $V_1(N^+)$-level structure ν on A, and a non-vanishing global section of $\underline{\omega}_{A/R_0}$.

Two test objects $(A/R_0, \iota, \nu, \omega)$ and $(A'/R_0, \iota', \nu', \omega')$ over R_0 are **isomorphic** if there is an isomorphism $(A/R_0, \iota, \nu) \to (A'/R_0, \iota', \nu')$, of QM abelian surfaces with $V_1(N^+)$-level structure, pulling ω' back to ω.

A **modular form of weight k and level $V_1(N^+)$ over R** is a rule F that assigns to every isomorphism class of test objects $(A/R_0, \iota, \nu, \omega)$ over every R-algebra R_0, a value $F(A/R_0, \iota, \nu, \omega) \in R_0$, such that

- (Compatibility with base change) If $\varphi : R_0 \to R'_0$ is a map of R-algebras, inducing $\varphi : A \to A \otimes_\varphi R'_0$, then

$$F((A/R_0, \iota, \nu) \otimes_\varphi R'_0, \omega) = \varphi(F(A/R_0, \iota, \nu, \varphi^*(\omega))).$$

- (Weight condition) For any $\lambda \in R_0$, one has

$$F(A/R_0, \iota, \nu, \lambda\omega) = \lambda^{-k} F(A/R_0, \iota, \nu, \omega).$$

Definition 1.6.3. A **modular form of weight k and level $V_1(N^+)$ over R** is a rule G that, for any R-algebra R_0, assigns to any isomorphism class of QM abelian surfaces over R_0 with $V_1(N^+)$-level structure $(A/R_0, \iota, \nu)$, a translation-invariant section of $\underline{\omega}_{A/R_0}^{\otimes k}$, subject to the following base-change axiom: if $\varphi : R_0 \to R'_0$ is a map of R-algebras one has

$$G((A/R_0, \iota, \nu) \otimes_\varphi R'_0) = \varphi^* G(A/R_0, \iota, \nu).$$

Given a modular form as in the third definition, we get a modular form as in the first definition by taking the section assigned to the universal QM abelian surface with level structure $\mathcal{A} \otimes R \to Sh \otimes R$. This is an equivalence because $\mathcal{A} \otimes R$ is universal. The last two definitions are related by

$$G(A, \iota, \nu) = F(A, \iota, \nu, \omega)\omega^{\otimes k},$$

where ω is any translation-invariant global section. This expression is independent of the choice of ω.

1.6.2 p-adic modular forms on Shimura curves

Let R be a p-adic ring (for p-adic ring we mean a complete and separated, with respect to the p-adic topology, $\mathbb{Z}_p$-algebra). Define the space $V_p(N^+, R)$ of p-**adic modular forms** of level $V_1(N^+)$ by

$$V_p(N^+, R) := \varprojlim_m H^0(\varprojlim_n I_n \otimes R/p^m R, \mathcal{O}_{\varprojlim_n I_n \otimes R/p^m R})$$

$$\cong \varprojlim_m \varinjlim_n H^0(I_n \otimes R/p^m R, \mathcal{O}_{I_n \otimes R/p^m R}),$$

where $\mathcal{O}$ is the structure sheaf. When $n = 0$ one can take the coordinate ring of the affine scheme obtained from $Sh \otimes R/p^m R$ by deleting the supersingular points, i.e., $H^0((Sh \otimes R/p^m R)^{\mathrm{ord}}, \mathcal{O}_{(Sh \otimes R/p^m R)^{\mathrm{ord}}})$. If $m = 0$, we take $H^0((I_{n/R}, \mathcal{O}_{I_{n/R}})$. Thus elements in $V_p(N^+, R)$ are formal functions on the tower I_n, i.e., $f \in V_p(N^+, R)$ is a rule that assigns to each quadruple $(A, \iota, \nu_{N^+}, \nu_p^\infty)$, where $(A, \iota, \nu_{N^+}, \nu_p^\infty)$ is a QM abelian

surface over an R-algebra R_0 with $V_1(N^+p^\infty)$-level structure, a value $f(A, \iota, \nu_{N^+}, \nu_p^\infty) \in R_0$, which depends only on the isomorphism class and that is compatible with base changes. We say that a p-adic modular form f is of weight $k \in \mathbb{Z}_p$, if for every $u \in \mathbb{Z}_p^\times$, we have

$$f(A, \iota, \nu_{N^+}, \nu_p^\infty) = u^{-k} f(A, \iota, \nu_{N^+}, \nu_p^\infty u),$$

where $(A, \iota, \nu_{N^+}, \nu_p^\infty)$ is a QM abelian surface over an R-algebra with $V_1(N^+p^\infty)$-level structure.

If f is a modular form with respect to B of weight k and level $V_1(N^+)$ over R as in 1.6.1, then we can see it as a p-adic modular form $\hat{f}$ as follows. The $V_1(N^+p^\infty)$-level structure on A/R_0 determines a point $P \in eT_pA_0^t(k)$, where A_0 is the reduction mod p of A. A point $P \in eT_pA_0^t(k)$ determines a differential $\omega_P \in \underline{\omega}_{A/R_0}$. Indeed, there is an isomorphism

$$T_p(A_0^t) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\hat{A}, \hat{\mathbb{G}}_m).$$

So, taking the homorphism α_P corresponding to the point P, one can consider the pullback $\omega_P := \alpha_P^*(dT/T) \in \underline{\omega}_{\hat{A}/R_0} = \underline{\omega}_{A/R_0}$, of the standard differential dT/T of $\hat{\mathbb{G}}_m$. See [Ka, §3.3] or the proof of [Brooks, Lemma 4.2] for details. One can define

$$\hat{f}(A, \iota, \nu_{N^+}, \nu_p^\infty) := f(A, \iota, \nu_{N^+}, \omega_P).$$

It follows from the definition that if f is a geometric modular form of weight k and level $V_1(N^+)$, then $\hat{f}$ is a p-adic modular form of weight k and level $V_1(N^+)$.

1.6.3 Jacquet–Langlands correspondence

The Jacquet–Langlands correspondence establishes a Hecke-equivariant bijection between automorphic forms on GL_2 and automorphic forms on multiplicative groups of quaternion algebras. In our setting, this can be stated as a correspondence between classical modular forms and quaternionic modular forms.

Theorem 1.6.4 (Jacquet–Langlands)**.** *There is a canonical (up to scaling) isomorphism*

$$M_k(Sh, \mathbb{C}) \xrightarrow[\mathrm{JL}]{\cong} S_k(\Gamma_1(N), \mathbb{C})^{D\text{-new}},$$

where $\Gamma_1(N)$ is the standard congruence group

$$\Gamma_1(N) := \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\},$$

and $S_k(\Gamma_1(N), \mathbb{C})^{D\text{-new}}$ is the space of classical cuspidal eigenforms with respect to $\Gamma_1(N)$, of weight k and that are new at D. This bijection is compatible with the Hecke-action and the Atkin-Lehner involutions on each of the spaces.

In particular, to each eigenform $f \in S_k(\Gamma_1(N), \mathbb{C})^{D\text{-new}}$ with Nebentypus ε_f with respect to the action of $\Gamma_0(N)$, corresponds a unique (up to scaling) quaternionic form $f_B = \mathrm{JL}(f) \in M_k(Sh, \mathbb{C})$ having the same Hecke eigenvalues as f for the good Hecke operators T_ℓ for $(\ell, D) = 1$ and the Atkin–Lehner involutions. Here

$$\Gamma_0(N) := \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N \right\}.$$

More precisely the Jacquet–Langlands correspondence asserts the existence of a holomorphic function f_B on the upper half plane, determined only up to a scalar multiple, such that f_B is a modular form for the discrete subgroup Γ_{1,N^+} of $\mathrm{GL}_2(\mathbb{R})$, of weight

k, with the same eigenvalues as f for the good Hecke operators and with Nebentypus ε_f for the action of Γ_{0,N^+}, where $\hat{\Gamma}_{0,N^+}$ is the open compact subgroup of $\hat{\mathcal{O}}_B^\times$ composed of the elements $b \in \hat{\mathcal{O}}_B^\times$ such that $\pi_{N^+}(b) \in \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in GL_2(\mathbb{Z}/N^+\mathbb{Z})\}$, and Γ_{0,N^+}^+ is the subgroup of matrices in $\Phi_\infty((\hat{\Gamma}_{0,N^+} \cap B)^\times)$ of determinant 1.

The function f_B gives rise canonically, as in [Brooks, §2.7], to a modular form in the sense of the geometric definition seen before, i.e., to a section of $\underline{\omega}_\mathbb{C} = e\pi_*\Omega_{\mathcal{A}\otimes\mathbb{C}/Sh\otimes\mathbb{C}}$. In particular, if we start from a classical modular forms for $\Gamma_0(N)$ we obtain a quaternionic modular form with trivial Nebentypus with respect to to the action of Γ_{0,N^+}.

Chapter 2

Deformation theory and t-expansions for modular forms

In order to associate with modular forms over Shimura curves power series expansions at CM points, we are interested in deformation theory. In particular, Serre–Tate deformation theory provides us with a way to do this. Thus, in this chapter we will study the deformation theory of QM abelian surfaces, which is closely related to the deformation theory of elliptic curves, as is well explained in [Buz]. Then we will define power series expansions for modular forms on Shimura curves.

As we will see, deformation theory is a tool to understand the local structure of a moduli space. We begin with an introduction to deformation functors associated with moduli problems.

2.1 Deformation functors

We start with a moduli functor $\mathcal{F}$, which in our case is a contravariant functor

$$\mathcal{F} : \mathrm{Sch}_{|\mathcal{W}(k)} \longrightarrow \mathcal{S}et$$

from the category of schemes over the ring of Witt vectors $\mathcal{W}(k)$ over $k := \overline{\mathbb{F}}_p$ to the category of sets. Since k is the residue field of $\mathcal{W}(k)$, there is a projection $\mathcal{W}(k) \twoheadrightarrow k$ that induces $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{W}(k))$, so we can see $\mathrm{Spec}(k)$ as a $\mathcal{W}(k)$-scheme. We want to analyze the local structure of $\mathcal{F}$ at a k-point of $\mathcal{F}$, i.e., an element of $\mathcal{F}(k) := \mathcal{F}(\mathrm{Spec}(k))$.

Consider the category $\mathcal{C}$ of local artinian rings with residue field isomorphic to k. To be precise, an object of $\mathcal{C}$ is a couple (A, α_A) where A is an artinian local ring with maximal ideal $\mathfrak{m}_A$ endowed with a fixed isomorphism $\alpha_A : A/\mathfrak{m}_A \xrightarrow{\cong} k$. Morphisms in this category are homomorphisms of local rings inducing the identity on k: a morphism ϕ between (A_1, α_1) and (A_2, α_2) in $\mathcal{C}$ is a ring homomorphism satisfying $\varphi(\mathfrak{m}_{A_1}) \subseteq \varphi(\mathfrak{m}_{A_2})$ or, equivalently, $\varphi^{-1}(\mathfrak{m}_{A_2}) = \mathfrak{m}_{A_1}$ and such that the triangle

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ \varphi\ } & A_2 \\
{\scriptstyle \alpha_{A_1}}\big\downarrow & \swarrow {\scriptstyle \alpha_{A_2}} & \\
k & &
\end{array}
$$

commutes. We will denote an object (A, α_A) of $\mathcal{C}$ only by A, disregarding the isomorphism but keeping it in mind and remembering that a morphism in $\mathcal{C}$ must induce the identity on k. The ring of Witt vectors $\mathcal{W}(k)$ with the natural projection is an object

of $\mathcal{C}$ and for any ring A in $\mathcal{C}$ there exists a unique morphism $\mathcal{W}(k) \to A$ (in $\mathcal{C}$). Thus, each ring in $\mathcal{C}$ has a canonical structure as a $\mathcal{W}(k)$-algebra and $\mathrm{Spec}(R)$ has a canonical structure as a $\mathcal{W}(k)$-scheme.

We associate with the functor $\mathcal{F}$ and a k-point $x \in \mathcal{F}(k)$ a covariant *deformation funtor*

$$\widehat{\mathcal{F}}_x : \mathcal{C} \longrightarrow \mathcal{S}et$$

given by

$$R \longmapsto \widehat{\mathcal{F}}_x(R) := \mathcal{F}(\mathrm{Spec}(R)) \times_{\mathcal{F}(k)} \{x\},$$

where the fiber product is determined by the diagram

$$
\begin{array}{ccc}
\mathcal{F}(\mathrm{Spec}(R)) \times_{\mathcal{F}(k)} \{x\} & \longrightarrow & \{x\} \\
\downarrow & & \downarrow \\
\mathcal{F}(\mathrm{Spec}(R)) & \xrightarrow{\mathcal{F}(\alpha_R)} & \mathcal{F}(k)
\end{array}
$$

in which α_R is the morphism that comes from the projection $R \twoheadrightarrow k$ induced by the isomorphism associated with R. There is a natural identification

$$\widehat{\mathcal{F}}_x(R) = \{y \in \mathcal{F}(\mathrm{Spec}(R)) \mid \mathcal{F}(\alpha_R)(y) = x\}.$$

The functor $\widehat{\mathcal{F}}_x$ is called the **completion of $\mathcal{F}$ at x**.

Suppose that the moduli functor $\mathcal{F}$ is represented by a $\mathcal{W}(k)$-scheme X (later on, our aim will be to consider a deformation functor associated with our Igusa tower over $Sh_{|_{\mathcal{W}(k)}} := Sh \otimes_{\mathbb{Z}_{(p)}} \mathcal{W}(k)$). Since X represents $\mathcal{F}$, there is a natural isomorphism

$$\mathcal{F} \cong \mathrm{Hom}_{\mathcal{W}(k)}(-, X),$$

so for any ring R in $\mathcal{C}$ we have that $\mathcal{F}(\mathrm{Spec}(R))$ coincides with the set of R-points $X(R) := \mathrm{Hom}_{\mathcal{W}(k)}(\mathrm{Spec}(R), X)$ of X. In light of this, we will usually not distinguish between X and $\mathcal{F}$. In this case, one can describe $\widehat{\mathcal{F}}_x$ as follows. Choose a k-point $x : \mathrm{Spec}(k) \to X$ in $X(k)$. The **formal completion of X at x** is the deformation functor

$$\widehat{X}_x : \mathcal{C} \longrightarrow \mathcal{S}et$$
$$R \longmapsto \widehat{X}_x(R) := X(R) \times_{X(k)} \{x\},$$

so $\widehat{X}_x(R) = \{y \in X(R) \mid y \circ \alpha_R = x\}$.

Because the only homomorphism $k \to k$ that respects the $\mathcal{W}(k)$-algebra structures is the identity, for every point $z \in X$ there is a unique k-point in $X(k)$ with image z. Indeed, each morphism $x' : \mathrm{Spec}(k) \to X$ factors through the canonical morphism

$$i_z : \mathrm{Spec}(k(z)) \to X,$$

where $k(z)$ is the residue field of $z = \mathrm{im}\, x'$. Then there is a sequence

$$\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k(z)) \xrightarrow{i_z} X \longrightarrow \mathrm{Spec}(\mathcal{W}(k))$$

that is induced by the sequence of $\mathcal{W}(k)$-algebras

$$\mathcal{W}(k) \longrightarrow k(z) \longrightarrow k.$$

2.1. Deformation functors

Then the first arrow $\mathcal{W}(k) \to k(z)$ must be local and factor through k, so there is a sequence
$$\mathcal{W}(k) \longrightarrow\!\!\!\!\!\to k \longrightarrow k(z) \longrightarrow k$$
and $k(z) \cong k$.

Starting with k-points $x_1, x_2 : \operatorname{Spec}(k) \to X$ with image z, we obtain a diagram

$$
\begin{array}{ccc}
 & & k \\
 & \nearrow & \\
 & \cong \uparrow\!\!\swarrow & \\
\mathcal{W}(k) \longrightarrow k(z) & & \\
 & \searrow & \\
 & \cong & \\
 & & k.
\end{array}
$$

Therefore the two isomorphisms from $k(z)$ to k must be the same, so $x_1 = x_2$. This argument shows that there is a bijection between k-points of X as a $\mathcal{W}(k)$-scheme and points of X with residue field isomorphic to k. For this reason we identify our fixed point $x \in X(k)$ with its image. This implies that
$$\widehat{X}(R) = \big\{ y \in X(R) \mid \operatorname{im} y = x \big\},$$
as the points y such that $y \circ \alpha_R = x$ are exactly the R-points with image x.

For every $z \in X$, there is a bijection
$$
\left\{
\begin{array}{c}
\operatorname{Spec}(R) \to X \text{ in} \\
\operatorname{Sch}_{|\mathcal{W}(k)} \text{ with image } z
\end{array}
\right\}
\xrightarrow{\cong}
\operatorname*{Hom}_{\substack{\text{loc. rings} \\ \mathcal{W}(k)\text{-alg.}}}(\mathcal{O}_{X,z}, R)
$$
$$
f \longmapsto \big(f^{\#} : \mathcal{O}_{X,z} \to \mathcal{O}_{\operatorname{Spec}(R),\mathfrak{m}_R} = R \big).
$$

Since R is complete, we have
$$
\operatorname*{Hom}_{\substack{\text{loc. rings} \\ \mathcal{W}(k)\text{-alg.}}}(\mathcal{O}_{X,z}, R) = \operatorname*{Hom}_{\substack{\text{loc. rings} \\ \mathcal{W}(k)\text{-alg.}}}\big(\widehat{\mathcal{O}}_{X,z}, R\big),
$$
where $\widehat{\mathcal{O}}_{X,z}$ is the completion of the local ring $\mathcal{O}_{X,z}$, which is the stalk at x of the structure sheaf $\mathcal{O}_X$ of X.

If our fixed point x is a smooth point, the ring $\widehat{\mathcal{O}}_{X,z}$ is a complete noetherian local ring with residue field k. If we denote by $\mathcal{C}'$ the category of complete noetherian local rings with residue field k, of which $\mathcal{C}$ is a full subcategory, then
$$
\operatorname*{Hom}_{\substack{\text{loc. rings} \\ \mathcal{W}(k)\text{-alg.}}}\big(\widehat{\mathcal{O}}_{X,x}, R\big) = \operatorname*{Hom}_{\mathcal{C}'}\big(\widehat{\mathcal{O}}_{X,x}, R\big).
$$

Because of the functoriality of the bijections above, $\widehat{X}_x$ is "quasi-represented" by $\widehat{\mathcal{O}}_{X,x}$:
$$
\widehat{X}_x \cong \operatorname{Hom}_{\mathcal{C}'}\big(\widehat{\mathcal{O}}_{X,x}, -\big).
$$

More precisely, $\widehat{X}_x$ is "pro-representable", in the following sense. Every functor of artinian rings $\mathcal{G} : \mathcal{C} \to \mathcal{S}et$ can be extended to a functor $\mathcal{G}' : \mathcal{C}' \to \mathcal{S}et$ by setting $\mathcal{G}'(R) := \varprojlim_n \mathcal{G}(R/\mathfrak{m}^n)$ for every R in $\mathcal{C}'$ with maximal ideal $\mathfrak{m}$. If $\mathcal{G}'$ is representable by an element R in $\mathcal{C}'$, then $\mathcal{G}$ is said to be *pro-representable*. An element x of $\mathcal{G}'(R)$ is called a *formal element* of $\mathcal{G}$. By definition, a formal element $x \in \mathcal{G}(R)$ can be represented as a system of elements $\{x_n\}_n$, where $x_n \in \mathcal{G}(R/\mathfrak{m}^n)$, compatible with respect to the maps $\mathcal{G}(R/\mathfrak{m}^{n+1}) \to \mathcal{G}(R/\mathfrak{m}^n)$ induced by the projections $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$. In our case, $\widehat{X}'_x$ coincide with $\widehat{X}_x$ extended to $\mathcal{C}'$ by the same rule $R \mapsto X(R) \times_{X(k)} \{x\}$, so we will not distinguish between all these functors and simply denote them all by $\widehat{X}_x$.

2.2 Serre–Tate deformation theory

Following [Ka], we introduce the Serre–Tate deformation theory for ordinary abelian varieties, which provides a way to attach power series expansions to modular forms on Shimura curves, replacing classical q-expansions for "elliptic" modular forms that are not available in our case.

Fix an algebraically closed field k of characteristic $p > 0$ (for our goals, we can take $k = \overline{\mathbb{F}}_p$) and consider an *ordinary* abelian variety A over k. Recall that an abelian variety A over k is said to be ordinary if $A[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^{\dim(A)}$. Let A^t be the dual abelian variety, which is isogenous to A and hence ordinary too. Consider the Tate modules

$$T_p A := \varprojlim_n A[p^n](k), \qquad T_p A^t := \varprojlim_n A^t[p^n](k)$$

of A and A^t. Because of the ordinarity assumption on A, $T_p A$ and $T_p A^t$ are free $\mathbb{Z}_p$-modules of rank $g := \dim A = \dim A^t$.

Definition 2.2.1. If R is an artinian local ring with maximal ideal $\mathfrak{m}_R$ and residue field k, a **deformation** of A to R is an abelian scheme $\mathcal{A}$ over R together with an identification $\mathcal{A} \times_R k \cong A$.

Following a construction due to Serre and Tate, we attach to such a deformation a $\mathbb{Z}_p$-bilinear form

$$q(\mathcal{A}/R; -, -) : T_p A \times T_p A^t \longrightarrow \widehat{\mathbb{G}}_m(R) = 1 + \mathfrak{m}_R,$$

where $\widehat{\mathbb{G}}_m := \mathrm{Spf}\big(k[T, S]/(TS - 1)\big)$ is the completion of the multiplicative group scheme $\mathbb{G}_m := \mathrm{Spec}\big(k[T, S]/(TS - 1)\big)$ over k. This bilinear map is constructed from the Weil pairings

$$e_{p^n} : A[p^n] \times A^t[p^n] \longrightarrow \mu_{p^n}$$

of k-group schemes, as defined in [Oda]. These pairings come from Cartier duality for the p-divisible groups $A[p^\infty]$ and $A^t[p^\infty](k)$ (duality of abelian schemes is compatible with Cartier duality). Here μ_{p^n} is $\mathrm{Spec}\big(k[T]/(T^{p^n} - 1)\big)$, the k-group scheme of p^n-th roots of unity, which can be seen inside $\mathbb{G}_m$ through the map $k[T, S]/(TS - 1) \to k[T]/(T^{p^n} - 1)$ defined by $T \mapsto T$ and $S \mapsto T^{p^n - 1}$. For each k-algebra R, $\mu_{p^n}(R)$ corresponds to the p^n-torsion of $\mathbb{G}_m(R)$. See, e.g., [Lip, p. 16] or [Feng] for details.

We briefly sketch the construction of the bilinear map $q(\mathcal{A}/R; -, -)$. Choose an integer $n \geq 0$ such that $\mathfrak{m}_R^n = 0$. Since $p \in \mathfrak{m}_R$, $\widehat{\mathcal{A}}(R) := \ker\big(\mathcal{A}(R) \to \mathcal{A}(k) = A(k)\big)$ is killed by p^n. Let $P \in A(k)$; for any lift $\tilde{P} \in \mathcal{A}(R)$ of P, since $\widehat{\mathcal{A}}(R)$ is killed by p^n, we have that $p^n \tilde{P}$ is independent of the choice of the lift $\tilde{P}$. The existence of a lift $\tilde{P} \in \mathcal{A}(R)$ of $P \in A(k)$ is guaranteed by the smoothness of $\mathcal{A}/R$ ([Liu, Corollary 2.13]). Therefore we obtain a map $A(k) \xrightarrow{"p^n"} \mathcal{A}(R)$. If we take $P \in A[p^n](k)$, then "p^n"$P \in \widehat{\mathcal{A}}(R)$, so we get

$$\text{``}p^n\text{''} : A[p^n] \longrightarrow \widehat{\mathcal{A}}(R).$$

Because of the compatibility of the maps "p^n" when $n \gg 0$, we obtain a homomorphism

$$\text{``}p^n\text{''} : T_p A \twoheadrightarrow A[p^n](k) \xrightarrow{"p^n"} \widehat{\mathcal{A}}(R)$$

that is independent of n.

Now, restricting the Weil pairings

$$e_{p^n} : \widehat{A}[p^n] \times A^t[p^n] \longrightarrow \mu_{p^n}$$

2.2. Serre–Tate deformation theory

for every $n \geq 1$, we obtain a perfect pairing, and then an isomorphism

$$\widehat{A}[p^n] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p}\big(A^t[p^n], \mu_{p^n}\big)$$

of k-group-schemes. Because of the compatibility of the pairings with respect to n, passing to the limit, we deduce an isomorphism

$$\widehat{A}(k) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p A^t, \widehat{\mathbb{G}}_m\big),$$

of formal groups over k.

Since R is artinian, the p-divisible group $\mathcal{A}[p^\infty]$ has a canonical structure of an extension, as given by

$$0 \longrightarrow \widehat{\mathcal{A}} \longrightarrow \mathcal{A}[p^\infty] \longrightarrow T_p A \times (\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0$$

of the constant p-divisible group $T_p A(k) \times (\mathbb{Q}_p/\mathbb{Z}_p)$ over R by $\widehat{\mathcal{A}}$, which is the unique toroidal formal group over R lifting $\widehat{A}$. Then the preceding two isomorphisms extend uniquely to isomorphisms of R-group schemes

$$\widehat{\mathcal{A}}[p^n](R) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p}\big(A^t[p^n], \mu_{p^n}\big)$$

and

$$\widehat{\mathcal{A}}(R) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p A^t, \widehat{\mathbb{G}}_m\big)$$

(see the proof of [Ka, Theorem 2.1]), giving pairings

$$e_{p^n,\mathcal{A}} : \widehat{\mathcal{A}}[p^n](R) \times A^t[p^n] \longrightarrow \mu_{p^n},$$

and

$$e_{\mathcal{A}} : \widehat{\mathcal{A}}(R) \times T_p A^t \longrightarrow \widehat{\mathbb{G}}_m.$$

Finally, the map $q(\mathcal{A}/R; -, -)$ is defined by

$$q\big(\mathcal{A}/R; P, Q^t\big) := e_{\mathcal{A}}\big(\text{``}p^n\text{''}P, Q^t\big),$$

for $P \in T_p A$ and $Q^t \in T_p A^t$.

Theorem 2.2.2 (Serre–Tate). *With notation as above, the construction*

$$\mathcal{A}/R \longmapsto q(\mathcal{A}/R; -, -) \in \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p A \otimes T_p A^t, \widehat{\mathbb{G}}_m(R)\big)$$

establishes a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{deformations of } A/k \text{ to } R \end{array} \right\} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p}(T_p A(k) \otimes T_p A^t(k), \widehat{\mathbb{G}}_m(R)).$$

Furthermore, this correspondence is functorial in R, i.e., if $\mathcal{F}$ is the deformation functor from the category $\mathcal{C}$ of artinian local rings with residue field k to the category of sets given by

$$\mathcal{F} : R \longmapsto \mathcal{F}(R) := \big\{\text{isomorphism classes of deformations of } A/k \text{ to } R\big\},$$

then there is an isomorphism of functors

$$\mathcal{F} \sim \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p A \otimes T_p A^t, \widehat{\mathbb{G}}_m\big).$$

Proof. This is [Ka, Theorem 2.1, 1) and 2)]. $\qquad\square$

The proof of the theorem rests on the fact that there is an equivalence

$$\left\{\begin{array}{c} \text{isomorphism classes of} \\ \text{deformations } \mathcal{A}/R \text{ of } A/k \end{array}\right\} \overset{\cong}{\longrightarrow} \left\{\begin{array}{c} \text{isomorphism classes of} \\ \text{deformations } \mathcal{A}[p^\infty]/R \text{ of } A[p^\infty]/k \end{array}\right\},$$

so deforming an ordinary abelian variety A/k is the same that deforming its p-divisible group $A[p^\infty]$.

Taking inverse limits, we can replace the category of artinian local rings with the category of complete noetherian local rings in the preceding discussion. We can do this because of the compatibility of these correspondences with inverse limits and of the fact that every complete noetherian local ring is the inverse limit of artinian local rings (if R is a complete noetherian local ring with maximal ideal $\mathfrak{m}$, then $R \cong \varprojlim_n R/\mathfrak{m}^n$). However, the procedure for computing the pairings $q_{\mathcal{A}/R}$ only makes sense for artinian local rings.

Passing to complete noetherian local ring is useful because the deformation functor is not representable by an artinian local ring in $\mathcal{C}$ but is pro-representable by a complete noetherian local ring. Namely, the deformation functor $\mathcal{F}$ is pro-represented by a complete local noetherian ring $\mathcal{R}^u$ that is non-canonically isomorphic to the power series ring $\mathcal{W}[[T_{ij}, 1 \leq i, j \leq g]]$, where we have set $\mathcal{W} := \mathcal{W}(k)$. Therefore the functor $\mathcal{F}$ can be seen as a formal scheme $\mathrm{Spf}(\mathcal{R}^u)$. Denote by $\widehat{\mathcal{A}}^u/\mathrm{Spf}(\mathcal{R}^u)$ the universal formal deformation of A/k, i.e., the formal element of $\mathcal{F}$ corresponding to the identity in $\mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathcal{R}^u, \mathcal{R}^u)$.

Given elements $P \in T_p A$, $P^t \in T_p A^t$, there is a map

$$\mathcal{F} \longrightarrow \widehat{\mathbb{G}}_m$$
$$\mathcal{A}/R \longmapsto q(\mathcal{A}/R; P, P^t).$$

If we pick $\mathbb{Z}_p$-bases $\{P_1, \ldots, P_g\}$ and $\{P_1^t, \ldots, P_g^t\}$ of $T_p A(k)$ and $T_p A^t(k)$, respectively, then we have g^2 functions

$$t_{ij} : \mathcal{F} \longrightarrow \widehat{\mathbb{G}}_m$$
$$\mathcal{A}/R \longmapsto q(\mathcal{A}/R; P_i, P_j^t)$$

called **Serre–Tate coordinates** and g^2 elements $t_{ij}(\widehat{\mathcal{A}}^u/\mathcal{R}^u) \in \mathcal{R}$. Writing $T_{ij} := t_{ij} - 1$, there is a ring isomorphism

$$\mathcal{R}^u \cong \mathcal{W}[[\{T_{ij}\}]].$$

We conclude with the following

Proposition 2.2.3. *Let $f : A \to B$ be a morphism of ordinary abelian varieties over k, let $f^t : B^t \to A^t$ be the dual morphism of f and let $\mathcal{A}/R$ and $\mathcal{B}/R$ be deformations of A/k and B/k to R. Then f lifts to a morphism $F : \mathcal{A} \to \mathcal{B}$ of deformations if and only if*

$$q\big(\mathcal{A}/R; P, f^t(Q^t)\big) = q\big(\mathcal{B}/R; f(P), Q^t\big)$$

for every $P \in T_p A(k)$ and $Q^t \in T_p B^t(k)$. Furthermore, if a lifting exists, then it is unique.

Proof. This is [Ka, Theorem 2.1, 4)]. $\qquad\square$

2.3   Serre–Tate coordinates for Shimura curves

Take now an *ordinary* QM abelian surface A over k with a $V_1(N^+)$-level structure. We want to deform our abelian surface not only as an abelian surface but also with its structures. Thus, we consider the subfunctor $\mathcal{M}$ of $\mathcal{F} = \mathrm{Spf}(\mathcal{R}^u)$ which sends an artinian local ring R with residue field k to the set of deformations of A to R, where by deformation of A to R we mean a deformation $\mathcal{A}$ of A to R together with an embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}_R(\mathcal{A})$ deforming the given embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}_k(A)$ and a $V_1(N^+)$-level structure on $\mathcal{A}$ deforming the given $V_1(N^+)$-level structure on A. The $V_1(N^+)$-level structure automatically lifts uniquely, as $A[N^+]$ is étale over R, so we can ignore it in our discussion.

Consider the idempotent e that acts as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p)$ on T_pA (i_K and Φ_p can be chosen to be compatible, by the choice of $\mathfrak{p}$ over $p = \mathfrak{p}\bar{\mathfrak{p}}$ split in K). We can find a $\mathbb{Z}_p$-basis $\{P_1, P_2\}$ of T_pA such that $eP_1 = P_1$ and $eP_2 = 0$, indeed $T_pA = eT_pA \oplus (1-e)T_pA$. Then $P_1^t \in (eT_pA)^t$.

Proposition 2.3.1. *The subfunctor $\mathcal{M}$ of $\mathcal{F}$ is pro-representable by a ring $\mathcal{R}^f$ that is a quotient of $\mathcal{R}^u$. In fact, $\mathcal{R}^f$ is the quotient of $\mathcal{R}^u$ by the closed ideal generated by the relations*

$$q\big(\widehat{\mathcal{A}}^u/\mathcal{R}^u; bP, Q^t\big) = q\big(\widehat{\mathcal{A}}^u/\mathcal{R}^u; P, b^*Q^t\big)$$

for any $b \in B, P \in T_pA, Q^t \in T_pA^t$. Furthermore, there is an isomorphism

$$\mathcal{R}^f \cong \mathcal{W}[[T_{11}]],$$

where $T_{11} = t_{11} - 1$ and t_{11} corresponds to $q\big(\widehat{\mathcal{A}}^u/\mathcal{R}^u; P_1, P_1^t\big)$.

Proof. This is a consequence of Proposition 2.2.3. For details, see [Brooks, Proposition 4.5] and [Mo, Proposition 3.3].   □

Thus, deformations of the QM abelian surface A/k depend only on the e-component eT_pA of T_pA.

By what we have seen in §2.1, since the deformation functor $\mathcal{M}$ is the deformation functor associated with $Sh^{\mathrm{ord}}_{|\mathcal{W}(k)}$, i.e., the ordinary part of $Sh_{|\mathcal{W}(k)}$, and the point $x \in Sh^{\mathrm{ord}}(k)$ corresponding to the fixed ordinary QM abelian surface A/k with $V_1(N^+)$-level structure, it follows that $\mathcal{M}$ is the formal completion $\widehat{Sh}^{\mathrm{ord}}_x$ of $Sh^{\mathrm{ord}}_{|\mathcal{W}}$ at x and so it is the formal spectrum $\mathrm{Spf}(\widehat{\mathcal{O}}_{Sh^{\mathrm{ord}},x})$, where $\mathcal{O}_{Sh^{\mathrm{ord}},x}$ is the local ring of Sh^{ord} at x.

2.4   Deformations of QM abelian surfaces

In the case of QM abelian surfaces, the coordinate ring of the deformation functor has only one coordinate obtained by choosing a point $P \in T_pA$ such that $eP = P$, as we have seen in the previous section. Also in the case of elliptic curves there is only one coordinate obtained by choosing a point $P \in T_pE$. Actually, there is a strict link between deformations of QM abelian surfaces and deformations of elliptic curves.

Take an ordinary QM abelian surface A/k where k is again $\overline{\mathbb{F}}_p$. Then, as already pointed out, its deformation theory is equivalent to the deformation theory of the p-divisible group $A[p^\infty]$ (see the proof of [Ka, Theorem 2.1]). The p-divisible group $A[p^\infty]$ attached to A inherits an action of $\mathcal{O}_B$ and hence of $\mathcal{O}_B \otimes \mathbb{Z}_p$, which is identified with $M_2(\mathbb{Z}_p)$ via the fixed isomorphism Φ_p. If we set $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p)$ (the idempotent e acts as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p)$ on $A[p^\infty]$), then $A[p^\infty]$ splits as $eA[p^\infty] \oplus (1 - e)A[p^\infty]$.

Moreover, $eA[p^\infty]$ and $(1-e)A[p^\infty]$ are isomorphic via multiplication by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since A is ordinary, there is an isomorphism $A[p^\infty] \cong E[p^\infty]^2$ for E/k an ordinary elliptic curve with $E[p^\infty] \cong eA[p^\infty]$ (see [Buz, Corollary 4.6]). Following [Buz], we want to recover the deformation theory of A from the deformation theory of an elliptic curve. Deforming $A[p^\infty]$ with its $\mathcal{O}_B$-action is the same as deforming $A[p^\infty]$ with its $M_2(\mathbb{Z}_p) \cong \mathcal{O}_B \otimes \mathbb{Z}_p$-action. According to Theorem 2.3.1, this is equivalent to deforming $eA[p^\infty]$, therefore the deformation theory of A/k (or $A[p^\infty]$) is equivalent to the deformation theory of E/k (or $E[p^\infty]$).

We want to relate the bilinear map $q_\mathcal{A}$, associated with a deformation $\mathcal{A}/R$ of a QM abelian surface A/k, to the map $q_\mathcal{E}$ associated with the deformation $\mathcal{E}/R$, corresponding to $\mathcal{A}/R$, of an elliptic curve E/k, when there is an isomorphism of p-divisible groups

$$\alpha : eA[p^\infty] \xrightarrow{\cong} E[p^\infty]$$

over k. So we start by comparing the Weil pairings. Since the Weil pairing comes from Cartier duality for p-divisible groups, there is a commutative diagram for the Weil pairings

$$
\begin{array}{ccc}
eA[p^n] \times (eA[p^n])^t & \xrightarrow{e_{p^n,A}} & \mu_{p^n} \\
\downarrow{\scriptstyle \alpha_n \times (\alpha_n^t)^{-1}} & & \uparrow{\scriptstyle =} \\
E[p^n] \times E^t[p^n] & \xrightarrow{e_{p^n,E}} & \mu_{p^n}
\end{array}
$$

where α_n is the n-component of α and the first line in the diagram is the restriction of the Weil pairing associated with A to $eA[p^n] \times (eA[p^n])^t$ (Cartier duality is compatible with duality of abelian schemes, so $(eA[p^n])^t \hookrightarrow (A[p^n])^t \cong A^t[p^n]$ and $(eA[p^n])^t \cong (E[p^n])^t \cong E^t[p^n]$). This means that for each $P \in eA[p^n](k)$ and $Q^t \in E^t[p^n](k)$, we have

$$e_{p^n,A}\big(P, \alpha_n^t(Q^t)\big) = e_{p^n,E}\big(\alpha_n(P), Q^t\big).$$

The same is true when we take inverse limits.

Considering the completions at the origin and restricting the pairings, we obtain

$$
\begin{array}{ccc}
e\widehat{A}[p^n] \times (eA[p^n])^t & \xrightarrow{e_{p^n,A}} & \mu_{p^n} \\
\downarrow{\scriptstyle \cong} & & \\
\widehat{eA}[p^n] \times (eA[p^n])^t & & \\
\downarrow{\scriptstyle \alpha_n \times (\alpha_n^t)^{-1}} & & \\
\widehat{E}[p^n] \times E^t[p^n] & \xrightarrow{e_{p^n,E}} & \mu_{p^n}
\end{array}
$$

(with the right vertical arrows $=$)

because the functor $G \mapsto \widehat{G}$ sending a p-divisible group to its completion at the origin is exact and the connected-étale sequence is functorial. Then passing to the limits yields pairings between Tate modules and the commutative diagram

$$
\begin{array}{ccc}
e\widehat{A}(k) \times (eT_pA)^t & \xrightarrow{E_A} & \widehat{\mathbb{G}}_m \\
\downarrow{\scriptstyle \alpha \times (\alpha^t)^{-1}} & & \uparrow{\scriptstyle =} \\
\widehat{E}(k) \times T_pE^t & \xrightarrow{E_E} & \widehat{\mathbb{G}}_m.
\end{array}
$$

When we extend these pairings to $e\widehat{\mathcal{A}}$ and $\widehat{\mathcal{E}}$, everything works well because of the functoriality of the structure of extensions of p-divisible groups and the fact that we are deforming also the action of $\mathcal{O}_B$ and so the action of e, so that $e\mathcal{A}[p^\infty] \cong \mathcal{E}[p^\infty]$.

2.5. Deformation at points of I

Observe everything works fine for the "p^n" maps as well, and there is a commutative diagram

$$
\begin{array}{ccccc}
T_p E & \longrightarrow & E[p^n] & \xrightarrow{\ "p^n"\ } & \widehat{\mathcal{E}}(R) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
eT_p A & \longrightarrow & eA[p^n] & \xrightarrow{\ "p^n"\ } & e\widehat{\mathcal{A}}(R),
\end{array}
$$

again because we are deforming also the action of $\mathcal{O}_B$ and so the action of e.

In conclusion, computing the bilinear map on $eT_pA \times (eT_pA)^t$ is the same as computing it on $T_pE \times T_pE^t$, that is

$$
q(\mathcal{A}; P, Q^t) = q\big(E, \alpha(P), (\alpha^t)^{-1}(Q^t)\big),
$$

for all $P \in eT_pA$ and $Q^t \in (eT_pA)^t \subseteq T_pA^t$.

2.5 Deformation at points of I

If A/k is an ordinary QM abelian surface with $A[p] \cong E[p]^2$ as $\mathcal{O}_B$-group schemes, where here $\mathcal{O}_B$ acts via the natural action of $\mathcal{O}_B \otimes \mathbb{F}_p \cong M_2(\mathbb{F}_p)$ on $E[p]^2$, then there is an induced isomorphism between the set of $V_1(p)$- level structures on A and the set of $\Gamma_1^{\mathrm{arith}}(p)$-level structures on E. In the same way, when $A[p^\infty] \cong E[p^\infty]^2$ there is a bijection between the set of $V_1(p^\infty)$- level structures on A and the set of $\Gamma_1^{\mathrm{arith}}(p^\infty)$-level structures on E. Thus, the deformation theory of a k-point in $I_n(k)$, or in the Igusa tower, is equivalent to the deformation theory of the associated elliptic curve viewed as a k-point of the scheme parameterizing elliptic curves with $\Gamma_1^{\mathrm{arith}}(N^+ p^n)$- or $\Gamma_1^{\mathrm{arith}}(N^+ p^\infty)$-level structures. In light of what we have seen in the previous section, we can use this equivalence to compute Serre–Tate coordinates.

2.6 t-expansions for modular forms

Let us start from an $\overline{\mathbb{F}}_p$-point x in the Igusa tower, i.e., the isomorphism class of a quadruple $(A/\overline{\mathbb{F}}_p, \iota, \nu_{N^+}, \nu_{p^\infty})$. Then the $V_1(p^\infty)$-level structure on $A_{|\overline{\mathbb{F}}_p}$ determines a point $P^t \in (eT_pA)^t$ (cf. [CH, §3.1]). Take $P \in eT_pA$ corresponding to P^t via the principal polarization. We fix the Serre–Tate coordinate t_x around x to be

$$
t_x := q(-; P, P^t).
$$

Denote by $\big(\mathcal{A}/\mathcal{W}[[T]], \iota, \boldsymbol{\nu}_{N^+}, \boldsymbol{\nu}_{p^\infty}\big)$ the universal deformation of x and note that we can evaluate every p-adic modular form $f \in V_p(N^+, \mathcal{W})$ at $\big(\mathcal{A}/\mathcal{W}[[T]], \iota, \boldsymbol{\nu}_{N^+}, \boldsymbol{\nu}_{p^\infty}\big)$. We call

$$
f(t_x) := f\big(\mathcal{A}/\mathcal{W}[[T]], \iota, \boldsymbol{\nu}_{N^+}, \boldsymbol{\nu}_{p^\infty}\big) \in \mathcal{W}[[T]],
$$

where $T := t_x - 1$, the *t-expansion of f at x*.

2.7 On Serre–Tate coordinates at CM points

In this section we want to obtain a result analogous to [CH, Lemma 3.2] in our setting.

If $\mathfrak{a}$ is a prime to cpN fractional ideal of $\mathcal{O}_c$ with $p \nmid c$, then $A_{\mathfrak{a}}$ has a model defined over $\mathcal{V} := \mathcal{W} \cap K^{\mathrm{ab}}$. Here $x(\mathfrak{a}) = (A_{\mathfrak{a}}, \iota_{\mathfrak{a}}, \nu_{\mathfrak{a}, N^+}, \nu_{\mathfrak{a}, p^\infty}) \in I(\mathcal{V})$ is as defined in §1.5.5.

Denote by t the Serre–Tate coordinate around $\overline{x}(\mathfrak{a}) := x(\mathfrak{a}) \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p$. For $u = 1, \dots, p^n - 1$ with $(u, p) = 1$, set

$$x(\mathfrak{a}) \star \alpha(u/p^n) := x(cp^n)^{\mathrm{rec}_K(a^{-1}u_{\mathfrak{p}}p_{\mathfrak{p}}^{-n})} \cdot u \in I(\mathcal{V}),$$

where $\mathrm{rec}_K : K^{\times} \backslash \hat{K}^{\times} \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the geometrically normalized reciprocity law map, $a \in \hat{K}^{(cp)\times}$ is such that $\mathfrak{a} = a\hat{\mathcal{O}}_c \cap K$ and the subscript $b_{\mathfrak{p}}$ for $b \in \mathbb{Z}_p^{\times}$ denotes its image in $\hat{K}^{\times}$ under the inclusions $\mathbb{Z}_p^{\times} \subseteq K_{\mathfrak{p}}^{\times} \subseteq \hat{K}^{\times}$.

Lemma 2.7.1. *With notation as above, one has that $\left(x(\mathfrak{a}) \star \alpha(u/p^n)\right) \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p = \overline{x}(\mathfrak{a})$ and $t\left(x(\mathfrak{a}) \star \alpha(u/p^n)\right) = \zeta_{p^n}^{-uN(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}}$.*

Proof. The p-divisible module $eA_x[p^{\infty}]$, with A_x the QM abelian surface corresponding to $x = x(\mathfrak{a}) \star \alpha(u/p^n)$, is exactly the p-divisible module associated with the point $x_{\mathfrak{a}} \star \mathbf{n}(up^{-n})$ considered in [CH, Lemma 3.2] (cf. [CH, §4.5]). Hence, x in the deformation space of $\overline{x}(\mathfrak{a})$ corresponds to $x_{\mathfrak{a}} \star \mathbf{n}(up^{-n})$ in the deformation space of $x_{\mathfrak{a}} \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p$, where $x_{\mathfrak{a}}$ is the CM point defined in [CH, §2.4].

Set $\overline{A}_{\mathfrak{a}} := A_{\mathfrak{a}} \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p$ and $\overline{E}_{\mathfrak{a}} := E_{\mathfrak{a}} \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p$, with $E_{\mathfrak{a}}$ the elliptic curve corresponding to the CM point $x_{\mathfrak{a}}$. Since the point $P^t \in (eT_p\overline{A}_{\mathfrak{a}})^t$ that is determined by the $V_1(p^{\infty})$-level structure is the same as the point that is determined by the $\Gamma_1^{\mathrm{arith}}(p^{\infty})$-level structure on $T_p\overline{E}_{\mathfrak{a}}^t$, the claim follows from the computations of §2.4 and [CH, Lemma 3.2]. $\square$

Chapter 3

Anticyclotomic p-adic L-functions

In this chapter we will define our p-adic L-function as a measure on $\mathrm{Gal}(K_{p^\infty}/K)$ with values in $\mathcal{W}$, which is again the ring of Witt vectors $\mathrm{Witt}(\overline{\mathbb{F}}_p)$, i.e., the ring of integer of the completion of the maximal unramified extension of $\mathbb{Q}_p$. In order to do this, we study some theory of p-adic measures on $\mathbb{Z}_p$ and of measures on $\mathrm{Gal}(K_{p^\infty}/K)$.

3.1 Measures on $\mathbb{Z}_p$

We start by giving the definition of a p-adic measure and stating some properties; for more details, the reader is referred to [Hi93, Chapter 3].

Let $\mathcal{C}(\mathbb{Z}_p, \mathcal{W})$ be the $\mathcal{W}$-algebra of continuous functions $\mathbb{Z}_p \to \mathcal{W}$.

Definition 3.1.1. A **p-adic measure** on $\mathbb{Z}_p$ with values in $\mathcal{W}$ is a $\mathcal{W}$-linear function $\mu : \mathcal{C}(\mathbb{Z}_p, \mathcal{W}) \to \mathcal{W}$ such that there exists a constant $B \geq 0$ with

$$\big|\mu(\varphi)\big|_p \leq B|\varphi|_p$$

for each $\varphi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$, where $|\varphi|_p := \sup_{x \in \mathbb{Z}_p} \big|\varphi(x)\big|_p$.

We will write $\int_{\mathbb{Z}_p} \varphi \, d\mu := \mu(\varphi)$ for the value of a measure μ on a continuous function φ. We denote by $M(\mathbb{Z}_p, \mathcal{W})$ the space of p-adic measures on $\mathbb{Z}_p$ with values in $\mathcal{W}$. When equipped with the norm

$$|\mu|_p := \sup_{|\varphi|_p = 1} \big|\mu(\varphi)\big|_p,$$

the space $M(\mathbb{Z}_p, \mathcal{W})$ is a p-adic Banach $\mathcal{W}$-module. In particular, if we have a sequence of continuous functions φ_n such that $\varphi_n \to \varphi$ in $\mathcal{C}(\mathbb{Z}_p, \mathcal{W})$ for $n \to \infty$ and a measure $\mu \in M(\mathbb{Z}_p, \mathcal{W})$, then

$$\big|\mu(\varphi_n) - \mu(\varphi)\big|_p \leq |\mu|_p \cdot |\varphi_n - \varphi|_p,$$

which means that

$$\lim_{n \to \infty} \mu(\varphi_n) = \mu\big(\lim_{n \to \infty} \varphi_n\big) = \mu(\varphi).$$

In other words, we can exchange the integral and limit operations.

Consider now the binomial polynomial given by

$$\binom{x}{n} := \begin{cases} \frac{x(x-1)\cdots(x-n+1)}{n!} & \text{if } n \geq 1, \\[2mm] 1 & \text{if } n = 0. \end{cases}$$

The function $x \mapsto \binom{x}{n}$ is continuous on $\mathbb{Z}_p$.

Proposition 3.1.2. *A measure μ is determined by the bounded sequence of numbers*

$$\left\{ \int_{\mathbb{Z}_p} \binom{x}{n} d\mu \right\}_{n \in \mathbb{N}} \subseteq \mathcal{W}.$$

Conversely, given a bounded sequence $\{c_n\}$ of numbers in $\mathcal{W}$, one can define uniquely a p-adic measure μ satisfying

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu = c_n$$

for every n.

Proof. This is [Hi93, §3.3, Theorem 1]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since $\binom{x}{n} \in \mathbb{Q}[x]$, the values $\int_{\mathbb{Z}_p} \binom{x}{n} d\mu$ are uniquely determined by $\int_{\mathbb{Z}_p} x^m d\mu$ for $m \geq 0$.

Now, given a measure $\mu \in M(\mathbb{Z}_p, \mathcal{W})$, consider the power series defined by

$$\Phi_\mu(t) := \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu \right) T^n \in \mathcal{W}[[T]], \quad \text{with } T := t - 1.$$

By Proposition 3.1.2, the measure μ is determined by Φ_μ and, moreover, the following result holds.

Proposition 3.1.3. *The map*

$$M(\mathbb{Z}_p, \mathcal{W}) \stackrel{\cong}{\longrightarrow} \mathcal{W}[[T]]$$
$$\mu \longmapsto \Phi_\mu$$

is an isomorphism.

Proof. See [Hi93, §3.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

There is an equality

$$\int_{\mathbb{Z}_p} x^m d\mu = \left(t \frac{d}{dt} \right)^m \Phi_\mu \mid_{t=1} \text{ for any } m \geq 0. \qquad (3.1.1)$$

When $z \in \mathcal{W}$ with $|z|_p < 1$, the infinite sum

$$(1 + z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n$$

is convergent for $x \in \mathbb{Z}_p$. Thus, one can define the *p*-adic exponential by

$$z^x := \sum_{n=0}^{\infty} \binom{x}{n} (z - 1)^n.$$

Then

$$\int_{\mathbb{Z}_p} z^x d\mu = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} (z - 1)^n d\mu = \Phi_\mu(z) \text{ for } z \in \mathcal{W} \text{ with } \mid z - 1 \mid_p < 1. \quad (3.1.2)$$

3.2. Measures on $\mathrm{Gal}(K_{c_0p^\infty}/K)$

The space of p-adic measures $M(\mathbb{Z}_p, \mathcal{W})$ is naturally a $\mathcal{C}(\mathbb{Z}_p, \mathcal{W})$-module in the following way: for $\phi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$ and $\mu \in M(\mathbb{Z}_p, \mathcal{W})$, we set

$$\int_{\mathbb{Z}_p} \varphi d\phi \cdot \mu := \int_{\mathbb{Z}_p} \varphi \phi d\mu$$

for any $\varphi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$. Furthermore, for $\phi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$ and $\mu \in M(\mathbb{Z}_p, \mathcal{W})$, we write

$$[\phi]\Phi_\mu(t) := \Phi_{\phi\mu}(t) = \int_{\mathbb{Z}_p} \phi(x) t^x d\mu \in \mathcal{W}[[t-1]].$$

Note that for $m \geq 0$

$$[x^m]\Phi_\mu = \Phi_{x^m\mu} = \left(t \frac{d}{dt} \right)^m \Phi_\mu \tag{3.1.3}$$

and for $z \in \mathcal{W}$ with $|z - 1|_p < 1$

$$\Phi_{z^x \cdot \mu}(t) = \Phi_\mu(tz) \in \mathcal{W}[[t-1]].$$

Consider now an element ζ in the group $\boldsymbol{\mu}_{p^n}$ of p^n-th roots of unity in the algebraic closure of $\mathrm{Frac}(\mathcal{W})$. Since the only p-power unit in characterist p is 1, then $\zeta - 1 \equiv 0$ mod the maximal ideal, so $|\zeta - 1|_p < 1$ and we can consider the exponential ζ^x for $x \in \mathbb{Z}_p$. Using the orthogonality relations

$$\sum_{\zeta \in \boldsymbol{\mu}_{p^n}} \zeta^{x-b} = \begin{cases} p^n & \text{if } x \equiv b \pmod{p^n \mathbb{Z}_p} \\ 0 & \text{if } x \not\equiv b \pmod{p^n \mathbb{Z}_p}, \end{cases}$$

we obtain that for $\phi : \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Frac}(\mathcal{W})(\boldsymbol{\mu}_{p^n})$ there is an equality

$$\phi(x) = p^{-n} \sum_{\zeta \in \boldsymbol{\mu}_{p^n}} \zeta^x \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \phi(b) \zeta^{-b}.$$

Thus, if we consider a locally constant function $\phi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$ that factors through $\mathbb{Z}_p/p^n\mathbb{Z}_p$, then

$$[\phi]\Phi_\mu(t) = \Phi_{\phi\mu}(t) = p^{-n} \sum_{b \in \mathbb{Z}/p^n\mathbb{Z}} \phi(b) \sum_{\zeta \in \boldsymbol{\mu}_{p^n}} \zeta^{-b} \Phi_\mu(\zeta t) \in \mathcal{W}[[t-1]] \tag{3.1.4}$$

for $\mu \in M(\mathbb{Z}_p, \mathcal{W})$. Observe that the notation $[\phi]\Phi_\mu$ coincides with that in [Bra, (8.1)] and [Hi93, §3.5], and corresponds to $\Phi_\mu \otimes \phi$ in [CH, §3.1]. Furthermore, there is an equality

$$\int_{\mathbb{Z}_p} \phi(x) x^m d\mu = \left(t \frac{d}{dt} \right)^m \left([\phi]\Phi_\mu \right)|_{t=1}. \tag{3.1.5}$$

3.2 Measures on $\mathrm{Gal}(K_{c_0p^\infty}/K)$

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_H$ be a complete set of representatives for $\mathrm{Pic}(\mathcal{O}_{c_0})$. As in [Bra, §8.2], there is an explicit coset decomposition

$$\mathrm{Gal}(K_{c_0p^\infty}/K) = \mathrm{Pic}\,\mathcal{O}_{c_0p^\infty} = \bigsqcup_{j=1}^{H} \mathfrak{a}_j^{-1} \mathbb{Z}_p^\times, \tag{3.2.6}$$

that allows us to construct a $\mathcal{W}$-valued measure μ on $\mathrm{Pic}\,\mathcal{O}_{c_0p^\infty}$ by constructing H distinct $\mathcal{W}$-valued measures $\mu_{\mathfrak{a}_j}$ on $\mathbb{Z}_p^\times$, so that for every continuous function φ : $\mathrm{Pic}\,\mathcal{O}_{c_0p^\infty} \to \mathcal{W}$ we have

$$\int_{\mathrm{Pic}\,\mathcal{O}_{c_0p^\infty}} \varphi d\mu_f = \sum_{\mathfrak{a}\in\mathrm{Pic}\,\mathcal{O}_{c_0}} \int_{\mathbb{Z}_p^\times} \varphi \mid [\mathfrak{a}] d\mu_{f,\mathfrak{a}},$$

where $\varphi \mid [\mathfrak{a}]$ is φ restricted to $\mathfrak{a}^{-1}\mathbb{Z}_p^\times$.

Therefore, a measure on $\mathrm{Gal}(K_{c_0p^\infty}/K)$ is equivalent to a collection of H measures on $\mathbb{Z}_p^\times$:

$$\mu \longleftrightarrow \{\mu_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathrm{Pic}(c_0)}.$$

3.3 Measure associated with a modular form

Let g be a p-adic modular form on Sh over $\mathcal{W}$ and let $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{c_0})$. Define a $\mathcal{W}$-valued measure $\mu_{g,\mathfrak{a}}$ on $\mathbb{Z}_p$ by

$$\int_{\mathbb{Z}_p} t^x d\mu_{g,\mathfrak{a}} = g(t_{\mathfrak{a}}) \in \mathcal{W}[[t_{\mathfrak{a}}]],$$

where $t_{\mathfrak{a}}$ is the Serre–Tate coordinate around $x(\mathfrak{a}) \otimes_{\mathcal{W}} \overline{\mathbb{F}}_p$, where $x(\mathfrak{a})$ is as defined in Section 1.5.5. Indeed, if $\mathfrak{a}$ is a prime-to-pN fractional ideal of $\mathcal{O}_c$ and $p \nmid c$, then $x(\mathfrak{a})$ has a model defined over $\mathcal{V} := \mathcal{W} \cap K^{\mathrm{ab}}$. If $\mu_{g,\mathfrak{a}}$ is supported on $\mathbb{Z}_p^\times$, then we can put them together to obtain a measure μ_g on $\mathrm{Gal}(K_{c_0p^\infty}/K)$.

3.4 p-depletion of a modular form

In order to obtain measures supported on $\mathbb{Z}_p^\times$, now we introduce the p-depletion of a modular form. We follow [Brooks, §3.6].

Recall the operators U and V. Take a QM abelian surface A with ordinary reduction over a p-adic field L. Then there is a unique p-torsion cyclic $\mathcal{O}_B$-submodule C of A which reduces mod p to the kernel of the Frobenius morphism, that is the canonical subgroup (cf. [Kas, Theorem 1.1]). Denote by $\phi_i : A \to A/C_i$, for $i = 0, \ldots, p$, the distinct p-isogenies of QM abelian surfaces on A ordered in such a way that C_0 is the canonical subgroup of A. If $t : \mu_{N^+} \times \mu_{N^+} \hookrightarrow A[N^+]$ is a $V_1(N^+)$-level structure on A, then, since $p \nmid N^+$, $t_i = \phi_i \circ t$ is a $V_1(N^+)$-level structure on A/C_i. Also if ω is a one-form on A, then there is a unique one-form ω_i on A/C_i such that $\phi_i^* \omega_i = \omega$. If g is a modular form, we can define another modular form $g \mid V$ by

$$g \mid V(A, t, \omega) := g(A/C_0, 1/pt_0, p\omega_0)$$

and also a modular form $g \mid U$ by

$$g \mid U(A, t, \omega) := \sum_{i=1}^{p} g(A/C_i, t_i, \omega_i).$$

If $[p]$ is the operator on modular forms that is given by $g \mid [p](A, t, \omega) = g(A, pt, 1/p\omega)$, then U and V are related to the usual T_p operator by

$$T_p = U + 1/p[p]V.$$

Furthermore, one has $VU = \mathrm{id}$ and the operators UV and $VU - UV$ are idempotent. The p-**depletion** of a modular form g is defined to be

$$g^{(p)} := g \mid (\mathrm{id} - UV) = g \mid (\mathrm{id} - T_p V + 1/p[p]V^2).$$

3.5 Hecke characters and *p*-adic Galois characters

Let K be our imaginary quadratic field and let m, n be integers.

Definition 3.5.1. A **Hecke character of K of infinity type** (m,n) is a continuous homomorphism

$$\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

satisfying

$$\chi(x \cdot z_\infty) = \chi(x) z_\infty^m \overline{z}_\infty^n,$$

for every $z_\infty \in K_\infty^\times$ and $x \in \hat{K}^\times$.

In particular, the infinite component of χ is given by $\chi_\infty(z) = z^m \overline{z}^n$. For each prime $\mathfrak{q}$ of K, denote by $\chi_\mathfrak{q} : K_\mathfrak{q}^\times \to \mathbb{C}^\times$ the $\mathfrak{q}$-component of χ. The *conductor* of χ is the largest integral ideal $\mathfrak{c}_f$ of K such that $\chi_\mathfrak{q}(u) = 1$ for each element $u \in 1 + \mathfrak{c}_f \mathcal{O}_{K,\mathfrak{q}}$. As it is known, one can identify a Hecke character χ with a character on fractional ideals of $\mathcal{O}_K$ prime to $\mathfrak{c}_f$ via the formula

$$\chi(\mathfrak{a}) = \prod_{\mathfrak{q}|\mathfrak{a} \, prime} \chi_\mathfrak{q}(\pi_\mathfrak{q})^{v_\mathfrak{q}(\mathfrak{a})},$$

with $\pi_\mathfrak{q}$ a uniformizer at $\mathfrak{q}$; the formula is independent of the choice of the uniformizer. So, if χ has conductor $\mathfrak{c}$ and $\mathfrak{a}$ is any fractional ideal prime to $\mathfrak{c}$, we write $\chi(\mathfrak{a})$ for $\chi(a)$, where $a \in \mathbb{A}_K^\times$ is an idele with $a\hat{\mathcal{O}}_K \cap K = \mathfrak{a}$ and $a_\mathfrak{q} = 1$ for all $\mathfrak{q} \mid \mathfrak{c}$.

A Hecke character χ is called **anticyclotomic** if χ is trivial on $\mathbb{A}_\mathbb{Q}^\times$. The *p*-**adic avatar** $\hat{\chi} : K^\times \backslash \hat{K}^\times \to \mathbb{C}_p^\times$ of a Hecke character χ of infinity type (m,n) is defined by

$$\hat{\chi}(x) = \chi(x) x_\mathfrak{p}^m x_{\overline{\mathfrak{p}}}^n$$

with $x \in \hat{K}^\times$ and $\mathfrak{p}$ the chosen prime above p which splits in K. Every *p*-adic Galois character $\rho : G_K \to \mathbb{C}_p^\times$ can be seen as a *p*-adic character $K^\times \backslash \hat{K}^\times \to \mathbb{C}_p^\times$ via the geometrically normalized reciprocity law map $\mathrm{rec}_K : K^\times \backslash \hat{K}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$.

A *p*-adic Galois character is said to be **locally algebraic** if it is the *p*-adic avatar of some Hecke character. A locally algebraic character is called of infinity type (m,n) if the associated Hecke character is of infinity type (m,n), and its conductor is the conductor of the associated Hecke character.

3.6 Construction of a measure

Consider now our modular form $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, with $k \geq 4$, and let $F/\mathbb{Q}_p$ be a finite extension containing the image of the Fourier coefficients of f. Via the Jacquet–Langlands correspondence we can see f as a modular form in $M_k(Sh, \mathcal{O}_F)$. Take the *p*-depletion $f^{(p)}$ of f and then consider it as a *p*-adic modular form $\hat{f}^{(p)}$ in $V_p(N^+, \mathcal{O}_F)$ of weight k.

Fix an anticyclotomic Hecke character ψ of infinity type $(k/2, -k/2)$, and let $c_0 \mathcal{O}_K$ be the prime to p part of the conductor of ψ. Take a finite extension of $\mathcal{W}$ obtained by adjoining values of the Hecke character ψ and, with an abuse of notation, we still denote it by $\mathcal{W}$. Let $\hat{\psi}$ be the *p*-adic avatar of ψ. The $\mathcal{W}$-valued measures $\mu_{\hat{f}^{(p)}, \mathfrak{a}}$ are given by

$$\psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \psi_\mathfrak{p} \mu_{\hat{f}_\mathfrak{a}^{(p)}},$$

where $\mu_{\hat{f}_{\mathfrak{a}}^{(p)}}$ is defined by

$$\int_{\mathbb{Z}_p} t^x d\mu_{\hat{f}_{\mathfrak{a}}^{(p)}} = \hat{f}_{\mathfrak{a}}^{(p)}(t_{\mathfrak{a}}) := \hat{f}^{(p)}(t_{\mathfrak{a}}^{N(\mathfrak{a}^{-1})\sqrt{(-D_K)}^{-1}}) \in \mathcal{W}[[t_{\mathfrak{a}} - 1]],$$

and $t_{\mathfrak{a}}$ is the Serre–Tate coordinate around $x(\mathfrak{a}) \otimes_{\mathcal{W}} \overline{\mathbb{F}}_p$.

Remark 3.6.1. The measure $\mu_{\hat{f}_{\mathfrak{a}}^{(p)}}$ associated with $\hat{f}_{\mathfrak{a}}^{(p)}$ is supported on $\mathbb{Z}_p^{\times}$. Indeed, $UV\hat{f}^{(p)} = 0$ because $VU = \text{id}$. Since UV acts on the expansion in Serre–Tate coordinates as $UVg(t) = 1/p \sum_{\zeta \in \boldsymbol{\mu}_p} g(\zeta t)$ (see [Brooks, Proposition 4.17]), taking $\phi = \mathbf{1}_{\mathbb{Z}_p^{\times}}$ to be the characteristic function of $\mathbb{Z}_p^{\times}$ and using (3.1.4) yields

$$[\phi]g(t) = p^{-1} \sum_{b \in \mathbb{Z}/p\mathbb{Z}} \phi(b) \sum_{\zeta \in \boldsymbol{\mu}_p} \zeta^{-b} g(\zeta t)$$

$$= [\mathbf{1}_{\mathbb{Z}_p}]g(t) - p^{-1} \sum_{\zeta \in \boldsymbol{\mu}_p} g(\zeta t).$$

Hence, $\mu_{\hat{f}^{(p)}, \mathfrak{a}}$ is supported on $\mathbb{Z}_p^{\times}$ as well.

Definition 3.6.2. Let ψ denote an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$ and conductor $c_0 \mathcal{O}_K$ with $(c_0, pN^+) = 1$. The **measure** $\mathscr{L}_{f,\psi}$ **associated with f and ψ** is the $\mathcal{W}$-valued measure given by

$$\mathscr{L}_{f,\psi}(\varphi) = \sum_{\mathfrak{a} \in \text{Pic}(\mathcal{O}_{c_0})} \int_{\mathbb{Z}_p^{\times}} \varphi|[\mathfrak{a}] d\mu_{\hat{f}^{(p)}, \mathfrak{a}}$$

for any continuous function $\varphi : \text{Gal}(K_{c_0 p^{\infty}}/K) \to \mathcal{W}$.

Therefore

$$\mathscr{L}_{f,\psi}(\varphi) = \sum_{\mathfrak{a} \in \text{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \int_{\mathbb{Z}_p^{\times}} \psi_{\mathfrak{p}} \varphi|[\mathfrak{a}] d\mu_{\hat{f}_{\mathfrak{a}}^{(p)}}$$

$$= \sum_{\mathfrak{a} \in \text{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \Phi_{\psi_{\mathfrak{p}} \varphi|[\mathfrak{a}] d\mu_{\hat{f}_{\mathfrak{a}}^{(p)}}} \big|_{t=1}$$

$$= \sum_{\mathfrak{a} \in \text{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} [\psi_{\mathfrak{p}} \varphi|[\mathfrak{a}]] \hat{f}_{\mathfrak{a}}^{(p)}(t_{\mathfrak{a}}) \big|_{t=1}$$

$$= \sum_{\mathfrak{a} \in \text{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} [\psi_{\mathfrak{p}} \varphi|[\mathfrak{a}]] \hat{f}_{\mathfrak{a}}^{(p)}(x(\mathfrak{a})).$$

Notice that the second equality holds because $\mu_{\hat{f}_{\mathfrak{a}}^{(p)}}$ is supported on $\mathbb{Z}_p^{\times}$. Indeed, for a measure μ supported on $\mathbb{Z}_p^{\times}$ one has $\int_{\mathbb{Z}_p^{\times}} 1 d\mu = \int_{\mathbb{Z}_p} 1 d\mu = \Phi_{\mu} \big|_{t=1}$.

Remark 3.6.3. We are interested in evaluating $\mathscr{L}_{f,\psi}$ at continuous functions that factor through $\text{Gal}(K_{p^{\infty}}/K)$. In other words, we will view $\mathscr{L}_{f,\psi}$ as a measure on $\text{Gal}(K_{p^{\infty}}/K)$.

Now we state a result that we will use later. Let $g \in V_p(N^+, \mathcal{W})$ be a p-adic modular form and let $g_{\mathfrak{a}}$ be defined as at the beginning of this section, so that $g_{\mathfrak{a}}(t) = g(t^{N(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}})$ with t the Serre–Tate coordinate around $x(\mathfrak{a}) \otimes_{\mathcal{W}} \overline{\mathbb{F}}_p$.

Proposition 3.6.4. *If $g \in V_p(N^+, \mathcal{W})$ and $\phi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathbb{C}^\times$ is a primitive Dirichlet character, then*

$$[\phi]g_{\mathfrak{a}}(x(\mathfrak{a})) = p^{-n}G(\phi) \sum_{u \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \phi^{-1}(u)g(x(\mathfrak{a}) \star \alpha(u/p^n)),$$

where $G(\phi) = \sum_{v \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \phi(v)\zeta_{p^n}^v$ is the Gauss sum of ϕ.

Proof. The statement follows by applying (3.1.4) and Lemma 2.7.1. $\qquad\square$

3.7 Interpolation properties

Working on $\hat{B}$ instead of $\mathrm{GL}_2(\hat{\mathbb{Q}})$ and adapting the computations from [Hs], one can obtain an analogue of [Hs, Theorem 3.4] in our quaternionic setting and use it, as in [CH], to get an interpolation formula for our p-adic L-function evaluated at Galois characters that are p-adic avatars of anticyclotomic Hecke characters of infinity type $(n, -n)$ with $n \geq 0$. In particular, one can relate our p-adic L-function to the Rankin–Selberg L-function associated with f and some anticyclotomic Hecke character χ of infinity type $(k/2 + n, k/2 - n)$ with $n \geq 0$, i.e., the L-function associated with the G_K-representation $V_{f,\chi} = V_f(k/2) \otimes \chi$.

In the statement of the following theorem, ψ is, as usual, an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$ and conductor $c_0\mathcal{O}_K$ with $(c_0, pN^+) = 1$.

Theorem 3.7.1. *Let $\hat{\phi}$ be the p-adic avatar of an anticyclotomic Hecke character ϕ of infinity type $(n, -n)$ with $n \geq 0$ and p-power conductor. Then there exists a non-zero constant $C(f, \psi, \phi, K)$ depending on f, ψ, ϕ, K such that $C(f, \psi, \phi, K) \cdot L(f, \psi\phi, k/2)$ is an algebraic number and*

$$\left(\mathscr{L}_{f,\psi}(\hat{\phi}) \right)^2 = C(f, \psi, \phi, K) \cdot L(f, \psi\phi, k/2),$$

where the equality holds via the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

A proof of this result will appear in a future project.

Chapter 4

Generalized Heegner cycles

Recall that K is an imaginary quadratic field, N is a positive integer with a factorization $N = N^+N^-$ where N^+ is a product of primes that are split in K and N^- is a square-free product of an even number of primes that are inert in K, B is again our indefinite rational quaternion algebra of discriminant $D = N^-$ and p is an odd prime that splits in K and B and such that $(p, N) = 1$.

Consider then our Shimura curve Sh, our fixed modular form f in $S_k(\Gamma_0(N))$ of weight $k = 2r + 2 \geq 4$, which can be seen by Jacquet–Langlands as a modular form on Sh, and the r-fold fiber product $\mathcal{A}^r$ of the universal QM abelian surface over Sh with itself.

Following the work of Brooks, [Brooks], we want to define generalized Heegner cycles associated with f lying over a Kuga–Sato variety over Sh. Indeed, these cycles will live in the Chow groups of the generalized Kuga–Sato variety $\mathcal{X}_r = \mathcal{A}^r \times A^r$, where A will be a fixed QM abelian surface. Then, to obtain cohomology classes from the generalized Heegner cycles, we will apply the p-adic Abel–Jacobi map. We will construct in this way a system of generalized Heegner classes indexed by fractional ideals of K.

4.1   Kuga–Sato varieties

Consider the r-fold fiber product $\mathcal{A}^r$ of the universal QM abelian surface $\mathcal{A}$ with itself over Sh, which is called the rth-**Kuga–Sato variety** over Sh.

We define the action of the Hecke operator T_ℓ, for $\ell \nmid N^+D$ on the Kuga–Sato variety $\mathcal{A}^r$ as follows. Recall the interpretation of the Hecke operator T_ℓ on Sh as correspondence. Let $Sh(\ell)$ be the Shimura curve classifying quadruples (A, ι, ν_{N^+}, C), where (A, ι, ν_{N^+}) is a QM abelian surface with $V_1(N^+)$-level structure endowed with a subgroup C of $A[\ell]$ stable under the action of $\mathcal{O}_B$ and cyclic as $\mathcal{O}_B$-module. $A[\ell]$ has $\ell + 1$ such $\mathcal{O}_B$-submodules, all of them of order ℓ^2. Consider the natural forgetful morphism of Shimura curves $\alpha : Sh(\ell) \to Sh$ and the morphism $\beta : Sh(\ell) \to Sh$ given by $(A, \iota, \nu_{N^+}, C) \mapsto (A/C, \iota_{\psi_C}, \nu_{N^+,\psi_C})$, where $\iota_{\psi_C}, \nu_{N^+,\psi_C}$ are induced by the isogeny $\psi_C : A \twoheadrightarrow A/C$. Then T_ℓ is defined by the correspondence

$$
\begin{array}{ccc}
 & Sh(\ell) & \\
 {}^{\alpha}\swarrow & & \searrow^{\beta} \\
Sh & & Sh,
\end{array}
$$

which means that $T_\ell = \alpha^* \circ \beta_*$, i.e., $T_\ell(x) = \sum_{y \in \alpha^{-1}(x)} \beta(y)$. In other words, we recover the definition given in Chapter 1.

Now take the fiber product $\mathcal{A}_\ell := \mathcal{A} \times_{Sh} Sh(\ell)$, which is the universal QM abelian surface over $Sh(\ell)$, equipped with a subgroup scheme $\mathcal{C}$ of $\mathcal{A}[\ell]$ that is an $\mathcal{O}_B$-module of order ℓ^2. Consider the quotient $\mathcal{Q} := \mathcal{A}_\ell/\mathcal{C}$ with induced QM and level structure and the fiber products $\mathcal{A}_\ell^r$ and $\mathcal{Q}^r$ over $Sh(\ell)$. The action of the Hecke operator T_ℓ on the Kuga–Sato variety $\mathcal{A}^r$ is defined by the commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{A}^r & \xleftarrow{\;p_1\;} & \mathcal{A}_\ell^r & \xrightarrow{\quad\psi\quad} & \mathcal{Q}^r & \xrightarrow{\;p_2\;} & \mathcal{A}^r \\
\downarrow & & \downarrow \searrow & & \swarrow & & \downarrow \\
Sh & \xleftarrow{\quad\alpha\quad} & & Sh(\ell) & \xrightarrow{\quad\beta\quad} & & Sh,
\end{array}
$$

where the two squares are cartesian, by the formula

$$
T_\ell = p_{1*} \circ \psi^* \circ p_2^*.
$$

Write $\overline{\mathcal{A}}^r$ for the base change of $\mathcal{A}^r$ to $\overline{\mathbb{Q}}$. The correspondence T_ℓ just defined induces an endomorphism of the étale cohomology groups $H^*_{\text{ét}}(\overline{\mathcal{A}}^r, -)$, which will still be denoted by T_ℓ.

The reader is advised to compare with [Sch] for the definition of Hecke operators on Kuga–Sato varieties over modular curves and with [EdVP] for the case of Shimura curves relative to "Γ_0-type" level structures.

4.2 Generalized Kuga–Sato varieties

Fix the QM abelian surface A with $V_1(N^+)$-level structure and CM by $\mathcal{O}_K$ defined in (1.5.5). Thanks to the assumption that p splits in K, the surface A is ordinary at p. Our **generalized Kuga–Sato variety** is the product $\mathcal{X}_r := \mathcal{A}^r \times A^r$. This enlarged Kuga–Sato variety will be the space where our arithmetic cycles will live. As a piece of notation, we shall write $\overline{\mathcal{X}}_r$ for the base change of $\mathcal{X}_r$ to $\overline{\mathbb{Q}}$.

The usual Hecke correspondence T_ℓ for a prime $\ell \nmid N^+D$ on the Kuga–Sato variety $\mathcal{A}^r$ induces a Hecke correspondence $T_\ell \times \text{id}$ on $\mathcal{X}_r$, which will still be denoted by T_ℓ.

4.3 Projectors on Kuga–Sato varieties

We will define our algebraic cycles as graphs of morphisms of QM abelian surfaces. In order to make them homologically trivial, we will need to modify them by certain projectors associated with the generalized Kuga–Sato variety. Consider the projectors $P \in \text{Corr}_{Sh}(\mathcal{A}^r)$ and $\varepsilon_A \in \text{Corr}(A^r)$ defined in [Brooks, §6.1]. Then

$$
PH^*_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p) \subseteq H^{k-1}_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p)
$$

and

$$
\varepsilon_A H^i_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p) = 0 \quad \text{if } i \neq k-2,
$$
$$
\varepsilon_A H^{k-2}_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p) = \text{Sym}^{2r} e H^1(\overline{A}, \mathbb{Z}_p).
$$

Consider the variety $\mathcal{X}_r$ together with the projector $\varepsilon = P\varepsilon_A \in \text{Corr}_{Sh}(\mathcal{X}_r)$. Thanks to properties of the projectors, one has

$$
\varepsilon H^i_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) = 0 \quad \text{if } i \neq 2k-3,
$$
$$
\varepsilon H^{2k-3}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) \cong PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}}_r, \mathbb{Z}_p) \otimes \text{Sym}^{2r} e H^1_{\text{ét}}(\overline{A}, \mathbb{Z}_p).
$$
$$\tag{4.3.1}$$

We can prove these relations using the Künneth decomposition. More precisely, the Künneth decomposition for $\mathcal{X}_r = \mathcal{A}_r \times A^r$ reads

$$H^i_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) \cong \bigoplus_{n+s=i} H^n_{\text{ét}}(\overline{\mathcal{A}}_r, \mathbb{Z}_p) \otimes H^s_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p).$$

Indeed, the Künneth exact sequence for $\mathcal{A}_r$ and A^r (for which we refer, for example, to [LEC, §22]) is

$$0 \longrightarrow \bigoplus_{n+s=i} H^n_{\text{ét}}(\overline{\mathcal{A}}_r, \mathbb{Z}_p) \otimes H^s_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p) \longrightarrow H^i_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) \longrightarrow$$

$$\longrightarrow \bigoplus_{n+s=i+1} \text{Tor}_1^{\mathbb{Z}_p}\left(H^n_{\text{ét}}(\overline{\mathcal{A}}_r, \mathbb{Z}_p), H^s_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p)\right) \longrightarrow 0.$$

Since A is an abelian variety, by [AV, Theorem 12.1] the p-adic cohomology of $\overline{A}^r$ is a free $\mathbb{Z}_p$-module, so the last term of the sequence above is 0. As a consequence, we obtain the desired Künneth decomposition.

Now we want to apply the projector ε and the twists. Since

$$PH^i_{\text{ét}}(\overline{\mathcal{A}}_r, \mathbb{Z}_p) = 0 \quad \text{if } i \neq k - 1$$

and

$$\varepsilon_A H^i_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p) = 0 \quad \text{if } i \neq k - 2,$$
$$\varepsilon_A H^{k-2}_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p) = \text{Sym}^{2r} e H^1(\overline{A}, \mathbb{Z}_p),$$

we deduce that

$$\varepsilon H^i_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) = 0 \quad \text{for } i \neq 2k - 3,$$

as in this case all the terms on the right hand side of the Künneth decomposition vanish after applying the projectors. For $i = 2k - 3$ the only term in the sum in the right hand side of the Künneth decomposition that does not vanish after the application of the projectors is $PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p) \otimes \varepsilon_A H^{k-2}_{\text{ét}}(\overline{A}^r, \mathbb{Z}_p)$, hence

$$\varepsilon H^{2k-3}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p) = PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p) \otimes \text{Sym}^{2r} e H^1_{\text{ét}}(\overline{A}, \mathbb{Z}_p),$$

as desired.

4.4 Galois representations and Kuga–Sato varieties over Shimura curves

Let V_f be the 2-dimensional Galois representation attached to $f \in S_k^{\text{new}}(\Gamma_0(N))$ by Deligne ([Del]) and let $V_f(k/2)$ be the self-dual twist of V_f. As explained, for example, in [Nek92, §2 and §3], $V_f(k/2)$ can be realized as a direct summand of the $(k - 1)$-st p-adic cohomology group of the Kuga–Sato variety over a certain modular curve.

Let ϕ be Euler's function and observe that the index of Γ_{1,N^+} in Γ_{0,N^+} divides $\phi(N^+)$ (see [Brooks, p. 4184]). From now on, we work under the following

Assumption 4.4.1. $p \nmid N\phi(N^+)$.

Consider now the r-th Kuga–Sato variety $\mathcal{A}^r$ over the Shimura curve Sh. A similar construction can be performed to obtain the representation $V_f(k/2)$ from the étale cohomology group $H^{k-1}_{\text{ét}}(\overline{\mathcal{A}^r}, \mathbb{Q}_p)(k/2)$ of $\mathcal{A}^r$. Namely, let F be the finite extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f, whose valuation ring will be denoted by $\mathcal{O}_F$. Moreover, let P be the projector from §4.3. Under Assumption 4.4.1, one can define a Galois-equivariant surjection

$$PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}^r}, \mathbb{Z}_p)(k/2) \longrightarrow\!\!\!\!\rightarrow T,$$

where T is a suitable Galois-stable $\mathcal{O}_F$-lattice inside $V_f(k/2)$, whose definition can be found, for example, in [Nek92] and [Ota] (see also [Nek93]). This can be done by adapting the arguments in [IS, §5 and Appendix 10.1] to our setting, which coincides with that of [Brooks]. Since the modifications required are straightforward, we leave them to the interested reader. See also [EdVP, §3.3] for further details.

4.5   The p-adic Abel–Jacobi map

The p-adic Abel–Jacobi map is a map that sends algebraic cycles of an algebraic variety, i.e., finite linear combinations of irreducible subvarieties with integer coefficients, into étale cohomology classes of the variety with respect to a p-adic sheaf.

Let us start with a smooth projective variety X of dimension d over a field F of characteristic 0. Let $C^s(X)$ denote the free abelian group generated by the irreducible closed subvarieties of X of codimension s. Elements of $C^s(X)$ are called *algebraic cycles* of codimension s on X. The quotient

$$\text{CH}^s(X/F) := C^s(X/F)/\sim_{\text{rat}}$$

of $C^s(X/F)$ by rational equivalence is the s-th Chow group of X. Two algebraic cycles $Y, W \in C^s(X/F)$ are said to be rationally equivalent if there exist an open subset $U \subset \mathbb{P}_1$, a cycle $Z \in C^s(X \times U)$ (Z can be seen as a family of algebraic cycles of X parametrized by points of U) and two distinct points t_1, t_2 of U such that $Y = Z \cap C^s(X \times \{t_1\})$ and $W = Z \cap C^s(X \times \{t_2\})$.

Consider the p-adic étale sheaf $\mathbb{Z}_p$ on $\overline{X} := X \otimes_F \overline{F}$ defined by the system of the locally constant étale sheaves $\mathbb{Z}/p^n\mathbb{Z}$. The p-adic cohomology group $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_p)$ is defined as the inverse limit

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_p) := \varprojlim_n H^i_{\text{ét}}(\overline{X}, \mathbb{Z}/p^n\mathbb{Z}).$$

Let

$$\text{cl}_X : \text{CH}^s(X/F) \to H^{2s}_{\text{ét}}(\overline{X}, \mathbb{Z}_p(s))^{G_F}$$

denote the cycle class map in étale cohomology, where $\mathbb{Z}_p(s)$ denote the s-th Tate twist of $\mathbb{Z}_p$ and G_F is the absolute Galois group $\text{Gal}(\overline{F}/F)$. See [LEC, §23] or [Mi80, Chapter VI, §9] for the definition of the cycle map and [LEC, Chapter I] for an introduction to étale and p-adic étale cohomology. We denote by

$$\text{CH}^s(X/F)_0 := \ker(\text{cl}_X)$$

the group of homologically trivial cycles of codimension s on X up to rational equivalence.

Take now an element of $\text{CH}^s(X/F)_0$ represented by an algebraic cycle Z, with Z/F smooth. Consider the Gysin sequence in p-adic étale cohomology from [LEC, Corollary

16.2]. Setting $X = \overline{X}$, $Z = \overline{Z}$, $U = \overline{X} \setminus \overline{Z}$, $r = s$ and $\mathcal{F} = \mathbb{Z}_p(s)$ in [LEC, Corollary 16.2], we obtain an exact sequence

$$0 \longrightarrow H^{2s-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) \longrightarrow H^{2s-1}_{\text{ét}}(\overline{X} \setminus \overline{Z}, \mathbb{Z}_p)(s) \longrightarrow H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p)(s)_0 \longrightarrow 0$$

in the category of continuous p-adic representations of G_F, where

$$H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p)(s)_0 := \ker\left(H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p)(s) \xrightarrow{i_*} H^{2s}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) \right)$$

is the kernel of the Gysin map i_* and $H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p)(s)$ is the cohomology group with support on $\overline{Z}$ (see [LEC, §18]). Recall that $\text{cl}_X(Z)$ is defined to be the image of 1 under the Gysin map

$$\mathbb{Z}_p = H^0_{\text{ét}}(\overline{Z}, \mathbb{Z}_p) \xrightarrow{\cong} H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p(s)) \xrightarrow{i_*} H^{2s}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s)$$

such that

$$1 \longmapsto \text{cl}^{\overline{X}}_{\overline{Z}}(Z) \longmapsto \text{cl}_X(Z)$$

(see [LEC, §23]), where $\text{cl}^{\overline{X}}_{\overline{Z}}(Z) \in H^{2s}_{\overline{Z}}(\overline{X}, \mathbb{Z}_p(s))$ is the class from which $\text{cl}_X(Z)$ comes from, and Z is homologically trivial, i.e., $\text{cl}_X(Z) = 0$. It follows that the map

$$\gamma : \mathbb{Z}_p \longrightarrow H^{2s}_{\overline{Z}}(\overline{X}, \mathbb{Z}_p(s))_0$$

given by

$$1 \longmapsto \text{cl}^{\overline{X}}_{\overline{Z}}(Z)$$

is well defined.

Pulling back the previous exact sequence by γ gives the following commutative diagram with exact rows, where the right square is cartesian:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2s-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) & \longrightarrow & E_Z & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\
& & \downarrow{=} & & \downarrow & & \downarrow{\gamma} & & \\
0 & \longrightarrow & H^{2s-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) & \longrightarrow & H^{2s-1}_{\text{ét}}(\overline{X} \setminus \overline{Z}, \mathbb{Z}_p)(s) & \longrightarrow & H^{2s}_{\text{ét}, \overline{Z}}(\overline{X}, \mathbb{Z}_p)(s)_0 & \longrightarrow & 0.
\end{array}
$$

The **p-adic étale Abel–Jacobi map** is the map

$$\text{AJ}_{p,F} : \text{CH}^s(X/F)_0 \longrightarrow \text{Ext}^1\left(\mathbb{Z}_p, H^{2s-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) \right) \cong H^1\left(F, H^{2s-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_p)(s) \right)$$

that sends the class of a homologically trivial cycle Z to the isomorphism class of the extension V_Z in the category of Galois representations, as determined by the last commutative diagram.

4.6 Generalized Heegner cycles

For any morphism $\varphi : (A, i, \nu_{N^+}) \to (A', i', \nu'_{N^+})$ of abelian surfaces over a field $F \supseteq \mathbb{Q}$ with QM by $\mathcal{O}_B$ and $V_1(N^+)$-level structure, we can consider its graph $\Gamma_\varphi \subseteq A \times A'$. Consider the point x in $Sh(F)$ corresponding to the class of (A', i', ν'_{N^+}). There is an embedding $A' = \mathcal{A}_x \hookrightarrow \mathcal{A}$ of the fiber of $\mathcal{A}$ above x in $\mathcal{A}$ that induces an embedding $i_x : (A')^r \hookrightarrow \mathcal{A}^r$. Via this embedding, we can view the (r)-power of the graph of φ inside $\mathcal{A}^r \times \mathcal{A}^r$:

$$\Gamma^r_\varphi \subseteq A^r \times A'^r = A'^r \times A^r \stackrel{i_x \times \text{id}}{\hookrightarrow} \mathcal{A}^r \times \mathcal{A}^r = \mathcal{X}_r.$$

We define the *generalized Heegner cycle* Δ_φ associated with φ as

$$\Delta_\varphi := \varepsilon \Gamma_\varphi^r \in \mathrm{CH}^{k-1}(\mathcal{X}_r/F),$$

where $\varepsilon = P\varepsilon_A \in \mathrm{Corr}_{Sh}^r(\mathcal{X}_r, \mathcal{X}_r)$ and F is a field such that (A', i', ν'_{N+}) and φ are defined over F.

One needs to apply the projector ε to make the cycle Δ_φ homologically trivial, that means that we want the image of Δ_φ via the cycle map

$$\mathrm{cl}_{\mathcal{X}_r} : \mathrm{CH}^{k-1}(\mathcal{X}_r/F)_0 \to H_{\text{ét}}^{2k-2}(\overline{\mathcal{X}}_r, \mathbb{Z}_p(k-1))$$

to be trivial in order to apply the Abel–Jacobi map. Indeed, the cycle Δ_φ is homologically trivial, because, thanks to equation (4.3.1), one has $\varepsilon H_{\text{ét}}^{2k-2}(\overline{\mathcal{X}}_r, \mathbb{Z}_p(k-1)) = 0$.

Therefore, we have that $\varepsilon \mathrm{CH}^{k-1}(\mathcal{X}_r/F) \subseteq \mathrm{CH}^{k-1}(\mathcal{X}_r/F)_0$, hence Δ_φ is cohomologically trivial and we can consider the image of this cycle under the p-adic Abel–Jacobi map

$$\mathrm{AJ}_{p,F} : \mathrm{CH}^{k-1}(\mathcal{X}_r/F)_0 \longrightarrow H^1\big(F, H_{\text{ét}}^{2k-3}(\overline{\mathcal{X}}_r, \mathbb{Z}_p(k-1))\big).$$

Applying the projector ε in the construction of the Abel–Jacobi map (as in [BDP, §3.1] for Kuga–Sato varieties over modular curves) we obtain a p-adic Abel–Jacobi map

$$\mathrm{AJ}_{p,F} : \mathrm{CH}^{k-1}(\mathcal{X}_r/F) \longrightarrow H^1\big(F, \varepsilon H_{\text{ét}}^{2k-3}(\overline{\mathcal{X}}_r, \mathbb{Z}_p(k-1))\big).$$

Then, considering the twist in (4.3.1), one has

$$\varepsilon H_{\text{ét}}^{2k-3}(\overline{\mathcal{X}}_r, \mathbb{Z}_p(k-1)) = PH_{\text{ét}}^{k-1}(\overline{\mathcal{A}}^r, \mathbb{Z}_p(k/2)) \otimes \mathrm{Sym}^{2r} e H_{\text{ét}}^1(\overline{A}, \mathbb{Z}_p)(r),$$

so in the following we will see the Abel–Jacobi map as taking values in

$$H^1\big(F, PH_{\text{ét}}^{k-1}(\overline{\mathcal{A}}^r, \mathbb{Z}_p)(k/2) \otimes \mathrm{Sym}^{2r} e H_{\text{ét}}^1(\overline{A}, \mathbb{Z}_p)(r)\big).$$

4.7 A distinguished family of generalized Heegner cycles

With notation as in §1.5.5, start with the fixed QM abelian surface A. For any integer c prime to N^+, take the multiplication-by-c isogeny

$$(A, i, \nu_{N+}) \xrightarrow{\phi_c} (A_c, i_c, \nu_{c,N+}),$$

which is an isogeny of QM abelian surfaces with $V_1(N^+)$-level structures. For each class $[\mathfrak{a}]$ in $\mathrm{Pic}\,\mathcal{O}_c$, where the representative $\mathfrak{a}$ is chosen to be integral and prime to N^+pc, consider the isogeny

$$\phi_\mathfrak{a} : A \longrightarrow A_\mathfrak{a},$$

defined as the composition

$$(A, i, \nu_{N+}) \xrightarrow{\phi_c} (A_c, i_c, \nu_{c,N+}) \xrightarrow{\varphi_\mathfrak{a}} (A_\mathfrak{a}, i_\mathfrak{a}, \nu_{\mathfrak{a},N+}),$$

and then the cycle

$$\Delta_{\phi_\mathfrak{a}} \in \mathrm{CH}^{k-1}(\mathcal{X}_r/K_c).$$

In fact, both $(A_\mathfrak{a}, i_\mathfrak{a}, \nu_{\mathfrak{a},N+})$ and the isogeny $\phi_\mathfrak{a}$ are defined over the ring class field K_c of K of conductor c.

4.8 Generalized Heegner classes

For any integer c prime to N^+, consider the Abel–Jacobi map

$$\mathrm{AJ}_{p,K_c} : \mathrm{CH}^{k-1}(\mathcal{X}_r/K_c)_0 \longrightarrow H^1(K_c, PH_{\text{ét}}^{k-1}(\overline{\mathcal{A}}^r, \mathbb{Z}_p)(k/2) \otimes \mathrm{Sym}^{2r} e H_{\text{ét}}^1(\overline{A}, \mathbb{Z}_p)(r)).$$

Because $V_f(k/2)$ can be realized as a quotient of $PH_{\text{ét}}^{k-1}(\overline{\mathcal{A}}^r, \mathbb{Z}_p(k/2))$, then we can see the Abel–Jacobi map as a map

$$\mathrm{CH}^{k-1}(\mathcal{X}_r/K_c)_{\mathbb{Q}} \longrightarrow H^1(K_c, T \otimes \mathrm{Sym}^{2r} e H_{\text{ét}}^1(\overline{A}, \mathbb{Z}_p)(r)),$$

where T is the Galois stable lattice in $V_f(k/2)$. Since $e H_{\text{ét}}^1(\overline{A}, \mathbb{Z}_p) \cong H_{\text{ét}}^1(\overline{E}, \mathbb{Z}_p)$, then we have a map

$$\Phi_{K_c} : \mathrm{CH}^{k-1}(\mathcal{X}_r/K_c) \longrightarrow H^1\big(K_c, T \otimes \mathrm{Sym}^{2r} H_{\text{ét}}^1(\overline{E}, \mathbb{Z}_p)(r)\big) \longrightarrow H^1\big(K_c, T \otimes S(E)\big),$$

where, as in [CH, §4.2], we set $S(E) := \mathrm{Sym}^{2r} T_p(\overline{E})(-r)$.

We define the *generalized Heegner class* $z_{\mathfrak{a}}$, associated with an ideal $\mathfrak{a}$ of $\mathcal{O}_c$, to be

$$z_{\mathfrak{a}} := \Phi_{K_c}(\Delta_{\phi_{\mathfrak{a}}}).$$

4.9 χ-components

As in [CH], we want to define "χ-components" of the generalized Heegner classes and construct classes $z_{c,\chi} \in H^1(K_c, T \otimes \chi)$, for χ an anticyclotomic Galois character. For this we will use $S(E) = \mathrm{Sym}^{2r} T_p(\overline{E})(-r)$ appearing in the image of the Abel–Jacobi map and we will do the same work as [CH]. So, closely following [CH, §4.4], let us start with a positive integer c_0 coprime to pN^+, and let $\chi : \mathrm{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_F^\times$ (possibly enlarging F so that $\mathrm{im}(\chi) \subseteq \mathcal{O}_F^\times$) be a locally algebraic anticyclotomic character of infinity type (j, j) with $-k/2 < j < k/2$ and conductor $c_0 p^s \mathcal{O}_K$.

Recall that $E = \mathbb{C}/\mathcal{O}_K$ (cf. §1.5); note that E is denoted with A in [CH]. Consider the abelian variety $W_{/K} := \mathrm{Res}_{K_1/K} E$, obtained by Weil restriction of E from K_1 to K, defined as the product

$$W = \prod_{\sigma \in \mathrm{Gal}(K_1/K)} E^\sigma,$$

where E^σ is the curve determined by the polynomials obtained by applying σ to the coefficients of the polynomials defining E. The Weil restriction W is again a CM abelian variety but over K and of dimension $[K_1 : K]$. The endomorphism ring of W, $M := \mathrm{End}_K(W) \otimes \mathbb{Q}$, is a product of CM fields over K and $\dim W = [M : K] = [K_1 : K]$ (see [Rub, §1] and [Wi] for a general introduction to the Weil restriction of abelian varities). Since

$$T_p(W) = \prod_{\sigma \in \mathrm{Gal}(K_1/K)} T_p(E^\sigma) = \mathrm{Ind}_{G_{K_1}}^{G_K} T_p(E),$$

viewing Tate modules as Galois representations, where Ind is the induced representation, we have an inclusion $T_p(E) \hookrightarrow T_p(W)$. Define the G_K-module

$$S(W) := \mathrm{Sym}^{2r} T_p(W)(-r) \otimes_{\mathbb{Z}_p} \mathcal{O}_F = \mathrm{Ind}_{G_{K_1}}^{G_K} S^r(E) \otimes_{\mathbb{Z}_p} \mathcal{O}_F.$$

By the discussion in [CH, §4.4], there exists a finite order anticyclotomic character χ_t such that χ can be realized as a direct summand of $S(W) \otimes \chi_t$ as G_K-modules. Denote by

$$e_\chi : S(W) \otimes \chi_t \longrightarrow\!\!\!\!\!\rightarrow \chi$$

the corresponding G_K-equivariant projection. The character χ_t is unique up to multiplication by a character of $\mathrm{Gal}(K_1/K)$ and it has the same conductor as χ.

Since $T_p(E) \hookrightarrow T_p(W)$ and then $S(E) \hookrightarrow S(W)$, we can see the classes $z_{\mathfrak{a}}$ as elements of $H^1(K_c, T \otimes S(W))$, where $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_c$ with c divisible by $c_0 p^s$.

For each integer c divisible by the conductor $c_0 p^s$ of χ, put

$$z_c \otimes \chi_t := z_c \in H^1(K_c, T \otimes S(W) \otimes \chi_t),$$

through the map $H^1(K_c, T \otimes S(W)) \to H^1(K_c, T \otimes S(W) \otimes \chi_t)$ and define

$$z_{c,\chi} := (\mathrm{id} \otimes e_\chi)(z_c \otimes \chi_t) \in H^1(K_c, T \otimes \chi), \qquad (4.9.2)$$

the χ-**component of the class** z_c.

4.10   Compatibility properties

First we study compatibility properties of the generalized Heegner classes defined in section 4.8, by examining the action of the Hecke operators, and proving results analoguos to [CH, Lemma 4.3 and Proposition 4.4].

Let $I(D_K)$ denote the group of fractional ideals of K that are prime to D_K and let $\tilde{\kappa}_E : I(D_K) \to M^\times$ be the CM character associated to W (as in [CH, §4.4]), satisfying the following properties:

- $\tilde{\kappa}_E(\alpha \mathfrak{a}) = \pm \alpha \cdot \tilde{\kappa}_E(\mathfrak{a})$ for all $\mathfrak{a} \in I(D_K)$ and $\alpha \in K^\times$ with α prime to D_K;
- For all $\mathfrak{a} \in I(D_K)$ and $x \in W[m]$ with $(m, N(\mathfrak{a})) = 1$, one has $\tilde{\kappa}_E(\mathfrak{a})(x) = \sigma_{\mathfrak{a}}(x)$, and if $\sigma_{\mathfrak{a}}$ is trivial on K_1, then $\tilde{\kappa}_E(\mathfrak{a}) \in K^\times$ and $\sigma_{\mathfrak{a}}(x) = [\tilde{\kappa}_E(\mathfrak{a})]x$ for all $x \in E[m]$.

Denote by $\kappa_E : G_K \to \mathcal{O}_F^\times$ the p-adic avatar of $\tilde{\kappa}_E$, possibly enlarging F so that $M \subseteq F$.

Proposition 4.10.1. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be fractional ideals in* $\mathcal{O}_c$ *prime to* $cN^+ D_K$*. Suppose that* $\mathfrak{a}\mathcal{O}_K$ *is trivial in* $\mathrm{Pic}\, \mathcal{O}_K$ *and put* $\alpha := \tilde{\kappa}_E(\mathfrak{a}) \in K^\times$*. Then*

$$(\mathrm{id} \times \alpha)^* \Delta_{\phi_{\mathfrak{b}}}^{\sigma_{\mathfrak{a}}} = \Delta_{\phi_{\mathfrak{a}\mathfrak{b}}},$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K^{ab}/K_1)$ *corresponds to* $\mathfrak{a}$ *through the Artin reciprocity map.*

Proof. Recall that the QM abelian surfaces A and $A_{\mathfrak{a}}$ are respectively the self-products of elliptic curves $E \times E$ ad $E_{\mathfrak{a}} \times E_{\mathfrak{a}}$ for each fractional ideal $\mathfrak{a}$. Note that the isogenies $\phi_{\mathfrak{a}} : A \twoheadrightarrow A_{\mathfrak{a}}$ of QM abelian surfaces are the self-product of the isogenies $E \twoheadrightarrow E_{\mathfrak{a}}$, denoted with $\varphi_{\mathfrak{a}}$ by Castella and Hsieh and used by them to define their Heegner cycles (cf. [CH, §4.1]), and also $\alpha \in K^\times$ acts on A (and $A_{\mathfrak{a}}$) as the matrix multiplication by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, hence as the multiplication by α in each component E (and $E_{\mathfrak{a}}$). Because of this, the proof of [CH, Lemma 4.3] works also in our case, working component by component. So we obtain that

$$(\mathrm{id} \times \alpha)^* \Gamma_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = (\alpha \times \mathrm{id})_* \Gamma_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = \Gamma_{\alpha \circ \phi_{\mathfrak{b}}}^{\sigma_{\mathfrak{a}}} = \Gamma_{\phi_{\mathfrak{a}\mathfrak{b}}} = \Gamma_{\mathfrak{a}\mathfrak{b}}.$$

Because $x(\mathfrak{b})^{\sigma_{\mathfrak{a}}} = (A_{\mathfrak{b}}, i_{\mathfrak{b}}, \nu_{\mathfrak{b},N^+})^{\sigma_{\mathfrak{a}}} = \mathfrak{a} \star (A_{\mathfrak{b}}, i_{\mathfrak{b}}, \nu_{\mathfrak{b},N^+}) = (A_{\mathfrak{a}\mathfrak{b}}, i_{\mathfrak{a}\mathfrak{b}}, \nu_{\mathfrak{a}\mathfrak{b},N^+}) = x_{\mathfrak{a}\mathfrak{b}}$ as points in Sh, we have that the immersions $i_{x(\mathfrak{b})^{\sigma_{\mathfrak{a}}}}$ and $i_{x(\mathfrak{a}\mathfrak{b})}$ are equal, thus, taking the r-power and applying the projector ε, we have that

$$(\mathrm{id} \times \alpha)^* \Delta_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = \Delta_{\mathfrak{a}\mathfrak{b}},$$

and we are done. $\qquad\square$

4.10. Compatibility properties

Proposition 4.10.2. *Suppose that $p \nmid c$. For all $n > 1$, one has*

$$T_p z_{cp^{n-1}} = p^{k-2} z_{p^{n-2}} + \mathrm{Cor}_{K_{cp^n}/K_{cp^{n-1}}}(z_{cp^n}).$$

For $\ell \nmid c$ that is inert in K, one has

$$T_\ell z_c = \mathrm{Cor}_{K_{c\ell}/K_c}(z_{c\ell}).$$

Proof. The operator T_p acts on the cycle $\Delta_{\phi_{cp^{n-1}}}$ coming from the isogeny $\phi_{cp^{n-1}}$: $A \to A_{cp^{n-1}}$ of QM abelian surfaces with $V_1(N^+)$-level structure and CM by K in the following way

$$T_p \Delta_{\phi_{cp^{n-1}}} = \sum_{i=1}^{p+1} \Delta_{\phi_i},$$

where the isogenies $\phi_i : A_i \to A_{cp^{n-1}}$ are p^2-isogenies of QM abelian surfaces with $V_1(N^+)$-level structures. These isogenies correspond to the $p+1$ sublattices of $\mathcal{O}_{cp^{n-1}} \times \mathcal{O}_{cp^{n-1}}$ that are invariant under the action of $\mathcal{O}_B$. Since such a sublattice L is determined by eL, one can work with sublattices of $\mathcal{O}_c$ of index p. Therefore one can rearrange the computation in the proof of [CH, Proposition 4.4] to obtain the formula in the statement, using an analogue of [CH, Lemma 4.2] in this case. The second part of the proposition can be proved analogously. $\square$

The following result is analogous to [CH, Proposition 4.5].

Proposition 4.10.3. *Let $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}_c$ prime to cN^+D_K. Then*

$$\chi_t(\sigma_\mathfrak{a})(\mathrm{id} \otimes e_\chi)z_c^{\sigma_\mathfrak{a}} = \chi(\sigma_\mathfrak{a})\chi_{\mathrm{cyc}}^{-r}(\sigma_\mathfrak{a})(\mathrm{id} \otimes e_\chi)z_\mathfrak{a},$$

where χ_{cyc} is the p-adic cyclotomic character.

Proof. Denote by $\sigma_\mathfrak{a} \in \mathrm{Gal}(K_c/K)$ the image of $\mathfrak{a}$ under the classical Artin reciprocity map. We have

$$\begin{aligned}
(\mathrm{id} \times \varphi_{\mathfrak{a}\mathcal{O}_K})_* \Gamma_{\phi_\mathfrak{a}} &= (\mathrm{id} \times \phi_\mathfrak{a})_* \big\{ (\phi_\mathfrak{a}(z), z) \mid z \in A \big\} \\
&= \big\{ (\varphi_\mathfrak{a}\phi_c(z), \varphi_{\mathfrak{a}\mathcal{O}_K} z) \mid z \in A \big\} \\
&= \Gamma_{\phi_c}^{\sigma_\mathfrak{a}},
\end{aligned}$$

as $\sigma_\mathfrak{a}(z) = \varphi_{\mathfrak{a}\mathcal{O}_K}(z)$ for any $z \in A(\mathbb{C})$ and $\sigma_\mathfrak{a}(z) = \varphi_\mathfrak{a}(z)$ for any $z \in A_c(\mathbb{C})$. Because $x_c^{\sigma_\mathfrak{a}} = x_\mathfrak{a}$, where $x_\mathfrak{a} = (A_\mathfrak{a}, i_\mathfrak{a}, \nu_{\mathfrak{a},N^+})$ and $x_c = x_{\mathcal{O}_c}$, one has $i_{x_c^{\sigma_\mathfrak{a}}} = i_{x_\mathfrak{a}}$. Hence, applying ε to $\big((\mathrm{id} \times \varphi_{\mathfrak{a}\mathcal{O}_K})_* \Gamma_{\phi_\mathfrak{a}}\big)^r = \big(\Gamma_{\phi_c}^{\sigma_\mathfrak{a}}\big)^r$, we obtain

$$(\mathrm{id} \times \varphi_{\mathfrak{a}\mathcal{O}_K})_* \Delta_{\phi_\mathfrak{a}} = \Delta_{\phi_c}^{\sigma_\mathfrak{a}}.$$

Then

$$z_c^{\sigma_\mathfrak{a}} = \Phi_{K_c}(\Delta_{\phi_c}^\mathfrak{a}) = \Phi_{K_c}((\mathrm{id} \times \varphi_\mathfrak{a})_* \Delta_{\phi_\mathfrak{a}}) = (\mathrm{id} \times \varphi_{\mathfrak{a}\mathcal{O}_K})_* \Phi_{K_c}(\Delta_{\phi_\mathfrak{a}}) = (\mathrm{id} \times \varphi_{\mathfrak{a}\mathcal{O}_K})_* z_\mathfrak{a}.$$

Observe that $\varphi_{\mathfrak{a}\mathcal{O}_K}$ acts on $\mathrm{Sym}^{2r} e T_p(\overline{A}) \cong \mathrm{Sym}^{2r} T_p(\overline{E})$ as its first component $\lambda_{\mathfrak{a}\mathcal{O}_K}$: $E \to E/E[\mathfrak{a}\mathcal{O}_K] = E_{\mathfrak{a}\mathcal{O}_K}$. From the proof of [CH, Proposition 4.5], we know that $\lambda_{\mathfrak{a}\mathcal{O}_K}$ acts on $\mathrm{Sym}^{2r} H_{\mathrm{ét}}^1(\overline{W}, \mathbb{Z}_p))$ as the push-forward $[\tilde{\kappa}_E(\mathfrak{a}\mathcal{O}_K)]_*$ of $\tilde{\kappa}_E(\mathfrak{a}\mathcal{O}_K) \in \mathrm{End}(W)$, which in turn induces the Galois action $\sigma_\mathfrak{a}$. Since e_χ commutes with the action of G_K, there are equalities

$$\begin{aligned}
\chi_{\mathrm{cyc}}^r(\sigma_\mathfrak{a})\chi_t(\sigma_\mathfrak{a})e_\chi\big(\sigma_\mathfrak{a} \otimes \mathrm{id} \otimes \mathrm{id} \cdot y\big) &= e_\chi\big((\sigma_\mathfrak{a} \otimes \chi_{\mathrm{cyc}}^r(\sigma_\mathfrak{a}) \otimes \chi_t(\sigma_\mathfrak{a}))(y)\big) \\
&= e_\chi(\sigma_\mathfrak{a} \cdot y) = \chi(\sigma_\mathfrak{a})e_\chi(y),
\end{aligned}$$

for any $y = y \otimes 1 \otimes 1 \in S(W) \otimes \chi_t = \mathrm{Sym}^{2r} H^1_{\text{ét}}(\overline{W}, \mathbb{Z}_p) \otimes \chi^r_{\mathrm{cyc}} \otimes \chi_t$. Therefore, viewing $z^{\sigma_\mathfrak{a}}_c, z_\mathfrak{a} \in H^1(K_c, T \otimes S(E)) \subseteq H^1(K_c, T \otimes S(W))$ as in §4.9 and letting $z^{\sigma_\mathfrak{a}}_c := z^{\sigma_\mathfrak{a}}_c \otimes 1$, $z_\mathfrak{a} := z_\mathfrak{a} \otimes 1 \in H^1(K_c, T \otimes S(W) \otimes \chi_t)$, we get

$$
\begin{aligned}
(\mathrm{id} \otimes e_\chi) z^{\sigma_\mathfrak{a}}_c &= (\mathrm{id} \otimes e_\chi)(\mathrm{id} \otimes [\tilde{\kappa}_E(\mathfrak{a}\mathcal{O}_K)]_*) z_\mathfrak{a} \\
&= \chi^{-r}_{\mathrm{cyc}}(\sigma_\mathfrak{a}) \chi^{-1}_t(\sigma_\mathfrak{a})(\mathrm{id} \otimes e_\chi)(\mathrm{id} \otimes \sigma_\mathfrak{a}) z_\mathfrak{a} \\
&= \chi^{-r}_{\mathrm{cyc}}(\sigma_\mathfrak{a}) \chi^{-1}_t(\sigma_\mathfrak{a}) \chi(\sigma_\mathfrak{a})(\mathrm{id} \otimes e_\chi) z_\mathfrak{a},
\end{aligned}
$$

which completes the proof. $\qquad\square$

Finally, we conclude with two more propositions. As one can rearrange the proofs of analogous results in [CH] to work also in our case, as we did in the proofs of the previous results, we will skip details.

Proposition 4.10.4. *Let τ be the complex conjugation. Then*

$$
z^\tau_{c,\chi} = w_f \cdot \chi(\sigma) \cdot (z_{c,\chi^{-1}})^\sigma,
$$

for some $\sigma \in \mathrm{Gal}(K_c/K)$, with $w_f = \pm 1$ the Atkin–Lehner eigenvalue of f.

Proof. Proceed as in the proof of [CH, Lemma 4.6]. $\qquad\square$

Proposition 4.10.5. *Let ℓ be a prime inert in K such that $\ell \nmid cND_K$. Let $\overline{\lambda}$ be a prime of $\overline{\mathbb{Q}}$ above ℓ. Denote by $K_{c\ell,\lambda}$ and $K_{c,\lambda}$ the completions of $K_{c\ell}$ and K_c, respectively, at the prime above ℓ determined by $\overline{\lambda}$, and write loc_λ for the localization map. Then*

$$
\mathrm{loc}_\lambda(z_{c\ell,\chi}) = \mathrm{Res}_{K_{c\ell,\lambda}/K_{c,\lambda}}\left(\mathrm{loc}_\lambda(z_{c,\chi})^{\mathrm{Frob}_\ell}\right).
$$

Proof. Proceed as in the proof of [CH, Lemma 4.7]. $\qquad\square$

Chapter 5

A p-adic Gross–Zagier formula

We want our p-adic L-function to satisfy a p-adic Gross–Zagier formula that relates Bloch–Kato logarithms of generalized Heegner cycles associated with characters $\chi : \mathrm{Gal}(K_{p\infty}/K) \to \mathcal{O}_{\mathbb{C}_p}^\times$ of infinity type $(j, -j)$ with $-k/2 < j < k/2$, with its values, i.e., we look for a formula of the shape

$$\mathscr{L}_f(\psi)(\hat{\phi}) = (\text{something}) \cdot \langle \log(z_\chi), * \rangle \, ,$$

where $\hat{\phi} : \mathrm{Gal}(K_{p\infty}/K) \to \mathcal{O}_{\mathbb{C}_p}^\times$ is the p-adic avatar of a Hecke character ϕ of infinity type $(-k/2 - j, k/2 + j)$ and $\chi = \hat{\psi}^{-1}\hat{\phi}^{-1}$. This formula provides a relation

$$\mathscr{L}_f(\psi) \longleftrightarrow \text{ generalized Heegner cycles.}$$

To establish a Gross–Zagier formula we will link our p-adic L-function to the differential operator $\theta = t\frac{d}{dt}$ on the Serre–Tate coordinates and then we will use some results of Brooks to obtain a key formula relating this operator θ applied to the modular form calculated on CM points with our generalized Heegner cycles.

5.1   The Bloch–Kato logarithm map

Let F, L be finite extensions of $\mathbb{Q}_p$ and V a F-vector space with a continuous linear $G_L := \mathrm{Gal}(\overline{L}/L)$-action. Set $DR_L(V) := H^0(L, B_{\mathrm{dR}} \otimes V) = (B_{\mathrm{dR}} \otimes V)^{G_L}$, where B_{dR} is the Fontaine's de Rham periods ring. If V is a de Rham representation (i.e., $\dim_F V = \dim_L DR_L(V)$), then we can consider the Bloch–Kato exponential map

$$\exp_{BK} : \frac{DR_L(V)}{\mathrm{Fil}^0 DR_L(V)} \xrightarrow{\cong} H^1_f(L, V),$$

that is the connecting homomorphism of the long exact sequence in cohomology coming from the short exact sequence

$$0 \longrightarrow V \longrightarrow B_{\mathrm{crys}}^{\phi=1} \otimes V \oplus \mathrm{Fil}^0 B_{\mathrm{dR}} \otimes V \longrightarrow B_{\mathrm{dR}} \longrightarrow 0.$$

See [BK, Definition 3.10, Corollary 3.8.4, Proposition 1.17]. Here $H^1_f(L, V)$ is the Bloch–Kato finite part

$$H^1_f(L, V) := \ker \left(H^1(L, V) \longrightarrow H^1_f(L, V \otimes B_{\mathrm{crys}}) \right),$$

where B_{crys} is Fontaine's crystalline period ring. Consider now the inverse of this map, the Bloch–Kato logarithm map

$$\log_{BK} : H^1_f(L, V) \xrightarrow{\cong} \frac{DR_L(V)}{\mathrm{Fil}^0 DR_L(V)}.$$

Since the long exact sequence in cohomology is functorial, the Bloch–Kato logarithm map is functorial as well, i.e., for any F-linear and G_L-equivariant morphism $V \to V'$ there is a commutative square

$$
\begin{array}{ccc}
H^1_f(L, V) & \xrightarrow{\log_{BK}} & \frac{DR_L(V)}{\mathrm{Fil}^0 DR_L(V)} \\
\downarrow & & \downarrow \\
H^1_f(L, V') & \xrightarrow{\log_{BK}} & \frac{DR_L(V')}{\mathrm{Fil}^0 DL_L(V')}.
\end{array}
$$

Denote by $V^* := \mathrm{Hom}_F(V, F)$ the dual of V and consider the perfect de Rham pairing

$$\langle -, - \rangle : DR_L(V) \times DR_L\big(V^*(1)\big) \longrightarrow \mathbb{C}_p.$$

Then we can consider the logarithm $\log_{BK}$ as a map

$$\log_{BK} : H^1_f(L, V) \xrightarrow{\cong} \frac{DR_L(V)}{\mathrm{Fil}^0 DL_F(V)} \cong \Big(\mathrm{Fil}^0 DR_L\big(V^*(1)\big)\Big)^{\vee}, \qquad (5.1.1)$$

which is again functorial.

5.2 The operator θ

Recall the differential operator $\theta := t\frac{d}{dt}$ on the power series ring $W[[t - 1]]$. For a negative exponent j, one can define

$$\theta^j := \lim_{i \to \infty} \theta^{j+(p-1)p^i}.$$

Indeed, this limit makes sense because of [Brooks, Proposition 4.18], which implies that $\theta^{j+(p-1)p^m} F \equiv \theta^{j+(p-1)p^n} F \mod p^{n+1}$ for $m \geq n \gg 0$ and $F \in W[[t - 1]]$, so $\big|\theta^{j+(p-1)p^m} F - \theta^{j+(p-1)p^n} F\big|_p \leq 1/p^{n+1}$.

For a positive integer m we know from equation (3.1.3) that

$$[x^m]F = \theta^m F,$$

but we are interested in the case when m is negative.

Fix an integer $j < 0$. For $n \gg 0$, one has $j + (p-1)p^n \geq 0$, therefore

$$\theta^{j+(p-1)p^n} F = [x^{j+(p-1)p^n}]F.$$

Denote by ϕ_n the continuous map in $\mathcal{C}(\mathbb{Z}_p, W)$ given by $z \mapsto z^{j+(p-1)p^n}$ for $n \gg 0$ and by ϕ the continuous map in $\mathcal{C}(\mathbb{Z}_p, W)$ given by $\phi(z) = z^j$, if $z \in \mathbb{Z}_p^\times$, and $\phi(z) = 0$ otherwise ($\mathbb{Z}_p^\times$ is open and closed in $\mathbb{Z}_p$). Consider $|\phi_n - \phi|_p = \sup_{x \in \mathbb{Z}_p} |\phi_n(x) - \phi(x)|_p$. If $x \in \mathbb{Z}_p^\times$ then

$$\big|\phi_n(x) - \phi(x)\big|_p = |x^j|_p \cdot |x^{(p-1)p^n} - 1|_p \leq 1/p^{n+1},$$

as $x^{(p-1)p^n} \equiv 1 \mod p^{n+1}$. If $x \in \mathbb{Z}_p \setminus \mathbb{Z}_p^\times = p\mathbb{Z}_p$, then $x = p^h z$ with $z \in \mathbb{Z}_p^\times$ and $h \geq 1$, so

$$\left| \phi_n(x) - \phi(x) \right|_p = \left| (p^h z)^{j+(p-1)p^n} \right|_p \leq 1/p^{j+(p-1)p^n}.$$

Hence $|\phi_n - \phi|_p \leq \max\left\{ 1/p^{n+1}, 1/p^{j+(p-1)p^n} \right\}$, which means that $\phi_n \to \phi$ in $\mathcal{C}(\mathbb{Z}_p, W)$ as $n \to \infty$. It follows that for a measure $\mu \in M(\mathbb{Z}_p, W)$ there are equalities

$$\lim_{n\to\infty} \mu(\phi_n) = \mu(\lim_{n\to\infty} \phi_n) = \mu(\phi),$$

so we can exchange the integral sign with the limit. Then

$$\theta^j F = \lim_{n\to\infty} \theta^{j+(p-1)p^n} F = \lim_{n\to\infty} [x^{j+(p-1)p^n}]F = \lim_{n\to\infty} \int_{\mathbb{Z}_p} x^{j+(p-1)p^n} t^x dF$$

$$= \lim_{n\to\infty} \int_{\mathbb{Z}_p} \phi_n(x) t^x dF = \int_{\mathbb{Z}_p} \lim_{n\to\infty} \phi_n(x) t^x dF$$

$$= \int_{\mathbb{Z}_p} \phi(x) t^x dF = \int_{\mathbb{Z}_p^\times} x^j t^x dF = [x^j]F.$$

Here by $\int_{\mathbb{Z}_p^\times}$ we mean $\int_{\mathbb{Z}_p}$ but with the characteristic function of $\mathbb{Z}_p^\times$ inside, so that we can write x^j in the integral. We conclude that, for any locally constant function $\phi \in \mathcal{C}(\mathbb{Z}_p, \mathcal{W})$, the formula

$$\int_{\mathbb{Z}_p} \phi(x) x^m dF = \theta^m([\phi]F)\big|_{t=1} \tag{5.2.2}$$

holds also for a negative integer m.

5.3 CM periods and de Rham cohomology classes

Recall that A has a model defined over $\mathcal{V} := \mathcal{W} \cap K^{\mathrm{ab}}$. Fix a non-vanishing global section ω_A of the line bundle $e\Omega_{A/\mathcal{V}}$ on A, defined over $\mathcal{V}$. Define a p-adic period $\Omega_p \in \mathbb{C}_p$ by the rule

$$\omega_A = \Omega_p \hat{\omega}_A$$

where $\hat{\omega}_A$ denote the section ω_P determined for A as in the last lines of 1.6.2 (which depends upon the p^∞- level structure on A).

Now, take a finite extension F of $\mathbb{Q}_p$ containing K_1 and recall the fixed non-vanishing differential ω_A in $eH^0(A, \Omega_{A/F})$. We can see it as an element of $H^1_{\mathrm{dR}}(A/F)$. This determines another element η_A, as in the last lines of [Brooks, §2.8], so that $\omega_A^i \eta_A^{2r-i}$ is a basis for $\mathrm{Sym}^{2r} eH^1_{dR}(A/F)$, when $i = 0, \ldots, 2r$. We can arrange ω_A and η_A in a way such that they correspond to the elements $\omega_E, \eta_E \in H^1_{\mathrm{dR}}(E/F)$ defined [CH, §4.5], through the isomorphism $eH^1_{\mathrm{dR}}(A/F) \cong H^1_{\mathrm{dR}}(E/F)$, keeping in mind that the elliptic curve E is denoted by A in [CH].

5.4 A key formula

We want now to prove the key formula that relates this operator θ applied to the modular form calculated on CM points with our generalized Heegner cycles. The proof of this formula is the same as the proof of [CH, (4.9)] but with [BDP, Proposition 3.24 and Lemmas 3.23, 3.22] replaced by [Brooks, Theorem 7.3, Lemma 8.6 and Proposition 7.4].

5.4.1 Preliminaries

We briefly recall some results from [Brooks] that will be used later. Our notation is as in [Brooks, §7.2], the only exception being that p-depletion is denoted here by $(\cdot)^{(p)}$ instead of $(\cdot)^\flat$. Note that in the notation of [Brooks] a triple (A, t, ω) consists of a QM abelian surface A, a $V_1(N^+)$-level structure t and a non-vanishing global section ω. In particular, the quaternionic multiplication is always understood.

The first result relates the operator θ applied to $f^{(p)}$ to the components $\tilde{g}_i^{(p)}$ of the Coleman primitive of $\omega_{f^{(p)}}$ associated with $f^{(p)}$.

Proposition 5.4.1 (Brooks). $\tilde{g}_i^{(p)} = i! \theta^{-1-i} f^{(p)}$.

Proof. This is [Brooks, Theorem 7.3]. □

Here $\tilde{g}_i^{(p)}$ is the i-component of the Coleman primitive of $\omega_f^{(p)}$ as in [Brooks, §7.2], which is a locally analytic p-adic modular form (in the sense of the last lines of [Brooks, §3.1]) of weight $k - 2 - 2i$.

The next result allows us to write the components of the Coleman primitive $\tilde{g}_i^{(p)}$ of $\omega_{f^{(p)}}$ in terms of the components of the Coleman primitive $\tilde{g}$ of ω_f.

Lemma 5.4.2 (Brooks). *If g is a locally analytic p-adic modular form of weight h such that*

$$T_p \tilde{g} = b_p \tilde{g}, \qquad \langle p \rangle \, \tilde{g} = \varepsilon_{\tilde{g}}(p) \tilde{g},$$

then

$$\tilde{g}^{(p)}(A', t', \omega') = \tilde{g}(A', t', \omega') - \frac{\epsilon_{\tilde{g}}(p) b_p}{p^h} \tilde{g}\big(\mathfrak{p} \star (A', t', \omega')\big) + \frac{\epsilon_{\tilde{g}}(p)}{p^{h+1}} \tilde{g}\big(\mathfrak{p}^2 \star (A', t', \omega')\big)$$

for every CM triple (A', t', ω').

Proof. This is [Brooks, Lemma 8.6]. □

Finally, we relate the components of the Coleman primitive of ω_f with the images of the Heegner cycles through the Abel–Jacobi map.

Lemma 5.4.3 (Brooks). *Let $\varphi : (A, t, \omega) \to (A', t', \omega')$ be an isogeny of degree d. Then*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \otimes \omega_A^i \eta_A^{2r-i}) = d^i \tilde{g}_i(A', t', \omega')$$

for $i = 0, \dots, 2r$.

Proof. This is [Brooks, Proposition 7.4]. □

Here F is a finite extension of $\mathbb{Q}_p$, containing K_1, such that Δ_φ is defined over F, and ω_f can be seen as an element in $\mathrm{Fil}^{k-1} PH_{\mathrm{dR}}^{k-1}(\mathcal{A}^r/F)$ as in [Brooks]. Recall the elements $\omega_A^i \eta_A^{2r-i} \in \mathrm{Sym}^{2r} eH_{dR}^1(A/F)$ for $i = 0, \dots, 2r$. Then

$$\omega_f \otimes \omega_A^i \eta_A^{2r-i} \in \mathrm{Fil}^{r+1} PH_{\mathrm{dR}}^{k-1}(\mathcal{A}^r/F) \otimes \mathrm{Sym}^{2r} eH_{\mathrm{dR}}^1(A/F) \cong \mathrm{Fil}^{k-1} \varepsilon H_{\mathrm{dR}}^{4r+1}(\mathcal{X}_r/F).$$

See [Brooks] and [BDP, pp. 1050–1053] for an explanation in the case of modular curves. Finally, AJ_F is the Abel–Jacobi map

$$\mathrm{AJ}_F : CH^{k-1}(\mathcal{X}_r/F)_0 \longrightarrow (\mathrm{Fil}^{k-1} \varepsilon H_{\mathrm{dR}}^{4r+1}(\mathcal{X}_r/F))^\vee$$

5.4. A key formula

defined in §6.3, which is the composition of the usual Abel–Jacobi map

$$CH^{k-1}(\mathcal{X}_r/F)_0 \longrightarrow H^1_f\Big(F, \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\Big)$$

with the Bloch–Kato logarithm map

$$\log_{\text{BK}} : H^1_f\Big(F, \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\Big) \longrightarrow \frac{DR_F\Big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\Big)}{\text{Fil}^0\Big(DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\big)\Big)}$$

for the Galois representation $V = \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)$. Actually, the image of the p-adic Abel–Jacobi map is contained in the subgroup $H^1_f(F, \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1))$ of $H^1(F, \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1))$, see [Nek00, Theorem 3.1].

Since the comparison isomorphism

$$\Phi : DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)\big) \xrightarrow{\cong} \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F),$$

for which we refer to [Xa, Theorem 9], is compatible with the filtrations, and since the Tate twist shifts the filtration, there are isomorphisms

$$\text{Fil}^j\Big(DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\big)\Big) \cong \text{Fil}^{j+k-1}\Big(DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)\big)\Big)$$

$$\cong \text{Fil}^{j+k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F).$$

It follows that there is a functorial isomorphism

$$\Phi : \frac{DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\big)}{\text{Fil}^0\, DR_F\big(\varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)\big)} \xrightarrow{\cong} \frac{\varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}{\text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}.$$

On the other hand, by Poincaré duality, there is an isomorphism

$$\frac{\varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}{\text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)} \cong \text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)^\vee. \tag{5.4.3}$$

Thus, writing $V := \varepsilon H^{4r+1}_{\text{ét}}(\overline{\mathcal{X}}_r, \mathbb{Q}_p)(k-1)$ in this case, we can view $\log_{BK}$ as a map

$$\log_{BK} : H^1_f(F, V) \xrightarrow{\cong} \frac{DR_F(V)}{\text{Fil}^0\, DR_F(V)} \cong \text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)^\vee.$$

For more details, see [Brooks, §6.3].

Remark 5.4.4. We can restate [Brooks, Proposition 7.4] as

$$\big\langle \log_{BK}(\xi_\varphi), \omega_f \otimes \omega^i_A \eta^{r-i}_A \big\rangle = d^i G_i(A', t', \omega'),$$

where ξ_φ is the image of Δ_φ under the usual Abel–Jacobi map. Indeed, isomorphism (5.4.3) sends $\log_{BK}(\xi_\varphi) \in \frac{\varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}{\text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}$ to $\big\langle \log_{BK}(\xi_\varphi), - \big\rangle$.

Since the comparison isomorphism respects the dualities, it is the same to compute $\langle \log_{BK}(\xi_\varphi), x \rangle \in \mathbb{C}_p$ where $\log_{BK}$ is considered to have image in $\frac{\varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}{\text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)}$ and $x \in \text{Fil}^{k-1}\, \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F)$, and $\langle \log_{BK}(\xi_\varphi), \Phi^{-1}(x) \rangle \in \mathbb{C}_p$, where $\log_{BK}$ is considered to have image in $\frac{DR_F(V)}{\text{Fil}^0\, DR_F(V)}$ and $\Phi^{-1}(x) \in \text{Fil}^0\, DR_L(V)$.

Remark 5.4.5. We want to apply the Bloch–Kato logarithm to the localization at $\mathfrak{p}$ of the generalized Heegner classes $z_{\mathfrak{a}}$, regarded as elements of $H^1(K_c, T \otimes \chi)$ via the map $\mathrm{id} \otimes e_\chi$, and also to the classes $z_\chi \in H^1(K, T \otimes \chi)$ that will be defined in (5.5.4). Recall the choice of the prime $\mathfrak{p}$ of K over p. If M is a finite extension of K, then we write $M_{\mathfrak{p}}$ for the completion of M at the prime over $\mathfrak{p}$ determined by the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and denote by $\mathrm{loc}_{\mathfrak{p}}$ the localization map

$$\mathrm{loc}_{\mathfrak{p}} : H^1(M, V) \longrightarrow H^1(M_{\mathfrak{p}}, V),$$

for any representation V of G_M. Thus, denote by $\log_{\mathfrak{p}}$ the composition of the localization at $\mathfrak{p}$ with $\log_{BK}$, so that $\log_{\mathfrak{p}}(z_{\mathfrak{a}})$ (respectively, $\log_{\mathfrak{p}}(z_\chi)$) is the image of the localization $\mathrm{loc}_{\mathfrak{p}}(z_{\mathfrak{a}}) \in H^1_f(K_{c,\mathfrak{p}}, T \otimes \chi)$ (respectively, of the localization $\mathrm{loc}_{\mathfrak{p}}(z_\chi) \in H^1_f(K_{\mathfrak{p}}, T \otimes \chi)$) via the Bloch–Kato logarithm. Since V_f can be realized as a quotient of $PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p)$, by the comparison isomorphism recalled above we get a map

$$PH^{k-1}_{\text{dR}}(\mathcal{A}^r/F) \cong \mathrm{DR}_F\big(PH^{k-1}_{\text{ét}}(\overline{\mathcal{A}}^r, \mathbb{Z}_p)\big) \longrightarrow \mathrm{DR}_F(V_f),$$

and hence also a map

$$\varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/F) \longrightarrow \mathrm{DR}_F\big(V_f(k/2) \otimes \chi\big).$$

Therefore we can view the $\omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}$ as elements of $\mathrm{DR}_F\big(V_f(k/2) \otimes \chi\big)$, which we will denote in the same way. Because of the functoriality of the logarithm and because the comparison isomorphism preserves the dualities, there is an equality

$$\big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle = \big\langle \log_{\mathfrak{p}}(\xi_{\phi_{\mathfrak{a}}}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle,$$

where $\xi_{\phi_{\mathfrak{a}}}$ is the image of $\Delta_{\mathfrak{a}}$ through the usual Abel–Jacobi map AJ_{p,K_c} and the $\omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}$ are viewed as elements of $\mathrm{Fil}^0 \mathrm{DR}_{K_{c,\mathfrak{p}}}(V_f(k/2) \otimes \chi)$ and of $\mathrm{Fil}^{k-1} \varepsilon H^{4r+1}_{\text{dR}}(\mathcal{X}_r/K_{c,\mathfrak{p}})$, respectively.

5.4.2 A technical lemma

Evaluating the p-adic L-function at the p-adic avatar of an anticyclotomic Hecke character of infinity type $(k/2 + j, -k/2 - j)$ with $-k/2 < j < k/2$ and conductor $p^n \mathcal{O}_K$ with $n \geq 1$, we get $\theta^{-j-k/2}\hat{f}^{(p)}(x(c_0 p^n)^{\sigma_{\mathfrak{a}}})$, with $x(c_0 p^n)$ the CM point defined in §1.5.5. Now we want to relate this expression to the image of the Abel–Jacobi map of certain Heegner classes.

Lemma 5.4.6. *Set* $z_{\mathfrak{a}}^{(p)} := z_{\mathfrak{a}} - a_p p^{2j-1} z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-1}}} + p^{4j+k-3} z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-2}}}$. *Then*

$$\theta^{-j-k/2}\hat{f}^{(p)}(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}) = \frac{(c_0 p^n N(\mathfrak{a}))^{-j-k/2+1}}{\Omega_p^{2j}(j+k/2-1)!} \cdot \big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}}^{(p)}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle.$$

Proof. We have

$$\theta^{-j-k/2}\hat{f}^{(p)}\big(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}\big) = \theta^{-j-k/2}f^{(p)}(A_{c_0 p^n}, \iota_{c_0 p^n}, \nu_{N^+,c_0 p^n}, \hat{\omega}_{c_0 p^n})^{\sigma_{\mathfrak{a}}}$$

$$= \left(\frac{1}{\Omega_p}\right)^{2j} \theta^{-j-k/2}f^{(p)}\big(\mathfrak{a} \star (A_{c_0 p^n}, \iota_{c_0 p^n}, \nu_{N^+,c_0 p^n}, \omega_{c_0 p^n})\big),$$

as the form $\theta^{-j-k/2}f^{(p)}$ has weight $-2j$ and, by definition, $\hat{\omega}_{c_0 p^n} = (1/\Omega_p)\omega_{c_0 p^n}$, with $\omega_{c_0 p^n}$ induced by ω_A. Let $\tilde{g}^{(p)}_{j+k/2-1}$ be the $j+k/2-1$-st component of the Coleman primitive of $\omega_{f^{(p)}}$. Applying Proposition 5.4.1 yields

$$\theta^{-j-k/2}\hat{f}^{(p)}(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}) = \frac{1}{\Omega_p^{2j}(j+k/2-1)!} \cdot \tilde{g}^{(p)}_{j+k/2-1}\big(\mathfrak{a}\star(A_{c_0 p^n}, \iota_{c_0 p^n}, \nu_{N^+, c_0 p^n}, \omega_{c_0 p^n})\big).$$

Writing $x_{\mathfrak{a}}$ for $\mathfrak{a}\star(A_{c_0 p^n}, \iota_{c_0 p^n}, \nu_{N^+, c_0 p^n}, \omega_{c_0 p^n}) = (A_{\mathfrak{a}}, \iota_{\mathfrak{a}}, \nu_{N^+, \mathfrak{a}}, \omega_{\mathfrak{a}})$, by Lemma 5.4.2 one has

$$\theta^{-j-k/2}\hat{f}^{(p)}\big(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}\big) = \frac{1}{\Omega_p^{2j}(j+k/2-1)!}\bigg[\, \tilde{g}_{j+k/2-1}(x_{\mathfrak{a}})$$
$$- \frac{a_p p^{-j-k/2}}{p^{-2j}}\, \tilde{g}_{j+k/2-1}(x_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-1}}})$$
$$+ \frac{1}{p^{-2j+1}}\, \tilde{g}_{j+k/2-1}(x_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-2}}})\bigg].$$

Then, applying Lemma 5.4.3, and keeping Remark 5.4.5 in mind, we get

$$\theta^{-j-k/2}\hat{f}^{(p)}\big(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}\big) =$$
$$= \frac{1}{\Omega_p^{2j}(j+k/2-1)!} \cdot \bigg[\, \frac{1}{(c_0 p^n N(\mathfrak{a}))^{j+k/2-1}}\big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle$$
$$- \frac{a_p p^{j-k/2}}{(c_0 p^{n-1} N(\mathfrak{a}))^{j+k/2-1}} \cdot \big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-1}}}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle$$
$$+ \frac{p^{2j-1}}{(c_0 p^{n-2} N(\mathfrak{a}))^{j+k/2-1}} \cdot \big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-2}}}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle\bigg].$$

Finally, if we set

$$z_{\mathfrak{a}}^{(p)} := z_{\mathfrak{a}} - a_p p^{2j-1} z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-1}}} + p^{4j+k-3} z_{\mathfrak{a}\mathcal{O}_{c_0 p^{n-2}}},$$

then we obtain

$$\theta^{-j-k/2}\hat{f}^{(p)}\big(x(c_0 p^n)^{\sigma_{\mathfrak{a}}}\big) = \frac{(c_0 p^n N(\mathfrak{a}))^{-j-k/2+1}}{\Omega_p^{2j}(j+k/2-1)!} \cdot \big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}}^{(p)}), \omega_f \otimes \omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle,$$

as was to be shown. $\qquad\square$

5.5   A p-adic Gross–Zagier formula

In this section we finally state and prove the p-adic Gross–Zagier formula that we are interested in, which relates our p-adic L-function to the Bloch–Kato logarithm of generalized Heegner classes.

First, we define cohomology class $z_\chi \in H^1(K, T \otimes \chi)$ associated with f and χ, which will be linked with the p-adic L-function by our p-adic Gross–Zagier type formula. Recall that $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ is our fixed modular form and $\chi : \mathrm{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_F^\times$ is a locally algebraic anticyclotomic character of infinity type (j, j) with $-k/2 < j < k/2$

and conductor $c_0 p^n \mathcal{O}_K$, where c_0 is prime to pN^+. Put

$$
\begin{aligned}
z_\chi : &= \mathrm{Cor}_{K_{c_0 p^n}/K}\, z_{c_0 p^n, \chi} \\
&= \sum_{\sigma \in \mathrm{Gal}(K_{c_0 p^n}/K)} \sigma \cdot (id \otimes e_\chi)(z_{c_0 p^n} \otimes \chi_t) \\
&= \sum_{\sigma \in \mathrm{Gal}(K_{c_0 p^n}/K)} (id \otimes e_\chi)\big(\chi_t(\sigma) z_{c_0 p^n}^\sigma\big) \\
&= \sum_{\mathfrak{a} \in \mathrm{Pic}\, \mathcal{O}_{c_0 p^n}} (id \otimes e_\chi)\big(\chi_t(\sigma_\mathfrak{a}) z_{c_0 p^n}^{\sigma_\mathfrak{a}}\big) \\
&= \sum_{\mathfrak{a} \in \mathrm{Pic}\, \mathcal{O}_{c_0 p^n}} \chi \varepsilon_{cyc}^{-r}(\sigma_\mathfrak{a})(id \otimes e_\chi)(z_\mathfrak{a}),
\end{aligned}
\tag{5.5.4}
$$

where the last equality holds by Proposition 4.10.3.

It is convenient to use, in the statement of the following theorem, the symbol $\doteq$ to indicate that the claimed equality holds up to an explicit non-zero multiplicative factor that is comparatively less important than the main terms. Recall that $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ with $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ distinct maximal ideals of $\mathcal{O}_K$.

Theorem 5.5.1. *Let ψ be an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$ and conductor $c_0 \mathcal{O}_K$ with $(c_0, Np) = 1$. If $\hat{\phi} : \mathrm{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_{\mathbb{C}_p}^\times$ is the p-adic avatar of an anticyclotomic Hecke character ϕ of infinity type $(k/2 + j, -k/2 - j)$ with $-k/2 < j < k/2$ and conductor $p^n \mathcal{O}_K$, $n \geq 1$, then*

$$
\frac{\mathscr{L}_{\mathfrak{p}, \psi}(f)(\hat{\phi}^{-1})}{\Omega_p^*} \doteq \big\langle \log_\mathfrak{p}(z_{f,\chi}), \omega_f \otimes \omega_A^{k/2+j-1} \eta_A^{k/2-j-1} \big\rangle,
$$

where, as before, $\chi := \hat{\psi}^{-1}\hat{\phi}$ and $\log_\mathfrak{p} := \log_{BK} \circ \mathrm{loc}_\mathfrak{p}$.

Proof. By definition of $\mathscr{L}_{f,\psi}$ (cf. Definition 3.6.2), one has

$$
\mathscr{L}_{f,\psi}(\hat{\phi}^{-1}) = \sum_{\mathrm{Pic}\, \mathcal{O}_{c_0}} \psi(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \int_{\mathbb{Z}_p^\times} \psi_\mathfrak{p} \hat{\phi}^{-1} \big| [\mathfrak{a}] d\mu_{\hat{f}_\mathfrak{a}^{(p)}}.
$$

Here $\hat{\phi}^{-1}\big|[\mathfrak{a}] : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ is given by $\big(\hat{\phi}^{-1}\big|[\mathfrak{a}]\big)(x) := \hat{\phi}^{-1}(x)\hat{\phi}^{-1}(\mathfrak{a})$, where x is viewed as an element of $\hat{K}^\times$ via the chosen embedding $\mathbb{Z}_p^\times \cong \mathcal{O}_{K,\mathfrak{p}} \hookrightarrow K_\mathfrak{p}^\times \hookrightarrow \hat{K}^\times$. It follows that

$$
\big(\hat{\phi}^{-1}\big|[\mathfrak{a}]\big)(x) = \phi^{-1}(x) x_\mathfrak{p}^{-k/2-j} x_{\bar{\mathfrak{p}}}^{k/2+j} \phi^{-1}(a) = \phi_\mathfrak{p}^{-1}(x) x^{-k/2-j} \phi^{-1}(a),
$$

as $a \in \hat{K}^{(c_0 p)\times}$ satisfies $a\hat{\mathcal{O}}_K \cap K = \mathfrak{a}$. Therefore

$$
\begin{aligned}
\mathscr{L}_{f,\psi}(\hat{\phi}^{-1}) &= \sum_{\mathrm{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) \phi^{-1}(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \int_{\mathbb{Z}_p^\times} \psi_\mathfrak{p} \phi_\mathfrak{p}^{-1}(x) x^{-k/2-j} d\mu_{\hat{f}_\mathfrak{a}^{(p)}} \\
&= \sum_{\mathrm{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a}) \phi^{-1}(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \int_{\mathbb{Z}_p} \psi_\mathfrak{p} \phi_\mathfrak{p}^{-1}(x) x^{-k/2-j} d\mu_{\hat{f}_\mathfrak{a}^{(p)}}.
\end{aligned}
$$

This is a consequence of the fact that the measure is supported on $\mathbb{Z}_p^\times$. Then, in light of what was explained in §5.2, we can bring out the differential operator $\theta = t_\mathfrak{a} \frac{d}{dt_\mathfrak{a}}$ from the integral of $x^{-k/2-j}$ and get

$$
\mathscr{L}_{f,\psi}(\hat{\phi}^{-1}) = \sum_{\mathrm{Pic}\, \mathcal{O}_{c_0}} \psi(\mathfrak{a}) \phi^{-1}(\mathfrak{a}) N(\mathfrak{a})^{-k/2} \theta^{-k/2-j} [\psi_\mathfrak{p} \phi_\mathfrak{p}^{-1}] \hat{f}_\mathfrak{a}^{(p)}\big(t_\mathfrak{a}(x_\mathfrak{a})\big),
$$

where $t_{\mathfrak{a}}$ is the Serre–Tate coordinate around the reduction modulo p of $x_{\mathfrak{a}}$.

Setting $\xi := \psi^{-1}\phi$, then we have

$$\mathscr{L}_{f,\psi}(\hat{\phi}^{-1}) = \sum_{\mathrm{Pic}(\mathcal{O}_{c_0})} \psi(\mathfrak{a})\phi^{-1}(\mathfrak{a})N(\mathfrak{a})^{-k/2}[\psi_{\mathfrak{p}}\phi_{\mathfrak{p}}^{-1}]\theta^{-k/2-j}\hat{f}_{\mathfrak{a}}^{(p)}\big(t_{\mathfrak{a}}(x_{\mathfrak{a}})\big)$$

$$= \sum_{\mathrm{Pic}(\mathcal{O}_{c_0})} \xi^{-1}(\mathfrak{a})N(\mathfrak{a})^{-k/2}[\xi_{\mathfrak{p}}^{-1}]\theta^{-k/2-j}\hat{f}_{\mathfrak{a}}^{(p)}\big(t_{\mathfrak{a}}(x_{\mathfrak{a}})\big).$$

Note that, since ξ^{-1} is a character of conductor $c_0 p^n \mathcal{O}_K$, $\xi_{\mathfrak{p}}^{-1}$ is a primitive Dirichlet character mod p^n via the isomorphism $(\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}^n)^\times \cong \mathcal{O}_{K,\mathfrak{p}}^\times/1+\mathfrak{p}^n$. By Proposition 3.6.4, since $(N(\mathfrak{a})\sqrt{-D_K})^{k/2+j}(\theta^{-j-k/2}\hat{f}^{(p)})_{\mathfrak{a}} = \theta^{-j-k/2}\hat{f}_{\mathfrak{a}}^{(p)}$, we obtain

$$\mathscr{L}_{f,\psi}(\hat{\phi}^{-1}) = \sum_{\mathrm{Pic}(\mathcal{O}_{c_0})} \xi^{-1}(\mathfrak{a})N(\mathfrak{a})^{-k/2}\big(N(\mathfrak{a})\sqrt{-D_K}\big)^{k/2+j}p^{-n}G(\xi_{\mathfrak{p}}^{-1})$$

$$\cdot \sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \xi_{\mathfrak{p}}(u)\theta^{-k/2-j}\hat{f}^{(p)}\big(x_{\mathfrak{a}}\star\alpha(u/p^n)\big).$$

For positive exponents one obtains $\big(N(\mathfrak{a})\sqrt{-D_K}\big)^{-m}(\theta^m\hat{f}^{(p)})_{\mathfrak{a}} = \theta^m\hat{f}_{\mathfrak{a}}^{(p)}$ by an easy computation; because $\theta^m = \lim_{i\to\infty}\theta^{m+(p-1)p^i}$, one has the same formula for negative exponents. Now, as in the proof of [CH, Theorem 4.9] and applying Lemma 5.4.6, we obtain

$$\frac{\mathscr{L}_{f,\psi}(\hat{\phi}^{-1})}{\Omega_p^{-2j}} = \frac{c_0^{-j-k/2+1}(\sqrt{-D_K})^{k/2+j}p^{n(-j-k/2)}G(\xi_{\mathfrak{p}}^{-1})\chi_{\mathfrak{p}}(p^n)}{(j+k/2-1)!}$$

$$\cdot \sum_{\mathrm{Pic}(\mathcal{O}_{c_0 p^n})} \chi\chi_{\mathrm{cyc}}^{1-k/2}(\sigma_{\mathfrak{a}})\big\langle \log_{\mathfrak{p}}(z_{\mathfrak{a}}^{(p)}), \omega_f\otimes\omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle.$$

Finally, by the argument in [CH, Theorem 4.9] and by (5.5.4), we get

$$\frac{\mathscr{L}_{f,\psi}(\hat{\phi}^{-1})}{\Omega_p^{-2j}} = C'\cdot\big\langle \log_{\mathfrak{p}}(z_\chi), \omega_f\otimes\omega_A^{j+k/2-1}\eta_A^{k/2-j-1}\big\rangle,$$

with

$$C' := \frac{c_0^{-j-k/2+1}(\sqrt{-D_K})^{k/2+j}p^{n(-j-k/2)}G(\xi_{\mathfrak{p}}^{-1})\chi_{\mathfrak{p}}(p^n)}{(j+k/2-1)!},$$

and the theorem is proved. $\qquad\square$

Chapter 6

Reciprocity law and Selmer groups

In this chapter we want to extend to our setting the reciprocity law of [CH, Theorem 5.7], relaxing the Heegner hypothesis and making use of generalized Heegner cycles on generalized Kuga–Sato varieties over Shimura curves and our p-adic L-function. This result will be important to prove (under certain assumptions) the vanishing of the Selmer group associated with the twisted representation $V_{f,\chi} := V_f(k/2) \otimes \chi$.

6.1 The algebraic anticyclotomic p-adic L-function $\mathcal{L}_{\mathfrak{p},\psi}$

In this section we will construct an algebraic p-adic L-function in terms of a big logarithm map and some Iwasawa cohomology classes coming from generalized Heegner classes. Assume for this section that our modular form $f \in S_k^{new}(\Gamma_0(N))$ is p-ordinary, i.e., that the p-th Fourier coefficient a_p is a unit of $\mathcal{O}_F$.

6.1.1 Perrin-Riou's big logarithm

Let G be a commutative compact p-adic Lie group and L a complete discretely valued extension of $\mathbb{Q}_p$. Recall that a *p-adic Lie group* is a group G endowed with a structure of a manifold over $\mathbb{Q}_p$ such that the group operation is locally analytic. For the definitions of a manifold over $\mathbb{Q}_p$ and of a locally analytic map, the reader is referred to [Schn].

Consider the noetherian topological O_L-algebra $\mathcal{O}_L[\![G]\!]$; if $L/\mathbb{Q}_p$ is a finite extension then it is compact. Put $\Lambda_L(G) := L \otimes_{\mathcal{O}_L} \mathcal{O}_L[\![G]\!]$, which is also noetherian; it is isomorphic to the continuous dual of the space $C(G, L)$ of continuous L-valued functions on G (cf. [LZ1, §2.2]). Now let $\mathcal{H}_L(G)$ denote the space of L-valued locally analytic distributions on G i.e., the continuous dual of the space $C^{la}(G, L)$ of L-valued locally analytic functions on G. There is an injective algebra homomorphism

$$\Lambda_L(G) \longrightarrow \mathcal{H}_L(G)$$

(see [Eme04, Proposition 2.2.7]), dual to the dense inclusion $C^{la}(G, L) \hookrightarrow C(G, L)$. We endow $\mathcal{H}_L(G)$ with its natural topology as an inverse limit of Banach spaces, with respect to which the map $\Lambda_L(G) \hookrightarrow \mathcal{H}_L(G)$ is continuous.

If L is a finite unramified extension of $\mathbb{Q}_p$ and G is the Galois group of a p-adic Lie extension $L_\infty = \cup_n L_n$ with L_n/L finite and Galois, then define the Iwasawa cohomology group

$$H^1_{\mathrm{Iw}}(L_\infty, V) := \left(\varprojlim_n H^1(L_n, T) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where V is a p-adic G_L-representation and T is a Galois stable lattice. The definition is independent from the choice of T. Now let $\hat{F}^{ur}$ the composite of $\hat{\mathbb{Q}}_p^{ur}$ with a finite extension F of $\mathbb{Q}_p$.

Suppose that V is a crystalline F-representation of G_L with non negative Hodge–Tate weights and that V has no quotient isomorphic to the trivial representation. Let $\mathfrak{F}$ be a relative height one Lubin–Tate formal group over $\mathcal{O}_L/\mathbb{Z}_p$ and let $\Gamma := \mathrm{Gal}(L(\mathfrak{F}_{p^\infty})/L) \cong \mathbb{Z}_p^\times$. Let B_{crys} be the Fontaine's crystalline period ring and put $\mathbf{D}_{\mathrm{crys},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_L}$. Assume that $V^{G_L(\mathfrak{F}_{p^\infty})} = 0$.

Theorem 6.1.1. *There exists a $\mathbb{Z}_p[\![\Gamma]\!]$-linear map*

$$\mathcal{L}_V : H^1_{\mathrm{Iw}}(L(\mathfrak{F}_{p^\infty}), V) \longrightarrow \mathcal{H}_{\hat{F}^{ur}}(\Gamma) \otimes_L \mathbf{D}_{\mathrm{crys},L}(V)$$

such that for any $\mathbf{z} \in H^1_{\mathrm{Iw}}(L(\mathfrak{F}_{p^\infty}), V)$ and any locally algebraic character $\chi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of Hodge-Tate weight j and conductor p^n there is an equality

$$\mathcal{L}_V(\mathbf{z})(\chi) = \varepsilon(\chi^{-1}) \cdot \frac{\Phi^n P(\chi^{-1}, \Phi)}{P(\chi, p^{-1}\Phi^{-1})} \cdot \begin{cases} \frac{(-1)^{-j-1}}{(-j-1)!} \cdot \log_{L,V(\chi^{-1})}(\mathbf{z}^{\chi^{-1}}) \otimes t^{-j} & \text{if } j < 0, \\ j! \cdot \exp^*_{L,V(\chi^{-1})*(1)}(\mathbf{z}^{\chi^{-1}}) \otimes t^{-j} & \text{if } j \geq 0, \end{cases}$$

where

- $\varepsilon(\chi^{-1})$ *and* $P(\chi^{\pm 1}, -)$ *are the ε-factor and the L-factor (see [LZ1, p. 8]);*

- Φ *denotes the crystalline Frobenius operator on* $\overline{\mathbb{Q}}_p \otimes_L \mathbf{D}_{\mathrm{crys},L}(V)$ *acting trivially on first factor;*

- $\mathbf{z}^{\chi^{-1}} \in H^1(L, V \otimes \chi^{-1})$ *is the specialization of $\mathbf{z}$ at χ^{-1}.*

Proof. This is [CH, Theorem 5.1]. $\qquad\square$

In the statement of this result, $\exp^*_{L,V(\chi^{-1})^*(1)}$ is the dual exponential map

$$\exp^*_{L,V(\chi^{-1})^*(1)} : H^1(L, V(\chi)) \to (\mathrm{Fil}^0 DR_L(V(\chi)))^\vee$$

of the Bloch–Kato exponential map of $V(\chi^{-1})^*(1)$.

We will apply Theorem 6.1.1 to some representation $\mathcal{F}^+V$ attached to a twist of $V_f(k/2)$ to obtain a map $\mathcal{L}_{\mathcal{F}^+V}$. Let ψ be an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$ and conductor $c\mathcal{O}_K$ with $p \nmid c$ and let $\hat{\psi} : \mathrm{Gal}(K_{cp^\infty}/K) \to \mathbb{C}_p^\times$ be its p-adic avatar. Let F be a finite extension of $\mathbb{Q}_p$ containing the Fourier coefficients of f and the image of $\hat{\psi}$, so $\hat{\psi} : \mathrm{Gal}(K_{cp^\infty}/K) \to \mathcal{O}_F^\times$. Since $p \nmid N$, if V_f is the F-linear Galois representation of $G_\mathbb{Q}$ associated with f, then $V_f \mid_{G_{\mathbb{Q}_p}}$ is crystalline. Because f is p-ordinary, there is an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \longrightarrow \mathcal{F}^+V_f \longrightarrow V_f \longrightarrow \mathcal{F}^-V_f \longrightarrow 0$$

with $\mathcal{F}^\pm V_f \cong F$ and $\mathcal{F}^+V_f$ unramified (see [Wil, Theorem 2.1.4] or [SC, §12.5.3]). Recall that $T \subseteq V_f(k/2)$ is a Galois-stable lattice; set

$$\begin{aligned} \mathcal{F}^+T &:= \mathcal{F}^+V_f(k/2) \cap T; \\ V &:= V_f(k/2) \otimes \hat{\psi}_{\mathfrak{p}}^{-1}, \quad \text{where } \hat{\psi}_{\mathfrak{p}} := \hat{\psi} \mid_{K_{\mathfrak{p}}}; \\ \mathcal{F}^\pm V &:= \mathcal{F}^\pm V_f(k/2) \otimes \hat{\psi}_{\mathfrak{p}}^{-1}. \end{aligned}$$

6.1. The algebraic anticyclotomic p-adic L-function $\mathcal{L}_{\mathfrak{p},\psi}$

Consider the dual representation $V^* := \mathrm{Hom}_F(V, F)$ of V and, with notation as above, define $\mathcal{F}^{\pm}V^* := \mathrm{Hom}_F(\mathcal{F}^{\mp}V, F)$. Let L_∞/L be the $\mathfrak{p}$-adic completion of K_{cp^∞}/K_c. The big logarithm $\mathcal{L}_{\mathcal{F}^+V}$, obtained on applying Theorem 6.1.1 to $\mathcal{F}^+V$ as a representation of G_L, is a map

$$\mathcal{L}_{\mathcal{F}^+V} : H^1_{\mathrm{Iw}}\big(L(\mathfrak{F}_{p^\infty}), \mathcal{F}^+V\big) \longrightarrow \mathcal{H}_{\hat{F}^{ur}}\big(\mathrm{Gal}(L(\mathfrak{F}_{p^\infty})/L)\big) \otimes_L \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V).$$

Since $L_\infty \subseteq L(\mathfrak{F}_{p^\infty})$, we can restrict $\mathcal{L}_{\mathcal{F}^+V}$ to the Galois group $\Gamma := \mathrm{Gal}(L_\infty/L) \cong \mathrm{Gal}(K_{cp^\infty}/K_c)$ to obtain a map

$$H^1_{\mathrm{Iw}}(L_\infty, \mathcal{F}^+V) \longrightarrow \mathcal{H}_{\hat{F}^{ur}}(\Gamma) \otimes_L \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V).$$

Recall the element $\omega_f \in DR_L(V_f)$ attached to f as in 5.4. Let $t \in B_{\mathrm{dR}}$ denotes Fontaine's p-adic analogue of $2\pi i$. Define the class

$$\omega_{f,\psi} := \omega_f \otimes t^{1-2r} \otimes \omega_\psi \in \mathbf{D}_{\mathrm{crys},L}(V^*),$$

where $\omega_\psi \in \mathbf{D}_{\mathrm{crys},L}(\hat{\psi}_{\mathfrak{p}}(k/2))$ is as in [CH, §5.3]. Denote again by $\omega_{f,\psi}$ its image under the projection $\mathbf{D}_{\mathrm{crys},L}(V^*) \twoheadrightarrow \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^-V^*)$.

There is a pairing

$$\langle -, - \rangle : \mathcal{H}_{\hat{F}^{ur}}(\Gamma) \otimes_L \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V) \times \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^-V^*) \longrightarrow \mathcal{H}_{\hat{F}^{ur}}(\Gamma).$$

Recall that $\mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V) = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} \mathcal{F}^+V)^{G_L}$ and $\mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^-V^*) = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} \mathcal{F}^-V^*)^{G_L}$ Finally, the composition of $\mathcal{L}_{\mathcal{F}^+V}$ with the map

$$\langle -, \omega_{f,\psi} \rangle : \mathcal{H}_{\hat{F}^{ur}}(\Gamma) \otimes_L \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V) \longrightarrow \mathcal{H}_{\hat{F}^{ur}}(\Gamma)$$

has image contained in the Iwasawa algebra $\Lambda_{\hat{F}^{ur}}(\Gamma) := \mathcal{O}_{\hat{F}^{ur}}[\![\tilde{\Gamma}]\!] \otimes \hat{F}^{ur}$. For details, see [CH, Lemma 5.5].

6.1.2 Iwasawa classes associated with generalized Heegner classes

Consider the Iwasawa cohomology group

$$H^1_{\mathrm{Iw}}(K_{cp^\infty}, T) := \left(\varprojlim_n H^1(\mathrm{Gal}(K'/K_{cp^n}), T) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where K' is the maximal extension of K unramified outside the primes above pNc (the representation T is unramified outside the prime above pN). Let α denote the root of the Hecke polynomial $x^2 - a_p x + p^{k-1}$ that is a p-adic unit. For each fractional ideal $\mathfrak{a}$ of $\mathcal{O}_c$ prime to $cNpD_K$, recall the cohomology class $z_\mathfrak{a}$ introduced in §4.8 and define the class

$$z_{\mathfrak{a},\alpha} := \begin{cases} z_\mathfrak{a} - \frac{p^{k-2}}{\alpha} \cdot z_{\mathfrak{a}\mathcal{O}_{c/p}} & \text{if } p \mid c \\ \frac{1}{\#\mathcal{O}_c^\times}\left(1 - \frac{p^{k/2-1}}{\alpha}\sigma_\mathfrak{p}\right)\left(1 - \frac{p^{k/2-1}}{\alpha}\sigma_{\overline{\mathfrak{p}}}\right) \cdot z_\mathfrak{a} & \text{if } p \nmid c, \end{cases}$$

which lives in $H^1\big(K_c, T \otimes S(E)\big)$. By Proposition 4.10.2, one knows that

$$\mathrm{Cor}_{K_{cp}/K_c}(z_{cp,\alpha}) = \alpha \cdot z_{c,\alpha}.$$

Now consider the projection e_χ for $\chi = \mathbf{1}$ and write $z_{\mathfrak{a},\alpha,\mathbf{1}}$ for the image of $z_{\mathfrak{a},\alpha}$ under $\mathrm{id} \otimes e_\mathbf{1} : H^1\big(K_c, T \otimes S(W)\big) \to H^1(K_c, T)$. Thus, it makes sense to consider the element

$$\boldsymbol{z}_{c,\alpha} := \varprojlim_n \alpha^{-n} z_{cp^n,\alpha,\mathbf{1}}$$

in the Iwasawa cohomology group $H^1_{\mathrm{Iw}}(K_{cp^\infty}, T)$.

There is an isomorphism

$$H^1_{\mathrm{Iw}}(K_{cp^\infty}, T) \cong H^1\big(K_c, T \otimes \mathcal{O}_F[\![\Gamma]\!]\big),$$

where $\Gamma := \mathrm{Gal}(K_{cp^\infty}/K_c)$ (see the proof of [LZ2, Proposition 2.4.2]). Put $\tilde{\Gamma}_c := \mathrm{Gal}(K_{cp^\infty}/K)$ and consider the map

$$H^1(K_{cp^\infty}, T) \cong H^1\big(K_c, T \otimes \mathcal{O}_F[\![\Gamma]\!]\big) \longrightarrow H^1\big(K_c, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!]\big);$$

we can view the classes $\boldsymbol{z}_{c,\alpha}$ as elements of $H^1\big(K_c, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!]\big)$. Then set

$$\boldsymbol{z}_f := \mathrm{Cor}_{K_c/K}(\boldsymbol{z}_{c,\alpha}) \in H^1\big(K, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!]\big), \tag{6.1.1}$$

where the subscript f is meant to remind that the class above, like the others already defined, depends on it.

For any character $\chi : \tilde{\Gamma}_c \to \mathcal{O}_{\mathbb{C}_p}^\times$, we can consider the twist $\boldsymbol{z}_f^\chi \in H^1(K, T \otimes \chi)$ of $\boldsymbol{z}_f$ through the χ-specialization map

$$H^1\big(K, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!]\big) \longrightarrow H^1\Big(K, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!] \otimes_{\mathcal{O}_F[\![\tilde{\Gamma}_c]\!]} \chi\Big) = H^1(K, T \otimes \chi),$$

where χ is extended to $\chi : \mathcal{O}_F[\![\tilde{\Gamma}_c]\!] \to \mathcal{O}_F$ in the obvious way, possibly enlarging F by adding the image of χ. Suppose that χ is non-trivial, of finite order and with conductor cp^n; then

$$\boldsymbol{z}_f^\chi = \alpha^{-n} z_\chi, \tag{6.1.2}$$

where $z_\chi \in H^1(K, T \otimes \chi)$ is as in (5.5.4). See [CH, Lemma 5.4] for details.

6.1.3 The algebraic anticyclotomic p-adic L-function

We want to apply the logarithm map $\mathcal{L}_{\mathcal{F}^+V}$ to the localization at $\mathfrak{p}$ of the classes $\boldsymbol{z}_{c,\alpha} \otimes \hat{\psi}^{-1}$, so we need to check that these classes actually lie in $H^1_{\mathrm{Iw}}(L_\infty, \mathcal{F}^+V) = H^1_{\mathrm{Iw}}(K_{cp^\infty, \mathfrak{p}}, \mathcal{F}^+V) \hookrightarrow H^1_{\mathrm{Iw}}(K_{cp^\infty, \mathfrak{p}}, V)$.

Similarly to what we said in §5.4.1, by [Nek00] one knows that $z_{\mathfrak{a}}$ lies in the Bloch–Kato Selmer group $\mathrm{Sel}(K_c, T \otimes S(E))$, indeed $z_{\mathfrak{a}}$ is the image, through a morphism of G_{K_c}-modules, of a cohomology class in $\mathrm{Sel}(K_c, \varepsilon H^{4r+1}_{\mathrm{\acute{e}t}}(\overline{\mathcal{X}}_r, \mathbb{Z}_p)(k-1))$, which is the Abel–Jacobi image of the generalized Heegner cycle $\Delta_{\mathfrak{a}}$. Recall that the Bloch–Kato Selmer group $\mathrm{Sel}(F, M)$ of a G_F-representation M, with F number field, is the subspace of elements x of $H^1(G_F, M)$ such that for all finite place v of F, the localization $\mathrm{loc}_v(x) \in H^1_f(F_v, M)$. See §6.4 for more precise definitions. Thus, $z_{cp^n, \alpha, \mathbf{1}} \in \mathrm{Sel}(K_{cp^n}, T)$ as well, so $\mathrm{loc}_{\mathfrak{p}}(z_{cp^n, \alpha, \mathbf{1}}) \in H^1_f(K_{cp^n, \mathfrak{p}}, T)$.

But $H^1_f(K_{cp^n, \mathfrak{p}}, T)$ is identified with the image of the map $H^1(K_{cp^n, \mathfrak{p}}, \mathcal{F}^+T) \to H^1(K_{cp^n, \mathfrak{p}}, T)$ (cf. [CH, §5.5]), so we can view $\mathrm{loc}_{\mathfrak{p}}(z_{cp^n, \alpha, \mathbf{1}}) \in H^1(K_{cp^n, \mathfrak{p}}, \mathcal{F}^+T)$. Since $\boldsymbol{z}_{c,\alpha}$ is the inverse limit of the classes $\alpha^{-n} z_{cp^n, \alpha, \mathbf{1}}$, one has

$$\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}_{c,\alpha}) \in H^1_{\mathrm{Iw}}(L_\infty, \mathcal{F}^+T).$$

We conclude that

$$\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}_{c,\alpha} \otimes \hat{\psi}^{-1}) \in H^1_{\mathrm{Iw}}(L_\infty, \mathcal{F}^+V).$$

Now, using notation similar to that in [CH], we can define

$$
\begin{aligned}
\mathcal{L}^*(\boldsymbol{z}_f \otimes \hat{\psi}^{-1}) &:= \mathrm{Cor}_{K_c/K}(\mathcal{L}_{\mathcal{F}+V}(\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}_{c,\alpha} \otimes \hat{\psi}^{-1}))) \\
&= \sum_{\sigma \in \tilde{\Gamma}_c/\Gamma} \mathcal{L}_{\mathcal{F}+V}(\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}_{c,\alpha}^{\sigma} \otimes \hat{\psi}^{-1})) \hat{\psi}(\sigma^{-1}) \\
&\in \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V) \otimes \Lambda_{\hat{F}^{ur}}(\tilde{\Gamma}_c),
\end{aligned}
$$

where $\Lambda_{\hat{F}^{ur}}(\tilde{\Gamma}_c) = \mathcal{O}_{\hat{F}^{ur}}[\![\tilde{\Gamma}_c]\!] \otimes \hat{F}^{ur}$. Finally, consider the restriction

$$
\mathcal{L}_{\psi}(\boldsymbol{z}_f) := \mathrm{Res}_{K_{p\infty}}(\mathcal{L}^*(\boldsymbol{z}_f \otimes \hat{\psi}^{-1})) \in \mathbf{D}_{\mathrm{crys},L}(\mathcal{F}^+V) \otimes \Lambda_{\hat{F}^{ur}}(\tilde{\Gamma}), \tag{6.1.3}
$$

where $\mathrm{Res}_{K_{p\infty}} : \tilde{\Gamma}_c = \mathrm{Gal}(K_{cp\infty}/K) \to \tilde{\Gamma} = \mathrm{Gal}(K_{p\infty}/K)$ is the restriction map.

6.2   Reciprocity law

We start by giving a sketch of the proof of the following theorem, which is analogous to that of [CH, Theorem 5.7].

Theorem 6.2.1. *Let* $\psi : K^{\times}/\mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ *be an anticyclotomic Hecke character of infinity type* $(k/2, -k/2)$ *and conductor* $c\mathcal{O}_K$ *with* $p \nmid c$ *and suppose that* f *is* p-*ordinary. Then*

$$
\langle \mathcal{L}_{\psi}(\boldsymbol{z}_f), \omega_f \otimes t^{1-k} \rangle = -c^{-j-k/2+1}(\sqrt{-D_K})^{k/2+j} \cdot \mathscr{L}_{f,\psi} \cdot \sigma_{-1,\mathfrak{p}} \in \Lambda_{\hat{F}^{ur}}(\tilde{\Gamma}),
$$

where $\sigma_{-1,\mathfrak{p}} := \mathrm{rec}_{\mathfrak{p}(-1)|_{K_{p\infty}}} \in \tilde{\Gamma}$ *is an element of order 2.*

Sketch of proof. For any $n > 1$, let $\hat{\phi} : \mathrm{Gal}(K_{p\infty}/K) \to \mathbb{C}_p^{\times}$ be the p-adic avatar of a Hecke character ϕ of infinity type $(k/2, -k/2)$ and conductor p^n. Moreover, define the finite order character $\chi := \hat{\psi}^{-1}\hat{\phi}$. Recall that, by (6.1.2), we have

$$
\boldsymbol{z}^{\chi} = \alpha^{-n} \cdot z_{\chi}.
$$

Now, since our ω_A and η_A are chosen to be compatible with the ones of [CH], so that $\omega_A \eta_A = t$ (see [CH, §5.3]), applying Theorem 5.5.1 with $j = 0$ and (6.1.2) yields the expression $\langle \log_{\mathfrak{p}}(\boldsymbol{z}_f^{\chi}) \otimes t^{k/2}, \omega_f \otimes t^{1-k} \rangle$ from $\mathscr{L}_{f,\psi}(\hat{\phi}^{-1})$. Now, performing the same computation as in the proof of [CH, Theorem 5.7] and applying Theorem 6.1.1 to the expression $\langle \mathcal{L}_{\psi}(\boldsymbol{z}_f), \omega_f \otimes t^{1-k} \rangle$, we obtain the formula of the statement evaluated at $\hat{\phi}^{-1}$ for any p-adic avatar $\hat{\phi}$ as above. By an argument that is formally identical to the one at the end of the proof of [CH, Theorem 5.7], one gets the desired equality. $\qquad \square$

Now we state the reciprocity law that is the counterpart of [CH, Corollary 5.8].

Theorem 6.2.2. *Let* $\chi : \mathrm{Gal}(K_{p\infty}/K) \to \mathcal{O}_F^{\times}$ *be a locally algebraic* p-*adic Galois character of infinity type* $(j, -j)$ *with* $j \geq k/2$ *and conductor* $cp^n\mathcal{O}_K$ *with* $p \nmid c$ *and suppose that* f *is* p-*ordinary. Then*

$$
\langle \exp^*(\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}^{\chi^{-1}})), \omega_f \otimes \omega_A^{-k/2-j+1}\eta_A^{-k/2+j+1} \rangle^2 = D(f, \psi, \chi\psi^{-1}, K) \cdot L(f, \chi, k/2),
$$

where $D(f, \psi, \chi\psi^{-1}, K)$ *is non-zero a constant depending on* f, χ, K *and* ψ, *which is a Hecke character of infinity type* $(k/2, -k/2)$ *and conductor* c.

Sketch of proof. Let $\hat{\psi} : \mathrm{Gal}(K_{p^\infty}/K) \to \mathbb{C}_p^\times$ be the p-adic avatar of a Hecke character ψ of infinity type $(k/2, -k/2)$ and conductor c, so that $\hat{\phi} := \chi\hat{\psi}^{-1}$ is a locally algebraic character of infinity type $(j - k/2, -j + k/2)$ and conductor p^n. The proof proceeds by using Theorem 6.1.1 to extract the expression $\langle \exp^*(\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{z}^{\chi^{-1}})), \omega_f \otimes \omega_A^{-k/2-j+1}\eta_A^{-k/2+j+1} \rangle^2$ from $\langle \mathcal{L}_{\mathfrak{p},\psi}(\boldsymbol{z}), \omega_f \otimes t^{1-k} \rangle (\hat{\phi})$. Now one can square and apply Theorem 6.2.1 to recover the square of the p-adic L-function $\mathscr{L}_{f,\psi}(\hat{\phi})$, and then use the interpolation formula of Theorem 3.7.1 to obtain the statement. The constant $D(f, \psi, \chi\psi^{-1}, K)$ turns out to be

$$D(f, \psi, \chi\psi^{-1}, K) = \alpha^{-2n}\epsilon(0, \phi_{\mathfrak{p}}^{-1}\phi_{\bar{\mathfrak{p}}}^{-1})^{-2}p^{2nr}\Gamma(j - k/2 + 1)^{-2}$$
$$\cdot \Omega_p^{-4j}c^{-2j-k+2}\sqrt{-D_K}^{k+2j}C(f, \psi, \chi\psi^{-1}, K).$$

For the details of the computation, see the proof of [CH, Corollary 5.8]. $\qquad\square$

6.3   The anticyclotomic Euler system method

In this section we apply the Kolyvagin-type method developed in [CH, Section 7] to our system of Heegner classes, in order to deduce results on the Selmer group of the representation $V_{f,\chi} := V_f(k/2) \mid_{G_K} \otimes\chi$, with $\chi : \mathrm{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_F^\times$ a locally algebraic p-adic Galois character of infinity type $(j, -j)$ with $j \geq k/2$ and conductor $c_0 p^s \mathcal{O}_K$ with $c = c_0 p^s$ and $(pN, c_0) = 1$.

First of all, we introduce the objects and the properties of the Kolyvagin method employed in [CH]. Then, we will apply it to our system of generalized Heegner cycles and, finally, we will deduce results on Selmer groups. As will be clear, we follow [CH] closely.

6.3.1   Anticyclotomic Euler systems

Let $\mathcal{G}_n := \mathrm{Gal}(K_n/K)$ and let $H^1(K_n, -)$ denote the cohomology group with respect to $\mathrm{Gal}(K^{\Sigma_n}/K_n)$, where Σ_n is the finite set containig the prime factors of pNc_0n and K^{Σ_n} is the maximal extension of K unramified outside the primes above Σ_n.

By [Nek92, Proposition 3.1], there is a $G_\mathbb{Q}$-equivariant $\mathcal{O}_F$-linear perfect pairing

$$\langle -, - \rangle : T \times T \longrightarrow \mathcal{O}_F(1)$$

that induces for each local field L the local Tate pairing

$$\langle -, - \rangle_L : H^1(L, T) \times H^1(L, T) \longrightarrow \mathcal{O}_F. \tag{6.3.4}$$

Here T is the $G_\mathbb{Q}$-stable $\mathcal{O}_F$-lattice inside $V_f(k/2)$ that was fixed before. Let ϖ be a uniformizer of $\mathcal{O}_F$ and let $\mathbb{F} := \mathcal{O}_F/(\varpi)$ be the residue field. For every integer $M \geq 1$, set $T_M := T/\varpi^M T$.

For us, ℓ will always denote a prime inert in K and λ will be the unique prime of K above ℓ; denote by Frob_ℓ the Frobenius element of λ in G_K. Let $H^1_f(K_\lambda, -)$ be the finite part of $H^1(K_\lambda, -)$ and put $H^1_s(K_\lambda, -) := H^1(K_\lambda, -)/H^1_f(K_\lambda, -)$, where K_λ is the completion of K at λ. Denote by $\mathrm{loc}_\ell : H^1(K, -) \to H^1(K_\lambda, -)$ the localization map at ℓ and write $\delta_n : H^1(K, -) \to H^1_s(K_\lambda, -)$ for the composition of loc_ℓ with the quotient map.

Let $\mathcal{S}$ be the set of square-free products of primes ℓ inert in K with $\ell \nmid 2pNc_0$. Let τ denote complex conjugation.

Definition 6.3.1. An **anticyclotomic Euler system** for T and χ is a collection $\mathbf{c} = \{c_n\}_{n \in \mathcal{S}}$ of classes $c_n \in H^1(K_{nc}, T \otimes \chi^{-1})$ such that for any $n = m\ell \in \mathcal{S}$ the following properties hold:

1. $\mathrm{Cor}_{K_{nc}/K_{mc}}(c_n) = a_\ell(f) \cdot c_m$;

2. $\mathrm{loc}_\ell(c_n) = \mathrm{Res}_{K_{mc,\lambda}/K_{nc,\lambda}}(\mathrm{loc}_\ell(c_m)^{\mathrm{Frob}_\ell})$;

3. if $\chi^2 = 1$ then $c_n^\tau = w_f \cdot \chi(\sigma) \cdot c_n^\sigma$ for some $\sigma \in \mathrm{Gal}(K_{nc}/K)$,

where $w_f \in \{\pm 1\}$ is the Atkin–Lehner eigenvalue of f.

6.3.2 Kolyvagin's derivative classes

Define the constant β as in [CH, (7.2)]. For any integer $M \geq 1$, denote by $\mathcal{S}_M \subseteq \mathcal{S}$ the set of square-free products of primes ℓ such that

1. ℓ is inert in K;
2. $\ell \nmid 2c_0 Np$;
3. $\varpi^M \mid \ell + 1, a_\ell(f)$;
4. $\varpi^{M+\beta+1} \nmid \ell + 1 \pm a_\ell(f)\ell^{1-r}$.

A prime number satisfying all these conditions is called an M-**admissible (Kolyvagin) prime**. By using the Čebotarev density theorem, it can be checked that there exist infinitely many M-admissible primes.

Put $G_n := \mathrm{Gal}(K_n/K_1) \cong \mathrm{Gal}(K_{nc}/K_c) \subseteq \mathcal{G}_{nc}$. Let $n \in \mathcal{S}_M$; since n is square-free, there is a splitting $G_n = \prod_{\ell \mid n} G_\ell$. Moreover, each $\ell \mid n$ is inert in K, so the group $G_\ell \cong \mathrm{Gal}(K_\ell/K_1)$ is cyclic of order $\ell + 1$. Fix a generator σ_ℓ for each G_ℓ and put

$$D_\ell := \sum_{i=1}^{\ell} i\sigma_\ell^i \in \mathbb{Z}[G_\ell],$$

$$D_n := \prod_{\ell \mid n} D_\ell \in \mathbb{Z}[G_n] \subseteq \mathcal{O}_F[\mathcal{G}_{nc}].$$

The telescoping identity

$$(\sigma_\ell - 1)D_\ell = (\ell + 1 - \mathrm{Tr}_\ell),$$

where $\mathrm{Tr}_\ell := \sum_{i=0}^{\ell} \sigma_\ell^i$, holds in $\mathbb{Z}[G_\ell]$.

Now we choose a positive integer M' such that $p^{M'}$ annihilates

1. the kernel and the cokernel of $\mathrm{Res}_{K/K_n} : H^1(K, T_M \otimes \chi^{-1}) \to H^1(K_n, T_M \otimes \chi^{-1})^{\mathcal{G}_n}$ for all $n, M \in \mathbb{Z}_+$;
2. the local cohomology groups $H^1(K_v, T_M \otimes \chi^{-1})$ for all places $v \mid c_0 N$.

One can prove that such an integer exists as in [Nek92, Proposition 6.3, Corollary 6.4 and Lemma 10.1].

Now consider an anticyclotomic Euler system $\mathbf{c} = \{c_n\}$ for T and χ. Denote by red_M the reduction $H^1(-, T \otimes \chi^{-1}) \to H^1(-, T_M \otimes \chi^{-1})$. For $n \in \mathcal{S}_M$, we want to apply the derivative operators D_n to the classes c_n. For each $n \in \mathcal{S}_M$ there is a unique class $\mathcal{D}_M(n) \in H^1(K_c, T_M \otimes \chi^{-1})$ such that

$$\mathrm{Res}_{K/K_n}(\mathcal{D}_M(n)) = p^{3M'} \mathrm{red}_M(D_n c_n),$$

because of the properties of M'. Define the derivative class by

$$P_{M,\chi^{-1}}(n) := \mathrm{Cor}_{K_c/K}(\mathcal{D}_M(n)) \in H^1(K, T_M \otimes \chi^{-1}). \tag{6.3.5}$$

6.3.3 Local conditions

Now we introduce Selmer groups imposing local conditions at p at the cohomology classes, through the choices of subspaces of $H^1(K_{\mathfrak{p}}, V_f(k/2) \otimes \chi^{-1}) \oplus H^1(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi^{-1})$, recall that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K.

Let $\mathcal{F} \subseteq H^1(K_{\mathfrak{p}}, V_f(k/2) \otimes \chi^{-1}) \oplus H^1(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi^{-1})$ be an F-subspace and let $\mathcal{F}^* \subseteq H^1(K_{\mathfrak{p}}, V_f(k/2) \otimes \chi) \oplus H^1(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi)$ be the orthogonal complement of $\mathcal{F}$ with respect to the local Tate pairing of equation (6.3.4). Assume that $\mathcal{F}^* = \mathcal{F}$ if $\chi^2 = 1$. Define $\mathcal{F}_T \subseteq H^1(K_{\mathfrak{p}}, V_f(k/2) \otimes \chi^{-1}) \oplus H^1(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi^{-1})$ to be the F-subspace obtained as the inverse image of $\mathcal{F}$ under the direct sum of the maps $H^1(K_{\mathfrak{p}}, T \otimes \chi^{-1}) \to H^1(K_{\mathfrak{p}}, V_f(k/2) \otimes \chi^{-1})$, $H^1(K_{\bar{\mathfrak{p}}}, T \otimes \chi^{-1}) \to H^1(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi^{-1})$ and $\mathcal{F}_M \subseteq H^1(K_{\mathfrak{p}}, T_M \otimes \chi^{-1}) \oplus H^1(K_{\bar{\mathfrak{p}}}, T_M \otimes \chi^{-1})$ as the image of $\mathcal{F}_T$ through the reduction map. Put $Y_M := T_M \otimes \chi^{-1}$. Now define

$$\mathrm{Sel}_{\mathcal{F}}^{(n)}(K, Y_M) := \left\{ x \in H^1(K, Y_M) \mid \begin{matrix} \mathrm{loc}_v(x) \in H_f^1(K_v, Y_M) & \text{if } v \nmid pn \\ \mathrm{loc}_p \in \mathcal{F}_M & \text{if } p \nmid n \end{matrix} \right\},$$

where $\mathrm{loc}_p = \mathrm{loc}_{\mathfrak{p}} \oplus \mathrm{loc}_{\bar{\mathfrak{p}}}$. Note that if $p \mid n$ then the choice of $\mathcal{F}_M$ is irrelevant. If $n = 1$ we abbreviate $\mathrm{Sel}_{\mathcal{F}}(K, Y_M) := \mathrm{Sel}_{\mathcal{F}}^{(1)}(K, Y_M)$. Define then

$$\mathrm{Sel}_{\mathcal{F}}(K, T \otimes \chi^{-1}) := \varinjlim_M \mathrm{Sel}_{\mathcal{F}}(K, Y_M).$$

If $\mathbf{c} = \{c_n\}_{n \in \mathcal{S}}$ is an anticyclotomic Euler system for T and χ, let

$$\mathbf{c}_K := \mathrm{Cor}_{K_c/K}(c_1) \in H^1(K, T \otimes \chi^{-1}).$$

Then

$$P_{M,\chi^{-1}}(1) = p^{3M'} \mathrm{red}_M(\mathbf{c}_K),$$

since the square

$$
\begin{array}{ccc}
H^1(K_c, T \otimes \chi^{-1}) & \xrightarrow{\mathrm{Cor}_{K_c/K}} & H^1(K, T \otimes \chi^{-1}) \\
\downarrow{\scriptstyle \mathrm{red}_M} & & \downarrow{\scriptstyle \mathrm{red}_M} \\
H^1(K_c, T_M \otimes \chi^{-1}) & \xrightarrow{\mathrm{Cor}_{K_c/K}} & H^1(K, T_M \otimes \chi^{-1})
\end{array}
$$

is commutative. Using [Nek92, Proposition 10.2] we obtain that

$$P_{M,\chi^{-1}}(n) \in \mathrm{Sel}_{\mathcal{F}}^{(np)}(K, Y_M).$$

Here we do not bother about $\mathcal{F}$ because $p \mid pn$.

Definition 6.3.2. An anticyclotomic Euler system $\mathbf{c} = \{c_n\}_{n \in \mathcal{S}}$ for T and χ has **local condition** $\mathcal{F}$ if it satisfies

4. $\mathbf{c}_K \in \mathrm{Sel}_{\mathcal{F}}(K, T \otimes \chi^{-1})$ and $\mathbf{c}_K^\tau \in \mathrm{Sel}_{\mathcal{F}^*}(K, T \otimes \chi^{-1})$, that is $\mathrm{loc}_p(\mathbf{c}_K) \in \mathcal{F}_T$ and $\mathrm{loc}_p(\mathbf{c}_K^\tau) \in \mathcal{F}_T^*$;

5. for every M and $n \in \mathcal{S}_M$, one has

$$P_{M,\chi^{-1}}(n) \in \mathrm{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1}),$$

that is, $\mathrm{loc}_p(P_{M,\chi^{-1}}(n)) \in \mathcal{F}_M$ (these two conditions are equivalent because $P_{M,\chi^{-1}}(n) \in \mathrm{Sel}_{\mathcal{F}}^{(np)}(K, Y_M)$).

6.3. The anticyclotomic Euler system method

Now we state an important technical result.

Theorem 6.3.3. *Let $\mathbf{c} = \{c_n\}_{n \in \mathcal{S}}$ be an anticyclotomic Euler system for T and χ with local condition $\mathcal{F}$. If $\mathbf{c}_K \neq 0$, then*

$$\mathrm{Sel}_{\mathcal{F}^*}(K, V \otimes \chi^{-1}) = F \cdot \mathbf{c}_K^\tau.$$

Proof. This is [CH, Theorem 7.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.3.4 Construction of Euler systems for generalized Heegner cycles

We keep notation and assumptions introduced at the beginning of this section, but now we assume that

- χ has infinity type $(j, -j)$ with $j \geq k/2$;

- f is ordinary at p.

Recall the cohomology class $\boldsymbol{z}_f \in H^1\big(K, T \otimes \mathcal{O}_F[\![\tilde{\Gamma}_c]\!]\big)$ defined in (6.1.1) and consider its χ-specialization $\boldsymbol{z}_f^\chi \in H^1(K, T \otimes \chi)$. Let us consider for $v = \mathfrak{p}, \bar{\mathfrak{p}}$ the subspace $\mathcal{L}_v$ of $H^1(K_v, V \otimes \chi)$ spanned by $\mathrm{loc}_v(\boldsymbol{z}_f^\chi)$ and put

$$\mathcal{L}_{v,T} := \mathcal{L}_v \cap H^1(K, T \otimes \chi) = \mathcal{O}_F \varpi^{-a_v} \mathrm{loc}_v(\boldsymbol{z}_f^\chi) + H^1(K_v, T)_{\mathrm{tor}},$$

where a_v is some positive integer and the subscript tor denotes the torsion subgroup. Set $\mathcal{L}^* := \mathcal{L}_{\mathfrak{p}}^* \oplus \mathcal{L}_{\bar{\mathfrak{p}}}^*$. Choose M' large enough so that $p^{M'} H^1(K_v, T)_{\mathrm{tor}} = 0$ for $v = \mathfrak{p}, \bar{\mathfrak{p}}$. Recall the cohomology classes $\boldsymbol{z}_{m,\alpha} \in H^1(K_{m\infty}, T)$ and for $n \in \mathcal{S}$ set

$$c_n := \boldsymbol{z}_{cn,\alpha}^{\chi^{-1}} \in H^1(K_{nc}, T \otimes \chi^{-1}),$$

where $\boldsymbol{z}_{cn,\alpha}^{\chi^{-1}}$ is the specialization at χ^{-1} obtained via the map

$$H^1_{\mathrm{Iw}}(K_{ncp^\infty}, T) \longrightarrow H^1(K_{nc}, T \otimes \chi^{-1}).$$

Finally, define

$$\mathbf{c} := \{c_n\}_{n \in \mathcal{S}} = \left\{ \boldsymbol{z}_{cn,\alpha}^{\chi^{-1}} \right\}_{n \in \mathcal{S}}. \tag{6.3.6}$$

We would like to prove that this collection of cohomology classes is an anticyclotomic Euler system with local condition $\mathcal{L}^*$.

Proposition 6.3.4. *The collection $\mathbf{c} := \{c_n\}_{n \in \mathcal{S}}$ is an anticyclotomic Euler system with local condition $\mathcal{L}^*$. Moroever, $\mathbf{c}_K = \boldsymbol{z}_f^{\chi^{-1}}$.*

Proof. If $\mathbf{c}$ is an anticyclotomic Euler system, then

$$\mathbf{c}_K = \mathrm{Cor}_{K_c/K}(c_1) = \mathrm{Cor}_{K_c/K}(\boldsymbol{z}_{c,\alpha}^{\chi^{-1}}) = \mathrm{Cor}_{K_c/K}(\boldsymbol{z}_{c,\alpha})^{\chi^{-1}} = \boldsymbol{z}_f^{\chi^{-1}}.$$

Analogously to (6.1.2), we obtain

$$\boldsymbol{z}_{n,\alpha}^\psi = \alpha^{-t} \cdot z_{n,\psi}$$

for each non-trivial finite order character $\psi : \mathrm{Gal}(K_{c_0 p^\infty}/K_{c_0}) \to \mathcal{O}_{\mathbb{C}_p}^\times$ of conductor $c = c_0 p^s$. By Propositions 4.10.2, 4.10.4, 4.10.5, we know that

1. $\mathrm{Cor}_{K_{nc}/K_{mc}}(z_{nc,\psi}) = a_\ell(f) \cdot z_{mc,\psi}$;
2. $\mathrm{loc}_\ell(z_{nc,\psi}) = \mathrm{Res}_{K_{mc,\lambda}/K_{nc,\lambda}}(\mathrm{loc}_\ell(z_{mc,\psi})^{\mathrm{Frob}_\ell})$;
3. $z_{nc,\psi}^\tau = w_f \cdot \psi(\sigma) \cdot z_{nc,\psi^{-1}}^\sigma$ for some $\sigma \in \mathrm{Gal}(K_{nc}/K)$.

Upon taking $\psi = \mathbf{1}$, we deduce these properties for the classes $\mathbf{z}_{nc,\alpha}$ and then for $\mathbf{z}_{nc,\alpha}^{\chi^{-1}}$ by specializing the relations at χ^{-1}. This proves that $\mathbf{c}$ is an anticyclotomic Euler system. The last thing we need to show is that $\mathbf{c}$ has local condition $\mathcal{L}^*$, which can be checked as in the proof of [CH, Proposition 7.8]. $\qquad \square$

Proposition 6.3.5. *If* $\mathrm{loc}_{\mathfrak{p}}\left(\mathbf{z}_f^{\chi^{-1}}\right) \neq 0$, *then* $\mathrm{Sel}\left(K, V_f(k/2) \otimes \chi\right) = 0$.

Proof. The proof is completely analogous to that of [CH, Theorem 7.9], so we will briefly sketch the arguments. For each choice of subspaces $\mathcal{F}_v \subset H^1\left(K_v, V_f(k/2) \otimes \chi\right)$ with $v = \mathfrak{p}, \overline{\mathfrak{p}}$, consider the "generalized Selmer group" given by

$$H^1_{\mathcal{F}_{\mathfrak{p}}, \mathcal{F}_{\overline{\mathfrak{p}}}}\left(K, V_f(k/2) \otimes \chi\right) := \left\{ x \in H^1(K, V_f(k/2) \otimes \chi) \;\middle|\; \begin{array}{ll} \mathrm{loc}_v(x) \in H^1_f(V_f(k/2) \otimes \chi) & \text{if } v \nmid p \\ \mathrm{loc}_v(x) \in \mathcal{F}_v & \text{if } v \mid p \end{array} \right\}.$$

Thanks to Proposition 6.3.4, we know that $\mathbf{c}$ is an anticyclotomic Euler system for T and χ with local condition $\mathcal{L}^*$ such that $\mathbf{c}_K = \mathbf{z}_f^{\chi^{-1}}$. Since $\mathrm{loc}_{\mathfrak{p}}(\mathbf{z}_f^{\chi^{-1}}) \neq 0$, it follows that $\mathbf{z}_f^{\chi^{-1}} \neq 0$, so Theorem 6.3.3 ensures that

$$H^1_{\mathcal{L}_{\mathfrak{p}}, \mathcal{L}_{\overline{\mathfrak{p}}}}\left(K, V_f(k/2) \otimes \chi\right) = \mathrm{Sel}_{\mathcal{L}}\left(K, V_f(k/2) \otimes \chi\right) = F \cdot (\mathbf{z}_f^{\chi^{-1}})^\tau = F \cdot \mathbf{z}_f^\chi.$$

We have that

$$H^1_{\mathcal{L}_{\mathfrak{p}}, 0}\left(K, V_f(k/2) \otimes \chi\right) \subseteq H^1_{\mathcal{L}_{\mathfrak{p}}, \mathcal{L}_{\overline{\mathfrak{p}}}}\left(K, V_f(k/2) \otimes \chi\right) = F \cdot \mathbf{z}_f^\chi$$

Since $\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathbf{z}_f^\chi)^\tau = \mathrm{loc}_{\mathfrak{p}}(\mathbf{z}_f^{\chi^{-1}})$, also $\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathbf{z}_f^\chi)^\tau \neq 0$, hence $\mathbf{z}_f^\chi \notin H^1_{\mathcal{L}_{\mathfrak{p}}, 0}(K, V_f(k/2) \otimes \chi)$. It follows that

$$H^1_{\mathcal{L}_{\mathfrak{p}}, 0}(K, V_f(k/2) \otimes \chi) = 0.$$

There is a Poitou–Tate exact sequence

$$0 \longrightarrow H^1_{0, \emptyset}(K, V_f(k/2) \otimes \chi^{-1}) \longrightarrow H^1_{\mathcal{L}_{\mathfrak{p}}^*, \emptyset}(K, V_f(k/2) \otimes \chi^{-1}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \mathcal{L}_{\mathfrak{p}}^*$$
$$\longrightarrow H^1_{\emptyset, 0}(K, V_f(k/2) \otimes \chi)^\vee \longrightarrow H^1_{\mathcal{L}_{\mathfrak{p}}, 0}(K, V_f(k/2) \otimes \chi)^\vee \longrightarrow 0,$$

where $\emptyset$ indicates that we are imposing no condition. Then $H^1_{\emptyset, 0}\left(K, V_f(k/2) \otimes \chi\right) = 0$. Since, by [CH, (6.2)], there is an equality

$$H^1_f\left(K_v, V_f(k/2) \otimes \chi\right) = \begin{cases} H^1\left(K_v, V_f(k/2) \otimes \chi\right) & \text{if } v = \mathfrak{p} \\ \{0\} & \text{if } v = \overline{\mathfrak{p}}, \end{cases}$$

we conclude that $\mathrm{Sel}\left(K, V_f(k/2) \otimes \chi\right) = H^1_{\emptyset, 0}\left(K, V_f(k/2) \otimes \chi\right) = 0$. $\qquad \square$

Now we construct another anticyclotomic Euler system associated with our generalized Heegner cycles. In the remainder of this subsection, we assume that

- χ has infinity type $(j, -j)$ with $-k/2 < j < k/2$.

6.3. The anticyclotomic Euler system method

Notice that this time we do not need to assume that f is ordinary at p. Recall the cohomology classes $z_{n,\chi^{-1}} \in H^1(K_n, T \otimes \chi^{-1})$ defined in (4.9.2). For $n \in \mathcal{S}$, set

$$c'_n := z_{cn,\chi^{-1}} \in H^1(K_{nc}, T \otimes \chi^{-1}),$$

and define

$$\mathbf{c}' := \{c'_n\}_{n \in \mathcal{S}} = \{z_{cn,\chi^{-1}}\}_{n \in \mathcal{S}}. \tag{6.3.7}$$

Denote by $\mathcal{L}_{BK}$ the direct sum of the Bloch–Kato finite subspaces

$$\mathcal{L}_{BK} := H^1_f\big(K_\mathfrak{p}, V_f(k/2) \otimes \chi^{-1}\big) \oplus H^1_f\big(K_{\bar{\mathfrak{p}}}, V_f(k/2) \otimes \chi^{-1}\big).$$

We would like to prove that this collection of cohomology classes is an anticyclotomic Euler system with local condition $\mathcal{L}_{BK}$.

Proposition 6.3.6. *The collection* $\mathbf{c}' := \{c'_n\}_{n \in \mathcal{S}}$ *is an anticyclotomic Euler system with local condition* $\mathcal{L}_{BK}$. *Moreover,* $\mathbf{c}'_K = z_{\chi^{-1}}$.

Proof. If $\mathbf{c}'$ is an anticyclotomic Euler system, then (5.5.4) implies that

$$\mathbf{c}'_K = \mathrm{Cor}_{K_c/K}(c'_1) = \mathrm{Cor}_{K_c/K}(z_{c,\chi^{-1}}) = z_{\chi^{-1}}.$$

By Propositions 4.10.2, 4.10.4, 4.10.5, we know that

1. $\mathrm{Cor}_{K_{nc}/K_{mc}}(z_{nc,\chi^{-1}}) = a_\ell(f) \cdot z_{mc,\chi^{-1}}$;
2. $\mathrm{loc}_\ell(z_{nc,\chi^{-1}}) = \mathrm{Res}_{K_{mc,\lambda}/K_{nc,\lambda}}(\mathrm{loc}_\ell(z_{mc,\chi^{-1}})^{\mathrm{Frob}_\ell})$;
3. $z^\tau_{nc,\chi^{-1}} = w_f \cdot \psi(\sigma) \cdot z^\sigma_{nc,\chi}$ for some $\sigma \in \mathrm{Gal}(K_{nc}/K)$.

It follows that $\mathbf{c}'$ is an anticyclotomic Euler system.

The last thing we need to show is that $\mathbf{c}'$ has local condition $\mathcal{L}_{BK}$. In analogy to what was remarked at the beginning of §6.1.3, the results in [Nek00] ensure that

$$z_{nc,\chi^{-1}} \in \mathrm{Sel}(K_{nc}, T \otimes \chi^{-1}), \quad z_{\chi^{-1}} \in \mathrm{Sel}(K, T \otimes \chi^{-1}) = \mathrm{Sel}_{\mathcal{L}_{BK}}(K, T \otimes \chi^{-1}).$$

Since τ induces an isomorphism

$$H^1_f(K_\mathfrak{p}, T \otimes \chi^{-1}) \oplus H^1_f(K_{\bar{\mathfrak{p}}}, T \otimes \chi^{-1}) \cong H^1_f(K_\mathfrak{p}, T \otimes \chi) \oplus H^1_f(K_{\bar{\mathfrak{p}}}, T \otimes \chi),$$

we deduce that also $\mathbf{c}'^\tau_K = z^\tau_{\chi^{-1}} \in \mathrm{Sel}_{\mathcal{L}^*_{BK}}(K, T \otimes \chi)$. Furthermore, one has

$$\mathrm{loc}_v\Big(\mathrm{Res}_{K_c/K_{nc}}\big(\mathcal{D}_M(n)\big)\Big) = \mathrm{loc}_v\Big(p^{3M'} \mathrm{red}_M\big(D_n z_{nc,\chi^{-1}}\big)\Big) \in H^1_f(K_{nc,v}, T_M \otimes \chi^{-1})$$

for $v = \mathfrak{p}, \bar{\mathfrak{p}}$. In light of [CH, Lemma 7.5], it follows that

$$\mathrm{loc}_v\big(\mathcal{D}_M(n)\big) \in H^1_f(K_{c,v}, T_M \otimes \chi^{-1}),$$

which implies that $\mathrm{loc}_p\big(P_{M,\chi^{-1}}(n)\big) \in \mathcal{L}_{BK,M}$. $\qquad\square$

6.4 Results on Selmer groups

In this final section, we use the anticyclotomic Euler system method to deduce results on the Selmer group of $V_{f,\chi}$. First of all, we recall notation and assumptions.

As usual, $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ is our newform of weight $k = 2r + 2 \geq 4$ and level N, $\chi : \text{Gal}(K_{c_0 p^\infty}/K) \to \mathcal{O}_F$ is a locally algebraic anticyclotomic character of infinity type $(j, -j)$ and conductor $c_0 p^s \mathcal{O}_K$ with $(c_0, pN) = 1$, $V_{f,\chi} := V_f(k/2)_{|G_K} \otimes \chi$ is the twist of $V_f(k/2)$ by χ, $L(f, \chi, s)$ is the associated Rankin L-series and $\text{Sel}(K, V_{f,\chi})$ is the Block–Kato Selmer group of $V_{f,\chi}$ over K. Assume that:

1. $p \nmid 2N\phi(N^+)$;
2. c_0 is prime to N;
3. either $D_K > 3$ is odd or $8 \mid D_K$;
4. $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K;
5. $N = N^+N^-$ where N^+ is a product of primes that split in K and N^- is a square free product of an *even* number of primes that are inert in K.

As before, the last condition can be expressed by saying that K satisfies a **generalized Heegner hypothesis** relative to N.

Recall now the definition of the Bloch–Kato Selmer group. If v is a place of K such that $v \nmid p$, we consider the inertia group $I_{K_v} \subseteq G_{K_v}$. The unramified subgroup of $H^1(K_v, V_{f,\chi})$ is defined by

$$H^1_{\text{ur}}(K_v, V_{f,\chi}) := \ker\Big(H^1(K_v, V_{f,\chi}) \longrightarrow H^1(I_{K_v}, V_{f,\chi})\Big).$$

Set $H^1_f(K_v, V_{f,\chi}) := H^1_{\text{ur}}(K_v, V_{f,\chi})$. If v is a place of K such that $v \mid p$, then we set

$$H^1_f(K_v, V_{f,\chi}) := \ker\Big(H^1(K_v, V_{f,\chi}) \longrightarrow H^1(K_v, V_{f,\chi} \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{cris}})\Big).$$

The **global Bloch–Kato Selmer group** $\text{Sel}(K, V_{f,\chi})$ of $V_{f,\chi}$ over K is the subspace of $H^1(K, V_{f,\chi})$ given by

$$\text{Sel}(K, V_{f,\chi}) := \left\{ x \in H^1(K, V_{f,\chi}) \ \middle|\ \begin{array}{ll} \text{loc}_v(x) \in H^1_{\text{ur}}(K_v, V_{f,\chi}) & \text{if } v \nmid p \\ \text{loc}_v(x) \in H^1_f(K_v, V_{f,\chi}) & \text{if } v \mid p \end{array} \right\}.$$

Now we can prove our theorems on Selmer groups, the first of which is a vanishing result.

Theorem 6.4.1. *Suppose that f is p-ordinary. If $L(f, \chi, k/2) \neq 0$, then*

$$\dim_F \text{Sel}(K, V_{f,\chi}) = 0.$$

Of course, we can equivalently assert that $\text{Sel}(K, V_{f,\chi}) = 0$.

Proof. Let $\epsilon(V_{f,\chi}) \in \{\pm 1\}$ be the sign of the functional equation for $L(f, \chi, s)$. Then

$$\epsilon(V_{f,\chi}) = -1 \iff -k/2 < j < k/2.$$

Indeed, $\epsilon(V_{f,\chi})$ is a product

$$\epsilon(V_{f,\chi}) = \prod_v \epsilon(1/2, \pi_{K_v} \otimes \chi_v)$$

of local signs, where π_K is the base change to K of the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated with f, π_{K_v} are the local factors of π_K and the local ϵ-factors $\epsilon(1/2, \pi_{K_v} \otimes \chi_v)$ are defined as follows. If F is a finite extension of $\mathbb{Q}_\ell$ and π' is an irreducible representation of $GL_n(F)$, then

$$\epsilon(s, \pi') := \epsilon(s, \pi', \psi_F),$$

where $\psi_F = \psi \circ \mathrm{Tr}_{F/\mathbb{Q}_\ell}$ is the standard additive character. (see, e.g., [Schm] for the definition). It follows that

$$\epsilon(V_{f,\chi}) = \prod_v \epsilon(1/2, \pi_{K_v} \otimes \chi_v, \psi_{K_v}).$$

Each of these local factors is equal either to 1 or to -1. Because N^- is an even product of inert primes, the product of the local factors at the finite places is equal to 1. Thus, the global ϵ-factor depends only on the infinite part. Furthermore, by [Ta, (3.2.5)], one has

$$\epsilon(\frac{1}{2}, \pi_\infty \otimes \chi_\infty, \psi_{K_\infty}) = \epsilon(\frac{1}{2}, \mu^{(\frac{k}{2} - \frac{1}{2} + j)}, \psi_{K_\infty}) \epsilon(\frac{1}{2}, \mu^{(-\frac{k}{2} + \frac{1}{2} + j)}, \psi_{K_\infty}) = i^{|k-1+2j| + |1-k+2j|},$$

where $\mu : z \in \mathbb{C}^\times \mapsto \frac{z}{\bar{z}} \in \mathbb{C}^\times$. Hence, one can check that

$$\epsilon(V_{f,\chi}) = -1 \iff -k/2 < j < k/2,$$
$$\epsilon(V_{f,\chi}) = +1 \iff j \leq -k/2 \text{ or } j \geq k/2.$$

Since $L(f, \chi, k/2) \neq 0$, we know that $\epsilon(V_{f,\chi}) = +1$, therefore either $j \leq -k/2$ or $j \geq k/2$. As before, let τ be complex conjugation; set $\chi^\tau(g) := \chi(\tau g \tau)$. There is an equality of L-functions $L(f, \chi, k/2) = L(f, \chi^\tau, k/2)$ and the action of τ induces an isomorphism $\mathrm{Sel}(K, V_{f,\chi}) \cong \mathrm{Sel}(K, V_{f,\chi^\tau})$. This shows that we can assume $j \geq k/2$.

Finally, since $L(f, \chi, k/2) \neq 0$, by Theorem 6.2.2 we know that $\mathrm{loc}_\mathfrak{p}(z_f^{\chi^{-1}}) \neq 0$, and then Proposition 6.3.5 gives $\mathrm{Sel}(K, V_{f,\chi}) = 0$. $\qquad \square$

Our second theorem on Selmer groups gives a one-dimensonality result.

Theorem 6.4.2. *If $\epsilon(V_{f,\chi}) = -1$ and $z_\chi \neq 0$, then*

$$\mathrm{Sel}(K, V_{f,\chi}) = F z_\chi.$$

Proof. Since $z_\chi \neq 0$, also $z_\chi^{-1} \neq 0$. Because $\epsilon(V_{f,\chi}) = -1$, we know that $-k/2 < j < k/2$. Then, by Proposition 6.3.6, the collection $\mathbf{c}'$ is an anticyclotomic Euler system for T and χ^{-1} with local condition $\mathcal{L}_{BK}$. Since $\mathbf{c}'_K = z_{\chi^{-1}}$, applying Theorem 6.3.3, we obtain that $\mathrm{Sel}(K, V_{f,\chi}) = \mathrm{Sel}_{\mathcal{L}^*_{BK}}(K, V_{f,\chi}) = F z_\chi$. $\qquad \square$

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