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Calculus of Variations

Euler's optimal profile problem

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Abstract

We study an old variational problem formulated by Euler as Proposition 53 of his *Scientia Navalis* by means of the direct method of the calculus of variations. Precisely, through relaxation arguments, we prove the existence of minimizers. We fully investigate the analytical structure of the minimizers in dependence of the geometric parameters and we identify the ranges of uniqueness and non-uniqueness.

Mathematics Subject Classification 49Q10 · 49K30

1 Introduction

L. Euler in his treatise *Scientia Navalis* (1749), which is considered to be one of the cornerstones of the eighteenth century naval architecture, at Proposition 53, formulated the following optimal profile problem (see [14,21]).

Among all curves AM which with the axis AP and perpendicular PM comprehend the same area, to find that one which with its symmetric branch on the opposite side of the axis AP will form the figure offering the least resistance in water when it moves in the direction PA along the axis (Fig. 1).

The problem can be viewed as a variant of the celebrated Newton's aerodynamic problem (Proposition 34 of Book 2 of the Principia, 1687, [27]) which relies in optimizing the shape of a solid of revolution, moving in a fluid along its axis, experiencing the least resistance, at parity of length and caliber. Actually, at Proposition 65 of the same treatise, Euler studies

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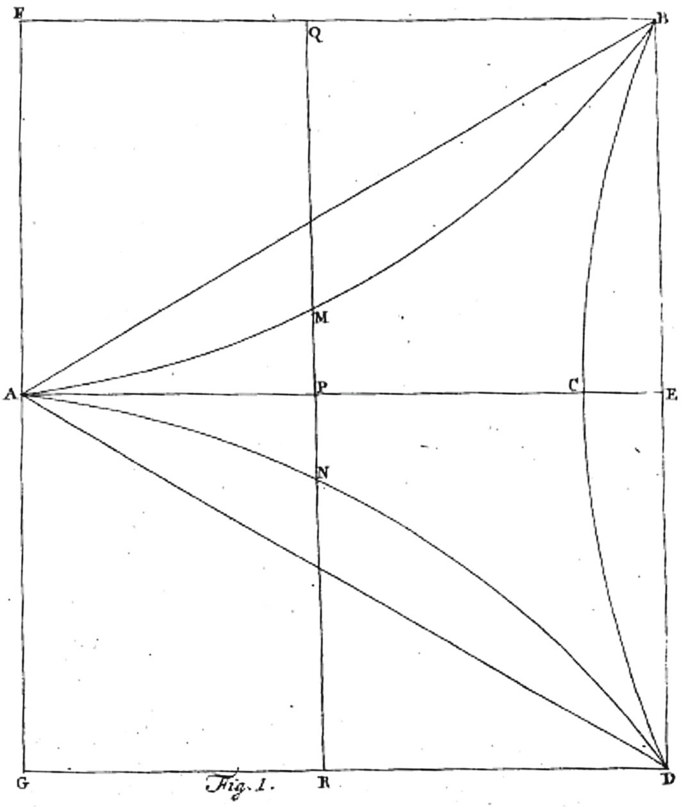

Fig. 1 L. Euler, *Scientia Navalis*, 1749

in different terms a very similar problem. Newton's problem of minimal resistance was the first *solved* problem in the calculus of variations (by Newton himself a decade before the *brachistochrone problem*, see [16]) and assumes a fluid like medium made by particles of equal mass moving at a constant velocity with a fixed direction, while the dynamic interaction between solid and fluid is only due to the perfectly elastic collisions between the fluid particles and the surface of the solid body. Though Newton's constitutive assumptions ruling the fluid-solid interaction seems too crude to copy the complex physical phenomena occurring at the interface (strongly influenced by the properties of the fluid and the dynamic features of the motion, [26]), certainly they capture the essential basic ingredients of the problem. Let us recall that the *drag problem* is one of the oldest problems in fluid mechanics and at present it still seems to be out of reach of analytical results, for realistic Reynolds numbers. On the other hand, from a mathematical perspective, the variational integral representing the resistance functional is neither coercive nor convex, hence a natural route to prove existence of a minimum via the direct method relies in imposing additional constraints on the admissible shapes. These arguments explain the reasons the oldest problem of the calculus of variations still provides continuous inspirations for new and challenging problems: we refer, for instance, to [2–5,7–12,15,17–20,22–25,28–30].

Unlike Newton's problem, the Euler optimal profile problem, as far as the authors know, has never been studied in the framework of modern calculus of variations, with the only exception of the paper [3] which deals with a constrained Newton's problem in a special class of admissible functions.

In analytical terms the problem admits the following formulation. Given $a > 0$, $h > 0$, $L \in (0, ah)$, find a curve $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma = (\gamma_1, \gamma_2)$, such that $\gamma(0) = (0, 0)$, $\gamma(1) = (a, h)$, and such that (with the notation $z_+ := z \vee 0$)

$$\mathcal{F}(\gamma) = \int_0^1 \frac{(\gamma_2')_+^3}{(\gamma_1')^2 + (\gamma_2')^2}\, dt \to \min, \tag{1.1}$$

subject to the area constraint

$$\int_0^1 \gamma_1(t)\gamma_2'(t)\, dt = ah - L. \tag{1.2}$$

In fact, problem (1.1)–(1.2) is a constrained Newton-like problem, since L represents the area of the region between the curve γ and the lines $y = 0$ and $x = a$, taking $\{0; x, y\}$ as a coordinate system in $\mathbb{R}^2$. L. Euler, after the problem statement (*Propositio 53, Scientia Navalis*, pg. 238) deduces the stationary conditions in terms of differential equations and G.H. Light (in [21]) proves that the extremal curves are precisely branches of hypocycloids of three cusps. In this paper we provide an exhaustive solution of the problem (1.1), (1.2), by exploiting the direct methods of the calculus of variations. It turns out that, in the generality of Euler's formulation, the problem doesn't admit a solution (see Example 2.2). Indeed, we prove the existence of global minimizers (Theorem 2.1) under the natural assumption $\gamma_1' \geq 0$. Then, we study their precise analytical structure in dependence of the given geometric parameters a, h, L. In most cases, the optimal profile is the union of the graph of a convex or concave function (which is exactly Euler's solution) and of a vertical segment (Theorem 2.3). Moreover, non-uniqueness of minimizers is shown to occur for certain ranges of the geometric parameters (Theorem 2.4).

These results, obtained through relaxation techniques, seem to capture the essential ideas of naval architecture: indeed, it is easy to recognize that a lot of boat profiles are quite similar to the solutions of the Euler's problem (see Fig. 1), suggesting that the global shapes realize a compromise between the dynamical performance and the total mass. On the other hand we guess that the non-uniqueness of solutions appearing for certain ranges of the parameters, suggests the possible occurrence of solutions exhibiting fine scale structures. Indeed, as it is well known [13] the skin of fast-swimming sharks is characterized (at the mesoscale) by the presence of riblet structures which are known to reduce skin friction drag in the turbulent-flow regime. In this respect, it would be quite natural to ask if a suitable modification of the Euler resistance could select a class of minimizers exhibiting at certain scales the riblet geometries which are responsible of the impressive drag reduction characterizing the shark's skin, contributing in the comprehension of this surprising natural morphology.

2 Statement of the problem and main results

2.1 Existence and uniqueness

Let $a > 0$, $h > 0$ and $L \in (0, ah)$. We shall introduce a suitable function space for the minimization of the resistance functional. Starting from the original formulation of the problem, a natural choice is the class of rectifiable simple curves connecting $(0,0)$ with (a,h). Admissible curves should be contained in $[0,a]\times[0,h]$ and should split such rectangle in two subsets with prescribed areas L and $ah-L$. A rectifiable simple curve is an equivalence class: the equivalence relation $\sim$ is given by orientation-preserving parametrizations, so that $\tilde\gamma \sim \gamma$ if a monotone nondecreasing mapping ϕ from $[0,1]$ onto itself exists such that $\tilde\gamma = \gamma\circ\phi$. We shall identify each rectifiable simple curve γ with an absolutely continuous parametrization (still denoted by γ) such that $|\gamma'(t)|\neq 0$ a.e in $(0,1)$. Therefore, we set

$$\mathcal{A}^0_{a,h,L} := \Big\{\gamma\in AC([0,1];[0,a]\times[0,h]) : \gamma(0)=(0,0),\ \gamma(1)=(a,h),$$
$$\gamma \text{ simple},\ |\gamma'(t)|\neq 0 \text{ for a.e. } t\in(0,1),\ \int_0^1 \gamma_1(t)\gamma_2'(t)\,dt = ah-L\Big\}.$$

We also consider the class

$$\mathcal{A}_{a,h,L} := \{\gamma\in\mathcal{A}^0_{a,h,L} : \gamma_1'(t)\geq 0 \text{ for a.e. } t\in(0,1)\}$$

and the minimization problem for functional $\mathcal{F}$ from (1.1), that is,

$$\min\left\{\mathcal{F}(\gamma):\ \gamma\in\mathcal{A}_{a,h,L}\right\}. \tag{2.1}$$

The following is our first main result.

Theorem 2.1 *Let $a>0$, $h>0$, $L\in(0,ah)$. The following properties hold.*

(i) *If $2L\notin(a^2,2ah-a^2)$ (in particular if $h\leq a$), then there exists a unique solution to problem (2.1).*
(ii) *If $2L\in(a^2,2ah-a^2)$, then there exist infinitely many solutions to problem (2.1).*

The choice of the subclass $\mathcal{A}_{a,h,L}$ is motivated by the fact that, without further constraints, the problem $\min\{\mathcal{F}(\gamma):\gamma\in\mathcal{A}^0_{a,h,L}\}$ admits no solution, as shown through the following

Example 2.2 Let $u:\mathbb{R}\to\mathbb{R}$ be a 1-periodic function defined as

$$u(t) := \begin{cases} t & \text{if } 0\leq t\leq \frac12 \\ 1-t & \text{if } \frac12\leq t\leq 1 \end{cases}$$

and, for every $n\in\mathbb{N}$, let $u_n(t)=u(nt)$, $t\in[0,1]$. Let us define $v_n\in AC[0,1]$ such that $v_n(0)=0$ and $v_n'(t)=\frac1n(u_n'(t))_+$ for a.e. $t\in(0,1)$. Then we set

$$x_n(t) = \begin{cases} u_n(t) & \text{if } 0\leq t\leq 1-\frac1n \\ \frac{n(t-1)+1}{2} & \text{if } 1-\frac1n < t\leq 1, \end{cases} \qquad y_n(t) = \begin{cases} v_n(t) & \text{if } 0\leq t\leq 1-\frac1n \\ \frac{t}{2} & \text{if } 1-\frac1n < t\leq 1 \end{cases} \tag{2.2}$$

and we define $\gamma^n(t)=(x_n(t),y_n(t))$, $t\in[0,1]$. See Fig. 2. We have $\gamma^n(0)=(0,0)$, $\gamma^n(1)=(\frac12,\frac12)$ and $|(\gamma^n)'(t)|\neq 0$ for a.e. $t\in(0,1)$. A direct computation shows that for every $n\in\mathbb{N}$ the area between the curve γ^n and the lines $y=0$ and $x=\frac12$ is

$$\int_0^1 x_n(t)y_n'(t)\,dt = \sum_{j=0}^{n-2}\int_{j/n}^{(2j+1)/(2n)} (nt-j)\,dt + \int_{(n-1)/n}^1 \frac{n(t-1)+1}{4}\,dt = \frac18.$$

Thus, for any $n\in\mathbb{N}$ we have $\gamma^n\in\mathcal{A}^0_{a,h,L}$ with $a=h=\frac12$ and $L=ah-L=\frac18$. Moreover, another direct computation shows that $\mathcal{F}(\gamma^n)\to 0$ as $n\to\infty$. Since $\mathcal{F}(\gamma)>0$ for every $\gamma\in\mathcal{A}^0_{a,h,L}$, it follows that no minimizer exists.

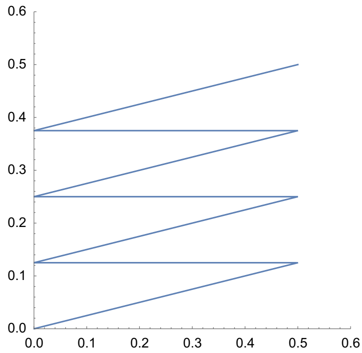

Fig. 2 The curve γ^n, for $n = 4$, in Example 2.2

It is not difficult to modify the above example in order to see that, for any other value of a, h, L, there holds $\inf\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}^0_{a,h,L}\} = 0$. Strong changing-sign oscillations of γ_1' are indeed energetically favorable.

2.2 Representation of solutions

In the uniqueness range of Theorem 2.1, the form of the solution can be obtained through an explicit parametrization. Towards this end, we need some more notation. Here and in the following let

$$g(z) := \frac{z_+^3}{1+z^2}, \quad z \in \mathbb{R}. \tag{2.3}$$

Let $\Psi : [0, 1]^2 \to \mathbb{R}$ and $\Phi : [0, 1]^2 \to \mathbb{R}$ be defined by

$$\Psi(\xi, \eta) := h + a \int_\xi^\eta \frac{1-t^2}{(1+t^2)^2} \frac{g'(t) - g'(\xi)}{g'(\eta) - g'(\xi)}\, dt - \frac{a\eta}{1+\eta^2}, \tag{2.4}$$

$$\Phi(\xi, \eta) := \frac{a^2\xi}{2} + \frac{a^2}{2} \int_\xi^\eta \left(\frac{g'(\eta) - g'(t)}{g'(\eta) - g'(\xi)}\right)^2 dt, \tag{2.5}$$

where the integral terms are understood to vanish in case $\xi = \eta$. Moreover, let

$$\mathcal{T} := \{(\xi, \eta) : 0 \le \xi \le \eta \le 1,\ \Phi(\xi, \eta) = L\}. \tag{2.6}$$

Then we have

Theorem 2.3 *Let $a > 0$, $h > 0$. Suppose that $0 < 2L \le (ah) \wedge a^2$. If $2L = (ah) \wedge a^2$, then the unique solution of problem (2.1) is given by the piecewise affine curve connecting the points $(0, 0)$, $(a, a \wedge h)$ and (a, h). Else if $2L < (ah) \wedge a^2$, then there exists a unique minimizer (ξ_*, η_*) of Ψ on $\mathcal{T}$, there holds $\xi_* < \eta_*$, and the unique solution to problem (2.1) is*

$$\gamma_*(t) = \begin{cases} (x_*(2t + \xi_*), y_*(2t + \xi_*)) & \text{if } t \in \left[0, \frac{\eta_* - \xi_*}{2}\right] \\ \left(a, h + \frac{2(h - h_*)}{2 - \eta_* + \xi_*}(t - 1)\right) & \text{if } t \in \left[\frac{\eta_* - \xi_*}{2}, 1\right] \end{cases} \tag{2.7}$$

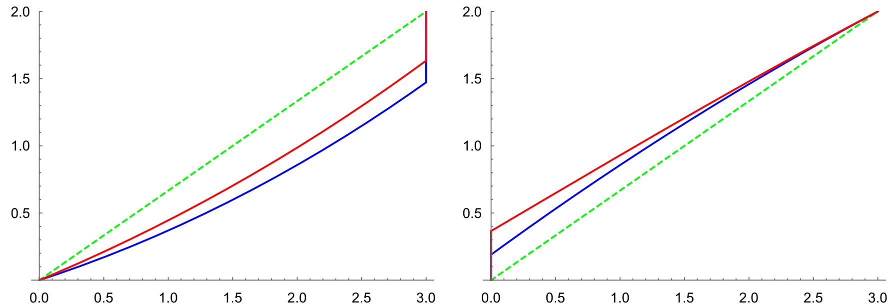

Fig. 3 Numerical simulation of hypocycloidal solutions for $a = 3$, $h = 2$ and different values of L. Left: $L = 2$ (blue), $L = 2.3$ (red), $L = 3$ (green). Right: $L = 3$ (green), $L = 3.4$ (blue), $L = 3.7$ (red) (color figure online)

where

$$x_*(\tau) := \frac{a(g'(\tau) - g'(\xi_*))}{g'(\eta_*) - g'(\xi_*)}, \quad y_*(\tau) := \int_{\xi_*}^{\tau} s x_*'(s)\, ds, \quad \tau \in [\xi_*, \eta_*] \tag{2.8}$$

and $h_* := y_*(\eta_*) < h$.

It has been argued in [21] that, whenever $t \in [0, \frac{\eta_* - \xi_*}{2}]$, the parametrization given in (2.7)–(2.8) is that of a branch of an hypocycloyd with three vertices and it is worth noticing that its trace is the graph of a convex function. In particular, if $2L < (ah) \wedge a^2$, the optimal profile is the union of the graph of such convex function and of a vertical segment of length $h - h_* > 0$.

We also notice that Theorem 2.3 covers only half of the uniqueness range of the parameters. The other half is $2L \geq (ah) \vee (2ah - a^2)$. However, the parameters fall in the latter range if L satisfying the assumptions of Theorem 2.3 is changed to $ah - L$. In particular, if $2L > (ah) \vee (2ah - a^2)$, then the corresponding optimal profile becomes the graph of a concave function joined to a vertical segment of strictly positive length. Indeed, given the solution γ_* in $\mathcal{A}_{a,h,L}$ from (2.7)–(2.8) and letting $t_* = \frac{\eta_* - \xi_*}{2}$, we will prove later on that the solution in $\mathcal{A}_{a,h,ah-L}$ is just obtained by reflection and precisely it is given by

$$\tilde{\gamma}_*(t) = \begin{cases} \gamma_*(t + t_*) - (a, h_*) & \text{if } t \in [0, 1 - t_*] \\ (a, h) - \gamma_*(1 - t) & \text{if } t \in [1 - t_*, 1]\,. \end{cases} \tag{2.9}$$

We refer to Fig. 3 for a plot of the solutions obtained with a numerical simulation.

Let us now discuss the non-uniqueness range of Theorem 2.1. We have the following

Theorem 2.4 *Let* $h > a > 0$ *and* $2L \in (a^2, 2ah - a^2)$. *Then* $\gamma \in \mathcal{A}_{a,h,L}$ *is solution to problem* (2.1) *if and only if* $\gamma_2'(t) \geq 0$ *for a.e.* $t \in (0, 1)$ *and* $\gamma_1'(t) = \gamma_2'(t)$ *for a.e.* t *in* $\{\gamma_1'(t) > 0\}$.

The piecewise affine curve γ° *connecting the points* $(0, 0)$, $(0, p)$, $(a, p + a)$ *and* (a, h), *where* $p := \frac{L}{a} - \frac{a}{2}$, *is a solution to problem* (2.1). *Moreover,* γ° *is the unique solution to problem* (2.1) *among all curves* γ *that further satisfy* $\{\gamma_1'(t) > 0\} = (t_1, t_2)$ *(up to a* $\mathcal{L}^1$*-negligible set) for some* $0 < t_1 < t_2 < 1$.

More piecewise affine solutions to problem (2.1) can be constructed as follows. Let $k \in \mathbb{N}$, $k \geq 5$. Let (x_j, y_j) be points in $\{(x, y) \in [0, a] \times [0, h] : x \leq y \leq h - a + x\}$, such that $0 = x_0 \leq x_1 \leq \ldots \leq x_k = a$, $0 = y_0 < y_1 < \ldots < y_k = h$, and such that for any $j = 1, \ldots k$ there holds either $x_j = x_{j-1}$ or $x_j - x_{j-1} = y_j - y_{j-1}$. We denote by $J_2(k)$

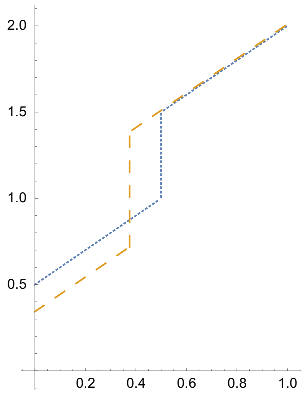

Fig. 4 Two solutions with $a = 1$, $h = 2$, $L = 1.25$

the set of indices in $\{1, \dots k\}$ such that $x_j = x_{j-1}$ and by $J_1(k)$ its complement in $\{1, \dots k\}$. Let $\hat{\gamma}(t) = (x_{j-1}, y_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}}(x_j - x_{j-1}, y_j - y_{j-1})$ for $t \in [t_{j-1}, t_j]$, $j = 1, \dots, k$. Then the energy of $\hat{\gamma}$ can be computed as

$$\mathcal{F}(\hat{\gamma}) = \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(\gamma_2')_+^3}{(\gamma_1')^2 + (\gamma_2')^2}\, dt = \sum_{j \in J_1(k)} \frac{y_j - y_{j-1}}{2} + \sum_{j \in J_2(k)} (y_j - y_{j-1}) = h - \frac{a}{2}, \tag{2.10}$$

where we have exploited the fact that $\sum_{j \in J_1(k)}(y_j - y_{j-1}) = a$ and $\sum_{j \in J_2}(y_j - y_{j-1}) = h - a$. Hence, we see that any piecewise affine curve made by vertical segments and slope 1 segments has the same energy of γ°: it is therefore solution to problem (2.1) as soon as the area constraint $\sum_{j \in J_1(k)}(y_j + y_{j-1})(x_j - x_{j-1}) = 2L$ is matched. See also Fig. 4.

Understanding L as a material design constraint, it is natural to look for its optimal value, in case there is some freedom in its choice. Letting $\mathcal{F}_{min}(a, h, L)$ be the minimal value corresponding to the solution of problem (2.1), we have the following result (see also Fig. 5).

Theorem 2.5 *The mapping* $(0, ah) \ni L \mapsto \mathcal{F}_{min}(a, h, L)$ *is continuous and symmetric around* $L = ah/2$. *If* $h \le a$, *then*

it is strictly decreasing on $(0, ah/2]$, *strictly increasing on* $[ah/2, ah)$, *and its range is* $[\frac{h^3}{a^2+h^2}, h)$. *Else if* $h > a$, *then it is strictly decreasing on* $(0, a^2/2]$, *constant on* $[a^2/2, ah - a^2/2]$, *strictly increasing on* $[ah - a^2/2, ah)$, *and its range is* $[h - a/2, h)$.

Let us conclude by remarking that the maximization problem is easier. Indeed, we have $\sup\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L}\} = +\infty$. For instance, if $a = h = \frac{1}{2}$ and $L = \frac{1}{8}$, this can be seen by taking the sequence of curves $\bar{\gamma}^n(t) := (y_n(t), x_n(t))$, $t \in [0, 1]$, where x_n and y_n are defined in (2.2). Again, the same behavior is clearly possible for any $a > 0$, $h > 0$, $L \in (0, ah)$. On the other hand, if we maximize $\mathcal{F}$ over $\mathcal{A}_{a,h,L}$ with the further constraint $\gamma_2'(t) \ge 0$ for a.e. $t \in (0, 1)$, we may consider the estimate

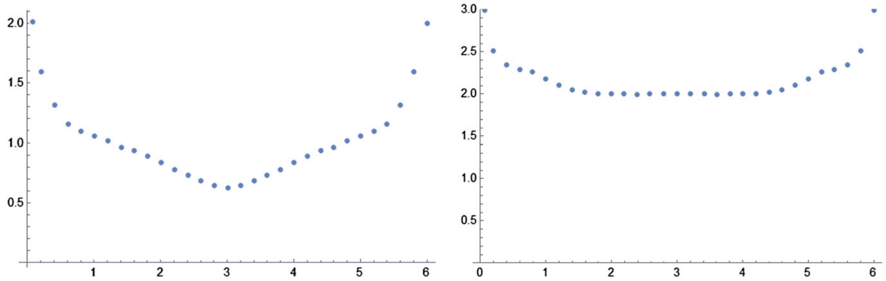

Fig. 5 Numerical simulation of optimal energy values as a function of L, with the choice of parameters $a = 3$, $h = 2$ (left) and $a = 2$, $h = 3$ (right)

$$\mathcal{F}(\gamma) = \int_0^1 \gamma_2'(t) - \frac{\gamma_2'(t)\, \gamma_1'(t)^2}{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, dt \le \int_0^1 \gamma_2'(t)\, dt = h$$

where equality holds if and only if $\gamma_1'(t) \wedge \gamma_2'(t) = 0$ for a.e. $t \in (0, 1)$. Hence, for any $a > 0$, $h > 0$ and $L \in (0, ah)$, the problem

$$\max\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L},\ \gamma_2'(t) \ge 0 \text{ for a.e. } t \in (0, 1)\}$$

has infinitely many solutions. Any piecewise affine curve made by alternating horizontal ad vertical segments is indeed a solution as soon as the area constraint is matched, as it realizes the maximal value h. Such construction is analogous to the one of piecewise affine minimizers in the nonuniqueness regime from Theorem 2.4. However, these piecewise affine maximizers are found for any value of $a > 0$, $h > 0$ and $L \in (0, ah)$.

Plan of the paper

Section 3 provides some basic properties of functional $\mathcal{F}$. In Sect. 4 we introduce the relaxed functional and we analyze the associated minimization problem. Section 5 delivers the proof of the main results.

Notation

Through the rest of the paper, without further explicit mention, it is always understood that the parameters are in the range $a > 0$, $h > 0$ and $L \in (0, ah)$.

3 Some properties of functional $\mathcal{F}$

Let us start with a very simple estimate.

Lemma 3.1 *There holds*

$$\inf\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L}\} < h.$$

Proof Let us suppose that $2L \ge ah$ (the other case is analogous). It is enough to test the functional on the following curve made by two segments

$$\gamma^r(t) = \begin{cases} (0, 2tr) & \text{if } t \in [0, 1/2] \\ (0, r) + (2t - 1)(a, h - r) & \text{if } t \in [1/2, 1], \end{cases}$$

where $r \in [0, h]$ is a parameter. Note that $\gamma^r \in \mathcal{A}_{a,h,L}$ if and only if $ar = 2L - ah$. A direct computation shows that

$$\mathcal{F}(\gamma^r) = r + \frac{(h-r)^3}{a^2 + (h-r)^2}.$$

The function $[0, h] \ni r \mapsto \mathcal{F}(\gamma^r)$ is strictly decreasing on $[0, r_*]$ and strictly increasing on $[r_*, h]$, where $r_* := (h-a)_+$, as easily checked. Moreover, $\mathcal{F}(\gamma^0) = \frac{h^3}{a^2+h^2} < h = \mathcal{F}(\gamma^h)$. In particular, such function is uniquely maximized for $r = h$ with value h. The result is proved. □

Remark 3.2 Let $\gamma \in \mathcal{A}_{a,h,L}$. We note that if $\gamma_1(t_1) = \gamma_1(t_2)$ and $\gamma_2(t_2) - \gamma_2(t_1) = h$ for some $0 \le t_1 < t_2 \le 1$, then $\mathcal{F}(\gamma) \ge h$. This happens in particular if $(0, h) \in \gamma([0, 1])$ or $(a, 0) \in \gamma([0, 1])$. Indeed, it is enough to compute the contribution to the functional coming from the interval $[t_1, t_2]$ where γ is a vertical segment, which is exactly h.

We will often make use of approximations by means of piecewise affine curves. Here, we provide the approximation construction.

Lemma 3.3 *For any $\epsilon > 0$ and any $\gamma \in \mathcal{A}_{a,h,L}$, there exists $\bar{\gamma} \in \mathcal{A}_{a,h,L}$ such that*

(i) *$\bar{\gamma}$ is piecewise affine*
(ii) *$\bar{\gamma}_1'(t) > 0$ for a.e. $t \in (0, 1)$*
(iii) *$\bar{\gamma}_2'(t) \ge 0$ for a.e. $t \in (0, 1)$ if the same holds for γ.*
(iv) *$|\mathcal{F}(\bar{\gamma}) - \mathcal{F}(\gamma)| < \epsilon$*
(v) *$\sup_{t\in[0,1]} |\bar{\gamma}(t) - \gamma(t)| < \epsilon$.*

In particular, there holds

$$\begin{aligned}&\inf\left\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L}\right\}\\ &\quad = \inf\left\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L},\ \gamma_1'(t) > 0 \textit{ for a.e. } t \in (0, 1),\ \gamma \textit{ piecewise affine}\right\}.\end{aligned}$$

Proof *Step 1* We approximate any $\gamma \in \mathcal{A}_{a,h,L}$ with a piecewise affine $\breve{\gamma}$ with nodes on the curve γ, such that $\breve{\gamma}(0) = (0, 0)$ and $\breve{\gamma}(1) = (a, h)$. This entails strong $W^{1,1}(0, 1)$ (hence uniform) approximation of both γ_1 and γ_2. In particular, for any $\delta > 0$, $\breve{\gamma}$ can be chosen such that

$$\left|\int_0^1 \breve{\gamma}_1(t)\,\breve{\gamma}_2'(t)\,dt - (ah - L)\right| = \left|\int_0^1 \breve{\gamma}_1'(t)\,\breve{\gamma}_2(t)\,dt - L\right| < \delta/2, \tag{3.1}$$

and

$$\sup_{t\in[0,1]} |\breve{\gamma}(t) - \gamma(t)| < \delta/2, \qquad |\mathcal{F}(\gamma) - \mathcal{F}(\breve{\gamma})| \le C \int_0^1 |\gamma'(t) - \breve{\gamma}'(t)|\,dt < \delta/2, \tag{3.2}$$

where $C = 3\sqrt{3}/4$ is the Lipschitz constant of the map $\mathbb{R}^2 \ni (x, y) \mapsto \frac{x^3}{x^2+y^2}$.

Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be the partition of $[0, 1]$ such that $\gamma(t_i)$, $i = 1, \ldots, n-1$ are the nodes of $\breve{\gamma}$. We mention that since $ah > L > 0$, if the partition is fine enough there are always grid points t_i, $i = 1, \ldots, n-1$, such that $0 < \gamma_2(t_i) < h$. Let $I \subset \{1, \ldots, n\}$ denote the subset of indices such that $\breve{\gamma}_1'(t) = 0$ on (t_{i-1}, t_i) if $i \in I$ and $\breve{\gamma}_1'(t) \ne 0$ on (t_{i-1}, t_i) otherwise. We assume wlog that I does not contain two consecutive integers. We introduce the piecewise affine curve $\hat\gamma$, such that $\hat\gamma(0) = (0,0)$ and $\hat\gamma(1) = (a,h)$, whose nodes are found at the points

$$\begin{aligned}
&\gamma(t_i) \quad \text{for } i \in \{1,\dots,n-2\}\setminus I \text{ (and also for } i = n-1 \text{ if } n \notin I),\\
&\gamma(t_i) + (2^{-n-2}(C\vee h)^{-1}\delta, 0) \quad \text{for } i \in \{1,\dots,n-1\}\cap I,\\
&\gamma(t_{n-1}) - (2^{-n-2}(C\vee h)^{-1}\delta, 0) \quad \text{if } n \in I.
\end{aligned}$$

For small enough δ the trace of $\hat\gamma$ is still contained in $[0,a]\times[0,h]$ and there holds $\hat\gamma_1'(t) > 0$ for a.e. $t \in (0,1)$. Clearly, if $\gamma_2'(t) \ge 0$ for a.e. $t \in (0,1)$, then $\hat\gamma$ and $\check\gamma$ enjoy this same property. It is readily seen that $\sup_{t\in[0,1]} |\check\gamma(t) - \hat\gamma(t)| \le \delta/2$, and by computing the sums of trapezoidal areas we get

$$\begin{aligned}
&\left|\int_0^1 \hat\gamma_1'(t)\,\hat\gamma_2(t)\,dt - \int_0^1 \check\gamma_1'(t)\,\check\gamma_2(t)\,dt\right|\\
&\quad = \frac{1}{2}\left|\sum_{i=1}^n (\check\gamma_2(t_{i-1}) + \check\gamma_2(t_i))(\check\gamma_1(t_i) - \check\gamma_1(t_{i-1}) - \hat\gamma_1(t_i) + \hat\gamma_1(t_{i-1}))\right|\\
&\quad \le h\sum_{i=1}^n |\check\gamma_1(t_i) - \check\gamma_1(t_{i-1}) - \hat\gamma_1(t_i) + \hat\gamma_1(t_{i-1})| \le \delta/2.
\end{aligned}$$

Moreover,

$$|\mathcal{F}(\hat\gamma) - \mathcal{F}(\check\gamma)| \le C\int_0^1 |\hat\gamma'(t) - \check\gamma'(t)|\,dt \le C\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\delta 2^{-n}}{t_i - t_{i-1}}\,dt \le \delta/2.$$

By combining the latter estimates with (3.1) and (3.2), we find

$$\sup_{t\in[0,1]} |\hat\gamma(t) - \gamma(t)| < \delta, \quad \left|\int_0^1 \hat\gamma_1'(t)\,\hat\gamma_2(t)\,dt - L\right| < \delta, \quad |\mathcal{F}(\hat\gamma) - \mathcal{F}(\gamma)| < \delta. \tag{3.3}$$

Therefore, by taking δ small enough we see that $\hat\gamma$ satisfies properties (i)–(v). Still, it does not necessarily belong to $\mathcal{A}_{a,h,L}$.

Step 2 In view of the previous step, we need to modify $\hat\gamma$ in order to match the area constraint. A parametrization for $\hat\gamma$ is

$$\hat\gamma(t) = \hat\gamma(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}}\,(\hat\gamma(t_i) - \hat\gamma(t_{i-1})) \quad \text{if } t \in [t_{i-1}, t_i], \qquad i = 1,\dots,n. \tag{3.4}$$

Let $\sigma \in [-1,1]$. We define a new piecewise affine curve depending on σ. Let $\gamma_\sigma(t_0) = (0,0)$, $\gamma_\sigma(t_n) = (a,h)$, and let $\gamma_\sigma(t_i) = (\hat\gamma_1(t_i), (1-|\sigma|)\hat\gamma_2(t_i) + \sigma_+ h)$, $i = 1,\dots n-1$.

Accordingly, let

$$\gamma_\sigma(t) = \gamma_\sigma(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}}\,(\gamma_\sigma(t_i) - \gamma_\sigma(t_{i-1})) \quad \text{if } t \in [t_{i-1}, t_i], \qquad i = 1,\dots,n. \tag{3.5}$$

The area in $[0,a]\times[0,h]$ that lies below the curve γ_σ is once more easily computed as sum of trapezoidal areas and there holds

$$\mathcal{I}(\sigma) := \int_0^1 (\gamma_\sigma)_1'(\gamma_\sigma)_2 = \sigma_+ ah + (1-|\sigma|)\int_0^1 \hat\gamma_1'\hat\gamma_2. \tag{3.6}$$

Since $\int_0^1 \hat\gamma_1'\hat\gamma_2 < ah$, we see from (3.6) that the map $[-1,1] \ni \sigma \mapsto \mathcal{I}(\sigma)$ is continuous strictly increasing. Moreover, it is readily seen using the second estimate in (3.3) and (3.6) that $\mathcal{I}(\frac{2\delta}{ah-L+\delta}) > L+\delta$ and that $\mathcal{I}(-\frac{2\delta}{L+\delta}) < L-\delta$. We conclude that there exists a unique value $\sigma_\delta \in (-\frac{2\delta}{L+\delta}, \frac{2\delta}{ah-L+\delta})$ such that $\mathcal{I}(\sigma_\delta) = L$, so that $\gamma_{\sigma_\delta} \in \mathcal{A}_{a,h,L}$.

It is clear that $\sup_{t\in[0,1]} |\hat\gamma(t) - \gamma_{\sigma_\delta}(t)| < |\sigma_\delta| h$. Eventually, by taking derivatives in (3.4) and (3.5) we get

$$
\begin{aligned}
|\mathcal{F}(\gamma_{\sigma_\delta}) - \mathcal{F}(\hat\gamma)| &\le C\int_0^1 |\gamma_{\sigma_\delta}'(t) - \hat\gamma'(t)|\,dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{|\sigma_\delta||\hat\gamma_2(t_i) - \hat\gamma_2(t_{i-1})|}{t_i - t_{i-1}}\,dt \\
&\le |\sigma_\delta| \sum_{i=1}^n |\gamma_2(t_i) - \gamma_2(t_{i-1})| \le |\sigma_\delta| \int_0^1 |\gamma'(t)|\,dt
\end{aligned}
$$

By taking (3.2) and the latter estimates into account, we get

$$
\sup_{t\in[0,1]} |\gamma_{\sigma_\delta}(t) - \gamma(t)| < \delta + |\sigma_\delta| h, \qquad |\mathcal{F}(\gamma_{\sigma_\delta}) - \mathcal{F}(\gamma)| < \delta + |\sigma_\delta| \int_0^1 |\gamma'(t)|\,dt.
$$

Since σ_δ vanishes as $\delta \downarrow 0$, if we define, for δ small enough, $\bar\gamma := \gamma_{\sigma_\delta}$ we obtain $\bar\gamma \in \mathcal{A}_{a,h,L}$ and (i)–(v) hold. □

4 Relaxation

In this section we gather some results about minimization of auxiliary functionals defined on BV functions of one variable, rather than parametric curves of the plane. We start by introducing some more notation.

Let g as in (2.3) and let

$$
g^{**}(z) := \begin{cases} g(z) & \text{if } z < 1 \\ z - \frac{1}{2} & \text{if } z \ge 1 \end{cases} \tag{4.1}
$$

be the convex envelope of g, i.e., the largest convex function that is smaller than or equal to g. In the following for every $u \in BV_{loc}(\mathbb{R})$, u' will denote the distributional derivative and $\dot u$, u_s' its absolutely continuous and singular part respectively.

Let

$$
\begin{aligned}
\mathcal{B}_{a,h,L} &:= \left\{ u \in W^{1,1}_{loc}(\mathbb{R}) : u(x) \equiv 0 \text{ if } x < 0,\ u(x) \equiv h \text{ if } x > a,\ 0 \le u \le h,\ \int_0^a u = L \right\}, \\
\mathcal{B}^+_{a,h,L} &:= \left\{ u \in \mathcal{B}_{a,h,L} : u' \ge 0 \right\}, \\
\mathcal{C}^+_{a,h,L} &:= \left\{ u \in BV_{loc}(\mathbb{R}) : u(x) \equiv 0 \text{ if } x < 0,\ u(x) \equiv h \text{ if } x > a,\ u' \ge 0,\ \int_0^a u = L \right\}.
\end{aligned}
$$

We further define the functionals

$$
\mathcal{G}(u) := \begin{cases} \displaystyle\int_0^a g(\dot u(x))\,dx & \text{if } u \in \mathcal{B}_{a,h,L} \\ +\infty & \text{otherwise in } BV_{loc}(\mathbb{R}), \end{cases}
$$

$$
\mathcal{J}(u) := \begin{cases} \displaystyle\int_0^a g^{**}(\dot u(x))\,dx & \text{if } u \in \mathcal{B}_{a,h,L} \\ +\infty & \text{otherwise in } BV_{loc}(\mathbb{R}), \end{cases}
$$

and the functionals

$$
\begin{aligned}
\mathcal{J}_+(u) &:= \begin{cases} \displaystyle\int_0^a g^{**}(\dot{u}(x))\,dx & \text{if } u \in \mathcal{B}^+_{a,h,L} \\ +\infty & \text{otherwise in } BV_{loc}(\mathbb{R}), \end{cases} \\
\overline{\mathcal{J}}_+(u) &:= \begin{cases} \displaystyle\int_0^a g^{**}(\dot{u}(x))\,dx + u'_s([0,a]) & \text{if } u \in \mathcal{C}^+_{a,h,L} \\ +\infty & \text{otherwise in } BV_{loc}(\mathbb{R}). \end{cases}
\end{aligned} \tag{4.2}
$$

We shall often use the shorthands $\inf \mathcal{G}$, $\inf \mathcal{J}$, $\inf \mathcal{J}_+$ $\inf \overline{\mathcal{J}}_+$ for the infimum over $BV_{loc}(\mathbb{R})$. We also write $\inf \mathcal{F}$ in place of $\inf\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L}\}$, which is the infimum of problem (2.1).

The first statement of this section is a suitable version of Lemma 3.3 for the new functionals.

Lemma 4.1 *Let $\epsilon > 0$. Let $u \in \mathcal{B}_{a,h,L}$. There exist a piecewise affine function $\bar{u} \in \mathcal{B}_{a,h,L}$ such that $|\mathcal{G}(u) - \mathcal{G}(\bar{u})| + |\mathcal{J}(u) - \mathcal{J}(\bar{u})| < \epsilon$. Moreover, $\bar{u} \in \mathcal{B}^+_{a,h,L}$ if $u \in \mathcal{B}^+_{a,h,L}$. In particular, there hold*

$$
\begin{aligned}
\inf \mathcal{G} &= \inf\{\mathcal{G}(u) : u \in \mathcal{B}_{a,h,L},\ u \text{ is piecewise affine}\}, \\
\inf \mathcal{J} &= \inf\{\mathcal{J}(u) : u \in \mathcal{B}_{a,h,L},\ u \text{ is piecewise affine}\}, \\
\inf \mathcal{J}_+ &= \inf\{\mathcal{J}_+(u) : u \in \mathcal{B}^+_{a,h,L},\ u \text{ is piecewise affine}\}.
\end{aligned}
$$

Proof By considering that both g from (2.3) and g^{**} from (4.1) are Lipschitz on $\mathbb{R}$, the proof follows the same line of that of Lemma 3.3. It is in fact an application of the same construction to the case of curves in $\mathcal{A}_{a,h,L}$ that are graphs of functions in $\mathcal{B}_{a,h,L}$, therefore we omit the details. □

The following result shows that it is convenient to consider nondecreasing functions.

Lemma 4.2 *There holds*

$$
\inf \mathcal{J} = \inf \mathcal{J}_+ = \inf\{\mathcal{J}(u) : u \in \mathcal{B}^+_{a,h,L}\}.
$$

Proof Thanks to Lemma 4.1, it is enough to show that for any piecewise linear function $u \in \mathcal{B}_{a,h,L}$, there exists a piecewise linear nondecreasing function $w \in \mathcal{B}_{a,h,L}$ such that $\mathcal{J}(w) \leq \mathcal{J}(u)$. This will be achieved is some steps.
Step 1 For $n \in \mathbb{N}$ we shall consider sequences of N points $(x_i, y_i)_{\{i=1,\ldots N\}} \in S$, where

$$
S := \{(x_i, y_i)_{\{i=1,\ldots N\}} : (x_1, \ldots x_N) \in [0,a]^N,\ (y_1, \ldots, y_N) \in [0,h]^N,\ 0 < x_1 < \ldots < x_N < a\}
$$

is a connected subset of the rectangle $[0,a]^N \times [0,h]^N$. To a sequence of points $(x_i, y_i)_{\{i=1,\ldots N\}} \in S$ we may associate a continuous piecewise linear function $u = u_{\{x_1,y_1,\ldots x_N,y_N\}}$, joining the endpoints $(0,0)$ and (a,h), with vertices located at the points (x_i, y_i), and such that $0 \leq u \leq h$. As a convention, we do not include the endpoints $(0,0)$ and (a,h) in the list of vertices, and we do not exclude that three or more consecutive points lie on the same line segment.

We notice that the energy of $u = u_{\{x_1,y_1,\ldots x_N,y_N\}}$ is

$$
\mathcal{J}(u) = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} g^{**}(u(t))\,dt = \sum_{i=0}^{N} (x_{i+1} - x_i)\, g^{**}\!\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right).
$$

In particular, $\mathcal{J}$ is continuous on S, as g^{**} is continuous on $\mathbb{R}$. We also notice that the area below the graph of $u = u_{\{x_1, y_1, \ldots x_N, y_N\}}$ is given by

$$\int_0^a u(t)\,dt = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\,(t - x_i) + y_i \right) dt$$
$$= \sum_{i=0}^{N} \left(\frac{1}{2}(y_{i+1} - y_i)(x_{i+1} - x_i) + (x_{i+1} - x_i) y_i \right)$$

and it is also a continuous function on S.

Let us moreover introduce a connected subset of S by

$$S' := \{(x_i, y_i)_{\{i=1,\ldots N\}} \in S : 0 \le y_1 \le y_2 \le \ldots \le y_N \le h\}, \tag{4.3}$$

so that the corresponding function $u_{\{x_1, y_1, \ldots x_N, y_N\}}$ is a monotone nondecreasing piecewise constant functions with N vertices.

Step 2 Now, let us fix $(\bar{x}_i, \bar{y}_i)_{\{i=1,\ldots N\}} \in S$ and the corresponding function $u = u_{\{\bar{x}_1, \bar{y}_1, \ldots \bar{x}_N, \bar{y}_N\}}$.

Let

$$\emptyset \ne V_1 := \operatorname{argmin}\{u(x) : x \in \{\bar{x}_1, \ldots \bar{x}_N\}\} \qquad \text{and} \qquad v_1 = \max V_1.$$

Then we recursively define

$$V_j = \operatorname{argmin}\{u(x) : x \in \{\bar{x}_1, \ldots \bar{x}_N\},\ x > v_{j-1}\} \qquad \text{and} \quad v_j = \max V_j,$$

for any $j \in \{2, \ldots N\}$ such that $\bar{x}_N > v_{j-1}$. Let $J := \max\{j \in \{1, \ldots N\} : \bar{x}_N > v_{j-1}\}$, so we necessarily have $v_J = \bar{x}_N$. Notice that by construction

$$0 < v_1 < \ldots < v_J = \bar{x}_N, \quad 0 \le u(v_1) < \ldots < u(v_J) \le h, \quad \{v_1, \ldots v_J\} \subseteq \{\bar{x}_1, \ldots \bar{x}_N\}. \tag{4.4}$$

In particular, the continuous piecewise linear function u_- having vertices exactly at the points $\{v_1, \ldots v_J\}$ (and endpoints at $(0, 0)$, (a, h)) is nondecreasing on $[0, a]$.

With the convention $v_0 = 0$ and $v_{J+1} = a$, on each interval $[v_j, v_{j+1}]$, $j \in (0, J)$, let us consider the line segment

$$\mathfrak{s}_j(x) = \frac{u(v_{j+1}) - u(v_j)}{v_{j+1} - v_j}\,(x - v_j) + u(v_j)$$

connecting $(v_j, u(v_j))$ and $(v_{j+1}, u(v_{j+1}))$. We claim that

$$u(x) \ge \mathfrak{s}_j(x) \quad \text{on } [v_j, v_{j+1}].$$

This is obvious if $u \equiv 0$ or $u \equiv h$ in $[v_j, v_{j+1}]$, and in fact it holds with equality on $[v_J, v_{J+1}]$ since $v_J = \bar{x}_N$. Otherwise, from (4.4) $\mathfrak{s}_j$ has positive slope and if by contradiction there is a point $p \in (v_j, v_{j+1})$ such that $u(p) < \mathfrak{s}_j(p)$, then since u is piecewise linear and joins $(v_j, u(v_j))$ with $(v_{j+1}, u(v_{j+1}))$, then u needs to have at least one vertex p' on the interval (v_j, v_{j+1}), such that

$$u(p') < \mathfrak{s}_j(p') < \mathfrak{s}_j(v_{j+1}) = u(v_{j+1}).$$

This is a contradiction, since by definition of V_{j+1} and v_{j+1} the value of u at v_{j+1} is minimal among all the vertex points v of u such that $v > v_j$. The claim is proved and since j is arbitrary we have $u_-(x) \le u(x)$ on $[0, a]$.

For the sake of consistency, if $J < N$ we complete te sequence $(v_i, u(v_i))_{\{i=1,\ldots J\}}$ by adding $N - J$ vertices on a uniform partition of the line segment connecting $(v_J, u(v_J))$ to (a, h), so that we obtain a sequence of points $(v_i, u(v_i))_{\{i=1,\ldots N\}} \in S$, and the associated piecewise linear function is still u_-.

All in all, we have constructed a sequence of N vertices $(v_i, u(v_i))_{\{i=1,\ldots N\}} \in S$, and the associated piecewise linear function u_- is nondecreasing with $u_-(0) = 0, u_-(a) = h$, it satisfies $0 \leq u_- \leq h$, and moreover its vertices are on the graph of u.

Eventually, with an analogous construction we provide another continuous piecewise constant function $0 \leq u_+ \leq h$, with $u^+(0) = 0, u^+(a) = h$, having a sequence of vertices in S which lie on the graph of u, such that u_+ is nondecreasing and $u_+(x) \geq u(x)$ for any $x \in [0, a]$. In particular, the set of vertices of u_+ and u_- belong to S' from (4.3).

Step 3 Given $(\bar{x}_i, \bar{y}_i)_{\{i=1,\ldots N\}} \in S$ and the associate piecewise linear function $u = u_{\{\bar{x}_1,\bar{y}_1,\ldots\bar{x}_N,\bar{y}_N\}}$ from the previous step, we consider the set

$$S'' := \{(x_i, y_i)_{\{i=1,\ldots N\}} \in S : (x_i, y_i) \in \text{graph}(u), i = 1, \ldots N\}.$$

We claim that S'' is a conncected subset of S. Indeed, let $(x_i, y_i)_{i=1,\ldots N} \in S$ and $(\tilde{x}_i, \tilde{y}_i)_{i=1,\ldots N} \in S$. Then for each $t \in [0, 1]$, we let

$$x_i(t) := (1-t)x_i + t\tilde{x}_i, \qquad y_i(t) := u(x_i(t)), \qquad i = 1, \ldots N$$

so that $[0, 1] \ni t \mapsto (x_i(t), y_i(t))_{\{i=1,\ldots N\}} \in [0, a]^N \times [0, h]^N$ is a continuous mapping and by its very definition we have $(x_i(t), y_i(t))_{\{i=1,\ldots N\}} \in S''$ for any $t \in [0, 1]$. This proves the claim.

Step 4 We consider again a generic piecewise linear mapping $u = u_{\{\bar{x}_1,\bar{y}_1,\ldots\bar{x}_N,\bar{y}_N\}} \in \mathcal{B}_{a,h,L}$, with vertices at $(\bar{x}_i, \bar{y}_i)_{\{i=1,\ldots N\}} \in S$. We consider the two piecewise linear nondecreasing mappings u_+, u_-, defined in Step 2. By the construction of u^+ and u_-, the respective sets of N vertices belong to $S' \cap S''$. Moreover, we recall that the area below the graph is continuous on S, as seen in Step 1. On the other hand, still from Step 2 we have $u_- \leq u \leq u_+$ therefore $\int_0^a u_- \leq \int_0^a u = L \leq \int_0^a u_+$. Since the set of vertices of u_+ and u_- belong to $S' \cap S''$, which is a connected subset of S by Step 3, and since the area is continuous on S, we deduce that there exists a set of vertices in $S' \cap S''$ which realizes the value L of the area. We let w the corresponding piecewise linear function, which therefore belongs to $\mathcal{B}_{a,h,L}$. If $0 \leq p < q \leq h$ correspond to any two consecutive vertices of w (or a vertex and an endpoint), since these points lie on the graph of u we have

$$\int_p^q u'(t)\,dt = \int_p^q w'(t)\,dt.$$

Since g_* is convex on $\mathbb{R}$, by the above equality we may invoke Jensen inequality and get

$$\int_p^q g^{**}(w(t))\,dt \leq \int_p^q g^{**}(u(t))\,dt.$$

We conclude that $\mathcal{J}(w) \leq \mathcal{J}(u)$, where w is a nondecreasing piecewise linear function in $\mathcal{B}_{a,h,L}$. □

The following is not a Γ-convergence result since in the limsup inequality the sequence u_j is not required to be converging to u. In any case, this will be sufficient for our later purposes.

Lemma 4.3 *The following two properties hold true:*

a) *for every* $u \in BV_{loc}(\mathbb{R})$ *and every sequence* $(u_j) \subset BV_{loc}(\mathbb{R})$ *such that* $u_j \to u$ *in* $w^* - BV_{loc}(\mathbb{R})$*, there holds*

$$\liminf_{j\to\infty} \mathcal{J}_+(u_j) \geq \overline{\mathcal{J}}_+(u); \tag{4.5}$$

b) *for every* $u \in BV_{loc}(\mathbb{R})$ *there exists a sequence* $(u_j) \subset BV_{loc}(\mathbb{R})$ *such that*

$$\limsup_{j\to\infty} \mathcal{J}_+(u_j) \leq \overline{\mathcal{J}}_+(u). \tag{4.6}$$

Proof We first prove a). Let $u_j \to u$ in $w^* - BV_{loc}(\mathbb{R})$ and assume without restriction that $\mathcal{J}_+(u_j)$ is a bounded sequence. Then $u \in \mathcal{C}^+_{a,h,L}$ and (4.5) follow from [6, Theorem 3.4.1, Corollary 3.4.2], see also [1].

In order to prove b) it will be enough to assume that $u \in \mathcal{C}^+_{a,h,L}$. If this is the case by recalling that u'_s has compact support we choose $c_1, c_2 \geq 0$ such that

$$c_1 + c_2 = u'_s([0,a]) \qquad \text{and} \qquad (2c_1 + 3c_2)\, a = 6\int_{-\infty}^{+\infty} x u'_s \tag{4.7}$$

and we introduce the function $\widetilde{u} \in BV_{loc}(\mathbb{R})$ defined by:

$$\begin{aligned} &\widetilde{u}(x) = u(x) \ \text{ if } \ x < 0, \\ &\widetilde{u}' = \dot{u}\, dx + c_1\delta_{a/3} + c_2\delta_{a/2} \qquad \text{in } \mathbb{R}. \end{aligned} \tag{4.8}$$

It is readily seen that $\widetilde{u}(0^+) = 0$ and by using (4.7), (4.8) we get

$$\widetilde{u}(a^-) = \int_0^a \widetilde{u}' = \int_0^a \dot{u}\, dx + c_1 + c_2 = \int_0^a \dot{u}\, dx + u'_s([0,a]) = u(a^+) = h,$$

and by taking into account (4.8) we get $\widetilde{u}(x) = h$ for every $x > a$. On the other hand, again by (4.8) and the relation $\int_0^a \widetilde{u} + x\widetilde{u}' = \int_0^a (x\widetilde{u})' = a\widetilde{u}(a^-)$, we get

$$\begin{aligned} \int_0^a \widetilde{u}\, dx &= a\widetilde{u}(a^-) - \int_0^a x\dot{\widetilde{u}}\, dx - \int_0^a x\widetilde{u}'_s = ah - \int_0^a x\dot{u}\, dx - \frac{ac_1}{3} - \frac{ac_2}{2} \\ &= ah - \int_0^a x\dot{u}\, dx - \int_{-\infty}^{+\infty} x u'_s = ah - \int_0^a x\dot{u}\, dx - \int_0^a x u'_s + a(h - u(a^-)) \\ &= au(a^-) - \int_0^a x\dot{u}\, dx - \int_0^a x u'_s = \int_0^a u\, dx = L. \end{aligned}$$

Since $\widetilde{u}'_s([0,a]) = u'_s([0,a])$ we get $\overline{\mathcal{J}}_+(u) = \overline{\mathcal{J}}_+(\widetilde{u})$ and it will be enough to find a sequence $(\widetilde{u}_j) \subset BV_{loc}(\mathbb{R})$ such that $\limsup_{j\to\infty} \mathcal{J}_+(\widetilde{u}_j) \leq \overline{\mathcal{J}}_+(\widetilde{u})$ to achieve the result.

Let us consider the nondecreasing $W^{1,1}(0, a/3)$ function w_1 satisfying $w_1(0) = 0$ and $w_1(a/3) = \widetilde{u}(a/3^-)$, that is obtained by restricting $\widetilde{u}$ to $(0, a/3)$. Similarly, by taking the restriction of $\widetilde{u}$ to $(a/3, a/2)$ (resp. to $(a/2, a)$), we obtain a nondecreasing function $w_2 \in W^{1,1}(a/3, a/2)$ with $w_2(a/3) = \widetilde{u}(a/3^+)$ and $w_2(a/2) = \widetilde{u}(a/2^-)$ (resp. a nondecreasig function $w_3 \in W^{1,1}(a/2, a)$ with $w_3(a/2) = \widetilde{u}(a/2^+)$ and $w_3(a) = h$). We let $a_0 := 0$, $a_1 := a/3$, $a_2 := a/2$, $a_3 := a$. Thanks to Lemma 4.1, for $i = 1, 2, 3$ we approximate w_i with nondecreasing piecewise affine functions $(w_{i,j})_{j\in\mathbb{N}}$ with same values at a_{i-1} and a_i and such that

$$\int_{a_{i-1}}^{a_i} w_{i,j}\, dx = \int_{a_{i-1}}^{a_i} \widetilde{u} \qquad j = 1, 2, \ldots$$

and

$$\lim_{j\to+\infty}\int_{a_{i-1}}^{a_i} g^{**}(\dot{w}_{i,j})\,dx = \int_{a_{i-1}}^{a_i} g^{**}(\dot{\widetilde{u}})\,dx.$$

Therefore, by defining $v_j := w_{i,j}$ on (a_{i-1}, a_i), $i = 1, 2, 3$ (extended to $\mathbb{R}$ with value 0 for $x < 0$ and with value h for $x > a$), we get $v_j \in \mathcal{C}^+_{a,h,L}$ and for any $j \in \mathbb{N}$ the function v_j is piecewise affine nondecreasing, it is continuous outside at most two jump points at $a/3$ and $a/2$, and

$$v_j(0) = \widetilde{u}(0) = 0,\ v_j(a/3^\pm) = \widetilde{u}(a/3^\pm),\ v_j(a/2^\pm) = \widetilde{u}(a/2^\pm),\ v_j(a) = \widetilde{u}(a) = h. \tag{4.9}$$

Moreover, there holds

$$\lim_{j\to+\infty}\int_0^a g^{**}(\dot{v}_j)\,dx = \int_0^a g^{**}(\dot{\widetilde{u}})\,dx \tag{4.10}$$

If $c_1 = c_2 = 0$, then $v_j \in \mathcal{B}^+_{a,h,L}$ and we let $\widetilde{u}_j = v_j$, thus the proof is concluded since (4.6) holds true. In general, as v_j may have jump points at $a/3$, $a/2$, we approximate it with a continuous piecewise affine function in $\mathcal{B}^+_{a,h,L}$ as follows.

We choose a decreasing vanishing sequence $(\lambda_j) \subset \mathbb{R}$ such that $\dot{v}_j$ is constant on $(a/3 - \lambda_j, a/3)$, $(a/3, a/3+\lambda_j)$, $(a/2-\lambda_j, a/2)$, $(a/2, a/2+\lambda_j)$ and we define for every $t \in [0, 1]$

$$\widetilde{v}_{j,t}(x) := \begin{cases} v^*_{j,t}(x) & \text{if } (t-1)\lambda_j < x - \frac{a}{3} < t\lambda_j \\ v^{**}_{j,t} & \text{if } (t-1)\lambda_j < x - \frac{a}{2} < t\lambda_j \\ v_j(x) & \text{otherwise in } \mathbb{R}, \end{cases}$$

where

$$v^*_{j,t}(x) := \lambda_j^{-1}\,(v_j(\tfrac{a}{3} + \lambda_j t) - v_j(\tfrac{a}{3} - (1-t)\lambda_j))\,(x - \tfrac{a}{3} + (1-t)\lambda_j) + v_j(\tfrac{a}{3} - (1-t)\lambda_j),$$

$$v^{**}_{j,t}(x) := \lambda_j^{-1}\,(v_j(\tfrac{a}{2} + \lambda_j t) - v_j(\tfrac{a}{2} - (1-t)\lambda_j))\,(x - \tfrac{a}{2} + (1-t)\lambda_j) + v_j(\tfrac{a}{2} - (1-t)\lambda_j).$$

It is readily seen that $\widetilde{v}_{j,t} \in W^{1,1}_{loc}(\mathbb{R})$, $\widetilde{v}_{j,t}(0) = 0$, $\widetilde{v}_{j,t}(a) = h$, that $\Phi_j(t) := \int_0^a \widetilde{v}_{j,t}$ is continuous on the whole $[0, 1]$ and that $\Phi_j(1) \le L \le \Phi_j(0)$. Hence, there exists $t_j \in [0, 1]$ such that $\int_0^a \widetilde{v}_{j,t_j} = L$, so that $\widetilde{u}_j := \widetilde{v}_{j,t_j} \in \mathcal{B}^+_{a,h,L}$ and

$$\mathcal{J}_+(\widetilde{u}_j) = \int_{(0,a)\setminus I_j} g^{**}(\dot{\widetilde{u}}_j)\,dx + \int_{I_j} g^{**}(\dot{\widetilde{u}}_j)\,dx,$$

where we have set $I_j := (\frac{a}{3} + (t-1)\lambda_j, \frac{a}{3} + t\lambda_j) \cup (\frac{a}{2} + (t-1)\lambda_j, \frac{a}{2} + t\lambda_j)$. By taking into account (4.1), (4.9), (4.10) and the fact that $\lim_{j\to+\infty} |I_j| = 0$ we get

$$\lim_{j\to+\infty}\int_{(0,a)\setminus I_j} g^{**}(\dot{\widetilde{u}}_j)\,dx = \int_0^a g^{**}(\dot{\widetilde{u}}(x))\,dx, \qquad \lim_{j\to+\infty}\int_{I_j} g^{**}(\dot{\widetilde{u}}_j)\,dx = \widetilde{u}'_s([0,a]),$$

thus $\lim_{j\to+\infty} \mathcal{J}_+(\widetilde{u}_j) = \overline{\mathcal{J}}_+(\widetilde{u})$ and b) follows. □

We next give an alternative representation for functional $\overline{\mathcal{J}}_+$ from (4.2) and show that it admits a minimizer.

Lemma 4.4 *For every $u \in \mathcal{C}^+_{a,h,L}$ we have*

$$\overline{\mathcal{J}}_+(u) = h + \int_0^a (g^{**}(\dot{u}(x)) - \dot{u}(x))\,dx.$$

Proof Since

$$u'_s([0,a]) = u'_s((0,a)) + u(0^+) + h - u(a^-)$$

and

$$u(a^-) - u(0^+) = \int_0^a \dot{u}\,dx + u'_s((0,a))$$

the result follows. □

Lemma 4.5 *The functional $\overline{\mathcal{J}}_+$ admits a minimizer over $\mathcal{C}^+_{a,h,L}$ and*

$$\inf \mathcal{J}_+ = \min\{\overline{\mathcal{J}}_+(u) : u \in \mathcal{C}^+_{a,h,L}\}.$$

Proof Let $(u_j) \subset \mathcal{B}^+_{a,h,L}$ be a sequence such that $\mathcal{J}_+(u_j) = \inf \mathcal{J}_+ + o(1)$ as $j \to +\infty$. Since $\dot{u}_j \geq 0$, $u_j(x) \equiv 0$ if $x \leq 0$, $u_j(x) \equiv h$ if $x \geq a$ then u_j are equibounded in $BV_{loc}(\mathbb{R})$ hence there exists $u \in \mathcal{C}^+_{a,h,L}$ such that , up to subsequences, $u_j \to u$ in $w^* - BV_{loc}(\mathbb{R})$. By a) of Lemma 4.3 we get

$$\inf \mathcal{J}_+ = \liminf_{j\to\infty} \mathcal{J}_+(u_j) \geq \overline{\mathcal{J}}_+(u)$$

and by b) of Lemma 4.3 for any other $\widetilde{u} \in \mathcal{C}^+_{a,h,L}$ there exists $\widetilde{u}_j \in \mathcal{B}^+_{a,h,L}$ such that

$$\limsup_{j\to\infty} \mathcal{J}_+(\widetilde{u}_j) \leq \overline{\mathcal{J}}_+(\widetilde{u}).$$

Therefore,

$$\overline{\mathcal{J}}_+(u) + o(1) \leq \inf \mathcal{J}_+ + o(1) \leq \mathcal{J}_+(\widetilde{u}_j) + o(1)$$

and by taking the limit the result is proved. □

We need now some fine properties of minimizers of $\overline{\mathcal{J}}_+$. To this aim we introduce for $\epsilon > 0$ the penalized functionals

$$\mathcal{J}_\epsilon(u) := \overline{\mathcal{J}}_+(u) + \int_0^a \epsilon \dot{u}^2\,dx + \frac{1}{\epsilon}\int_0^a (u_-^2 + (u-h)_+^2 + \dot{u}_-^2)\,dx$$
$$+\frac{1}{\epsilon}\left(\int_0^a (0 \vee u \wedge h)\,dx - L\right)^2,$$

defined for $u \in \mathcal{H}$ and extended with value $+\infty$ if $u \in BV_{loc}(\mathbb{R})\backslash\mathcal{H}$, where

$$\mathcal{H} := \{u \in W^{1,2}_{loc}(\mathbb{R}) : u(x) \equiv 0 \text{ if } x < 0,\ u(x) \equiv h \text{ if } x > a\}.$$

Minimizing sequences for $\mathcal{J}_\varepsilon$ are equibounded in $W^{1,2}_{loc}(\mathbb{R})$, therefore (up to subsequences) converging weakly in $W^{1,2}_{loc}(\mathbb{R})$ and strongly in $L^2_{loc}(\mathbb{R})$. By taking into account the convexity and nonnegativity of $x \mapsto x_-^2$ and $x \mapsto g^{**}(x)$, it is readily seen that the limit points minimize $\mathcal{J}_\varepsilon$ over $\mathcal{H}$. We next show that Lemma 4.3 holds also for $\mathcal{J}_\varepsilon$.

Lemma 4.6 *Let $\epsilon_j \to 0$ be a decreasing sequence, then*
a) *for every $u \in BV_{loc}(\mathbb{R})$ and every sequence $(u_j) \subset BV_{loc}(\mathbb{R})$ such that $u_j \to u$ in $w^* - BV_{loc}(\mathbb{R})$, there holds*

$$\liminf_{j\to\infty} \mathcal{J}_{\epsilon_j}(u_j) \geq \overline{\mathcal{J}}_+(u);$$

b) *for every* $u \in BV_{loc}(\mathbb{R})$ *there exists a sequence* $(u_j) \subset BV_{loc}(\mathbb{R})$ *such that*

$$\limsup_{j\to\infty} \mathcal{J}_{\epsilon_j}(u_j) \le \overline{\mathcal{J}}_+(u).$$

Proof a) is straightforward by sequential lower semicontinuity of $\overline{\mathcal{J}}_+(u)$ an b) is obvious if $u \notin \mathcal{C}^+_{a,h,L}$. If $u \in \mathcal{C}^+_{a,h,L}$, we choose $c_1, c_2 \ge 0$ such that (4.7) holds. We define $\widetilde{u} \in BV_{loc}(\mathbb{R})$ as in (4.8): as seen in the proof of Lemma 4.3, there holds $\widetilde{u}(0+) = 0$, $\widetilde{u}'_s([0,a]) = u'_s([0,a])$, hence $\overline{\mathcal{J}}_+(u) = \overline{\mathcal{J}}_+(\widetilde{u})$ and it is now enough to approximate $\overline{\mathcal{J}}_+(\widetilde{u})$. We let $\delta_j \to 0^+$ such that $\epsilon_j\delta_j^{-1} \to 0$ and we define

$$\widetilde{u}_j(x) = \begin{cases} x\delta_j^{-1}\widetilde{u}(\delta_j) & \text{if } 0 \le x \le \delta_j \\ \delta_j^{-1}(\widetilde{u}(\frac{a}{3}^+) - \widetilde{u}(\frac{a}{3} - \delta_j))\,(x - \frac{a}{3} + \delta_j) + \widetilde{u}(\frac{a}{3} - \delta_j) & \text{if } -\delta_j \le x - \frac{a}{3} \le 0 \\ \delta_j^{-1}(\widetilde{u}(\frac{a}{2}^+) - \widetilde{u}(\frac{a}{2} - \delta_j))\,(x - \frac{a}{2} + \delta_j) + \widetilde{u}(\frac{a}{2} - \delta_j) & \text{if } -\delta_j \le x - \frac{a}{2} \le 0 \\ \delta_j^{-1}(h - \widetilde{u}(a - \delta_j))\,(x - a) + h & \text{if } a - \delta_j \le x \le a \\ \widetilde{u}(x) & \text{otherwise in } \mathbb{R}. \end{cases}$$

Then $\widetilde{u}_j \in \mathcal{H}$ and $\limsup_{j\to\infty} \mathcal{J}_{\epsilon_j}(\widetilde{u}_j) \le \overline{\mathcal{J}}_+(\widetilde{u})$ follows by arguing as in Lemma 4.3. □

The next lemma introduces the Euler–Lagrange equation for functional $\overline{\mathcal{J}}_+$, which will be a key step for the proof of Theorem 2.3.

Lemma 4.7 *Let* $j \in \mathbb{N}$. *Let* $\epsilon_j \to 0$ *be a decreasing sequence and let* $u_j \in \operatorname{argmin}_{\mathcal{H}} \mathcal{J}_{\epsilon_j}$. *Then:*

(i) $\dot{u}_j$ *is continuous and monotone in* $(0, a)$*;*
(ii) $\dot{u}_j \ge 0$ *a.e. in* $(0, a)$ *and* $\int_0^a \dot{u}_j = h$ *for any* $j \in \mathbb{N}$*;*
(iii) *there exists a (not relabeled) subsequence* (u_j) *such that* $u_j \to u_*$ *in* $w^* - BV_{loc}(\mathbb{R})$ *as* $j \to +\infty$ *and* u_* *minimizes* $\overline{\mathcal{J}}_+$ *over* $\mathcal{C}^+_{a,h,L}$*;*
(iv) *either* $\dot{u}_* \ge 1$ *a.e. in* $(0, a)$ *or* $u_* \in W^{1,\infty}(0, a)$ *with* $0 \le \dot{u}_* \le 1$ *a.e. in* $(0, a)$ *and in the latter case we have for suitable* $\overline{\lambda}$, $\overline{\mu} \in \mathbb{R}$

$$g'(\dot{u}_*) = \overline{\lambda}x + \overline{\mu} \quad \text{a.e. in } (0, a).$$

Proof Let $u_j \in \operatorname{argmin} \mathcal{J}_{\epsilon_j}$. Then $0 \le u_j \le h$ in $\mathbb{R}$ (indeed, if this was not the case, $0 \vee u_j \wedge h$ would provide a lower value for $\mathcal{J}_{\varepsilon_j}$). Since $0 \le u_j \le h$, by the Du-Bois-Raymond equation, there exist a real constant μ_j such that

$$h_j(\dot{u}_j) := -2\epsilon_j^{-1}\dot{u}_j^- + 2\epsilon_j\dot{u}_j + (g^{**})'(\dot{u}_j) = \lambda_j x + \mu_j, \qquad x \in (0, a) \tag{4.11}$$

where $\lambda_j = 2\epsilon_j^{-1}((\int_0^a u_j) - L)$. Since h_j is a continuous strictly increasing function, from (4.11) we have $\dot{u}_j = h_j^{-1}(\lambda_j x + \mu_j)$ and we see that $\dot{u}_j$ is continuous and monotone on the whole $(0, a)$ thus proving (i).

If $|\{\dot{u}_j < 0\}| > 0$ then there exists an interval $[\alpha_j, \beta_j] \subset [0, a]$ such that $\dot{u}_j < 0$ in (α_j, β_j). Since $0 > \int_{\alpha_j}^{\beta_j} \dot{u}_j = u_j(\beta_j) - u_j(\alpha_j)$ and since $u_j(0) = 0 \le u(x) \le h = u_j(a)$ in $\mathbb{R}$, we can exclude both $\alpha_j = 0$ and $\beta_j = h$. Therefore $0 < \alpha_j < \beta_j < h$ and $\dot{u}_j^+(\alpha_j) = \dot{u}_j^-(\beta_j) = 0$, hence

$$2\epsilon_j^{-1}\dot{u}_j + 2\epsilon_j\dot{u}_j = \lambda_j x + \mu_j$$

in (α_j, β_j) which implies $\lambda_j\alpha_j + \mu_j = \lambda_j\beta_j + \mu_j = 0$, that is $\lambda_j = \mu_j = 0$ so $\dot{u}_j \equiv 0$ in (α_j, β_j), a contradiction. Since $\dot{u}_j \geq 0$ a.e. in $(0, a)$ we get

$$0 \leq \int_0^a |\dot{u}_j| = \int_0^a \dot{u}_j = h$$

and (ii) is proven.

By (ii) we get, up to subsequences, that $u_j \to u_*$ in $w^* - BV_{loc}(\mathbb{R})$ and by point a) of Lemma 4.6

$$\liminf_{j\to\infty} \mathcal{J}_{\epsilon_j}(u_j) \geq \overline{\mathcal{J}}_+(u_*).$$

If now $u \in \mathcal{C}^+_{a,h,L}$, we construct $\widetilde{u}_j$ from u as done in the proof of Lemma 4.6, which then entails along with the minimality of u_j

$$\limsup_{j\to\infty} \mathcal{J}_{\epsilon_j}(u_j) \leq \limsup_{j\to\infty} \mathcal{J}_{\epsilon_j}(\widetilde{u}_j) \leq \overline{\mathcal{J}}_+(u).$$

Hence, $\overline{\mathcal{J}}_+(u_*) \leq \overline{\mathcal{J}}_+(u)$ and (iii) is proven.

We eventually prove (iv). Since $(g^{**})'(\dot{u}_j) \leq 1$ by (4.11) and (ii) we get

$$0 \leq \lambda_j x + \mu_j \leq 1 + 2\epsilon_j \dot{u}_j,$$

hence by integrating both members of previous inequality in $[0, a]$ and in $[0, a/3]$ and by assuming without restriction that $2h\epsilon_j \leq a$ we get

$$0 \leq a\lambda_j + 2\mu_j \leq 4 \qquad \text{and} \qquad 0 \leq a\lambda_j + 6\mu_j \leq 24.$$

Then, by taking into account (ii) we have, up to subsequences, $\lambda_j \to \overline{\lambda}$, $\mu_j \to \overline{\mu}$ and $\epsilon_j\dot{u}_j \to 0$ in $L^1(0, a)$ so by recalling (4.11) we get

$$1 = (g^{**})'(\dot{u}_j) = \lambda_j x + \mu_j - 2\epsilon_j\dot{u}_j \qquad \text{on the set } \{\dot{u}_j > 1\} \tag{4.12}$$

and $\lambda_j x + \mu_j - 2\epsilon_j\dot{u}_j \to \overline{\lambda}x + \overline{\mu}$ in $L^1(0, a)$. By (i) $\dot{u}_j$ is monotone and continuous and without restriction we may assume (up to subsequences) that $\{\dot{u}_j > 1\} = (s_j, a)$ for some $s_j \in (0, a)$ and that $s_j \to s \in [0, a]$.

If $s < a$ then by (4.12) we get $\overline{\lambda}x + \overline{\mu} \equiv 1$ in (s, a), that is $\overline{\mu} = 1$, $\overline{\lambda} = 0$. Therefore, since

$$(g^{**})'(\dot{u}_j)\mathbf{1}_{(0,s_j)} = (\lambda_j x + \mu_j - 2\epsilon_j\dot{u}_j)\mathbf{1}_{(0,s_j)},$$

by taking into account the form of $(g^{**})'$ and the fact that $\dot{u}_j \leq 1$ on $(0, s_j)$, we get $\dot{u}_j \to 1$ a.e. on each compact subset of $(0, s)$ that is $u'_* = \dot{u}_* = 1$ a.e on $(0, s)$. On the other hand since for j large enough $\dot{u}_j > 1$ on each compact subset of (s, a) we get $\dot{u}_* \geq 1$ a.e. on (s, a) thus proving that $\dot{u}_* \geq 1$ a.e. on $(0, a)$ in this case.

If $s = a$ then $|\{\dot{u}_j > 1\}| \to 0$ and for every $0 < \beta < a$, we have $0 \leq \dot{u}_j \leq 1$ in $(0, \beta)$ for j large enough. Thus (up to subsequences), we find $v \in L^\infty((0, \beta))$ with $\|v\|_{L^\infty((0,\beta))} \leq 1$ such that $\dot{u}_j \to v$ in $w^* - L^\infty((0, \beta))$, so $u'_* = v = \dot{u}_*$ on $(0, \beta)$. This holds for every $0 < \beta < a$, that is, $u'_* = \dot{u}_*$ on $(0, a)$ and $u_* \in W^{1,\infty}(0, a)$, $0 \leq \dot{u}_* \leq 1$ a.e. in $(0, a)$.

In addition by recalling that $\dot{u}_j \to \dot{u}_*$ in $w^* - L^\infty((0, \beta))$ and $u_j(x) = \int_0^\beta \dot{u}_j(t)\mathbf{1}_{(0,x)}\, dt$ for every $x \in (0, \beta)$ we get $u_j(x) \to u_*(x)$ in $(0, \beta)$ which, by taking into account that u_j is convex, entails $\dot{u}_j \to \dot{u}_*$ a.e. in $(0, \beta)$ and (iv) completely follows from (4.11) by passing to the limit as $j \to \infty$. □

Next we discuss property (iv) of Lemma 4.7 in relation to the parameters range.

Lemma 4.8 *There exists a minimizer u_* of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$ such that $u_* \in W^{1,\infty}(0,a)$ and $0 \le \dot u_* \le 1$ a.e. in $(0,a)$. Moreover, if $2L \notin [a^2, 2ah - a^2]$, then any minimizer u of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$ satisfies $u \in W^{1,\infty}(0,a)$ and $0 \le \dot u \le 1$ a.e. in $(0,a)$.*

Proof Case I: $0 < h < a$ (hence $2L < a^2$). By (iv) of Lemma 3.10 there exists $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$ such that either $\dot u_* \ge 1$ a.e. in $(0,a)$ or $u_* \in W^{1,\infty}(0,a)$ with $0 \le \dot u_* \le 1$ a.e. in $(0,a)$. If the first case occurs then by taking into account that $u_*' \ge 0$ and $u_*(0^+) \ge 0$ we get $u_*(a^-) \ge \int_0^a \dot u_* \ge a > h$, a contradiction. Hence, $u_* \in W^{1,\infty}(0,a)$, $0 \le \dot u_* \le 1$ a.e. in $(0,a)$ and $\overline{\mathcal{J}}_+(u_*) = \mathcal{J}_+(u_*)$, thus proving the thesis.

Case II: $h = a$ and $2L < a^2$. Choose $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$ as in the previous case: if $\dot u_* \ge 1$ a.e. in $(0,a)$ then by taking into account that $u_*' \ge 0$ and $u_*(0^+) \ge 0$ we get $u_*(x) \ge x$ hence $L = \int_0^a u_* \ge a^2/2$, a contradiction. The thesis follows by arguing as before.

Case III: $h \ge a$ and $a^2 \le 2L \le a(2h-a)$. It is readily seen that there exists $u_* \in W^{1,\infty}(0,a)$ such that $\dot u_* = 1$ a.e. in $(0,a)$ and $\int_0^a u_* = L$. Since $g^{**}(1) - 1 \le g^{**}(z) - z$ for every $z \in \mathbb{R}$ we get $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$ and a direct computation shows that $\overline{\mathcal{J}}_+(u_*) = \mathcal{J}_+(u_*)$, thus proving the thesis.

Case IV: $h \ge a$ and $a(2h-a) < 2L < 2ah$. Assume by contradiction that $\dot u_* \ge 1$ a.e. in $(0,a)$: then either $u_*(0^+) > h - a$ or $u_*(0^+) \le h - a$. In the first case we easily get $u_*(x) > h - a + x$, hence $u_*(a^-) > h$, a contradiction. In the second one we claim that $u_*(x) \le h - a + x$: if this is true we get

$$a(2h-a) < 2L = 2\int_0^a u_*(x)\,dx \le 2\int_0^a (h-a+x)\,dx = a(2h-a),$$

a contradiction. To prove the claim it is enough to observe that if there exists $\overline{x} \in (0,a)$ such that $u_*(\overline{x}) > h - a + \overline{x}$ then by taking into account that $\dot u_* \ge 1$ a.e. in $(0,a)$ we get $u_*(x) \ge u_*(\overline{x}) + x - \overline{x} > h - a + x$ for every $x \ge \overline{x}$ hence $u_*(a-) > h$, a contradiction. Therefore $u_* \in W^{1,\infty}(0,a)$ and $0 \le u_* \le 1$ a.e. in $(0,a)$ also in this last case. □

The following is the version of Theorem 2.1 for functional $\overline{\mathcal{J}}_+$.

Lemma 4.9 *Suppose that $2L \notin (a^2, 2ah - a^2)$. Then $\overline{\mathcal{J}}_+$ admits a unique minimizer over $\mathcal{C}^+_{a,h,L}$. Otherwise, $\overline{\mathcal{J}}_+$ admits infinitely many minimizers over $\mathcal{C}^+_{a,h,L}$.*

Proof By Lemma 4.4, there holds $\overline{\mathcal{J}}_+(u) = h + \int_0^a \psi(\dot u(x))\,dx$ for every $u \in BV_{loc}(\mathbb{R})$, where $\psi(x) := g^{**}(x) - x$ is a convex function on $\mathbb{R}$ which is strictly convex on $[0,1]$.

Suppose first that $2L \notin [a^2, 2ah - a^2]$. By Lemma 4.8, $\overline{\mathcal{J}}_+$ admits a minimizer u_* over $\mathcal{C}^+_{a,h,L}$, which necessarily satisfies $u_* \in W^{1,\infty}(0,a)$ with $0 \le \dot u_* \le 1$ a.e. in $(0,a)$. On the other hand $\dot u_* = 1$ a.e. in $(0,a)$ is not admissible in this range of the parameters a, h, L (see Lemma 4.8), thus $\dot u_* < 1$ on a set of positive measure in $(0,a)$. Since ψ is strictly convex in $[0,1]$, if $v_* \in \mathcal{C}^+_{a,h,L}$ was another minimizer of $\overline{\mathcal{J}}_+$, not coinciding a.e. with u_*, we could consider $\mathcal{C}^+_{a,h,L} \ni w_* := \frac{1}{2}u_* + \frac{1}{2}v_*$: by the strict convexity of ψ in $[0,1]$, Jensen inequality would give $\overline{\mathcal{J}}_+(w_*) < \overline{\mathcal{J}}_+(u_*)$, contradicting minimality of u_*. Therefore, the minimizer u_* of $\overline{\mathcal{J}}_+$ is unique. If $h \ge a$ and either $2L = a^2$ or $2L = a(2h-a)$, the $\mathcal{C}^+_{a,h,L}$ piecewise affine function u_* having slope 1 on $(0,a)$ is the unique minimizer of $\overline{\mathcal{J}}_+$. Indeed, in this case it is clear that if $v_* \in \mathcal{C}^+_{a,h,L}$ satisfies $\dot v_* \ge 1$ a.e. in $(0,a)$, then $v_* = u_*$ a.e. $\mathbb{R}$. Therefore any admissible competitor v_*, not coinciding *a.e.* with u_*, needs to satisfy $\dot v_* < 1$ on a set of positive measure in $(0,a)$, thus it is not a minimizer due to the former Jensen inequality argument.

Else suppose that both the conditions $h > a$ and $a^2 < 2L < a(2h - a)$ hold true. Since we are in Case III from the proof of Lemma 4.8, we see that Lemmas 4.8 and 4.7 entail existence of a minimizer u_* of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$ such that $\dot{u}_* = 1$ a.e. in $(0, a)$. In this range of parameters, there necessarily holds $0 < u_*(0^+) < u_*(a^-) < h$ (in order to match the area constraint). Therefore, we may consider the family $u_\epsilon(x) := (1 + \epsilon)(x - a/2) + u_*(a/2)$, $x \in (0, a)$, and for any $\epsilon > 0$ small enough u_ϵ fits the strip $[0, h]$. After having extended u_ϵ to $\mathbb{R}$ in such a way that it belongs to $\mathcal{C}^+_{a,h,L}$, from the representation of $\overline{\mathcal{J}}_+$ given by $\overline{\mathcal{J}}_+(u) = h - \int_0^a \psi(\dot{u}(x))\, dx$, it is clear that $\overline{\mathcal{J}}_+(u_\varepsilon)$ does not depend on ϵ, as ψ is constant on $[1, +\infty)$ and the slope of u_ε is greater than 1 for any $\epsilon > 0$. □

Remark 4.10 In case competitors with $\dot{u} > 1$ a.e. in $(0, a)$ are present, a large nonuniquenss phenomenon occurs. Solutions are not restricted to functions such that $\dot{u}$ is constant in $(0, a)$ as in the proof of Lemma 4.9. For instance, it is clear that any other continuous piecewise affine curve with slopes greater or equal than 1 on $(0, a)$, as soon as it satisfies the constraints that define $\mathcal{C}^+_{a,h,L}$, is a minimizer of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$. Any other graph enjoying the same properties will attain the minimum. However, by Jensen inequality we obtain that the solution defined by $\dot{u} = 1$ in $(0, a)$ is unique among those elements u of $\mathcal{C}^+_{a,h,L}$ that satisfy $u \in W^{1,\infty}(0, a)$ and $\dot{u} \le 1$ a.e. in $(0, a)$.

This section ends with some further properties of minimizers of functional $\overline{\mathcal{J}}_+$.

Lemma 4.11 *Suppose that* $2L \notin (a^2, 2ah - a^2)$. *Then the unique minimizer* u *of* $\overline{\mathcal{J}}_+$ *over* $\mathcal{C}^+_{a,h,L}$ *provided by* Lemma 4.9 *is either convex on* $(0, a)$ *with* $u(0^+) = 0$ *or concave on* $(0, a)$ *with* $u(a^-) = h$.

Proof By points (i) and (iii) of Lemma 4.7, u can be obtained as $w^* - BV_{loc}(\mathbb{R})$ limit of $W^{1,2}_{loc}(\mathbb{R})$ functions u_j that are convex for all j or concave for all j. Up to subsequences, u_j converge to u pointwise in $(0, a)$ and u itself is therefore either concave or convex.

If $\dot{u}(x) = 1$ for any $x \in (0, a)$, by Lemma 4.9 we are necessarily in the case $2L = a^2$ or in the case $2L = a(2h - a)$ and the proof is concluded. Else suppose that u is concave and that there exists $0 < c < a$ such that $u' < 1$ a.e. in $(0, c)$. Suppose by contradiction that $u(a^-) < h$. Let us consider a piecewise affine approximation of $\bar{u}$ of u, with nodes on the graph of u, such that $\int_0^a u - \int_0^a \bar{u} = \varepsilon$. By Jensen inequality, due to the strict convexity of $\psi(x) := g^{**}(x) - x$ on $(0, 1)$, we have $\overline{\mathcal{J}}_+(\bar{u}) < \overline{\mathcal{J}}_+(u)$. On the other hand, if ε is small enough we have that $v := \bar{u} + (\varepsilon/a)\mathbf{1}_{(0,a)}$ belongs to $\mathcal{C}^+_{a,h,L}$ and $\overline{\mathcal{J}}_+(v) = \overline{\mathcal{J}}_+(\bar{u})$. This contradicts the minimality of u. In case u is convex and $u' < 1$ on a set of positive measure, an analogous argument shows that $u(0^+) = 0$. □

Corollary 4.12 *Let* $h \le a$. *If* $2L \le ah$ *(resp.* $2L \ge ah$*), then the unique minimizer* u *of* $\overline{\mathcal{J}}_+$ *over* $\mathcal{C}^+_{a,h,L}$ *provided by* Lemma 4.9 *is convex on* $(0, a)$ *with* $u(0^+) = 0$ *(resp. concave on* $(0, a)$ *with* $u(a^-) = h$*). In particular,* $u(x) = 0 \vee (hx/a) \wedge h$ *if* $2L = ah$.

Else suppose that $h > a$. *If* $2L \le a^2$ *(resp.* $2L \ge a(2h - a)$*), then the unique minimizer* u *of* $\overline{\mathcal{J}}_+$ *over* $\mathcal{C}^+_{a,h,L}$ *provided by* Lemma 4.9 *is convex with* $u(0^+) = 0$ *(resp. concave with* $u(a^-) = h$*). In particular, if* $2L = a^2$ *then* $u(x) = x$ *in* $(0, a)$.

Proof Let $h \le a$. Suppose that $2L < ah$. Suppose by contradiction that u is concave on $(0, a)$. Letting $w(x) := 0 \vee (hx/a) \wedge h$, since $u(a^-) = h$ by Lemma 4.11 and since u is concave, it is clear that $u \ge w$ in $(0, a)$. This entails $\int_0^a u \ge \int_0^a w = ah/2 > L$, a contradiction. In case $2L > ah$ the argument is analogous.

The same reasoning also applies for proving the result in case $h > a$. □

Remark 4.13 It is worth noticing that by symmetry reasons, if $u \in \mathcal{C}^+_{a,h,L}$ is a minimizer and it is convex in $(0, a)$, then $v(x) := h - u(a - x)$ satisfies $\overline{\mathcal{J}}_+(v) = \overline{\mathcal{J}}_+(u)$ and it is a minimizer in $\mathcal{C}^+_{a,h,ah-L}$ which is concave in $(0, a)$. Therefore all significant cases of Corollary 4.12 can be reduced to $2L \le (ah) \wedge a^2$ (as in Theorem 2.3).

5 Proof of the main results

We go back to the analyis of functional $\mathcal{F}$. The next two results give its relation with the auxiliary functionals from Sect. 4.

Lemma 5.1 *Let $\gamma \in \mathcal{A}_{a,h,L}$ be a piecewise affine curve such that $\gamma_1'(t) > 0$ for a.e. $t \in (0, 1)$. Then there exists a piecewise affine function $u \in \mathcal{B}_{a,h,L}$ such that $\mathcal{G}(u) = \mathcal{F}(\gamma)$.*

Conversely, let $u \in \mathcal{B}_{a,h,L}$ be piecewise affine. Then there exists $\gamma \in \mathcal{A}_{a,h,L}$ with $\gamma_1'(t) > 0$ for a.e. $t \in (0, 1)$ such that $\mathcal{F}(\gamma) = \mathcal{G}(u)$.

In particular there holds

$$\inf\left\{\mathcal{F}(\gamma) : \gamma \in \mathcal{A}_{a,h,L},\ \gamma_1'(t) > 0 \text{ for a.e. } t \in (0,1),\ \gamma \text{ piecewise affine}\right\} = \inf\left\{\mathcal{G}(u) : u \in \mathcal{B}_{a,h,L},\ u \text{ piecewise linear}\right\}.$$

Proof It is enough to exploit the fact that the values of $\mathcal{F}(\gamma)$ and $\int_0^1 \gamma_1(t)\gamma_2'(t)\,dt$ are invariant by reparametrization. If $\gamma \in \mathcal{A}_{a,h,L}$ is piecewise affine with $\gamma_1' > 0$ a.e. in $(0, 1)$, then γ can be reparametrized as the graph of a continuous piecewise affine map u on $[0, a]$, that is, as $[0, a] \ni t \mapsto (t, u(t))$. This is done by defining $u := \gamma_2 \circ \gamma_1^{-1}$. Note that u is absolutely continuous, as the composition of an absolutely continuous function and an absolutely continuous strictly increasing function. By changing variables, since $(\gamma_1^{-1})'(x) = 1/\gamma_1'(\gamma_1^{-1}(x))$ for a.e. $x \in (0, a)$, we get

$$\begin{aligned}\mathcal{G}[u] &= \int_0^a \frac{((\gamma_2 \circ \gamma_1^{-1})'(x))_+^3}{1 + (\gamma_2 \circ \gamma_1^{-1}(x))^2}\,dx = \int_0^a \frac{(\gamma_2'(\gamma_1^{-1}(x)))_+^3 / \gamma_1'(\gamma_1^{-1}(x))}{(\gamma_1'(\gamma_1^{-1}(x)))^2 + (\gamma_2'(\gamma_1^{-1}(x)))^2}\,dx \\ &= \int_0^1 \frac{(\gamma_2'(t))_+^3\,dt}{\gamma_1'(t)^2 + \gamma_2'(t)^2},\end{aligned}$$

$$\begin{aligned}\int_0^a u(x)\,dx &= \int_0^a \gamma_2(\gamma_1^{-1}(x))\,dx = \int_0^1 \gamma_2(t)\gamma_1'(t)\,dt \\ &= ah - \int_0^1 \gamma_1(t)\gamma_2'(t)\,dt = L,\end{aligned}$$

showing that indeed $u \in \mathcal{B}_{a,h,L}$ and $\mathcal{G}(u) = \mathcal{F}(\gamma)$.

Similarly, if $u \in \mathcal{B}_{a,h,L}$ is a piecewise affine map, we may consider the curve $[0, 1] \ni t \mapsto \gamma(t) := (at, u(at))$. It is immediate to check that $\gamma \in \mathcal{A}_{a,h,L}$ is piecewise affine with $\gamma_1'(t) > 0$ in $(0, 1)$ and that $\mathcal{F}(\gamma) = \mathcal{G}(u)$. □

Lemma 5.2 *There holds* $\inf \mathcal{G} = \inf \mathcal{J} = \inf \mathcal{F} = \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$.

Proof Take $u_* \in BV_{loc}(\mathbb{R})$ from Lemma 4.8, such that $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$ and $0 \le \dot{u}_* \le 1$ a.e. in $(0, a)$. It is easy to check that minimality of u_* implies that $\dot{u}_* > 0$ on a set of postive measure in $(0, a)$, and since the inequality $g^{**}(z) < z$ holds in $(0, 1]$, by Lemma 4.4 we get

$$\overline{\mathcal{J}}_+(u_*) = h + \int_0^a (g^{**}(\dot{u}(x)) - \dot{u}(x))\,dx < h.$$

Lemma 4.2 and Lemma 4.5 entail

$$h > \overline{\mathcal{J}}_+(u_*) = \inf \mathcal{J}_+ = \inf \mathcal{J}.$$

On the other hand, by definition of g^{**} in (4.1) it is clear that $\mathcal{G} \geq \mathcal{J}$, so that by the above equalities we get $\inf \mathcal{J} \leq \inf \mathcal{G}$. We are left to prove the opposite inequality.

Let $0 \leq t_1 < t_2 \leq 1$ and let $\gamma_* : [0,1] \to [0,a] \times [0,h]$ be defined by

$$\gamma_*(t) := \begin{cases} (0, t\, t_1^{-1} u_*(0^+)) & \text{if } t \in [0, t_1) \\ (a(t_2 - t_1)^{-1}(t - t_1), u_*(a(t_2 - t_1)^{-1}(t - t_1))) & \text{if } t \in [t_1, t_2] \\ (a, u_*(a^-) + (h - u_*(a^-))(1 - t_2)^{-1}(t - t_2)) & \text{if } t \in (t_2, 1]. \end{cases} \tag{5.1}$$

It is readily seen that $\gamma_* \in \mathcal{A}_{a,h,L}$ and that

$$\begin{aligned} \mathcal{F}(\gamma_*) &= u_*(0^+) + h - u_*(a^-) + \int_{t_1}^{t_2} a(t_2 - t_1)^{-1} \frac{u_*'(a(t_2 - t_1)^{-1}(t - t_1))^3}{1 + u_*'(a(t_2 - t_1)^{-1}(t - t_1))^2}\, dt \\ &= \int_0^a (g(\dot{u}_*(x)) - \dot{u}_*(x))\, dx + h \end{aligned}$$

where we can replace g with g^{**} in the last line due to $0 \leq \dot{u}_* \leq 1$. Therefore, by Lemma 4.4 we get $\mathcal{F}(\gamma_*) = \overline{\mathcal{J}}_+(u_*) < h$. We next take $\varepsilon > 0$ and a piecewise affine curve $\bar{\gamma} \in \mathcal{A}_{a,h,L}$ such that $\bar{\gamma}_1'(t) > 0$ for a.e. $t \in (0,1)$ and $|\mathcal{F}(\gamma_*) - \mathcal{F}(\bar{\gamma})| < \varepsilon$, which is possible by Lemma 3.3. By Lemma 5.1 there is $\bar{u} \in \mathcal{B}_{a,h,L}$ such that $\mathcal{G}(\bar{u}) = \mathcal{F}(\bar{\gamma})$. Summing up we have

$$\inf \mathcal{G} \leq \mathcal{G}(\bar{u}) = \mathcal{F}(\bar{\gamma}) < \mathcal{F}(\gamma_*) + \varepsilon = \overline{\mathcal{J}}_+(u_*) + \varepsilon,$$

and by arbitrariness of ε we get $\inf \mathcal{G} \leq \overline{\mathcal{J}}_+(u_*) = \inf \mathcal{J}$.

We have shown that $\inf \mathcal{J} = \inf \mathcal{G}$. Lemma 3.3, Lemma 5.1 and Lemma 4.1 imply that $\inf \mathcal{G} = \inf \mathcal{F}$, concluding the proof. □

Before proceeding to the proof of Theorem 2.1, we need three more technical lemmas.

Lemma 5.3 *Let $\gamma \in \mathcal{A}_{a,h,L}$ and suppose that there exist t_1, t_2, with $0 \leq t_1 < t_2 \leq 1$, such that $\gamma_2(t_2) < \gamma_2(t_1)$. Then $\mathcal{F}(\gamma) > \inf \mathcal{F}$.*

Proof We let $\epsilon > 0$ and we let $\hat{\gamma} \in \mathcal{A}_{a,h,L}$ be a piecewise affine approximation of γ with $\hat{\gamma}_1'(t) > 0$ a.e. in $(0,1)$, such that $|\mathcal{F}(\gamma) - \mathcal{F}(\hat{\gamma})| < \epsilon$ and such that $|\hat{\gamma}_2(t_i) - \gamma_2(t_i)| < \epsilon$, $i = 1, 2$. We let $(x_p, y_p) := (\hat{\gamma}_1(t_1), \hat{\gamma}_2(t_1))$ and $(x_q, y_q) := (\hat{\gamma}_1(t_2), \hat{\gamma}_2(t_2))$. We may assume wlog that

$$y_q < \hat{\gamma}_2(t) < y_p \quad \text{for any } t \in (t_1, t_2),$$

otherwise we could define

$$\tilde{t}_1 := \max\{t \in [t_1, t_2] : \hat{\gamma}_2(t) \geq \hat{\gamma}_2(t_1)\}, \quad \tilde{t}_2 := \min\{t \in [\tilde{t}_1, t_2] : \hat{\gamma}_2(t) \leq \hat{\gamma}_2(t_2)\}$$

and subsequently redefine $(x_p, y_p) := (\hat{\gamma}_1(\tilde{t}_1), \hat{\gamma}_2(\tilde{t}_1))$ and $(x_q, y_q) := (\hat{\gamma}_1(\tilde{t}_2), \hat{\gamma}_2(\tilde{t}_2))$.

Notice that $\hat{\gamma}$ coincides on $[0, a]$ with the graph of a piecewise affine function $\hat{u} \in \mathcal{B}_{a,h,L}$. Hence, by the proof of Lemma 4.2 there exists a new piecewise affine curve $u \in \mathcal{B}^+_{a,h,L}$ having ordered vertices at the points $(0,0) = (s_0, u(s_0)),\ (s_1, u(s_1)), \ldots, (s_k, u(s_k)) = (a, h)$ along the curve $\hat{\gamma}$. We let $S := \{s_0, s_1, \ldots, s_k\}$. Jensen inequality ensures that

$$\int_{s_j}^{s_{j+1}} g^{**}(u'(x))\, dx \leq \int_{s_j}^{s_{j+1}} g^{**}(\hat{u}'(x))\, dx, \qquad j = 0, \ldots, k-1. \tag{5.2}$$

We let

$$x_1 = \max\{s \in S, s \le x_p\}, \quad x_4 = \min\{s \in S, s \ge x_q\}, \quad y_1 = u(x_1), \quad y_4 = u(x_4).$$

Supposing that $\{s \in S : x_p < s < x_q\} \neq \emptyset$, we further define

$$x_2 = \min\{s \in S, s > x_p\}, \quad x_3 = \max\{s \in S, s < x_q\}, \quad y_2 = u(x_2), \quad y_3 = u(x_3),$$

so that $y_1 \le y_2 \le y_3 \le y_4$. Else if $\{s \in S : x_p < s < x_q\} = \emptyset$, we define $x_2 = x_3 = \frac{x_p+x_q}{2}$ and $y_2 = y_3 = \frac{y_p+y_q}{2}$.

By construction, there always holds $y_q \le y_2 \le y_3 \le y_p$.

By repeated use of Jensen inequality and since from (4.1) we have $g^{**}(z) = 0$ for $z \le 0$, there hold

$$\begin{aligned}
&\int_{x_1}^{x_2} g^{**}(u') + \int_{x_3}^{x_4} g^{**}(u') \le (x_2 - x_1)\, g^{**}\left(\tfrac{y_2-y_1}{x_2-x_1}\right) + (x_4 - x_3)\, g^{**}\left(\tfrac{y_4-y_3}{x_4-x_3}\right), \\
&\int_{x_1}^{x_2} g^{**}(\hat{u}') \ge \int_{x_1}^{x_p} g^{**}(\hat{u}') \ge (x_p - x_1)\, g^{**}\left(\tfrac{y_p-y_1}{x_p-x_1}\right) \ge (x_2 - x_1)\, g^{**}\left(\tfrac{y_p-y_1}{x_2-x_1}\right), \\
&\int_{x_3}^{x_4} g^{**}(\hat{u}') \ge \int_{x_q}^{x_4} g^{**}(\hat{u}') \ge (x_4 - x_q)\, g^{**}\left(\tfrac{y_4-y_q}{x_4-x_q}\right) \ge (x_4 - x_3)\, g^{**}\left(\tfrac{y_4-y_q}{x_4-x_3}\right),
\end{aligned} \tag{5.3}$$

where the mapping $(0, +\infty) \times \mathbb{R} \ni (x, y) \mapsto x g^{**}(y/x)$ is understood to be extended by continuity to $x = 0$ (with value y_+), and we used the fact that $[0, +\infty) \ni x \mapsto x g^{**}(y/x)$ is nonincreasing for any $y \in \mathbb{R}$. Thanks to (5.2) and (5.3) we get

$$\begin{aligned}
\mathcal{J}(\hat{u}) - \mathcal{J}(u) \ge\ & (x_2 - x_1)\left(g^{**}\left(\tfrac{y_p-y_1}{x_2-x_1}\right) - g^{**}\left(\tfrac{y_2-y_1}{x_2-x_1}\right)\right) \\
&+ (x_4 - x_3)\left(g^{**}\left(\tfrac{y_4-y_q}{x_4-x_3}\right) - g^{**}\left(\tfrac{y_4-y_3}{x_4-x_3}\right)\right).
\end{aligned} \tag{5.4}$$

We define $\varphi(x) := \min\left\{\frac{x}{2}, \frac{x^3}{a^2+h^2}\right\}$ for $x \ge 0$. Again the definition of g^{**} in (4.1) entails

$$u\left(g^{**}\left(\tfrac{v_1}{u}\right) - g^{**}\left(\tfrac{v_2}{u}\right)\right) \ge \varphi(v_1 - v_2)\, \mathbf{1}_{v_2 \ge 0} \tag{5.5}$$

for all $u \in (0, a)$, $v_1 \in (0, h)$, $v_2 \in (0, h)$ with $v_1 \ge v_2$.

If $y_2 \ge y_1$ and $y_4 \ge y_3$, from (5.4) and (5.5) we get

$$\mathcal{J}(\hat{u}) - \mathcal{J}(u) \ge \varphi(y_p - y_2) + \varphi(y_3 - y_q) \ge \varphi(\max\{y_p - y_2, y_3 - y_q\}) \ge \varphi\left(\tfrac{y_p-y_q}{2}\right),$$

where we have used $y_3 \ge y_2$ which entails $2\max\{y_p - y_2, y_3 - y_q\} \ge y_p - y_2 + y_3 - y_q \ge y_p - y_q$. Else we notice that $y_2 < y_1$ or $y_4 < y_3$ may happen only if $\{s \in S : x_p < s < x_q\} = \emptyset$, in which case $y_p - y_2 = y_3 - y_q = \frac{y_p-y_q}{2}$. Moreover, in such case since $y_1 \le y_4$ and $y_2 = y_3$ it is clear that the two inequalities $y_2 < y_1$ and $y_4 < y_3$ do not simultaneously hold. Therefore, even in this case from (5.4) and (5.5) we get $\mathcal{J}(\hat{u}) - \mathcal{J}(u) \ge \varphi\left(\frac{y_p-y_q}{2}\right)$.

We finally notice that

$$y_p - y_q \ge \hat{\gamma}_2(t_2) - \hat{\gamma}_2(t_1) \ge \gamma_2(t_2) - \gamma_2(t_1) - 2\varepsilon,$$

where $\gamma_2(t_2) - \gamma_2(t_1)$ is, by assumption, a prescribed positive value (independent of ϵ). We conclude that for any small enough ϵ

$$\mathcal{F}(\gamma) \ge \mathcal{F}(\hat{\gamma}) - \epsilon = \mathcal{G}(\hat{u}) - \epsilon \ge \mathcal{J}(\hat{u}) - \epsilon \ge \mathcal{J}(u) + \varphi\left(\tfrac{y_p-y_q}{2}\right) - \epsilon > \mathcal{J}(u).$$

Since we have shown in Lemma 5.2 that $\inf \mathcal{J} = \inf \mathcal{F}$, the result follows. □

Before stating the next lemma, as further notation we introduce the class

$$\mathcal{A}^+_{a,h,L} := \left\{ \gamma \in \mathcal{A}_{a,h,L} : \gamma_2'(t) \geq 0 \text{ for a.e. } t \in (0,1) \right\}. \tag{5.6}$$

Lemma 5.4 *Suppose that* $2L \notin (a^2, 2ah - a^2)$.

Let $\gamma \in \mathcal{A}^+_{a,h,L}$. *If* $0 < t_1 < t_2 < 1$ *exist such that* $\gamma_1'(t) = 0$ *on* (t_1, t_2), *then* $\mathcal{F}(\gamma) > \inf \mathcal{F}$.

Proof For $\gamma_2(t_1) < s < \gamma_2(t_2)$, we define t_s as the unique number in (t_1, t_2) such that $\gamma_2(t_s) = \gamma_2(t_1) + s$, $h_s := \gamma_2(t_2) - \gamma_2(t_s)$ and

$$\gamma^s(t) := \begin{cases} (0, ts(t_s - t_1)^{-1}) & \text{if } t \in [0, t_s - t_1] \\ \gamma(t - t_s + t_1) + (0, s) & \text{if } t \in (t_s - t_1, t_s] \\ \gamma(t + t_2 - t_s) - (0, h_s) & \text{if } t \in (t_s, t_s + 1 - t_2) \\ (a, h - h_s) + (t_2 - t_s)^{-1}(t - 1 + t_2 + t_s)(0, h_s) & \text{if } t \in [t_s + 1 - t_2, 1]. \end{cases}$$

It is clear that for any s, $\mathcal{F}(\gamma^s) = \mathcal{F}(\gamma)$, since γ^s is just obtained from γ by rearrangement of pieces (by translations). It is also clear that s can be (uniquely) chosen such that $\gamma_s \in \mathcal{A}^+_{a,h,L}$. Let r denote such value of s and let $\breve{\gamma} := \gamma^r$. We next define suitable approximations by means of Lemma 3.3. We let

$$\breve{\gamma}^N(t) := \begin{cases} (0, tr(t_r - t_1)^{-1}) & \text{if } t \in [0, t_r - t_1] \\ \gamma^N(t) & \text{if } t \in (t_r - t_1, t_r + 1 - t_2) \\ (a, h - h_r) + (t_2 - t_r)^{-1}(t - 1 + t_2 + t_r)(0, h_r) & \text{if } t \in [t_r + 1 - t_2, 1], \end{cases}$$

where

$$\gamma^N : [t_r - t_1, t_r + 1 - t_2] \to [0, a] \times [r, h - h_r]$$

are piecewise affine approximations of $\breve{\gamma}\big|_{[t_r - t_1, t_r + 1 - t_2]}$, with same initial point $\breve{\gamma}(t_r - t_1) = (0, r)$, same end point $\breve{\gamma}(t_r + 1 - t_2) = (a, h - h_r)$, with $(\gamma_2^N)'(t) \geq 0$ and $(\gamma_1^N)'(t) > 0$ for a.e. $t \in (t_r - t_1, t_r + 1 - t_2)$. These approximating curves are constructed by means of Lemma 3.3, so that $\breve{\gamma}^N \in \mathcal{A}_{a,h,L}$, $\breve{\gamma}_N \to \breve{\gamma}$ uniformly on $[t_r - t_1, t_r + 1 - t_2]$ and $\mathcal{F}(\breve{\gamma}^N) \to \mathcal{F}(\breve{\gamma})$ as $N \to +\infty$. Since γ_1^N is strictly increasing we may define the piecewise affine function $u_N := \gamma_2^N \circ (\gamma_1^N)^{-1}$ on $[0, a]$, that we extend to $\mathbb{R}$ by setting $u_N(x) = 0$ if $x < 0$ and $u_N(x) = h$ if $x > a$. By changing variables as done in the proof of Lemma 5.1 we get

$$\begin{aligned} \overline{\mathcal{J}}_+(u_N) &= r + h - h_r + \int_0^a g^{**}(\dot{u}_N) \leq r + h - h_r + \int_0^a \frac{\dot{u}_N(x)^3}{1 + \dot{u}_N(x)^2}\, dx \\ &= r + h - h_r + \int_{t_r - t_1}^{t_r + 1 - t_2} \frac{(\gamma_2^N)'(t)^3}{(\gamma_1^N)'(t)^2 + (\gamma_2^N)'(t)^2}\, dt = \mathcal{F}(\breve{\gamma}^N) \end{aligned} \tag{5.7}$$

and

$$\int_0^a u_N = \int_{t_r - t_1}^{t_r + 1 - t_2} \gamma_2^N (\gamma_1^N)' = \int_0^1 \breve{\gamma}_2^N (\breve{\gamma}_1^N)',$$

$$\int_0^a |u_N'(x)|\, dx = \int_{t_r - t_1}^{t_r + 1 - t_2} |(\gamma_2^N)'(t)|\, dt \leq h.$$

Thanks to the latter estimate, u_N admits a $w^* - BV_{loc}(\mathbb{R})$ limit u, which satisfies $u(0^+) \geq r$ and $u(a^-) \leq h - h_r$. Up to extraction of a not relabeled subsequence, the convergence also holds strongly in $L^1(0, a)$, thus $\int_0^a u = \lim_{N \to +\infty} \int_0^a u_n = L$ so that $u \in \mathcal{C}^+_{a,h,L}$. The lower semicontinuity of $\overline{\mathcal{J}}_+$ and (5.7) entail

$$\overline{\mathcal{J}}_+(u) \leq \liminf_{N \to +\infty} \overline{\mathcal{J}}_+(u^N) \leq \liminf_{N \to +\infty} \mathcal{F}(\breve{\gamma}^N) = \mathcal{F}(\breve{\gamma}) = \mathcal{F}(\gamma).$$

But $u(0^+) > 0$ and $u(a^-) < h$, thus u is not a minimizer of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$ due to Lemma 4.11. We conclude that $\mathcal{F}(\gamma) > \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$. By Lemma 5.2, the result follows. □

Lemma 5.5 *Let $\gamma \in \mathcal{A}^+_{a,h,L}$. Then $\mathcal{F}(\gamma) \geq h - a/2$ and equality holds if and only if $\gamma_1'(t) = \gamma_2'(t)$ for a.e. $t \in \{\gamma_1'(t) > 0\}$.*

Proof Let $\gamma \in \mathcal{A}^+_{a,h,L}$. We have

$$\begin{aligned}\mathcal{F}(\gamma) &= \int_0^1 \left(\gamma_2'(t) + \frac{\gamma_2'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} - \gamma_2'(t)\right) dt \\ &= h - \int_0^1 \frac{\gamma_1'(t)\gamma_2'(t)}{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, \gamma_1'(t)\, dt \geq h - \frac{1}{2}\int_0^1 \gamma_1'(t)\, dt = h - \frac{a}{2}.\end{aligned}$$

Equality holds if and only if $\gamma_1' = \gamma_2'$ a.e. on $\{\gamma_1'(t) > 0\}$, since the Young inequality $2\alpha\beta \leq \alpha^2 + \beta^2$ is an equality if and only if $\alpha = \beta$. □

We are ready for the proof of the main results.

Proof of Theorem 2.1 Let us start by proving existence. Take $u_* \in \mathcal{C}^+_{a,h,L}$ from Lemma 4.8, such that $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$, $u_* \in W^{1,\infty}(0,a)$ and $0 \leq \dot{u}_* \leq 1$ a.e. in $(0,a)$. We have seen in the proof of Lemma 5.2 that there exists $\gamma_* \in \mathcal{A}_{a,h,L}$ such that $\overline{\mathcal{J}}_+(u_*) = \mathcal{F}(\gamma_*) = \inf \mathcal{F}$. This concludes the proof. We also stress that from (5.1) we deduce $\gamma_* \in \mathcal{A}^+_{a,h,L}$, which is the class defined in (5.6). In fact, any solution to problem (2.1) belongs to $\mathcal{A}^+_{a,h,L}$ by Lemma 5.3.

Let us prove (i). Suppose that $2L \notin (a^2, 2ah - a^2)$. Let γ be an element of $\mathcal{A}^+_{a,h,L}$ that solves problem (2.1), so that $\mathcal{F}(\gamma) = \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$. Taking advantage of Lemma 5.4, there exist $0 \leq t_1 < t_2 \leq 1$ such that γ_1 is constant on $[0, t_1]$ and $[t_2, 1]$, and it is strictly increasing on $[t_1, t_2]$. We let $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2)$ denote the restriction of γ to $[t_1, t_2]$ and we define a monotonic $BV_{loc}(\mathbb{R})$ function by $u(x) = \bar{\gamma}_2 \circ \bar{\gamma}_1^{-1}(x)$ for $x \in (0,a)$ (extended to $\mathbb{R}$ by $u(x) = 0$ if $x < 0$ and $u(x) = h$ if $x > a$). By invoking Lemma 3.3 as done in the proof of Lemma 5.4, we introduce piecewise affine approximations $\gamma^N \in \mathcal{A}^+_{a,h,L}$ of γ, with $(\gamma_1^N)'(t) > 0$ for a.e. $t \in (0,1)$, so that $\mathcal{F}(\gamma^N) \to \mathcal{F}(\gamma)$ and $\gamma^N \to \gamma$ uniformly on $[0,1]$ as $N \to +\infty$. As a consequence, letting $u_N := \gamma_2^N \circ (\gamma_1^N)^{-1}$ in $(0,a)$ (extended to $\mathbb{R}$ by $u_N(x) = 0$ if $x < 0$ and $u^N(x) = h$ if $x > a$) there also holds $u_N \to u$ pointwise a.e. in $\mathbb{R}$ as $N \to +\infty$. By changing variables we get

$$\int_0^a u_N = \int_0^a \gamma_2^N((\gamma_1^N)^{-1}(x))\, dx = \int_0^1 \gamma_2^N(t)(\gamma_1^N)'(t)\, dt = \int_0^1 \gamma_2(t)\gamma_1'(t)\, dt = L,$$

so that $u_N \in \mathcal{C}^+_{a,h,L}$ for any N, and (by using $g^{**} \leq g$)

$$\overline{\mathcal{J}}_+(u_N) \leq \int_0^a g^{**}(\dot{u}_N) = \int_0^1 \frac{(\gamma_2^N)'(t)^3}{(\gamma_1^N)'(t)^2 + (\gamma_2^N)'(t)^2}\, dt = \mathcal{F}(\gamma^N).$$

A $w^* - BV_{loc}(\mathbb{R})$ limit point of u_N necessarily coincides with u since $w^* - BV_{loc}(\mathbb{R})$ and pointwise a.e. limit coincide. By passing to the limit with the $w^* - BV_{loc}(\mathbb{R})$ lower semicontinuity of $\overline{\mathcal{J}}_+$ we get $\overline{\mathcal{J}}_+(u) \leq \mathcal{F}(\gamma)$. But Lemma 5.2 and Theorem 2.1 yield $\mathcal{F}(\gamma) = \inf \mathcal{F} = \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$. We conclude that u coincides with the unique minimizer u_* of $\overline{\mathcal{J}}_+$ over $\mathcal{C}^+_{a,h,L}$ provided by Lemma 4.9. Hence the curve γ necessarily coincides with the graph of u_* on $(0, a)$ plus the possible vertical segments at $x = 0$ or $x = a$. This concludes the proof of (i).

Eventually, let us prove the statement (ii). Suppose that $h > a$ and $a^2 < 2L < a(2h - a)$. All the piecewise affine curves γ in $\mathcal{A}^+_{a,h,L}$ that are constructed in Sect. 2 after the statement of Theorem 2.4 satisfy $\mathcal{F}(\gamma) = h - a/2$ as seen in (2.10). Therefore, they solve problem 2.1 thanks to Lemma 5.5. □

Let us now give a precise characterization of solutions in the nonuniqueness range, by proving Theorem 2.4.

Proof of Theorem 2.4 Suppose that $h > a$ and $a^2 < 2L < a(2h - a)$. By Lemma 5.3 any solution to problem (2.1) belongs to $\mathcal{A}^+_{a,h,L}$. As $\gamma^\circ \in \mathcal{A}^+_{a,h,L}$ and $\mathcal{F}(\gamma^\circ) = h - a/2$, we conclude that γ° solves problem (2.1) as a consequence of Lemma 5.5. More generally, still by Lemma 5.5, γ is solution to problem (2.1) if and only if $\gamma \in \mathcal{A}^+_{a,h,L}$ and $\gamma_1' = \gamma_2'$ a.e. on $\{\gamma_1' > 0\}$. It is clear that γ° is the unique curve in the latter class such that the set $\{\gamma_1'(t) > 0\}$ is an interval (t_1, t_2) for some $0 < t_1 < t_2 < 1$. □

Remark 5.6 Suppose that $h > a$ and $a^2 < 2L < a(2h - a)$. γ° corresponds indeed to the unique minimizer of $\overline{\mathcal{J}}_+$ among functions u in $\mathcal{C}^+_{a,h,L}$ such that $u \in W^{1,\infty}(0, a)$ with $\dot{u} = 1$ on $(0, a)$, see Lemma 4.9 and Remark 4.10.

The proof of Theorem 2.3 relies on a careful application of the Euler–Lagrange equation and it requires some preliminary lemmas. We shall provide a parametrization in terms of $\dot{u}$ as originally done by Euler in the solution of Proposition 53 in Scientia Navalis [14]. Without loss of generality, as we have pointed out in Lemma 4.11 and in Remark 4.13, we may consider only the case of convex solutions. We start by proving the following

Lemma 5.7 *Assume that* $2L \le (ah) \wedge a^2$ *holds true and let* $\Psi,\ \Phi,\ \mathcal{T}$ *as in* (2.4), (2.5) *and* (2.6) *respectively. Then*

$$\min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+ = \min_{\mathcal{T}} \Psi.$$

Proof Since $2L \le (ah) \wedge a^2$, then by Lemma 4.9 there exists a unique $u_* \in \operatorname{argmin}_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+$, $u_* \in W^{1,\infty}(0, a)$ and $0 \le \dot{u}_* \le 1$. By Corollary 4.12 u_* is convex in $(0, a)$, $u_*(0) = 0$ and finally by Lemma 4.7 there exist $\overline{\lambda},\ \overline{\mu} \in \mathbb{R}$ such that

$$g'(\dot{u}_*) = \overline{\lambda} x + \overline{\mu} \quad \text{a.e. in } (0, a) \tag{5.8}$$

hence $\overline{\mu} = g'(\dot{u}_*(0))$ and $\overline{\lambda} = a^{-1}(g'(\dot{u}_*(a)) - g'(\dot{u}_*(0))) \ge 0$ since g' is increasing in $[0, 1]$ and $\dot{u}_*(a) \ge \dot{u}_*(0)$ by convexity of u. Moreover, due to continuity of the right hand side, (5.8) holds everywhere in $[0, a]$ and $\dot{u}_*$ is continuous therein; therefore the set

$$\mathcal{K} := \left\{ u \in C^1([0, a]) : u(0) = 0,\ \int_0^a u = L,\ u \text{ convex},\ 0 \le \dot{u} \le 1,\ g'(\dot{u}(x)) = \overline{\lambda} x + \overline{\mu} \right\}$$

is nonempty and

$$\min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+ = \min_{\mathcal{K}} \overline{\mathcal{J}}_+.$$

If $u \in \mathcal{K}$ and $\dot{u}(0) < \dot{u}(a)$ then $\dot{u}$ is strictly increasing and by setting $t := \dot{u}(x)$, taking into account that $g'(\dot{u}(x)) = \overline{\lambda} x + \overline{\mu}$ and that $u(0) = 0$, it is readily seen that the curve $\sigma(x) := (x, u(x)),\ x \in [0, a]$, is equivalent to the one parametrized by

$$x(t) := \dot{u}^{-1}(t) = \frac{a(g'(t) - g'(\dot{u}(0))}{g'(\dot{u}(a)) - g'(\dot{u}(0))}, \quad y(t) := \int_{\dot{u}(0)}^{t} s x'(s)\, ds, \quad t \in [\dot{u}(0), \dot{u}(a)]$$

and a direct computation using Lemma 4.4 shows that

$$\begin{aligned} \overline{\mathcal{J}}_+(u) &= h + \int_0^a (g^{**}(\dot{u}(x)) - \dot{u}(x))\, dx = h - \int_0^a \frac{\dot{u}(x)}{1+\dot{u}(x)^2}\, dx \\ &= h - \int_{\dot{u}(0)}^{\dot{u}(a)} \frac{t}{1+t^2} x'(t)\, dt \\ &= h - \frac{a\dot{u}(a)}{1+\dot{u}(a)^2} + \int_{\dot{u}(0)}^{\dot{u}(a)} \frac{1-t^2}{(1+t^2)^2} \frac{g'(t) - g'(\dot{u}(0))}{g'(\dot{u}(a)) - g'(\dot{u}(0))}\, dt = \Psi(\dot{u}(0), \dot{u}(a)). \end{aligned} \tag{5.9}$$

Moreover, by using again the change of variable $t := \dot{u}(x)$, taking into account that $u(0) = 0$, $x(\dot{u}(0)) = 0$, $x(\dot{u}(a)) = a$, the area constraint becomes

$$\begin{aligned} L &= \int_0^a u(x)\, dx = \int_0^a (a - u(x))\dot{u}(x)\, dx = \int_{\dot{u}(0)}^{\dot{u}(a)} (a - x(t))\, t x'(t)\, dt \\ &= \frac{a^2}{2}\dot{u}(0) + \frac{1}{2}\int_{\dot{u}(0)}^{\dot{u}(a)} (a - x(t))^2\, dt = \frac{a^2}{2}\dot{u}(0) + \frac{a^2}{2}\int_{\dot{u}(0)}^{\dot{u}(a)} \left(\frac{g'(\dot{u}(a)) - g'(t)}{g'(\dot{u}(a)) - g'(\dot{u}(0))}\right)^2 dt \\ &= \Phi(\dot{u}(0), \dot{u}(a)). \end{aligned} \tag{5.10}$$

On the other hand if $\dot{u}(0) = \dot{u}(a)$ then $\dot{u}(x) \equiv \dot{u}(0)$ and since $\int_0^a u = L$ we get $\dot{u}(x) \equiv 2La^{-2}$ and

$$\overline{\mathcal{J}}_+(u) = \Psi(2La^{-2}, 2La^{-2}), \qquad \Phi(2La^{-2}, 2La^{-2}) = L. \tag{5.11}$$

That is, (5.9) holds true for every $u \in \mathcal{K}$. Hence

$$\min_{\mathcal{K}} \overline{\mathcal{J}}_+(u) = \min\{\Psi(\dot{u}(0), \dot{u}(a)) : u \in \mathcal{K}\}$$

and by noticing that $\{(\dot{u}(0), \dot{u}(a)) : u \in \mathcal{K}\} \subset \mathcal{T}$ we get

$$\min_{\mathcal{K}} \overline{\mathcal{J}}_+(u) = \min\{\Psi(\dot{u}(0), \dot{u}(a)) : u \in \mathcal{K}\} \geq \min_{\mathcal{T}} \Psi.$$

We claim that if $(\xi_*, \eta_*) \in \operatorname{argmin}_{\mathcal{T}} \Psi$ then there exists $u_* \in \mathcal{K}$ such that $(\dot{u}_*(0), \dot{u}_*(a)) = (\xi_*, \eta_*)$: indeed if $\xi_* = \eta_*$ then by (2.5) we get $\Phi(\xi_*, \xi_*) = a^2\xi_*/2 = L$, that is $\xi_* = \eta_* = 2L/a^2$ and therefore it is enough to choose $u_*(x) = \xi_* x$; otherwise we have $\xi_* < \eta_*$ and we may define a parametrized curve by

$$x_*(t) := \frac{a(g'(t) - g'(\xi_*))}{g'(\eta_*) - g'(\xi_*)}; \quad y_*(t) := \int_{\xi_*}^{t} s x'(s)\, ds, \quad t \in [\xi_*, \eta_*]. \tag{5.12}$$

It is readily seen that $x_*(t)$ is strictly increasing from $[\xi_*, \eta_*]$ onto $[0, a]$ and by denoting with φ its inverse we define $u_*(x) := y_*(\varphi(x))$. A direct computation shows that u_* is differentiable in $(0, a)$ and $\dot{u}_*(x) = \varphi(x)$ therein, so it is easy to see that $u_* \in \mathcal{K}$ and $\dot{u}_*(0) = \varphi(0) = \xi_*$, $\dot{u}_*(a) = \varphi(a) = \eta_*$ thus proving the claim. Therefore

$$\min\{\Psi(\dot{u}(0), \dot{u}(a)) : u \in \mathcal{K}\} \leq \min_{\mathcal{T}} \Psi$$

and the proof is achieved. □

Lemma 5.8 *Assume that $2L \leq (ah) \wedge a^2$. Then there exists a unique $(\xi_*, \eta_*) \in \operatorname{argmin}_{\mathcal{T}} \Psi$. Moreover, if $2L < (ah) \wedge a^2$, then $\xi_* < \eta_*$.*

Proof Let (ξ_*, η_*) be a minimizer of Ψ over $\mathcal{T}$. Following the proof of Lemma 5.7, there exists $u \in \mathcal{K}$ such that $(\xi_*, \eta_*) = (\dot{u}(0), \dot{u}(a))$ and moreover

$$\min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+ = \min_{\mathcal{T}} \Psi = \Psi(\xi_*, \eta_*) = \Psi(\dot{u}(0), \dot{u}(a)) = \overline{\mathcal{J}}_+(u)$$

But Lemma 4.9 shows that there exists a unique minimizer u_* of $\overline{\mathcal{J}}$ over $\mathcal{C}^+_{a,h,L}$ (and $u_* \in \mathcal{K}$ as seen in the proof of Lemma 5.7). Therefore u necessarily coincides with u_*. Thus $(\xi_*, \eta_*) = (\dot{u}_*(0), \dot{u}_*(a))$ and this proves uniqueness.

Assume now by contradiction that $2L < (ah) \wedge a^2$ and $\xi_* = \eta_*$. Then by (2.5) $\xi_* = \eta_* = 2L/a^2$. If we consider a couple (ξ, η) that satisfies

$$\eta \in [2L/a^2, 1], \quad \xi \in [0, 2L/a^2], \quad (a^2/2 - L)\xi + L\eta = L, \tag{5.13}$$

then it is readily seen that $(\xi, \eta) \in \mathcal{T}$: indeed by setting

$$u_{\xi,\eta}(x) := \begin{cases} \xi x & \text{if } x \in [0, a - \sqrt{2L}] \\ \eta(x - a + \sqrt{2L}) + \xi(a - \sqrt{2L}) & \text{if } x \in [a - \sqrt{2L}, a], \end{cases}$$

and by reasoning as done in (5.10) we get

$$\Phi(\xi, \eta) = \int_0^a u_{\xi,\eta}(x)\, dx = (a^2/2 - L)\xi + L\eta = L.$$

At the same time, by computing as in (5.9),

$$\Psi(\xi, \eta) = h - \int_0^a \frac{\dot{u}_{\xi,\eta}(x)}{1 + \dot{u}_{\xi,\eta}(x)^2}\, dx = h - (a - \sqrt{2L})\frac{\xi}{1 + \xi^2} - \frac{\eta\sqrt{2L}}{1 + \eta^2}.$$

If we set

$$\phi(\eta) := \Psi\left(\frac{2L(1 - \eta)}{a^2 - 2L}, \eta\right), \qquad \eta \in [2L/a^2, 1],$$

we have $\phi(2L/a^2) = \Psi(\xi_*, \eta_*)$ and by taking into account that $2L < a^2$ a direct computation shows that

$$\phi'(2L/a^2) = \left(\frac{2L(a - \sqrt{2L})}{a^2 - 2L} - \sqrt{2L}\right)\frac{1 - 4L^2/a^4}{(1 + 4L^2/a^4)^2} < 0.$$

Hence there exist $1 \geq \overline{\eta} > 2L/a^2$ and $0 \leq \overline{\xi} = \frac{2L(1-\overline{\eta})}{a^2-2L} < 2L/a^2$ such that the couple $(\overline{\xi}, \overline{\eta})$ satisfies the constraint (5.13) and

$$\phi(\overline{\eta}) = \Psi(\overline{\xi}, \overline{\eta}) < \Psi(\xi^*, \eta^*),$$

thus contradicting minimality of (ξ_*, η_*). □

The previous results suggests the following parametric representation of the minimizer.

Lemma 5.9 *Assume that* $0 < 2L \leq (ah) \wedge a^2$. *Let* u_* *be the unique minimizer of* $\overline{\mathcal{J}}_+$ *over* $\mathcal{C}^+_{a,h,L}$ *provided by* Lemma 4.9. *Then either* $2L = (ah) \wedge a^2$ *and* $u_*(x) = 2La^{-2}x$ *for any* $x \in (0, a)$, *or the curve* $\sigma(x) := (x, u_*(x))$, $x \in [0, a]$, *is equivalent to the one parametrized by* (5.12), *where* (ξ_*, η_*) *is the unique minimizer of* Ψ *on* $\mathcal{T}$, $\xi_* < \eta_*$, *and* $h_* := y_*(\eta_*) = u_*(a^-) < h$.

Proof By Corollary 4.12 if $2L = (ah) \wedge a^2$ then $u_*(x) = 2La^{-2}x$ for any $x \in (0, a)$. Assume now that $2L < (ah) \wedge a^2$: if $(\xi_*, \eta_*) \in \operatorname{argmin}_{\mathcal{T}} \Psi$ then by Lemma 5.8 $\xi_* < \eta_*$, the unique minimizer u_* can be parametrized as in (5.12) and in particular (ξ_*, η_*) is the unique minimizer of Ψ on $\mathcal{T}$. We have only to prove that $h_* < h$. Indeed

$$
\begin{aligned}
L = \int_0^a u_*(x)\,dx &= a u_*(a^-) - \int_0^a x u_*'(x)\,dx \\
&= a h_* - \int_{\xi_*}^{\eta_*} t x_*(t) x_*'(t)\,dt = a h_* - \eta_* \frac{a^2}{2} + \frac{1}{2}\int_{\xi_*}^{\eta_*} x_*(t)^2\,dt
\end{aligned}
$$

and

$$
h_* = \int_0^a u_*'(x)\,dx = \int_{\xi_*}^{\eta_*} t x_*'(t)\,dt = a\eta_* - \int_{\xi_*}^{\eta_*} x_*(t)\,dt.
$$

By gathering together the two last relations we get

$$
h_* = \frac{2L}{a} + \int_{\xi_*}^{\eta_*} \left(x_*(t) - \frac{x_*(t)^2}{a} \right) dt \le \frac{2L}{a} < h,
$$

thus concluding the proof. □

Proof of Theorem 2.3 By Lemma 5.2 and Lemma 5.7
we get

$$
\inf \mathcal{F} = \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+ = \min_{\mathcal{T}} \Psi. \tag{5.14}
$$

Let $\gamma_* \in \mathcal{A}_{a,h,L}$ as in (2.7), x_*, y_*, u_* as in Lemma 5.9: since $h_* := u_*(a^-)$ and $0 \le \dot{u}_* \le 1$, a direct computation shows that

$$
\mathcal{F}(\gamma_*) = \int_0^a \frac{\dot{u}_*^3}{1+\dot{u}_*^2}\,dx + h - h_* = h + \int_0^a \left(\frac{\dot{u}_*^3}{1+\dot{u}_*^2} - \dot{u}_* \right) dx = \overline{\mathcal{J}}_+(u_*) = \min_{\mathcal{C}^+_{a,h,L}} \overline{\mathcal{J}}_+
$$

and the result follows easily by taking (5.14) into account. □

Remark 5.10 If we change L to $ah - L$, the unique solution is given by $\tilde{\gamma}_*$ from (2.9). Indeed, by considering the construction of the solution, this is a consequence of Remark 4.13.

The following simple lemma will be used for proving Theorem 2.5.

Lemma 5.11 *Let $\mathcal{S} := \{(\xi, \eta) : 0 \le \xi \le \eta \le 1\}$ and let $\Phi : \mathcal{S} \to \mathbb{R}$ be the function defined by (2.5). Then $\partial_\xi \Phi(\xi, \eta) > 0$ for any $(\xi, \eta) \in \mathcal{S}$ such that $0 < \xi < \eta$, $\partial_\eta \Phi(0, \eta) > 0$ for any $\eta \in (0, 1)$, and $q'(\xi) > 0$ for any $\xi \in (0, 1)$, where $q(\xi) := \Phi(\xi, \xi)$. Moreover, $\Phi(\mathcal{S}) = [0, a^2/2]$.*

Proof A computation exploiting (2.5) shows that for any $(\xi, \eta) \in \mathcal{S}$ such that $0 < \xi < \eta$ there holds

$$
\partial_\xi \Phi(\xi, \eta) = \frac{a^2 g''(\eta)}{(g'(\eta) - g'(\xi))^3} \int_\xi^\eta (g'(\eta) - g'(t))^2\,dt.
$$

Similarly, for any $\eta \in (0, 1)$ there holds

$$
\partial_\eta \Phi(0, \eta) = \frac{a^2 g''(\eta)}{g'(\eta)} \int_0^\eta \left(1 - \frac{g'(t)}{g'(\eta)}\right) dt.
$$

Positivity follows by considering the explicit expression of g from (2.3). The statement about q is obvious since $q(\xi) = \frac{1}{2} a^2 \xi$. Φ is continuous on $\mathcal{S}$ with $\Phi(0,0) = 0$ and $\Phi(1,1) = a^2/2$, thus having checked the sign of the derivatives, we conclude that $\Phi(\mathcal{S}) = [0, a^2/2]$. □

Proof of Theorem 2.5 Let us first prove the continuity of $(0, \frac{1}{2}((ah) \wedge a^2)] \ni L \mapsto \mathcal{F}_{min}(a,h,L)$. Let $\mathcal{S}$ as in Lemma 5.11. If $0 < 2L \leq (ah) \wedge a^2$ we take a sequence $(L_j) \subset (0, \frac{1}{2}((ah) \wedge a^2)) \setminus \{L\}$ such that $L_j \to L$ as $j \to \infty$. By taking advantage of Lemma 5.7 we take a couple (ξ_j, η_j) that minimizes Ψ over $\mathcal{T}_j := \{(\xi,\eta) : 0 \leq \xi \leq \eta \leq 1,\ \Phi(\xi,\eta) = L_j\}$, so that $\mathcal{F}_{min}(a,h,L_j) = \Psi(\xi_j,\eta_j)$. We extract a subsequence (not relabeled) such that $\xi_j \to \xi_\infty$, $\eta_j \to \eta_\infty$ as $j \to \infty$. By continuity of Φ over $\mathcal{S}$ we have $(\xi_\infty, \eta_\infty) \in \mathcal{T} := \{(\xi,\eta) : 0 \leq \xi \leq \eta \leq 1,\ \Phi(\xi,\eta) = L\}$.

We claim that (ξ_∞,η_∞) is a minimizer of Ψ over $\mathcal{T}$. Indeed, let us assume by contradiction that it is not, and by using Lemma 5.7 let us take a minimizer $(\hat{\xi},\hat{\eta})$ of Ψ over $\mathcal{T}$, thus $\Psi(\hat{\xi},\hat{\eta}) = \mathcal{F}_{min}(a,h,L)$ and $\Psi(\xi_\infty,\eta_\infty) - \Psi(\hat{\xi},\hat{\eta}) =: \sigma > 0$. For $\varepsilon > 0$, let $\hat{B}_\varepsilon$ denote the ε-neighbour of $(\hat{\xi},\hat{\eta})$ in $\mathcal{S}$. Thanks to Lemma 5.11, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the image of $\hat{B}_\varepsilon$ through the continuous function Φ contains the interval $(L-\delta, L+\delta) \cap [0, a^2/2]$. By using the continuity of Ψ in $\mathcal{S}$, let $\varepsilon > 0$ be small enough such that $|\Psi(\hat{\xi},\hat{\eta}) - \Psi(\xi,\eta)| < \sigma/2$ for any $(\xi,\eta) \in \hat{B}_\varepsilon$. Therefore, we can find j large enough and $(\tilde{\xi},\tilde{\eta}) \in \hat{B}_\varepsilon$ such that $\Phi(\tilde{\xi},\tilde{\eta}) = L_j$ (hence $(\tilde{\xi},\tilde{\eta}) \in \mathcal{T}_j$) and such that $|\Psi(\xi_\infty,\eta_\infty) - \Psi(\xi_j,\eta_j)| < \sigma/2$. Summarizing, we have the three relations

$$\Psi(\xi_\infty,\eta_\infty) - \Psi(\hat{\xi},\hat{\eta}) = \sigma, \quad |\Psi(\hat{\xi},\hat{\eta}) - \Psi(\tilde{\xi},\tilde{\eta})| < \sigma/2, \quad |\Psi(\xi_\infty,\eta_\infty) - \Psi(\xi_j,\eta_j)| < \sigma/2,$$

and such relations imply $\Psi(\xi_j,\eta_j) > \Psi(\tilde{\xi},\tilde{\eta})$, contradicting the minimality of (ξ_j,η_j) for Ψ on $\mathcal{T}_j$. The claim is proved, and since the minimizer of Ψ over $\mathcal{T}$ is unique by Lemma 5.8, the whole sequence (ξ_j,η_j) converges to (ξ_∞,η_∞), yielding

$$\lim_{j\to\infty} \mathcal{F}_{min}(a,h,L_j) = \lim_{j\to\infty} \Psi(\xi_j,\eta_j) = \Psi(\xi_\infty,\eta_\infty) = \mathcal{F}_{min}(a,h,L).$$

This proves the continuity of the map $L \mapsto \mathcal{F}_{min}(a,h,L)$ on $(0, \frac{1}{2}((ah) \wedge a^2))$ and the left continuity at $\frac{1}{2}((ah) \wedge a^2)$.

Let us also remark that if $h \leq a$, (5.11) yields $\Psi(h/a, h/a) = \frac{h^3}{a^2+h^2}$, therefore by the above left continuity we get $\lim_{L\uparrow ah/2} \mathcal{F}_{min}(a,h,L) = \frac{h^3}{a^2+h^2}$. Similarly, if $h > a$, still by (5.11) we have $\Psi(1,1) = h - a/2$, hence $\lim_{L\uparrow a^2/2} \mathcal{F}_{min}(a,h,L) = h - a/2$. In case $h > a$, we also have $\mathcal{F}_{min}(a,h,L) = h - a/2$ for any $L \in [a^2/2, ah/2]$ as a consequence of the characterization of the optimal energy in the nonuniqueness range, see Theorem 2.4.

We next notice that $\Phi(\xi,\eta) = 0$ implies $\xi = \eta = 0$ and the elementary estimate $\Phi(\xi,\eta) \geq \frac{a^2}{2}(\xi \vee (\eta - \xi))$ on $\mathcal{S}$ shows that $\mathcal{T}$ shrinks to the origin as L goes to 0. Since $\Psi(0,0) = h$ we obtain by (5.14) that $\lim_{L\downarrow 0} \mathcal{F}_{min}(a,h,L) = h$ by continuity of Φ and Ψ over $\mathcal{S}$.

All in all, we have proven the continuity of the map $L \mapsto \mathcal{F}_{min}(a,h,L)$ in $(0, ah/2)$, the left continuity at $ah/2$ and $\lim_{L\downarrow 0} \mathcal{F}_{min}(a,h,L) = h$. The symmetry around $L = ah/2$ follows from Remark 5.10, and then it implies continuity on $(0, ah)$.

Let us eventually discuss the monotonicity. Let $h \leq a$. Of course we have $\mathcal{F}_{min}(a,h,L) < h$ (see Lemma 3.1 and Remark 3.2). If $L < ah/2$ is increased to a close value $\overline{L}$, still with $2\overline{L} \leq ah$, from the curve that realizes the value $\mathcal{F}_{min}(a,h,L)$ we take a piecewise affine interpolating curve whose subtended area is $\overline{L}$. The energy goes down by convexity (slopes are smaller than 1). Therefore $\mathcal{F}_{min}(a,h,\overline{L}) < \mathcal{F}_{min}(a,h,L)$, proving the monotonicity. The range is $[\frac{h^3}{a^2+h^2}, h)$, as we have already obtained the continuity and the limit values at $L = 0$ and $L = ah/2$. About the case $2L \geq ah$, we obtain the desired monotonicity by making use of the symmetry of the optimal energy values around $L = ah/2$. Let now $h > a$: we cross the nonuniqueness regime as L grows from 0 to ah. The argument is the same, also taking into account that $\mathcal{F}_{min}(a, h, L) = h - a/2$ for any $L \in [a^2/2, ah - a^2/2]$ as seen in the proof of Theorem 2.4. □

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