Modeling and Estimation of Thermal Flows Based on Transport and Balance Equations *

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Heat transfer in counter-flow heat exchangers is modeled by using transport and balance equations with the temperatures of cold fluid, hot fluid, and metal pipe as state variables distributed along the entire pipe length. Using such models, boundary values problems can be solved to estimate the temperatures over all the length by means of measurements taken only at the boundaries. Conditions for the stability of the estimation error given by the difference between the temperatures and their estimates are established by using a Lyapunov approach. Toward this end, a method to construct nonlinear Lyapunov functionals is addressed by relying on a polynomial diagonal structure. This stability analysis is extended in case of presence of bounded modeling uncertainty. The theoretical findings are illustrated with numerical results, which show the effectiveness of the proposed approach.

1. Introduction

The mathematical modeling of heat exchangers is really important since these devices are employed in chemical process, petroleum refineries, power systems, and heating/refrigeration of air conditioning plants [1, 2]. In this paper, we address the modeling of a counter-flow heat exchanger based on transport and balance PDEs and use such models for the purpose to monitor the thermodynamic process by solving boundary values problems that exploit only few temperature measurements at the boundaries. A rigorous stability analysis is addressed to prove the effectiveness of the resulting temperature estimates from the theoretical and numerical points of view in line with the widespread of methods based on Lyapunov approach in a number of different applications [3–6].

Heat exchangers for industrial processes allow to recover lost energy from hot fluid streams or to heat a cold fluid for the purpose of air conditioning. Finite-dimensional models of such processes are used for fouling detection by using Kalman filtering methods [1, 2]. As compared with these models, the proposed modeling framework is in infinite dimension since it relies on hyperbolic PDEs. Based on such equations, first we will focus on a model with the dynamics of the temperatures of cold fluid, hot fluid, and metal pipe. In addition, a reduced model is considered with only the temperatures of cold and hot fluid as state variables in line with [7]. Since such temperatures

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cannot be determined by direct, distributed measurements, it is fundamental to estimate them by using a few number of measures. Toward this end, using such models, we address the solution of boundary values problems to estimate the distributed state, which provides such estimate by having at disposal measurements of temperature only at the boundaries. We will refer to also as boundary-output Luenberger observers or simply state observers (see [8] and the reference therein). Stability conditions on the estimation error (i.e., the difference between the temperatures and their estimates) will be presented to construct such estimators in case of perfect modeling and assuming bounded uncertainty. In such a case, we will rely on the notion of quadratic boundedness (QB, for short) [9]. Finally, numerical results will be given to showcase the theoretical findings.

The stability of the estimation error is analyzed in two different settings. First, we will consider perfect modeling and hence asymptotic stability results will be provided. Next, we will assume the presence of additive noises, which account for bounded modeling uncertainties and stability is studied by using QB. Generally speaking, QB allows to deal with positively invariant sets and derive upper bounds on the trajectories of the state of a dynamic system subject to bounded disturbances [9]. Thus, QB has been successfully used for both output feedback control [10] and state estimation [11]. Here we will extend the use of QB for the purpose of estimation in the considered infinite dimensional context. It is worth noting that, in lieu of searching for exact solutions of the flow equations (see, e.g., [12, 13]), here we deal with the construction of boundary-output observers with guaranteed stability properties on the estimation error.

All the stability conditions that will be presented demand the selection of nonlinear Lyapunov functionals. In this paper, we will address this task by using the SOS (sum-of-squares) approach [14, 15], which was originally developed for the analysis and control of polynomial systems described by ODEs (see [16] and the references therein) and recently applied also to systems described by PDEs [17–19]. SOS methods are quite computationally efficient since they are based on semidefinite programming (SDP) [20, 21]. In this respect, the SOS approach allows to find Lyapunov functionals just like the methods based on LMIs (linear matrix inequalities) enable to compute Lyapunov functions for linear systems based on ODE models [22]. As compared with the search of classical Lyapunov functions to investigate local stability [23], we do not rely on linearized models but provide conditions of global stability.

The paper is organized as follows. Section 2 reports the basic definitions that will be used in the following. Modeling of thermal flows in one-dimensional heat exchangers is faced in Section 3. The proposed estimation approach is presented in Section 4, where stability is analyzed in a noise-free modeling framework. This analysis is extended to models affected by bounded disturbances in Section 5. A brief account on the use of the SOS method to find the Lyapunov functionals and thus ensure stability is given in Section 6. Section 7 illustrates the numerical results, while conclusions are drawn in Section 8.

2. Notations and Definitions

The set of real numbers is denoted by $\mathbb{R}$ and hence $\mathbb{R}_+$ will be used for the set of strictly positive real numbers. The set of the integer numbers equal to or greater than zero is denoted by $\mathbb{N}$.

The minimum and maximum eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. Moreover, $P > 0$ ($P < 0$) means that it is also positive (negative) definite. Given a generic matrix M, $|M| := \left(\lambda_{\max}(M^\top M)\right)^{1/2} = \left(\lambda_{\max}(MM^\top)\right)^{1/2}$ and hence, for a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $|x| := (x^\top x)^{1/2}$ is its Euclidean norm. $\mathbb{R}[x]$ denotes the ring of real polynomials in $x \in \mathbb{R}^n$; $\mathbb{R}[x]^d$ is the set of real polynomials of degree equal or less than $d \in \mathbb{N}$.

Given the compact set $\Omega \subset \mathbb{R}^n$, $\mathscr{L}_2(\Omega)$ denotes the Hilbert space of square integrable functions $\gamma : \Omega \times [0, +\infty) \to \mathbb{R}^m$ with norm $|\gamma(\cdot,t)|_{\mathscr{L}_2} = \left(\int_\Omega |\gamma(x,t)|^2 dx\right)^{1/2} < \infty$ for all $t \geq 0$. The solution $\phi(x,t) \in \mathscr{L}_2(\Omega)$ of a PDE with initial condition $\phi(x,0) = \phi_0(x) \in \mathscr{L}_2(\Omega)$ is said to be

- $\mathscr{L}_2$ stable (to zero) if, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|\phi_0|_{\mathscr{L}_2} < \delta_\varepsilon \Rightarrow |\phi(\cdot,t)|_{\mathscr{L}_2} < \varepsilon$$

  for all $t \geq 0$;
- $\mathscr{L}_2$ asymptotically stable (to zero) if it is stable and

$$\lim_{t \to +\infty} |\phi(\cdot,t)|_{\mathscr{L}_2} = 0;$$

- $\mathscr{L}_2$ exponentially stable (to zero) if there exists $\lambda > 0$ such that

$$|\phi(\cdot,t)|_{\mathscr{L}_2} \leq c\,|\phi_0|_{\mathscr{L}_2} \exp(-\lambda t)$$

  for some $c > 0$ and all $t \geq 0$.

3. Modeling Heat Transfer in Heat Exchangers

Consider the one-dimensional model of counter-flow heat exchangers with three distributed state variables, i.e., $T_c(x,t)$, $T_h(x,t)$, and $T_m(x,t)$ for $x \in [0,L]$ and $t \geq 0$, as depicted in Fig. 1; such variables represent the temperatures of cold fluid, hot fluid, and pipe, respectively (the subscript indicating that is made of some metal in such a way to maximize the heat transfer). Based on the balance of enthalpy for the hot fluid, we get

$$\rho_h c_h S_h \frac{\partial T_h}{\partial t} + \dot{m}_h c_h \frac{\partial T_h}{\partial x} = U_h p_h (T_m - T_h) \tag{3.1}$$

where ρ_h is the density (in kg/m^3), c_h is the specific heat (in J/(kg K)), $\dot{m}_h$ is the mass flow rate (in kg/s), U_h is the transfer coefficient (in W/(m^2 K)), S_h (in m^2) is the internal surface of the pipe, and p_h its perimeter (in m). Concerning the cold fluid, the balance of enthalpy is given by

$$\rho_c c_c S_c \frac{\partial T_c}{\partial t} - \dot{m}_c c_c \frac{\partial T_c}{\partial x} = U_c p_c (T_m - T_c) \tag{3.2}$$

where ρ_c the density (in kg/m^3), c_c is the specific heat (in J/(kg K)), U_c is the transfer coefficient (in W/(m^2 K)). Moreover, S_c (in m^2) is the external surface of the pipe, and p_c its perimeter (in m). Finally, for the metal pipe it follows that

$$\rho_m c_m S_m \frac{\partial T_m}{\partial t} = U_h p_h (T_h - T_m) + U_c p_c (T_c - T_m) \tag{3.3}$$

where ρ_m is the density (in kg/m^3) of the metal, c_m is the specific heat (in J/(kg K)) of the metal, and $S_m := S_c - S_h$ (in m^2) is the surface of the pipe section.

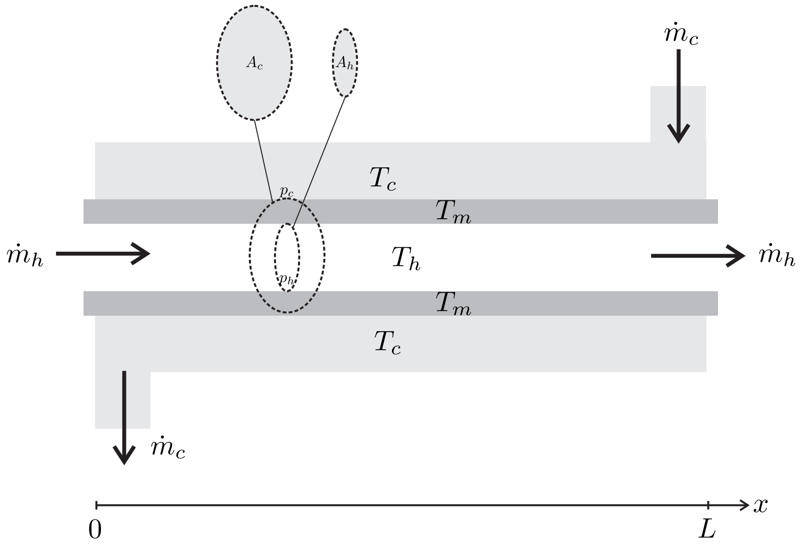

Fig. 1. Sketch of the heat exchanger model.

After dividing each of the three previous equations for the first coefficient in each respective l.h.s., we get

$$\frac{\partial T_h}{\partial t} + a_1 \frac{\partial T_h}{\partial x} = b_1 (T_m - T_h) \tag{3.4a}$$

$$\frac{\partial T_c}{\partial t} - a_2 \frac{\partial T_c}{\partial x} = b_2 (T_m - T_c) \tag{3.4b}$$

$$\frac{\partial T_m}{\partial t} = b_3 (T_h - T_m) + b_4 (T_c - T_m) \tag{3.4c}$$

where

$$a_1 = \frac{\dot{m}_h}{\rho_h S_h} \quad a_2 := \frac{\dot{m}_c}{\rho_c S_c} \quad b_1 := \frac{U_h p_h}{\rho_h c_h S_h} b_2 := \frac{U_c p_c}{\rho_c c_c S_c} \quad b_3 := \frac{U_h p_h}{\rho_m c_m S_m} \quad b_4 := \frac{U_c p_c}{\rho_m c_m S_m}$$

or, more concisely, as follows

$$\partial_t T + A_1 \, \partial_x T = B_1 \, T \tag{3.5}$$

where $T(x,t) := \mathrm{col}\,(T_h(x,t), T_c(x,t), T_m(x,t)) \in \mathbb{R}^3$ and

$$A_1 := \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B_1 := \begin{pmatrix} -b_1 & 0 & b_1 \\ 0 & -b_2 & b_2 \\ b_3 & b_4 & -b_3 - b_4 \end{pmatrix} .$$

The system (3.5) is solved with initial conditions

$$T_h(x,0) = T_h^0(x),\, T_c(x,0) = T_c^0(x),\, T_m(x,0) = T_m^0(x)\,,\, x \in [0,L] \tag{3.6}$$

and Dirichlet boundary conditions

$$T_h(0,t) = \overline{T}_h(0,t),\, T_c(L,t) = \overline{T}_c(L,t) \tag{3.7}$$

for all $t \geq 0$. The boundary conditions on T_m are not necessary since the equation (3.4c) does not contain spatial derivatives of T_m.

Likewise in [7], a reduced model can be easily derived from (3.5) by neglecting the dynamics of the temperature in the pipe, i.e., assuming that such a temperature is fixed by the values of the temperatures of cold and hot fluids, i.e., imposing that the r.h.s. of (3.3) is null. Therefore, we obtain

$$\begin{aligned} \frac{\partial T_h}{\partial t} + a_1 \frac{\partial T_h}{\partial x} &= -\frac{b_1 b_4}{b_3 + b_4} T_h + \frac{b_1 b_4}{b_3 + b_4} T_c \\ \frac{\partial T_c}{\partial t} - a_2 \frac{\partial T_c}{\partial x} &= \frac{b_2 b_3}{b_3 + b_4} T_h - \frac{b_2 b_3}{b_3 + b_4} T_c \end{aligned}$$

or, more compactly,

$$\partial_t T + A_2 \, \partial_x T = B_2 \, T \tag{3.8}$$

where $T(x,t) := \mathrm{col}\,(T_h(x,t), T_c(x,t)) \in \mathbb{R}^2$ and

$$A_2 := \begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix} \quad B_2 := \frac{1}{b_3 + b_4} \begin{pmatrix} -b_1 b_4 & b_1 b_4 \\ b_2 b_3 & -b_2 b_3 \end{pmatrix}$$

with initial conditions

$$T_h(x,0) = T_h^0(x),\, T_c(x,0) = T_c^0(x)\,,\, x \in [0,L] \tag{3.9}$$

and Dirichlet boundary conditions

$$T_h(0,t) = \overline{T}_h(0,t),\, T_c(L,t) = \overline{T}_c(L,t) \tag{3.10}$$

for all $t \geq 0$.

The model (3.5) is of hyperbolic type but not strictly hyperbolic since the matrix A_1 has a zero eigenvalue. By contrast, (3.8) is strictly hyperbolic since the eigenvalues of A_2 have non null real part. In the next section, we will consider the problem to reconstruct all the temperatures by using a generic hyperbolic model.

4. Estimation for Heat Exchangers with Stable Estimation Error

Consider the hyperbolic equation

$$\partial_t T + A \, \partial_x T = B \, T \tag{4.1}$$

where $T(x,t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ diagonal and $B \in \mathbb{R}^{n \times n}$. Consider also the observer for (4.1) given by

$$\partial_t \hat{T} + A \, \partial_x \hat{T} = B \, \hat{T} \tag{4.2}$$

where $\hat{T}(x,t) \in \mathbb{R}^n$ is the estimate of $T(x,t)$. Such PDEs need some suitable boundary conditions, which will be given later. The stability of the estimation error can be guaranteed under suitable conditions as follows.

Theorem 4.1. *The estimation error $e(x,t) := T(x,t) - \hat{T}(x,t) \in \mathbb{R}^n$ is $\mathscr{L}_2$ asymptotically stable if there exists a diagonal matrix $P(x) \in \mathbb{R}^{n\times n}$ with $P(x) > 0$ for all $x \in [0,L]$ such that*

$$e(0,t)^\top AP(0)\,e(0,t) - e(L,t)^\top AP(L)\,e(L,t) \leq 0 \tag{4.3}$$

and

$$A\,\partial_x P(x) + B^\top P(x) + P(x)B < 0 \tag{4.4}$$

for all $x \in [0,L]$.

Proof. Consider the Lyapunov functional

$$V(t) := \int_0^L e(x,t)^\top P(x)\,e(x,t)\,dx \tag{4.5}$$

and note that the asymptotic stability of the estimation error is ensured if

$$\dot{V}(t) < 0 \tag{4.6}$$

for all $e(x,t) \in \mathbb{R}^n$ since $P(x) > 0$ for all $x \in [0,L]$. Since P does not depend on time, we get

$$\dot{V}(t) = \int_0^L -\partial_x e^\top A^\top Pe - e^\top PA\,\partial_x e + e^\top (B^\top P + PB)e\,dx$$

and, owing to the diagonal structure of A and P,

$$-\partial_x e^\top A^\top Pe - e^\top PA\partial_x e = -\partial_x(e^\top APe) + e^\top A\partial_x Pe\,.$$

Thus, it follows

$$\dot{V}(t) = -e(L,t)^\top AP(L)\,e(L,t) + e(0,t)^\top AP(0)\,e(0,t) + \int_0^L e^\top (A\partial_x P + B^\top P + PB)e\,dx$$

and hence (4.6) holds owing to (4.3) and (4.4). □

Note that if, instead of (4.4), we consider

$$A\,\partial_x P(x) + B^\top P(x) + P(x)B + \alpha P(x) < 0 \tag{4.7}$$

with $\alpha > 0$ for all $x \in [0,L]$, the estimation error is $\mathscr{L}_2$ exponentially stable with rate of decrease equal to α. Thus, a design goal may consist in maximizing α, which can be regarded as a sort of generalized eigenvalue problem for Lyapunov functionals instead of Lyapunov functions [22].

In case of the reduced model (3.8) (i.e., with $A = A_2$ and $B = B_2$) the stability conditions of Theorem 4.1 can be greatly simplified according to [7] by using

$$P(x) := \mathrm{diag}(\exp(\mu_1 x), \exp(\mu_2 x)) \tag{4.8}$$

and boundary conditions

$$\hat{T}_h(0,t) = T_h(0,t) + \ell_1(T_c(0,t) - \hat{T}_c(0,t)) \tag{4.9a}$$
$$\hat{T}_c(L,t) = T_c(L,t) + \ell_2(T_h(L,t) - \hat{T}_h(L,t)) \tag{4.9b}$$

for all $t \geq 0$ with $\ell_1, \ell_2, \mu_1, \mu_2 \in \mathbb{R}$ to be suitably chosen. More specifically, in [7] it is proposed to satisfy (4.4) by using an LMI-based discretized approach to get μ_1, μ_2 and then select ℓ_1, ℓ_2 such that

$$a_2\ell_2^2\exp(\mu_2 L) - a_1\exp(\mu_1 L) \leq 0$$
$$a_1\ell_1^2 - a_2 \leq 0$$

by choosing, for example,

$$\ell_1 = \sqrt{\frac{a_2}{a_1}} \quad \ell_2 = \sqrt{\frac{a_1}{a_2}}\exp((\mu_1-\mu_2)L/2)\,. \tag{4.10}$$

The construction of Lyapunov functionals is not an easy task to solve in general and will be addressed later in Section 6.

5. Estimation Under Modeling Uncertainties

Let us focus on estimation in the presence of bounded disturbances. More specifically, instead of (4.1) we consider

$$\partial_t T + A\,\partial_x T = B\,T + D\,w \tag{5.1}$$

where $D \in \mathbb{R}^{n\times q}$ and $w(x,t) \in \mathbb{R}^q$ such that $|w_i(x,t)| \leq 1$, $i = 1,\ldots,q$ for all $x \in [0,L]$ and $t \geq 0$ without loss of generality (different upper bounds can be taken into account by suitably scaling the coefficients of D), where q is an integer number no larger than n. Estimation will be performed by using the same observer adopted in the noise-free case, i.e., (4.2).

The presence of the system noises prevents from ensuring asymptotic stability but, since such disturbances are bounded, an invariant set for the estimation error exists and can be studied by using QB [9].

Toward this end, first of all we need to define QB in the $\mathscr{L}_2$ sense. More specifically, the estimation error is said to be $\mathscr{L}_2$ quadratically bounded with Lyapunov functional $V(t)$ as defined in (4.5) if

$$V(t) > L \Rightarrow \dot{V}(t) < 0\,, \forall w \in [-1,1]^q\,. \tag{5.2}$$

As a matter of fact, any positive constant can be chosen instead of L, but with such a choice we reduce the notational burden. This does not entail loss of generality since the conditions ensuring QB are homogeneous in the design parameters, which scales with L and thus allows for this simplification.

Owing to (5.2), the set

$$\mathscr{E} := \left\{ e \in \mathscr{L}_2([0,L]) : \int_0^L e(x)^\top P(x)\,e(x)\,dx \leq L \right\}$$

turns out to be positively invariant and it is attractive (i.e., if the error is out of $\mathscr{E}$, it approaches $\mathscr{E}$ asymptotically).

Theorem 5.1. *The estimation error is quadratically bounded if there exist a diagonal matrix $P(x) \in \mathbb{R}^{n\times n}$ with $P(x) > 0$ for all $x \in [0,L]$ and $\alpha \in \mathbb{R}^q_{>0}$, $\beta > 0$ such that* (4.3) *and*

$$\begin{pmatrix} A\,\partial_x P(x) + B^\top P(x) + P(x)B + \beta P(x) & P(x)D \\ D^\top P(x) & -\mathrm{diag}(\alpha) \end{pmatrix} < 0 \tag{5.3}$$

$$\sum_{i=1}^{q} \alpha_i - \beta \le 0\,. \tag{5.4}$$

hold for all $x \in [0,L]$.

Proof. Likewise in the proof of Theorem 4.1, we easily compute the time derivative of the Lyapunov functional (4.5) and get that $\dot V(t) < 0$ subject to (4.3) is equivalent to

$$e^\top (A\,\partial_x P + B^\top P + PB)e + w^\top D^\top Pe + e^\top PDw < 0 \tag{5.5}$$

Since $-e^\top Pe + 1 > 0$ implies $V(t) > L$ and $-w_i^2 + 1 > 0$ holds, using the well-known S-procedure (see [22, p. 23]) we obtain that that QB holds if

$$e^\top (A\,\partial_x P + B^\top P + PB + \beta P)e + w^\top D^\top Pe + e^\top PDw - \sum_{i=1}^{q} \alpha_i w_i^2 + \sum_{i=1}^{q} \alpha_i - \beta < 0 \tag{5.6}$$

for some $\alpha \in \mathbb{R}^q_{>0}$ and $\beta > 0$ or equivalently if (5.3) and (5.4) hold. □

Clearly, (5.3) implies

$$A\,\partial_x P(x) + B^\top P(x) + P(x)B + \beta P(x) < 0$$

for all $x \in [0,L]$, which ensures that the estimation error in the absence of noises is $\mathscr{L}_2$ exponentially stable with rate of decrease equal to β. Thus, a convenient design goal may consist in maximizing β, as pointed out in [7]. Another, quite popular objective of design is related to the $\mathscr{L}_2$ gain between disturbance and estimation error.

Theorem 5.1 provides conditions ensuring that the set $\mathscr{E}$ is attractive. Thus, one can design the estimator in such a way to keep $\mathscr{E}$ as small as possible. Toward this end, note that

$$\lambda_{\min}(P(x))\,|e|^2 \le e^\top P(x)\,e\,,\; x \in [0,L]$$

for all $e \in \mathscr{E}$ and hence, as a design objective, we may maximize the minimum eigenvalue of $P(x)$ since

$$|e(x,t)| \le \frac{1}{\sqrt{\lambda_{\min}(P(x))}}$$

for all $t \ge 0$, $x \in [0,L]$. The inequality above allows to prove the boundedness in the $\mathscr{L}_2$ sense as

$$|e(\cdot,t)|_{\mathscr{L}_2} \le \frac{L}{\sqrt{\min\limits_{x\in[0,L]} \lambda_{\min}(P(x))}}$$

for all $t \ge 0$.

To compare the effectiveness of a given observer in terms of rejection of the noise, it is quite popular to rely on the notion of the $\mathscr{L}_2$ gain by assuming finite-energy noises, i.e., $w(\cdot,t) \in \mathscr{L}_2(\Omega)$.

More specifically, we say that the proposed observer admits and $\mathscr{L}_2$ gain between disturbance and estimation error equal to $\gamma > 0$ if

$$\int_0^T |e(\cdot,t)|_{\mathscr{L}_2}\, dt \le \gamma^2 \int_0^T |w(\cdot,t)|_{\mathscr{L}_2}\, dt$$

for $T > 0$. It is straightforward to show that the bound of the $\mathscr{L}_2$ gain holds if

$$\begin{pmatrix} A\,\partial_x P(x) + B^\top P(x) + P(x)B + \frac{1}{\gamma^2} I & P(x)D \\ D^\top P(x) & -I \end{pmatrix} < 0\,. \tag{5.7}$$

Note also that this condition is quite similar to (5.3). Moreover, minimizing γ is equivalent to maximize $1/\gamma^2$ in accordance with the goal of maximizing β.

6. Search of Lyapunov Functionals Using the SOS Approach

Stability conditions such as (4.4), (4.7), (5.3), or (5.7) (each together with $P(x) > 0$) demand the satisfaction of LMIs in $P(x)$ for all $x \in [0,L]$, i.e., the solution of semi-infinite programming problems. Such problems arise in a variety of applications and are usually approximately solved in discretized form on a sufficiently fine mesh of points (see, e.g., [24]). The main difficulty to address semi-infinite programming problems concerns both the choice of a sufficiently large number of points and especially the local minima trapping, by which nonlinear programming solvers may be affected in trying to find the solution. A more appealing way to solve such problems consists in resorting to the SOS paradigm, which enables to turn the construction of a Lyapunov functional into a convex problem that can be efficiently solved by using SDP without encountering such issues due to local minima.

The idea behind such an approach is the SOS decomposition of a candidate Lyapunov functional as well as of the opposite of its time derivative by using a positivity certification, which does not depend on the characteristics of the chosen polynomial [21, 25]. More specifically, the following result holds.

Theorem 6.1. *A polynomial $p(x) \in \mathbb{R}[x]^{2d}$ in $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ has sum-of-squares decomposition (or is said to be SOS) if and only if there exists a real symmetric and positive semidefinite matrix $Q \in \mathbb{R}^{s(d)\times s(d)}$ such that$p(x) = v_d(x)^\top Q\, v_d(x)$, where $v_d(x)$ is the vector of all the monomials in the components of $x \in \mathbb{R}^n$ of degree equal to or less than $d \in \mathbb{N}$, i.e.,*

$$v_d(x) := (1, x_1, \dots, x_n, x_1x_2, \dots, x_{n-1}x_n, x_n^2, \dots, x_1^d, \dots, x_n^d)$$

of dimension

$$s(d) := \binom{n+d}{d}\,.$$

Proof. See [26, Proposition 2.1, p. 17]. □

Consider, for example, the estimation of the state variables of (3.4) by using the results of Theorem 4.1. Based on Theorem 6.1, a procedure can be adopted to find suitable SOS polynomials as diagonal elements of $P(x) = \mathrm{diag}\,(P_{11}(x), P_{22}(x), P_{33}(x))$, all to be taken in $\mathbb{R}[x]^{2d}$ with $x \in \mathbb{R}$ for increasing values of d. Toward this end, we say that $p(x) \in \mathbb{R}[x]^{2d}$ is ε-SOS polynomial if $p(x) - \varepsilon \sum_{i=1}^n x_i^2 \in \mathbb{R}[x]^{2d}$ with $d \in \mathbb{N}$, $d \ge 1$ is SOS for some "small" tolerance $\varepsilon > 0$ [27]. In addition, a polynomial, square matrix $M(x)$ with $M_{ij}(x) \in \mathbb{R}[x]^{2d}$ for $i, j = 1, \ldots, m$ is said to be an ε-SOS matrix if $y^\top M(x) y$ is ε-SOS in $\mathbb{R}[x, y]$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}^m$.

Given $d \in \mathbb{N}$ with $d \geq 1$ and $\varepsilon > 0$, the problem to solve when dealing, for example, with the stability conditions of Theorem 4.1 is the following:

Problem 6.1. *Find* $P_{11}(x), P_{22}(x), P_{33}(x) \in \mathbb{R}[x]^{2d}$ *such that*

$$P_{11}(x), P_{22}(x), P_{33}(x) \text{ are } \varepsilon\text{-SOS} \tag{6.1}$$

$$\begin{aligned} &-A \operatorname{diag}\left(P'_{11}(x), P'_{22}(x), P'_{33}(x)\right) - B^\top \operatorname{diag}\left(P_{11}(x), P_{22}(x), P_{33}(x)\right) \\ &-\operatorname{diag}\left(P_{11}(x), P_{22}(x), P_{33}(x)\right) B \quad \text{is a } \varepsilon\text{-SOS matrix} \end{aligned} \tag{6.2}$$

for all $x \in [0, L]$.

If Problem 6.1 admits a solution, we can follow the same reasoning that leads to (4.10) to ensure the satisfaction of (4.3) by choosing

$$\ell_1 = \sqrt{\frac{a_2 P_{22}(0)}{a_1 P_{11}(0)}} \quad \ell_2 = \sqrt{\frac{a_1 P_{11}(L)}{a_2 P_{22}(L)}}\,. \tag{6.3}$$

Similar problems may be formulated by replacing the SOS description of (6.2) with those corresponding to (4.7), (5.3), and (5.7).

7. Numerical Results

For the sake of brevity, in the following we will describe only the numerical solution of the boundary value problem that results from the discretization of the model (3.5) since this includes that of the reduced model (3.8) as a special case. We have discretized the domain $[0, L]$ with M equally spaced points x_j for $j = 1, \ldots, M$, both (3.8) and (4.2) are treated by using a finite difference method. The equation (3.4a) and (3.4b) composed by an advection term ($a_1 \partial_x T_h$ and $-a_2 \partial_x T_c$) and a reaction one (the right side of the equalities) are semi-discretized in space by an upwind finite difference scheme [28–30]. The choice of an upwind scheme is more appropriate for the convective term in which there is a propagation direction. The boundary conditions (3.10) for (3.8) (and (4.9) for (4.2)) are given for T_h and T_c only in the left ($x = 0$) and right ($x = L$) part of the domain, respectively. The term $a_1 \partial_x T_h$ in (3.4a) is therefore discretized by using backward finite difference since a_1 is positive, whereas the term $-a_2 \partial_x T_c$ in (3.4b) is discretized by forward one since $-a_2$ is negative.

Setting by $(T_h)_j(t)$ the solution at the discretization point x_j at time t (and analogously for T_c and T_m), we obtain the following system of ODEs:

$$(T_h)'_j(t) = -a_1 \left[(T_h)_j(t) - (T_h)_{j-1}(t)\right] + b_1 \left[(T_m)_j(t) - (T_h)_j(t)\right],\ j = 2, \ldots, M \tag{7.1a}$$

$$(T_c)'_j(t) = a_2 \left[(T_c)_{j+1}(t) - (T_c)_j(t)\right] + b_2 \left[(T_m)_j(t) - (T_c)_j(t)\right],\ j = 1, \ldots, M-1 \tag{7.1b}$$

$$(T_m)'_j(t) = b_3 \left[(T_h)_j(t) - (T_m)_j(t)\right] + b_4 \left[(T_c)_j(t) - (T_m)_j(t)\right],\ j = 1, \ldots, M \tag{7.1c}$$

$$(T_h)'_1(t) = 0 \tag{7.1d}$$

$$(T_c)'_M(t) = 0 \tag{7.1e}$$

with initial conditions

$$(T_h)_j(0) = T_h^0(x_j),\ (T_c)_j(0) = T_c^0(x_j)\,.$$

The boundary conditions (3.10) are translated in the system by setting

$$(T_h)_1(t) = \overline{T}_h(0,t),\, (T_c)_M(t) = \overline{T}_c(L,t)\,.$$

The temperatures T_h and T_c at the initial time are chosen constant and equal to the values at the boundary points where their measures are available, i.e., equals to $\overline{T}_h$ in $x=0$ and $\overline{T}_c$ in $x=L$ such as

$$T_h^0(x_j) = \overline{T}_h(0,0),\, T_c^0(x_j) = \overline{T}_c(L,0),\, j=1,\dots,M$$

with a constant value belonging to the interval $[T_h, T_c]$ taken as initial conditions for T_m, i.e.,

$$T_m^0(x_j) = \overline{T}_m,\, j=1,\dots,M\,.$$

The scheme for the observer (4.2) is the same by replacing $(T_h)_j(t)$ with $(\hat{T}_h)_j(t)$, $(T_c)_j(t)$ with $(\hat{T}_c)_j(t)$ and so on, except for the boundary conditions, which are the following:

$$\begin{aligned}(\hat{T}_h)_1(t) &= \overline{T}_h(t) + l_1[(T_c)_1(t) - (\hat{T}_c)_1(t)]\\ (\hat{T}_c)_M(t) &= \overline{T}_c(t) + l_2[(T_h)_M(t) - (\hat{T}_h)_M(t)]\,.\end{aligned}$$

For the observer, the initial conditions in the simulation are chosen as:

$$\begin{aligned}\hat{T}_h^0(x) &= \overline{T}_h(0,0) + 3\sin(1.2\pi x)\\ \hat{T}_c^0(x) &= 6\sin(1.2\pi x) - 9.539x + 0.535x^2 + 18\\ \hat{T}_m^0(x) &= \overline{T}_m\,.\end{aligned}$$

The system of ODEs (7.1) is solved by using a fourth-order Runge-Kutta method (implementated in the *ode45* function of *Matlab*) with variable time step. In the first instants of the simulation the fast dynamics of the convective part prevails, while later the reaction term predominates. To accurately simulate the model, it is therefore necessary to select a very small time step in the first time instants, while it may be chosen larger afterwards but, of course, always respecting the stability condition of the numerical scheme. Implicit methods (e.g., the scheme implemented in the routine *ode15s* of *Matlab*) are tested, since they generally allow to use a larger time step. The running time is resulted longer, due to the higher complexity to complete each iteration. Then, explicit methods seem to be the most appropriate choice for this problem.

In the numerical case study, we have fixed $L=5$ m, $a_1=1$ m/s, $a_2=1$ m/s, $b_1=0.01$ 1/s, $b_2=0.01$ 1/s, $b_3=0.01$ 1/s, and $b_4=0.01$ 1/s. Concerning the proposed estimation approach with (3.5) and (3.8), we have solved Problem 6.1 by using the SOS toolbox [27] and the numerical solver Yalmip [31] with $\varepsilon=10^{-6}$. In the first case (i.e., $A=A_1$ and $B=B_1$) we have got a solution to such a problem with $d=1$ given by

$$\begin{aligned}P(x) = \mathrm{diag}\big(&(1.096\cdot 10^{-4}x + 1.017)^2 + (0.0017x - 1.096\cdot 10^{-4})^2\,,\\ &(2.8477\cdot 10^{-4}x + 0.9864)^2 + (0.0017x + 2.8477\cdot 10^{-4})^2\,,\\ &(1.4496\cdot 10^{-4}x + 1.0093)^2 + (0.0017x + 1.4496\cdot 10^{-4})^2\big)\end{aligned}$$

and select ℓ_1, ℓ_2 according to (6.3). The same has been done for the reduced model with $A=A_2$ and $B=B_2$ by obtaining

$$\begin{aligned}P(x) = \mathrm{diag}\big(&(1.9955\cdot 10^{-4}x + 0.9546)^2 + (0.001x + 1.9955\cdot 10^{-4})^2\,,\\ &(4.1525\cdot 10^{-4}x + 0.9124)^2 + (0.001x + 4.1525\cdot 10^{-4})^2\big)\end{aligned}$$

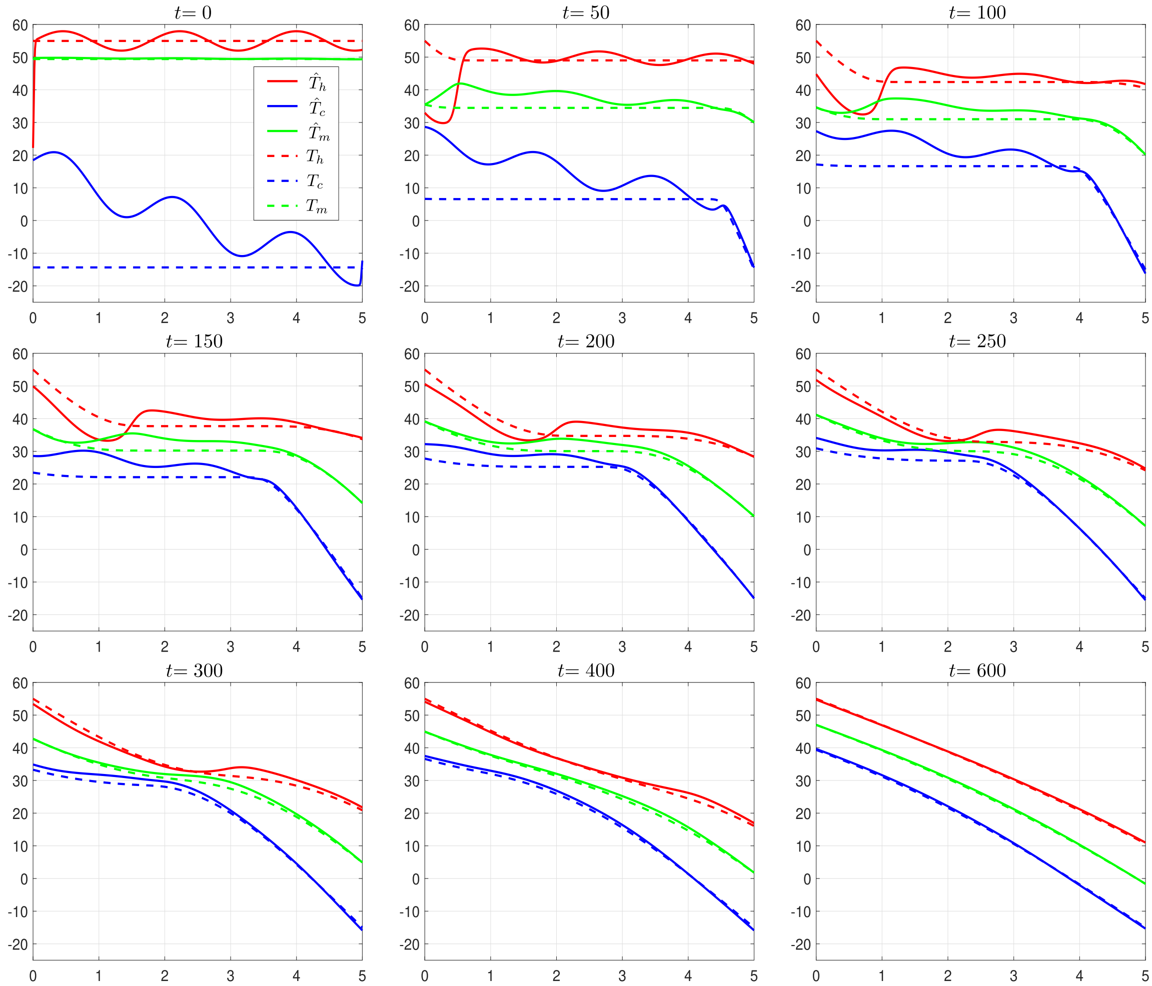

Fig. 2. Simulation run in a noise-free case at different time steps for the model (3.5).

and taking ℓ_1, ℓ_2 again according to (6.3). We have chosen $M = 500$, $\overline{T}_h(0,0) = 55$, $\overline{T}_c(L,0) = -15$, $\overline{T}_m = 50$.[a]

Fig. 2 shows the behavior of the temperatures T_h, T_c, and T_m and their estimates given by $\hat{T}_h$, $\hat{T}_c$, and $\hat{T}_m$ at different time instants, all based on (3.5). Fig. 3 shows the behavior of the temperatures T_h, T_c and the corresponding estimates $\hat{T}_h$, $\hat{T}_c$, based on the reduced model (3.8).

8. Conclusions

Two models have been presented to account for the temperature dynamics in heat exchangers. Such models are based on hyperbolic PDEs with the most complete having the temperatures of cold fluid, hot fluid, and pipe as state variables, whereas the reduced one relies only on the temperatures of cold and hot fluids. Estimators resulting from the solution of boundary values problems based

[a]The simulation code is available upon request to Angelo Alessandri, email: alessandri@dime.unige.it.

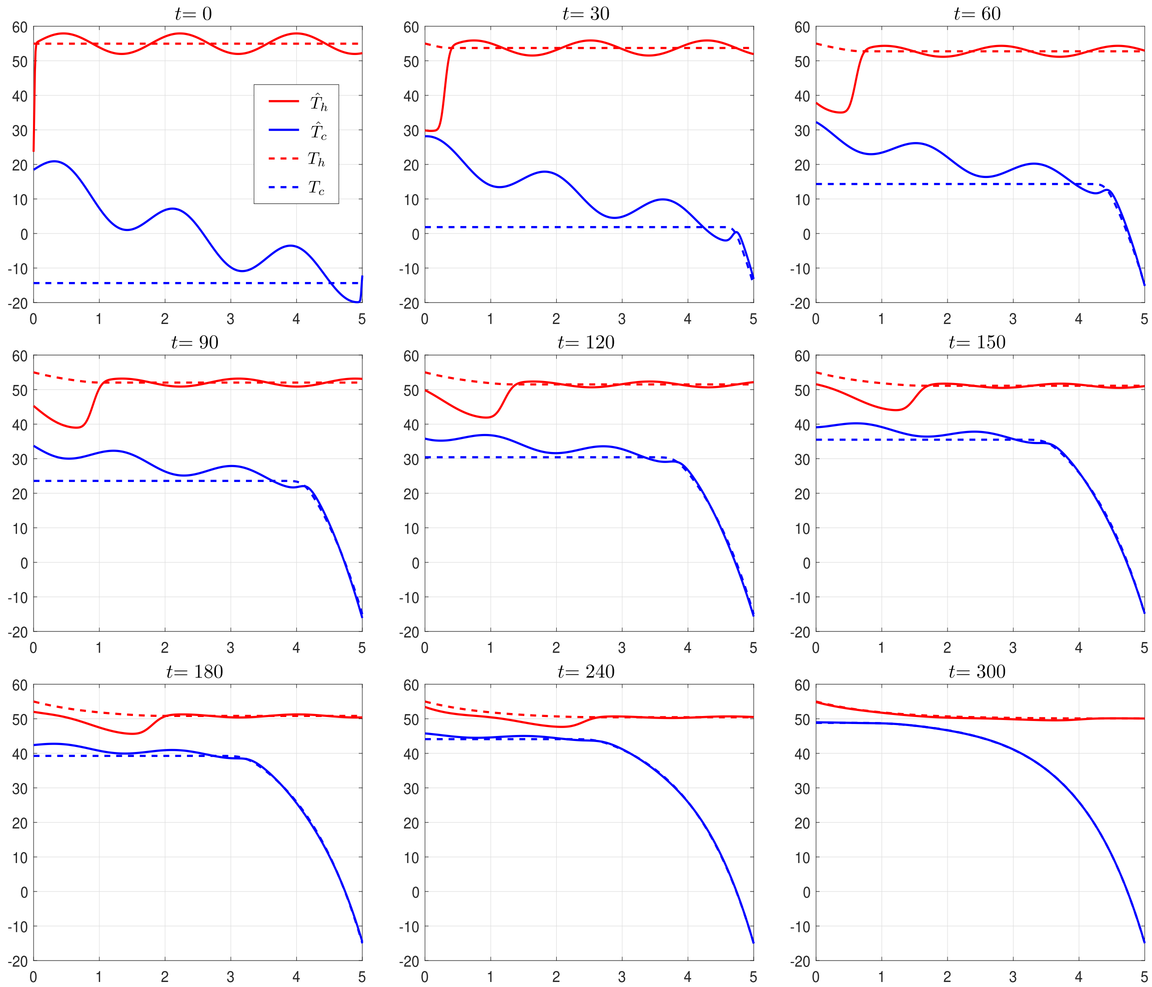

Fig. 3. Simulation run in a noise-free case at different time steps for the reduced model (3.8).

on such models have been proposed and provided with a rigorous stability analysis with the Lyapunov approach. To this end, the existence of nonlinear polynomial Lyapunov functionals has been addressed by using the SOS approach.

Future work will concern the investigation on the stability of systems described by nonlinear hyperbolic equations (see, e.g., [32]). Another direction of research is the study of stability of cascaded hyperbolic and other nonlinear PDEs such as the normal flow equation to deal with multi-phase problems, which turn out to be increasingly difficult but with a wide range of potential applications [33].

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