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## USING THREE-PARAMETER SINE FITTING FOR SCALING MODE SHAPES WITH OMAH

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### ABSTRACT

Scaling mode shapes obtained by operational modal analysis (OMA) is sometimes desirable, for example if the modal model is going to be used for model calibration, or for some structural health monitoring applications. A new method for scaling mode shapes from OMA, utilizing harmonic excitation and called OMAH, was recently proposed by the authors. In the present paper, we briefly describe this new technique, and then investigate a particular method, the three-parameter sine fit technique, for estimating the harmonic force and responses under different signal-to-noise (SNR) cases. This technique is shown to provide good performance even in severe SNR cases, and the method also has the advantage that the accuracy of the estimate of the harmonics can be estimated alongside the harmonic estimates, and can be improved by adding more data to the measurement. This allows to develop an iterative technique, allowing to estimate the harmonics iteratively, until a sufficient accuracy is obtained. The method is demonstrated on simulated OMA measurement data.

*Keywords: operational modal analysis, mode shape scaling, OMAH*

### 1. INTRODUCTION

Obtaining a scaled modal model is essential in many applications, for example to be able to use it for designing tuned absorbers, or for some structural health monitoring or model calibration applications. Operational modal analysis (OMA), naturally results in unscaled modal models, as there is no knowledge about the loads acting on the structure. Thus, different methods for scaling mode shapes obtained by OMA have earned a lot of attention lately. These methods can essentially be divided into two subclasses

- methods based on making repeated OMA tests with different configurations (mass and/or stiffness), [13, 3, 12, 6, 10], or

- methods based on applying broadband excitation to the structure in addition to the data acquisition for the OMA analysis, so-called OMAX techniques [9, 8, 14, 15, 7].

Whereas the methods in the first group above are very interesting, as they do not require any artificial excitation, they are also rather elaborate and slow to apply. Some of the methods in the second group, on the other hand, require artificial excitation through an actuator capable of providing broadband excitation. For large structures, this may be difficult to achieve. Some studies, however, such as [7], have utilized human interaction on footbridges, with reasonable results, thus eliminating having to install an artificial actuator.

A special application of the OMAX method, utilizing a shaker providing harmonic excitation rather than broadband, was recently introduced by the authors [5]. It was shown that the method worked well by demonstrating it on a staircase. The method, which was given the name OMAH for *operational modal analysis with harmonic scaling*, has the advantage that it is readily applicable to structures of almost any size, such as highrise buildings and bridges, since actuators able to produce harmonic excitation are relatively inexpensive, even when producing relatively large forces. A harmonic force also has the advantage that it can be estimated even at a very low level compared to the natural vibrations present. This will be further investigated and discussed in the present paper.

## 2. OMAH THEORY

The OMAH technique is based on the expression of a general frequency response between two points,  $p$ , and  $q$ , given by [4]

$$H_{p,q}(j\omega) = \sum_{r=1}^{N_m} \frac{\psi_r^p \psi_r^q}{m_r(j\omega - s_r)(j\omega - s_r^*)} \quad (1)$$

where  $\omega$  is the circular frequency,  $j$  is the imaginary unit,  $N_m$  is the number of modes of the structure,  $m_r$  is the modal mass of mode  $r$ ,  $s_r$  the pole of mode  $r$ , and  $*$  denotes complex conjugation. Finally,  $\psi_r^p$  and  $\psi_r^q$  denote the mode shape coefficients of mode  $r$  in points  $p$  and  $q$ , respectively. The poles  $s_r$  are expressed by  $s_r = -\zeta_r \omega_r + j\omega_r \sqrt{1 - \zeta_r^2}$ , where  $\omega_r$  and  $\zeta_r$  are the eigenfrequency and non-dimensional damping ratio of mode  $r$ , respectively.

The modal mass,  $m_r$  in Eq. (1) is unknown when the modal parameters are obtained by OMA. Scaling of a particular mode shape,  $r$ , can (in the simplest way) thus be obtained by knowledge of the mode shape coefficients  $\psi_r^p$  and  $\psi_r^q$ , the pole,  $s_r$ , from an OMA test, and the frequency response  $H_{p,q}(j\omega)$ , from an extra measurement where the structure is excited by a harmonic force, by solving for the modal mass  $m_r$  in Eq. (1). Assuming the structure is excited by a harmonic force  $F(t) = F_q(j\omega) \exp(j\omega t)$ , and the response in degree of freedom (DOF)  $p$  is  $u_p(t) = U_p(j\omega) \exp(j\omega t)$ , then the task is to find the complex constants  $F_q(j\omega)$  and  $U_p(j\omega)$ . A suitable method for this purpose is the so-called *three-parameter sine fit* technique [1, 11]. It should be noted that [5] includes a more general discussion of how to solve for the modal mass; to keep the present paper concise, we refer to that paper for details.

### 2.1. Three-parameter sine fit

The three-parameter sine fit method assumes that the frequency is known, and that we thus have three parameters to fit for a sine, namely amplitude, phase, and mean value. Without loss of generality, we will assume that the mean is zero, as we are dealing with vibration signals. This is easily established by removing any offset prior to estimating the sines (of force and response). We therefore assume that we have an unknown signal of the general form

$$y(t) = a \cos(\omega t) + b \sin(\omega t) + e(t) \quad (2)$$

where  $\omega$  is the frequency of the sine,  $a$  and  $b$  are the unknown Fourier series coefficients we wish to estimate, and  $e(t)$  contains the natural response of the structure, and possibly higher harmonic components, if there is harmonic distortion in the excitation. If we want the complex amplitude/phase values, they are readily obtained by  $\sqrt{a^2 + b^2} \cdot \exp(j \arctan(b/a))$ . If we measure  $N$  samples of the signal  $y(n \cdot \Delta t)$  with a sampling frequency of  $f_s = 1/\Delta t$ , then we can define

$$\mathbf{A} = \begin{bmatrix} \cos(\omega \cdot 0\Delta t) & \sin(\omega \cdot 0\Delta t) \\ \cos(\omega \cdot 1\Delta t) & \sin(\omega \cdot 1\Delta t) \\ \cos(\omega \cdot 2\Delta t) & \sin(\omega \cdot 2\Delta t) \\ \vdots & \vdots \\ \cos(\omega \cdot (N-1)\Delta t) & \sin(\omega \cdot (N-1)\Delta t) \end{bmatrix} \quad (3)$$

and the unknown coefficient vector  $\mathbf{x} = [a \quad b]^T$ , and the measurement vector

$$\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N-1) \end{bmatrix}. \quad (4)$$

The model in Eq. (2), can now be written as:

$$\mathbf{Ax} = \mathbf{y} + \mathbf{e} \quad (5)$$

which we can solve Eq. (5) for the estimates  $\hat{\mathbf{x}}$  by a least squares solution. In addition, the result of the least squares solution can be used to estimate the remaining noise, by:

$$\hat{\mathbf{e}} = \mathbf{y} - \mathbf{A}\hat{\mathbf{x}} \quad (6)$$

and the variance of  $\hat{\mathbf{e}}$ , which we denote  $\sigma_{\hat{\mathbf{e}}}^2$ , can then be readily estimated by:

$$\sigma_{\hat{\mathbf{e}}}^2 = \frac{\hat{\mathbf{e}}^T \hat{\mathbf{e}}}{N}. \quad (7)$$

This, and the power of the ideal sine which is  $(\hat{a}^2 + \hat{b}^2)/2$ , can now be used to estimate the signal-to-noise ratio, SNR, as

$$\text{SNR} = \frac{\hat{a}^2 + \hat{b}^2}{2\sigma_{\hat{\mathbf{e}}}^2} \quad (8)$$

The coefficients  $\hat{a}$  and  $\hat{b}$  can be shown to be Gaussian with mean  $a$  and  $b$ , respectively, that is, they are unbiased estimates [11]. Furthermore, the variance of each estimate is:

$$\sigma_{\hat{a}}^2 = \sigma_{\hat{b}}^2 = \frac{2\sigma_{\hat{\mathbf{e}}}^2}{N} \quad (9)$$

Let us assume that we want to estimate the coefficients  $a$  and  $b$  with a certain maximum normalized uncertainty,  $\varepsilon$ . Using 95 % confidence level, thus  $(\hat{a} + 2\sigma_{\hat{a}})/\hat{a} \leq 1 + \varepsilon$ , which means the normalized uncertainty is:

$$\varepsilon_a = \frac{2\sigma_{\hat{a}}}{\hat{a}} = \frac{4\sigma_e}{N\hat{a}} \quad (10)$$

by using Eq. (9), and similarly for  $\varepsilon_b$ .

SNR	$\sigma_e$	$E[\hat{a}]$	$\sigma_a$	$E[\hat{b}]$	$\sigma_b$	Eq. (9).
0	1.000	1.000	0.032	1.000	0.032	0.032
-20	1.000	0.101	0.031	0.101	0.033	0.032
-40	1.000	0.009	0.031	0.011	0.031	0.032

**Table 1:** Means and standard deviations of the estimates of  $\hat{a}$  and  $\hat{b}$ , and compared to the theoretical standard deviation of those estimates, Eq. (9). Note that the true values of the amplitude estimates are 1.0, 0.1, and 0.01, respectively for the three SNR cases. The simulation was based on  $N = 2000$  samples and 1000 realizations were computed

It may in some cases be more interesting to look at the amplitude estimate of the sine we apply, which is given by

$$\hat{A} = \sqrt{\hat{a} + \hat{b}} \quad (11)$$

This equation is nonlinear, which means that the estimate  $\hat{A}$  will be biased. Therefore it may make more sense to look at the mean square error (MSE) of this estimate, rather than only the variance. In [11] it is shown that the MSE is given approximately by

$$\text{MSE}[\hat{A}] \approx \frac{2\sigma_e^2}{N} \quad (12)$$

that is, the mean square error is approximately equal to the variance of the amplitude estimates given by Eq. (9). The approximation in Eq. (12) is furthermore a conservative error, so for bad SNR cases the error is smaller than this value.

If we wish to have a certain maximum uncertainty in  $\hat{A}$ , then we can use the formula for the normalized relative error which is

$$\varepsilon_A = \frac{E[\hat{A}] - A}{A} \approx \frac{1}{2N \cdot \text{SNR}}. \quad (13)$$

Finally it should be noted that the three-parameter sine fit method is very sensitive to the frequency, which must thus be accurately known. In most cases this will not be a limitation, as long as the measurement system is capable of generating the sine excitation signal. In other cases the iterative four-parameter sine fit method can be alternatively used [2]

### 3. DISCUSSION

To validate the equations for the performance of the estimates  $\hat{a}$  and  $\hat{b}$ , we define a Gaussian signal, assuming a sampling frequency of  $f_s = 20\text{Hz}$ . We then add a harmonic with frequency 1 Hz, so that the Nyquist frequency is 10 times the frequency of the sine, and with a (power) SNR of 0, -20 dB, and -40 dB, meaning that the RMS level of the sine is equal to, 1/10, and 1/100 of the RMS level of the random response, respectively. This is equivalent to amplitudes  $a = b = 1$ ,  $a = b = 0.1$ , and  $a = b = 0.01$ , respectively. We collect 100 periods of the harmonic, which in our example means that  $N = 2000$ . The whole process is repeated in a Monte Carlo simulation, repeating each simulation 1000 times, so that values of the mean and standard deviation of the estimates of the unknown coefficients  $\hat{a}$  and  $\hat{b}$  can be computed. In Table 1 the results of this simulation are presented. It can be seen that the standard deviations of the estimates  $\hat{a}$  and  $\hat{b}$  follow Eq. (9).

Eq (10) means that by using a sufficient amount of samples,  $N$ , to estimate the amplitude vector  $\hat{x}$ , the relative error in the estimates  $\hat{a}$  and  $\hat{b}$  can be made arbitrarily small, regardless of the size of the noise  $e(t)$ . The procedure is to first make a measurement, then computing the estimates  $\hat{a}$  and  $\hat{b}$  by solving

SNR	$\sigma_e$	$E[\hat{A}]$	$\sigma_A$	Eq. (12)	$\varepsilon_A$ [%]
0	1.000	1.414	0.032	0.0316	2.24
-20	1.000	0.146	0.032	0.0316	22.42
-40	1.000	0.041	0.021	0.0316	224.20

**Table 2:** Means and standard deviations of the estimates of  $\hat{A}$ , compared to the theoretical standard deviation of those estimates, Eq. (12). The rightmost column gives the computed normalized relative error, which should be compared to Eq. (13) which in this case should be 2.5, 25, and 250 %, respectively. Note that the true values of the amplitude estimates are  $\sqrt{2}$ ,  $0.1 \cdot \sqrt{2}$ , and,  $0.01 \cdot \sqrt{2}$ , respectively for the three SNR cases. The simulation was based on  $N = 2000$  samples and 1000 realizations were computed

Eq. (5), and then finding the variance of the random signal  $e(t)$  by Eqs. (6) and (7). A sufficient number of samples,  $N_{\min}$ , for an arbitrary relative error  $\varepsilon_a$  is then easily computed by Eq. (10). If this number turns out to be greater than the computed minimum number of samples, a new measurement has to be made.

In Table 2, results are presented from the simulation mentioned above, but looking at the amplitude estimate instead of the individual amplitudes of the cosine and sine components. In the rightmost column we have also added the obtained normalized error in percent. It can be seen that this error is close to the error according to Eq. (13).

In applications of OMAH for scaling mode shapes from OMA, usually it is only necessary to apply a force in one (or very few) DOF(s) for each mode shape to be scaled. Such DOFs should, furthermore, of course be selected from DOFs with large mode shape coefficients in the mode shape to be scaled. It may in some cases, however, not be appropriate to force the structure with excessive force levels, significantly adding to the response levels of the structure. This may be either for safety reasons, or because the actuator used is not large enough to produce a high enough force. It is therefore reasonable to assume that cases with relatively low SNR should be expected. If we, furthermore, as an example assume that a normalized error of, say, 5 % are acceptable for both the force and response amplitudes, then the number of samples needed are given by  $N = 1/(2\varepsilon_A \cdot \text{SNR})$ , which, for power SNR ratios of 0.01 and  $10^{-4}$ , results in 1000 and 100 000 samples, respectively. If we assume we sample with, say 50 times the lowest natural frequency, then this corresponds to 20 and 2000 cycles of the lowest natural frequency. It is obvious that the SNR ratio should be kept as high as possible, but that, if that is not possible, arbitrary accuracy can be obtained anyway, at the expense of measurement time.

#### 4. CONCLUSIONS

In the present paper we have outlined a recently proposed method for scaling mode shapes using harmonic excitation, and using the three-parameter sine fit method for estimating the harmonic excitation and response signals. It has been shown that the method works even with very poor signal-to-noise ratios SNRs, with arbitrary accuracy. Formulas have been given which allow to calculate suitable measurement time (or, number of samples needed for the amplitude estimation), given different SNR cases.

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