Stabilization of Diffusive Systems Using Backstepping and the Circle Criterion

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Abstract

Pool boiling is a pretty well-known benchmark of diffusive system with both practical and theoretical interest. We have investigated Neumann-boundary feedback control by combining the backstepping approach and the circle criterion. Conditions for local/global stability with nonlinear boundary conditions as input are investigated by explicitly solving the backstepping kernel equations. Simulation results are reported to illustrate the findings of the theoretical investigation and compare all the considered approaches.

Keywords: Pool boiling, diffusive system, stabilization

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1. Introduction

Pool-boiling (PB) systems have recently attracted a lot of attention from the research community for the purpose of both modeling [1, 2, 3, 4] and control [5, 6]. Such diffusive systems are of interest to develop more performing conditioning plants but also for increasing component integration in electronic setups to enhance efficiency by heat removal as well as in the semiconductor manufacturing, where the lithographic process is severely affected by temperature fluctuations and loss of homogeneity. Boiling heat tranfer achieves a higher flux when compared with single phase cooling [7, 8, 9]. However, some issues may arise such as the burnout risk, i.e., an uncontrolled increase in temperature when the boiling process approaches the critical heat flux (CHF). A safe use of boiling needs efficient, closed-loop methods to deal with the unpredictability of the boiling curve and to control the unstable behavior of the process when exceeding the CHF.

The motivations to study the control of PB systems are also theoretical. The capability to control such systems allows to address the problem of optimizing efficiency with cost reduction. From the point of view of theory, it represents a challenging example of a parabolic boundary control problem with a nonlinear behavior and an unstable equilibrium. Concerning the connection with the control of parabolic systems, it is worth to recall that the boundary feedback stabilization of parabolic systems was previously considered by Triggiani [10, 11]. Specifically in [11], the system is splitted in a stable infinite-dimensional part and an unstable-finite dimensional one, to be stabilized by the pole-shifting theorem.

Following [12], in this paper we present a new approach to the control of PB systems based on backstepping and the circle criterion. The backstepping method has been successfully applied to construct stabilizing feedback regulators and relies on the use of a Volterra transformation that turns a distributed parameter system into an equivalent one with known stability properties [13, 14, 15, 16]. Here, three feedback laws will be analyzed with two of them designed according to backstepping by showing that global stability holds thanks to the circle criterion.

The circle criterion is usually adopted to prove the stability of dynamic systems in the presence of nonlinear interconnections. Its original formulation is reported in [17]. In [18] this criterion is analyzed in

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relation with the notion of absolute stability and has been extended to infinite dimensional systems in [19]. Clearly, for such systems the additional difficulty is the need of accounting for well-posedness. Toward this end, in this paper an extension of the theory of C^0-semigroups dealing with systems with inputs, will be exploited [20, 21, 22].

We will show that the local stability results based on backstepping can be extended to become of global type by using arguments based on the circle criterion for infinite-dimensional systems [19, 23]. This complements the previous theoretical achievements reported in [5, 24, 6], where global results are presented by using Lyapunov arguments for one-dimensional and two-dimensional PB systems.

The considered PB system is described in Section 2. The closed loop stabilization by the backstepping method and the local stability results are presented in Section 3. The extension from local to global stability properties is addressed in Section 4, by properly applying the circle criterion. Theoretical and numerical results are analyzed in Section 5. Section 6 presents a final discussion and the conclusions.

2. Description of the PB system

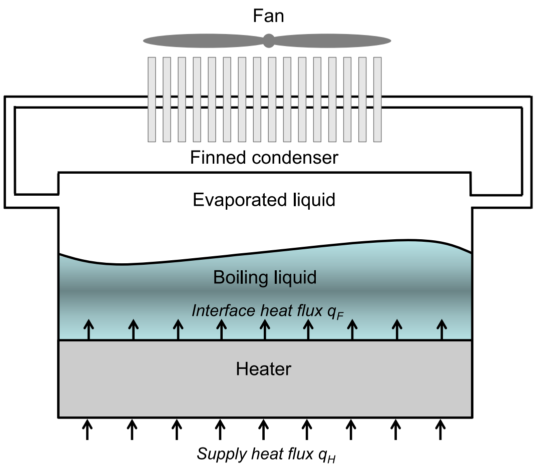

Figure 1: Schematic representation of a PB system.

The PB system depicted in Fig. 1 can be regarded as a solid heater positioned under a liquid pool [25, 7, 8, 9, 5]. A heat source is located below the heater; it provides an heat flux q_H entering the conducting solid, while cooling is achieved by the interface heat flux q_F.

Boiling is a phase change process taking place when the temperature at the solid-fluid interface (denoted by T_F) exceeds the saturation temperature of the fluid T_{SAT}, defined as the temperature at which the liquid boils into its vapour phase for a corresponding saturation pressure. The local boiling curve in Fig. 2 summarizes the complex dynamics resulting from the solid-liquid interaction, and it characterizes q_F as a function of the interface temperature T_F. It describes the dependence of the local interface heat flux q_F on the local interface superheat ΔT, the latter being the difference between the interface temperature T_F and the saturation temperature T_{SAT} of the fluid. Here the term local has to be intended at a suitable mesoscopic level, so that the quantities above, q_F and T_F, are in fact local averages in space and time, over intervals larger than the bubble sizes and the bubble lifetimes.

The PB conditioning system is required to operate in the nucleate boiling regime (Region II: $T_F < T_C$): bubbles detach from the surface and rise in the liquid versus the free surface. Both bubble frequency and heat flux increase with T_F. Nucleate boiling transits to film boiling (Region IV: $T_F > T_M$) upon exceeding the critical heat flux (CHF), when a vapour blanket, which thermally insulates the heater, appears at the heater-liquid interface. Transition boiling (Region III: $T_C < T_F < T_M$) connects the two previous boiling modes, and its instability is the sum of the reduced heat flux, due to the insulating vapour film, and the temperature increase. Even if the boiling curve is not well defined in this region, it will be assumed to be

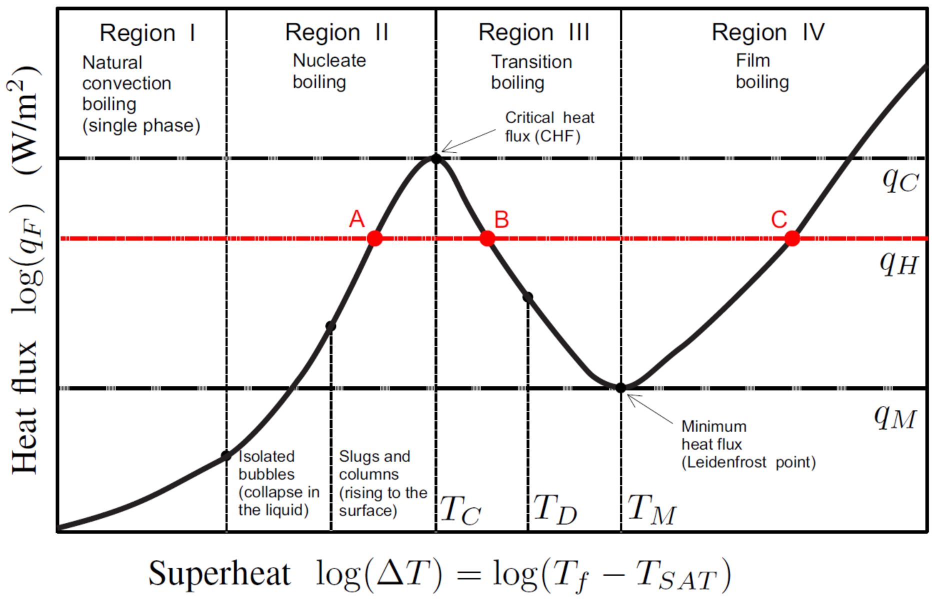

Figure 2: Typical local boiling curve.

continuous, with a typical transition boiling temperature T_D, approximately matching the flex point of the curve. For a fixed heat supply q_H (the red dotted line), the system admits three equilibria: A and C are stable, B unstable. Efficient conditioning requires the system to operate in the fully developed nucleate boiling regime and not far from the CHF, in order to maximize the interface heat flux responsible for the cooling action. This operating point, however, exposes the system to a burnout risk: whenever the CHF temperature is exceeded and assuming the heat supply remains constant, the transition boiling mode is entered, and its instability quickly settles the system to the equilibrium C. This point is located at a much higher temperature, even over the material melting point, and an unrecoverable fail of the heater is plausible in this condition. Moreover, the burnout risk is increased both by the unpredictability of the boiling curve in the transition regime, and by its dependence on the liquid pressure. Indeed a pressure increase could translate into a positive shift of the CHF and of the whole boiling curve along the temperature axis.

Only an active temperature control can avoid the evolution of transition boiling to one of the two stable boiling modes, and can prevent the burnout occurrence. This motivates the investigation of feedback laws stabilizing the system around the unstable transition boiling equilibrium B [5, 24, 6]. The heater-only model introduced in [25, 7, 8, 9, 5], relies on the synthesizing capability of the local boiling curve. Its mesoscopic level allows to smooth out microscopic fluctuations, yet capturing the behaviour of heterogeneous or non uniform states, i.e., states with “hot” and “cold” spots on the boiling surface. This is a key feature when the heater is modeled as a two or three-dimensional solid. For the purpose of this work, we consider the one-dimensional model that approximates physical behaviors with negligible temperature fluctuations in the two other spatial dimensions. This turns out to be the case when the heater thickness H is large and/or the thermal diffusivity k high.

Let $\mathcal{H} = [0, H]$ be the spatial domain and $\Gamma = \partial\mathcal{H} = \Gamma_H \cup \Gamma_F = \{\xi \in \mathcal{H} | \xi = 0\} \cup \{\xi \in \mathcal{H} | \xi = H\}$ its boundary. The state of the system is the superheat profile in the heater, denoted by T and simply referred to as temperature. Instead, it is convenient to deal with the temperature offset with respect to T_{SAT}. The evolution of T is described by the heat equation

$$\begin{array}{l} T_t(\xi,t) = \alpha\, T_{\xi\xi}(\xi,t) \\ T(\xi,0) = T_0(\xi) \\ -kT_\xi\big|_{\Gamma_H} = q_H + u(t) \\ kT_\xi\big|_{\Gamma_F} = -q_F(T_F) \end{array} \tag{1}$$

with $\xi \in \mathcal{H}$ and $t \geq 0$ and where k is the thermal conductivity, ρ the density, c_p the specific heat and $\alpha := k/\rho\, c_p$ the heater thermal diffusivity. The Neumann boundary conditions account for the heat in-flux and the out-flux from the heater to the fluid, modeled by the local boiling curve. The in-flux acting on Γ_H is the sum of a steady contribute q_H and a variable contribute $u(t)$ representing the control used to stabilize the system. The out-flux is expressed by the local boiling curve $q_F(T_F)$, where $T_F(t) = T(H,t)$ for all $t \geq 0$.

The choice of the output signal is less natural. Measurements of temperature can be done by thermocouples, placed in some points on the heater. However measurements at the heater-fluid interface, just one point in the non-dimensional model, are the most practical ones. That is why the system output will be assumed to be the scalar value:

$$y(t) = T_F(t)\,. \tag{2}$$

2.1. Derivation of the non-dimensional PB model

From (1) and (2) a non-dimensional model can be derived by scaling variables and parameters by characteristic values; let us redefine:

$$\xi' = \frac{\xi}{H}\pi,\ t' = \frac{t}{\tau},\ T' = \frac{T}{T_D},\ u' = \frac{u}{q_H},\ {q_F}' = \frac{q_F}{q_C} \tag{3}$$

where $\tau > 0$ is the time scale of the system evolution, q_H the typical heat supply, T_D the temperature during transition boiling, and q_C the critical heat flux. These last two values characterize the boling curve and affect its scaling as well. The non-dimensional equations, dropping primes in favor of readability, become

$$\begin{array}{l} T_t(\xi,t) = \kappa\, T_{\xi\xi}(\xi,t) \\ T(\xi,0) = T_0(\xi) \\ T_\xi\big|_{\Gamma_H} = -\frac{1}{\Lambda}(1+u(t)) \\ T_\xi\big|_{\Gamma_F} = -\frac{\Pi}{\Lambda} q_F(T_F) \\ y(t) = T_F(t) \end{array} \tag{4}$$

with domain $\mathcal{H} = [0,\pi]$, $t \geq 0$ and boundary $\Gamma = \Gamma_H \cup \Gamma_F = \{\xi = 0\} \cup \{\xi = \pi\}$; the non-dimensional parameters are

$$\kappa := \frac{\alpha\tau}{H^2},\ \Lambda := \frac{kT_D}{q_H H},\ \Pi := \frac{q_C}{q_H}\,. \tag{5}$$

The boiling curve $q_F(T_F)$ is scaled too and the normalized value $T_F = 1$ corresponds now to the typical transition boiling temperature. Even if this curve is quantitatively different from the local boiling curve, nevertheless it allows to study the generic dynamics of boiling systems and their control strategies.

2.2. Equilibria of the PB system

The equilibrium or steady-state solution of the system (4) is the solution of the Laplace equation:

$$\begin{array}{l} \overline{T}_{\xi\xi}(\xi) = 0 \\ \overline{T}_\xi\Big|_{\Gamma_H} = -\frac{1}{\Lambda} \\ \overline{T}_\xi\Big|_{\Gamma_F} = -\frac{\Pi}{\Lambda}\, q_F(\overline{T}_F) \end{array} \tag{6}$$

where $\overline{T}_F = \overline{T}_F|_{\Gamma_F}$. The solution of (6) is straightforward:

$$\overline{T}(\xi) = \frac{\pi - \xi}{\Lambda} + \overline{T}_F\,. \tag{7}$$

The steady state temperature $\overline{T}_F$ can be computed by substituting the solution (7) in the boundary condition at the heater-fluid interface, or, equivalently, the solution being a line, forcing the boundary conditions at the top and the bottom of the heater to be equal. The resulting equation is given by

$$q_F(\overline{T}_F) = \Pi^{-1} \tag{8}$$

and its solution provides the temperatures corresponding to the three intersections - denoted by A, B, and C - between the normalized boiling curve and the heat-supply characteristic line. As anticipated, there exist three possible equilibria, corresponding to three interface temperatures $\overline{T}_{F,1}$, $\overline{T}_{F,2}$ and $\overline{T}_{F,3}$, one for each boiling mode, i.e., nucleate boiling, transition boiling, and film boiling, respectively.

The linearization around some of the three equilibrium points can be performed as follows. Toward this end, let us denote by $\overline{T}_F$ the temperature corresponding to one of such of them, we define

$$x(\xi, t) = T(\xi, t) - \overline{T}_F$$

Since $\Pi^{-1} = q_F(\overline{T}_F)$, we define

$$u'(t) = -\frac{1}{\Lambda}u(t),\; f(x) = -\frac{\Pi}{\Lambda}(q_F(x + \overline{T}_F) - q_F(\overline{T}_F)\,.$$

When $\overline{T}_F = \overline{T}_{F,2}$, the transition boiling equilibrium, f is the cubic-like function in Fig. 3, which is assumed to be globally Lipschitz continuous and to satisfy the sector condition, which it is usally denoted by $f \in [a, b]$, i.e.,

$$ay^2 \leq f(y)\, y \leq by^2 \tag{9}$$

with $a, b > 0$. Reasonable values for a and b are, respectively, $a = -1$ and $b = q$, where $q = f_y(0)$ with $f \in [-1, q]$.

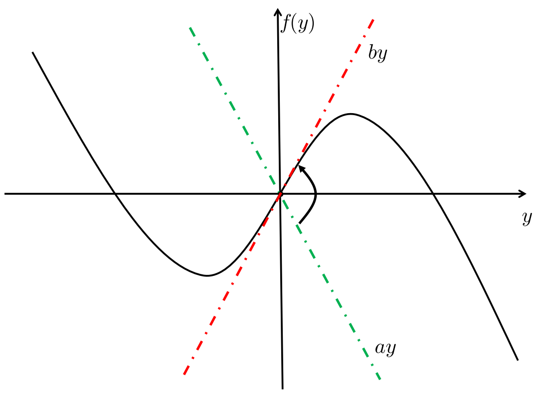

Figure 3: Typical shape of the nonlinearity f, satisfying the sector condition $f \in [a, b]$, with a and b slopes.

2.3. State equation of the PB system

Based on the aforesaid, if we drop primes in favour of readability, the PB system turns out to be described by

$$\begin{aligned} &x_t(\xi, t) = \kappa\, x_{\xi\xi}(\xi, t)\\ &x(\xi, 0) = x_0(\xi)\\ &x_\xi(0, t) = u(t)\\ &x_\xi(\pi, t) = f(y(t))\\ &y(t) = x(\pi, t) \end{aligned} \tag{10}$$

over $\mathcal{H} \times [0, +\infty)$, where $x \in X = L^2[0, \pi]$ (for well-posedness see [12] and the references therein).

A block diagram corresponding to (10) is reported in Fig. 4. where also the controller $K : X \to \mathbb{R}$ is depicted. Note the resulting interconnected system is a Lure system, which will be a key ingredient for the application of the circle criterion in Section 4.

A linearized version of the PB system is also exploited in Section 3. In order to turn the nonlinear system (10) into a linear one, the nonlinear boundary condition in $\xi = \pi$ is replaced by the linear one, i.e.,

$$x_\xi(\pi, t) = q\, y(t),\; q = f_y(0)\,. \tag{11}$$

The sign of q in (11) depends on the equilibrium corresponding to the linearization point: it is positive ($q > 0$) for the transition boiling equilibrium and negative ($q < 0$) for the nucleate and film boiling ones.

3. Closed-loop local stabilization

In order to face the design process of the controller, we first consider the stabilization of the linearized PB system. This simplified problem can be treated with by the available methods for linear system theory. Below, three different feedback laws K making the closed-loop linearized system stable, will be constructed: one of them is a simple proportional feedback, the two others are of backstepping type. When applied to the nonlinear PB system (10) such feedback laws can ensure only local stability in a neighborhood of the transition boiling equilibrium. These local stability properties will be extended to global ones in Section 4.

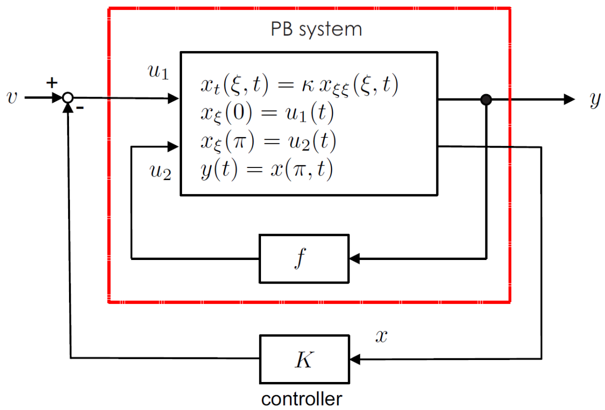

Figure 4: Block diagram for the controlled PB system, where v denotes a reference.

When dealing with systems described by PDEs, even the feedback operator can undermine well-posedness. Toward this end, we need to rely on the abstract differential representation of the linear part of the PB system in the Kalman state space form; the adopted notations follow [22]. The input is composed of two components u_1 and u_2. Hence, we define the Hilbert spaces

$$X = L^2[0, \pi],\; U = U_1 \oplus U_2 = \mathbb{R}^2,\; Y = \mathbb{R}\,.$$

The semigroup generator of the system is the operator $A : X_1 \to X$ with $\psi \mapsto A\psi = \psi_{\xi\xi}$, where

$$X_1 = \left\{ \psi \in H^2(0, \pi) \,\middle|\, \psi_\xi(0) = 0, \psi_\xi(\pi) = 0 \right\}.$$

A is the Neumann Laplacian, a regular Sturm-Liouville operator, and the generator of an analytic semigroup. As for the control operator B, let us first consider the solution operator $S : U \to Z$ of the elliptic boundary value problem

$$\frac{d^2 z(\xi)}{d\xi^2} - \beta z(\xi) = 0\,,\; \beta \in \rho(A)\,,\; z \in Z \tag{12a}$$

$$\left(\frac{dz(\xi)}{d\xi}(0), \frac{dz(\xi)}{d\xi}(\pi)\right) = (u_1, u_2) \in \mathbb{R}^2 \tag{12b}$$

where $\rho(A)$ denotes the resolvent set of A and $Z = H^2[0, \pi]$. According to [21, Remark 10.1.5], $S = (\beta I - A)^{-1}B$, so that B is in fact the operator

$$B : U \to X_{-1},\ B = (\beta I - A)S \tag{13}$$

where X_{-1} is the extrapolation space defined as the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$, actually a space of distributions. Indeed, B can be rewritten (see [26]) in the more intuitive form

$$[u_1, u_2] \mapsto B[u_1, u_2] = [-\delta(\xi), \delta(\pi - \xi)] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{14}$$

As $B(U) \subset X_{-1} \setminus X$, B is an unbounded control operator. Finally, the observation operator C is the Sobolev trace operator:

$$C : X_1 \to Y,\ \psi \in X_1 \mapsto y = \psi(\pi)\,. \tag{15}$$

Based on the aforesaid, let us introduce the following definitions, which also account for the possibility of unbounded control and observation operators, such as the one above.

Definition 1. *Consider the state equation $\dot{x}(t) = Ax(t) + Bu(t)$, with state space X, input space U and $D(A) = X_1$. The system is said*

(i) *boundedly stabilizable if a state feedback operator $K \in \mathcal{L}(X, U)$ exists such that $A_K = A + BK$, with domain $D(A_K) = D(A)$, generates an exponentially stable semigroup;*
(ii) *regularly stabilizable if a state feedback operator $K \in \mathcal{L}(X_1, U)$ exists such that*
  (a) *(A, B, K) is a regular triple;*
  (b) *I is an admissible feedback operator for $K_\Lambda(sI - A)^{-1}B$;*
  (c) *$A_K = A + BK_\Lambda$ (with its natural domain) generates an exponentially stable semigroup.*
(iii) *(A, B) is said completely boundedly or regularly stabilizable if for arbitrary $\omega \in \mathbb{R}$ there exists an operator K as above and a constant $M > 0$ such that*

$$\|T_K(t)\| \leq M\ \exp(\omega t)\,,\ \forall t \geq 0$$

where $T_K(t) = e^{(A+BK)t}$.

The interested reader is referred to [22] for the notion of admissible feedback operator. Even if the notion of regular stabilizability was introduced in [27], the naming convention in Def. 1 is taken from [28].

In the following, three different types of feedback will be considered for the linearized PB system.

3.1. Proportional feedback: K_P

Consider the state feedback operator

$$K_P : X_1 \to \mathbb{R}, \qquad x \mapsto Fx(\pi) \tag{16}$$

where F is a static gain to be selected as proportional law [5, 24, 6]. In order for K_P to stabilize the system, the operator $A_{K_P} = A + BK_P$ corresponding to the closed-loop system

$$\begin{aligned} \dot{x}(t) &= A_{K_P}x(t) \\ x(0) &= x_0(\xi) \end{aligned} \tag{17}$$

has to generate an exponentially stable semigroup. A_{K_P} is defined as follows:

$$D(A_{K_P}) = \{\psi \in H^2(0, \pi)\ \ | \tag{18a}$$

$$\psi_\xi(0) - F\psi(\pi) = 0, \psi_\xi(\pi) - q\psi(\pi) = 0\}$$

$$A_{K_P} : D(A_{K_P}) \to X, \qquad \psi \mapsto A_{K_P}\psi = \psi_{\xi\xi}\,. \tag{18b}$$

This is a (non regular) Sturm-Liouville operator and its stability properties are related to its spectral bound. In order for the closed-loop system to be exponentially stable, the unique positive eigenvalue λ_1 of the PB system has to be moved in the open left half-plane, and, as shown in [24], this is possible for some F if

$$q < 2/\pi\,. \tag{19}$$

If (19) is not satisfied, the proportional feedback does not stabilize the system, irrespective of the gain value F.

3.2. First backspepping-based feedback: K_{BS1}

The full-state backstepping-based feedback considered in the following is motivated by the goal of removing the limitation due to (19). The proposed approach by Krstic and coworkers [13, 14, 15] consists in the search of an invertible transformation of the state space and in turning the system in a stable autonomous target system. Toward this end, let us consider the Volterra-like operator

$$z(\xi,t) = x(\xi,t) - \int_\xi^\pi K(\xi,\eta)x(\eta,t)d\eta \tag{20}$$

where K is an upper triangular kernel to be determined and z denotes the state of the target system.

The first choice for the target system is the exponentially stable system

$$\begin{aligned} &\dot z(\xi,t) = \kappa z_{\xi\xi}(\xi,t)\\ &z(\xi,0) = z_0(\xi)\\ &z_\xi(0,t) = 0\\ &z_\xi(\pi,t) = -Q\,z(\pi,t) \end{aligned} \tag{21}$$

with $Q > 0$. In order for (10) and (11) to match (21), the kernel K in (20) has to satisfy the kernel equations

$$\begin{aligned} &K_{\xi\xi}(\xi,\eta) - K_{\eta\eta}(\xi,\eta) = 0\\ &K_\eta(\xi,\pi) - q\,K(\xi,\pi) = 0\\ &K(\xi,\xi) = p \end{aligned} \tag{22}$$

where p is a positive constant to be fixed over the upper triangular domain $\Omega = \{(\xi,\eta) : 0 \le \xi \le \eta \le \pi\}$.

Even if the target system is different, the kernel equations (22) are similar to the ones derived in [14] with solution

$$K_1(\xi,\eta) = -(q+Q)\,\exp\left(q(\eta-\xi)\right)\,.$$

Finally, the feedback law u corresponding to the kernel K_1 is

$$u(t) = \int_0^\pi K_{1\xi}(0,\eta)x(\eta,t)d\eta - K_1(0,0)x(0,t)$$

and hence

$$u(t) = (q+Q)\left[x(0,t) + q\int_0^\pi \exp\left(q\eta\right)\,x(\eta,t)\,d\eta\right]\,. \tag{23}$$

For the sake of brevity, let us refer to $u = K_{BS1}x$ in operator form.

With the control (23) the closed-loop PB system is exponentially stable in $L^2[0,\pi]$ but with a decrease rate that cannot be arbitrarily fixed by adjusting the control parameter Q, i.e., the system is not completely stabilizable by this feedback law.

3.3. Second backspepping-based feedback: K_{BS2}

The second choice for the target system is:

$$\begin{aligned} & z_t(\xi,t) = \kappa z_{\xi\xi}(\xi,t) - Cz(\xi,t) \\ & z(\xi,0) = z_0(\xi) \\ & z_\xi(0,t) = 0 \\ & z_\xi(\pi,t) = qz(\pi,t) \end{aligned} \tag{24}$$

with $C > 0$. The unstable boundary condition is still present, however exponential stability is achieved by the additional addendum $-Cz(\xi,t)$ in the state equation whenever $C \geq kq^2 + k/2$ (see [13]).

To match (24), we need to solve the kernel equations

$$\begin{aligned} & K_{\xi\xi}(\xi,\eta) - K_{\eta\eta}(\xi,\eta) = \lambda K(\xi,\eta) \\ & K_\eta(\xi,\pi) - qK(\xi,\pi) = 0 \\ & K(\xi,\xi) = (\lambda\xi)/2, \qquad \lambda = C/k \end{aligned} \tag{25}$$

over the upper triangular domain $\Omega = \{(\xi,\eta) : 0 \leq \xi \leq \eta \leq \pi\}$. The solution K_2 of (25), yet with a scaled and mirrored domain, can be found in [13]. With the needed adjustments, it is $K_2(\xi,\eta) = H(\pi-\xi,\pi-\eta)$ where

$$\begin{aligned} H(\xi,\eta) = & -\lambda\xi \frac{I_1(\sqrt{\lambda(\xi^2-\eta^2)})}{\sqrt{\lambda(\xi^2-\eta^2)}} - \frac{q\lambda}{\sqrt{\lambda+q^2}} \\ & \times \int_0^{\xi-\eta} \exp(q\tau/2) \sinh\left(\frac{\sqrt{\lambda+q^2}}{2}\tau\right) \\ & \times I_0(\sqrt{\lambda(\xi+\eta)(\xi-\eta-\tau)})d\tau, \end{aligned} \tag{26}$$

and I_0 and I_1 are modified Bessel functions of the first kind. Finally, the second backstepping-based feedback is

$$u(t) = \int_0^\pi K_{2\xi}(0,\eta)x(\eta,t)d\eta - K_2(0,0)x(0,t)\,. \tag{27}$$

Shortly, in operator form it will be denoted by $u = K_{BS2}x$. The resulting closed-loop system is completely stabilizable by a proper choice of the control parameter C, and the convergence is stronger, both in $L^2[0,\pi]$ and $H^1[0,\pi]$.

feedback law	local stability property	global stability property
K_P	exponential stab. if $q \in (0, 2/\pi)$	exponential stab. if $f_y(0) \in (0, 0.5)$
K_{BS1}	exponential stab. for any $q > 0$	exponential stab. if $f_y(0) \in (0, 1)$
K_{BS2}	complete stab. for any $q > 0$	only conjecture on complete stab. for any $q > 0$

Table 1: Summary of the stability properties of feedback laws for the PB system.

4. Global stabilization using the circle criterion

Now let us show that the three approaches presented so far enjoy global stability by using the infinite dimensional version of the circle criterion [19, 23]. Its application is made straightforward by the particular Lure structure of the PB system. This is emphasized in Fig. 5, where the block diagram of the PB system is properly rearranged by exploiting the symmetric roles played by the controller and the nonlinear boundary condition.

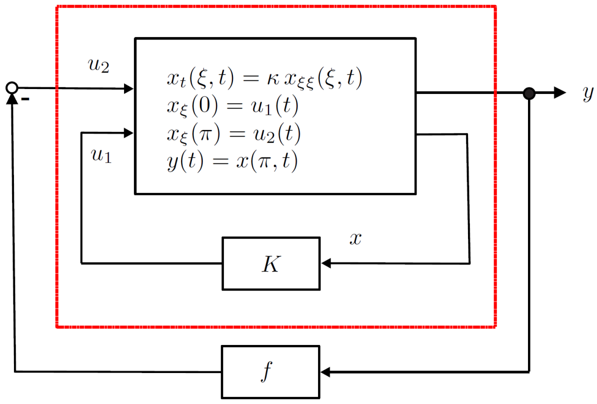

Figure 5: The Lure structure of the controlled PB system, with its linear part inside the red box, and with K being one among K_P, K_{BS1}, or K_{BS2}.

The latter, which was part of the PB system in Fig. 4, is now moved outside the system, on the feedback path, as shown in Fig. 5. At the same time, the controller K, i.e., one of the three linear feedbacks considered in Section 3, is put inside the "red box," which becomes a linear system since the nonlinearity f stays outside. The resulting system in Fig. 5 clearly exhibits a Lure structure, made by the interconnection of a linear system on the forward path and a nonlinearity in the feedback path.

As the linear system is a scalar single-input single-output one, the circle criterion just reduces to the classical circle criterion for finite-dimensional systems [19]. In particular, the Lure system in Fig. 5, with $ay^2 \leq f(y)y \leq by^2$, is exponentially stable when the following three conditions are met:

1. The linear part of the system, with $U = Y = \mathbb{R}$, is both optimizable and estimatable[2].
2. The transfer function T of the linear part of the system is H^∞[3].
3. $T(i\omega) \in \Delta(a,b)$, where $\Delta(a,b)$ is the open disk in $\mathbb{C}$ with centre in $\mathbb{R}$ and $-1/a$ and $-1/b$ in its boundary (circle criterion).

The first two hypotheses are easily verified. Moreover, the considered feedback makes the system stable even when $q = 0$, and this implies $T \in H^\infty$. The last condition requires to compute $T(s)$ for each feedback, i.e., the ratio

$$T(s) = \frac{\hat{x}(\pi,s)}{\hat{u}_2(s)} \tag{28}$$

by using the Laplace transform [29, 30]. Again relying on the Laplace transforms, we get

$$s\hat{x}(\xi,s) = \hat{x}_{\xi\xi}(\xi,s) \tag{29}$$

$$\hat{x}_\xi(0,s) = \hat{u}_1(s), \quad \hat{x}_\xi(\pi,s) = \hat{u}_2(s)\,. \tag{30}$$

Once solved (29) with the two boundary conditions (30), its solution $\hat{x}(\xi,s)$ can be placed in the feedback law, as follows:

$$\hat{u}_1(s) = F\hat{x}(\pi,s) \tag{31}$$

for the proportional feedback K_P. For the other two control laws, we have

$$\hat{u}_1(s) = \int_0^\pi K_{i\xi}(0,\eta)\hat{x}(\eta,s)d\eta - K_i(0,0)\hat{x}(0,s) \tag{32}$$

[2]Roughly speaking, a system is optimizable if, for every input belonging to L^2, the system admits a solution in L^2; it is estimatable if the output belongs to L^2 [19]

[3]The space of the H^∞ transfer functions from U to Y is given by the holomorphic mapping from any nonempty half space of the complex plane into the set of the bounded functions from U to Y [19]

with $i \in \{1, 2\}$, when the backstepping-based feedback BSi is considered. In the latter case, the integration in (32) can be carried on. The resulting equation is thus solved in the unknown $\hat{u}_1(s)$ and its expression is substituted in $\hat{x}(\xi, s)$ so as to find the ratio (28).

The above outlined computation can be accomplished for the first two feedback laws, K_P and K_{BS1} with the support of Matlab symbolic toolbox [12]. The case of K_{BS2} is more difficult, as its complex analytic form does not allow the integration in (32). Thus, a polynomial approximation of K_2 can be used to obtain an expression for T_{BS2}.

5. Analysis of the theoretical and numerical results

We show below how the application of the circle criterion may be a key instrument to study the closed-loop stability of the PB system. Indeed, global stability is evaluated for each feedback law considered. The diagrams in Fig.s 6-8 assume the sector condition $f \in [-1, q]$, and they try to highlight the maximum value of q matching the circle condition.

Table 1 provides a concise comparison of the conditions for both local and global stability of the PB system, namely the ranges of q and $f_y(0)$ such that the closed-loop system is stable, at least for a choice of the control parameters F, Q, and C.

For the proportional feedback K_P, the local stability region reported in Table 1 has been derived in [24] by spectral methods. As expected, global stability is achievied in a subregion of the latter, constrained by the circle condition (see Fig.6). Moreover, in [24] even a condition for global stability was derived by Lyapunov methods, i.e., $f_y(0) \in (0, 1/\pi)$; if compared with the one in Table 1, it looks more severe, thus confirming the effectiveness of the circle criterion.

K_{BS1} extends the global stability region of K_P from $f_y(0) \in (0, 0.5)$ to $f_y(0) \in (0, 1)$, yet preserving a simple analytic form. Fig. 7 highlights the constraint imposed by the circle condition on the choice of K_{BS1}, where a high value of the control parameter Q is required. When $f_y(0)$ is greater than 1 even an increase of Q fails to ensure a stable closed-loop behaviour.

To enlarge again the global stability region the feedbak K_{BS2} is required. We already pointed out how it is the only control law, among the three considered, achieving complete stability of the linearized PB system. Not surprisingly, K_{BS2} also exhibits the larger global stability region. For instance, Fig. 8 shows the circle condition is met when $f_y(0) = 1.5$, with the choice $C = 12$ for the control parameter. We conjecture this feedback law is able to achieve global stability even for large slopes $f_y(0)$ of the boiling curve. However, the large value of C required as well has the exponential growth of the kernel K_2 make increasingly challenging the computation of the transfer function.

Finally, the theoretical results obtained so far have been confirmed via simulations of a PB system given by (10) with $\kappa = 1$ with Comsol Multiphisics 4.2. The initial state $x_0(\xi)$ in all the simulation runs is chosen equal to a triangle shaped function. The stabilizing effect of the three feedback laws considered is compared against different performance indexes (see Fig. 9). In particular, the transient behavior of $x(\pi)$, the state value on the opposite side of the controlled boundary, $\|x\|_{L^2(0,\pi)}$ and $\|x\|_{H^1(0,\pi)}$ are plotted and compared (see Fig. 10-12). Note that, even when the circle condition is not met, the PB system turns out to be stable.

6. Conclusions

Three feedback laws for the PB system have been investigated: specifically, in addition to K_P and K_{BS2} already reported in the literature, a new, yet quite simple feedback (i.e.,K_{BS1}) is derived by using the backstepping method. For the proportional feedback K_P, the results obtained match the ones already derived by using Lyapunov-based approaches [24, 6]. Not surprisingly, K_{BS2} is the only feedback achieving complete stability.

This work shows how the application of the circle criterion may be a key instrument to study the global closed-loop stability of one-dimensional diffusive systems and an effective alternative to direct Lyapunov methods; a possible extension to the two-dimensional case will be evaluated. Future work will be devoted to the construction of observers for such systems to design an output feedback control scheme that allows

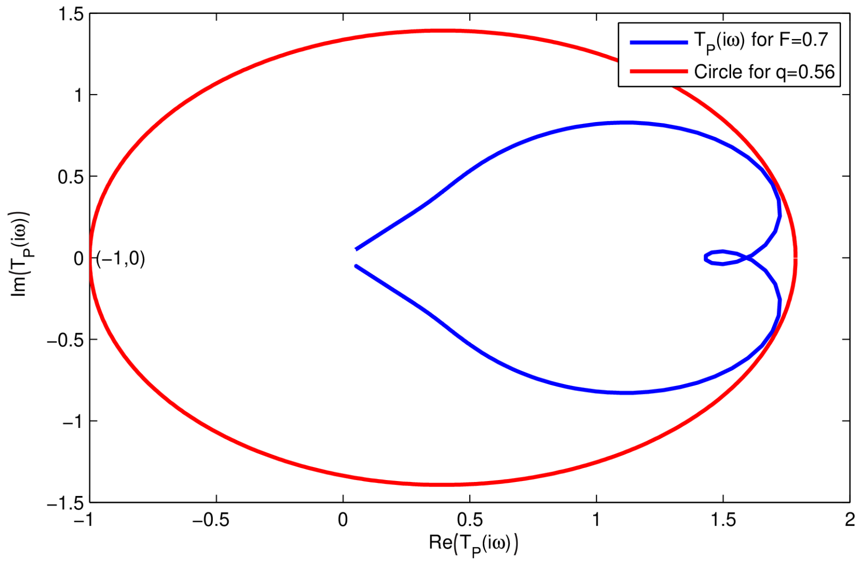

Figure 6: Plot of $T_P(i\omega)$; sector condition: $f \in [-1, 0.56]$.

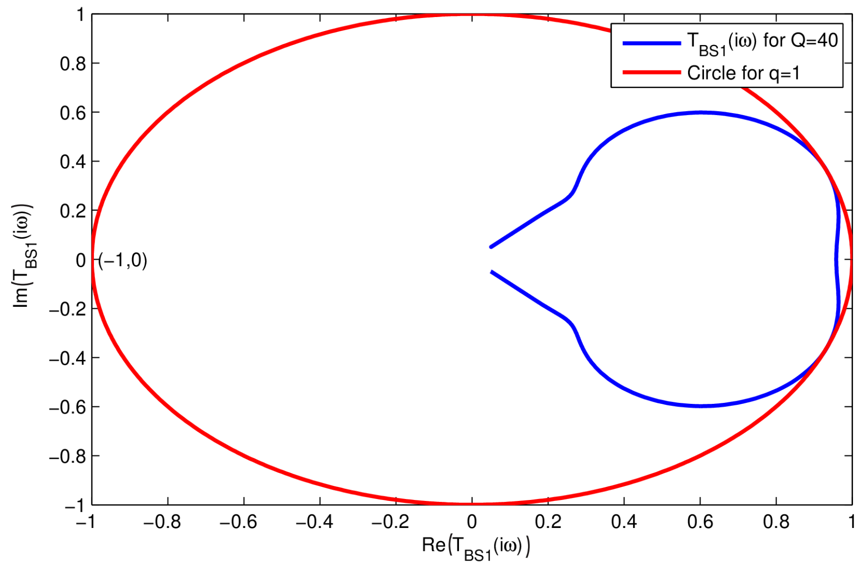

Figure 7: Plot of $T_{BS1}(i\omega)$; sector condition: $f \in [-1, +1]$.

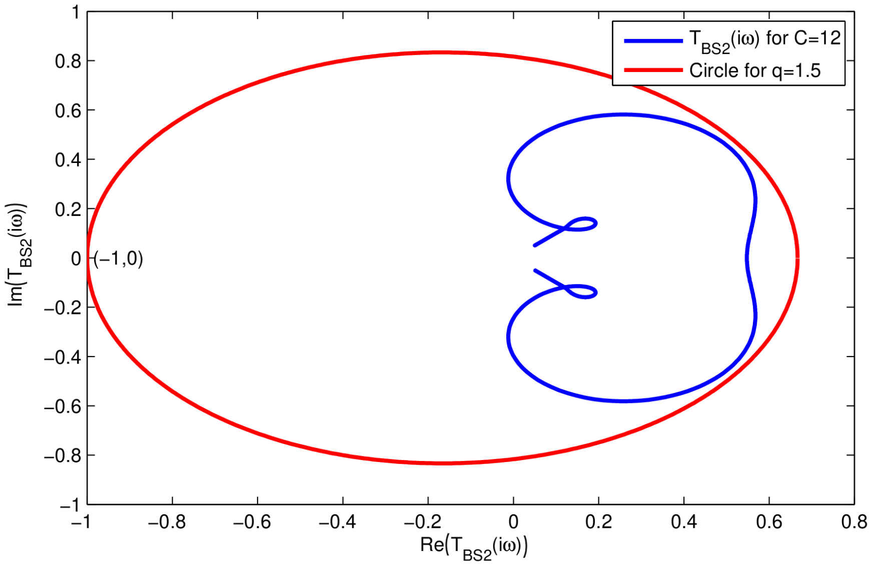

Figure 8: Plot of $T_{BS2}(i\omega)$; sector condition: $f \in [-1, +1.5]$.

to avoid the knowledge of the full state and hence enjoying an increased practical interest. Another topic of interest will the the investigation of the robustness of closed-loop PB systems subject to external disturbances by using input-to-state stability.

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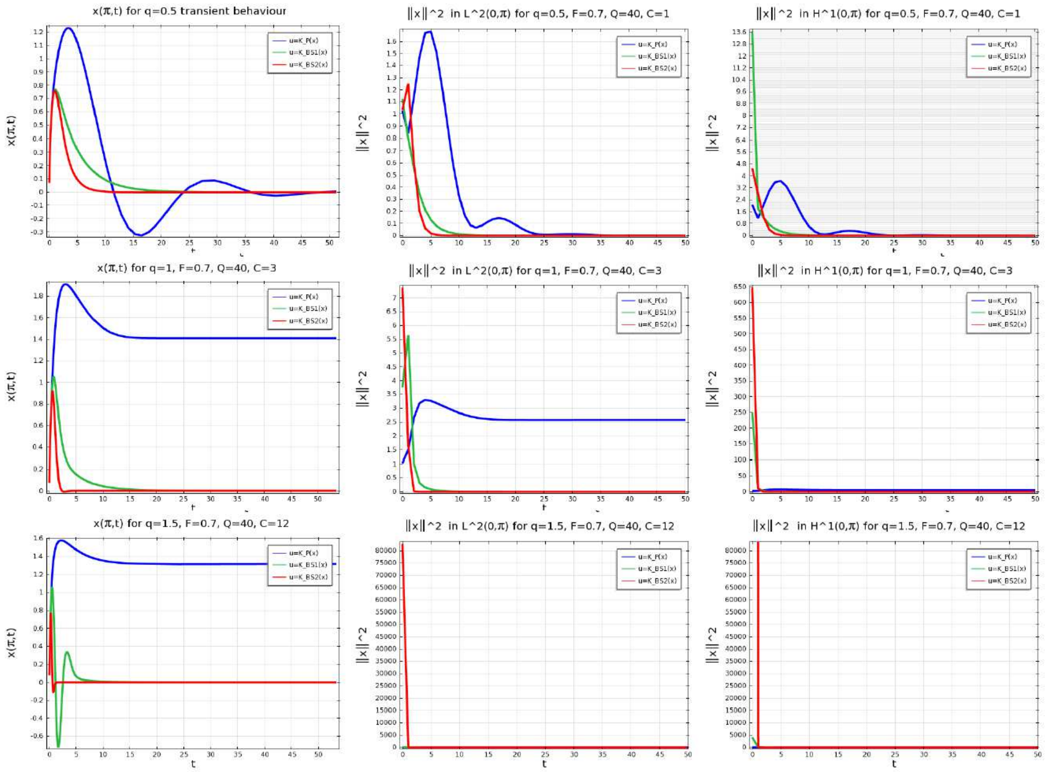

Figure 9: Comparison of the transient behaviour of $\|x\|^2_{L^2(0,\pi)}$, $\|x\|^2_{L^2(0,\pi)}$ and $\|x\|^2_{H^1(0,\pi)}$ for $q = 0.5$ and $C = 1$ (first row), for $q = 1$ and $C = 3$ (second row) and for $q = 1.5$ and $C = 12$ (third row).

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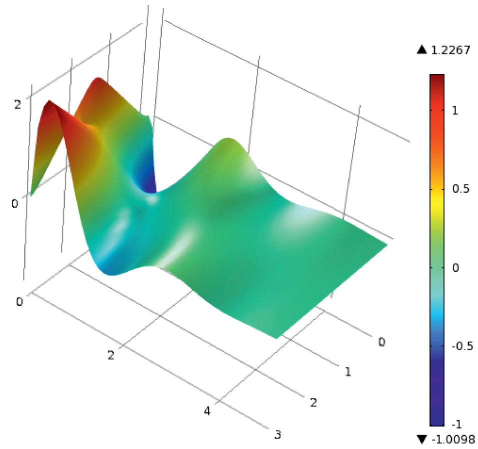

Figure 10: Transient plot of $x(\xi, t)$ for K_P with $q = 0.5$ and $F = 0.7$.

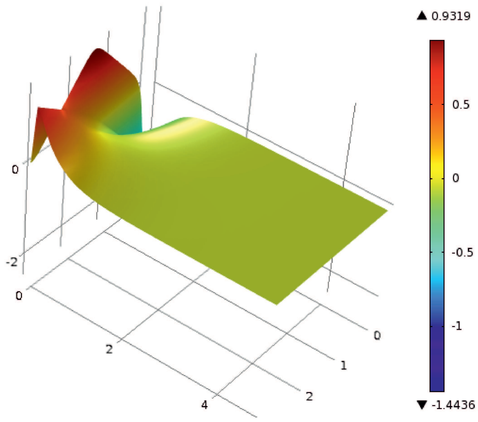

Figure 11: Transient plot of $x(\xi, t)$ for K_{BS1} with $q = 0.5$ and $Q = 40$.

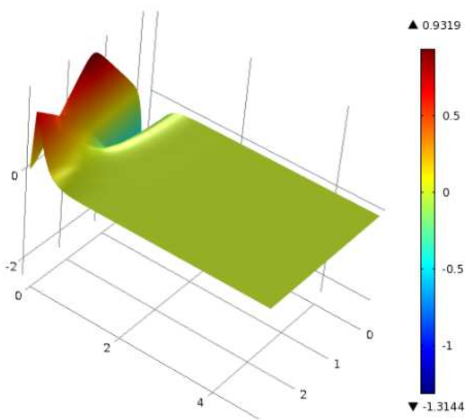

Figure 12: Transient plot of $x(\xi, t)$ for K_{BS2} with $q = 0.5$ and $C = 1$.