A PHASE-FIELD APPROACH TO
EULERIAN INTERFACIAL ENERGIES

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ABSTRACT. We analyze a phase-field approximation of a sharp-interface model for two-phase materials proposed by M. Šilhavý [41, 42]. The distinguishing trait of the model resides in the fact that the interfacial term is Eulerian in nature, for it is defined on the deformed configuration. We discuss a functional frame allowing for existence of phase-field minimizers and Γ-convergence to the sharp-interface limit. As a by-product, we provide additional detail on the admissible sharp-interface configurations with respect to the analysis in [41, 42].

1. INTRODUCTION

This paper addresses the equilibrium of a two-phase elastic medium, whose stored energy takes the form

$$\mathcal{F}_0(y, \zeta) = \mathcal{F}^{\text{bulk}}(y, \zeta) + \mathcal{F}_0^{\text{int}}(y, \zeta)$$
$$:= \int_\Omega \Big((\zeta \circ y) \, W_1(\nabla y) + (1 - \zeta \circ y) \, W_0(\nabla y) \Big) \, \mathrm{d}x + \gamma \operatorname{Per}\big(\{\zeta = 1\}, y(\Omega)\big). \qquad (1.1)$$

Here, $y : \Omega \to \mathbb{R}^3$ stands for the deformation of the medium with respect to its *reference* configuration $\Omega \subset \mathbb{R}^3$ and W_0, W_1 are the elastic energy densities of the two pure phases [39]. The Eulerian phase indicator $\zeta : y(\Omega) \to \{0, 1\}$ is defined on the *deformed* configuration $y(\Omega)$ instead. Note that solely pure phases are allowed. The stored energy of the medium includes an elastic bulk part $\mathcal{F}^{\text{bulk}}(y, \zeta)$, consisting of an integral on the reference configuration, and an interface contribution $\mathcal{F}_0^{\text{int}}(y, \zeta)$, featuring the perimeter of the phase $\{\zeta = 1\}$ in $y(\Omega)$, where $\gamma > 0$ is a surface-tension coefficient. With respect to classical hyperelastic theory, the distinguishing feature in (1.1) is that the interface is measured *in the deformed configuration*. Variational theories featuring Eulerian interfacial energy terms can be traced back at least to [21] and have been considered, for instance, in [17, 26, 28, 29, 30], among others. See also [33] for an atomistic-informed discussion.

The choice of the elastic energy $\mathcal{F}_0$ is inspired by the notion of *interface polyconvex energy*, introduced by M. Šilhavý in the series of contributions [40, 41, 42]. The explicit form in (1.1) is in fact just a first example in the wider class considerer therein, where the general interfacial term reads

$$\int_{\partial E \setminus \partial \Omega} \Psi(n, \nabla_S y \times n, (\operatorname{cof} \nabla_S y) n) \, \mathrm{d}S. \qquad (1.2)$$

1

Here, $\mathrm{d}S$ is the infinitesimal area element (in the reference configuration), and $\Psi : \mathbb{R}^{15} \to \mathbb{R}$ is a positively 1-homogeneous convex function depending on the normal n to the interface, on the surface gradient $\nabla_S y$ of the deformation, and on the cofactor of the surface gradient. More precisely, $\Psi = \Psi(n, \mathbb{F} \times n, \mathrm{cof}\,\mathbb{F}n)$, where $\mathbb{F} \in \mathbb{R}^{3\times 3}$ is a placeholder of the surface gradient of the deformation and $\mathbb{F} \times n : \mathbb{R}^3 \to \mathbb{R}^3$ is defined for all $a \in \mathbb{R}^3$ as $(\mathbb{F} \times n)a := \mathbb{F}(n \times a)$. Note that $\mathbb{F}n = 0$, because n inevitably lives in the kernel of $\mathbb{F}$. A rigorous definition would ask to cope with the possible nonsmoothness of y, the existence of the surface gradient $\nabla_S y$, and also whether n does exist at the phase interface, which in turn relates with the regularity of phase 1 in the reference configuration, for $E = y^{-1}(\{\zeta = 1\})$ in (1.2). The specific interfacial term in (1.1) corresponds to the choice [42, Ex. 5.7]

$$\tilde{\Psi}(n, \mathbb{F} \times n, \mathrm{cof}\,\mathbb{F}n) := \gamma|\mathrm{cof}\,\mathbb{F}n|. \tag{1.3}$$

Indeed, it is a standard matter to check that $(\mathrm{cof}\,\nabla_S y)n = (\mathrm{cof}\,\nabla y)n$. Then, a formal application of the change-of-variables formula for surface integrals [11] gives

$$\int_{\partial E \backslash \partial \Omega} \tilde{\Psi}(n, \mathbb{F} \times n, \mathrm{cof}\,\mathbb{F}n)\,\mathrm{d}S = \gamma \int_{\partial E \backslash \partial \Omega} |(\mathrm{cof}\,\nabla y)n|\,\mathrm{d}S = \gamma \int_{\partial y(E) \backslash \partial y(\Omega)} \mathrm{d}S^y. \tag{1.4}$$

As $\mathrm{d}S^y$ is the infinitesimal area element in the deformed configuration $y(\Omega)$, we have checked that, along with choice (1.3), the interfacial energy term measures indeed the surface of the interface in the deformed configuration. This is consistent with the definition of $\mathcal{F}_0^{\mathrm{int}}$ from (1.1).

Our main results are the existence of minimizers of $\mathcal{F}_0$ (Theorem 2.3) and the viability of a phase-field approach (Theorem 2.4) to such sharp-interface model via the diffuse-interface energies for $\varepsilon > 0$

$$\mathcal{F}_\varepsilon(y, \zeta) = \mathcal{F}^{\mathrm{bulk}}(y, \zeta) + \mathcal{F}_\varepsilon^{\mathrm{int}}(y, \zeta) := \mathcal{F}^{\mathrm{bulk}}(y, \zeta) + \int_{y(\Omega)} \left(\frac{\varepsilon}{2}|\nabla \zeta|^2 + \frac{1}{\varepsilon}\Phi(\zeta) \right) \mathrm{d}\xi. \tag{1.5}$$

Note that the diffuse-interface term $\mathcal{F}_\varepsilon^{\mathrm{int}}(y, \zeta)$ is still Eulerian, but the phase indicator ζ takes now values in the interval $[0, 1]$. Here and throughout the paper, ξ stands for the variable in the deformed configuration $y(\Omega)$. The function Φ in (1.5) is a classical double-well potential with minima at 0 and 1, and $\int_0^1 \sqrt{2\Phi(s)}\,\mathrm{d}s = \gamma$. By checking the Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_0$ we essentially deliver a version of the Modica-Mortola Theorem [32] *in the deformed configuration*, see also [31]. Instrumental to this is the discussion of the interplay of deformations and perimeters in deformed configurations, which constitutes the main technical contribution of our paper (Theorem 2.2).

Measuring the interface in the deformed configuration gives rise to a variational model of mixed Lagrangian-Eulerian type. Variational formulations featuring both Lagrangian and Eulerian terms are currently attracting increasing attention. A prominent case is that of magnetoelastic materials [16], where Lagrangian mechanical terms and Eulerian magnetic effects combine [6, 7, 27, 38]. Mixed Lagrangian-Eulerian formulations arise in the modeling of nematic polymers [5, 6], where the Eulerian variable is the nematic director orientation, and in piezoelectrics [37], involving the Eulerian polarization instead.

An interplay of Lagrangian and Eulerian effects occurs already in case of space dependent forcings, like in the variable-gravity case [18], as well as in specific finite-plasticity settings [43], where elastic and plastic deformations are composed. Most notably, such mixed formulations arise naturally in the study of fluid-structure interaction, where the deformed body defines the (complement of the) fluid domain [36].

The plan of the paper is as follows. We present in detail our assumptions on the ingredients of the models in Section 2. In particular, we specify the class of admissible deformations and state a characterization of sets of finite perimeter with respect to deformed configurations (Theorem 2.2). Subsection 2.4 contains the statements of our main existence and approximation results. These are put in relation with the former theory by M. Šilhavý in Subsection 2.5. We check in Section 3 that admissible deformations are actually homeomorphisms, so that, in particular, the deformed configuration is well defined. The proof of the Characterization Theorem 2.2 is presented in Section 4, along with a suite of results on perimeters in deformed configurations. The existence of minimizers to $\mathcal{F}_0$ (Theorem 2.3) is proved in Section 5. Eventually, Section 6 proves the Γ-convergence of the phase-field diffuse-interface energies $\mathcal{F}_\varepsilon$ to the sharp-interface limit $\mathcal{F}_0$ (Theorem 2.4).

2. Main results

We devote this section to specifying the functional frame (Subsections 2.1-2.3) and stating our main results (Subsection 2.4). The relation of our results with the former existence theory by M. Šilhavý is also discussed (Subsection 2.5).

We first introduce some basic notation. We denote by $B(a, \varepsilon) := \{z \in \mathbb{R}^n \mid |z - a| < \varepsilon\}$ the open ball of radius $\varepsilon > 0$ centered at $a \in \mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is an open set, $C^m(\Omega; \mathbb{R}^k)$ denotes the space of continuous maps on Ω with values in $\mathbb{R}^k$ that admit continuous derivatives up to the order $m \geq 0$. $C_c^m(\Omega; \mathbb{R}^k)$ is the subspace of compactly supported maps. For $p \in [1, +\infty)$, $W^{1,p}(\Omega; \mathbb{R}^k)$ denotes the standard Sobolev space, and $W_{\mathrm{loc}}^{1,p}(\Omega; \mathbb{R}^k)$ denotes its local counterpart. The space of finite vector Radon measures on Ω with values in $\mathbb{R}^k$ is denoted by $\mathcal{M}(\Omega, \mathbb{R}^k)$ and it is normed by the total variation $|\cdot|(\Omega)$. $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^k)$ denotes the space of locally finite vector Radon measures. Furthermore, $BV(\Omega; \mathbb{R}^k)$ stands for the space of maps with bounded variation. See e.g. [2] for references. With slight abuse of notation, we occasionally replace $\mathbb{R}^k$ in the target space by a set. For a measurable set $E \subset \Omega$, we denote the n-dimensional Lebesgue measure by $|E|$ and the m-dimensional Hausdorff measure by $\mathcal{H}^m(E)$. By χ_E we denote the characteristic function of E. The perimeter of E in Ω is classically defined as [2, Def. 3.35]

$$\mathrm{Per}(E, \Omega) := \sup \left\{ \int_E \mathrm{div}\varphi \, \mathrm{d}x \mid \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ \|\varphi\|_\infty \leq 1 \right\}.$$

Given $y : E \to \mathbb{R}^3$, we will use the notation $E^y := y(E)$.

2.1. **Finite distorsion and finite perimeter.** Let us start by defining the function classes that we are going to be dealing with.

Definition 2.1 (Finite distorsion). Let $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ be an open set. A Sobolev map $y \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^n)$ with $\det \nabla y \geq 0$ almost everywhere in Ω is said to be of *finite distorsion* if $\det \nabla y \in L_{\text{loc}}^1(\Omega)$ and there is a function $K : \Omega \to [1, +\infty]$ with $K < +\infty$ almost everywhere in Ω such that $|\nabla y|^n \leq K \det \nabla y$. For a mapping y of finite distorsion, the *(optimal) distorsion function* $K_y : \Omega \to \mathbb{R}$ is defined as

$$K_y := \begin{cases} |\nabla y|^n / \det \nabla y & \text{if } \det \nabla y \neq 0 \\ 1 & \text{if } \det \nabla y = 0. \end{cases}$$

The relation of our theory to the former one by M. Šilhavý is encoded in the following characterization result for sets of finite perimeters in the actual configuration. Although it will be later applied just for $n = 3$, we state the characterization here for general dimension, for we believe that it could be of independent interest.

Theorem 2.2 (Characterization of sets of finite perimeter). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $n \geq 2$. Suppose that $E \subset \Omega$ is a measurable set and that $y \in W_{\text{loc}}^{1,n}(\Omega; \mathbb{R}^n)$ is a homeomorphism of finite distorsion. Then $\text{Per}(E^y, \Omega^y) < \infty$ if and only if there exists a finite Radon measure $p_{y,E} \in \mathcal{M}(\Omega; \mathbb{R}^n)$ such that there holds*

$$\int_E \text{cof}\,(\nabla y) : \nabla \psi \, \mathrm{d}x = \int_\Omega \psi \cdot \mathrm{d}p_{y,E} \qquad \forall \psi \in C_{\text{c}}^\infty(\Omega; \mathbb{R}^n). \tag{2.1}$$

In this case, $\text{Per}(E^y, \Omega^y) = |p_{y,E}|(\Omega)$.

A proof of this characterization is provided in Section 4.

In the following, we call $p_{y,E}$ a *interfacial measure* if it is a finite Radon measure and it fulfills (2.1) for some y and E within the assumption frame of Theorem 2.2. This naming is hinting to the relevance that such measures enjoy within the theory by M. Šilhavý [40, 41], see Subsection 2.5 below, i.e., they measure the elastic energy of interfaces between phases. Theorem 2.2 proves in particular that, given an admissible deformation y, interfacial measures correspond one-to-one to sets of finite perimeter in the deformed configuration Ω^y.

Notice in particular that, by taking y to be the identity map on Ω, Theorem 2.2 reduces to the classical characterization of sets E of finite perimeter in Ω [2, Thm. 3.36], namely those sets such that there exists a finite measure p_E with

$$\int_E \text{div}\psi \, \mathrm{d}x = \int_\Omega \psi \cdot \mathrm{d}p_E \qquad \forall \psi \in C_{\text{c}}^\infty(\Omega; \mathbb{R}^n).$$

2.2. **Admissible states.** From now on let the open, bounded, and Lipschitz domain $\Omega \subset \mathbb{R}^3$ indicate the reference configuration. The body undergoes a deformation $y : \Omega \to \mathbb{R}^3$, which is assumed to be a Sobolev mapping of finite distorsion. We will in fact ask that y is orientation-preserving, i.e., $\det \nabla y > 0$ almost everywhere in Ω. It is well known

that positivity of $\det \nabla y$ ensures only the local injectivity of y [11]. However, it is shown by Ciarlet and Nečas [12] that if $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ for some $p > 3$, $\det \nabla y > 0$ almost everywhere, and additionally the so-called *Ciarlet-Nečas condition*

$$\int_\Omega \det \nabla y(x) \, \mathrm{d}x \leq |\Omega^y| \tag{2.2}$$

holds, then almost every point in Ω^y has only one preimage. Under such assumptions, as we will thoroughly discuss in Section 3, everywhere injectivity (so that the deformation is a homeomorphism) can be further enforced by requiring that the distorsion function K_y is in $L^q(\Omega)$ for some $q > 2$.

Therefore, we define the set of *admissible deformations* as

$$\mathbb{Y} := \left\{ y \in W^{1,p}(\Omega; \mathbb{R}^3) \mid \det \nabla y > 0 \text{ a.e. }, \int_\Omega \det \nabla y(x) \, \mathrm{d}x \leq |y(\Omega)|, \, K_y \in L^q(\Omega) \right\} \tag{2.3}$$

where $p > 3$ and $q > 2$ are fixed. We shall check in Section 3 that admissible deformations are homeomorphisms, see Theorem 3.5. In particular, the deformed configuration Ω^y is an open set.

We consider a material with two different phases (e.g., two martensitic variants of a shape memory alloy) which we indicate with the subscripts 0 and 1. To indicate the portion $E \subset \Omega$ of the reference configuration where one finds phase 1, one defines $z : \Omega \to \{0,1\}$ and $\zeta : \Omega^y \to \{0,1\}$ to be the characteristic functions of E and E^y, respectively. In particular, we have that $z = \zeta \circ y$.

The set of *admissible states* (y, ζ) is defined as

$$\mathbb{Q} := \{(y, \zeta) \mid y \in \mathbb{Y}, \, \zeta \in BV(\Omega^y; \{0,1\})\}.$$

Similarly, we define the set of admissible states for the phase-field approximation as

$$\overline{\mathbb{Q}} := \{(y, \zeta) \mid y \in \mathbb{Y}, \, \zeta \in BV(\Omega^y; [0,1])\}.$$

2.3. Assumptions on the bulk energy.

We assume that W_0 and W_1 are *polyconvex* [3], i.e., for $F \in \mathbb{R}^{3 \times 3}$

$$W_i(F) := \begin{cases} h_i(F, \mathrm{cof}\, F, \det F) & \text{if } \det F > 0, \\ \infty & \text{otherwise} \end{cases} \tag{2.4}$$

for some convex functions $h_i : \mathbb{R}^{19} \to \mathbb{R}$, $i = 0, 1$. In addition, we assume W_i to be coercive, frame-indifferent, and unbounded as $\det F \to 0+$. More precisely, for $i = 0, 1$, we assume that there exist $C > 0$ such that

$$W_i(F) \geq C \left(|F|^p + \frac{|F|^{3q}}{(\det F)^q} - 1 \right) \qquad \forall F \in \mathbb{R}^{3 \times 3}, \, p > 3, \, q > 2, \tag{2.5}$$

$$W_i(RF) = W_i(F) \qquad \forall R \in \mathrm{SO}(3), F \in \mathbb{R}^{3 \times 3}, \tag{2.6}$$

$$W_i(F) \to \infty \quad \text{as} \quad \det F \to 0_+ \tag{2.7}$$

where SO(3) is the special orthogonal group $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid RR^T = I, \ \det R = 1\}$. The third term on the right-hand side of (2.5) ensures that deformation gradients $F = \nabla y$ with finite energy will have a q-integrable distorsion function $F \mapsto |F|^3 / \det F$. Notice that $F \mapsto |F|^3 / \det F$ is polyconvex on the set of matrices with positive determinant.

Eventually, we specify boundary conditions by imposing admissible deformations to match a given deformation y_0 at the boundary $\partial\Omega$. To this aim, we assume that

$$\exists (y_0, \zeta_0) \in \mathbb{Q} \ \text{ with } \ \mathcal{F}_0(y_0, \zeta_0) < \infty \tag{2.8}$$

and define

$$\mathbb{Q}_{y_0} := \{(y, \zeta) \in \mathbb{Q} \mid y = y_0 \text{ on } \partial\Omega\}.$$

Analogously, we consider

$$\overline{\mathbb{Q}}_{y_0} := \{(y, \zeta) \in \overline{\mathbb{Q}} \mid y = y_0 \text{ on } \partial\Omega\}.$$

Letting $\Gamma_0 \subset \partial\Omega$ be relatively open in $\partial\Omega$ with $\mathcal{H}^2(\Gamma_0) > 0$, we also define

$$\mathbb{Q}_{(y_0, \Gamma_0)} := \{(y, \zeta) \in \mathbb{Q} \mid y = y_0 \text{ on } \Gamma_0\}.$$

2.4. **Main results.** We are now in the position of stating the main results of the paper, which concern existence for the sharp-interface minimization problem and convergence of the phase-field approximation.

Theorem 2.3 (Existence of minimizers). *Under assumptions* (2.4)-(2.8) *the functional* $\mathcal{F}_0$ *admits a minimizer on* $\mathbb{Q}_{(y_0, \Gamma_0)}$.

A proof of this statement is in Section 5.

Our second main result delivers a Modica-Mortola-type approximation via the functionals $\mathcal{F}_\varepsilon$ from (1.5), corresponding indeed to diffuse-interface models. Under the additional assumption that the current configuration Ω^y is a Lipschitz domain (which is not necessarily true for general $y \in W^{1,p}(\Omega; \mathbb{R}^3)$) we have the following

Theorem 2.4 (Phase-field approximation). *Under assumptions* (2.4)-(2.8), *for any* $\varepsilon > 0$ *the functional* $\mathcal{F}_\varepsilon$ *admits a minimizer on* $\overline{\mathbb{Q}}_{y_0}$. *If* Ω^{y_0} *is a Lipschitz domain and* $\varepsilon_k \to 0$, *then, for every sequence* (y_k, ζ_k) *of minimizers of* $\mathcal{F}_{\varepsilon_k}$ *on* $\overline{\mathbb{Q}}_{y_0}$, *there exists* $(y, \zeta) \in \mathbb{Q}_{y_0}$ *such that, up to not relabeled subsequences,*

   i) $y_k \to y$ *weakly in* $W^{1,p}(\Omega; \mathbb{R}^3)$, *and* $\|\zeta_k - \zeta\|_{L^1(\Omega^y)} \to 0$ *as* $k \to \infty$.
   ii) (y, ζ) *minimizes* $\mathcal{F}_0$ *on* $\mathbb{Q}_{y_0}$.

2.5. **Relation with Šilhavý's theory.** Before moving on, let us comment on our results in light of the theory by M. Šilhavý [40, 42]. To this end, we need to clarify the definition of the general interfacial-energy term in (1.2), which requires introducing some measure theoretic setting. We recall that the *reduced boundary* of a finite perimeter set E in Ω is defined as the set of points x of Ω such that $x \in \mathrm{supp}\, |\nabla \chi_E|$ and such that the limit

$n_E(x) := \lim_{\varepsilon \to 0} \frac{-\nabla \chi_E(B(x,\varepsilon))}{|\nabla \chi_E|(B(x,\varepsilon))}$ exists and satisfies $|n_E(x)| = 1$ (see [2, Def. 3.54]). We say that n_E is the *outer measure-theoretic unit normal* to E. We let

$$\mathcal{Q} := \{(y, z) \mid y \in W^{1,p}(\Omega), \det \nabla y > 0 \text{ a.e. in } \Omega, z \in BV(\Omega; \{0, 1\})\}.$$

For any pair $(y, z) \in \mathcal{Q}$, let $E := \{z = 1\}$, let S denote the reduced boundary of the finite perimeter set E in Ω, and let n_E denote the corresponding outer measure-theoretic unit normal. Following [41, Def. 3.1], we denote by $\mathcal{Q}_0 \subset \mathcal{Q}$ the set of all pairs $(y, z) \in \mathcal{Q}$ for which there exists a finite Radon measure $m_{y,E} := (a_{y,E}, h_{y,E}, p_{y,E}) \in \mathcal{M}(\Omega; \mathbb{R}^{15})$ such that $a_{y,E} := n_E \mathcal{H}^2_{|S}$ and such that there hold (2.1) and

$$\int_E \nabla y \, (\operatorname{curl} \psi) \, \mathrm{d}x = \int_\Omega \psi \, \mathrm{d}h_{y,E} \qquad \forall \psi \in C_{\mathrm{c}}^\infty(\Omega; \mathbb{R}^3). \qquad (2.9)$$

Consider a positively 1-homogeneous convex function $\Psi : \mathbb{R}^{15} \to \mathbb{R}$ such that

$$\Psi(A) \geq C|A| \quad \text{for some } C > 0 \text{ and all } A \in \mathbb{R}^{15}. \qquad (2.10)$$

If $|m_{y,E}|$ denotes the total variation of $m_{y,E}$, the interfacial energy is then defined as

$$\mathcal{F}_{\text{Šilhavý}}^{\text{int}}(y, z) := \begin{cases} \displaystyle\int_\Omega \Psi\left(\frac{\mathrm{d}m_{y,E}}{\mathrm{d}|m_{y,E}|}\right) \mathrm{d}|m_{y,E}| & \text{for } (y, z) \in \mathcal{Q}_0, \\ +\infty & \text{otherwise.} \end{cases} \qquad (2.11)$$

On the other hand, the bulk energy in the reference configuration is defined as

$$\widetilde{\mathcal{F}}^{\text{bulk}}(y, z) := \int_\Omega \Big(z \, W_1(\nabla y) + (1 - z) \, W_0(\nabla y) \Big) \, \mathrm{d}x$$

where W_i are assumed to satisfy (2.4), (2.6)-(2.7), and $W_i(F) \geq C|F|^p$ for $i = 0, 2$ and some $p > 3$. Under such assumptions on W_i and (2.10), Šilhavý proves that $\widetilde{\mathcal{F}}^{\text{bulk}}(y, z) + \mathcal{F}_{\text{Šilhavý}}^{\text{int}}(y, z)$ admits a minimizer on $\{(y, z) \in \mathcal{Q}_0 \mid y = y_0 \text{ on } \partial\Omega\}$, see [41, Thm. 3.3] and [42, Thm. 1.2]. Our Characterization Theorem 2.2 shows in particular that, under the further assumption of y being a homeomorphism, the perimeter of the image set $E^y = \{z = 1\}^y$ is finite in Ω^y. More specifically, Theorem 2.2 provides a characterization of those deformations that admit an interfacial measure $p_{y,E} \in \mathcal{M}(\Omega; \mathbb{R}^3)$.

The existence result of Theorem 2.3 refers to the specific case (1.3) within the larger class (1.2). As such, the global coercivity assumption (2.10) is not required.

3. Admissible deformations are homeomorphisms

The aim of this section is to check that the continuous representative of the class of the admissible deformations $y \in \mathbb{Y}$ (2.3) is injective, hence a homeomorphism between Ω and Ω^y, see Theorem 3.5 below. We have $p > 3$ and $q > 2$ in the definition (2.3) of $\mathbb{Y}$. Through this section only, we allow for the case $p = 3$ (recalling that a $W^{1,3}(\Omega; \mathbb{R}^3)$ mapping of finite distorsion has a continuous representative [23, Thm. 2.3]). We break

down the argument into Lemmas, which we believe to be of an independent interest. Let us start with a definition.

Definition 3.1 (almost-everywhere injectivity)**.** We say that $y : \Omega \to \mathbb{R}^3$ is *almost-everywhere injective* if there exists $\omega \subset \Omega$ such that $|\omega| = 0$ and $y(x_1) \neq y(x_2)$ for every $x_1, x_2 \in \Omega \setminus \omega$ satisfying $x_1 \neq x_2$.

Given $y : \Omega \to \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, and a subset $\omega \subset \Omega$, we define the *Banach indicatrix* $N(\xi, y, \omega)$ by

$$N(\xi, y, \omega) := \#\{x \in \omega \mid y(x) = \xi\} , \tag{3.1}$$

where the right-hand-side denotes the cardinality (i.e., the number of elements) of the set. The map $y : \Omega \to \mathbb{R}^3$ is said to satisfy *Lusin's condition N* if it maps negligible sets to negligible sets, namely $|\omega^y| = 0$ for all $\omega \subset \Omega$ such that $|\omega| = 0$. Moreover, it satisfies *Lusin's condition N^{-1}* if the preimage of any negligible set is negligible, namely $|y^{-1}(\omega)| = 0$ for all $\omega \subset \Omega^y$ such that $|\omega| = 0$.

Any continuous map $y \in W^{1,p}(\Omega; \mathbb{R}^3)$, $p > 3$, satisfies the Lusin's condition N [23, Thm. 4.2] and the same holds for $p = 3$ if y is a mapping of finite distorsion [23, Thm. 4.5]. This implies the validity of the area formula with equality [23, Thm. A.35]. Moreover, $p > 3$ (or $p = 3$ and finite distorsion) implies that y is differentiable almost everywhere [23, Cor. 2.25]. If in addition $\det \nabla y > 0$ almost everywhere in Ω, then y satisfies Lusin's condition N^{-1} as well [8, Thm. 8.3, Lem. 8.3-8.4]. This in particular implies that the continuous representative of $y \in \mathbb{Y}$ fulfils both Lusin's N and N^{-1} condition (even in case $p = 3$).

Let us present a first result on almost-everywhere injectivity, see [20, Prop. 3.2] for a similar argument.

Lemma 3.2 (Ciarlet-Nečas implies almost-everywhere injectivity)**.** *Let $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ be continuous, $p \geq 3$, and $\det \nabla y > 0$ almost everywhere in Ω. If the Ciarlet-Nečas condition (2.2) holds, then y is almost-everywhere injective in the sense of Definition 3.1.*

Proof. Since y has finite distorsion, by the above discussion y satisfies Lusin's condition N. Hence, the area formula holds with equality. The Ciarlet-Nečas condition (2.2) implies that

$$|\Omega^y| \leq \int_{y(\Omega)} N(\xi, y, \Omega) \, \mathrm{d}\xi = \int_\Omega \det \nabla y \, \mathrm{d}x \leq |\Omega^y|,$$

which entails $N(\xi, y, \Omega) = 1$ for almost every $\xi \in \Omega^y$. The set $\omega := \{\xi \in y(\Omega) \mid N(\xi, y, \Omega) > 1\}$ is hence negligible. Since by [8, Thm. 8.3, Lem. 8.3-8.4] y satisfies Lusin's condition N^{-1}, we get that $|\{x \in \Omega \mid y(x) \in \omega\}| = 0$ as well, which corresponds to the statement. $\square$

Maps that are almost-everywhere injective still include rather nonphysical situations, for a dense, countable set of points could be mapped to a single point. We refer to [25] for

an example of a map which is continuous, injective almost everywhere but not everywhere. We shall hence present a result in the direction of everywhere injectivity.

Lemma 3.3 (a.e. injectivity and openness imply injectivity). *Let $y : \Omega \to \mathbb{R}^3$ be continuous, almost-everywhere injective, open (maps open sets to open sets), and fulfill Lusin's condition N. Then, y is everywhere injective in Ω.*

Proof. Assume by contradiction that y is not everywhere injective, i.e. that there exist $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$ such that $y(x_1) = y(x_2) =: a$. The openness of y implies that Ω^y is open. We can hence find $\varepsilon > 0$ such that $B(a, \varepsilon) \subset \Omega^y$. Continuity implies that $y^{-1}(B(a, \varepsilon)) \subset \Omega$ is open. As $x_1, x_2 \in y^{-1}(B(a, \varepsilon))$ one can find two open disjoint neighborhoods U, V such that $x_1 \in U$, $x_2 \in V$ and $U^y \cap V^y \ni a$. As U^y and V^y are both open their intersection is also open and therefore $|U^y \cap V^y| > 0$, i.e. $N(\xi, y, \Omega) > 1$ for every $\xi \in U^y \cap V^y$. On the other hand, the pre-image of $U^y \cap V^y$ must have a positive measure because y satisfies Lusin's condition N. This contradicts almost-everywhere injectivity and concludes the proof. $\qquad\square$

Let us now recall a sufficient condition for the openness of a map.

Lemma 3.4 ([23, Thm. 3.4]). *Let $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ be continuous, $p \geq 3$. Assume that $K_y \in L^q(\Omega)$ for some $q > 2$. Then y is either constant or open.*

We are finally in the position of stating the main result of this section (in the following result, $p = 3$ in the definition (2.3) of $\mathbb{Y}$ is allowed).

Theorem 3.5 (Admissible deformations are homeomorphisms). *The continuous representative of $y \in \mathbb{Y}$ is everywhere injective on Ω.*

Proof. Let $y \in \mathbb{Y}$ be the continuous representative of the equivalence class. Lemma 3.4 implies that y is either constant or open. However, it cannot be constant because it is almost everywhere injective by Lemma 3.2. Hence, it is open. By Lemma 3.3, y is everywhere injective on Ω. By the Invariance of Domain Theorem y is a homeomorphism between Ω and Ω^y. $\qquad\square$

4. Šilhavý measure and perimeter: Proof of Theorem 2.2

Within this section, Ω is assumed to be an open subset of $\mathbb{R}^n$, $n \geq 2$. In particular, we are not restricting here to $n = 3$. We are interested in properties of Sobolev homeomorphisms y in relation to sets of finite perimeter. In case y is bi-Lipschitz, sets of finite perimeter are mapped onto sets of finite perimeter, see [2, Theorem 3.16] whereas the same property does not hold for y in $W^{1,p}$ with $p < \infty$. The aim of this section is that of proving Theorem 2.2, which characterizes pairs (y, E) (y is a Sobolev map and $E \subset \Omega$ is a measurable set) such that E^y is of finite perimeter in Ω^y. We start by preparing some preliminary result.

Proposition 4.1 (Perimeter = total variation of the interfacial measure). *Assume that $E \subset \Omega$ is measurable, $y \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is a homeomorphism, and there exists a vector Radon measure $p_{y,E} \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ such that (2.1) holds. Then, $\mathrm{Per}(E^y, \Omega^y) = |p_{y,E}|(\Omega)$. In particular, if we assume that $p_{y,E}$ is finite, we get that the perimeter of E^y in Ω^y is finite as well.*

Proof. A homeomorphism in $W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ satisfies the Lusin's condition N [35, Thm. 3] and is almost-everywhere differentiable [23, Cor. 2.2.5]. Thanks to the Lusin's condition N, the area formula holds with equality and gives

$$\mathrm{Per}(E^y, \Omega^y) = \sup \left\{ \int_{E^y} \mathrm{div}\varphi(\xi)\, \mathrm{d}\xi \mid \varphi \in C^\infty_c(\Omega^y; \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}$$

$$= \sup \left\{ \int_E \mathrm{div}\varphi(y(x)) \det\nabla y(x)\, \mathrm{d}x \mid \varphi \in C^\infty_c(\Omega^y; \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

Note that the identity

$$(\mathrm{div}\varphi) \circ y \det\nabla y = \mathrm{cof}\,\nabla y : \nabla(\varphi \circ y) \tag{4.1}$$

holds almost everywhere in Ω. Here ":" denotes the dot product in $\mathbb{R}^{3\times 3}$. Indeed, we may write $\mathrm{div}\varphi = \nabla\varphi : I$ (where I is the identity matrix), and relation (4.1) follows from the chain-rule formula $\nabla(\varphi \circ y) = (\nabla\varphi \circ y)\nabla y$, which is valid almost everywhere in Ω, and from the matrix identity $(\mathrm{cof}\,A)A^T = I \det A$. Therefore, we get

$$\mathrm{Per}(E^y, \Omega^y) = \sup \left\{ \int_E \mathrm{cof}\,(\nabla y) : \nabla(\varphi \circ y)\, \mathrm{d}x \mid \varphi \in C^\infty_c(\Omega^y; \mathbb{R}^3),\ \|\varphi\|_\infty \leq 1 \right\}. \tag{4.2}$$

As $y \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, we have $\mathrm{cof}\,\nabla y \in L^r_{\mathrm{loc}}(\Omega)$ with $r = n/(n-1)$. Formula (2.1) can be extended by continuity to all test functions in the class $W^{1,n}(\Omega; \mathbb{R}^n) \cap C^0_c(\Omega; \mathbb{R}^n)$ since $p_{y,E}$ is a measure and the conjugated exponent of r is n. Fix now $\varphi \in C^\infty_c(\Omega^y; \mathbb{R}^3)$ and notice that there holds $\varphi \circ y \in C^0_c(\Omega; \mathbb{R}^n)$, as y is a homeomorphism and hence $y^{-1}(\mathrm{supp}(\varphi))$ is compact in Ω. Moreover, since $y \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, we have that $\varphi \circ y \in W^{1,n}(\Omega; \mathbb{R}^n)$. Therefore, $\varphi \circ y$ is an admissible test function for equality (2.1).

From (4.2) and the extension of (2.1) to $W^{1,n}(\Omega; \mathbb{R}^n) \cap C^0_c(\Omega; \mathbb{R}^n)$ we obtain

$$\mathrm{Per}(E^y, \Omega^y) = \sup \left\{ \int_\Omega (\varphi \circ y) \cdot \mathrm{d}p_{y,E} \mid \varphi \in C^\infty_c(\Omega^y; \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}. \tag{4.3}$$

On the other hand, the total variation of $p_{y,E}$ is, by definition,

$$|p_{y,E}|(\Omega) = \sup \left\{ \int_\Omega f \cdot \mathrm{d}p_{y,E}(x) \mid f \in C^0_c(\Omega; \mathbb{R}^n),\ \|f\|_\infty \leq 1 \right\}. \tag{4.4}$$

From (4.3) and (4.4) it immediately follows that

$$\mathrm{Per}(E^y, \Omega^y) \leq |p_{y,E}|(\Omega). \tag{4.5}$$

In order to establish the reverse inequality, one has to prove that any $f \in C^\infty_c(\Omega; \mathbb{R}^n)$ can be uniformly approximated by functions of the form $\varphi \circ y$, with $\varphi \in C^\infty_c(\Omega^y; \mathbb{R}^n)$.

Fix $f \in C_c^0(\Omega; \mathbb{R}^n)$ and $K := \mathrm{supp}(f)$. Then K^y is compact in Ω^y. On K^y, define the function $g := f \circ y^{-1}$, which can be extended to $g \in C_c^0(\Omega^y; \mathbb{R}^n)$ by setting $g = 0$ outside K^y. For all $\varepsilon > 0$ choose now $\varphi_\varepsilon \in C_c^\infty(\Omega^y; \mathbb{R}^n)$ with $\sup_{\Omega^y} |g - \varphi_\varepsilon| < \varepsilon$. Then, one has that $\sup_\Omega |f - \varphi_\varepsilon \circ y| < \varepsilon$, which provides the desired approximation. $\qquad\square$

The proof of Theorem 2.4 follows from checking the converse statement of Proposition 4.1. In order to achieve this, a crucial role is played by the following result on Sobolev homeomorphisms of finite distorsion due to Csörnyei, Hencl, and Malý [13], see also [22, 24].

Proposition 4.2 ([13, Theorem 1.2]). *Let $y \in W_{\mathrm{loc}}^{1,n-1}(\Omega; \mathbb{R}^n)$ be a homeomorphism of finite distorsion. Then $y^{-1} \in W_{\mathrm{loc}}^{1,1}(\Omega^y; \mathbb{R}^n)$ and is of finite distorsion.*

Taking advantage of the latter result, we can now proceed to the proof of Theorem 2.2.

Proof of Theorem 2.2. Given Proposition 4.1, we are left with the converse statement. Namely, for all $E \subset \Omega$ measurable and all $y \in W_{\mathrm{loc}}^{1,n}(\Omega; \mathbb{R}^n)$ homeomorphism of finite distorsion with $\mathrm{Per}(E^y, \Omega^y) < \infty$ we should find a finite Radon measure (the interfacial measure) such that relation (2.1) holds.

Let $\psi \in C_c^\infty(\Omega; \mathbb{R}^n)$ with $\|\psi\|_\infty \le 1$ be given. Since y is a homeomorphism, we have that $\psi \circ y^{-1} \in C_c^0(\Omega^y; \mathbb{R}^n)$. By Proposition 4.2, we also get $\psi \circ y^{-1} \in W^{1,1}(\Omega^y; \mathbb{R}^n)$. Let $\varepsilon > 0$ and $\varphi_\varepsilon \in C_c^\infty(\Omega^y; \mathbb{R}^n)$ be defined by $\varphi_\varepsilon := (\psi \circ y^{-1}) * \rho_\varepsilon$, where $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$ and ρ is the standard unit symmetric mollifier in $\mathbb{R}^n$. Notice that, by choosing ε_0 small enough one has that the support of φ_ε is compact in Ω^y for any $0 < \varepsilon < \varepsilon_0$. Moreover, $\|\varphi_\varepsilon\|_\infty \le 1$ and φ_ε converge strongly to $\psi \circ y^{-1}$ in $W^{1,1}(\Omega^y; \mathbb{R}^n)$ as $\varepsilon \to 0$. As y satisfies the Lusin's condition N the area formula holds with equality, hence

$$\int_{E^y} \mathrm{div}(\psi \circ y^{-1})\,\mathrm{d}\xi = \int_{E^y} I : (\nabla\psi) \circ y^{-1} \nabla y^{-1}\,\mathrm{d}\xi = \int_E (\det \nabla y)\, I : \nabla\psi\,(\nabla y^{-1} \circ y)\,\mathrm{d}x. \quad (4.6)$$

Since $\nabla y^{-1}(y(x)) = (\nabla y(x))^{-1}$ holds at any differentiability point x of y such that $\det \nabla y(x) > 0$, hence almost everywhere in the set $\{\det \nabla y > 0\}$, from (4.6) we deduce

$$\begin{aligned}
\int_{E^y} \mathrm{div}(\psi \circ y^{-1})\,\mathrm{d}\xi &= \int_{\{\det \nabla y > 0\}} (\det \nabla y)\, I : \nabla\psi\,(\nabla y)^{-1}\,\mathrm{d}x \\
&= \int_{\{\det \nabla y > 0\}} \det \nabla y\,(\nabla y)^{-T} : \nabla\psi\,\mathrm{d}x = \int_E \mathrm{cof}\,\nabla y : \nabla\psi\,\mathrm{d}x.
\end{aligned} \quad (4.7)$$

Notice that the last equality in (4.7) follows from the fact that y is of finite distorsion, which implies $\mathrm{cof}\,\nabla y = 0$ almost everywhere on $\{\det \nabla y = 0\}$. Similarly, by the area formula and by (4.1) we obtain

$$\int_E \mathrm{cof}\,\nabla y : \nabla(\varphi_\varepsilon \circ y)\,\mathrm{d}x = \int_E \det \nabla y\, I : (\nabla\varphi_\varepsilon) \circ y\,\mathrm{d}x = \int_{E^y} \mathrm{div}\varphi_\varepsilon\,\mathrm{d}\xi. \quad (4.8)$$

Since $\mathrm{div}\varphi_\varepsilon$ converges to $\mathrm{div}(\psi \circ y^{-1})$ in $L^1(\Omega^y)$ as $\varepsilon \to 0$, from (4.8) we get

$$\lim_{\varepsilon \to 0} \int_E \mathrm{cof}\,\nabla y : \nabla(\varphi_\varepsilon \circ y)\,\mathrm{d}x = \int_{E^y} \mathrm{div}(\psi \circ y^{-1})\,\mathrm{d}\xi.$$

By combining the latter with (4.2) and (4.7), with we deduce

$$\int_E \mathrm{cof}\,\nabla y : \nabla\psi\,\mathrm{d}x = \lim_{\varepsilon \to 0}\int_E \mathrm{cof}\,\nabla y : \nabla(\varphi_\varepsilon \circ y)\,\mathrm{d}x$$

$$\leq \sup\left\{\int_E \mathrm{cof}\,\nabla y : \nabla(\varphi \circ y)\,\mathrm{d}x \mid \varphi \in C_\mathrm{c}^\infty(\Omega^y;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1\right\}$$

$$= \mathrm{Per}(E^y, \Omega^y).$$

We have hence checked that

$$\sup\left\{\int_E \mathrm{cof}\,\nabla y : \nabla\psi\,\mathrm{d}x \mid \psi \in C_\mathrm{c}^\infty(\Omega;\mathbb{R}^n),\ \|\psi\|_\infty \leq 1\right\} \leq \mathrm{Per}(E^y, \Omega^y) < \infty.$$

This implies that the distributional divergence of $\chi_E\mathrm{cof}\,\nabla y$ is a finite measure on Ω. $\square$

The interfacial measure $p_{E,y}$ given by Theorem 2.2 is the distributional divergence of $-\chi_E\mathrm{cof}\,\nabla y$. Therefore, in order to have that $\mathrm{Per}(E^y, \Omega^y) < \infty$, Theorem 2.2 requires $\chi_E\mathrm{cof}\,\nabla y$ to be a divergence measure field. By strengthening the assumptions one may obtain improved characterizations of the divergence of such fields, see for instance [1, 10, 40]. In particular, we can prove the following.

Proposition 4.3 (Support of the interfacial measure). *Under the assumptions of Proposition 4.1 let* $\mathrm{Per}(E, \Omega) < \infty$. *Then,* $p_{E,y}$ *is concentrated on the closure of the reduced boundary of* E *in* Ω.

Proof. Let $y_\varepsilon := y * \rho_\varepsilon$, with $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$ and ρ be the standard mollifier. Since y_ε is smooth, χ_E is a function of bounded variation, and the cofactor is divergence-free, we readily have that $\mathrm{div}(\chi_E\mathrm{cof}\,\nabla y_\varepsilon) = \mathrm{cof}\,\nabla y_\varepsilon \nabla\chi_E$ is a measure concentrated on the reduced boundary of E in Ω. Notice that $\{\mathrm{cof}\,\nabla y_\varepsilon\}$ converges weakly to $\mathrm{cof}\,\nabla y$ in $L_\mathrm{loc}^{n/(n-1)}(\Omega;\mathbb{R}^{n\times n})$ (see e.g. [11]) , so that integration by parts entails

$$-\int_\Omega \psi \cdot \mathrm{d}(\mathrm{div}(\chi_E\mathrm{cof}\,\nabla y_\varepsilon)) = \int_\Omega \chi_E\mathrm{cof}\,\nabla y_\varepsilon : \nabla\psi\,\mathrm{d}x \to \int_E \mathrm{cof}\,\nabla y : \nabla\psi\,\mathrm{d}x = \int_\Omega \psi \cdot \mathrm{d}p_{E,y}$$

as $\varepsilon \to 0$, for every $\psi \in C_\mathrm{c}^\infty(\Omega;\mathbb{R}^n)$. For all $\varepsilon > 0$, the measure $\mathrm{div}(\chi_E\mathrm{cof}\,\nabla y_\varepsilon)$ is concentrated on the reduced boundary of E in Ω. We hence conclude that $p_{E,y}$ is concentrated on the closure of the reduced boundary. $\square$

In case $y^{-1} \in W_\mathrm{loc}^{1,n}(\Omega^y;\mathbb{R}^n)$ the characterization of Theorem 2.4 can be applied to the inverse deformation y^{-1}. Note that such regularity of the inverse follows for instance for mappings with L^{n-1} distorsion, see [24]. Therefore, We have the following

Corollary 4.4 (Characterization for the inverse deformation). *Suppose that $E \subset \Omega$ is a measurable set and that $y \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is a homeomorphism of finite distorsion with $K_y \in L^{n-1}(\Omega)$. Then, $\mathrm{Per}(E, \Omega) < \infty$ if and only if the distribution $p_{E^y, y^{-1}} := -\mathrm{div}(\chi_{E^y} \mathrm{cof}\, \nabla y^{-1})$ is a finite Radon measure on Ω^y.*

5. Existence of minimizers: Proof of Theorem 2.3

The aim of this section is to discuss the existence of minimizers of both $\mathcal{F}_0$ and $\mathcal{F}_\varepsilon$ on the respective sets of admissible deformations. This in particular proves Theorem 2.3 as well as the existence statement in Theorem 2.4.

We start by establishing some preliminary result on the convergence of the deformed domains and phase configurations associated to a $\mathbb{Y}$-converging sequence of deformations. A crucial tool in this direction is the semicontinuity of the perimeter in the deformed configuration, when both the ambient sets Ω^{y_k} and the finite perimeter sets $F_k \subset \Omega^{y_k}$ vary along a sequence, see Proposition 5.4. This will prove to be essential for the Γ-limit result stated in Section 6.

We shall make use of the following equiintegrability result for inverse Jacobians of mappings of integrable distorsion, which is inspired by the work of Onninen and Tengvall [34].

Lemma 5.1 (Equiintegrability of $\det \nabla y_k^{-1}$). *Let $y_k \in W^{1,1}_{loc}(\Omega; \mathbb{R}^3)$ be homeomorphisms with uniformly L^q-integrable distorsion for $q > 2$ (namely, $\|K_{y_k}\|_{L^q(\Omega)}$ is bounded independently of k). Then, $\det \nabla y_k^{-1}$ are equiintegrable on Ω^{y_k}.*

Proof. From [34, Theorem 1.4] we have that

$$\int_{\Omega^{y_k}} |\nabla y_k^{-1}|^3 \log^s(\mathrm{e} + |\nabla y_k^{-1}|) \, \mathrm{d}\xi \le C \int_\Omega K_{y_k}^q \, \mathrm{d}\xi,$$

where $s = 2(q-2)$ and C is a constant depending only on q. Notice that by the elementary inequality $|\det F| \le 6|F|^3$, we have

$$|\nabla y_k^{-1}|^3 \log^s(\mathrm{e} + |\nabla y_k^{-1}|) \ge \frac{1}{6} 3^{-s} \det \nabla y_k^{-1} \log^s\left(\mathrm{e}^3 + \frac{1}{6} \det \nabla y_k^{-1}\right).$$

We conclude that

$$\int_{\Omega^{y_k}} \det \nabla y_k^{-1} \log^s\left(\mathrm{e}^3 + \frac{1}{6} \det \nabla y_k^{-1}\right) \mathrm{d}\xi \le C' \int_\Omega K_{y_k}^q \, \mathrm{d}\xi,$$

where C' depends only on q. The latter right-hand side is uniformly bounded. This entails that the superlinear function of the determinant on the left-hand side is uniformly bounded as well. This implies the equiintegrability of the sequence of the determinants of the inverses. $\square$

Lemma 5.2 (Convergence of deformed configurations). *Let $y, y_k \in \mathbb{Y}$ such that $y_k \to y$ weakly in $W^{1,p}$, $p > 3$ (hence uniformly). Then,*

(i) *For any open sets A, O such that $A \subset\subset \Omega^y \subset\subset O$, one has $A \subset \Omega^{y_k} \subset O$ for k large enough. In particular, $|\Omega^y \Delta \Omega^{y_k}| \to 0$;*

(ii) *If $\|K_{y_k}\|_{L^q(\Omega)} \leq c$ uniformly, by letting $O^k := \Omega^y \cap \Omega^{y_k}$, there holds*

$$|\Omega \setminus (y^{-1}(O_k) \cap y_k^{-1}(O_k))| \to 0.$$

Proof. Ad (i): Let V be open and such that $A \subset\subset V \subset\subset \Omega^y$. Since $\overline{A}$ and ∂V are disjoint compact sets, we have that $d(\overline{A}, \partial V) =: 2\delta > 0$. Let $U = y^{-1}(V) \subset\subset \Omega$ and $V_k = y_k(U)$. Since $y, y^k \in \mathbb{Y}$ are homeomorphisms on $\overline{U}$, we have $\partial V = y(\partial U)$ and $\partial V_k = y_k(\partial U)$. As $p > 3$, we have that $y_k \to y$ in $C(\overline{\Omega}; \mathbb{R}^3)$, thus $\|y - y_k\|_\infty < \delta$ for k large enough. Hence, for any boundary point $\xi \in \partial V_k$, we have that $d(\xi, \partial V) < \delta$ for k large, which yields $\overline{A} \subset V_k \subset \Omega^{y_k}$ owing to $d(\overline{A}, \partial V) = 2\delta$.

As $O \supset\supset \Omega^y$ we deduce as above that $d(\partial O, \overline{\Omega^y}) =: 2\delta$ for some $\delta > 0$, which immediately yields the inclusion $\overline{\Omega^y} + B(0, \delta) \subset O$. Then, since $\|y - y_k\|_\infty < \delta$ we have that

$$\Omega^{y_k} \subset \Omega^y + B(0, \delta) \subset \overline{\Omega^y} + B(0, \delta) \subset O.$$

In order to check that $|\Omega^y \Delta \Omega^{y_k}| \to 0$, observe that Ω^y can be approximated in measure by open sets $A_\ell \subset\subset \Omega^y$ (Ω^y can be approximated by internal compact sets). Moreover, $\overline{\Omega^y}$ can be approximated in measure by external open sets $O_\ell \supset \overline{\Omega^y}$. Since Ω is a bounded Lipschitz domain, by Lusin's N property for (a $W^{1,p}$ extension of) y and the fact $y(\partial\Omega) \supset \partial(\Omega^y)$, it follows that $|\partial(\Omega^y)| = 0$, i.e. $|\overline{\Omega^y}| = |\Omega^y|$.

Ad (ii): Since $y^{-1}(O_k) \subset \Omega$ and $y_k^{-1}(O_k) \subset \Omega$, it is sufficient to prove

$$|y^{-1}(O_k)| \to |\Omega|, \qquad |y_k^{-1}(O_k)| \to |\Omega|.$$

Firstly, $|\Omega^y \setminus O^k| \to 0$ by (i). Hence, since $\det \nabla y^{-1} \in L^1(\Omega^y)$,

$$|y^{-1}(O_k)| = \int_{O^k} \det \nabla y^{-1} \, d\xi \to \int_{\Omega^y} \det \nabla y^{-1} \, d\xi = |\Omega|.$$

Secondly,

$$|y_k^{-1}(O_k)| = \int_{O^k} \det \nabla y_k^{-1} \, d\xi = \int_{\Omega^{y_k}} \det \nabla y_k^{-1} \, d\xi - \int_{\Omega^{y_k} \setminus O^k} \det \nabla y_k^{-1} \, d\xi =$$

$$= |\Omega| - \int_{\Omega^{y_k} \setminus \Omega^y} \det \nabla y_k^{-1} \, d\xi.$$

By Lemma 5.1, the determinants ∇y_k^{-1} are equiintegrable. Since $|\Omega^{y_k} \setminus \Omega^y| \to 0$, the statement follows. $\square$

Lemma 5.3 (Convergence of the phases). *Let $y, y_k \in \mathbb{Y}$ such that $y_k \to y$ weakly in $W^{1,p}$, for $p > 3$, and have uniformly L^q-bounded distorsion, for $q > 2$. Let $\zeta \in L^\infty(\Omega^y, [0,1])$ and $\zeta_k \in L^\infty(\Omega^{y_k}, [0,1])$. Finally, let $z = \zeta \circ y$, $z_k = \zeta_k \circ y_k \in L^\infty(\Omega; [0,1])$ and $O_k := \Omega^y \cap \Omega^{y_k}$. Then,*

$$\|\zeta - \zeta_k\|_{L^1(O_k)} \to 0 \quad \Rightarrow \quad \|z - z_k\|_{L^1(\Omega)} \to 0.$$

Proof. By introducing the shorthand $E_k := y^{-1}(O^k) \cap y_k^{-1}(O^k) \subset \Omega$, we start by observing that

$$\|z_k - z\|_{L^1(\Omega)} \le |\Omega \setminus E_k| + \|z_k - z\|_{L^1(E_k)}.$$

As $|\Omega \setminus E_k| \to 0$ by Lemma 5.2, we are left to prove that $+\|z_k - z\|_{L^1(E_k)} \to 0$. One uses the triangle inequality to write

$$\|z_k - z\|_{L^1(E_k)} \le I_k^{(1)} + I_k^{(2)},$$

with

$$I_k^{(1)} := \|\zeta_k \circ y_k - \zeta \circ y_k\|_{L^1(E_k)}, \qquad I_k^{(2)} := \|\zeta \circ y_k - \zeta \circ y\|_{L^1(E_k)}$$

The L^q-bound on the distortion and Lemma 5.1 entail that the sequence $\det \nabla y_k^{-1}$ is equiintegrable. Let $\rho : [0, +\infty) \to [0, +\infty)$ (monotonically increasing) be a modulus of equiintegrability for the sets $\{\det \nabla y_k^{-1}\}_{k \ge 1} \cup \{\det \nabla y^{-1}\}$, i.e., for any measurable set $A \subset \mathbb{R}^3$ we ask for $\lim_{t \to 0+} \rho(t) = 0$ and

$$\int_{\Omega^y \cap A} \det \nabla y^{-1} \, d\xi \ \vee \ \sup_k \int_{\Omega^{y_k} \cap A} \det \nabla y_k^{-1} \, d\xi \le \rho(|A|).$$

Now, fix $\delta > 0$ and change variable $x \mapsto \xi$ in the integral in $I_k^{(1)}$ getting

$$I_k^{(1)} = \int_{y_k(E_k)} \det \nabla y_k^{-1} |\zeta_k - \zeta| \, d\xi \le \rho(|A_k(\delta)|) + \delta|\Omega|, \tag{5.1}$$

where $A_k(\delta) := \{\xi \in O^k \mid |\zeta_k(\xi) - \zeta(\xi)| > \delta\}$. Since $\|\zeta - \zeta_k\|_{L^1(O_k)} \to 0$ one has that $|A_k(\delta)| < \delta$ for k large enough.

In order to control $I_k^{(2)}$, let $\overline{\zeta}_\delta \in C^0(\overline{\Omega^y}, \mathbb{R})$ be a (uniformly) continuous L^1 approximation of ζ such that $\|\overline{\zeta}_\delta - \zeta\|_{L^1(\Omega^y)}$ is so small that

$$|\overline{A}(\delta)| < \delta \quad \text{for} \quad \overline{A}(\delta) := \{\xi \in \Omega^y \mid |\overline{\zeta}_\delta(\xi) - \zeta(\xi)| > \delta\}$$

We write $I_k^{(2)} \le J_k^{(1)} + J_k^{(2)} + J_k^{(3)}$, with

$$J_k^{(1)} = \|\zeta \circ y_k - \overline{\zeta}_\delta \circ y_k\|_{L^1(E_k)}, \ J_k^{(2)} = \|\overline{\zeta}_\delta \circ y_k - \overline{\zeta}_\delta \circ y\|_{L^1(E_k)}, \ J_k^{(3)} = \|\overline{\zeta}_\delta \circ y - \zeta \circ y\|_{L^1(E_k)}.$$

Now, similarly to (5.1), we can write

$$J_k^{(1)} + J_k^{(3)} \le 2\rho(|\overline{A}(\delta)|) + 2\delta|\Omega| \le 2\big(\rho(\delta) + \delta|\Omega|\big). \tag{5.2}$$

Finally, since $\overline{\zeta}_\delta$ is uniformly continuous and $|\Omega| < +\infty$, if ω_δ is the modulus of uniform continuity of $\overline{\zeta}_\delta$, we get

$$J_k^{(2)} \le \omega_\delta(\|y - y_k\|_\infty)|\Omega|. \tag{5.3}$$

By combining (5.1) and (5.3) and using the fact that δ is arbitrary, we obtain the statement. $\square$

The following result concerns the semicontinuity of the perimeter of sets in the deformed configuration along sequences of suitably converging sets and deformations. This is based on the characterization result from Theorem 2.2.

Proposition 5.4 (Lower semicontinuity of the perimeter). *Let $(y_k, \zeta_k) \in \mathbb{Q}$, $y \in \mathbb{Y}$, $\zeta \in L^\infty(\Omega^y, \{0; 1\})$ with y, y_k satisfying the assumptions of Lemma 5.3. Let $F = \{\xi \in \Omega^y \mid \zeta(\xi) = 1\}$, $F_k = \{\xi \in \Omega^{y_k} \mid \zeta_k(\xi) = 1\}$ and assume $|F_k \Delta F| \to 0$. If $I := \liminf_{k \to +\infty} \mathrm{Per}(F_k, \Omega^{y_k}) < \infty$, then*

$$\mathrm{Per}(F, \Omega^y) \leq I \quad \text{and} \quad (y, \zeta) \in \mathbb{Q};$$

Proof. Letting $E = y^{-1}(F)$, and $E_k = y_k^{-1}(F_k)$, we have by Theorem 2.2 that

$$\mathrm{Per}(F_k, \Omega^{y_k}) = |p_{y_k, E_k}|.$$

By applying Lemma 5.3 to $\zeta = \chi_F, \zeta_k = \chi_{F_k}$ we deduce that $\chi_{E_k} \to \chi_E$ in $L^1(\Omega)$. Moreover, since $\nabla y_k \to \nabla y$ weakly in $L^p(\Omega)$, the convergence $\mathrm{cof}\, \nabla y_k \to \mathrm{cof}\, \nabla y$ holds weakly in $L^{p/2}(\Omega)$. Therefore, for any test function $\psi \in C_c^\infty(\Omega; \mathbb{R}^3)$, as $k \to \infty$ we have

$$\int_\Omega \psi \cdot \mathrm{d}p_{y_k, E_k} = \int_\Omega \chi_{E_k} \mathrm{cof}\, \nabla y_k : \nabla \psi \, \mathrm{d}x \to \int_\Omega \chi_E \mathrm{cof}\, \nabla y : \nabla \psi \, \mathrm{d}x =: p_{y, E}(\psi),$$

where the last equality is a definition of the distribution on the right side. By the lower semicontinuity of the total variation, we have that $|p_{y,E}| \leq I$. We conclude by Theorem 2.2 as $\mathrm{Per}(F, \Omega^y) = |p_{y,E}|$. $\qquad\square$

After this preparatory discussion, we eventually move to the existence proof for minimizers. First we show that the diffuse-interface functional $\mathcal{F}_\varepsilon$ admits a minimizer for every $\varepsilon > 0$. Such existence result is part of the statement of Theorem 2.4. Indeed, we restate it here in a slightly more general form, in which the Dirichlet boundary condition is imposed only on a subset of the boundary of positive $\mathcal{H}^2$-measure, as it is customary in elasticity theory.

Proposition 5.5 (Existence for the diffuse-interface model). *Under assumptions (2.4)-(2.7), let $\Gamma_0 \subset \partial\Omega$ be relatively open in $\partial\Omega$ with $\mathcal{H}^2(\Gamma_0) > 0$. Moreover, let $\varepsilon > 0$, $y_0 \in \mathbb{Y}$ and $\zeta_0 \in W^{1,2}(\Omega^{y_0}; [0, 1])$ be such that the set $\widetilde{\mathbb{Q}}_{(y_0, \Gamma_0)} := \{(y, \zeta) \mid y \in \mathbb{Y}, \zeta \in W^{1,2}(\Omega^y; [0, 1]), y = y_0 \text{ on } \Gamma_0\}$ is nonempty and $\mathcal{F}_\varepsilon(y_0, \zeta_0) < \infty$. Then, there is a minimizer of $\mathcal{F}_\varepsilon$ on $\widetilde{\mathbb{Q}}_{(y_0, \Gamma_0)}$.*

Proof. Let $(y_k, \zeta_k) \in \widetilde{\mathbb{Q}}_{(y_0, \Gamma_0)}$ be a minimizing sequence for $\mathcal{F}_\varepsilon$. The coercivity (2.5) and the generalized Friedrichs inequality imply that one can extract a not relabeled subsequence such that $y_k \to y$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$. The boundary condition and the Ciarlet-Nečas condition (2.2) are readily preserved in the limit. Moreover, one has that the distorsion $K_y \in L^q(\Omega)$ as the function $F \to |F|^3 / \det F$ is polyconvex and $F_k = \nabla y_k$ are weakly converging. We conclude that $y \in \mathbb{Y}$ and $y = y_0$ on Γ_0.

For every $\delta > 0$, let $O_\delta := \{\xi \in \Omega^y \mid \mathrm{dist}(\xi, \partial\Omega^y) > \delta\} \subset\subset \Omega^y$. By Lemma 5.2 we have that $\Omega^y = \cup_\delta O_\delta$ and $O_\delta \subset \Omega^{y_k}$ for k large. Denote by η_k and H_k the zero extensions on $\mathbb{R}^3$ of ζ_k and $\nabla \zeta_k$ respectively. The coercivity of $\mathcal{F}_\varepsilon^{\mathrm{int}}$ implies that one can extract not relabeled subsequences such that $\eta_k \to \eta$ weakly* in $L^\infty(\mathbb{R}^3)$ and $H_k \to H$ weakly in

$L^2(\mathbb{R}^3)$. Set now $\zeta := \eta|_{\Omega^y}$. For every $\xi_0 \in O_\delta$ and $B(\xi_0, r) \subset O_\delta$ we have that $\eta_k \to \eta$ weakly in $W^{1,2}(B(\xi_0, r))$. This implies that $H = \nabla\eta = \nabla\zeta$ almost everywhere in $B(\xi_0, r)$. Moreover, by possibly extracting again one has that $\eta_k \to \eta$ strongly in $L^2(B(\xi_0, r))$. As every $\xi \in \Omega^y$ belongs to some O_δ for δ small enough, we get that $H = \nabla\zeta$ almost everywhere in Ω^y. It is also easy to see that $\eta = 0$, $H = 0$ almost everywhere on the complement of Ω^y due to the uniform convergence of y_k. Indeed, if $\xi_0 \notin \overline{\Omega^y}$ then there are two open disjoint neighborhoods of ξ_0 and $\overline{\Omega^y}$. Let $O \supset \overline{\Omega^y}$ be the open neighborhood of $\overline{\Omega^y}$. Then for k large enough $\Omega^{y_k} \subset O$ (Lemma 5.2), i.e. $\eta_k = 0$, $H_k = 0$ in a neighborhood of ξ_0. Consequently, $\eta(\xi_0) = 0$, $H(\xi_0) = 0$ at least if ξ_0 is a Lebesgue point of η and H.

The latter argument shows that $\eta_k \to \eta$ pointwise almost everywhere in the complement of $\overline{\Omega^y}$. Up to possibly extracting again, we hence have that $\eta_k \to \eta$ pointwise almost everywhere in $\mathbb{R}^3$ as well. In fact, the pointwise convergence in $\overline{\Omega^y}$ follows since $\eta_k \to \eta$ strongly in $L^2(B(\xi_0, r))$ for any $B(\xi_0, r) \subset\subset \Omega^y$ and $|\eta - \eta_k| \le 1$ almost everywhere.

Using the Fatou Lemma, we find

$$\liminf_{k\to\infty} \mathcal{F}_\varepsilon^{\mathrm{int}}(y_k, \zeta_k) = \liminf_{k\to\infty} \int_{\mathbb{R}^3} \left(\frac{\varepsilon}{2}|H_k|^2 + \frac{1}{\varepsilon}\Phi(\eta_k)\right) \mathrm{d}\xi \ge \int_{\mathbb{R}^3} \left(\frac{\varepsilon}{2}|H|^2 + \frac{1}{\varepsilon}\Phi(\eta)\right) \mathrm{d}\xi$$

$$= \int_{\Omega^y} \left(\frac{\varepsilon}{2}|\nabla\zeta|^2 + \frac{1}{\varepsilon}\Phi(\zeta)\right) \mathrm{d}\xi = \mathcal{F}_\varepsilon^{\mathrm{int}}(y, \zeta) \tag{5.4}$$

which shows the weak lower semicontinuity of the interfacial energy.

To show the weak lower semicontinuity of the bulk contribution, we write it as

$$\widetilde{\mathcal{F}}^{\mathrm{bulk}}(y, z) = \int_\Omega \left(z(x)W_1(\nabla y(x)) + (1 - z(x))W_0(\nabla y(x))\right) \mathrm{d}x,$$

Notice that the integrand is continuous in z and convex in ∇y and in its minors. Let now $z_k := \zeta_k \circ y_k$ and recall from Lemma 5.3 entails that $z_k \to z = \zeta \circ y$ in $L^1(\Omega)$. By applying [19, Cor. 7.9] we get that $\liminf_{k\to\infty} \widetilde{\mathcal{F}}^{\mathrm{bulk}}(y_k, z_k) \ge \widetilde{\mathcal{F}}^{\mathrm{bulk}}(y, z)$. Consequently,

$$\liminf_{k\to\infty} \mathcal{F}^{\mathrm{bulk}}(y_k, \zeta_k) = \liminf_{k\to\infty} \widetilde{\mathcal{F}}^{\mathrm{el}}(y_k, z_k) \ge \widetilde{\mathcal{F}}^{\mathrm{bulk}}(y, z) = \mathcal{F}^{\mathrm{bulk}}(y, \zeta). \tag{5.5}$$

Together with (5.4), the latter proves that (y, ζ) is a minimizer of $\mathcal{F}_\varepsilon$ on $\widetilde{\mathbb{Q}}_{(y_0, \Gamma_0)}$ by means of the direct method [14]. $\qquad\square$

We conclude this Section by providing a proof of Theorem 2.3.

Proof of Theorem 2.3. Let $(y_k, \zeta_k) \in \mathbb{Q}_{(y_0, \Gamma_0)}$ be a minimizing sequence for $\mathcal{F}_0$. As in the proof of Proposition 5.5, we can assume, up to extraction of a not relabeled subsequence, that $y_k \to y$ weakly in $W^{1,p}$ for some $y \in \mathbb{Y}$.

Letting $F_k = \{\zeta_k = 1\}$, we can identify the sequence of states with (y_k, F_k). Since the interface energy is bounded along the sequence (y_k, F_k), the sets F_k have uniformly bounded perimeters, namely, $\mathrm{Per}(F_k, \Omega^{y_k}) \le c$. For $\ell \in \mathbb{N}$, let $O^\ell := \{x \in \Omega^y | \mathrm{dist}(x, \partial\Omega^y) > 2^{-\ell}\} \subset\subset \Omega^y$. As $O^\ell \subset \Omega^{y_k}$ for k large enough due to Lemma 5.2, for any given $\ell \in \mathbb{N}$ we

have that $\limsup_k \operatorname{Per}(F_k, O^\ell) \le c$. We can hence find a measurable set $G^\ell \subset O^\ell$ and a not relabeled subsequence F_h such that
$$|(F_h \Delta G^\ell) \cap O^\ell| \to 0 \quad \text{for} \quad h \to \infty.$$
For all $\ell' > \ell$ we can further extract a subsequence $F_{h'}$ from F_h above in such a way that $|(F_{h'} \Delta G^{\ell'}) \cap O^{\ell'}| \to 0$ and $G^{\ell'} \cap O^\ell = G^\ell$. From the nested family of subsequences corresponding to $\ell = 1, 2, \dots$ we extract by a diagonal argument a further subsequence $F_{k'}$. By setting $F := \cup_\ell G^\ell$ and, owing to $O^\ell \nearrow \Omega^y$, we get that
$$|(F_{k'} \Delta F) \cap \Omega^y| \to 0.$$
Now, the set F has finite perimeter in Ω^y as a consequence of Proposition 5.4. By letting $\zeta = \chi_F|_{\Omega^y}$ we then have that $(y, \zeta) \in \mathbb{Q}_{y_0}$.

One is left to check that $\mathcal{F}_0(y, \zeta) \le \liminf \mathcal{F}_0(y_k, \zeta_k)$, which follows from the lower semicontinuity of $\mathcal{F}_0$. Indeed, the lower semicontinuity of bulk part of $\mathcal{F}_0$ follows by the argument of Proposition 5.5. As concerns the interface term, one just needs to recall Proposition 5.4. □

6. Convergence of phase-field approximations: Proof of Theorem 2.4

This section is devoted to the proof of the convergence Theorem 2.4. The argument relies on Γ-convergence [9, 15]. In particular, we prove a Γ-$\liminf$ inequality for the interfacial part in Proposition 6.1 and construct a recovery sequence in Proposition 6.2. Let us start by the former.

Proposition 6.1 (Γ-$\liminf$ inequality). *Let* $(y_k, \zeta_k), (y, \zeta) \in \overline{\mathbb{Q}}$ *be such that*

  i) $\liminf_{k \to +\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k) < \infty$ *for some sequence* $\varepsilon_k \to 0$,
  ii) $y_k \to y$ *weakly in* $W^{1,p}(\Omega; \mathbb{R}^3)$, $p > 3$,
  iii) $\lim_{k \to +\infty} \|\zeta_k - \zeta\|_{L^1(O^k)} = 0$, *with* $O^k := \Omega^{y_k} \cap \Omega^y$.

Then, there exists $E^y \subset \Omega^y$ *measurable such that*
$$\zeta = \chi_{E^y} \quad \text{and} \quad \gamma \operatorname{Per}(E^y, \Omega^y) \le \liminf_{k \to +\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k).$$

In particular, one has that $(y, \zeta) \in \mathbb{Q}$.

Proof. Moving from Proposition 5.4, the proof proceeds along the lines of the classical Modica-Mortola Γ-convergence result [32]. As $\liminf_{k \to +\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y, \zeta) < \infty$ and $\Phi(s) = 0$ only for $s = 0, 1$, we have that $\zeta = \chi_F$, for some measurable set $F \subset \Omega^y$. By using the coarea formula we deduce that
$$\mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k) = \int_{\Omega^{y_k}} \left(\frac{\varepsilon_k}{2} |\nabla \zeta_k|^2 + \frac{1}{\varepsilon_k} \Phi(\zeta_k) \right) \mathrm{d}\xi$$
$$\ge \int_{\Omega^{y_k}} \sqrt{2\Phi(\zeta_k)}\, |\nabla \zeta_k|\, \mathrm{d}\xi = \int_0^1 \sqrt{2\Phi(s)}\, \operatorname{Per}(\{\zeta_k > s\}, \Omega^{y_k})\, \mathrm{d}s$$

Given any $\delta \in (0,1)$ and $s \in [\delta, 1-\delta]$ one has that

$$|\{\xi \in \Omega^{y_k} \mid \zeta_k > s\} \Delta F| \leq \frac{1}{\delta} \|\zeta_k - \zeta\|_{L^1(O^k)} + |\Omega^{y_k} \Delta \Omega^y|$$

Therefore, by applying Lemma 5.2 we get

$$|\{\xi \in \Omega^{y_k} \mid \zeta_k > s\} \Delta F| \to 0 \quad \forall s \in [\delta, 1-\delta].$$

Owing to Proposition 5.4 we obtain

$$\mathrm{Per}(F, \Omega^y) \leq \liminf_{k \to +\infty} \mathrm{Per}(\{\zeta_k > s\}, \Omega^{y_k}) \quad \forall s \in [\delta, 1-\delta].$$

Hence, as $\delta \in (0,1)$, by applying the Fatou Lemma one gets

$$
\begin{aligned}
\liminf_{k \to +\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k) &\geq \int_0^1 \sqrt{2\Phi(s)} \liminf_{k \to +\infty} \mathrm{Per}(\{\zeta_k > s\}, \Omega^{y_k}) \, \mathrm{d}s \\
&\geq \int_\delta^{1-\delta} \sqrt{2\Phi(s)} \liminf_{k \to +\infty} \mathrm{Per}(\{\zeta_k > s\}, \Omega^{y_k}) \, \mathrm{d}s \\
&\geq \int_\delta^{1-\delta} \sqrt{2\Phi(s)} \cdot \mathrm{Per}(F, \Omega^y) \, \mathrm{d}s
\end{aligned}
$$

and the assertion follows as $\int_\delta^{1-\delta} \sqrt{2\Phi(s)} \, \mathrm{d}s \to \gamma$ for $\delta \to 0$. $\qquad \square$

The existence of a recovery sequence is a direct consequence of the classical Modica-Mortola theorem [32] as soon as we assume that Ω^y is a Lipschitz domain. Although this Lipschitz continuity could fail to hold for general deformations, we can enforce it by asking $y_0(\Omega)$ to be a Lipschitz domain where y_0 is the imposed boundary deformation, see, e.g., [4] for a similar argument. Note that the Lipschitz assumption on Ω^y was not needed for the $\Gamma\text{-}\liminf$ inequality of Proposition 6.1.

Proposition 6.2 (Recovery sequence). *If $(y, \zeta) \in \overline{\mathbb{Q}}_{y_0}$, $y_0(\Omega) \subset \mathbb{R}^3$ being a Lipschitz domain, and $F = \{\zeta = 1\}$, there exists a sequence $\zeta_k \subset W^{1,2}(\Omega^y; [0,1])$ such that*

$$\lim_{k \to \infty} \|\zeta_k - \zeta\|_{L^1(\Omega^y)} = 0 \quad and \quad \gamma \mathrm{Per}(F, \Omega^y) + \mathcal{F}^{\mathrm{bulk}}(y, \zeta) = \lim_{k \to \infty} \mathcal{F}_{\varepsilon_k}(y, \zeta_k).$$

Proof. The sequence ζ_k is delivered by the classical Modica-Mortola construction [32] applied to the functional $\mathcal{F}^{\mathrm{int}}_\varepsilon(y, \zeta)$ with y fixed. In fact, once the interface part convergence, the bulk part also follows because $\mathcal{F}^{\mathrm{bulk}}(y, \zeta)$ is strongly continuous in ζ. $\qquad \square$

We eventually combine the $\Gamma\text{-}\liminf$ inequality of Proposition 6.1 and the recovery-sequence construction of Proposition 6.2 in order to prove Theorem 2.4.

Proof of Theorem 2.4. Existence of minimizers (y_k, ζ_k) for $\mathcal{F}_{\varepsilon_k}$ has already been checked in Proposition 5.5. Let (y_0, ζ_{0k}) be the recovery sequence for (y_0, ζ_0) whose existence is proved in Proposition 6.2. By comparing with (y_0, ζ_{0k}) one gets that

$$\mathcal{F}^{\mathrm{el}}(y_k, \zeta_k) + \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k) = \mathcal{F}_{\varepsilon_k}(y_k, \zeta_k) \leq \mathcal{F}_{\varepsilon_k}(y_0, \zeta_{0k}) < C < \infty$$

where we have used the fact that $\mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_0, \zeta_{0k}) \to \mathcal{F}^{\mathrm{int}}_0(y_0, \zeta_0)$. The latter bound and the coercivity (2.5) ensures that $y_k \to y$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$ and $|\Omega^y \Delta \Omega^{y_k}| \to 0$ by Lemma 5.2, for some not relabeled subsequence. On the other hand, since Ω^{y_k} contains any open set $A \subset\subset \Omega^y$ for large k, the latter bound on $\mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k)$ yields strong $L^1(A)$ compactness for the sequence ζ_k. This implies the existence of $\zeta \in L^\infty(\Omega^y; [0,1])$ such that $\|\zeta_k - \zeta\|_{L^1(O^k)} \to 0$ for some not relabeled subsequence, as in the proof of Theorem 2.3. Proposition 6.1 ensures that ζ is a characteristic function and

$$\mathcal{F}^{\mathrm{int}}_0(y, \zeta) \leq \liminf_{k \to \infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k, \zeta_k).$$

Moreover, for all $(\tilde{y}, \tilde{\zeta}) \in \mathbb{Q}_{y_0}$ Proposition 6.2 ensures that there exists a recovery sequence $\tilde{\zeta}_k$ such that $\mathcal{F}_{\varepsilon_k}(\tilde{y}, \tilde{\zeta}_k) \to \mathcal{F}_0(\tilde{y}, \tilde{\zeta})$. As the bulk term $\mathcal{F}^{\mathrm{bulk}}$ is lower semicontinuous, we conclude that

$$\mathcal{F}_0(y, \zeta) \leq \liminf_{k \to \infty} \mathcal{F}_{\varepsilon_k}(y_k, \zeta_k) \leq \liminf_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\tilde{y}, \tilde{\zeta}_k) = \mathcal{F}_0(\tilde{y}, \tilde{\zeta}).$$

Hence, (y, ζ) minimizes $\mathcal{F}_0$ on $\mathbb{Q}_{y_0}$. $\qquad\square$

Acknowledgements

References

[1] Ambrosio, L., Crippa, G., Maniglia, S.: Traces and fine properties of a BD class of vector fields and applications. *Ann. Fac. Sci. Toulouse Math. (6)*, **14** (2005), 527–561
[2] Ambrosio, L., Fusco, N., Pallara, D.: *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford mathematical monographs. Oxford University Press, Oxford, 2000.
[3] Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **63** (1977), 337–403.
[4] Ball, J.M.: Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. Roy. Soc. Edinburgh, Series A*, **88** (1981), 315–328.
[5] Barchiesi, M., DeSimone, A.: Frank energy for nematic elastomers: a nonlinear model. *ESAIM Control Optim. Calc. Var.* **21** (2015), 372–377.
[6] Barchiesi, M., Henao, D., Mora-Corral, C.: Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity. *Arch. Ration. Mech. Anal.* **224** (2017), 743–816.
[7] Bielski, W., Gambin, B.: Relationship between existence of energy minimizers of incompressible and nearly incompressible magnetostrictive materials. *Rep. Math. Phys.* **66** (2010), 147–157.
[8] Bojarski, B., Iwaniec, T.: Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **8** (1983), 257–324.
[9] Braides, A.: *Γ-convergence for beginners*. Oxford Lecture Series in Mathematics and its Applications, Vol. 22. Oxford University Press, Oxford, 2002.

[10] Chen, G.Q., Torres, M., Ziemer, W. P.: Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Comm. Pure Appl. Math.* **62** (2009), 242–304.

[11] Ciarlet, P.G.: *Mathematical Elasticity*, Vol. I: Three-dimensional Elasticity, North-Holland, Amsterdam, 1988.

[12] Ciarlet, P.G., Nečas, J: Injectivity and self-contact in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **97** (1987), 171–188.

[13] Csörnyei, M., Hencl, S., Malý, J.: Homeomorphisms in the Sobolev space $W^{1,n-1}$. *J. Reine Angew. Math.* **644** (2010), 221–235.

[14] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* 2nd. ed., Springer, Berlin, 2008.

[15] Dal Maso, G.: *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[16] DeSimone, A., James, R. D.: A constrained theory of magnetoelasticity. *J. Mech. Phys. Solids*, **50** (2002), 283–320.

[17] Fischer, F.D., Waitz, T., Vollath, D., Simha, N.K.: On the role of surface energy and surface stress in phase-transforming nanoparticles. *Progress Mat. Sci.* **53** (2008), 481–527.

[18] Fitzpatrick, R.: *An introduction to celestial mechanics.* Cambridge University Press, Cambridge, 2012.

[19] Fonseca, I., Leoni, G.: *Modern Methods in the Calculus of Variations: L^p-Spaces.* Springer, New York, 2007.

[20] Giacomini, A., Ponsiglione, M.: Non-interpenetration of matter for SBV deformations of hyperelastic brittle materials. *Proc. Roy. Soc. Edinburgh Sect. A*, **138** (2008), 1019–1041.

[21] Gurtin, M.E., Murdoch, A.: A continuum theory of elastic material surfaces. *Arch. Ration. Mech. Anal.* **57** (1975), 291–323.

[22] Hencl, S., Koskela, P.: Regularity of the inverse of a planar Sobolev homeomorphism. *Arch. Ration. Mech. Anal.* **180** (2006), 75–95 (2006)

[23] Hencl, S., Koskela, P.: *Lectures on mappings of finite distortion.* Lecture Notes in Mathematics, Vol. 2096, Springer, 2014.

[24] Hencl, S., Koskela, P., Malý, J.: Regularity of the inverse of a Sobolev homeomorphism in space. *Proc. Roy. Soc. Edinburgh Sect. A* **136A** (2006), 1267–1285.

[25] Izzo, A.: Existence of continuous functions that are one-to-one almost everywhere. *Math. Scand.* **118** (2016), 269–276.

[26] Javili, A., McBride, A., Steinmann, P.: Thermomechanics of solids with lower-dimensional energetics: On the importance of surface, interface, and curve structures at the nanoscale. A unifying review. *Appl. Mech. Rev.* **65** (2013), 010802 (31 pages).

[27] Kružík, M., Stefanelli, U., Zeman, J.: Existence results for incompressible magnetoelasticity. *Discrete Contin. Dyn. Syst.* **35** (2015), 2615–2623.

[28] Levitas, V.I., Javanbakht, M.: Surface tension and energy in multivariant martensitic transformations: phase-field theory, simulations, and model of coherent interface. *Phys. Rev. Lett.* **105** (2010), 165701.

[29] Levitas, V.I.: Phase field approach to martensitic phase transformations with large strains and interface stresses. *J. Mech. Phys. Solids*, **70** (2014), 154–189.

[30] Levitas, V.I., Warren, J.A.: Phase field approach with anisotropic interface energy and interface stresses: Large strain formulation. *J. Mech. Phys. Solids*, **91** (2016), 94–125.

[31] Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.* **98** (1987), 123–142.

[32] Modica, L., Mortola, S.: Un esempio di Γ-convergenza. (Italian). *Boll. Un. Mat. Ital. B*, **14** (1977), 285–299.

[33] Nix, W.D., Gao, H.J.: An atomistic interpretation of interface stress. *Scr. Mater.* **39** (1998), 1653–1661.

[34] Onninen, J. , Tengvall, V.: Mappings of L^p-integrable distortion: regularity of the inverse. *Proc. Roy. Soc. Edinburgh Sect. A*, **146** (2016), 647–663.

[35] Reshetnyak, Y. G.: Some geometrical properties of functions and mappings with generalized derivatives. *Sibirsk. Math. Zh.* **7** (1966), 886–919.

[36] Richter, T.: *Fluid-structure interactions. Models, analysis and finite elements.* Lecture Notes in Computational Science and Engineering, 118. Springer, Cham, 2017.

[37] Rosato, D., Miehe, C.: Dissipative ferroelectricity at finite strains. Variational principles, constitutive assumptions and algorithms. *Internat. J. Engrg. Sci.* **74** (2014), 162–189.

[38] Rybka, P., Luskin, M.: Existence of energy minimizers for magnetostrictive materials. *SIAM J. Math. Anal.* **36** (2005), 2004–2019.

[39] Šilhavý, M.: *The Mechanics and Thermodynamics of Continuous Media.* Texts and Monographs in Physics. Springer, Berlin, 1997.

[40] Šilhavý, M.: Divergence measure fields and Cauchy's stress theorem. *Rend. Sem. Mat. Univ. Padova*, **113** (2005), 15–45.

[41] Šilhavý, M.: Phase transitions with interfacial energy: interface null Lagrangians, polyconvexity, and existence. In: Hackl, K. (ed.) *IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials*, pp. 233–244. Springer, Dordrecht (2010).

[42] Šilhavý, M.: Equilibrium of phases with interfacial energy: A variational approach. *J. Elast.* **105** (2011), 271–303.

[43] Stefanelli, U.: Existence for dislocation-free finite plasticity. *ESAIM Control Optim. Calc. Var.* https://doi.org/10.1051/cocv/2018014

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