# Twisting Noncommutative Geometries with Applications to High Energy Physics

by

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## Abstract

With the bare essentials of noncommutative geometry (defined by a spectral triple), we first describe how it naturally gives rise to gauge theories. Then, we quickly review the notion of twisting (in particular, minimally) noncommutative geometries and how it induces a Wick rotation, that is, a transition of the metric signature from euclidean to lorentzian. We focus on comparatively more tractable examples of spectral triples; such as the ones corresponding to a closed riemannian spin manifold, $\mathsf{U}(1)$ gauge theory, and electrodynamics. By minimally twisting these examples and computing their associated fermionic actions, we demonstrate how to arrive at physically relevant actions (such as the Weyl and Dirac actions) in Lorentz signature, even though starting from euclidean spectral triples. In the process, not only do we extract a physical interpretation of the twist, but we also capture exactly how the Wick rotation takes place at the level of the fermionic action.

Thesis Supervisor: Dr. Pierre Martinetti

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*"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation. There are at present fundamental problems in theoretical physics awaiting solution, e.g., the relativistic formulation of quantum mechanics and the nature of atomic nuclei (to be followed by more difficult ones such as the problem of life), the solution of which problems will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence to get the necessary new ideas by direct attempts to formulate the experimental data in mathematical terms. The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities (by a process like Eddington's Principle of Identification)."*

– P.A.M. Dirac, *Quantised singularities in the electromagnetic field, Proceedings of the Royal Society of London A* **133**: 821 (1931).

# Introduction

Understanding the intrinsic nature of spacetime is not only one of the most fundamental quests for theoretical physicists, but also a mountainous challenge for mathematicians. In the light of how the advent of general relativity was facilitated and brought forth by Riemann's very broad extension and abstract generalization of euclidean differential geometry of surfaces in $\mathbb{R}^3$, it is quite likely that accommodating Planck scale physics calls for a revision of our notions about geometry – towards geometric objects that are much more flexible than differentiable manifolds.

*Noncommutative geometry* (NCG) [C94] is such an approach to generalize riemannian geometry by giving a purely spectral/operator-algebraic characterization of geometry [C13]; in the same sense as Gel'fand duality (§A) provides an algebraic characterization of topology. The mathematical object encapsulating such a characterization of geometry in a generalized sense is a *spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consisting of a unital $*$-algebra $\mathcal{A}$ of bounded operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator $\mathcal{D}$ with compact resolvent on $\mathcal{H}$ such that the commutator $[\mathcal{D}, a]$ is bounded for any $a \in \mathcal{A}$. Spectral triples naturally give rise to gauge theories.

One of the most spectacular achievements of NCG for particle physics is the full derivation of its most important gauge theory, i.e. the Standard Model (SM) lagrangian, along with all its delicacies including the Higgs potential, spontaneous symmetry breaking, neutrino mixing, etc. [CCM] and the Einstein-Hilbert action [CC96, CC97]. In fact, during the early developmental stages, SM was regarded as one of the guiding principles behind the blueprints of the framework [C95]. NCG provides a purely geometric/gravitational description of the SM, where gravity is naturally present with minimal coupling to matter [C96, CC10, CS].

NCG offers various ways to build models even beyond the SM, see e.g. [CS, DKL] for a recent review. One of them involves twisting the spectral triple of the SM by an algebra automorphism [DLM1, DLM2, DM], in the sense of Connes and Moscovici [CMo]. This provides a mathematical justification to the extra scalar field introduced in [CC12] to both fit the mass of the Higgs and stabilize the electroweak vacuum. A significant difference from the construction based on spectral triples without first-order condition [CCS1, CCS2] is that the twist does not only yield an extra scalar field $\sigma$, but also a supplementary real one-form field $X_\mu$,[1] whose meaning was rather unclear so far.

[1] In [DM] this field was improperly called *vector field*.

Although Connes' work provides a spectral characterization of compact riemannian spin manifolds [C13] along with the tools for their noncommutative generalization [C96], extending this program to the pseudo-riemannian case is notoriously difficult, and there has, so far, been no completely convincing model of pseudo-riemannian spectral triples. However, several interesting results in this context have been obtained recently, see e.g. [BB, BBB, DPR, Fr, FE], there is nevertheless no reconstruction theorem for pseudo-riemannian manifolds in view, and it is still unclear how the spectral action should be handled in a pseudo-riemannian signature.

Quite unexpectedly, the twist of the SM, which has been introduced in a purely riemannian context, seems to have some link with Wick rotation. In fact, the inner product induced by the twist on the Hilbert space of euclidean spinors on a four-dimensional manifold $\mathcal{M}$, coincides with the Kreĭn product of lorentzian spinors [DFLM]. This is not so surprising, for the twist $\rho$ coincides with the automorphism that exchanges the two eigenspaces of the grading operator (in physicist's words: that exchanges the left and the right components of spinors). And this is nothing but the inner automorphism induced by the first Dirac matrix $\gamma^0 = c(dx^0)$. This explains why, by twisting, one is somehow able to single out the $x_0$ direction among the four riemannian dimensions of $\mathcal{M}$. However, the promotion of this $x_0$ to a "time direction" is not fully accomplished, at least not in the sense of Wick rotation [D'AKL]. Indeed, regarding the Dirac matrices, the inner automorphism induced by $\gamma^0$ does not implement the Wick rotation $\mathcal{W}$ (which maps the spatial Dirac matrices as $\gamma^j \to \mathcal{W}(\gamma^j) := i\gamma^j$) but actually its square:

$$\rho(\gamma^j) = \gamma^0\gamma^j\gamma^0 = -\gamma^j = \mathcal{W}^2(\gamma^j), \qquad \text{for} \quad j = 1, 2, 3. \tag{1}$$

Nevertheless, a transition from the euclidean to the lorentzian does occur, and the $x_0$ direction gets promoted to a time direction, but this happens at the level of the fermionic action. This is one of the main results of this thesis, summarized in Prop. 4.10 and Prop. 4.19.

More specifically, starting with the twisting of a *euclidean* manifold, then that of a $U(1)$ gauge theory, and finally the twisting of the spectral triple of electrodynamics in euclidean signature [DS]; we show how the fermionic action for twisted spectral triples, proposed in [DFLM], actually yields the Weyl (Prop. 4.10) and the Dirac (Prop. 4.19) equations in *Lorentz signature*. In addition, the zeroth component of the extra one-form field $X_\mu$ acquires a clear interpretation as an energy.

The following two aspects of the fermionic action for twisted spectral triples explain the above-mentioned change of the metric signature from euclidean to lorentzian:

1. First, in order to guarantee that the fermionic action is symmetric when evaluated on Graßmann variables,[2] one restricts the bilinear form that defines the action to the $+1$-eigenspace $\mathcal{H}_\mathcal{R}$ of the unitary $\mathcal{R}$ implementing the twist on the Hilbert space $\mathcal{H}$; whereas in the non-twisted case, one restricts the bilinear form to the $+1$-eigenspace of the grading $\gamma$, in order to solve the fermion doubling problem [LMMS]. This different choice of eigenspace had been noticed in [DFLM] but the physical consequences were

[2] which is an important requirement for the whole physical interpretation of the action formula, also in the non-twisted case [CCM])

not drawn. As already emphasized above, in the models relevant for physics, $\mathcal{R} = \gamma^0$, and once restricted to $\mathcal{H}_{\mathcal{R}}$, the bilinear form no longer involves derivative in the $x_0$ direction. In other words, the restriction to $\mathcal{H}_{\mathcal{R}}$ projects the euclidean fermionic action to what will constitute its spatial part in lorentzian signature.

2. Second, the twisted fluctuations of the Dirac operator of a 4-dimensional riemannian manifold are not necessarily zero [DM, LM1]; in contrast with the non-twisted case where those fluctuations are always vanishing. and these are parametrized by the above-mentioned real one-form field $X_\mu$. By interpreting the zeroth component of this field as an energy, one recovers a derivative in the $x_0$ direction – but now in Lorentz signature.

In addition to the second point above, at least for the spectral action of the twisted SM, the contribution of the $X_\mu$ field is minimized when the field itself vanishes, i.e. there is no twist or $\rho = \mathrm{id}$ [DM]. Thus, one may view the twist as a vacuum fluctuation around the noncommutative riemannian geometry or its spontaneous symmetry breaking to a lorentzian (twisted) noncommutative geometry.

The manuscript has been organized as follows.

Chapter 1 defines the notion of a *noncommutative geometry* in terms of spectral triples (Def. 1.1). In §1.1, we first list out the five axioms a spectral triple with a commutative algebra must obey to satisfy Connes' reconstruction theorem (Theorem 1.3) for riemannian manifolds. Two extra axioms added to the list take into account the spin structure and then we are also able to give a purely spectral characterization to riemannian spin geometries (§1.2) and define a *canonical* spectral triple associated to them (Def. 1.9). Considering the real structure, in §1.3, four of the above-mentioned seven axioms are modified to be more flexible and suitable to accommodate noncommutative algebras and, thus, generalizing the geometry defined by the spectral triples to a noncommutative setting.

As discussed in chapter 2, spectral triples naturally give rise to gauge theories. There exists a more general notion of equivalence than isomorphism between algebras known as Morita equivalence (§2.1). This notion of equivalence between algebras when lifted to the level of spectral triples – in a manner consistent with the real structure – gives rise to generalized gauge fields (§2.2). These gauge fields obtained as perturbations of the Dirac operator (encoding the metric information as it defines the distance formula in noncommutative geometry [C96]) are referred to as the *inner fluctuations* (of the metric).

The gauge transformations of generalized gauge fields (or, gauge potentials in physical gauge theories) thus obtained are arrived at via a change of connection on the bimodule (that implements Morita self-equivalence of real spectral triples) induced by an adjoint action of the group of unitaries of the algebra (§2.3). Then, in §2.3.1, we defined two of the most important gauge-invariant functionals on spectral triples: the spectral action and the fermionic action.

In §2.4, we define one of the most important classes of noncommutative geometries from the standpoint of physics – almost-commutative geometries, which will be very useful for our purposes later in this thesis. We then give a few examples of the physically relevant models describes by such geometries: such as $U(1)$ gauge theory (§2.4.1), electrodynamics (§2.4.2), the Standard Model of particle physics and its extensions (§2.4.3).

In §3.1, we first review the known material regarding spectral triples twisted by using algebra automorphisms and their compatibility with the real structure. We then recall the notions of a covariant Dirac operator, inner fluctuations (§3.1.1), and gauge transformations (§3.1.2) for twisted real spectral triples. We further discuss how the twist naturally induces a new inner product on the initial Hilbert space (§3.1.3), which helps to define a corresponding gauge-invariant fermionic action in the twisted case (§3.1.4).

Furthermore, in §3.2, we outline the construction named 'minimal twist by grading' that associates a *minimally* twisted counterpart to any given graded spectral triple – meaning the twisted spectral triple has the same Hilbert space and Dirac operator as that of the initial one, but the algebra is doubled in order to make the twisting possible [LM1].

Chapter 4 is the main and new contribution of the thesis, which is primarily concerned with making use of the methods summarized in the previous chapter, which entails applying the procedure of minimal twist by grading to three very simple yet concrete examples of spectral triples and computing the corresponding fermionic actions:

1. *A closed riemannian spin manifold.* We investigate the minimal twist of a four-dimensional manifold in §4.1 and show that the twisted fluctuations of corresponding Dirac operator are parametrized by a real one-form field $X_\mu$ (§4.1.1) – first discovered in [DM]. We further recall how to deal with gauge transformations in the twisted case (§4.1.2) – along the lines of [LM2]; and then compute the fermionic action showing that it yields a lagrangian density very similar to that of the Weyl lagrangian in lorentzian signature, as soon as one interprets the zeroth component of $X_\mu$ as the time component of the energy-momentum four-vector of fermions (§4.1.3). However, this lagrangian does not possess enough spinorial degrees of freedom to produce Weyl equations.

2. *A* $U(1)$ *gauge theory.* The previous example dictates that one must consider a spectral triple with slightly more room for the Hilbert space. So, in §4.2, we double its size by taking two copies of the background manifold – which describes a $U(1)$ gauge theory [DS]. We then compute the twisted-covariant Dirac operator (§4.2.1) and obtain the Weyl equations from the associated fermionic action (§4.2.2).

3. *The gauge theory of electrodynamics.* In §4.3, we apply the same construction as above to the spectral triple of electrodynamics proposed in [DS]. We write down its minimal twist and calculate the twisted fluctuations for both the free and the finite parts of the Dirac operator in §4.3.1. The gauge transformations are derived in §4.3.2 and, finally, the Dirac equation in Lorentz signature is obtained in §4.3.3.

This results not only in finding a physical interpretation for the twist, but also roots the link between the twist and Wick rotation – as depicted in [DFLM] – on a much more firm ground.

Chapter 5 addresses some issues that opened up due to this work and are yet to be settled. For instance, although we showed that the $\rho$-inner product and, hence, the fermionic action of a minimally twisted manifold are both invariant under Lorentz boosts (§5.1); how exactly Lorentz transformations arise within the framework of (twisted) noncommutative geometry is however rather unclear so far.

Another relevant question that naturally arises in this context is that of the spectral action. In §5.2, we compute the Lichnerowicz formula for the twisted-covariant Dirac operator of a closed riemannian spin manifold (with non-zero curvature), which is the very first step towards writing down the heat-kernel expansion for the spectral action.

Finally, we conclude with some outlook and perspective.

The appendices contain a brief recollection of Gel'fand duality (§A), the definitions of and notations for the Clifford algebras and Cliffords algebra bundles (§B), the modular theory of Tomita and Takesaki (§C), and all the required notations for $\gamma$-matrices (in chiral representation) and for the Weyl and Dirac equations – both in euclidean space and minkowskian spacetime (§D).

# Chapter 1

# The Axioms of Noncommutative Geometry

*"It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor a priori, whether it is possible."*

– Bernhard Riemann, *On the Hypotheses Which Lie at the Bases of Geometry* (1854)
(Original: *Über die Hypothesen, welche der Geometrie zu Grunde liegen*)

In §1.1, we look at the commutative case – that is, riemannian geometry – and list out the five axioms for Connes' reconstruction theorem [C13]. The reconstruction theorem is at the heart of the subject and it lays down the foundation for a nontrivial generalization of riemannian geometry by giving a way to translate the geometric information on riemannian manifolds into a spectral/operator-algebraic language and vice-versa. With two additional axioms (§1.2), the theorem also holds for riemannian spin manifolds [C96]. Finally, in §1.3, we state a slightly modified version of four out of the seven axioms to make them suitable for the said generalization. These seven (including the modified ones) will form the set of axioms a spectral triple must fulfill to define a noncommutative geometry.

**Definition 1.1** (from [CMa]). *A* **spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *consists of*

1. *a unital $*$-algebra $\mathcal{A}$ (see definition in App. A),*
2. *a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ acts as bounded operators, via a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$,*
3. *a (not necessarily bounded) self-adjoint operator $\mathcal{D} : \mathcal{H} \to \mathcal{H}$ such that its resolvent $(i + \mathcal{D})^{-1}$ is compact and its commutator with $\mathcal{A}$ is bounded, that is, $[\mathcal{D}, a] \in \mathcal{B}(\mathcal{H}),\ \forall a \in \mathcal{A}$.*

*A spectral triple is said to be* **commutative** *if $\mathcal{A}$ is commutative.*

## 1.1 Connes' reconstruction theorem

Riemannian geometry is described by a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfying the following five axioms:

**Axiom 1** (Dimension). *$\mathcal{D}$ is an infinitesimal of the order $n \in \mathbb{N}$, i.e. the $k$-th characteristic value of its resolvent $\mathcal{D}^{-1}$ is $O(k^{-\frac{1}{n}})$. The spectral triple is said to be of* KO-**dimension** $n$.

Axiom 1 fixes the dimension of the riemannian manifold that the spectral triple describes.

**Axiom 2** (First-order or order-one condition). *Omitting the symbol $\pi$ of the representation of the algebra $\mathcal{A}$,[1] one has that*

$$\big[[\mathcal{D}, a], b\big] = 0, \qquad \forall a, b \in \mathcal{A}. \tag{1.1}$$

Axiom 2 ensures that $\mathcal{D}$ is a first-order differential operator.

**Axiom 3** (Regularity condition). *For any $a \in \mathcal{A}$, both $a$ and $[\mathcal{D}, a]$ are in the smooth domain of the derivation $\delta(\cdot) := \big[|\mathcal{D}|, \cdot\big]$, where $|\mathcal{D}| := \sqrt{\mathcal{D}\mathcal{D}^*}$ denotes the absolute value of $\mathcal{D}$. That is, for any $m \in \mathbb{N}$,*

$$a \in \mathrm{Dom}(\delta^m), \quad [\mathcal{D}, a] \in \mathrm{Dom}(\delta^m), \qquad \forall a \in \mathcal{A}, \tag{1.2}$$

*where*

$$\mathrm{Dom}(\delta) \equiv \Big\{T \in \mathcal{B}(\mathcal{H}) \;\Big|\; T\,\mathrm{Dom}\,|\mathcal{D}| \subset \mathrm{Dom}\,|\mathcal{D}|,\; \delta(T) \in \mathcal{B}(\mathcal{H})\Big\}. \tag{1.3}$$

*In other words, both $\delta^m(a)$ and $\delta^m([\mathcal{D}, a])$ are bounded operators.*

Axiom 3 is an algebraic formulation of the smoothness of coordinates. For the next axiom we need the following definition.

**Definition 1.2.** *An $n$-dimensional* **Hochschild cycle** *is a finite sum of the elements of $\mathcal{A}^{\otimes(n+1)} := \mathcal{A} \otimes \mathcal{A} \otimes \ldots \otimes \mathcal{A}$ ($n+1$ times), given by $c = \sum_j (a_j^0 \otimes a_j^1 \otimes \ldots \otimes a_j^n)$, such that the contraction $bc = 0$, where, by definition, $b$ is linear and satisfies*

$$\begin{aligned} b(a^0 \otimes a^1 \otimes \ldots \otimes a^n) = (a^0 a^1 \otimes a^2 \otimes \ldots \otimes a^n) - (a^0 \otimes a^1 a^2 \otimes \ldots \otimes a^n) + \cdots \\ + (-1)^k (a^0 \otimes \ldots \otimes a^k a^{k+1} \otimes \ldots \otimes a^n) + \cdots \\ + (-1)^n (a^n a^0 \otimes \ldots \otimes a^{n-1}). \end{aligned} \tag{1.4}$$

[1]For brevity of notation, from now on and wherever applicable, we use $a$ to mean its representation $\pi(a)$. Thus, $a^*$ denotes $\pi(a^*) = \pi(a)^\dagger$, where $^*$ is the involution of $\mathcal{A}$ and $^\dagger$ is the hermitian conjugation on $\mathcal{H}$.

A Hochschild cycle is the algebraic formulation of a differential form. For a commutative algebra $\mathcal{A}$, it can be constructed easily by taking any $a^j$ and considering the following sum running over all the permutations $\sigma$ of $\{1, \ldots, n\}$ :

$$c = \sum \varepsilon(\sigma)(a^0 \otimes a^{\sigma(1)} \otimes a^{\sigma(2)} \otimes \ldots \otimes a^{\sigma(n)}), \tag{1.5}$$

which corresponds to the familiar differential form $a^0 da^1 \wedge da^2 \wedge \ldots \wedge da^n$, requiring no previous knowledge of the tangent bundle whatsoever, yet providing the volume form.

**Axiom 4** (Orientability condition). *Let $\pi_{\mathcal{D}}(c)$ be the representation of a Hochschild cycle $c$ on $\mathcal{H}$, induced by the following linear map:*

$$\pi_{\mathcal{D}}(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 [\mathcal{D}, a_1] \ldots [\mathcal{D}, a_n], \qquad \forall a_j \in \mathcal{A}. \tag{1.6}$$

*There exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$, such that for odd $n$, we have $\pi_{\mathcal{D}}(c) = 1$, whereas for even $n$, we have that $\pi_{\mathcal{D}}(c) = \gamma$ is a $\mathbb{Z}_2$-grading satisfying:*

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma\mathcal{D} = -\mathcal{D}\gamma, \qquad \gamma a = a\gamma, \quad \forall a \in \mathcal{A}. \tag{1.7}$$

**Axiom 5** (Finiteness and absolute continuity). *The space $\mathcal{H}^\infty := \cap_m \mathrm{Dom}(\mathcal{D}^m)$ is a finite projective $\mathcal{A}$-module (2.12), with a natural hermitian structure $(\cdot, \cdot)$ (2.17)-(2.18), given by*

$$\langle \zeta, a\eta \rangle = ⨍ a(\zeta, \eta)|\mathcal{D}|^{-n}, \qquad \forall a \in \mathcal{A}, \ \forall \zeta, \eta \in \mathcal{H}^\infty, \tag{1.8}$$

*where $⨍$ denotes the noncommutative integral given by the Dixmier trace.*

The following reconstruction theorem (see [C13, §11] for the proof) provides us a purely spectral/operator-algebraic characterization of riemannian geometries and, hence, facilitates their generalization to a noncommutative setting.

**Theorem 1.3.** *For a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ that respects Axioms 1, 2, 4, 5, a stronger[2] regularity condition (Axiom 3), and is equipped with an antisymmetric Hochschild cycle $c$; there exists a compact oriented smooth manifold $\mathcal{M}$ such that $\mathcal{A}$ is the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$.*

Moreover, the converse of the above statement also holds. That is, given any compact oriented smooth manifold $\mathcal{M}$, one can associate to it a strongly regular[2] commutative spectral triple $(C^\infty(\mathcal{M}), \mathcal{H}, \mathcal{D})$ respecting the Axioms 1, 2, 4, 5, and equipped with an antisymmetric Hochschild cycle $c$.

[2]All the elements of the endomorphism algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ are regular.

## 1.2 Riemannian spin geometry

With two more axioms, Theorem 1.3 can be extended to incorporate spin structures and, thus, gives a spectral characterization of riemannian spin manifolds, which are of importance for our purposes.

Before moving on to the axioms and the extension of the theorem, we quickly review some standard definitions (from [Su, §4.2]) of spin geometry [LM]. For the definitions and notations related to Clifford algebras see App. B.

**Definition 1.4.** *A* **spin$^c$ structure** $(\mathcal{M}, \mathcal{S})$ *on a riemannian manifold* $(\mathcal{M}, g)$ *consists of a vector bundle* $\mathcal{S} \to \mathcal{M}$ *such that*

$$\mathsf{End}(\mathcal{S}) \simeq \begin{cases} \mathbb{C}l(\mathcal{T}\mathcal{M}), & \mathcal{M} \text{ even-dimensional} \\ \mathbb{C}l(\mathcal{T}\mathcal{M})^0, & \mathcal{M} \text{ odd-dimensional} \end{cases}, \tag{1.9}$$

*as algebra bundles, where* $\mathbb{C}l(\mathcal{T}\mathcal{M})$ *is the complex Clifford algebra bundle over* $\mathcal{M}$ *and* $\mathbb{C}l(\mathcal{T}\mathcal{M})^0$ *its even part. A riemannian manifold with a spin$^c$ structure is called a* **spin$^c$ manifold**.

**Definition 1.5.** *If there exists a spin$^c$ structure* $(\mathcal{M}, \mathcal{S})$*, then* $\mathcal{S} \to \mathcal{M}$ *is called a* **spinor bundle** *and the continuous sections* $\psi \in \Gamma(\mathcal{S})$ *of the spinor bundle are called* **spinors**. *The completion of the space* $\Gamma(\mathcal{S})$ *in the norm with respect to the inner product*

$$\langle \psi_1, \psi_2 \rangle := \int_{\mathcal{M}} dx\sqrt{g}\,(\psi_1, \psi_2)(x), \tag{1.10}$$

*with* $dx\sqrt{g}$ *being the riemannian volume form; is called the* **Hilbert space** $L^2(\mathcal{M}, \mathcal{S})$ **of square integrable spinors**.

**Definition 1.6.** *Given a spin$^c$ structure* $(\mathcal{M}, \mathcal{S})$. *If there exists an anti-unitary operator* $\mathcal{J} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ *commuting with:*

1. *the action of real-valued continuous functions on* $\Gamma(\mathcal{S})$*, and*

2. *the algebra* $\mathtt{Cliff}^{-1}(\mathcal{M})$ (B.17) *or its even part* $\mathtt{Cliff}^{-1}(\mathcal{M})^0$ *if* $\mathcal{M}$ *is odd-dimensional,*

*then,* $\mathcal{J}$ *is called the* **charge conjugation** *operator. Also,* $(\mathcal{S}, \mathcal{J})$ *defines a* **spin structure** *and* $\mathcal{M}$ *is, then, referred to as a* **spin manifold**.

The following Axiom 6 is a K-theoretic reformulation (i.e. in terms of K-homologies) of the Poincaré duality, which will not be used within the scope of this thesis. So we state it here without explanation and refer to [C94] for the details.

**Axiom 6** (Poincaré duality). *The intersection form* $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ *is invertible.*

**Axiom 7** (Real structure). *There exists an antilinear isometry* $J : \mathcal{H} \to \mathcal{H}$ *such that*

$$J a J^{-1} = a^*, \qquad \forall a \in \mathcal{A}, \tag{1.11}$$

*and*

$$J^2 = \epsilon \mathbb{I}_{\mathcal{H}}, \qquad J\mathcal{D} = \epsilon' \mathcal{D} J, \qquad J\gamma = \epsilon'' \gamma J, \tag{1.12}$$

*where the signs* $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ *are determined by the* KO*-dimension* $n$ *modulo* 8*, as per Table 1.1, which come from the classification of the irreducible representations of Clifford algebras (see [Su, §4.1]).*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\epsilon$ | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |
| $\epsilon'$ | 1 | −1 | 1 | 1 | 1 | −1 | 1 | 1 |
| $\epsilon''$ | 1 | | −1 | | 1 | | −1 | |

Table 1.1: The signs of $\epsilon, \epsilon', \epsilon''$ for a spectral triple of KO-dimension $n$ (mod 8).

**Definition 1.7.** *A spectral triple is said to be* **real** *if it is endowed with a real structure* $J$*, as defined in the Axiom 7.*

> **Theorem 1.8.** *For a commutative real spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *satisfying the Axioms 1–5 and a weaker*[3] *Poincaré duality (Axiom 6), there exists a smooth oriented compact spin*[4] *manifold* $\mathcal{M}$ *such that* $\mathcal{A} = C^\infty(\mathcal{M})$*.*

**Definition 1.9.** *A* **canonical triple** *is the real spectral triple*

$$\big(\mathcal{A} := C^\infty(\mathcal{M}),\ \mathcal{H} := L^2(\mathcal{M}, \mathcal{S}),\ \mathcal{D} := \eth\big), \tag{1.13}$$

*associated to an* $n$*-dimensional closed spin manifold* $\mathcal{M}$*. The algebra* $C^\infty(\mathcal{M})$ *acts on the Hilbert space* $L^2(\mathcal{M}, \mathcal{S})$ *by point-wise multiplication*

$$\big(\pi_{\mathcal{M}}(f)\big)(x) := f(x)\psi(x). \tag{1.14}$$

$\eth$ *is the Dirac operator*

$$\eth := -i \sum_{\mu=1}^{n} \gamma^\mu \nabla^{\mathcal{S}}_\mu, \qquad \nabla^{\mathcal{S}}_\mu := \partial_\mu + \omega^{\mathcal{S}}_\mu, \tag{1.15}$$

[3] The multiplicity of the action of the bicommutant $\mathcal{A}''$ of $\mathcal{A}$ in $\mathcal{H}$ is $2^{n/2}$, see Theorem 11.5 of [C13].

[4] Without the real structure, the theorem continues to hold but rather for a spin$^c$ manifold; see the Remark 5 about Theorem 1 in [C96].

*where $\gamma$'s are the self-adjoint Euclidean Dirac matrices (see §D) and $\nabla^{\mathcal{S}}$ is the covariant derivative associated to the spin connection $\omega^{\mathcal{S}}$ on the spinor bundle. The real structure $J$ is given by the charge conjugation operator $\mathcal{J}$.*

One checks, from the Table 1.1, that for the canonical triple (1.13),

$$\text{KO-dim}(\mathcal{M}) = \dim(\mathcal{M}) = n. \tag{1.16}$$

## 1.3 Generalization to the noncommutative case

In general, the algebra $\mathcal{A}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ may not necessarily be commutative, in which case the Axioms 1, 3 and 5 stay unchanged while the remaining axioms are modified.

**Definition 1.10.** *The* **opposite algebra** *$\mathcal{A}^\circ$ associated to an algebra $\mathcal{A}$ is isomorphic to $\mathcal{A}$ (as a vector space), but it is endowed with an* **opposite product** $\bullet$ *given by*

$$a^\circ \bullet b^\circ := (ba)^\circ, \qquad \forall a^\circ, b^\circ \in \mathcal{A}^\circ, \tag{1.17}$$

*where $a \mapsto a^\circ$ is the isomorphism between the vector spaces underlying $\mathcal{A}$ and $\mathcal{A}^\circ$.*

Tomita-Takesaki theory (App. C) asserts, for a weakly closed $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ (endowed with a cyclic and separating vector), the existence of a canonical *modular involution* $J : \mathcal{H} \to \mathcal{H}$, which is an antilinear isometry

$$J : \mathcal{A} \to \mathcal{A}' \simeq J\mathcal{A}J^{-1}, \qquad \mathcal{A} \ni a \mapsto Ja^*J^{-1} \in \mathcal{A}', \tag{1.18}$$

defining a $\mathbb{C}$-linear anti-isomorphism from the algebra to its commutant (C.4). One identifies this with the antilinear isometry in Axiom 7, which is the real structure $J$, and requires it to induce an isomorphism between $\mathcal{A}$ and $\mathcal{A}^\circ$ given by the map

$$\mathcal{A} \ni a \mapsto a^\circ := Ja^*J^{-1} \in \mathcal{A}^\circ, \tag{1.19}$$

which defines a right representation of $\mathcal{A}$ or, equivalently a left representation of $\mathcal{A}^\circ$, on $\mathcal{H}$, thus

$$\psi a := a^\circ \psi = Ja^*J^{-1}\psi, \qquad \psi \in \mathcal{H}. \tag{1.20}$$

The opposite product $\bullet$ (1.17) in the opposite algebra $\mathcal{A}^\circ$ is 'opposite' to the product $\cdot$ in the algebra $\mathcal{A}$ in the following sense. For $a, b \in \mathcal{A}$, the map (1.19) sends $a \cdot b \in \mathcal{A}$ to

$$(a \cdot b)^\circ = J(a \cdot b)^* J^{-1} = Jb^*a^*J^{-1} = (Jb^*J^{-1})(Ja^*J^{-1}) = b^\circ \bullet a^\circ. \tag{1.21}$$

Consequently, the Axioms 7 and 2 are adapted accordingly as below and will be used extensively later.

**Axiom 7'** (Commutant property, zeroth-order or order-zero condition). *One has that*

$$\left[a, b^{\circ}\right] = ab^{\circ} - b^{\circ}a = 0, \qquad \forall a, b \in \mathcal{A}. \tag{1.22}$$

**Axiom 2'** (First-order or order-one condition). *One has that*

$$\left[[\mathcal{D}, a], b^{\circ}\right] = 0 \quad \textit{or, equivalently,} \quad \left[[\mathcal{D}, b^{\circ}], a\right] = 0, \qquad \forall a, b \in \mathcal{A}, \tag{1.23}$$

*where the equivalence follows from Axiom 7'.*

The commutativity in (1.22) provides $\mathcal{H}$ with the structure of an $(\mathcal{A} \otimes \mathcal{A}^{\circ})$-module or, equivalently, an $\mathcal{A}$-bimodule:

$$(a \otimes b^{\circ})\psi = aJb^{*}J^{-1}\psi, \qquad \forall a, b \in \mathcal{A}, \forall \psi \in \mathcal{H}. \tag{1.24}$$

Similar to Axiom 6, we shall not make use of the corresponding modification, i.e. the following Axiom 6', which reformulates Poincaré duality in the framework of Atiyah's $\mathsf{KR}$-theory. Nevertheless, for completeness, we give the statement without explanation and refer to [C95] for the details.

**Axiom 6'.** *The Kasparov cup product with the class $\mu \in \mathsf{KR}^{n}(\mathcal{A} \otimes \mathcal{A}^{\circ})$ of the Fredholm module associated to the spectral triple provides an isomorphism:*

$$\mathsf{K}_{*}(\mathcal{A}) \xrightarrow{\cap \mu} \mathsf{K}^{*}(\mathcal{A}) \tag{1.25}$$

*from $\mathsf{K}$-cohomology to $\mathsf{K}$-homology.*

**Axiom 4'.** *There exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\circ})$, such that for odd $n$, we have $\pi_{\mathcal{D}}(c) = 1$, whereas for even $n$, we have that $\pi_{\mathcal{D}}(c) = \gamma$ is a $\mathbb{Z}_2$-grading with*

$$\gamma = \gamma^{*}, \quad \gamma^{2} = 1, \quad \gamma\mathcal{D} = -\mathcal{D}\gamma, \qquad \gamma a = a\gamma, \quad \forall a \in \mathcal{A}. \tag{1.26}$$

Thus, an $n$-dimensional (metric) **noncommutative geometry** is defined by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ respecting the Axioms 1, 2', 3, 4', 5, 6' and 7'. Such spectral triples will be the key objects that we will work with.

# Chapter 2

# Gauge Theories from Spectral Triples

Morita equivalence is a generalized notion of isomorphisms between algebras (§2.1). The existence of Morita equivalence generates *inner fluctuations* in noncommutative geometry, which are interpreted as generalized gauge fields. In other words, Morita equivalence of algebras when lifted to spectral triples (§2.2) gives rise to gauge theories on noncommutative geometries. More concretely, exporting the geometry of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on to an algebra $\mathcal{B}$ Morita equivalent to $\mathcal{A}$ in a manner consistent with the real structure gives rise to the fluctuation of the metric (§2.2.3).

Since in Def. 1.1, we defined spectral triples for unital $*$-algebras; from here onwards we will work with unital algebras. The material presented in this section is well-known and has been mostly adapted from [C94, C96, CMa, LM2, Su].

## 2.1 Morita equivalence of algebras

**Definition 2.1.** *A* **left** $A$**-module** ${}_A\mathcal{E}$ *is a vector space* $\mathcal{E}$ *with a left representation of the algebra* $A$ *given by a bilinear map*

$$A \times \mathcal{E} \to \mathcal{E}, \quad (a, \zeta) \mapsto a\zeta, \qquad \forall a \in A, \forall \zeta \in \mathcal{E}, \tag{2.1}$$

*such that*

$$(a_1 a_2)\zeta = a_1(a_2\zeta), \qquad \forall a_1, a_2 \in A, \forall \zeta \in \mathcal{E}. \tag{2.2}$$

*A* **right** $B$**-module** $\mathcal{E}_B$ *is a vector space* $\mathcal{E}$ *with a right representation of the algebra* $B$ *given by a bilinear map*

$$\mathcal{E} \times B \to \mathcal{E}, \quad (\zeta, b) \mapsto \zeta b, \qquad \forall b \in B, \forall \zeta \in \mathcal{E}, \tag{2.3}$$

*such that*

$$\zeta(b_1 b_2) = (\zeta b_1)b_2, \qquad \forall b_1, b_2 \in B, \forall \zeta \in \mathcal{E}. \tag{2.4}$$

*An* A-B-**bimodule** ${}_A\mathcal{E}_B$ *is both a left* A*-module* ${}_A\mathcal{E}$ *and a right* B*-module* $\mathcal{E}_B$ *with the actions of the left and the right representations commuting with each other:*

$$a(\zeta b) = (a\zeta)b, \qquad \forall a \in A, \forall b \in B, \forall \zeta \in \mathcal{E}. \tag{2.5}$$

**Definition 2.2.** *A* **module homomorphism** *is a linear map* $\varphi : \mathcal{E} \to \mathcal{E}'$ *respecting the algebra representation on the module* $\mathcal{E}$*, in the following manner*

$$\begin{aligned}
&\text{for } {}_A\mathcal{E}: && \varphi(a\zeta) = a\varphi(\zeta), && \forall a \in A, \forall \zeta \in \mathcal{E},\\
&\text{for } \mathcal{E}_B: && \varphi(\zeta b) = \varphi(\zeta)b, && \forall b \in B, \forall \zeta \in \mathcal{E},\\
&\text{for } {}_A\mathcal{E}_B: && \varphi(a\zeta b) = a\varphi(\zeta)b, && \forall a \in A, \forall b \in B, \forall \zeta \in \mathcal{E}.
\end{aligned} \tag{2.6}$$

*Let* $\mathrm{Hom}_A(\mathcal{E}, \mathcal{E}') := \{\varphi : \mathcal{E} \to \mathcal{E}'\}$ *denote the* **space of** A**-linear module homomorphisms***. Then, the* **algebra of** A**-linear endomorphisms** *of* $\mathcal{E}$ *is given by* $\mathrm{Hom}_A(\mathcal{E}, \mathcal{E}) =: \mathrm{End}_A(\mathcal{E})$.

**Definition 2.3.** *A* **balanced tensor product** *of modules* $\mathcal{E}_A$ *and* ${}_A\mathcal{E}'$ *is defined as*

$$\mathcal{E} \otimes_A \mathcal{E}' := \mathcal{E} \otimes \mathcal{E}' / \left\{\textstyle\sum_j (\zeta_j a_j \otimes \zeta'_j - \zeta_j \otimes a_j \zeta'_j);\ \forall a_j \in A, \forall \zeta_j \in \mathcal{E}, \forall \zeta'_j \in \mathcal{E}'\right\}, \tag{2.7}$$

*where the quotient ensures the* A*-linearity of the tensor product:*

$$\zeta a \otimes_A \zeta' = \zeta \otimes_A a\zeta', \qquad \forall a \in A, \forall \zeta \in \mathcal{E}, \forall \zeta' \in \mathcal{E}'. \tag{2.8}$$

**Definition 2.4.** *The algebras* A *and* B *are called* **Morita equivalent** *to each other if there exist bimodules* ${}_A\mathcal{E}_B$ *and* ${}_B\mathcal{E}'_A$ *such that*

$$\mathcal{E} \otimes_B \mathcal{E}' \simeq A \qquad \text{and} \qquad \mathcal{E}' \otimes_A \mathcal{E} \simeq B, \tag{2.9}$$

*as* A*-bimodule and* B*-bimodule, respectively.*

**Example 1.** *For any* $n \in \mathbb{N}$, $B = M_n(A)$ *is Morita equivalent to* A*, where both of the bimodules* $\mathcal{E}$ *and* $\mathcal{E}'$*, implementing Morita equivalence, are given by*

$$A^n := \underbrace{A \oplus \cdots \oplus A}_{n \text{ times}}, \tag{2.10}$$

*viewed as an* $A$*-*$M_n(A)$*-bimodule and an* $M_n(A)$*-*$A$*-bimodule, respectively, so that*

$$A^n \otimes_{M_n(A)} A^n \simeq A \qquad \text{and} \qquad A^n \otimes_A A^n \simeq M_n(A). \tag{2.11}$$

*In particular, for* $n = 1$, $B = M_1(A) = A$*. That is, an algebra* A *is Morita equivalent to itself, with the bimodules implementing Morita equivalence being* $\mathcal{E} = \mathcal{E}' = A$.

**Definition 2.5.** *An $A$-module $\mathcal{E}$ is called* **finite projective** *(or, finitely generated projective), if there exists an idempotent matrix $\wp = \wp^2 \in M_n(A)$, for some $n \in \mathbb{N}$, such that*

$$\mathcal{E}_A \simeq \wp A^n \qquad or \qquad {}_A\mathcal{E} \simeq A^n \wp, \tag{2.12}$$

*where $A^n$ is the bimodule given by (2.10).*

Further, it follows that $\mathcal{E}_A$ is a finite projective module iff

$$\mathsf{End}_A(\mathcal{E}) \simeq \mathcal{E} \otimes_A \mathsf{Hom}_A(\mathcal{E}, A), \tag{2.13}$$

with $\mathsf{Hom}_A(\mathcal{E}, A)$ as a left $A$-module.

Morita equivalence of algebras can be characterized in terms of endomorphism algebras of finite projective modules as follows.

Two unital algebras $A$ and $B$ are Morita equivalent iff

$$B \simeq \mathsf{End}_A(\mathcal{E}), \tag{2.14}$$

for a finite projective module $\mathcal{E}$.

Thus, (2.14), with (2.13), implies that all the algebras Morita equivalent to a unital algebra $A$ are of the form $\mathcal{E} \otimes_A \mathsf{Hom}(\mathcal{E}, A)$ for some finite projective module $\mathcal{E}$. We notice, in particular, if $B = A$, then $\mathcal{E} = A$.

### 2.1.1 $*$-algebras and Hilbert bimodules

The above discussion on algebras and modules specializes to $*$-algebras (A.2), with additionally requiring $\wp$ in (2.12) to be an *orthogonal projection*, that is, $\wp^* = \wp$.

**Definition 2.6.** *For a right $\mathcal{A}$-module $\mathcal{E}_\mathcal{A}$, the* **conjugate module** ${}_\mathcal{A}\mathcal{E}^\circ$ *is a left $\mathcal{A}$-module*

$$\mathcal{E}^\circ := \left\{ \overline{\zeta} \mid \zeta \in \mathcal{E} \right\}, \qquad with \quad a\overline{\zeta} = \overline{\zeta a^*}, \qquad \forall a \in \mathcal{A}. \tag{2.15}$$

In case $\mathcal{E}_\mathcal{A}$ is a finite projective module, then it follows that $\mathcal{E}^\circ \simeq \mathsf{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$, as a left $\mathcal{A}$-module, and (2.13) gives

$$\mathsf{End}_\mathcal{A}(\mathcal{E}) \simeq \mathcal{E} \otimes_\mathcal{A} \mathcal{E}^\circ, \tag{2.16}$$

which, following (2.14), implies that all the $*$-algebras Morita equivalent to a unital $*$-algebra $\mathcal{A}$ are of the form $\mathcal{E} \otimes_\mathcal{A} \mathcal{E}^\circ$.

**Definition 2.7.** *On a finite projective right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$, there is a sesquilinear map*[1] $\langle\cdot,\cdot\rangle_{\mathcal{E}} : \mathcal{E}\times\mathcal{E}\to\mathcal{A}$ *called the* **hermitian structure***, which defines an $\mathcal{A}$-valued inner product on $\mathcal{E}$ satisfying*

$$\begin{aligned} \langle\zeta_1 a_1, \zeta_2 a_2\rangle_{\mathcal{E}} &= a_1^*\langle\zeta_1,\zeta_2\rangle_{\mathcal{E}}a_2, && \forall a_1,a_2\in\mathcal{A},\ \forall\zeta_1,\zeta_2\in\mathcal{E};\\ \langle\zeta_1,\zeta_2\rangle_{\mathcal{E}}^* &= \langle\zeta_2,\zeta_1\rangle_{\mathcal{E}}, && \forall\zeta_1,\zeta_2\in\mathcal{E};\\ \langle\zeta,\zeta\rangle_{\mathcal{E}} &\geqslant 0, && \forall\zeta\in\mathcal{E},\quad (\textit{equal iff } \zeta=0). \end{aligned} \tag{2.17}$$

*and it is obtained by restricting the hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{A}^n}$ on $\mathcal{A}^n$ (2.10), given by*

$$\langle\zeta,\eta\rangle_{\mathcal{A}^n} = \textstyle\sum_{j=1}^n \zeta_j^*\eta_j, \qquad \forall\zeta,\eta\in\mathcal{A}^n, \tag{2.18}$$

*to $\mathcal{E}_{\mathcal{A}}\simeq\wp\mathcal{A}^n$ (2.12), for some self-adjoint idempotent matrix $\wp\in M_n(\mathcal{A})$.*

A finite projective right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ is **self-dual** with respect to the hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ on it [Ri, Prop. 7.3], that is

$$\forall\varphi\in \mathsf{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A})\quad \exists!\ \xi_\varphi\in\mathcal{E}:\quad \varphi(\zeta)=(\xi_\varphi,\zeta)_{\mathcal{E}}, \qquad \forall\zeta\in\mathcal{E}. \tag{2.19}$$

Similarly, there exists a hermitian structure ${}_{\mathcal{E}}\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E}\to\mathcal{A}$ on a finite projective left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{E}$, which is linear in the first entry, antilinear in the second, and obtained by restriction to $\mathcal{A}^n$, since ${}_{\mathcal{A}}\mathcal{E}\simeq\mathcal{A}^n\wp$ (2.12).

**Definition 2.8.** *For $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a* **Hilbert bimodule** *$\mathcal{E}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ with a $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}} : \mathcal{E}\times\mathcal{E}\to\mathcal{B}$ on $\mathcal{E}$ satisfying*

$$\begin{aligned} \langle\zeta_1, a\zeta_2\rangle_{\mathcal{E}} &= \langle a^*\zeta_1,\zeta_2\rangle_{\mathcal{E}}, && \forall a\in\mathcal{A},\ \forall\zeta_1,\zeta_2\in\mathcal{E};\\ \langle\zeta_1,\zeta_2 b\rangle_{\mathcal{E}} &= \langle\zeta_1,\zeta_2\rangle_{\mathcal{E}}b, && \forall b\in\mathcal{B},\ \forall\zeta_1,\zeta_2\in\mathcal{E};\\ \langle\zeta_1,\zeta_2\rangle_{\mathcal{E}}^* &= \langle\zeta_2,\zeta_1\rangle_{\mathcal{E}}, && \forall\zeta_1,\zeta_2\in\mathcal{E};\\ \langle\zeta,\zeta\rangle_{\mathcal{E}} &\geqslant 0, && \forall\zeta\in\mathcal{E},\quad (\textit{equal iff } \zeta=0). \end{aligned} \tag{2.20}$$

In particular, a $*$-algebra $\mathcal{A}$ is Morita equivalent to itself, where the finite projective module $\mathcal{E}$ implementing Morita equivalence is taken to be the algebra $\mathcal{A}$ itself, carrying the hermitian structure:

$$\langle a,b\rangle_{\mathcal{A}} = a^*b \quad\text{or}\quad {}_{\mathcal{A}}\langle a,b\rangle = ab^*, \qquad \forall a,b\in\mathcal{A}. \tag{2.21}$$

Morita equivalent algebras have equivalent representation theories: given Morita equivalent $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a right $\mathcal{A}$-module $\mathcal{F}_{\mathcal{A}}$ is converted to a right $\mathcal{B}$-module via $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{E}\simeq\mathcal{B}$, for some $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$.

With a representation (A.3) of $\mathcal{A}$, the Hilbert space $\mathcal{H}$ is, in fact, a Hilbert bimodule ${}_{\mathcal{A}}\mathcal{H}_{\mathbb{C}}$ with the $\mathbb{C}$-valued inner product given by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

[1] antilinear in the first entry and linear in the second

### 2.1.2 Connections: from modules to Hilbert spaces

**Definition 2.9.** *A derivation $\delta$ of an algebra $\mathcal{A}$ taking values in an $\mathcal{A}$-bimodule $\Omega$ is a map $\delta : \mathcal{A} \to \Omega$ satisfying*

$$\delta(ab) = \delta(a) \cdot b + a \cdot \delta(b), \qquad \forall a, b \in \mathcal{A}, \tag{2.22}$$

*where $\cdot$ indicates both the left and the right $\mathcal{A}$-module structure of $\Omega$.*

**Definition 2.10.** *An $\Omega$-valued connection $\nabla$ on a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$ satisfying the Leibniz rule, i.e.*

$$\nabla(\zeta a) = \nabla(\zeta) \cdot a + \zeta \otimes \delta(a), \qquad \forall a \in \mathcal{A}, \forall \zeta \in \mathcal{E}, \tag{2.23}$$

*where the right action of $\mathcal{A}$ on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ is inherited from the right $\mathcal{A}$-module structure of $\Omega$ as follows*

$$(\zeta \otimes \omega) \cdot a := \zeta \otimes (\omega \cdot a), \qquad \forall \omega \in \Omega. \tag{2.24}$$

*Similarly, an $\Omega$-valued connection $\nabla$ on a left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \Omega \otimes_{\mathcal{A}} \mathcal{E}$ such that*

$$\nabla(a\zeta) = a \cdot \nabla(\zeta) + \delta(a) \otimes \zeta, \qquad \forall a \in \mathcal{A}, \forall \zeta \in \mathcal{E}, \tag{2.25}$$

*where the left action of $\mathcal{A}$ on $\Omega \otimes_{\mathcal{A}} \mathcal{E}$ is being inherited from the left module structure of $\Omega$ as follows*

$$a \cdot (\omega \otimes \zeta) := (a \cdot \omega) \otimes \zeta, \qquad \forall \omega \in \Omega. \tag{2.26}$$

When both $\mathcal{A}$ and $\Omega$ are acting on a Hilbert space $\mathcal{H}$ (on the left), the connection $\nabla$ can then be moved from the right module $\mathcal{E}_{\mathcal{A}}$ to $\mathcal{H}$, by virtue of the action of $\Omega$ on $\mathcal{H}$:

$$\mathcal{E} \otimes_{\mathbb{C}} \Omega \times \mathcal{H} \to \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H}, \qquad (\zeta \otimes \omega)\psi = \zeta \otimes (\omega\psi), \tag{2.27}$$

via the following map

$$\nabla : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H} \to \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H}, \qquad \nabla(\zeta \otimes \psi) := \nabla(\zeta)\psi, \tag{2.28}$$

defined with a slight abuse of notation. Similar to (2.28), for the right action of $\mathcal{A}$ and left action of $\Omega$ on $\mathcal{H}$, the action

$$\mathcal{H} \times \Omega \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}, \qquad \psi(\omega \otimes \zeta) = (\psi\omega) \otimes \zeta, \tag{2.29}$$

defines

$$\nabla : \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}, \qquad \nabla(\psi \otimes \zeta) := \psi\nabla(\zeta). \tag{2.30}$$

However, these maps (2.28 or 2.30) do not automatically extend over to the tensor products $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ or $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}$ (respectively) – on account of failing to be $\mathcal{A}$-linear, which is captured by the derivation $\delta$ (2.22) generating $\Omega$, provided that the action of $\mathcal{A}$ and $\Omega$ be compatible [LM2, Prop. 3.1, 3.2]:

1. If the left action of both $\mathcal{A}$ and $\Omega$ on $\mathcal{H}$ are such that $(\omega \cdot a)\psi = \omega(a\psi)$, then $\nabla$ in (2.28) satisfies (Leibniz rule)

$$\nabla(\zeta a)\psi = \nabla(\zeta)a\psi + \zeta \otimes \delta(a)\psi. \tag{2.31}$$

2. If the right action of $\mathcal{A}$ and the left action of $\Omega$ on $\mathcal{H}$ are such that $(a \cdot \omega)\psi = \omega(\psi a)$, then $\nabla$ in (2.30) satisfies (Leibniz rule)

$$\psi\nabla(a\zeta) = \psi a\nabla(\zeta) + \delta(a)\psi \otimes \zeta. \tag{2.32}$$

## 2.2 Morita equivalence of spectral triples

We are given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and an algebra $\mathcal{B} \simeq \mathsf{End}_{\mathcal{A}}(\mathcal{E})$ Morita equivalent to $\mathcal{A}$ via a finite projective module $\mathcal{E}$, as in (2.14). The task at hand is the construction of a spectral triple $(\mathcal{B}, \mathcal{H}', \mathcal{D}')$, that is, to export the geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on to an algebra Morita equivalent to $\mathcal{A}$.

### 2.2.1 Morita equivalence via right $\mathcal{A}$-module

Let us say that $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent via a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}_r$. Since ${}_{\mathcal{A}}\mathcal{H}_{\mathbb{C}}$ carries a left $\mathcal{A}$-module structure induced by the representation of the algebra $\mathcal{A}$, the product

$$\mathcal{E}_r \otimes_{\mathcal{A}} \mathcal{H} =: \mathcal{H}_r, \tag{2.33}$$

is a Hilbert $\mathcal{B}$-$\mathbb{C}$-bimodule with a left action of $\mathcal{B}$ inherited from $\mathcal{E}_r$, by extension:

$$b(\eta \otimes \psi) := b\eta \otimes \psi, \qquad \forall b \in \mathcal{B}, \forall (\eta \otimes \psi) \in \mathcal{H}_r; \tag{2.34}$$

and the $\mathbb{C}$-valued inner product given by

$$\langle \eta_1 \otimes \psi_1, \eta_2 \otimes \psi_2 \rangle_{\mathcal{H}_r} = \langle \psi_1, \langle \eta_1, \eta_2 \rangle_{\mathcal{E}_r} \psi_2 \rangle_{\mathcal{H}}, \qquad \forall \eta_1, \eta_2 \in \mathcal{E}_r, \forall \psi_1, \psi_2 \in \mathcal{H}. \tag{2.35}$$

The following naïve attempt at furnishing an adaption $\mathcal{D}_r$ of the action of $\mathcal{D}$ over to $\mathcal{H}_r$:

$$\mathcal{D}_r(\eta \otimes \psi) := \eta \otimes \mathcal{D}\psi, \tag{2.36}$$

fails due to its incompatibility with $\mathcal{A}$-linearity (2.8) of the balanced tensor product over $\mathcal{A}$, since its action on the elementary tensors generating the ideal, that is,

$$\mathcal{D}_r(\eta a \otimes \psi - \eta \otimes a\psi) = \eta a \otimes \mathcal{D}\psi - \eta \otimes \mathcal{D}a\psi = -\eta \otimes [\mathcal{D}, a]\psi \tag{2.37}$$

is not necessarily zero. However, (2.37) suggests for $\mathcal{D}_r$ to be a well-defined operator on $\mathcal{H}_r$ that it must rather act as follows:

$$\mathcal{D}_r(\eta a \otimes \psi - \eta \otimes a\psi) = \eta a \otimes \mathcal{D}\psi - \eta \otimes \mathcal{D}a\psi + \eta \otimes [\mathcal{D}, a]\psi. \tag{2.38}$$

Recalling Def. 2.10 of a connection $\nabla$ on a right $\mathcal{A}$-module $\mathcal{E}_r$ taking values in the $\mathcal{A}$-bimodule $\Omega^1_{\mathcal{D}}(\mathcal{A})$ generated by the derivation $\delta(a) = [\mathcal{D}, a]$, one has (2.31)

$$\eta \otimes [\mathcal{D}, a]\psi = \nabla(\eta a)\psi - \nabla(\eta)a\psi. \tag{2.39}$$

In that light, (2.38) becomes

$$\begin{aligned} \mathcal{D}_r(\eta a \otimes \psi) - \mathcal{D}_r(\eta \otimes a\psi) &= \eta a \otimes \mathcal{D}\psi - \eta \otimes \mathcal{D}a\psi + \nabla(\eta a)\psi - \nabla(\eta)a\psi \\ &= \eta a \otimes \mathcal{D}\psi + \nabla(\eta a)\psi - \big(\eta \otimes \mathcal{D}a\psi + \nabla(\eta)a\psi\big), \end{aligned} \tag{2.40}$$

implying that the correct action of $\mathcal{D}_r$ on $\mathcal{H}_r$ must be as follows

$$\mathcal{D}_r(\eta \otimes \psi) := \eta \otimes \mathcal{D}\psi + \nabla(\eta)\psi, \tag{2.41}$$

which is $\mathcal{A}$-linear by (2.31).

Now, if $\mathcal{E}_r$ is finite projective (2.12), any $\Omega^1_{\mathcal{D}}(\mathcal{A})$-valued connection $\nabla$ is of the form $\nabla_0 + \omega$, where $\nabla_0 := \wp \circ \delta$ is the Graßmann connection, i.e.

$$\forall \eta = \wp \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathcal{E}_r, \quad \text{with } \eta_{j=1,\dots,n} \in \mathcal{A}; \qquad \nabla_0\eta = \wp \begin{pmatrix} \delta(\eta_1) \\ \vdots \\ \delta(\eta_n) \end{pmatrix}, \tag{2.42}$$

and $\omega$ is an $\mathcal{A}$-linear map $\mathcal{E}_r \to \mathcal{E}_r \otimes \Omega^1_{\mathcal{D}}(\mathcal{A})$ such that

$$\omega(\eta a) = \omega(\eta) \cdot a, \qquad \forall a \in \mathcal{A},\ \forall \eta \in \mathcal{E}_r. \tag{2.43}$$

**Theorem 2.11** (Theorem 6.15, [Su]). *Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and a connection $\nabla$ on finite projective $\mathcal{E}_r$, then $(\mathcal{B}, \mathcal{H}_r, \mathcal{D}_r)$ is a spectral triple provided $\nabla$ is hermitian, that is, it satisfies*

$$\langle \eta_1, \nabla\eta_2 \rangle_{\mathcal{E}_r} - \langle \nabla\eta_1, \eta_2 \rangle_{\mathcal{E}_r} = \delta\langle \eta_1, \eta_2 \rangle_{\mathcal{E}_r}, \qquad \forall \eta_1, \eta_2 \in \mathcal{E}_r. \tag{2.44}$$

**Morita self-equivalence via $\mathcal{E}_r = \mathcal{A}$**

The algebra $\mathcal{A}$ is Morita equivalent to itself via $\mathcal{E}_r = \mathcal{A}$ and any connection $\nabla$ on this right $\mathcal{A}$-module $\mathcal{A}$ is of the form $\delta + \omega$, for some $\omega \in \Omega^1_{\mathcal{D}}(\mathcal{A})$. (2.41) becomes

$$\mathcal{D}_r(a \otimes \psi) = a \otimes \mathcal{D}\psi + (\delta + \omega)(a)\psi, \tag{2.45}$$

which, identifying $a \otimes \psi \in \mathcal{H}_r$ with $a\psi \in \mathcal{H}$ and recalling that $\delta(a) = [\mathcal{D}, a]$, becomes

$$\begin{aligned} \mathcal{D}_r(a\psi) &= a\mathcal{D}\psi + [\mathcal{D}, a]\psi + \omega a\psi \\ &= (\mathcal{D} + \omega)(a\psi), \end{aligned} \tag{2.46}$$

that is, on $\mathcal{H}$, one has that

$$\mathcal{D}_r = \mathcal{D} + \omega, \tag{2.47}$$

which has a compact resolvent and bounded commutator with $\mathcal{A}$, since $\omega$ is bounded. Thus, $(\mathcal{A}, \mathcal{H}, \mathcal{D}_r)$ is a spectral triple, given $\omega$ is self-adjoint [BMS], and said to be **Morita equivalent** to the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [LM2, §3.2].

If the initial spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ comes with a real structure $J$, the latter does not necessarily get inherited by the Morita equivalent spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_r)$ for $J\mathcal{D}_r = \epsilon'\mathcal{D}_r J$ holds iff $\omega = \epsilon' J\omega J^{-1}$, which, in general, may not be the case. In fact, it follows from (1.11) and (1.19) that

$$J\omega J^{-1} = \epsilon' \sum_j (a_j^*)^\circ \left[\mathcal{D}, (b_j^*)^\circ\right], \tag{2.48}$$

for some $\omega = \sum_j a_j[\mathcal{D}, b_j] \in \Omega^1_{\mathcal{D}}(\mathcal{A})$.

### 2.2.2 Morita equivalence via left $\mathcal{A}$-module

Let us say that $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent via a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{E}_l$. Since ${}_{\mathbb{C}}\mathcal{H}_{\mathcal{A}}$ is endowed with a right $\mathcal{A}$-module structure induced by the representation of the opposite algebra $\mathcal{A}^\circ$ (Def. 1.10), the product

$$\mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_l =: \mathcal{H}_l, \tag{2.49}$$

is a Hilbert $\mathbb{C}$-$\mathcal{B}$-bimodule with a right action of $\mathcal{B}$ inherited from $\mathcal{E}_l$, by extension:

$$(\psi \otimes \eta)b := \psi \otimes \eta b, \qquad \forall b \in \mathcal{B}, \forall(\psi \otimes \eta) \in \mathcal{H}_l; \tag{2.50}$$

and $\mathbb{C}$-valued inner product

$$\langle \psi_1 \otimes \eta_1, \psi_2 \otimes \eta_2 \rangle_{\mathcal{H}_l} = \langle \psi_1 \langle \eta_1, \eta_2 \rangle_{\mathcal{E}_l}, \psi_2 \rangle_{\mathcal{H}}, \qquad \forall \eta_1, \eta_2 \in \mathcal{E}_l, \forall \psi_1, \psi_2 \in \mathcal{H}. \tag{2.51}$$

As before, the naïve attempt at furnishing an adaption $\mathcal{D}_\iota$ of the action of $\mathcal{D}$ over to $\mathcal{H}_\iota$:

$$\mathcal{D}_\iota(\psi \otimes \eta) := \mathcal{D}\psi \otimes \eta \tag{2.52}$$

does not work for it is incompatible with the $\mathcal{A}$-linearity (2.8) of the balanced tensor product over $\mathcal{A}$ as its action on the elementary tensors generating the ideal, that is,

$$\begin{aligned}\mathcal{D}_\iota(\psi \otimes a\eta - \psi a \otimes \eta) &= \mathcal{D}\psi \otimes a\eta - \mathcal{D}(\psi a) \otimes \eta,\\ &= \big((\mathcal{D}\psi)a - \mathcal{D}(a^\circ\psi)\big) \otimes \eta,\\ &= \big(a^\circ(\mathcal{D}\psi) - \mathcal{D}(a^\circ\psi)\big) \otimes \eta,\\ &= -\big[\mathcal{D}, a^\circ\big]\psi \otimes \eta,\end{aligned} \tag{2.53}$$

is not necessarily vanishing. However, (2.53) suggests for $\mathcal{D}_\iota$ to be a well-defined operator on $\mathcal{H}_\iota$ that it must rather act as follows:

$$\mathcal{D}_\iota(\psi \otimes a\eta - \psi a \otimes \eta) = \mathcal{D}\psi \otimes a\eta - \mathcal{D}(\psi a) \otimes \eta + [\mathcal{D}, a^\circ]\psi \otimes \eta. \tag{2.54}$$

Recalling Def. 2.10 of a connection $\nabla^\circ$ on a left $\mathcal{A}$-module $\mathcal{E}_\iota$ taking values in the $\mathcal{A}$-bimodule $\Omega^1_{\mathcal{D}}(\mathcal{A}^\circ)$ generated by the derivation $\delta^\circ(a) = [\mathcal{D}, a^\circ]$, one has (2.32)

$$[\mathcal{D}, a^\circ]\psi \otimes \eta = \psi\nabla^\circ(a\eta) - \psi a\nabla^\circ(\eta). \tag{2.55}$$

In that light, (2.54) becomes

$$\begin{aligned}&\mathcal{D}_\iota(\psi \otimes a\eta) - \mathcal{D}_\iota(\psi a \otimes \eta)\\ &\quad= \mathcal{D}\psi \otimes a\eta - \mathcal{D}(\psi a) \otimes \eta + \psi\nabla^\circ(a\eta) - \psi a\nabla^\circ(\eta)\\ &\quad= \mathcal{D}\psi \otimes a\eta + \psi\nabla^\circ(a\eta) - \big(\mathcal{D}(\psi a) \otimes \eta + \psi a\nabla^\circ(\eta)\big),\end{aligned} \tag{2.56}$$

implying that the correct action of $\mathcal{D}_\iota$ on $\mathcal{H}_\iota$ must be

$$\mathcal{D}_\iota(\psi \otimes \eta) := \mathcal{D}\psi \otimes \eta + \psi\nabla^\circ(\eta), \tag{2.57}$$

which is $\mathcal{A}$-linear by (2.32).

Further, if $\mathcal{E}_\iota$ is finite projective (2.12), any $\Omega^1_{\mathcal{D}}(\mathcal{A}^\circ)$-valued connection $\nabla^\circ$ is of the form $\nabla^\circ_0 + \boldsymbol{\omega}^\circ$, where $\nabla^\circ_0 = \delta^\circ \circ \wp$ is the Graßmann connection, i.e.

$$\begin{aligned}\forall\eta = \big(\eta_1, \ldots, \eta_n\big)\wp \in \mathcal{E}_\iota, &\qquad \text{with} \quad \eta_{j=1,\ldots,n} \in \mathcal{A},\\ \nabla^\circ_0\eta = \big(\delta^\circ(\eta_1), \ldots, \delta^\circ(\eta_n)\big)\wp,&\end{aligned} \tag{2.58}$$

and $\boldsymbol{\omega}^\circ$ is an $\mathcal{A}$-linear map $\mathcal{E}_\iota \to \Omega^1_{\mathcal{D}}(\mathcal{A}^\circ) \otimes_{\mathcal{A}} \mathcal{E}_\iota$ such that

$$\boldsymbol{\omega}^\circ(a\eta) = a \cdot \boldsymbol{\omega}^\circ(\eta), \qquad \forall a \in \mathcal{A},\ \forall\eta \in \mathcal{E}_\iota. \tag{2.59}$$

**Morita self-equivalence via $\mathcal{E}_l = \mathcal{A}$**

The algebra $\mathcal{A}$ is Morita equivalent to itself via $\mathcal{E}_l = \mathcal{A}$ and any connection $\nabla^\circ$ on this left $\mathcal{A}$-module $\mathcal{A}$ is of the form $\delta^\circ + \omega^\circ$, for some $\omega^\circ \in \Omega^1_{\mathcal{D}}(\mathcal{A}^\circ)$. Then, (2.57) becomes

$$\begin{aligned} \mathcal{D}_l(\psi \otimes a) &= \mathcal{D}\psi \otimes a + \big(\delta^\circ(a) + \omega^\circ(a)\big)\psi \otimes 1, \\ &= \big((\mathcal{D}\psi)a + [\mathcal{D}, a^\circ]\psi + a \cdot \omega^\circ\psi\big) \otimes 1, \\ &= \big(a^\circ\mathcal{D}\psi + (\mathcal{D}a^\circ - a^\circ\mathcal{D})\psi + \omega^\circ a^\circ\psi\big) \otimes 1, \\ &= \big(\mathcal{D}a^\circ\psi + \omega^\circ a^\circ\psi\big) \otimes 1 = (\mathcal{D} + \omega^\circ)(\psi \otimes a). \end{aligned} \tag{2.60}$$

Thus, identifying $\psi \otimes a \in \mathcal{H}_l$ with $\psi a \in \mathcal{H}$, one obtains the following action on $\mathcal{H}$:

$$\mathcal{D}_l = \mathcal{D} + \omega^\circ. \tag{2.61}$$

Following from (1.11) and (1.19), any $\omega^\circ = \sum_j a_j^\circ[\mathcal{D}, b_j^\circ] \in \Omega^1_{\mathcal{D}}(\mathcal{A}^\circ)$ has a left action on $\mathcal{H}$ given by the bounded operator

$$\omega^\circ = \epsilon' J\omega J^{-1} \tag{2.62}$$

for $\omega = \sum_j a_j^*[\mathcal{D}, b_j^*] \in \Omega^1_{\mathcal{D}}(\mathcal{A})$. Therefore, $(\mathcal{A}, \mathcal{H}, \mathcal{D}_l)$ is a spectral triple, given $\omega$ is self-adjoint [BMS], said to be Morita equivalent to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [LM2, §3.2].

Yet again, the real structure $J$ of the initial spectral triple does not adapt to $(\mathcal{A}, \mathcal{H}, \mathcal{D}_l)$ for $J\mathcal{D}_l = \epsilon'\mathcal{D}_l J$ holds iff $\omega = \epsilon' J\omega J^{-1}$, which is not necessarily true.

### 2.2.3 Inner fluctuations by Morita self-equivalence

In order to overcome the incompatibility (mentioned at the end of §2.2.1 and §2.2.2) of the real structure with the construction of Morita equivalent spectral triples or, in other words, to construct Morita self-equivalent real spectral triples – one combines the above two constructions of §2.2.1 and §2.2.2 together, cf. [LM2, §A.1.3].

One begins with a real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J)$. Implementing Morita self-equivalence of $\mathcal{A}$ via a right $\mathcal{A}$-module $\mathcal{E}_r = \mathcal{A}$, one obtains (§2.2.1)

$$\big(\mathcal{A},\, \mathcal{H},\, \mathcal{D}\big) \xrightarrow[\text{via } \mathcal{E}_r = \mathcal{A}]{\text{Morita self-equivalance}} \big(\mathcal{A},\, \mathcal{H},\, \mathcal{D} + \omega_r\big). \tag{2.63}$$

Then, applying Morita self-equivalence of $\mathcal{A}$ via a left $\mathcal{A}$-module $\mathcal{E}_l = \mathcal{A}$ again gives (§2.2.2)

$$\big(\mathcal{A},\, \mathcal{H},\, \mathcal{D} + \omega_r\big) \xrightarrow[\text{via } \mathcal{E}_l = \mathcal{A}]{\text{Morita self-equivalance}} \big(\mathcal{A},\, \mathcal{H},\, \mathcal{D}' := \mathcal{D} + \omega_r + \omega_l^\circ\big), \tag{2.64}$$

where $\omega_l^\circ = \epsilon' J\omega_l J^{-1}$. Both $\omega_r$ and $\omega_l$ are self-adjoint one-forms in $\Omega^1_{\mathcal{D}}(\mathcal{A})$, but not necessarily related.

The real structure $J$ of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ also becomes the real structure for $(\mathcal{A}, \mathcal{H}, \mathcal{D}')$ iff there exists $\omega \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ such that [LM2, Prop. A.5]

$$\mathcal{D}' = \mathcal{D}_\omega := \mathcal{D} + \omega + \epsilon' J \omega J^{-1}. \tag{2.65}$$

Thus, Morita self-equivalence of real spectral triples induces *inner fluctuations* (2.65) of the Dirac operator (metric). The operator $\mathcal{D}_\omega$ is then referred to as the *gauged* or *covariant* or *fluctuated* Dirac operator and the self-adjoint one-forms $\omega \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ are identified as *generalized gauge fields*.

## 2.3 Gauge transformations

A *gauge transformation* on a module is a change of connection (2.67) on that module, induced by an adjoint action of a unitary endomorphism on it.

When the module is taken to be the algebra of a spectral triple itself (implementing Morita self-equivalence), a gauge transformation is tantamount to transforming the fluctuated Dirac operator under an adjoint action of a unitary of the algebra. This can equivalently be encoded in a law of transformation of the generalized gauge fields and subsequently gives the transformation rules for the gauge potentials in physical theories.

### On hermitian modules

Gauge transformations on a hermitian $\mathcal{A}$-module $\mathcal{E}$ are induced by the group of its unitary endomorphisms [LM2, §A.2.1]

$$\mathcal{U}(\mathcal{E}) := \left\{ u \in \mathsf{End}_{\mathcal{A}}(\mathcal{E}) \mid uu^* = u^*u = \mathsf{id}_{\mathcal{E}} \right\}, \tag{2.66}$$

which acts on an $\Omega$-valued connection $\nabla$ on $\mathcal{E}$ as

$$\nabla \to \nabla^u := u \nabla u^*, \qquad \forall u \in \mathcal{U}(\mathcal{E}), \tag{2.67}$$

where the action on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ or $\Omega \otimes_{\mathcal{A}} \mathcal{E}$, for a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ or a left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{E}$, respectively, is implemented as

$$u \otimes_{\mathcal{A}} \mathsf{id}_\Omega \qquad \text{or} \qquad \mathsf{id}_\Omega \otimes_{\mathcal{A}} u. \tag{2.68}$$

It follows that $\nabla^u$ in (2.67) is also an $\Omega$-valued connection on $\mathcal{E}$ [LM2, Prop. A.7].

If $\mathcal{E}$ is finite projective (2.12), then for any connection

$$\nabla = \nabla_0 + \omega \;\longrightarrow\; \nabla^u = \nabla_0 + \omega^u \qquad \Rightarrow \qquad \omega \to \omega^u, \tag{2.69}$$

that is, the transformation law for a gauge potential $\omega$ solely captures a gauge transformation. Explicitly, the Graßmann connection $\nabla_0 = \wp \circ \delta$, for a derivation $\delta$ of $\mathcal{A}$ in $\Omega$. Further, the group $\mathcal{U}(\mathcal{E})$ (2.66) consists of unitary matrices commuting with $\wp$, that is,

$$\mathcal{U}(\mathcal{E}) := \{u \in M_n(\mathcal{A}) \mid u\wp = \wp u,\ uu^* = u^*u = \mathrm{id}_{\mathcal{E}}\}, \tag{2.70}$$

acting by the usual matrix multiplication,

$$u\zeta := \begin{cases} \wp u\zeta, & \forall \zeta \in \mathcal{E}_{\mathcal{A}} \\ \zeta u^* \wp, & \forall \zeta \in {}_{\mathcal{A}}\mathcal{E} \end{cases}, \tag{2.71}$$

which implements the gauge transformation (2.69) as follows [LM2, Prop. A.8]:

$$\omega^u\zeta := \begin{cases} \wp u\delta(u^*)\zeta + u\omega u^*\zeta, & \forall \zeta \in \mathcal{E}_{\mathcal{A}} \\ \zeta\delta(u)u^*\wp + u\omega u^*\zeta, & \forall \zeta \in {}_{\mathcal{A}}\mathcal{E} \end{cases}, \tag{2.72}$$

where $\delta(u), \delta(u^*) \in M_n(\Omega)$ for unitaries $u, u^* \in \mathcal{A}$.

### On real spectral triples

Consider a real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J)$ with a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ taken to be the algebra $\mathcal{A}$ itself (implementing Morita self-equivalence) and a derivation $\delta(\cdot) = [\mathcal{D}, \cdot]$ of $\mathcal{A}$ in $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Then, the first case of (2.72) gives

$$\omega^u = u[\mathcal{D}, u^*] + u\omega u^*, \tag{2.73}$$

which induces [LM2, Rem. A.9]

$$\mathcal{D}_\omega \mapsto \mathcal{D}_{\omega^u} = \mathcal{D} + \omega^u + \epsilon' J\omega^u J^{-1}. \tag{2.74}$$

The same effect as above, i.e. the gauge transformation $\mathcal{D}_\omega \mapsto \mathcal{D}_{\omega^u}$ (2.74), can be achieved by the adjoint action of the group $\mathcal{U}(\mathcal{A})$ of unitaries of $\mathcal{A}$

$$\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A} \mid u^*u = uu^* = 1\}, \tag{2.75}$$

on the Hilbert space $\mathcal{H} \ni \psi$, defined as

$$(\mathrm{Ad}\ u)\psi := u\psi u^* = uJuJ^{-1}\psi, \tag{2.76}$$

recalling that $\mathcal{A}$ has both a left and a right representation on $\mathcal{H}$ thanks to the real structure $J$. On the Dirac operator $\mathcal{D}$, (2.76) induces, following Axioms 2' and 7', the transformation [CMa, Prop. 1.141]

$$\mathcal{D} \mapsto (\mathrm{Ad}\ u)\mathcal{D}(\mathrm{Ad}\ u)^{-1} = \mathcal{D} + u[\mathcal{D}, u^*] + \epsilon' Ju[\mathcal{D}, u^*]J^{-1}, \tag{2.77}$$

which is basically the relation (2.74) for $\omega = 0$ so that $\omega^u = u[\mathcal{D}, u^*]$ as per (2.73). On the gauged Dirac operator $\mathcal{D}_\omega$ (2.65), one has [CMa, Prop. 1.141]

$$\mathcal{D}_\omega \mapsto (\mathsf{Ad}\ u)\mathcal{D}_\omega(\mathsf{Ad}\ u)^{-1} = \mathcal{D}_{\omega^u}, \tag{2.78}$$

with $\mathcal{D}_{\omega^u}$ as in (2.74) and $\omega^u$ as (2.73).

### 2.3.1 Gauge-invariants

Now that we have discussed, in the previous section, the generalized gauge fields carrying the action of the group of unitaries of the algebra; we can define their gauge-invariants functionals on a spectral triple, viz. the (bosonic) spectral action and the fermionic action.

#### Spectral action

A general formalism for spectral triples is the *spectral action principle* [CC96, CC97, CC06a, CC06b], which proposes a universal action functional on spectral triples that depends only on the spectrum of the Dirac operator $\mathcal{D}$ or – more generally, if the inner fluctuations are turned on – that of its fluctuation $\mathcal{D}_\omega$.

A straight forward way to construct such an action is to count the eigenvalues that are smaller than a fixed energy scale $\Lambda$. Thus, one defines the *spectral action* as the fucntional

$$\boxed{S^b[\mathcal{D}_\omega] := \mathsf{Tr}\ f\left(\frac{\mathcal{D}_\omega^2}{\Lambda^2}\right),} \tag{2.79}$$

where $f$ is a positive and even real cutoff function taken to be the smooth approximation of the characteristic function on the interval $[0,1]$ such that the action $S^b[\mathcal{D}_\omega]$ vanishes sufficiently rapidly as the real cutoff parameter $\Lambda$ approaches infinity.

It can be expanded asymptotically (in power series of $\Lambda$) – using heat kernel expansion techniques given that $\mathcal{D}_\omega^2$ is a generalized Laplacian up to an endomorphism term (generalized Lichnerowicz formula) – as follows

$$\mathsf{Tr}\ f\left(\frac{\mathcal{D}_\omega^2}{\Lambda^2}\right) = \sum_{n\geqslant 0} f_{4-n}\Lambda^{4-n} a_n\left(\frac{\mathcal{D}_\omega^2}{\Lambda^2}\right), \tag{2.80}$$

where $f_n$ are the momenta of $f$ (given by $f_k := \int_0^\infty f(v)v^{k-1}dv$, for $k > 0$ and $f_0 = f(0)$), and $a_n$ are the Seeley-de Witt coefficients (non-zero only for even $n$) [Gi, Va], which yield the gauge theoretic lagrangians of the model.

$S^b[\mathcal{D}]$ is the fundamental action functional that can be used both at the classical level to compare different geometric spaces and at the quantum level in the functional integral formulation (after Wick rotation to euclidean signature) [CMa, §11].

However, when applied to the inner fluctuations $\mathcal{D}_\omega$, the action $S^b[\mathcal{D}_\omega]$ only yields the bosonic content on the theory (hence, the superscript $b$). For instance, on classical riemannian manifolds where inner fluctuations vanish, $S^b[\eth]$ gives the Einstein-Hilbert action of pure gravity. Thus, noncommutative geometries naturally contain gravity, while the other gauge bosons appear due to the noncommutativity of the algebra of the spectral triples. The coupling with the fermions is accounted for by adding to the spectral action an extra term called the fermionic action.

### Fermionic action

The *fermionic action* associated to a real graded spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J, \gamma)$, defined as

$$S^f[\mathcal{D}_\omega] := \mathfrak{A}_{\mathcal{D}_\omega}(\tilde{\psi}, \tilde{\psi}), \tag{2.81}$$

is a gauge-invariant quantity of Graßmann nature [CCM, Ba], constructed from the following bilinear form

$$\mathfrak{A}_{\mathcal{D}_\omega}(\phi, \psi) := \langle J\phi, \mathcal{D}_\omega \psi \rangle, \qquad \forall \phi, \psi \in \mathcal{H}, \tag{2.82}$$

defined by the covariant Dirac operator $\mathcal{D}_\omega := \mathcal{D} + \omega + \epsilon' J \omega J^{-1}$, where $\omega$ is a self-adjoint element of the set of generalized one-forms [C96]

$$\Omega^1_{\mathcal{D}}(\mathcal{A}) := \left\{ \sum\nolimits_i a_i [\mathcal{D}, b_i], \quad a_i, b_i \in \mathcal{A} \right\}. \tag{2.83}$$

Here, $\tilde{\psi}$ is a Graßmann vector in the Fock space $\widetilde{\mathcal{H}}_+$ of classical fermions, corresponding to the positive eigenspace $\mathcal{H}_+ \subset \mathcal{H}$ of the grading operator $\gamma$, i.e.

$$\widetilde{\mathcal{H}}_+ := \{\tilde{\psi}, \ \psi \in \mathcal{H}_+\}, \quad \text{where} \quad \mathcal{H}_+ := \{\psi \in \mathcal{H}, \ \gamma\psi = \psi\}. \tag{2.84}$$

Both actions $S^b$ and $S^f$ are invariant under a gauge transformation, i.e. the simultaneous adjoint action of the unitary group $\mathcal{U}(\mathcal{A})$ (2.75) both on $\mathcal{H}$ as (2.76) and on the fluctuated Dirac operator $\mathcal{D}_\omega$ as (2.78).

**Remark 1.** *Since the bilinear form (2.82) is anti-symmetric for* KO*-dim.* 2 *and* 4 *(cf. Lem.* 3.5 *below),* $\mathfrak{A}_{\mathcal{D}_\omega}(\psi, \psi)$ *vanishes when evaluated on vectors, but it is non-zero when evaluated on Graßmann vectors, see [CMa, § I.16.2].*

*In particular, the fermionic action associated to the spectral triple of the Standard Model (which has* KO*-dim.* 2*) contains the coupling of the fermionic matter with the fields (scalar, gauge, and gravitational).*

## 2.4 Almost-commutative geometries

Almost-commutative geometries are a special class of noncommutative geometries arising by taking the product[2] of the canonical triple (1.13) of an oriented closed spin manifold $\mathcal{M}$ with a *finite geometry* $\mathcal{F}$ defined by a finite-dimensional unital spectral triple $(\mathcal{A}_\mathcal{F}, \mathcal{H}_\mathcal{F}, \mathcal{D}_\mathcal{F})$. The resulting *product geometry*, denoted by $\mathcal{M} \times \mathcal{F}$, is then given by the spectral triple [C96]:

$$\big(\mathcal{A} := C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathcal{F}, \quad \mathcal{H} := L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_\mathcal{F}, \quad \mathcal{D} := \eth \otimes \mathbb{I}_\mathcal{F} + \gamma_\mathcal{M} \otimes \mathcal{D}_\mathcal{F}\big), \quad (2.85)$$

where $\mathbb{I}_\mathcal{F}$ is the identity in $\mathcal{H}_\mathcal{F}$, $\gamma_\mathcal{M}$ is the grading on $\mathcal{M}$, and the representation $\pi_0$ of $\mathcal{A}$ on $\mathcal{H}$ is the tensor product

$$\pi_0 := \pi_\mathcal{M} \otimes \pi_\mathcal{F} \quad (2.86)$$

of the multiplicative representation $\pi_\mathcal{M}$ (1.14) of $C^\infty(\mathcal{M})$ on spinors with the representation $\pi_\mathcal{F}$ of $\mathcal{A}_\mathcal{F}$ on $\mathcal{H}_\mathcal{F}$. If $\mathcal{F}$ is graded and real with grading $\gamma_\mathcal{F}$ and real structure $J_\mathcal{F}$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is also graded and real, respectively, with

$$\gamma = \gamma_\mathcal{M} \otimes \gamma_\mathcal{F}, \qquad J = \mathcal{J} \otimes J_\mathcal{F}. \quad (2.87)$$

Here, the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$ accounts for the fermionic content of the theory and its dimension reflects the number of elementary fermions in the model. An orthonormal basis can be chosen for $\mathcal{H}_\mathcal{F}$ where the basis vectors represent these fermions. The finite Dirac operator $\mathcal{D}_\mathcal{F}$ is a square matrix acting on $\mathcal{H}_\mathcal{F}$ and its entries encode the fermionic masses. $\gamma_\mathcal{F}$ and $J_\mathcal{F}$ being the parity and the charge conjugation operator of the finite space $\mathcal{F}$, respectively, switch between right/left-handed particles and particles/antiparticles.

In the following subsections, we give some examples of almost-commutative geometries and the corresponding gauge theories they describe. Primarily, we briefly recall the spectral triples of a $U(1)$ gauge theory and that of electrodynamics, which we will make explicit use of later. We also mention the spectral triples of Yang-Mills theory and the Standard Model of particle physics.

### 2.4.1 $U(1)$ gauge theory

One of the simplest finite noncommutative spaces is that consisting of two points only – the two-point space. The graded and real finite spectral triple $\mathcal{F}_2$ associated to a two-point space, given by the data:

$$\mathcal{A}_\mathcal{F} = \mathbb{C}^2, \ \mathcal{H}_\mathcal{F} = \mathbb{C}^2, \ \mathcal{D}_\mathcal{F} = 0; \quad \gamma_\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ J_\mathcal{F} = \begin{pmatrix} 0 & cc \\ cc & 0 \end{pmatrix}, \quad (2.88)$$

[2] The product of two graded real spectral triples is defined in the sense of the direct product of two manifolds, cf. [Su, §4].

when considered for the almost-commutative geometry $\mathcal{M} \times \mathcal{F}_2$, defined by (2.85), describes a $U(1)$ gauge theory [DS, §3]. Here, the grading $\gamma_{\mathcal{F}}$ and the real structure $J_{\mathcal{F}}$ of (2.88) are in the orthonormal basis $\{e, \overline{e}\}$ of $\mathcal{H}_{\mathcal{F}} = \mathbb{C}^2$ with $e$ being the basis element of $\mathcal{H}_{\mathcal{F}}^+$ (representing an electron) and $\overline{e}$ of $\mathcal{H}_{\mathcal{F}}^-$ (representing an anti-electron, i.e. a positron), where $\mathcal{H}_{\mathcal{F}}^{\pm}$ denote the $\pm 1$-eigenspace of the grading operator $\gamma_{\mathcal{F}}$. So, we have

$$\begin{aligned} \gamma_{\mathcal{F}} e &= e, \qquad \gamma_{\mathcal{F}} \overline{e} = \overline{e}, \\ J_{\mathcal{F}} e &= \overline{e}, \qquad J_{\mathcal{F}} \overline{e} = e. \end{aligned} \tag{2.89}$$

For the product geometry $\mathcal{M} \times \mathcal{F}_2$, the algebra $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2 \ni a := (f, g)$ acts on the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2$ via the representation $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, defined by

$$\pi_0(a) := \begin{pmatrix} \pi_{\mathcal{M}}(f) & 0 \\ 0 & \pi_{\mathcal{M}}(g) \end{pmatrix}, \qquad \forall f, g \in C^\infty(\mathcal{M}), \tag{2.90}$$

with $\pi_{\mathcal{M}}$ as in (1.14). The KO-dim. of (2.88) is 6, then for a 4-dim. manifold $\mathcal{M}$, the almost-commutative geometry $\mathcal{M} \times \mathcal{F}_2$ has KO-dim. $6 + 4 \bmod 8 = 2$.

The inner fluctuations of $\mathcal{M} \times \mathcal{F}_2$ are parametrized by a $U(1)$ gauge field $Y_\mu \in C^\infty(\mathcal{M}, \mathbb{R}) \simeq C^\infty\big(\mathcal{M}, i\,u(1)\big)$ as [DS, Prop. 3.3]

$$\mathcal{D} \mapsto \mathcal{D}_\omega := \mathcal{D} + \gamma^\mu Y_\mu \otimes \gamma_{\mathcal{F}}, \tag{2.91}$$

where $\mathcal{D} = \eth \otimes \mathbb{I}_{\mathcal{F}}$ (here, setting $\mathcal{D}_{\mathcal{F}} = 0$ is the only choice for $\mathcal{F}_2$ to have a real structure [DS, Prop. 3.1]). Thus, this gauge field $Y_\mu$ implements the action of a unitary $u := e^{i\theta} \in C^\infty\big(\mathcal{M}, U(1)\big)$ of $\mathcal{A}$ on $\mathcal{D}_\omega$, by conjugation:

$$Y_\mu \mapsto Y_\mu - i u \partial_\mu u^* = Y_\mu - \partial_\mu \theta, \qquad \text{for } \theta \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{2.92}$$

**Remark 2.** *Although the almost-commutative geometry $\mathcal{M} \times \mathcal{F}_2$ successfully describes a $U(1)$ gauge theory, it falls short of a complete description of classical electrodynamics, as discussed at the end of [DS, §3]. This is due to the following two reasons:*

1. *Since the finite Dirac operator $\mathcal{D}_{\mathcal{F}}$ is zero, the electrons cannot be massive.*

2. *The finite-dimensional Hilbert space $\mathcal{H}_{\mathcal{F}}$ does not possess enough room to capture the required spinor degrees of freedom. More precisely, the fermionic action (2.81) of $\mathcal{M} \times \mathcal{F}_2$ describes only one arbitrary Dirac spinor, whereas two of those, independent of each other, are needed to describe a free Dirac field, cf. [Col, pg. 311].*

*However, none of the above arises as an issue if one only wishes to obtain the Weyl action, since the Weyl fermions are massless anyway, and they only need half of the spinor degrees of freedom as compared to the Dirac fermions.*

### 2.4.2 Electrodynamics

Electrodynamics is one of the simplest field theories in physics. A slight modification of the example of $\mathcal{M} \times \mathcal{F}_2$ given in §2.4.1 can overcome the two obstructions of Rem. 2 and provide a unified (at the classical level) description of gravity and electromagnetism.

Such a modification entails doubling the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$ from $\mathbb{C}^2$ to $\mathbb{C}^4$; which not only allows for a non-zero finite Dirac operator $\mathcal{D}_\mathcal{F}$, but also gives the correct spinor degrees of freedom in the fermionic action. We refer to [DS, Su] for details.

The spectral triple of electrodynamics is given by the product geometry $\mathcal{M} \times \mathcal{F}_{ED}$ of a 4-dim. compact riemannian spin manifold $\mathcal{M}$ (with grading $\gamma_\mathcal{M} = \gamma^5$ and real structure $\mathcal{J}$) and the graded real finite spectral triple $\mathcal{F}_{ED}$ defined by the following data [DS, §4.1]:

$$\mathcal{F}_{ED} := \left(\mathcal{A}_\mathcal{F} = \mathbb{C}^2,\ \mathcal{H}_\mathcal{F} = \mathbb{C}^4,\ \mathcal{D}_\mathcal{F};\ \gamma_\mathcal{F},\ J_\mathcal{F}\right), \tag{2.93}$$

where, in the orthonormal basis $\left\{e_L, e_R, \overline{e_L}, \overline{e_R}\right\}$ of $\mathcal{H}_\mathcal{F} = \mathbb{C}^4$ (denoting both the left and right handed electrons and positrons) and for a constant parameter $d \in \mathbb{C}$, we have

$$\mathcal{D}_\mathcal{F} = \begin{pmatrix} 0 & d & 0 & 0 \\ \overline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} \\ 0 & 0 & d & 0 \end{pmatrix}, \quad \gamma_\mathcal{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_\mathcal{F} = \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}, \tag{2.94}$$

so that

$$\begin{aligned} \gamma_\mathcal{F} e_L &= e_L, & \gamma_\mathcal{F} e_R &= -e_R, & \gamma_\mathcal{F} \overline{e_L} &= -\overline{e_L}, & \gamma_\mathcal{F} \overline{e_R} &= \overline{e_R}, \\ J_\mathcal{F} e_L &= \overline{e_L}, & J_\mathcal{F} e_R &= \overline{e_R}, & J_\mathcal{F} \overline{e_L} &= e_L, & J_\mathcal{F} \overline{e_R} &= e_R. \end{aligned}$$

The algebra $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2 \ni a := (f, g)$ of $\mathcal{M} \times \mathcal{F}_{ED}$ acts on its Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4$ via the representation

$$\pi_0(a) := \begin{pmatrix} f\mathbb{I}_4 & 0 & 0 & 0 \\ 0 & f\mathbb{I}_4 & 0 & 0 \\ 0 & 0 & g\mathbb{I}_4 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_4 \end{pmatrix}, \qquad \forall f, g \in C^\infty(\mathcal{M}). \tag{2.95}$$

The KO-dim. of $\mathcal{F}_{ED}$ (2.93) is same as that of $\mathcal{F}_2$ (2.88), i.e. 6. Therefore, the KO-dim. of $\mathcal{M} \times \mathcal{F}_{ED}$ is 2 (mod 8). The inner fluctuations of $\mathcal{D} = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes \mathcal{D}_\mathcal{F}$

$$\begin{aligned} \mathcal{D} \to \mathcal{D}_\omega &= \mathcal{D} + \gamma^\mu \otimes B_\mu, \\ \text{where} \qquad B_\mu &:= \mathrm{diag}\left(Y_\mu, Y_\mu, -Y_\mu, -Y_\mu\right); \end{aligned} \tag{2.96}$$

are parametrized by a single $U(1)$ gauge field $Y_\mu$ carrying an adjoint action of the group $C^\infty\left(\mathcal{M}, U(1)\right)$ of unitaries of $\mathcal{A}$ on $\mathcal{D}_\omega$ (2.96), implemented by (2.92) [DS, §4.2].

The full action (spectral plus fermionic) of the almost-commutative geometry $\mathcal{M} \times \mathcal{F}_{ED}$ yields the lagrangian for electrodynamics (on a curved background manifold $\mathcal{M}$) – identifying $Y_\mu$ (2.96) as the $U(1)$ gauge potential of electrodynamics – along with a purely gravitational lagrangian [DS, §4.3].

### 2.4.3 Other physical models

The great potential and flexibility of noncommutative geometry – in particular, its applicability to theoretical high energy physics – can be realized by looking at the plethora of physically relevant models that can be described within its framework.

#### Yang-Mills theory

Electrodynamics, as discussed in the previous section, is an abelian $U(1)$ gauge theory, which can be further generalized to the non-abelian cases. For instance, a non-abelian $SU(n)$ gauge theory – also known as Yang-Mills theory among physicists, is described within the almost-commutative geometry $\mathcal{M} \times \mathcal{F}_{YM}$ defined by the data:

$$\big(C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C}),\ L^2(\mathcal{M}, \mathcal{S}) \otimes M_n(\mathbb{C}),\ \eth \otimes \mathbb{I}_n;\ \ \mathcal{J} \otimes (\cdot)^*,\ \gamma_{\mathcal{M}} \otimes \mathbb{I}_n\big). \tag{2.97}$$

The spectral triple (2.97) describes the Einstein-Yang-Mills theory [CC97], which not only adapts to ($N = 2$ and $N = 4$) supersymmetry [Ch94], but can also be extended to accommodate topologically non-trivial gauge configurations [BS].

#### Standard Model

Another very important non-abelian gauge theory is the Standard Model of particle physics, with structure (gauge) group $U(1) \times SU(2) \times SU(3)$, whose full lagrangian can be derived, together with the Higgs potential and the Einstein-Hilbert action of gravity with a minimal coupling; from the following spectral triple [CCM]:

$$\big(\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{SM},\quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_{\mathcal{F}},\quad \mathcal{D} = \eth \otimes \mathbb{I}_{\mathcal{F}} + \gamma_{\mathcal{M}} \otimes \mathcal{D}_{\mathcal{F}}\big), \tag{2.98}$$

where the Standard Model algebra [CC08]

$$\mathcal{A}_{SM} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{2.99}$$

acts on the space

$$\mathcal{H}_{\mathcal{F}} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_{\bar{L}} \oplus \mathcal{H}_{\bar{R}} = \mathbb{C}^{96} \tag{2.100}$$

of elementary fermions: 8 fermions (up and down quarks with 3 colors each plus electron and neutrino) for 3 generations and 2 chiralities left/right $(L, R)$ plus their antiparticles (denoted by bar). So, each of the four subspaces in (2.100) is of dim. 24 and, thus, isomorphic to $\mathbb{C}^{24}$.

$\mathcal{D}_\mathcal{F}$ is a $96 \times 96$ matrix acting on $\mathcal{H}_\mathcal{F} = \mathbb{C}^{96}$ whose entries correspond to the 31 real parameters[3] of the Standard Model [CMa, §13.5]. The grading is $\gamma = \gamma^5 \otimes \gamma_\mathcal{F}$ and the real structure is $J = \mathcal{J} \otimes J_\mathcal{F}$, where

$$\begin{aligned} \gamma_\mathcal{F} &:= \mathsf{diag}\big(\mathbb{I}_{24},\, -\mathbb{I}_{24},\, -\mathbb{I}_{24},\, \mathbb{I}_{24}\big), \\ J_\mathcal{F} &:= \begin{pmatrix} 0 & \mathbb{I}_{48} \\ \mathbb{I}_{48} & 0 \end{pmatrix} \mathsf{cc}. \end{aligned} \tag{2.101}$$

The KO-dim. of $\mathcal{M} \times \mathcal{F}_{SM}$ (2.98) is $4 + 6 \bmod 8 = 2$. The representation $\pi_0$ of $\mathcal{A}$ on $\mathcal{H}$ for one generation[4] is written as, cf. [CCM]:

$$\pi_0(f \otimes a) = \pi_\mathcal{M}(f) \otimes \pi_{SM}(a), \qquad \forall f \in C^\infty(\mathcal{M}),\; a \in \mathcal{A}_{SM}, \tag{2.102}$$

where $\pi_\mathcal{M}$ is as in (1.14) and $\pi_{SM}$ for $a := (\lambda, q, m) \in \mathcal{A}_{SM}$ is given by

$$\pi_{SM}(a) := \pi_L(q) \oplus \pi_R(\lambda) \oplus \pi_{\bar{L}}(\lambda, m) \oplus \pi_{\bar{R}}(\lambda, m), \tag{2.103}$$

with $\lambda \in \mathbb{C}$ acting on $\mathcal{H}_R \oplus \mathcal{H}_{\bar{L}} \oplus \mathcal{H}_{\bar{R}}$, quaternions $q \in \mathbb{H}$ on $\mathcal{H}_L$, and matrices $m \in M_3(\mathbb{C})$ on $\mathcal{H}_{\bar{L}} \oplus \mathcal{H}_{\bar{R}}$. The individual representations in (2.103), identifying $\mathbb{H}$ with its usual representation as $M_2(\mathbb{C})$, explicitly are:

$$\begin{aligned} \pi_L(q) &:= q \otimes \mathbb{I}_4, \\ \pi_R(\lambda) &:= \mathsf{diag}(\lambda, \bar{\lambda}) \otimes \mathbb{I}_4, \\ \pi_{\bar{L}}(\lambda, m) = \pi_{\bar{R}}(\lambda, m) &:= \mathbb{I}_2 \otimes \mathsf{diag}(\lambda, m), \end{aligned} \tag{2.104}$$

where $\mathbb{I}_4$ in the first two eqs. indicates that $\mathbb{C}$ and $\mathbb{H}$ preserve color and do not mix the leptons (electrons and neutrinos) with the quarks (up and down), and $\mathbb{I}_2$ in the last eq. indicates that $\mathbb{C}$ and $M_3(\mathbb{C})$ preserve the flavor: $\lambda$ acts on the antileptons whereas $m$ mixes the color of the antiquarks.

### Beyond Standard Model

It is also possible to construct Grand Unified Theories in the framework of noncommutative geometry, such as the $SO(10)$ model [CF1, CF2, W]. Among other extensions, there is the Pati-Salam model [AMST, CCS2, CCS3], whose symmetry spontaneously breaks down to the Standard Model.

Further, there are ways to modify and/or relax some of the axioms of noncommutative geometry to produce more flexible geometries that are capable of serving as platforms to

[3] Yukawa couplings of the fermions, the Dirac and Majorana masses of the neutrinos, the quark mixing angles of the CKM matrix, and the neutrino mixing angles of the PMNS matrix.

[4] In (2.100), $\mathcal{H}_\mathcal{F} = \mathbb{C}^{32}$ and all four of its subspaces are each isomorphic to $\mathbb{C}^8$. Then, the representation for all the three generations is just a direct sum of similar representations (2.102) for each generation.

explore the physics beyond that of the Standard Model, see e.g. [CCS1] and [DLM1, DM]. One such method of twisting will be the subject-matter of this thesis, which we will be exploring in the next chapters.

# Chapter 3

# Minimally Twisted Spectral Triples

Twisted spectral triples were first introduced – from a purely mathematical motivation – by Connes and Moscovici [CMo] in the context of operator algebras to extend the local index formula for algebras of type III.[1] Such algebras characteristically exhibit no nontrivial trace and, hence, are incompatible with the requirement of the boundedness of the commutator $[\mathcal{D}, a]$ in the Def. 1.1 of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. So, the 'twist' basically comprises of trading off this requirement for the condition that there exists of an automorphism $\rho$ of $\mathcal{A}$ such that the 'twisted' commutator, defined as

$$[\mathcal{D}, a]_\rho := \mathcal{D}a - \rho(a)\mathcal{D}, \tag{3.1}$$

is bounded for any $a \in \mathcal{A}$. Twisted commutators are well-defined on the domain of $\mathcal{D}$ and extend to bounded operators on $\mathcal{H}$.

Later, noncommutative geometries twisted in this manner found applications to high energy physics in describing extensions of Standard Model, such as the Grand Symmetry Model [DLM1, DM].

In §3.1, we define twisting real spectral triples using algebra automorphisms [LM1] and, in §3.1.2, state the laws of gauge transformations for them [LM2]. We recall how the twist $\rho$ naturally induces a $\rho$-inner product $\langle \cdot, \cdot \rangle_\rho$ on the Hilbert space $\mathcal{H}$ (§3.1.3), which allows to define a fermionic action suitable for real twisted spectral triples (§3.1.4) [DFLM].

The key difference from the Def. (2.81) of the fermionic action in the usual (i.e. non-twisted) case is that one no longer restricts to the positive eigenspace $\mathcal{H}_+$ of the grading $\gamma$, but rather to that of the unitary $\mathcal{R}$ implementing the twist $\rho$.

In §3.2, we highlight the 'twist by grading' procedure – which canonically associates a twisted partner to any graded spectral triple whose representation is sufficiently faithful – and the notion of twisting a spectral triple minimally [LM1].

[1] in the sense of the type classification of the von Neumann algebras, cf. [C94, CMo]

## 3.1 Twisting by algebra automorphisms

**Definition 3.1** (from [CMo]). *A* **twisted spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})_\rho$ *consists of a unital* $*$*-algebra* $\mathcal{A}$ *acting faithfully on a Hilbert space* $\mathcal{H}$ *as bounded operators, a self-adjoint operator* $\mathcal{D}$ *with compact resolvent on* $\mathcal{H}$ *referred to as the* **Dirac operator**, *and an automorphism* $\rho$ *of* $\mathcal{A}$ *such that the* **twisted commutator**, *defined as*

$$[\mathcal{D}, a]_\rho := \mathcal{D}a - \rho(a)\mathcal{D}, \tag{3.2}$$

*is bounded for any* $a \in \mathcal{A}$.

As for usual spectral triples, a **graded** or **even** twisted spectral triple is one endowed with a $\mathbb{Z}_2$-grading $\gamma$ on $\mathcal{H}$, that is, a self-adjoint operator $\gamma : \mathcal{H} \to \mathcal{H}$, satisfying (1.7).

The **real structure** (Axiom 7) easily adapts to the twisted case [LM1]. One considers an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ (1.11) satisfying (1.12), where – as in the non-twisted case – the signs $\epsilon, \epsilon', \epsilon''$ determine the KO-**dimension** of the twisted spectral triple. Additionally, $J$ is required to implement an isomorphism (1.19) between $\mathcal{A}$ and its opposite algebra $\mathcal{A}^\circ$ such that Axiom 7' is satisfied. However, in the twisted case, Axiom 2' is modified to be compatible with the twist as follows [DM, LM1]:

$$\left[[\mathcal{D}, a]_\rho, b^\circ\right]_{\rho^\circ} := [\mathcal{D}, a]_\rho b^\circ - \rho^\circ(b^\circ)[\mathcal{D}, a]_\rho = 0, \qquad \forall a, b \in \mathcal{A}, \tag{3.3}$$

where $\rho^\circ$ is the automorphism induced by $\rho$ on $\mathcal{A}^\circ$ via

$$\rho^\circ(b^\circ) = \rho^\circ(Jb^*J^{-1}) := J\rho(b^*)J^{-1}. \tag{3.4}$$

**Definition 3.2** (from [LM1]). *A* **real twisted spectral triple** *is a graded twisted spectral triple with a real structure* $J$ (1.11) *satisfying* (1.12)*, Axiom 7', and the 'twisted' first order condition given by* (3.3)*.*

In case the automorphism $\rho$ coincides with an inner automorphism of $\mathcal{B}(\mathcal{H})$, that is

$$\pi(\rho(a)) = \mathcal{R}\pi(a)\mathcal{R}^\dagger, \qquad \forall a \in \mathcal{A}, \tag{3.5}$$

where $\mathcal{R} \in \mathcal{B}(\mathcal{H})$ is unitary, then $\rho$ is said to be **compatible with the real structure** $J$, as soon as [DFLM, Def. 3.2]

$$J\mathcal{R} = \epsilon''' \mathcal{R}J, \qquad \text{for} \quad \epsilon''' = \pm 1. \tag{3.6}$$

The inner automorphism $\rho$ and, hence, the unitary $\mathcal{R}$ are not necessarily unique. In that case, $\rho$ is compatible with the real structure $\mathcal{J}$ if there exists at least one $\mathcal{R}$ satisfying the conditions mentioned above.

**Remark 3.** *In the original definition of the twist [CMo, (3.4)], the automorphism* $\rho$ *is not required to be a* $*$*-automorphism, but rather to satisfy the regularity condition* $\rho(a^*) = \rho^{-1}(a)^*$. *If, however, one requires* $\rho$ *to be a* $*$*-automorphism, then the regularity condition renders*

$$\rho^2 = \mathrm{Id}. \tag{3.7}$$

Other modifications of spectral triples by twisting the real structure have been proposed in [BCDS] and some interesting relations with the twisted spectral triples mentioned above have been worked out in [BDS].

### 3.1.1 Twisted fluctuation of the metric

The fluctuations of the Dirac operator in the non-twisted case – as discussed in §2.2.3 – can be extended for twisted spectral triples. Initially done by analogy in [DM], the twisted fluctuations have successfully been put on the same footing in [LM2], as Connes' original "fluctuations of the metric" [C96]. This essentially entails transporting a real twisted spectral triple over to a Morita equivalent algebra. Particularly, for Morita self-equivalence, one has the following.

**Definition 3.3** (from [LM1]). *Given a real twisted spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J)_\rho$, *a twisted fluctuation of* $\mathcal{D}$ *by* $\mathcal{A}$ *is a self-adjoint operator of the form*

$$\mathcal{D}_{\omega_\rho} := \mathcal{D} + \omega_\rho + \epsilon' J \omega_\rho J^{-1}, \tag{3.8}$$

*for some twisted one-form*

$$\omega_\rho \in \Omega^1_{\mathcal{D}}(\mathcal{A}, \rho) := \left\{ \textstyle\sum_j a_j \left[\mathcal{D}, b_j\right]_\rho, \;\; a_j, b_j \in \mathcal{A} \right\}. \tag{3.9}$$

*The operator* $\mathcal{D}_{\omega_\rho}$ *is then referred to as the twisted-covariant Dirac operator.*

The set $\Omega^1_{\mathcal{D}}$ (3.9) of twisted one-forms is an $\mathcal{A}$-bimodule with a twisted action on the left:

$$a \cdot \omega_\rho \cdot b = \rho(a)\omega_\rho b, \qquad \forall a, b \in \mathcal{A}, \tag{3.10}$$

so that the twisted commutator $[\mathcal{D}, \cdot]_\rho =: \delta_\rho(\cdot)$ is a derivation of $\mathcal{A}$ in $\Omega^1_{\mathcal{D}}$ [CMo, Prop. 3.4]:

$$\delta_\rho(ab) = \rho(a) \cdot \delta_\rho(b) + \delta_\rho(a) \cdot b. \tag{3.11}$$

Thus, $\Omega^1_{\mathcal{D}}(\mathcal{A}, \rho)$ is the $\mathcal{A}$-bimodule generated by $\delta_\rho$ and it acts as bounded operator on $\mathcal{H}$, since so do both $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]_\rho$.

Here, it is also worth noticing that the self-adjointness is being imposed on the twist-fluctuated operator $\mathcal{D}_{\omega_\rho}$ (3.8), which has the same domain as that of $\mathcal{D}$, and not necessarily on the twisted one-form $\omega_\rho$ (3.9). We shall further emphasize and explore this point in detail right after Lem. 4.2 below.

### 3.1.2 Twisted gauge transformations

For twisted spectral triples, the gauge transformations have been worked out in [LM2], which – as in the non-twisted case (§2.3) – involve a change of the connection in the bimodule implementing the Morita equivalence. These are induced by the simultaneous action of the unitary group $\mathcal{U}(\mathcal{A})$ (2.75) on both the Hilbert space $\mathcal{H}$ and the space $\mathsf{L}(\mathcal{H})$ of linear operators in $\mathcal{H}$.

On $\psi \in \mathcal{H}$, a unitary $u \in \mathcal{U}(\mathcal{A})$ acts via the usual adjoint action (2.76) of $\mathcal{A}$. However, on $\mathsf{T} \in \mathsf{L}(\mathcal{H})$, the action is twisted and implemented by the map

$$\mathsf{T} \mapsto \big(\mathsf{Ad}\ \rho(u)\big)\mathsf{T}\big(\mathsf{Ad}\ u^*\big), \quad \text{with} \quad \mathsf{Ad}\ \rho(u) := \rho(u)J\rho(u)J^{-1}, \tag{3.12}$$

and this evaluated for the twisted-covariant Dirac operator $\mathcal{D}_{\omega_\rho}$ gives [LM2, §4]

$$\mathcal{D}_{\omega_\rho} \mapsto \big(\mathsf{Ad}\ \rho(u)\big)\mathcal{D}_{\omega_\rho}\big(\mathsf{Ad}\ u^*\big) =: \mathcal{D}_{\omega_\rho^u}, \tag{3.13}$$

where, one has

$$\omega_\rho \mapsto \omega_\rho^u := \rho(u)\omega_\rho u^* + \rho(u)[\mathcal{D}, u^*]_\rho, \tag{3.14}$$

which is the twisted analogue of how the one-forms of the usual spectral triples transform as (2.73) in noncommutative geometry [C96].

Although gauge transformations leave the fluctuated Dirac operator $\mathcal{D}_\omega$ self-adjoint, the twist-fluctuated Dirac operator $\mathcal{D}_{\omega_\rho}$ is not self-adjoint under twisted gauge transformations. However, there does exist a more natural property than self-adjointness that is preserved under the twisted gauge transformations: **$\rho$-adjointness** defined with respect to the **$\rho$-inner product** induced by the twist $\rho$ on the Hilbert space $\mathcal{H}$, cf. [DFLM, Rem. 2.1].

### 3.1.3 $\rho$-inner product

Given a Hilbert space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle$ and an automorphism $\rho$ of $\mathcal{B}(\mathcal{H})$, a $\rho$-*inner product* $\langle\cdot,\cdot\rangle_\rho$ is an inner product satisfying [DFLM, Def. 3.1]

$$\langle\psi, \mathcal{O}\phi\rangle_\rho = \langle\rho(\mathcal{O})^\dagger\psi, \phi\rangle_\rho, \qquad \forall\mathcal{O} \in \mathcal{B}(\mathcal{H}),\ \forall\psi, \phi \in \mathcal{H}, \tag{3.15}$$

where $^\dagger$ denotes the hermitian adjoint with respect to $\langle\cdot,\cdot\rangle$. One denotes

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger \tag{3.16}$$

to be the $\rho$-*adjoint* of the operator $\mathcal{O}$.

If $\rho$ is an inner automorphism and implemented by a unitary operator $\mathcal{R} \in \mathcal{B}(\mathcal{H})$, that is, $\rho(\mathcal{O}) = \mathcal{R}\mathcal{O}\mathcal{R}^\dagger$ for any $\mathcal{O} \in \mathcal{B}(\mathcal{H})$. Then, a canonical $\rho$-inner product is given by

$$\langle\psi, \phi\rangle_\rho = \langle\psi, \mathcal{R}\phi\rangle, \qquad \forall\psi, \phi \in \mathcal{H}. \tag{3.17}$$

The $\rho$-adjointness is not necessarily an involution. If $\rho$ is a $*$-automorphism (e.g. when $\rho$ is an inner automorphism), then $^+$ is an involution iff (3.7) holds, since

$$(\mathcal{O}^+)^+ = \rho(\mathcal{O}^+)^\dagger = \rho\big((\mathcal{O}^+)^\dagger\big) = \rho\big(\rho(\mathcal{O})\big). \tag{3.18}$$

The same condition arises for a twisted spectral triple, when one defines the $\rho$-adjointness solely at the algebraic level, i.e. $a^+ := \rho(a)^*$, without assuming that $\rho \in \mathsf{Aut}(\mathcal{A})$ extends to an automorphism of $\mathcal{B}(\mathcal{H})$.

Indeed, assuming the regularity condition in Rem. 3 (written as $\rho(b)^* = \rho^{-1}(b^*)$ for any $b = a^* \in \mathcal{A}$), one then gets

$$(a^+)^+ = (\rho(a)^*)^+ = (\rho^{-1}(a^*))^+ = \rho(\rho^{-1}(a^*))^* = \rho(\rho(a)^*)^* = \rho^2(a). \tag{3.19}$$

### 3.1.4 Fermionic action

In twisted spectral geometry, the fermionic action is defined by replacing $\mathcal{D}_\omega$ in (2.81) with the twist-fluctuated Dirac operator $\mathcal{D}_{\omega_\rho}$ (3.8) and by replacing the inner product with the $\rho$-inner product (3.15) or, in particular, with (3.17) when the compatibility condition (3.6) holds [DFLM, §4.1]. Thus, instead of (2.82), one defines

$$\mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}(\phi, \psi) := \langle J\phi, \mathcal{D}_{\omega_\rho}\psi\rangle_\rho = \langle J\phi, \mathcal{R}\mathcal{D}_{\omega_\rho}\psi\rangle, \qquad \forall \phi, \psi \in \mathsf{Dom}(\mathcal{D}_{\omega_\rho}). \tag{3.20}$$

In the case when the twist $\rho$ is compatible with the real structure $J$, in the sense of (3.6), the bilinear form (3.20) is invariant under the 'twisted' gauge transformation, given by the simultaneous actions of (2.76) and (3.13) [DFLM, Prop. 4.1].

However, the antisymmetry of the form $\mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}$ is not guaranteed, unless one restricts to the positive eigenspace $\mathcal{H}_\mathcal{R}$ of $\mathcal{R}$ [DFLM, Prop. 4.2]

$$\mathcal{H}_\mathcal{R} := \big\{\chi \in \mathsf{Dom}(\mathcal{D}_{\omega_\rho}), \quad \mathcal{R}\chi = \chi\big\}, \tag{3.21}$$

which led to the following:

**Definition 3.4.** *For a real twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J)_\rho$, the fermionic action is*

$$S^f_\rho(\mathcal{D}_{\omega_\rho}) := \mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}(\tilde{\psi}, \tilde{\psi}), \tag{3.22}$$

*where $\tilde{\psi}$ is the Graßmann vector associated to $\psi \in \mathcal{H}_\mathcal{R}$.*

In the spectral triple of the Standard Model, the restriction to $\mathcal{H}_+$ is there to solve the fermion doubling problem [LMMS]. It also selects out the physically meaningful elements of the space $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_\mathcal{F}$, i.e. those spinors whose chirality in $L^2(\mathcal{M}, \mathcal{S})$ coincides with their chirality as elements of the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$.

In the twisted case, the restriction to $\mathcal{H}_{\mathcal{R}}$ is there to guarantee the antisymmetry of the bilinear form $\mathfrak{A}^{\rho}_{\mathcal{D}_{\omega_\rho}}$ (3.20). However, the eigenvectors of $\mathcal{R}$ may not have a well-defined chirality. If fact, they cannot have it when the twist comes from the grading (see §3.2), since the unitary $\mathcal{R}$ implementing the twist, given by (3.31), anticommutes with the chirality $\gamma = \text{diag}\,(\mathbb{I}_{\mathcal{H}_+}, -\mathbb{I}_{\mathcal{H}_-})$, so that we have

$$\mathcal{H}_+ \cap \mathcal{H}_{\mathcal{R}} = \{0\}. \tag{3.23}$$

From a physical point-of-view, by restricting to $\mathcal{H}_{\mathcal{R}}$ rather than $\mathcal{H}_+$, one loses a clear interpretation of the elements of the Hilbert space: a priori, an element of $\mathcal{H}_{\mathcal{R}}$ is not physically meaningful since its chirality is not well-defined. However, we shall demonstrate in what follows that – at least in two examples: the almost-commutative geometry of a $U(1)$ gauge theory and that of electrodynamics – the restriction to $\mathcal{H}_{\mathcal{R}}$ is actually meaningful, for it allows us to obtain the Weyl and Dirac equations in the lorentzian signature, even though one starts with a riemannian manifold.

We conclude this subsection with two easy but useful lemmas. The first one recalls how the symmetry property of the bilinear form $\mathfrak{A}_{\mathcal{D}} = \langle J\cdot, \mathcal{D}\cdot\rangle$ does not depend on the explicit form of the Dirac operator $\mathcal{D}$, but solely on the signs $\epsilon, \epsilon'$ in (1.12). And, the second one emphasizes that once restricted to $\mathcal{H}_{\mathcal{R}}$, the bilinear forms (2.82) and (3.20) differ only by a sign.

**Lemma 3.5.** *Let $J$ be an antilinear isometry on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ such that $J^2 = \epsilon\mathbb{I}$, and $\mathcal{D}$ be a self-adjoint operator on $\mathcal{H}$ such that $J\mathcal{D} = \epsilon'\mathcal{D}J$. Then, we have*

$$\langle J\phi, \mathcal{D}\psi\rangle = \epsilon\epsilon'\langle J\psi, \mathcal{D}\phi\rangle, \qquad \forall \phi,\psi \in \mathcal{H}. \tag{3.24}$$

**Proof.** By definition, an antilinear isometry satisfies $\langle J\phi, J\psi\rangle = \overline{\langle\phi,\psi\rangle} = \langle\psi,\phi\rangle$. Thus,

$$\begin{aligned}\langle J\phi, \mathcal{D}\psi\rangle &= \epsilon\langle J\phi, J^2\mathcal{D}\psi\rangle = \epsilon\langle J\mathcal{D}\psi, \phi\rangle \\ &= \epsilon\epsilon'\langle \mathcal{D}J\psi, \phi\rangle = \epsilon\epsilon'\langle J\psi, \mathcal{D}\phi\rangle. \qquad \blacksquare\end{aligned}$$

In particular, for KO-dim. 2 and 4 one has $\epsilon = -1, \epsilon' = 1$, so $\mathfrak{A}_{\mathcal{D}}$ is antisymmetric. The same is true for $\mathfrak{A}_{\mathcal{D}_\omega}$ in (2.82), because the covariant operator $\mathcal{D}_\omega$ also satisfies the same rules of sign (1.12) as $\mathcal{D}$. On the other hand, for KO-dim. 0 and 6 one has $\epsilon = \epsilon' = 1$, and so $\mathfrak{A}_{\mathcal{D}}$ is symmetric.

**Lemma 3.6.** *Given $\mathcal{D}$, and $\mathcal{R}$ compatible with $J$ in the sense of (3.6), one has*

$$\mathfrak{A}^{\rho}_{\mathcal{D}}(\phi,\psi) = \epsilon'''\,\mathfrak{A}_{\mathcal{D}}(\phi,\psi), \qquad \forall\phi,\psi \in \mathcal{H}_{\mathcal{R}}. \tag{3.25}$$

**Proof.** For any $\phi, \psi \in \mathcal{H}_\mathcal{R}$, we have

$$\begin{aligned}\mathfrak{A}^\rho_\mathcal{D}(\phi,\psi) &= \langle J\phi, \mathcal{R}\mathcal{D}\psi\rangle = \langle \mathcal{R}^\dagger J\phi, \mathcal{D}\psi\rangle \\ &= \epsilon'''\langle J\mathcal{R}^\dagger\phi, \mathcal{D}\psi\rangle = \epsilon'''\langle J\phi, \mathcal{D}\psi\rangle,\end{aligned} \tag{3.26}$$

where we used (3.6) as $\mathcal{R}^\dagger J = \epsilon''' J\mathcal{R}^\dagger$ and (3.21) as $\mathcal{R}^\dagger\phi = \phi$. ∎

## 3.2 Minimal twist by grading

The twisted spectral triples that are recently employed in physics have been built by minimally twisting a usual spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The idea of minimal twisting is to substitute the commutator $[\mathcal{D}, \cdot]$ with the twisted commutator $[\mathcal{D}, \cdot]_\rho$, while keeping the Hilbert space $\mathcal{H}$ and the Dirac operator $\mathcal{D}$ intact, since they encode the fermionic content of the theory and there has, so far, been no experimental indications of the existence of extra fermions beyond those of the Standard Model.

Such a substitution yields new fields [DLM1, DM] that not only make the theoretical mass of the Higgs boson compatible with its experimental value [CC12], but also offer a way out of the problem of the instability (or, meta-stability) of the electroweak vacuum at intermediate energies, as mentioned in the Introduction. However, for physically relevant spectral triples, both $[\mathcal{D}, \cdot]$ and $[\mathcal{D}, \cdot]_\rho$ cannot be bounded simultaneously and so one needs to enlarge the algebra [LM1].

**Definition 3.7** (from [LM1]). *A* minimal twist *of a spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *by a unital* $*$*-algebra* $\mathcal{B}$ *is a twisted spectral triple* $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, \mathcal{D})_\rho$ *where the initial representation* $\pi_0$ *of* $\mathcal{A}$ *on* $\mathcal{H}$ *is related to the representation* $\pi$ *of* $\mathcal{A}\otimes\mathcal{B}$ *on* $\mathcal{H}$ *by*

$$\pi(a\otimes\mathbb{I}_\mathcal{B}) = \pi_0(a), \qquad \forall a \in \mathcal{A}, \tag{3.27}$$

*where* $\mathbb{I}_\mathcal{B}$ *is the identity of the algebra* $\mathcal{B}$.

If the initial spectral triple is graded, then a canonical minimal twist can be obtained naturally as follows. The grading $\gamma$ commutes with the representation of the algebra $\mathcal{A}$, so the latter is a direct sum of two representations on the positive and negative eigenspaces, respectively, $\mathcal{H}_+$ and $\mathcal{H}_-$, of the grading $\gamma$, see (2.84).

Therefore, one has enough room on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ to represent the algebra $\mathcal{A}$ twice. It is tantamount to taking $\mathcal{B} = \mathbb{C}^2$ in the Def. 3.7 above, with $\mathcal{A}\otimes\mathbb{C}^2 \simeq \mathcal{A}\oplus\mathcal{A} \ni (a, a')$ represented on $\mathcal{H}$ as

$$\pi(a, a') := \mathfrak{p}_+\pi_0(a) + \mathfrak{p}_-\pi_0(a') = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a') \end{pmatrix}, \tag{3.28}$$

where

$$\mathfrak{p}_\pm := \frac{1}{2}\left(\mathbb{I}_{\mathcal{H}} \pm \gamma\right) \qquad \text{and} \qquad \pi_\pm(a) := \pi_0(a)_{|\mathcal{H}_\pm} \tag{3.29}$$

are, respectively, the projections on $\mathcal{H}_\pm$ and the restrictions on $\mathcal{H}_\pm$ of $\pi_0$.

If $\pi_\pm$ are faithful,[2] then $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, \mathcal{D})_\rho$ with the **flip automorphism** $\rho$ given by

$$\rho(a, a') := (a', a), \qquad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2, \tag{3.30}$$

is indeed a twisted spectral triple, with grading $\gamma$. Furthermore, if the initial spectral triple is real, then so is this minimal twist, with the same real structure [LM1].

The flip $\rho$ (3.30) is a $*$-automorphism satisfying (3.7) and coinciding on $\pi(\mathcal{A} \otimes \mathbb{C}^2)$ with the inner automorphism of $\mathcal{B}(\mathcal{H})$ implemented by the unitary operator

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_{\mathcal{H}_+} \\ \mathbb{I}_{\mathcal{H}_-} & 0 \end{pmatrix}, \tag{3.31}$$

where $\mathbb{I}_{\mathcal{H}_\pm}$ is the identity operator in $\mathcal{H}_\pm$.

[2] The requirement that $\pi_\pm$ are faithful was not explicit in [LM1]. If it does not hold, then $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_\rho$ still satisfies all the properties of a twisted spectral triple, except that $\pi$ in (3.28) might not be faithful.

# Chapter 4

# Lorentzian Fermionic Actions from Euclidean Spectral Triples

In this chapter, we study three examples of minimally twisted spectral triples: a manifold, $U(1)$ gauge theory, and electrodynamics – with their corresponding fermionic actions. This yields the Weyl and the Dirac action in lorentzian signature, respectively, in the last two cases.

We recall that the canonical $\rho$-inner product (3.17) associated to the minimal twist of a 4-dimensional closed **riemannian** spin manifold turns out to coincide with the **lorentzian** Kreĭn product on the Hilbert space of lorentzian spinors [DFLM, §3.2]. One of the main results of this thesis is to demonstrate that a similar transition of metric signature – from the euclidean to the lorentzian – also occurs at the level of the fermionic action [MS].

In §4.1, we will first investigate how this idea comes about, by looking at the simplest example of the minimal twist of a closed riemannian spin manifold $\mathcal{M}$ and computing the associated fermionic action.

Then, in §4.2, we demonstrate how we obtain a lorentzian Weyl action from the minimally twisted $U(1)$ gauge theory. Similarly, in §4.3, we also derive the lorentzian Dirac action from the spectral triple of electrodynamics [DS] – by minimally twisting it.

Since we intend to arrive at the physically relevant Weyl and Dirac actions, we chose to work in dimension 4, assuming gravity is negligible (hence the flat metric). This is tantamount to choosing in (1.12) the following signs:

$$\epsilon = -1, \qquad \epsilon' = 1, \qquad \epsilon'' = 1. \tag{4.1}$$

## 4.1 Minimal twist of a manifold

The minimal twist of the canonical triple (1.13) of a closed riemannian spin manifold $\mathcal{M}$ is the following real twisted spectral triple:

$$\left(\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}), \quad \mathcal{D} = \eth\right)_\rho \tag{4.2}$$

with the inner product on the Hilbert space $L^2(\mathcal{M}, \mathcal{S})$ given by

$$\langle \psi, \phi \rangle = \int_{\mathcal{M}} d\mu \; \psi^\dagger \phi, \tag{4.3}$$

where $d\mu$ is the volume form. With the self-adjoint euclidean Dirac matrices $\gamma$'s (see App. D), the real structure is given by [DM]

$$\mathcal{J} = i\gamma^0\gamma^2 cc = i \begin{pmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} cc, \tag{4.4}$$

where $cc$ denotes complex conjugation and the grading is

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \tag{4.5}$$

The representation (3.28) for the algebra $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ on the Hilbert space, decomposed as $L^2(\mathcal{M}, \mathcal{S}) = L^2(\mathcal{M}, \mathcal{S})_+ \oplus L^2(\mathcal{M}, \mathcal{S})_-$, is given by

$$\pi_{\mathcal{M}}(f, f') = \begin{pmatrix} f\,\mathbb{I}_2 & 0 \\ 0 & f'\,\mathbb{I}_2 \end{pmatrix}, \tag{4.6}$$

where each of the two copies of $C^\infty(\mathcal{M})$ acts independently and faithfully by the pointwise multiplication on the eigenspaces $L^2(\mathcal{M}, \mathcal{S})_\pm$ of the grading $\gamma^5$.

The automorphism $\rho$ of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ is the flip

$$\rho(f, f') = (f', f), \qquad \forall f, f' \in C^\infty(\mathcal{M}), \tag{4.7}$$

which coincides with the inner automorphism of $\mathcal{B}(\mathcal{H})$ implemented by the following unitary matrix

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}. \tag{4.8}$$

This $\mathcal{R}$ is nothing but the first Dirac matrix $\gamma^0$ and it is compatible with the real structure $\mathcal{J}$, in the sense of (3.6), with

$$\epsilon''' = -1. \tag{4.9}$$

**Lemma 4.1.** *For any* $a = (f, f') \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ *and* $\mu = 0, 1, 2, 3$, *one has*

$$\gamma^\mu a = \rho(a)\gamma^\mu, \qquad \gamma^\mu \rho(a) = a\gamma^\mu, \qquad \gamma^\mu \mathcal{J} = -\epsilon' \mathcal{J}\gamma^\mu. \tag{4.10}$$

**Proof.** The first equation is checked by direct calculation, using the explicit form of $\gamma^\mu$, along with (4.6). Omitting the symbol of representation, by $\pi_\mathcal{M}(\rho(a))$, we mean

$$\rho(a) = \begin{pmatrix} f' \mathbb{I}_2 & 0 \\ 0 & f\, \mathbb{I}_2 \end{pmatrix}. \tag{4.11}$$

The second equation then follows from (3.7). And, finally, the third one is obtained from (1.12), noticing that $\mathcal{J}$ commutes with $\partial_\mu$, having constant components:

$$0 = \mathcal{J}\eth - \epsilon' \eth \mathcal{J} = i\,(\mathcal{J}\gamma^\mu + \epsilon'\gamma^\mu \mathcal{J})\,\partial_\mu. \qquad \blacksquare$$

As an immediate corollary, one checks the boundedness of the twisted commutator, thus

$$\begin{aligned} \left[\eth, a\right]_\rho &= \left(\gamma^\mu \partial_\mu a - \rho(a)\gamma^\mu \partial_\mu\right) \\ &= -i\gamma^\mu \left[\partial_\mu, a\right] \\ &= -i\gamma^\mu(\partial_\mu a), \qquad\qquad \forall a \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2. \end{aligned} \tag{4.12}$$

### 4.1.1 The $X_\mu$ field

Following the standard terminology of the non-twisted case, given a twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})_\rho$, the substitution of $\mathcal{D}$ with $\mathcal{D}_{\omega_\rho}$ is referred to as the **twisted fluctuation**.

The Dirac operator $\eth$ of a four-dimensional manifold $\mathcal{M}$ has non-vanishing self-adjoint twisted fluctuation (3.8) of the form [LM1, Prop. 5.3]:

$$\eth \to \eth_X := \eth + \mathbf{X}, \tag{4.13}$$

where

$$\mathbf{X} := -i\gamma^\mu X_\mu, \qquad \text{with} \quad X_\mu := f_\mu \gamma^5, \tag{4.14}$$

for some $f_\mu \in C^\infty(\mathcal{M}, \mathbb{R})$.

In contrast, the self-adjoint (non-twisted) fluctuations of the Dirac operator $\eth$ are always vanishing, irrespective of the dimension of the manifold $\mathcal{M}$ [C96].

However, in [LM1] one imposes the self-adjointness of $\eth_X$, without necessarily requiring $\omega_\rho$ to be self-adjoint. One might wonder that the non-vanishing of $\mathbf{X}$ is then an artifact of such a choice and that $\mathbf{X}$ might vanish as soon as one also imposes $\omega_\rho = \omega_\rho^\dagger$. The following lemma clarifies this issue.

**Lemma 4.2.** *The twisted one-forms* $\omega_\rho$ (3.9) *and the twisted fluctuations* $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ *of the minimally twisted canonical triple* (4.2) *are of the form*

$$\omega_\rho = \mathbf{W} := -i\gamma^\mu W_\mu, \qquad \textit{with} \quad W_\mu = \mathrm{diag}\big(h_\mu \mathbb{I}_2,\, h'_\mu \mathbb{I}_2\big), \tag{4.15}$$

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \mathbf{X} := -i\gamma^\mu X_\mu, \qquad \textit{with} \quad X_\mu = \mathrm{diag}\big(f_\mu \mathbb{I}_2,\, f'_\mu \mathbb{I}_2\big), \tag{4.16}$$

*for some* $h_\mu, h'_\mu \in C^\infty(\mathcal{M})$ *with* $f_\mu := 2\,\mathrm{Re}\{h_\mu\}$ *and* $f'_\mu := 2\,\mathrm{Re}\{h'_\mu\}$. *They are self-adjoint, respectively, iff*

$$h'_\mu = -\bar{h}_\mu, \qquad \textit{and} \qquad f'_\mu = -f_\mu. \tag{4.17}$$

**Proof.** For some $a_i := (f_i, f'_i) \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, using (4.10) along with $[\nabla^{\mathcal{S}}_\mu, f_i] = (\partial_\mu f_i)$, one gets

$$\begin{aligned} \big[\eth, a_i\big]_\rho &= -i\big(\gamma^\mu \nabla^{\mathcal{S}}_\mu a_i - \rho(a_i)\gamma^\mu \nabla^{\mathcal{S}}_\mu\big) \\ &= -i\big(\gamma^\mu \nabla^{\mathcal{S}}_\mu a_i - \gamma^\mu a_i \nabla^{\mathcal{S}}_\mu\big) \\ &= -i\gamma^\mu \big[\nabla^{\mathcal{S}}_\mu, a_i\big] \qquad (4.18) \\ &= -i\gamma^\mu (\partial_\mu a_i) \\ &= -i\gamma^\mu \begin{pmatrix} (\partial_\mu f_i)\mathbb{I}_2 & 0 \\ 0 & (\partial_\mu f'_i)\mathbb{I}_2 \end{pmatrix}. \qquad (4.19) \end{aligned}$$

Then, with some $b_i := (g_i, g'_i) \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, one has

$$\omega_\rho = \sum_i b_i [\eth, a_i]_\rho = -i\gamma^\mu \sum_i \rho(b_i)(\partial_\mu a_i) = -i\gamma^\mu W_\mu, \tag{4.20}$$

where $W_\mu$ is defined in (4.15), with

$$h_\mu := \sum_i g'_i(\partial_\mu f_i) \qquad \text{and} \qquad h'_\mu := \sum_i g_i(\partial_\mu f'_i). \tag{4.21}$$

The adjoint is

$$\omega_\rho^\dagger = (-i\gamma^\mu W_\mu)^\dagger = iW_\mu^\dagger \gamma^\mu = i\gamma^\mu \rho(W_\mu^\dagger), \tag{4.22}$$

where the last equality follows from (4.10) applied to $W_\mu$ viewed as an element of the algebra $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$. Thus, $\omega_\rho$ is self-adjoint iff $\gamma^\mu \rho(W_\mu^\dagger) = -\gamma^\mu W_\mu$, that is, going back to the explicit form of $\gamma^\mu$, equivalent to

$$\sigma^\mu \bar{h}_\mu = -\sigma^\mu h'_\mu, \qquad \text{and} \qquad \tilde{\sigma}^\mu \overline{h'}_\mu = -\tilde{\sigma}^\mu h_\mu. \tag{4.23}$$

Multiplying the first equation by $\sigma^\lambda$ and using $\mathrm{Tr}(\sigma^\lambda \sigma^\mu) = 2\delta^{\mu\lambda}$, (4.23) then implies that $\bar{h}_\mu = -h'_\mu$, from where (4.17) follows. Hence, $\omega_\rho = \omega_\rho^\dagger$ is equivalent to the first equation of (4.17). Further, we have

$$\begin{aligned}\mathcal{J}\omega_\rho\mathcal{J}^{-1} &= \mathcal{J}(-i\gamma^\mu W_\mu)\mathcal{J}^{-1} = i\mathcal{J}(\gamma^\mu W_\mu)\mathcal{J}^{-1} \\ &= -i\gamma^\mu \mathcal{J} W_\mu \mathcal{J}^{-1} = -i\gamma^\mu W_\mu^\dagger,\end{aligned}$$

using $\mathcal{J}\gamma^\mu = -\gamma^\mu\mathcal{J}$ – from (4.1) and (4.10) – along with $\mathcal{J}W^\mu = W_\mu^\dagger\mathcal{J}$ – from (4.4) and the explicit form (4.15) of $W_\mu$. Therefore,

$$\omega_\rho + J\omega_\rho J^{-1} = -i\gamma^\mu(W_\mu + W_\mu^\dagger), \tag{4.24}$$

which is nothing but (4.16), identifying

$$\begin{aligned}X_\mu &:= W_\mu + W_\mu^\dagger \\ &= \mathsf{diag}\big((h_\mu + \bar{h}_\mu)\mathbb{I}_2, (h'_\mu + \overline{h'}_\mu)\mathbb{I}_2\big).\end{aligned} \tag{4.25}$$

One checks, in a similar way as above, that $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ is self-adjoint iff the second eq. of (4.17) holds. ■

Imposing that $\omega_\rho \neq 0$ be self-adjoint, that is imposing (4.17) with $h_\mu \neq 0$, does not imply that $X_\mu$ vanishes. It does vanish, if $h_\mu$ is purely imaginary, for then $h_\mu + \bar{h}_\mu = 0$ and (4.17) imposes that $h'_\mu$ is also purely imaginary, consequently, the sum $(h'_\mu + \overline{h'}_\mu)$ also vanishes, hence $X_\mu = 0$. However, $h_\mu$ is not necessarily purely imaginary, in which case the self-adjointness of $\omega_\rho$ does not forbid a non-zero twisted fluctuation.

### 4.1.2 Gauge transformation

For a minimally twisted manifold, not only is the fermionic action (3.22) invariant under a gauge transformation (2.76, 3.13), but also the twist-fluctuated Dirac operator $\mathcal{D}_{\omega_\rho}$ (in dim. 0 and 4) [LM2, Prop. 5.4]. It is interesting to check this explicitly by studying how the field $h_\mu$ parametrizing the twisted one-form $\omega_\rho$ in (4.15) transforms. This will also be useful later in the example of electrodynamics.

A unitary $u$ of the algebra $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ is of the form $(e^{i\theta}, e^{i\theta'})$ with $\theta, \theta' \in C^\infty(\mathcal{M}, \mathbb{R})$. It (and its twist) acts on $\mathcal{H}$ according to (4.6) as (omitting the symbol of representation):

$$\begin{aligned}u &= \begin{pmatrix} e^{i\theta}\mathbb{I}_2 & 0 \\ 0 & e^{i\theta'}\mathbb{I}_2 \end{pmatrix}, \\ \rho(u) &= \begin{pmatrix} e^{i\theta'}\mathbb{I}_2 & 0 \\ 0 & e^{i\theta}\mathbb{I}_2 \end{pmatrix}.\end{aligned} \tag{4.26}$$

**Proposition 4.3.** *Under a gauge transformation with unitary $u \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, the fields $h_\mu$ and $h'_\mu$ parametrizing the twisted one-form $\omega_\rho$ in (4.15) transform as*

$$h_\mu \to h_\mu - i\partial_\mu\theta, \qquad h'_\mu \to h'_\mu + i\partial_\mu\theta'. \tag{4.27}$$

**Proof.** Under a gauge transformation, a twisted one-form $\omega_\rho \in \Omega^1_\mathcal{D}(\mathcal{A},\rho)$ transforms as [LM2, Prop. 4.2]

$$\omega_\rho \to \omega^u_\rho := \rho(u)\big([\mathcal{D}, u^*]_\rho + \omega_\rho u^*\big). \tag{4.28}$$

For $\mathcal{D} = \eth = -i\gamma^\mu\partial_\mu$ and $\omega_\rho = -i\gamma^\mu W_\mu$, we have

$$\begin{aligned}\omega^u_\rho &= -i\rho(u)\big([\gamma^\mu\partial_\mu, u^*]_\rho + \gamma^\mu W_\mu u^*\big)\\ &= -i\rho(u)\gamma^\mu(\partial_\mu + W_\mu)u^*\\ &= -i\gamma^\mu(u\partial_\mu u^* + W_\mu),\end{aligned}$$

where we have used (4.19) for $a_i = u^*$, namely

$$\left[\gamma^\mu\partial_\mu, u^*\right]_\rho = \gamma^\mu(\partial_\mu u^*), \tag{4.29}$$

as well as (4.10) for $a = u$, together with $uW_\mu u^* = W_\mu$, since $u$ commutes with $W_\mu$. Therefore, $W_\mu \to W_\mu + u\partial_\mu u^*$, which with the explicit representation (4.15) of $W_\mu$ and (4.26) of $u$, respectively, reads

$$\begin{pmatrix} h_\mu\mathbb{I}_2 & 0\\ 0 & h'_\mu\mathbb{I}_2\end{pmatrix} \longrightarrow \begin{pmatrix}(h_\mu - i\partial_\mu\theta)\mathbb{I}_2 & 0\\ 0 & (h'_\mu + i\partial_\mu\theta')\mathbb{I}_2\end{pmatrix}, \qquad \blacksquare$$

Although $h_\mu$ and $h'_\mu$ transform in a nontrivial manner, their real parts remain invariant, as we have

$$h_\mu + \bar{h}_\mu \quad\longrightarrow\quad h_\mu - i\partial_\mu\theta + \bar{h}_\mu + i\partial_\mu\theta \;=\; h_\mu + \bar{h}_\mu,$$

and similarly for $h'_\mu$. Since it is the real parts that enter in the Def. (4.15) of $\mathbf{X}$, this explains why the latter is invariant under a gauge transformation.

Notice that this is true whether $\mathbf{X}$ is self-adjoint or not. In case $\omega_\rho$ is not self-adjoint, the imaginary part

$$g_\mu := \text{Im}\{h_\mu\} = \frac{1}{2i}(h_\mu - \bar{h}_\mu)$$

of $h_\mu$ is not invariant under a gauge transformation, but transforms as

$$g_\mu \to g_\mu - \partial_\mu\theta.$$

We return to this point while discussing the gauge transformations for the example of electrodynamics, where a similar phenomenon occurs in (4.92)–(4.93).

### 4.1.3 Fermionic action with no spinor freedom

First we work out how the the positive eigenspace $\mathcal{H}_{\mathcal{R}}$ (3.21) of the unitary matrix $\mathcal{R} = \gamma^0$, as in (4.8), looks like.

**Lemma 4.4.** *An eigenvector $\phi \in \mathcal{H}_{\mathcal{R}}$ is of the form $\phi := \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$, where $\varphi$ is a Weyl spinor.*

**Proof.** The $+1$-eigenspace of $\mathcal{R} = \gamma^0$ is spanned by

$$\upsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \upsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, a generic vector $\phi \in \mathcal{H}_{\mathcal{R}}$ is given by

$$\phi = \phi_1 \upsilon_1 + \phi_2 \upsilon_2 =: \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}, \qquad \text{with} \quad \varphi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \qquad \blacksquare$$

Now, we compute the fermionic action (3.22) of the minimally twisted manifold (1.13).

**Proposition 4.5.** *Let $\eth_X$ be the twist-fluctuated Dirac operator (4.13). The symmetric form (3.20) is*

$$\mathfrak{A}^{\rho}_{\eth_X}(\phi, \xi) = 2 \int_{\mathcal{M}} d\mu \left[ \bar{\varphi}^{\dagger} \sigma_2 \left( i f_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \zeta \right], \tag{4.30}$$

*where $\varphi$, $\zeta$ are, respectively, the Weyl components of the Dirac spinors $\phi, \xi \in \mathcal{H}_{\mathcal{R}}$, and $f_0$ is the zeroth component of $f_\mu$ in (4.14).*

**Proof.** One has the following relations:

$$\mathcal{J}\phi = i\gamma^0\gamma^2 cc \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} = i \begin{pmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} \bar{\varphi} \\ \bar{\varphi} \end{pmatrix} = i \begin{pmatrix} \tilde{\sigma}^2\, \bar{\varphi} \\ \sigma^2\, \bar{\varphi} \end{pmatrix}, \tag{4.31}$$

$$\eth\xi = -i\gamma^\mu \partial_\mu \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \partial_\mu \zeta \\ \partial_\mu \zeta \end{pmatrix} = -i \begin{pmatrix} \sigma^\mu \partial_\mu \zeta \\ \tilde{\sigma}^\mu \partial_\mu \zeta \end{pmatrix}, \tag{4.32}$$

$$\mathbf{X}\xi = -i\gamma^\mu X_\mu \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} f_\mu \mathbb{I}_2 & 0 \\ 0 & -f_\mu \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = -i \begin{pmatrix} -f_\mu \sigma^\mu \zeta \\ f_\mu \tilde{\sigma}^\mu \zeta \end{pmatrix}. \tag{4.33}$$

Hence, noticing that $(\tilde{\sigma}^2)^\dagger = \sigma^2$ and ${\sigma^2}^\dagger = -\sigma^2$ (see App. D), and using

$$\begin{aligned} \sigma^\mu + \tilde{\sigma}^\mu &= 2\mathbb{I}_2 \delta^{\mu 0}, \\ \sigma^\mu - \tilde{\sigma}^\mu &= -2i\delta^{\mu j}\sigma^j, \end{aligned} \tag{4.34}$$

one gets

$$\begin{aligned}
\mathfrak{A}_{\eth}(\phi,\xi) = \langle \mathcal{J}\phi, \eth\xi\rangle &= -\begin{pmatrix}\bar{\varphi}^\dagger\tilde{\sigma}^{2\dagger} & \bar{\varphi}^\dagger\sigma^{2\dagger}\end{pmatrix}\begin{pmatrix}\sigma^\mu\partial_\mu\zeta \\ \tilde{\sigma}^\mu\partial_\mu\zeta\end{pmatrix} \\
&= -\int_{\mathcal{M}} d\mu\left(\bar{\varphi}^\dagger\sigma^2\sigma^\mu\partial_\mu\zeta - \bar{\varphi}^\dagger\sigma^2\tilde{\sigma}^\mu\partial_\mu\zeta\right) \\
&= -\int_{\mathcal{M}} d\mu\left(\bar{\varphi}^\dagger\sigma^2(\sigma^\mu - \tilde{\sigma}^\mu)\partial_\mu\zeta\right) \\
&= 2\int_{\mathcal{M}} d\mu\left(\bar{\varphi}^\dagger\sigma_2 \textstyle\sum_{j=1}^3 \sigma_j\partial_j\zeta\right); \qquad (4.35)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A}_{\mathbf{X}}(\phi,\xi) = \langle \mathcal{J}\phi, \mathbf{X}\xi\rangle &= -\begin{pmatrix}\bar{\varphi}^\dagger\tilde{\sigma}^{2\dagger} & \bar{\varphi}^\dagger\sigma^{2\dagger}\end{pmatrix}\begin{pmatrix}-f_\mu\sigma^\mu\zeta \\ f_\mu\tilde{\sigma}^\mu\zeta\end{pmatrix} \\
&= \int_{\mathcal{M}} d\mu\left(\bar{\varphi}^\dagger\sigma^2 f_\mu\sigma^\mu\zeta + \bar{\varphi}^\dagger\sigma^2 f_\mu\tilde{\sigma}^\mu\zeta\right) \\
&= \int_{\mathcal{M}} d\mu\left(\bar{\varphi}^\dagger\sigma^2 f_\mu(\sigma^\mu + \tilde{\sigma}^\mu)\partial_\mu\zeta\right) \\
&= -2i\int_{\mathcal{M}} d\mu\left(f_0\bar{\varphi}^\dagger\sigma_2\zeta\right); \qquad (4.36)
\end{aligned}$$

where in (4.35) and (4.36) we have used $\sigma^2 = -i\sigma_2$. The result then follows from Lem. 3.6 and (4.9), as

$$\begin{aligned}
\mathfrak{A}^{\rho}_{\eth_X}(\phi,\xi) &= -\mathfrak{A}_{\eth_X}(\phi,\xi) \\
&= -\mathfrak{A}_{\eth}(\phi,\xi) - \mathfrak{A}_X(\phi,\xi). \qquad \blacksquare
\end{aligned}$$

The fermionic action is then obtained by substituting $\phi = \xi$ in (4.30) and replacing the components $\zeta$ of $\xi$ by the associated Graßmann variable $\tilde{\zeta}$, as follows

$$S^f_\rho(\mathcal{D}_{\omega_\rho}) = 2\int_{\mathcal{M}} d\mu\left[\tilde{\bar{\zeta}}^\dagger\sigma_2\left(if_0\mathbb{I}_2 - \textstyle\sum_{j=1}^3\sigma_j\partial_j\right)\tilde{\zeta}\right]. \qquad (4.37)$$

The most interesting observation regarding this action (4.37) is the disappearance of the derivative in the $x_0$ direction, and the appearance of the zeroth-component of the real field $f_\mu$ parametrizing the twisted fluctuation $\mathbf{X}$, instead. This derivative, however, can be restored by interpreting $-if_0\zeta$ as $\partial_0\zeta$, that is, assuming

$$\zeta(x_0, x_i) = \exp(-if_0x_0)\,\zeta(x_i). \qquad (4.38)$$

Denoting by $\sigma^\mu_M = \{\mathbb{I}_2, \sigma_j\}$ the upper-right components of the minkowskian Dirac matrices (see (D.4)), the integrand in the fermionic action $S^f_\rho$ (4.37) then reads (with summation on the index $\mu$)

$$-\bar{\tilde{\zeta}}^{\dagger}\sigma_M^2\left(\sigma_M^{\mu}\partial_{\mu}\right)\tilde{\zeta}, \tag{4.39}$$

which reminds us of the Weyl lagrangian densities (D.10):

$$S^F = i\Psi_r^{\dagger}\left(\sigma_M^{\mu}\partial_{\mu}\right)\Psi_r, \tag{4.40}$$

but with an extra $\sigma_M^2$ matrix factor that prevents us from simultaneously identifying $\tilde{\zeta}$ with $\Psi_r$ and $-\bar{\tilde{\zeta}}^{\dagger}\sigma_M^2$ with $i\Psi_r^{\dagger}$.

To make such an identification possible, one needs more spinorial degrees of freedom, which can be obtained by considering a tensor product of the manifold $\mathcal{M}$ with a two-point space $\mathcal{F}_2$.

## 4.2 Minimal twist of a $U(1)$ gauge theory

Following the minimal twist by grading procedure of §3.2, the minimal twist of the spectral triple of a $U(1)$ gauge theory (§2.4.1) is given by the algebra $\mathcal{A}\otimes\mathbb{C}^2$, where $\mathcal{A} := C^{\infty}(\mathcal{M})\otimes\mathbb{C}^2$, represented on the Hilbert space $\mathcal{H} := L^2(\mathcal{M},\mathcal{S})\otimes\mathbb{C}^2$ as

$$\pi(a,a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g'\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_2 \end{pmatrix} =: \begin{pmatrix} F & 0 \\ 0 & G' \end{pmatrix}, \tag{4.41}$$

for $a := (f,g),\ a' := (f',g') \in \mathcal{A}$; along with its twist represented as

$$\pi(\rho(a,a')) = \pi(a',a) = \begin{pmatrix} f'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix} =: \begin{pmatrix} F' & 0 \\ 0 & G \end{pmatrix}. \tag{4.42}$$

In both of the equations above, we have denoted

$$\begin{aligned} F &:= \pi_{\mathcal{M}}(f,f'), \qquad & F' &:= \pi_{\mathcal{M}}(f',f), \\ G &:= \pi_{\mathcal{M}}(g,g'), \qquad & G' &:= \pi_{\mathcal{M}}(g',g), \end{aligned} \tag{4.43}$$

where $\pi_{\mathcal{M}}$ is the representation (4.6) of $C^{\infty}(\mathcal{M})\otimes\mathbb{C}^2$ on $L^2(\mathcal{M},\mathcal{S})$.

### 4.2.1 Twisted fluctuation

Following the notation of (4.14), given $Z_\mu = \pi_\mathcal{M}(f_\mu, f'_\mu)$ and $Z'_\mu = \pi_\mathcal{M}(f'_\mu, f_\mu)$ for some $f_\mu, f'_\mu \in C^\infty(\mathcal{M})$, we denote the following

$$\boldsymbol{Z} := -i\gamma^\mu Z_\mu, \qquad \boldsymbol{Z}' := -i\gamma^\mu Z'_\mu, \qquad \overline{\boldsymbol{Z}} := -i\gamma^\mu \overline{Z}_\mu. \tag{4.44}$$

Notice that $\overline{\boldsymbol{Z}}$ *is not* the complex conjugate of $\boldsymbol{Z}$, since in (4.44) the complex conjugation does not act on the Dirac matrices. This guarantees that $^-$ and $'$ commute not only for $Z_\mu$ but also for $\boldsymbol{Z}$, that is,

$$\overline{Z'_\mu} = (\overline{Z}_\mu)', \qquad \text{and} \qquad (\overline{\boldsymbol{Z}})' = \overline{\boldsymbol{Z}'}. \tag{4.45}$$

Thus, the notation $\overline{\boldsymbol{Z}}'$ is then unambiguous and denotes indistinctly both the members of the second eq. in (4.45).

**Lemma 4.6.** *For any* $F$, $G$, $Z_\mu$, *as in (4.43, 4.44), one has*

$$F[\eth, G]_\rho = -i\gamma^\mu F'\partial_\mu G, \qquad \mathcal{J}\boldsymbol{Z}\mathcal{J}^{-1} = \overline{\boldsymbol{Z}}, \qquad \boldsymbol{Z}^\dagger = -\overline{\boldsymbol{Z}}'. \tag{4.46}$$

**Proof.** Eq. (4.10) for $a = F'$ gives $F\gamma^\mu = \gamma^\mu F'$, whereas eq. (4.19) for $a = G$ yields $[\eth, G]_\rho - i\gamma^\mu\partial_\mu G$. Thus, we have the first of (4.46):

$$F[\eth, G]_\rho = -iF\gamma^\mu\partial_\mu G = -i\gamma^\mu F'\partial_\mu G.$$

The second relation in (4.46) follows from

$$\mathcal{J}\boldsymbol{Z}\mathcal{J}^{-1} = i\mathcal{J}\gamma^\mu Z_\mu\mathcal{J}^{-1} = -i\gamma^\mu\mathcal{J}Z_\mu J^{-1} = -i\gamma^\mu\overline{Z}_\mu \quad = \quad \overline{\boldsymbol{Z}}, \tag{4.47}$$

where we used (4.10), as well as (recalling that in KO-dime. 4, one has $\mathcal{J}^{-1} = -\mathcal{J}$)

$$\begin{aligned}
\mathcal{J}Z_\mu\mathcal{J}^{-1} &= -i\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix} cc \begin{pmatrix} f_\mu\,\mathbb{I}_2 & 0\\ 0 & f'_\mu\,\mathbb{I}_2\end{pmatrix} i\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix} cc,\\
&= -\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\overline{f}_\mu\,\mathbb{I}_2 & 0\\ 0 & \overline{f'}_\mu\,\mathbb{I}_2\end{pmatrix}\begin{pmatrix}\bar{\tilde{\sigma}}^2 & 0\\ 0 & \bar{\sigma}^2\end{pmatrix}\\
&= \begin{pmatrix}\overline{f}_\mu\,\mathbb{I}_2 & 0\\ 0 & \overline{f'}_\mu\,\mathbb{I}_2\end{pmatrix} \quad = \quad \overline{Z}_\mu,
\end{aligned}$$

noticing that $\bar{\tilde{\sigma}}^2 = \tilde{\sigma}^2$ and $\bar{\sigma}^2 = \sigma^2$, so $\tilde{\sigma}^2\bar{\tilde{\sigma}}^2 = \sigma^2\bar{\sigma}^2 = -\mathbb{I}_2$. Finally, the third eq. of (4.46) follows from

$$\boldsymbol{Z}^\dagger = iZ_\mu^\dagger\gamma^\mu = i\overline{Z}_\mu\gamma^\mu = i\gamma^\mu(\overline{Z}_\mu)' = i\gamma^\mu\overline{Z'_\mu} = -\overline{\boldsymbol{Z}}', \tag{4.48}$$

where we notice that $Z_\mu^\dagger = \overline{Z}_\mu$, from the explicit form (4.6) of $\pi_\mathcal{M}$. ■

**Proposition 4.7.** *For* $a = (f, g)$, $a' = (f', g')$, $b = (v, w)$, $b' = (v', w')$ *in* $\mathcal{A}$, *let*

$$\omega_\rho := \pi(a, a')\left[\eth \otimes \mathbb{I}_2,\ \pi(b, b')\right]_\rho$$

*be a twisted one-form. Then,*

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \mathbf{X} \otimes \mathbb{I}_2 + i\mathbf{Y} \otimes \gamma_{\mathcal{F}}, \tag{4.49}$$

*where* $\mathbf{X} = -i\gamma^\mu X_\mu$ *and* $\mathbf{Y} = -i\gamma^\mu Y_\mu$ *with*

$$X_\mu = \pi_{\mathcal{M}}(f_\mu, f'_\mu), \qquad Y_\mu = \pi_{\mathcal{M}}(g_\mu, g'_\mu), \tag{4.50}$$

*where* $f_\mu, f'_\mu$ *and* $g_\mu, g'_\mu$ *denote, respectively, the real and the imaginary parts of*

$$z_\mu := f'\partial_\mu v + \bar{g}\partial_\mu \bar{w}', \qquad \text{and} \qquad z'_\mu = f\partial_\mu v' + \bar{g}'\partial_\mu \bar{w}'. \tag{4.51}$$

**Proof.** We first set the following notation:

$$\begin{aligned} V &:= \pi_{\mathcal{M}}(v, v'), \qquad V' := \pi_{\mathcal{M}}(v', v), \\ W &:= \pi_{\mathcal{M}}(w, w'), \qquad W' := \pi_{\mathcal{M}}(w', w). \end{aligned} \tag{4.52}$$

From (4.41, 4.42), we have

$$\left[\eth \otimes \mathbb{I}_2,\ \pi(b, b')\right]_\rho = \begin{pmatrix} [\eth, V]_\rho & 0 \\ 0 & [\eth, W']_\rho \end{pmatrix}, \tag{4.53}$$

so that, for $(a, a')$ as in (4.41), and using (4.46) we get

$$\begin{aligned} \omega_\rho &:= \begin{pmatrix} F & 0 \\ 0 & G' \end{pmatrix}\begin{pmatrix} [\eth, V]_\rho & 0 \\ 0 & [\eth, W']_\rho \end{pmatrix} \\ &= \begin{pmatrix} -i\gamma^\mu P_\mu & 0 \\ 0 & -i\gamma^\mu Q'_\mu \end{pmatrix} = \begin{pmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q}' \end{pmatrix}, \end{aligned} \tag{4.54}$$

with

$$P_\mu := F'\partial_\mu V, \qquad \text{and} \qquad Q'_\mu := G\partial_\mu W'. \tag{4.55}$$

The explicit form of the real structure $J$ and its inverse $J^{-1}$, that is,

$$J = \mathcal{J} \otimes J_F = \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}, \qquad J^{-1} = \begin{pmatrix} 0 & \mathcal{J}^{-1} \\ \mathcal{J}^{-1} & 0 \end{pmatrix}, \tag{4.56}$$

along with the second relation of (4.46), yield

$$J\omega_\rho J^{-1} = \begin{pmatrix} \mathcal{J}\mathbf{Q}'\mathcal{J}^{-1} & 0 \\ 0 & \mathcal{J}\mathbf{P}'\mathcal{J}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{Q}}' & 0 \\ 0 & \overline{\mathbf{P}} \end{pmatrix}. \tag{4.57}$$

Summing up (4.54) and (4.57), one obtains (4.61)

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \begin{pmatrix} \mathbf{Z} & 0 \\ 0 & \overline{\mathbf{Z}} \end{pmatrix}, \tag{4.58}$$

with $\mathbf{Z} := \mathbf{P} + \overline{\mathbf{Q}}' = -i\gamma^\mu Z_\mu$ and

$$\begin{aligned} Z_\mu &= P_\mu + \overline{Q}'_\mu = F'\partial_\mu V + \overline{G}\partial_\mu \overline{W}' \\ &= \begin{pmatrix} (f'\partial_\mu v + \bar{g}\partial_\mu \bar{w}')\mathbb{I}_2 & 0 \\ 0 & (f\partial_\mu v' + \bar{g}'\partial_\mu \bar{w})\mathbb{I}_2 \end{pmatrix}, \end{aligned} \tag{4.59}$$

where the last equality follows from the explicit form (4.52) of $V, W'$ and (4.43) of $F', G$, respectively. Then, with (4.51), this gives

$$\begin{aligned} Z_\mu &= \pi_{\mathcal{M}}(z_\mu, z'_\mu) \\ &= \pi_{\mathcal{M}}(f_\mu, f'_\mu) + i\pi_{\mathcal{M}}(g_\mu, g'_\mu) = X_\mu + iY_\mu. \end{aligned} \tag{4.60}$$

Similarly, $\overline{\mathbf{Z}} = -i\gamma^\mu \overline{Z}_\mu$ with $\overline{Z}_\mu = X_\mu - iY_\mu$. Hence, (4.58) becomes

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \begin{pmatrix} -i\gamma^\mu(X_\mu + iY_\mu) & 0 \\ 0 & -i\gamma^\mu(X_\mu - iY_\mu) \end{pmatrix}, \tag{4.61}$$

which is nothing but (4.49). ■

**Proposition 4.8.** *The self-adjoint twisted fluctuations of the Dirac operator for the* $U(1)$ *gauge theory (§2.4.1) are parametrized by two real fields* $f_\mu, g_\mu \in C^\infty(\mathcal{M}, \mathbb{R})$*, and are of the form:*

$$\eth_X \otimes \mathbb{I}_2 \ + \ g_\mu\gamma^\mu \otimes \gamma_{\mathcal{F}}, \tag{4.62}$$

*where* $\eth_X$ *is the twisted-covariant Dirac operator (4.13) of the manifold* $\mathcal{M}$.

**Proof.** A generic twisted fluctuation (4.58)[1] is self-adjoint iff $\mathbf{Z} = \mathbf{Z}^\dagger$ and $\overline{\mathbf{Z}} = \overline{\mathbf{Z}}^\dagger$. By (4.45), and the third eq. of (4.46), both conditions are equivalent to $\mathbf{Z} = -\overline{\mathbf{Z}}'$, that is,

$$-i\gamma^\mu\left(Z_\mu + \overline{Z}'_\mu\right) = 0.$$

As discussed in the argument following eq. (4.23), this is equivalent to $Z_\mu = -\overline{Z}'_\mu$, which using the explicit form (4.59) boils down to $z_\mu = -\bar{z}'_\mu$, that is,

$$f_\mu = -f'_\mu, \qquad \text{and} \qquad g_\mu = g'_\mu. \tag{4.63}$$

[1]Technically, one should add a summation index $i$ and redefine it as $\mathbf{Z} := \sum_i \mathbf{Z}_i$.

Then, in (4.50) one has the following explicit forms:

$$\begin{aligned} X_\mu &= \pi_{\mathcal{M}}(f_\mu, -f_\mu) = f_\mu \gamma^5, \\ Y_\mu &= \pi_{\mathcal{M}}(g_\mu, g_\mu) = g_\mu \mathbb{I}_4, \end{aligned} \tag{4.64}$$

so that (4.49) reads

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = -i\gamma^\mu f_\mu \gamma^5 \otimes \mathbb{I}_2 + g_\mu \gamma^\mu \otimes \gamma_{\mathcal{F}}. \tag{4.65}$$

Hence, the result follows by adding $\eth \otimes \mathbb{I}_2$ to (4.65). ■

Prop. 4.8 shows that the self-adjointness can directly be read into the bold notation. Meaning, (4.63) shows that $\mathbf{Z} = \mathbf{X} \otimes \mathbb{I}_2 + i\mathbf{Y} \otimes \gamma_{\mathcal{F}}$ is self-adjoint iff $\mathbf{X}' = -\overline{\mathbf{X}}$ and $\mathbf{Y}' = \overline{\mathbf{Y}}$, that is, from the third eq. in (4.46), iff $\mathbf{X} = \mathbf{X}^\dagger$ and $\mathbf{Y} = -\mathbf{Y}^\dagger$.

### 4.2.2 Weyl action in Lorentz signature

Here we show that the fermionic action associated to the twisted-covariant operator $\eth_X \otimes \mathbb{I}_2$ (assuming that $g_\mu = 0$) yields the Weyl equations in the lorentzian signature.

For the $U(1)$ gauge theory, the unitary operator implementing the action of the twist $\rho$ on the Hilbert space $\mathcal{H}$ is given by the matrix $\mathcal{R} = \gamma^0 \otimes \mathbb{I}_2$, which has eigenvalues $\pm 1$ and is compatible with the real structure $J$, in the sense of (3.6), with the sign $\epsilon''' = -1$.

Similar to Lem. 4.4, a generic element $\eta$ in the $+1$-eigenspace $\mathcal{H}_{\mathcal{R}}$ of $\mathcal{R}$ is written as

$$\eta = \xi \otimes e + \phi \otimes \bar{e}, \qquad \text{with} \quad \xi := \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}, \quad \phi := \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}, \tag{4.66}$$

where $\xi, \phi \in L^2(\mathcal{M}, \mathcal{S})$ are Dirac spinors with corresponding Weyl components $\zeta, \varphi$.

**Proposition 4.9.** *Let $\eta, \eta' \in \mathcal{H}_{\mathcal{R}}$, with $\zeta', \varphi'$ being the Weyl components of the Dirac spinors $\xi', \phi'$ – as in the decomposition (4.66) of $\eta'$. Then,*

$$\mathfrak{A}^{\rho}_{\eth_X \otimes \mathbb{I}_2}(\eta, \eta') = 2\int_{\mathcal{M}} d\mu \left[ \bar{\zeta}^\dagger \sigma_2 \left( i f_0 \mathbb{I}_2 - \sum_j \sigma_j \partial_j \right) \varphi' + \bar{\varphi}^\dagger \sigma_2 \left( i f_0 \mathbb{I}_2 - \sum_j \sigma_j \partial_j \right) \zeta' \right].$$

**Proof.** For $\eta \in \mathcal{H}_{\mathcal{R}}$ (4.66), recalling that $J_{\mathcal{F}} e = \bar{e}$ and $J_{\mathcal{F}} \bar{e} = e$, one has

$$\begin{aligned} J\eta &= \mathcal{J}\xi \otimes \bar{e} + \mathcal{J}\phi \otimes e, \\ (\eth_X \otimes \mathbb{I}_2)\eta' &= \eth_X \xi' \otimes e + \eth_X \phi' \otimes \bar{e}. \end{aligned}$$

Then, using Lem. 3.6 with $\epsilon''' = -1$ yields

$$\begin{aligned}\mathfrak{A}^{\rho}_{\eth_X \otimes \mathbb{I}_2}(\eta, \eta') &= -\langle J\eta, (\eth_X \otimes \mathbb{I}_2)\eta' \rangle \\ &= -\langle \mathcal{J}\xi, \eth_X \phi' \rangle - \langle \mathcal{J}\phi, \eth_X \xi' \rangle \\ &= -\mathfrak{A}_{\eth_X}(\xi, \phi') - \mathfrak{A}_{\eth_X}(\phi, \xi') \\ &= \mathfrak{A}^{\rho}_{\eth_X}(\xi, \phi') + \mathfrak{A}^{\rho}_{\eth_X}(\phi, \xi'),\end{aligned}$$

where the inner product in the first line is on $\mathcal{H}$, the ones in the second line are on $L^2(\mathcal{M}, S)$, and the second equality is due to (4.1.3). Thus, the result then follows from Prop. 4.5. ■

The twisted fermionic action $S^f_\rho$ is then obtained substituting $\eta' = \eta$ in the Prop. 4.9 and promoting $\zeta$ and $\varphi$ to their corresponding Graßmann variables $\tilde{\zeta}$ and $\tilde{\varphi}$, respectively. The bilinear form $\mathfrak{A}^{\rho}_{\eth_X}$ then becomes symmetric when evaluated on these Graßmann variables – as in the proof of [DS, Prop. 4.3]. Hence,

$$S^f_\rho(\eth_X \otimes \mathbb{I}_2) = 2\, \mathfrak{A}^{\rho}_{\eth_X}(\tilde{\phi}, \tilde{\xi}) = 4 \int_{\mathcal{M}} d\mu \left[ \bar{\tilde{\varphi}}^\dagger \sigma_2 \left( if_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \tilde{\zeta} \right]. \tag{4.67}$$

**Proposition 4.10.** *Identifying the physical Weyl spinors $\Psi_l, \Psi_r$ as*

$$\Psi_l := \tilde{\zeta}, \quad \Psi_l^\dagger := -i\bar{\tilde{\varphi}}^\dagger \sigma_2 \quad \text{or} \quad \Psi_r := \tilde{\zeta}, \quad \Psi_r^\dagger := i\bar{\tilde{\varphi}}^\dagger \sigma_2, \tag{4.68}$$

*the lagrangian*

$$\mathcal{L}^f_\rho := \bar{\tilde{\varphi}}^\dagger \sigma_2 \left( if_0 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \tilde{\zeta}$$

*in the fermionic action (4.67) describes, for a non-zero constant function $f_0$, a plane wave solution of the Weyl equation, with $x^0$ being the time coordinate.*

**Proof.** With the first identification in (4.68), $\mathcal{L}^f_\rho$ coincides with the Weyl lagrangian $\mathcal{L}^l_M$ (D.10), as soon as one imposes

$$\partial_0 \Psi_l = if_0 \Psi_l, \quad \text{i.e.} \quad \Psi_l(x_0, x_j) = \Psi_l(x_j) e^{if_0 x_0},$$

which is the plane-wave solution (D.11) with $f_0, x_0$ being identified with $E, t$ and $\Psi_l(x_j) = \Psi_0 e^{-ip_j x^j}$. The second identification in (4.68) yields the other Weyl lagrangian $\mathcal{L}^r_M$, imposing

$$\partial_0 \Psi_r = -if_0 \Psi_r, \quad \text{i.e.} \quad \Psi_r(x_0, x_i) = \Psi_r(x_i) e^{-if_0 x_0},$$

which is again the plane wave solution (D.11), where one identifies $f_0$ with $-E$ and $\Psi_r(x_j) = \Psi_0 e^{-ip_j x^j}$. ■

The above Prop. 4.10 adds weight to the observation made at the end of the previous section, right after (4.37). That is, without fluctuation, the fermionic action $S^f_\rho(\eth \otimes \mathbb{I}_2)$ of a minimally twisted $U(1)$ gauge theory yields the spatial part of the Weyl lagrangian, which is nothing but the lagrangian in (4.67) with $f_0 = 0$. For a non-zero constant $f_0$, the twisted fluctuation not only brings back a fourth component, but it also allows its interpretation as a time direction. This further provides a clear interpretation of the zeroth component $f_0$ (of the real field $f_\mu$ that parametrizes the twisted fluctuation) as an energy.

These above two examples discussed so far – that is, the manifold and the $U(1)$ gauge theory – indicate that the main difference between the non-twisted and the twisted fermionic actions does not lie so much in the twisting of the inner product to a $\rho$-inner product, but rather in the restriction to different subspaces, viz. $\mathcal{H}_\mathcal{R}$ instead of $\mathcal{H}_+$. Indeed, by Lem. 3.6, the twisting of the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ to $\langle \cdot, \cdot \rangle_\rho$ solely brings forth to a global sign. However, as highlighted in the following remark: it is the restriction to $\mathcal{H}_\mathcal{R}$ instead of $\mathcal{H}_+$ that explains the change of signature.

**Remark 4.** *The disappearance of a derivative has no analogue in the non-twisted case, i.e. for $\psi \in \mathcal{H}_+$:*

- *the usual fermionic action (2.81) vanishes on a manifold, since $\eth\psi \in \mathcal{H}_-$ while $\mathcal{J}\psi \in \mathcal{H}_+$;*
- *in case of a $U(1)$ gauge theory, $\mathcal{H}_+$ is spanned by $\{\xi \otimes e, \phi \otimes \bar{e}\}$ with $\xi = \begin{pmatrix} \zeta \\ 0 \end{pmatrix}$, $\phi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$. Then*

$$S^f(\eth \otimes \mathbb{I}_2) = 2\langle \mathcal{J}\tilde{\phi}, \eth\tilde{\xi} \rangle = -2 \int_\mathcal{M} d\mu \big( \tilde{\bar{\varphi}}^\dagger \sigma^2 \tilde{\sigma}^\mu \partial_\mu \tilde{\zeta} \big). \tag{4.69}$$

*Up to the identification (4.68), the integrand is the euclidean version*

$$\mathcal{L}^l_E := i\Psi^\dagger_l \tilde{\sigma}^\mu \partial_\mu \Psi_l$$

*of the Weyl lagrangian $\mathcal{L}^l_M$.*

According to the result of §4.1.2, we anticipate the invariance of the real field $f_\mu$ under a gauge transformation. We check this for the case of the spectral triple of electrodynamics in section §4.3.2. We will also discuss the meaning of the other field, viz. $g_\mu$, which parametrizes the twisted fluctuation in Prop. 4.8. As in the non-twisted case, this will be identified with the $U(1)$ gauge field of electrodynamics.

## 4.3 Minimal twist of electrodynamics

In this subsection, we first write down the minimal twist of electrodynamics (§2.4.2) following the recipe prepared in §3.2. Then, we compute the twisted fluctuation in §4.3.1 and investigate the gauge transformations in §4.3.2. Finally, we compute the fermionic action in §4.3.3 and derive the Dirac equation in lorentzian signature.

A minimally twisted spectral triple of electrodynamics is obtained by doubling its algebra $\mathcal{A}_{ED} := C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ to $\mathcal{A} = \mathcal{A}_{ED} \otimes \mathbb{C}^2$ along with its flip automorphism $\rho$ (3.30), with the representation $\pi_0$ of $\mathcal{A}$ defined by (3.28). Explicitly, the grading $\gamma$ is given by the tensor product[2]

$$\gamma = \gamma^5 \otimes \gamma_{\mathcal{F}} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{4.70}$$

so that the projections $\mathfrak{p}_\pm = \frac{1}{2}\left(\mathbb{I}_{16} \pm \gamma\right)$ on the eigenspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ are

$$\begin{aligned} \mathfrak{p}_+ &= \mathsf{diag}(\mathbb{I}_2, 0_2, 0_2, \mathbb{I}_2, 0_2, \mathbb{I}_2, \mathbb{I}_2, 0_2), \\ \mathfrak{p}_- &= \mathsf{diag}(0_2, \mathbb{I}_2, \mathbb{I}_2, 0_2, \mathbb{I}_2, 0_2, 0_2, \mathbb{I}_2). \end{aligned} \tag{4.71}$$

Therefore, for $(a, a') \in \mathcal{A}$, where $a := (f, g)$, $a' := (f', g')$ with $f, g, f', g' \in C^\infty(\mathcal{M})$, one has the representation,

$$\pi(a, a') = \mathfrak{p}_+ \pi_0(a) + \mathfrak{p}_- \pi_0(a'),$$

[2] The product has been taken in the following sense:

$$A \otimes B = \begin{pmatrix} b_{11}A & \dots & b_{1n}A \\ \vdots & \ddots & \vdots \\ b_{m1}A & \dots & b_{mn}A \end{pmatrix},$$

where $B := (b_{ij})$ is an $m \times n$ matrix.

explicitly given by

$$\pi(a,a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f\mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix} \tag{4.72}$$
$$=: \begin{pmatrix} F & 0 & 0 & 0 \\ 0 & F' & 0 & 0 \\ 0 & 0 & G' & 0 \\ 0 & 0 & 0 & G \end{pmatrix},$$

where $F, F', G$ and $G'$ are given as in (4.43). The image of $(a, a') \in \mathcal{A}$ under the flip $\rho$ is represented by

$$\pi\big(\rho(a,a')\big) = \pi(a', a) = \begin{pmatrix} F' & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G' \end{pmatrix}. \tag{4.73}$$

The unitary $\mathcal{R} \in \mathcal{B}(\mathcal{H})$ implementing the action of $\rho$ on $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4$ is

$$\mathcal{R} = \gamma^0 \otimes \mathbb{I}_4 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \otimes \mathbb{I}_4, \tag{4.74}$$

which, as before, is compatible with the real structure, in the sense of (3.6), with $\epsilon''' = -1$.

### 4.3.1 Twisted fluctuation

The twisted commutator $[\mathcal{D}, a]_\rho$ being linear in $\mathcal{D}$, we treat the free part $\eth \otimes \mathbb{I}_4$ and the finite part $\gamma^5 \otimes \mathcal{D}_\mathcal{F}$ of the Dirac operator $\mathcal{D}$ separately. The results are summarized in Prop. 4.15.

#### The free part

The self-adjoint twisted fluctuations of the free part $\eth \otimes \mathbb{I}_4$ of the Dirac operator $\mathcal{D}$ are parametrized by two real fields (Prop. 4.13). One we relate with the anticipated $X_\mu$ field arising from the minimal twist of the manifold $\mathcal{M}$ (4.13) and the other one with the $U(1)$ gauge field $Y_\mu$ (2.96) of electrodynamics. To arrive there, we need two lemmas that we discuss below.

The following lemma gives a general expression for a twisted one-form $\omega_{\rho_\mathcal{M}}$ associated to the free Dirac operator $\eth \otimes \mathbb{I}_4$.

**Lemma 4.11.** *For $a = (f, g)$, $b = (v, w)$ in $\mathcal{A}_{ED}$ with similar definitions for $a', b'$, one has*

$$\omega_{\rho_\mathcal{M}} := \pi(a, a')\left[\eth \otimes \mathbb{I}_4,\ \pi(b, b')\right]_\rho = \begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix}, \tag{4.75}$$

*where we use the notation (4.44) for*

$$\begin{aligned} P_\mu &:= F'\partial_\mu V, \qquad P'_\mu := F\partial_\mu V', \\ Q_\mu &:= G'\partial_\mu W, \qquad Q'_\mu := G\partial_\mu W', \end{aligned} \tag{4.76}$$

*with $F, F', G, G'$ as in (4.43), and $V, V', W, W'$ as in (4.52).*

**Proof.** Using (4.72, 4.73) written for $(b, b')$, one computes the twisted commutator as follows

$$\left[\eth \otimes \mathbb{I}_4,\ \pi(b, b')\right]_\rho =: \begin{pmatrix} [\eth, V]_\rho & 0 & 0 & 0 \\ 0 & [\eth, V']_\rho & 0 & 0 \\ 0 & 0 & [\eth, W']_\rho & 0 \\ 0 & 0 & 0 & [\eth, W]_\rho \end{pmatrix}. \tag{4.77}$$

The result simply follows by multiplying (4.77) with (4.73) and then using (4.46). ∎

The lemma below gives a general expression for the twisted fluctuation $\left(\omega_{\rho_\mathcal{M}} + J\omega_{\rho_\mathcal{M}}J^{-1}\right)$ associated to the free part $\eth \otimes \mathbb{I}_4$ of the Dirac operator $\mathcal{D}$.

**Lemma 4.12.** *With the same notations as in Lem. 4.11, one has*

$$\mathcal{Z} := \omega_{\rho_\mathcal{M}} + J\omega_{\rho_\mathcal{M}}J^{-1} = \begin{pmatrix} \mathbf{Z} & 0 & 0 & 0 \\ 0 & \mathbf{Z}' & 0 & 0 \\ 0 & 0 & \bar{\mathbf{Z}} & 0 \\ 0 & 0 & 0 & \bar{\mathbf{Z}}' \end{pmatrix}, \tag{4.78}$$

*denoting*

$$\begin{aligned} \mathbf{Z} &:= \mathbf{P} + \bar{\mathbf{Q}}', \qquad \mathbf{Z}' := \mathbf{P}' + \bar{\mathbf{Q}}, \\ \bar{\mathbf{Z}} &:= \bar{\mathbf{P}} + \mathbf{Q}', \qquad \bar{\mathbf{Z}}' := \bar{\mathbf{P}}' + \mathbf{Q}. \end{aligned} \tag{4.79}$$

**Proof.** Using the explicit form of $J = \mathcal{J} \otimes J_{\mathcal{F}}$ with $J_{\mathcal{F}}$ as in (2.94), one gets, from (4.75) and Lem. 4.6, the following

$$J\omega_{\rho_{\mathcal{M}}}J^{-1} = \begin{pmatrix} 0 & 0 & \mathcal{J} & 0 \\ 0 & 0 & 0 & \mathcal{J} \\ \mathcal{J} & 0 & 0 & 0 \\ 0 & \mathcal{J} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathcal{J}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{J}^{-1} \\ \mathcal{J}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{J}^{-1} & 0 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} \mathcal{J}\mathbf{Q}'\mathcal{J}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{J}\mathbf{Q}\mathcal{J}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{J}\mathbf{P}\mathcal{J}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{J}\mathbf{P}'\mathcal{J}^{-1} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{Q}}' & 0 & 0 & 0 \\ 0 & \bar{\mathbf{Q}} & 0 & 0 \\ 0 & 0 & \bar{\mathbf{P}} & 0 \\ 0 & 0 & 0 & \bar{\mathbf{P}}' \end{pmatrix}.$$

Adding this up with $\omega_{\rho_{\mathcal{M}}}$ (4.75), the result follows. ■

In the following proposition we constrain the form of the twisted fluctuation $\mathcal{Z}$ discussed above, by imposing self-adjointness on it.

**Proposition 4.13.** *Any self-adjoint twisted fluctuation* (4.78) *of the free Dirac operator* $\eth \otimes \mathbb{I}_4$ *is of the form*

$$\mathcal{Z} = \mathbf{X} \otimes \mathbb{I}' + i\mathbf{Y} \otimes \mathbb{I}'', \tag{4.80}$$

*where*

$$X_\mu := f_\mu \gamma^5 \qquad \text{and} \qquad Y_\mu := g_\mu \mathbb{I}_4$$

*are parametrized, respectively, by real fields* $f_\mu, g_\mu \in C^\infty(\mathcal{M}, \mathbb{R})$ *and*

$$\mathbb{I}' := \mathsf{diag}(1, -1, 1, -1), \qquad \mathbb{I}'' := \mathsf{diag}(1, 1, -1, -1).$$

**Proof.** $\mathcal{Z}$ as given by (4.78) is self-adjoint iff

$$\mathbf{Z} = \mathbf{Z}^\dagger, \qquad \mathbf{Z}' = \mathbf{Z}'^\dagger, \qquad \bar{\mathbf{Z}} = \bar{\mathbf{Z}}^\dagger, \qquad \bar{\mathbf{Z}}' = \bar{\mathbf{Z}}'^\dagger. \tag{4.81}$$

From (4.45) and the third eq. in (4.46), all of these four conditions are equivalent to $\mathbf{Z} = \mathbf{Z}^\dagger$, which is a condition that we have already encountered in the proof of Prop. 4.7, and it yields, cf. (4.64):

$$\begin{aligned} Z_\mu &= X_\mu + iY_\mu \\ &= \begin{pmatrix} (f_\mu + ig_\mu)\mathbb{I}_2 & 0 \\ 0 & -(f_\mu - ig_\mu)\mathbb{I}_2 \end{pmatrix}, \end{aligned} \tag{4.82}$$

where $X_\mu := f_\mu \gamma^5$ and $Y_\mu := g_\mu \mathbb{I}_4$ with $f_\mu$ and $g_\mu$ as defined in (4.51). Going back to (4.78), one obtains

$$\mathcal{Z} = \begin{pmatrix} \mathbf{Z} & 0 & 0 & 0 \\ 0 & -\bar{\mathbf{Z}} & 0 & 0 \\ 0 & 0 & \bar{\mathbf{Z}} & 0 \\ 0 & 0 & 0 & -\mathbf{Z} \end{pmatrix}$$

$$= \begin{pmatrix} -i\gamma^\mu Z_\mu & 0 & 0 & 0 \\ 0 & i\gamma^\mu \overline{Z}_\mu & 0 & 0 \\ 0 & 0 & -i\gamma^\mu \overline{Z}_\mu & 0 \\ 0 & 0 & 0 & i\gamma^\mu Z_\mu \end{pmatrix}$$

$$= \begin{pmatrix} -i\gamma^\mu (X_\mu + iY_\mu) & 0 & 0 & 0 \\ 0 & i\gamma^\mu (X_\mu - iY_\mu) & 0 & 0 \\ 0 & 0 & -i\gamma^\mu (X_\mu - iY_\mu) & 0 \\ 0 & 0 & 0 & i\gamma^\mu (X_\mu + iY_\mu) \end{pmatrix}$$

$$= -i\gamma^\mu X_\mu \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + i(-i\gamma^\mu Y_\mu) \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad \blacksquare$$

We conclude the discussion on the free part with the following remark on self-adjointness of the twisted one-form $\omega_{\rho\mathcal{M}}$ vs. that of the twisted fluctuation $\mathcal{Z}$.

**Remark 5.** *Imposing self-adjointness of the twisted one-form $\omega_{\rho_\mathcal{M}}$ amounts to the following constraints:*

$$\mathbf{P}^\dagger = \mathbf{P}, \qquad \mathbf{Q}^\dagger = \mathbf{Q}. \tag{4.83}$$

*These conditions imply, but are not equivalent to, imposing the self-adjointness of the twisted fluctuation $\mathcal{Z}$, that is,*

$$\mathbf{Z}^\dagger = \mathbf{Z}. \tag{4.84}$$

*As discussed right after Lem. 4.2 for the minimal twist of a manifold, the relevant point is that the stronger condition (4.83) does not imply that the twisted fluctuation $\mathcal{Z}$ be zero. The final form of the twist-fluctuated operator is the same, whether one requires (4.83) or (4.84). What differs is the relations among the real fields $f_\mu, g_\mu$ and the components of $(a, a'), (b, b')$ appearing in the definition of the twisted-one form $\omega_{\rho\mathcal{M}}$.*

### The finite part

In the spectral triple of electrodynamics, the finite part $\gamma^5 \otimes \mathcal{D}_\mathcal{F}$ of the Dirac operator $\mathcal{D}$ §2.4.2 does not fluctuate [DS], for it commutes with the representation $\pi_0$ (2.95) of the algebra $\mathcal{A}_{ED}$. The same is true in case of the minimal twist of electrodynamics also – as shown in the following proposition.

**PROPOSITION 4.14.** *The finite Dirac operator $\gamma^5 \otimes \mathcal{D}_\mathcal{F}$ has no twisted fluctuation.*

**PROOF.** With the representations (4.72, 4.73), one calculates that

$$
\begin{aligned}
&\left[\gamma^5 \otimes \mathcal{D}_\mathcal{F},\ \pi(a, a')\right]_\rho \\
&= (\gamma^5 \otimes \mathcal{D}_\mathcal{F})\, \pi(a, a') - \pi(a', a)\, (\gamma^5 \otimes \mathcal{D}_\mathcal{F}) \\
&= \begin{pmatrix} 0 & d\gamma^5 & 0 & 0 \\ \bar{d}\gamma^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d}\gamma^5 \\ 0 & 0 & d\gamma^5 & 0 \end{pmatrix} \begin{pmatrix} F & 0 & 0 & 0 \\ 0 & F' & 0 & 0 \\ 0 & 0 & G' & 0 \\ 0 & 0 & 0 & G \end{pmatrix} - \begin{pmatrix} F' & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G' \end{pmatrix} \begin{pmatrix} 0 & d\gamma^5 & 0 & 0 \\ \bar{d}\gamma^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d}\gamma^5 \\ 0 & 0 & d\gamma^5 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & d[\gamma^5, F'] & 0 & 0 \\ \bar{d}[\gamma^5, F] & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d}[\gamma^5, G] \\ 0 & 0 & d[\gamma^5, G'] & 0 \end{pmatrix} = 0,
\end{aligned}
$$

where $F$, $F'$, $G$, $G'$ (4.43) being diagonal, commute with $\gamma^5$. ■

The results of §4.3.1 summarize as follows:

**PROPOSITION 4.15.** *The Dirac operator $\mathcal{D} = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes \mathcal{D}_\mathcal{F}$ of electrodynamics (§2.4.2), under the minimal twist (4.72–4.74), twist-fluctuates to*

$$\mathcal{D}_\mathcal{Z} := \mathcal{D} + \mathcal{Z}, \quad \textit{where} \quad \mathcal{Z} := \mathbf{X} \otimes \mathbb{I}' + i\mathbf{Y} \otimes \mathbb{I}'', \tag{4.85}$$

*as given in Prop. 4.13.*

The explicit form of $\mathbf{Y}$ is the same as that of the gauge potential $Y_\mu$ (2.92) of electrodynamics in the non-twisted case. This is confirmed in the next section, where we show that $\mathbf{Y}$ transforms exactly as the $U(1)$ gauge potential of electrodynamics.

The $\mathbf{X}$ field is similar to that of the minimally twisted manifold. We show below that this field is gauge-invariant and induces a transition of signature from the euclidean to the lorentzian, in the same way as exhibited in §4.2.2.

**REMARK 6.** *Expectedly, substituting $\rho = \mathrm{Id}$, one returns to the non-twisted case of electrodynamics (§2.4.2): the triviality of $\rho$ is tantamount to equating (4.72) with (4.73), that is to identify the 'primed' functions $(f', g', \cdots)$ with their 'un-primed' partners $(f, g, \cdots)$. Hence, $\mathbf{Z}' = \mathbf{Z}$.*

*Imposing the self-adjointness, the third eq. (4.46) gives $\mathbf{Z} = -\overline{\mathbf{Z}}$. Going back to (4.82), this yields $f_\mu = 0$. Therefore, $X_\mu$ vanishes and only the $U(1)$ gauge field $\mathbf{Y}$ survives.*

### 4.3.2 Gauge transformation

Along the lines of §4.1.2, here we discuss the transformations of the fields parametrizing the twisted fluctuation $\mathcal{Z}$. Let $u := (e^{i\alpha}, e^{i\beta})$ and $u' := (e^{i\alpha'}, e^{i\beta'})$ be two unitaries of the algebra $\mathcal{A}_{ED}$, with $\alpha, \alpha', \beta, \beta' \in C^\infty(\mathcal{M}, \mathbb{R})$. A unitary of $\mathcal{A}_{ED} \otimes \mathbb{C}^2$ is of the form $(u, u')$, with the representation

$$\pi(u,u') = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & 0 \\ 0 & 0 & B' & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \qquad \text{and} \qquad \pi(\rho(u,u')) = \pi(u',u) = \begin{pmatrix} A' & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B' \end{pmatrix}, \tag{4.86}$$

where, similar to (4.43), we have denoted

$$\begin{aligned} A &:= \pi_{\mathcal{M}}(e^{i\alpha}, e^{i\alpha'}), \qquad & A' &:= \rho(A) = \pi_{\mathcal{M}}(e^{i\alpha'}, e^{i\alpha}), \\ B &:= \pi_{\mathcal{M}}(e^{i\beta}, e^{i\beta'}), \qquad & B' &:= \rho(B) = \pi_{\mathcal{M}}(e^{i\beta'}, e^{i\beta}). \end{aligned} \tag{4.87}$$

**Proposition 4.16.** *Under a gauge transformation (2.78) with a unitary $(u, u')$ (4.86), the fields $z_\mu, z'_\mu$ parametrizing the twisted-covariant operator $\mathcal{D}_{\mathcal{Z}}$ of Prop. 4.15 transform as*

$$z_\mu \to z_\mu - i\partial_\mu \vartheta, \qquad z'_\mu \to z'_\mu - i\partial_\mu \vartheta' \tag{4.88}$$

*for $\vartheta := \alpha - \beta'$ and $\vartheta' = \alpha' - \beta$ in $C^\infty(\mathcal{M}, \mathbb{R})$.*

**Proof.** Since the finite part $\gamma_{\mathcal{F}} \otimes \mathcal{D}_{\mathcal{F}}$ twist-commutes with the algebra, in the law of transformation of the gauge potential (4.28), it is then enough to consider only the free part $\eth \otimes \mathbb{I}_4$. Thus, $\omega_{\rho\mathcal{M}}$ in (4.78) transforms to

$$\begin{aligned} \omega_{\rho_{\mathcal{M}}}^{(u,u')} &= \rho(u,u')\left(\left[\eth \otimes \mathbb{I}_4, (u,u')^*\right]_\rho + \omega_{\rho_{\mathcal{M}}}(u,u')^*\right) \\ &= (u',u)\left(\eth \otimes \mathbb{I}_4 + \omega_{\rho_{\mathcal{M}}}\right)(u,u')^*, \end{aligned} \tag{4.89}$$

where, as in (4.29), we made use of $\left[\eth \otimes \mathbb{I}_4, (u,u')^*\right]_\rho = (\eth \otimes \mathbb{I}_4)(u,u')^*$. With the representation (4.86) of $(u, u')$ and $\omega_{\rho_{\mathcal{M}}}$ from Lem. 4.11, the above transformation (4.89) reads as

$$\begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix} \to \begin{pmatrix} A'(\eth + \mathbf{P})\bar{A} & 0 & 0 & 0 \\ 0 & A(\eth + \mathbf{P}')\bar{A}' & 0 & 0 \\ 0 & 0 & B(\eth + \mathbf{Q}')\bar{B}' & 0 \\ 0 & 0 & 0 & B'(\eth + \mathbf{Q})\bar{B} \end{pmatrix},$$

where recalling that matrices $A'$ and $B'$ twist-commute with $\gamma^\mu$ and $A$ commutes with $P_\mu$ (and, $B$ with $Q_\mu$), one obtains

$$P_\mu \to P_\mu + A\partial_\mu \bar{A}, \qquad \text{and} \qquad Q'_\mu \to Q_\mu + B'\partial_\mu \bar{B}'. \tag{4.90}$$

implying, for $Z_\mu = P_\mu + Q'_\mu$, that

$$Z_\mu \quad \to \quad Z_\mu + \left(A\partial_\mu \bar{A} + B'\partial_\mu \bar{B}'\right), \tag{4.91}$$

With the representations (4.82) of $Z_\mu$ (recalling that $z_\mu = f_\mu + ig_\mu$), and (4.87) of $A, B$, the transformation (4.91) reads

$$\begin{pmatrix} z_\mu \mathbb{I}_2 & 0 \\ 0 & z'_\mu \mathbb{I}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} (z_\mu - i\partial_\mu \vartheta)\, \mathbb{I}_2 & 0 \\ 0 & z'_\mu - i\partial_\mu \vartheta' \mathbb{I}_2 \end{pmatrix}. \qquad \blacksquare$$

By imposing that both $\mathcal{Z}$ and its gauge transform are self-adjoint, that is, by Lem. 4.6: $z'_\mu = -\bar{z}_\mu$ and $z'_\mu - i\partial_\mu \vartheta' = -\overline{z_\mu - i\partial_\mu \vartheta}$, one is forced to identify $\vartheta' = \vartheta$. Then, the law of transformation of $z_\mu$ in terms of its real and imaginary components reads:

$$f_\mu + ig_\mu \quad \longrightarrow \quad f_\mu + i(g_\mu - \partial_\mu \vartheta), \tag{4.92}$$

which implies for the fields $X_\mu = f_\mu \gamma^5$ and $Y_\mu = g_\mu \mathbb{I}_4$ of Prop. 4.13 that $X_\mu$ stays invariant, while $Y_\mu$ undergoes a nontrivial transformation, induced by

$$g_\mu \to g_\mu - \partial_\mu \vartheta, \qquad \vartheta \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{4.93}$$

In the light of (2.92), this identifies $g_\mu$ as the $U(1)$ gauge field of electrodynamics.

### 4.3.3 Dirac action in Lorentz signature

To calculate the twisted fermionic action, we first identify the eigenvectors of the unitary $\mathcal{R}$ implementing the twist.

**Lemma 4.17.** *Any $\eta$ in the $+1$-eigenspace $\mathcal{H}_\mathcal{R}$ (3.21) of the unitary $\mathcal{R}$ (4.74) is of the form*

$$\eta = \Phi_1 \otimes e_l + \Phi_2 \otimes e_r + \Phi_3 \otimes \overline{e_l} + \Phi_4 \otimes \overline{e_r}, \tag{4.94}$$

*with $\Phi_{k=1,\dots,4} := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$ where $\Phi_k \in L^2(\mathcal{M}, \mathcal{S})$ are Dirac spinors with Weyl components $\varphi_k$, while $\{e_l, e_r, \overline{e_l}, \overline{e_r}\}$ denotes the orthonormal basis for the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F} = \mathbb{C}^4$.*

**Proof.** $\mathcal{R}$ has eigenvalues $\pm 1$ and its eigenvectors corresponding to the eigenvalue $+1$ are:

$$\varepsilon_1 = \upsilon_1 \otimes e_l, \qquad \varepsilon_2 = \upsilon_2 \otimes e_l, \qquad \varepsilon_3 = \upsilon_1 \otimes e_r, \qquad \varepsilon_4 = \upsilon_2 \otimes e_r,$$
$$\varepsilon_5 = \upsilon_1 \otimes \overline{e_l}, \qquad \varepsilon_6 = \upsilon_2 \otimes \overline{e_l}, \qquad \varepsilon_7 = \upsilon_1 \otimes \overline{e_r}, \qquad \varepsilon_8 = \upsilon_2 \otimes \overline{e_l},$$

where

$$\upsilon_1 := \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad \upsilon_2 := \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$

denote the eigenvectors of $\gamma^0$, as in Lem. 4.4. Therefore, we have

$$\begin{aligned}
\eta &= \textstyle\sum_{j=1}^{8} \phi_j \varepsilon_j \\
&= (\phi_1\upsilon_1 + \phi_2\upsilon_2) \otimes e_l + (\phi_3\upsilon_1 + \phi_4\upsilon_2) \otimes e_r + (\phi_5\upsilon_1 + \phi_6\upsilon_2) \otimes \overline{e_l} + (\phi_7\upsilon_1 + \phi_8\upsilon_2) \otimes \overline{e_r} \\
&= \Phi_1 \otimes e_l + \Phi_2 \otimes e_r + \Phi_3 \otimes \overline{e_l} + \Phi_4 \otimes \overline{e_r},
\end{aligned}$$

denoting

$$\Phi_k := \begin{pmatrix}\varphi_k\\ \varphi_k\end{pmatrix} \text{ with } \varphi_1 := \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \varphi_2 := \begin{pmatrix}\phi_3\\ \phi_4\end{pmatrix}, \varphi_3 := \begin{pmatrix}\phi_5\\ \phi_6\end{pmatrix}, \varphi_4 := \begin{pmatrix}\phi_7\\ \phi_8\end{pmatrix}. \qquad \blacksquare$$

The twisted-covariant Dirac operator (4.85) contains two extra terms that were not present in the $U(1)$ gauge theory: $\gamma^5 \otimes \mathcal{D}_{\mathcal{F}}$ and the term in $\mathbf{Y}$. The following lemma is useful to compute the contribution of these two terms to the fermionic action.

**Lemma 4.18.** *For Dirac spinors* $\phi := \begin{pmatrix}\varphi\\ \varphi\end{pmatrix}$ *and* $\xi := \begin{pmatrix}\zeta\\ \zeta\end{pmatrix}$ *in* $L^2(\mathcal{M}, \mathcal{S})$*, one has*

$$\begin{aligned}
\langle \mathcal{J}\phi, i\mathbf{Y}\xi\rangle &= 2i \int_{\mathcal{M}} d\mu\, \left[\bar{\varphi}^\dagger \sigma_2 \big(\textstyle\sum_{j=1}^{3} \sigma_j g_j\big)\zeta\right], \\
\langle \mathcal{J}\phi, \gamma^5\xi\rangle &= -2 \int_{\mathcal{M}} d\mu\, \left(\bar{\varphi}^\dagger \sigma_2 \zeta\right).
\end{aligned} \tag{4.95}$$

**Proof.** Using (4.80) for $Y_\mu$ and (D.1) for the Dirac matrices, one gets

$$i\mathbf{Y}\phi = \gamma^\mu Y_\mu \begin{pmatrix}\varphi\\ \varphi\end{pmatrix} = \begin{pmatrix}0 & \sigma^\mu\\ \tilde{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}g_\mu \mathbb{I}_2 & 0\\ 0 & g_\mu \mathbb{I}_2\end{pmatrix}\begin{pmatrix}\varphi\\ \varphi\end{pmatrix} = \begin{pmatrix}g_\mu \sigma^\mu \varphi\\ g_\mu \tilde{\sigma}^\mu \varphi\end{pmatrix}.$$

Along with (4.31), recalling that $\sigma^{2\dagger} = i\sigma_2$ and $\tilde{\sigma}^{2\dagger} = -i\sigma_2$, yields

$$\begin{aligned}
(\mathcal{J}\phi)^\dagger (i\mathbf{Y}\xi) &= -i \begin{pmatrix}\tilde{\sigma}^2 \bar{\varphi}\\ \sigma^2 \bar{\varphi}\end{pmatrix}^\dagger \begin{pmatrix}g_\mu \sigma^\mu \zeta\\ g_\mu \tilde{\sigma}^\mu \zeta\end{pmatrix} &= -i\bar{\varphi}^\dagger \left(\tilde{\sigma}^{2\dagger}\sigma^\mu + \sigma^{2\dagger}\tilde{\sigma}^\mu\right) g_\mu \zeta \\
&= \bar{\varphi}^\dagger \sigma_2 \big(-\sigma^\mu + \tilde{\sigma}^\mu\big) g_\mu \zeta &= 2i\bar{\varphi}^\dagger \sigma_2 \left(\textstyle\sum_{j=1}^{3} \sigma_j g_j\right) \zeta,
\end{aligned}$$

where we used (4.34) and obtained the first equation of (4.95). The second one follows from

$$(\mathcal{J}\phi)^\dagger(\gamma^5\xi) = -i\begin{pmatrix}\tilde{\sigma}^2\bar{\varphi}\\ \sigma^2\bar{\varphi}\end{pmatrix}^\dagger\begin{pmatrix}\zeta\\ -\zeta\end{pmatrix} = -\bar{\varphi}^\dagger\sigma_2\zeta - \bar{\varphi}^\dagger\sigma_2\zeta = -2\bar{\varphi}^\dagger\sigma_2\zeta. \qquad \blacksquare$$

**Proposition 4.19.** *The fermionic action of the minimal twist of $\mathcal{M}\times\mathcal{F}_{ED}$ is*

$$\begin{aligned} S^f_\rho(\mathcal{D}_\mathcal{Z}) &= \mathfrak{A}^\rho_{\mathcal{D}_\mathcal{Z}}(\tilde{\Phi},\tilde{\Phi}) = 4\int_\mathcal{M} d\mu\, \mathcal{L}^f_\rho, \\ \text{where}\quad \mathcal{L}^f_\rho &:= \bar{\tilde{\varphi}}_1^\dagger\sigma_2\Big(if_0 - \sum_j \sigma_j\mathfrak{D}_j\Big)\tilde{\varphi}_3 - \bar{\tilde{\varphi}}_2^\dagger\sigma_2\Big(if_0 + \sum_j \sigma_j\mathfrak{D}_j\Big)\tilde{\varphi}_4 \\ &\quad + \Big(\bar{d}\bar{\tilde{\varphi}}_1^\dagger\sigma_2\tilde{\varphi}_4 + d\bar{\tilde{\varphi}}_2^\dagger\sigma_2\tilde{\varphi}_3\Big), \end{aligned} \tag{4.96}$$

*with $\mathfrak{D}_\mu := \partial_\mu - ig_\mu$ being the covariant derivative associated to the electromagnetic four-potential $g_\mu$ (4.93). Further, identifying the physical spinors as*

$$\Psi = \begin{pmatrix}\Psi_l\\ \Psi_r\end{pmatrix} := \begin{pmatrix}\tilde{\varphi}_3\\ \tilde{\varphi}_4\end{pmatrix}, \qquad \Psi^\dagger = \left(\Psi_l^\dagger,\ \ \Psi_r^\dagger\right) := \Big(-i\bar{\tilde{\varphi}}_1^\dagger\sigma_2,\ \ i\bar{\tilde{\varphi}}_2^\dagger\sigma_2\Big), \tag{4.97}$$

*the lagrangian (4.96) describes a plane-wave solution of the Dirac equation, in lorentzian signature, and with the temporal gauge.*

**Proof.** Let $\mathfrak{A}^\rho_{\mathcal{D}_\mathcal{Z}}$ be the antisymmetric bilinear form (3.20) defined by the twisted-covariant Dirac operator (4.85):

$$\mathcal{D}_\mathcal{Z} = \eth\otimes\mathbb{I}_4 + \mathbf{X}\otimes\mathbb{I}' + i\mathbf{Y}\otimes\mathbb{I}'' + \gamma^5\otimes\mathcal{D}_\mathcal{F},$$

which breaks down into the following four terms:

$$\mathfrak{A}^\rho_{\mathcal{D}_\mathcal{Z}} = \mathfrak{A}^\rho_{\eth\otimes\mathbb{I}_4} + \mathfrak{A}^\rho_{\mathbf{X}\otimes\mathbb{I}'} + \mathfrak{A}^\rho_{i\mathbf{Y}\otimes\mathbb{I}''} + \mathfrak{A}^\rho_{\gamma^5\otimes\mathcal{D}_\mathcal{F}}. \tag{4.98}$$

For $\Phi,\xi\in\mathcal{H}_\mathcal{R}$ with $\Phi$ as in (4.94) and $\Xi$ with components $\xi_i = \begin{pmatrix}\zeta_i\\ \zeta_i\end{pmatrix} \in L^2(\mathcal{M},\mathcal{S})$, one has

$$\begin{aligned} J\Phi &= \mathcal{J}\phi_1\otimes\overline{e_l} + \mathcal{J}\phi_2\otimes\overline{e_r} + \mathcal{J}\phi_3\otimes e_l + \mathcal{J}\phi_4\otimes e_r, \\ (\eth\otimes\mathbb{I}_4)\Xi &= \eth\xi_1\otimes e_l + \eth\xi_2\otimes e_r + \eth\xi_3\otimes\overline{e_l} + \eth\xi_4\otimes\overline{e_r}, \\ (\mathbf{X}\otimes\mathbb{I}')\Xi &= \mathbf{X}\xi_1\otimes e_l - \mathbf{X}\xi_2\otimes e_r + \mathbf{X}\xi_3\otimes\overline{e_l} - \mathbf{X}\xi_4\otimes\overline{e_r}, \\ (i\mathbf{Y}\otimes\mathbb{I}'')\Xi &= i\mathbf{Y}\xi_1\otimes e_l + i\mathbf{Y}\xi_2\otimes e_r - i\mathbf{Y}\xi_3\otimes\overline{e_l} - i\mathbf{Y}\xi_4\otimes\overline{e_r}, \\ (\gamma^5\otimes\mathcal{D}_\mathcal{F})\Xi &= \gamma^5\zeta_1\otimes\bar{d}e_r + \gamma^5\zeta_2\otimes de_l + \gamma^5\zeta_3\otimes d\overline{e_r} + \gamma^5\zeta_4\otimes\bar{d}\overline{e_l} \end{aligned} \tag{4.99}$$

where the first and the last eqs. come from the explicit forms (2.94) of $J_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{F}}$, respectively, while the third and the fourth ones follow from the explicit form (4.80) of $\mathbf{X}$ and $\mathbf{Y}$, respectively. These eqs. allow to reduce each of the four terms in (4.98) to a bilinear form on $L^2(\mathcal{M}, \mathcal{S})$ rather than on the tensor product $L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4$. More precisely, omitting the summation symbol on the index $j$ and recalling Lem. 3.6 with $\epsilon''' = -1$, one computes as below:

$$\begin{aligned}
\mathfrak{A}^{\rho}_{\eth\otimes\mathbb{I}_4}(\Phi, \Xi) &= -\mathfrak{A}_{\eth\otimes\mathbb{I}_4}(\Phi, \Xi) \quad = \quad -\langle J\Phi, (\eth \otimes \mathbb{I}_4)\Xi\rangle \\
&= -\langle \mathcal{J}\phi_1, \eth\xi_3\rangle - \langle \mathcal{J}\phi_2, \eth\xi_4\rangle - \langle \mathcal{J}\phi_3, \eth\xi_1\rangle - \langle \mathcal{J}\phi_4, \eth\xi_2\rangle \\
&= -\mathfrak{A}_{\eth}(\phi_1, \xi_3) - \mathfrak{A}_{\eth}(\phi_2, \xi_4) - \mathfrak{A}_{\eth}(\phi_3, \xi_1) - \mathfrak{A}_{\eth}(\phi_4, \xi_2); \qquad (4.100) \\
\mathfrak{A}^{\rho}_{\mathbf{X}\otimes\mathbb{I}'}(\Phi, \Xi) &= -\mathfrak{A}_{\mathbf{X}\otimes\mathbb{I}'}(\Phi, \Xi) \quad = \quad -\langle J\Phi, (\mathbf{X} \otimes \mathbb{I}')\Xi\rangle \\
&= -\langle \mathcal{J}\phi_1, \mathbf{X}\zeta_3\rangle + \langle \mathcal{J}\phi_2, \mathbf{X}\zeta_4\rangle - \langle \mathcal{J}\phi_3, \mathbf{X}\zeta_1\rangle + \langle \mathcal{J}\phi_4, \mathbf{X}\zeta_2\rangle \\
&= -\mathfrak{A}_{\mathbf{X}}(\phi_1, \zeta_3) + \mathfrak{A}_{\mathbf{X}}(\phi_2, \zeta_4) - \mathfrak{A}_{\mathbf{X}}(\phi_3, \zeta_1) + \mathfrak{A}_{\mathbf{X}}(\phi_4, \zeta_2); \qquad (4.101) \\
\mathfrak{A}^{\rho}_{i\mathbf{Y}\otimes\mathbb{I}''}(\Phi, \Xi) &= -\mathfrak{A}_{i\mathbf{Y}\otimes\mathbb{I}''}(\Phi, \Xi) \quad = \quad -\langle J\Phi, (i\mathbf{Y} \otimes \mathbb{I}'')\xi\rangle \\
&= \langle \mathcal{J}\phi_1, i\mathbf{Y}\xi_3\rangle + \langle \mathcal{J}\phi_2, i\mathbf{Y}\xi_4\rangle - \langle \mathcal{J}\phi_3, i\mathbf{Y}\xi_1\rangle - \langle \mathcal{J}\phi_4, i\mathbf{Y}\xi_2\rangle \\
&= \mathfrak{A}_{\mathbf{Y}}(\phi_1, \xi_3) + \mathfrak{A}_{\mathbf{Y}}(\phi_2, \xi_4) - \mathfrak{A}_{\mathbf{Y}}(\phi_3, \xi_1) - \mathfrak{A}_{\mathbf{Y}}(\phi_4, \xi_2); \qquad (4.102) \\
\mathfrak{A}^{\rho}_{\gamma^5\otimes\mathcal{D}_{\mathcal{F}}}(\Phi, \Xi) &= -\mathfrak{A}_{\gamma^5\otimes\mathcal{D}_{\mathcal{F}}}(\Phi, \Xi) \quad = \quad -\langle J\Phi, (\gamma^5 \otimes \mathcal{D}_{\mathcal{F}})\Xi\rangle \\
&= -\bar{d}\langle \mathcal{J}\phi_1, \gamma^5\xi_4\rangle - d\langle \mathcal{J}\phi_2, \gamma^5\xi_3\rangle - d\langle \mathcal{J}\phi_3, \gamma^5\xi_2\rangle - \bar{d}\langle \mathcal{J}\phi_4, \gamma^5\xi_1\rangle \\
&= -\bar{d}\,\mathfrak{A}_{\gamma^5}(\phi_1, \xi_4) - d\,\mathfrak{A}_{\gamma^5}(\phi_2, \xi_3) - d\,\mathfrak{A}_{\gamma^5}(\phi_3, \xi_2) - \bar{d}\,\mathfrak{A}_{\gamma^5}(\phi_4, \xi_1). \qquad (4.103)
\end{aligned}$$

Substituting $\Xi = \Phi$, and then promoting the spinor $\Phi$ to a Graßmann spinor $\tilde{\Phi}$, the sum of eqs. (4.100), (4.101), and (4.103) is

$$\begin{aligned}
-2\,\mathfrak{A}_{\eth}(\tilde{\phi}_1, \tilde{\phi}_3) - 2\,\mathfrak{A}_{\eth}(\tilde{\phi}_2, \tilde{\phi}_4) - 2\,\mathfrak{A}_{\mathbf{X}}(\tilde{\phi}_1, \tilde{\phi}_3) + 2\,\mathfrak{A}_{\mathbf{X}}(\tilde{\phi}_2, \tilde{\phi}_4) \\
-2\bar{d}\,\mathfrak{A}_{\gamma^5}(\phi_1, \phi_4) - 2d\,\mathfrak{A}_{\gamma^5}(\phi_2, \phi_3);
\end{aligned} \qquad (4.104)$$

where we have used the fact that the bilinear forms $\mathfrak{A}_{\eth}$, $\mathfrak{A}_{\mathbf{X}}$, and $\mathfrak{A}_{\gamma^5}$ are antisymmetric on vectors (by Lem. 3.5, since $\eth$, $\mathbf{X}$, and $\gamma^5$ are all commuting with $\mathcal{J}$ in KO-dim. 4), and so they are symmetric when evaluated on the corresponding Graßmann variables. On the other hand, (4.102) is symmetric on vectors (since in KO-dim. 4: $i\mathbf{Y}\mathcal{J} = g_\mu\gamma^\mu\mathcal{J} = -\mathcal{J}g_\mu\gamma^\mu = -\mathcal{J}i\mathbf{Y}$), while antisymmetric in Graßmann variables. Thus, it yields

$$2\mathfrak{A}_{\mathbf{Y}}(\phi_1, \phi_3) + 2\,\mathfrak{A}_{\mathbf{Y}}(\phi_2, \phi_4). \qquad (4.105)$$

The lagrangian (4.96) follows after substituting all the bilinear forms in (4.104, 4.105) with their explicit expressions given in (4.35, 4.36) and Lem. 4.18.

Upon the identification (4.97), one finds that $\mathcal{L}^f_\rho$ coincides with the Dirac lagrangian in lorentzian signature (D.8) (with the covariant derivative $\mathfrak{D}_\mu$ to take into account the coupling with the electromagnetic field, but in the temporal Weyl gauge $\mathfrak{D}_0 = \partial_0$)

$$\mathcal{L}_M = i\Psi_l^\dagger\big(\mathfrak{D}_0 - \sigma_j\mathfrak{D}_j\big)\Psi_l + i\Psi_r^\dagger\big(\mathfrak{D}_0 + \sigma_j\partial_j\big)\Psi_r - m\big(\Psi_l^\dagger\Psi_r + \Psi_r^\dagger\Psi_l\big), \tag{4.106}$$

as soon as one imposes that $\partial_0\Psi = if_0\Psi$, i.e.

$$\Psi(x_0, x_j) = \Psi(x_j)e^{if_0x_0}. \tag{4.107}$$

The mass terms also match up correctly if one imposes the parameter $d \in \mathbb{C}$ to be purely imaginary as $d := -im$. This is in agreement with the non-twisted electrodynamics, cf. [DS, Rem. 4.4]. ■

The above Prop. 4.19 extends the analysis done for the Weyl equation, in §4.2.2, to the Dirac equation. It confirms the interpretation of the zeroth component of the real field $f_\mu$, arising in the twisted fluctuation, as an energy. It also shows that the other field $g_\mu$ is well-identified with the electromagnetic gauge potential, as in the non-twisted case.

But this does not say anything about the other components $f_i$ for $i = 1, 2, 3$ since they do not appear in the lagrangian (4.96). It is tempting to identify them with the momenta. This, in fact, makes sense if one implements a Lorentz transformation, as discussed in the next section.

Another motivation to study the action of the Lorentz transformations on the twisted fermionic action is that the temporal Weyl gauge we ended with, is not Lorentz invariant. One must check whether the interpretation of the twisted fermionic action provided by Prop. 4.19 is robust enough to survive Lorentz transformations.

# Chapter 5

# Open Questions

In this chapter, we touch upon some open issues that arise as a result of this thesis. We show in §5.1 that the $\rho$-inner product and, hence, the fermionic action associated to a minimally twisted manifold is invariant under Lorentz boosts. That being said, the origin of Lorentz transformations (or, equally, the Lorentz group) within the context of (twisted) noncommutative geometry is not yet fully understood.

In §5.2, we work out the squared twisted-covariant Dirac operator $\eth_X^2$ for the minimally twisted manifold with curvature. It is the first step towards writing a heat-kernel expansion for spectral action associated to this twisted spectral triple. Here, we give the explicit expression for the endomorphism term that accounts for the potential terms in the spectral action. This formula gives the impression that there is a coupling between the curvature and the $X_\mu$ field that appears in the non-vanishing twisted fluctuation $\eth_X$ of the Dirac operator $\eth$.

These will be the subject of future works and a full exploration into these lines will appear elsewhere.

## 5.1 Lorentz invariance of fermionic action

A Lorentz boost $S[\Lambda]$ in the Dirac spinor representation $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ is given by

$$S[\Lambda] = \begin{pmatrix} \Lambda_+ & 0_2 \\ 0_2 & \Lambda_- \end{pmatrix} \qquad \text{with} \quad \Lambda_\pm = \exp\left(\pm \frac{\mathbf{b} \cdot \boldsymbol{\sigma}}{2}\right), \tag{5.1}$$

where $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector and $\mathbf{b} \in \mathbb{R}^3$ is the boost parameter. Under such a boost, a lorentzian spinor $\psi_M$ and the lorentzian Dirac operator $\eth_M$ transform as

$$\begin{aligned} \psi_M &\to S[\Lambda]\psi_M, \\ \eth_M &\to S[\Lambda]\, \eth_M\, S[\Lambda]^{-1}. \end{aligned} \tag{5.2}$$

We define the action of a boost on the minimal twist $(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2,\ L^2(\mathcal{M}, \mathcal{S}),\ \eth)$ of a closed euclidean manifold $\mathcal{M}$ of dimension 4, as follows

$$\begin{aligned} \psi &\to \psi_\Lambda = S[\Lambda]\psi, \qquad \forall \psi \in L^2(\mathcal{M}, \mathcal{S}), \\ \eth &\to \eth_\Lambda := \rho(S[\Lambda])\, \eth\, S[\Lambda], \end{aligned} \tag{5.3}$$

where $\rho(S[\Lambda]) = \mathcal{R} S[\Lambda] \mathcal{R}^\dagger$ for $\mathcal{R} = \gamma^0$ given in (4.8), i.e.

$$\rho(S[\Lambda]) = \gamma^0 \begin{pmatrix} \Lambda_+ & 0_2 \\ 0_2 & \Lambda_- \end{pmatrix} \gamma^0 = \begin{pmatrix} \Lambda_- & 0_2 \\ 0_2 & \Lambda_+ \end{pmatrix} = S[\Lambda]^{-1}. \tag{5.4}$$

The boost components $\Lambda_\pm$ can be decomposed into their 'even' and 'odd' parts as shown in the following lemma.

**Lemma 5.1.** *For the Lorentz boost components (5.1):*

$$\Lambda_\pm := \exp(\pm \mathbf{a} \cdot \boldsymbol{\sigma}) \qquad \textit{with} \quad \mathbf{a} := (b/2)\mathbf{n}, \tag{5.5}$$

*where $(b/2)$ is the rapidity and $\mathbf{n}$ is the direction of the boost, one has*

$$\Lambda_\pm = \Lambda_e \pm \Lambda_o \qquad \textit{with} \quad \begin{cases} \Lambda_e & := (\cosh |\mathbf{a}|)\, \mathbb{I}_2 \\ \Lambda_o & := (\sinh |\mathbf{a}|)\, \mathbf{n} \cdot \boldsymbol{\sigma} \end{cases}. \tag{5.6}$$

**Proof.** Using $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}_2$, one gets

$$\begin{aligned} (\pm \mathbf{a} \cdot \boldsymbol{\sigma})^2 &= (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)^2 \\ &= (a_1^2 + a_2^2 + a_3^2)\, \mathbb{I}_2 \;=\; |\mathbf{a}|^2\, \mathbb{I}_2. \end{aligned}$$

Collecting the terms with even and odd powers in the expansion of $\exp(\pm \mathbf{a} \cdot \boldsymbol{\sigma})$, one obtains

$$\begin{aligned} \Lambda_\pm &= \sum_{n=0}^{\infty} \frac{(\pm \mathbf{a} \cdot \boldsymbol{\sigma})^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(\pm \mathbf{a} \cdot \boldsymbol{\sigma})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(\pm \mathbf{a} \cdot \boldsymbol{\sigma})^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{|\mathbf{a}|^{2k}}{(2k)!}\, \mathbb{I}_2 \pm \sum_{k=0}^{\infty} \frac{|\mathbf{a}|^{2k}}{(2k+1)!}\, (\mathbf{a} \cdot \boldsymbol{\sigma}) \\ &= \sum_{k=0}^{\infty} \frac{|\mathbf{a}|^{2k}}{(2k)!}\, \mathbb{I}_2 \pm \sum_{k=0}^{\infty} \frac{|\mathbf{a}|^{2k+1}}{(2k+1)!}\, (\mathbf{n} \cdot \boldsymbol{\sigma}) \\ &= (\cosh |\mathbf{a}|)\, \mathbb{I}_2 \pm (\sinh |\mathbf{a}|)\, \mathbf{n} \cdot \boldsymbol{\sigma} \\ &=: \Lambda_e \pm \Lambda_o. \qquad \blacksquare \end{aligned}$$

The above decomposition (5.6) comes very handy in proving an important property of the real structure $\mathcal{J}$ and the Lorentz boost $S[\Lambda]$ given below.

**Proposition 5.2.** *The Lorentz boosts twist-commute with the real structure $\mathcal{J}$ (4.4) of a four-dimensional riemannian manifold, i.e.*

$$\mathcal{J}S[\Lambda] - \rho(S[\Lambda])\mathcal{J} = 0. \tag{5.7}$$

**Proof.** For $\mathcal{J} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} cc$ and $S[\Lambda]$ as in (5.1), we have

$$\begin{aligned} \mathcal{J}S[\Lambda] &= \begin{pmatrix} -\sigma_2\overline{\Lambda}_+ & 0 \\ 0 & \sigma_2\overline{\Lambda}_- \end{pmatrix} cc, \\ \text{and} \qquad S^{-1}[\Lambda]\mathcal{J} &= \begin{pmatrix} -\Lambda_-\sigma_2 & 0 \\ 0 & \Lambda_+\sigma_2 \end{pmatrix} cc, \end{aligned} \tag{5.8}$$

where we use $cc\Lambda_\pm = \overline{\Lambda}_\pm cc$, with the bar denoting the complex conjugation. For the decomposition (5.6), we have

$$\overline{\Lambda}_\pm = \overline{\Lambda}_e \pm \overline{\Lambda}_o = \Lambda_e \pm \overline{\Lambda}_o, \tag{5.9}$$

where

$$\overline{\Lambda}_o = \sum_{k=0}^{\infty} \frac{|\mathbf{a}|^{2k}}{(2k+1)!}\, \overline{(\mathbf{a}\cdot\boldsymbol{\sigma})}, \tag{5.10}$$

with

$$\overline{(\mathbf{a}.\boldsymbol{\sigma})} = a_1\sigma_1 - a_2\sigma_2 + a_3\sigma_3.$$

Further, recalling that

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\sigma_2 = -\sigma_2\overline{(\mathbf{a}\cdot\boldsymbol{\sigma})}$$

due to $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}_2$, we notice that

$$\Lambda_o\sigma_2 = -\sigma_2\overline{\Lambda}_o, \tag{5.11}$$

whence

$$\begin{aligned} \Lambda_\pm\sigma_2 &= \Lambda_e\sigma_2 \pm \Lambda_o\sigma_2 = \sigma_2\Lambda_e \mp \sigma_2\overline{\Lambda}_o \\ &= \sigma_2(\Lambda_e \mp \overline{\Lambda}_o) = \sigma_2\overline{(\Lambda_e \mp \Lambda_o)} = \sigma_2\overline{\Lambda}_\mp. \end{aligned}$$

Then, (5.8) reads as $\mathcal{J}S[\Lambda] = S^{-1}[\Lambda]\mathcal{J}$, and the result follows from (5.4). ∎

As we recalled in the Introduction, the $\rho$-inner product of euclidean spinors coincides with the Kreĭn product of lorentzian spinors. For the action (5.3) of the boost to be coherent, it should leave this product invariant. This is indeed the case as we shall see in what follows.

**Lemma 5.3.** *The canonical $\rho$-inner product (3.17) of a minimally twisted four-dimensional riemannian manifold is invariant under action (5.3) of the Lorentz boost $S[\Lambda]$.*

**Proof.** The product $\langle\psi,\phi\rangle_\rho$ is mapped to (omitting the argument $\Lambda$ of S)

$$\begin{aligned}\langle\psi_\Lambda,\phi_\Lambda\rangle_\rho &= \langle S\psi, S\phi\rangle_\rho = \langle\psi, S^+S\phi\rangle_\rho\\ &= \langle\psi,\rho(S)^\dagger S\phi\rangle_\rho = \langle\psi,(S^{-1})^\dagger S\phi\rangle_\rho\\ &= \langle\psi, S^{-1}S\phi\rangle_\rho = \langle\psi,\phi\rangle_\rho,\end{aligned}$$

using (5.4) and the fact that $S^{-1}$ is Hermitian. ■

**Corollary 5.3.1.** *The fermionic action of a minimally twisted four-dimensional euclidean manifold is boost invariant.*

**Proof.** Using $\mathcal{J}S = S^{-1}\mathcal{J}$ from Prop. 5.2 and (5.4), one has that

$$\begin{aligned}\mathcal{J}\psi_\Lambda &= \mathcal{J}S\psi = S^{-1}\mathcal{J}\psi,\\ \text{and}\qquad \eth_\Lambda\phi_\Lambda &= \rho(S)\,\eth\, S^{-1}S\phi = S^{-1}\eth\phi.\end{aligned}\tag{5.12}$$

By Lem. 5.3, one then gets

$$\langle\mathcal{J}\psi_\Lambda,\eth_\Lambda\phi_\Lambda\rangle_\rho = \langle S^{-1}\mathcal{J}\psi, S^{-1}\eth\phi\rangle_\rho = \langle\mathcal{J}\psi,\eth\phi\rangle_\rho. \qquad ■$$

## 5.2 Lichnerowicz formula for minimally twisted manifold

To compute the Seeley-de Witt coefficients in the asymptotic expansion (2.80) of the spectral action $S^b$ (2.79), one uses the standard local formula for the heat-kernel expansion [Gi, §4.8] on the square $\mathcal{D}_\omega^2$ of the fluctuated Dirac operator $\mathcal{D}_\omega$.

Let us first recall here the statement of [Gi, Lem. 4.8.1].

**Theorem 5.4.** *Given a differential operator* $D$ *acting on the sections of a vector bundle* $\mathcal{V}$ *on a compact riemannian manifold* $(\mathcal{M}, g)$ *with the leading symbol given by the metric tensor. That is,* $D$ *has the local form*

$$D = -\left(g^{\mu\nu}\mathbb{I}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B\right), \tag{5.13}$$

*where* $g^{\mu\nu}$ *is the inverse metric,* $\mathbb{I}$ *the identity matrix, and* $A^\mu$ *and* $B$ *are endomorphisms of* $\mathcal{V}$*. Then,* $D$ *can uniquely be written as*

$$D = \nabla^*\nabla - E, \tag{5.14}$$

*where* $\nabla$ *is a connection on* $\mathcal{V}$ *with the associated laplacian* $\nabla^*\nabla$ *and* $E$ *is an endomorphism of* $\mathcal{V}$*. Explicitly, one has that*

$$\nabla_\mu := \partial_\mu + \omega_\mu, \qquad \omega_\mu := \frac{1}{2}g_{\mu\lambda}\left(\alpha^\lambda + \Gamma^\lambda \cdot \mathrm{id}\right), \quad \textit{with } \Gamma^\lambda := g^{\mu\nu}\Gamma^\lambda_{\mu\nu}, \tag{5.15}$$

*where* $\mathrm{id}$ *is the identity endomorphism of* $\mathcal{V}$ *and* $\Gamma^\lambda_{\mu\nu}$ *are the Christoffel symbols of the Levi-Civita connection of the metric* $g$*; and*

$$E = B + \left(\Gamma^\nu \cdot \mathrm{id} - g^{\mu\nu}\nabla_\mu\right)\omega_\nu. \tag{5.16}$$

Now, we fix following the notation:

$$\eth_{\mathcal{S}} := -i\gamma^\mu\nabla^{\mathcal{S}}_\mu, \qquad \text{where} \quad \nabla^{\mathcal{S}}_\mu := \partial_\mu + \omega^{\mathcal{S}}_\mu, \tag{5.17}$$

and similarly

$$\eth_X := -i\gamma^\mu\nabla^X_\mu, \qquad \text{where} \quad \nabla^X_\mu := \partial_\mu + \omega^X_\mu, \tag{5.18}$$

with

$$\omega^X_\mu := \omega^{\mathcal{S}}_\mu + X_\mu \qquad \text{and} \qquad \boldsymbol{\omega}_X := -i\gamma^\mu\omega^X_\mu. \tag{5.19}$$

Thus, we have that

$$\nabla^X_\mu = \nabla^{\mathcal{S}}_\mu + X_\mu. \tag{5.20}$$

With that under the belt, we now give an expression for $\eth_X^2$ as an elliptic operator of the laplacian type (5.13), in order to write down a generalized Lichnerowicz formula (5.14) for it, using Theorem 5.4.

**Proposition 5.5.** *The squared twisted-covariant Dirac operator can be written as*

$$\eth_X^2 = -\big(g^{\mu\nu}\partial_\mu\partial_\nu + \alpha^\nu\partial_\nu + \beta\big), \tag{5.21}$$

*where*

$$\alpha^\nu = i\big(\eth\gamma^\nu\big) + i\big\{\omega_X, \gamma^\nu\big\}, \qquad \beta = -\big(\eth\omega_X\big) - \omega_X^2. \tag{5.22}$$

**Proof.** We have

$$\eth_X^2 \;=\; \big(\eth + \omega_X\big)^2 \;=\; \eth^2 + \omega_X\eth + \eth\omega_X + \omega_X^2. \tag{5.23}$$

The first term of (5.23) is

$$\begin{aligned} \eth^2 &= \big(-i\gamma^\mu\partial_\mu\big)\big(-i\gamma^\nu\partial_\nu\big) \\ &= -\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu \\ &= -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - \gamma^\mu\big(\partial_\mu\gamma^\nu\big)\partial_\nu \\ &= -g^{\mu\nu}\partial_\mu\partial_\nu - i\big(\eth\gamma^\nu\big)\partial_\nu, \end{aligned} \tag{5.24}$$

where the last equality holds by using the identity $\gamma^\mu\gamma^\nu = \frac{1}{2}\big[\gamma^\mu, \gamma^\nu\big] + g^{\mu\nu}\mathbb{I}$ and the symmetry $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ as following:

$$\begin{aligned} \big[\gamma^\mu, \gamma^\nu\big]\partial_\mu\partial_\nu &= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - \gamma^\nu\gamma^\mu\partial_\mu\partial_\nu \\ &= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - \gamma^\mu\gamma^\nu\partial_\nu\partial_\mu \\ &= \gamma^\mu\gamma^\nu\big[\partial_\mu, \partial_\nu\big] \\ &= 0. \end{aligned}$$

The second and third terms of (5.23), respectively, are

$$\begin{aligned} \omega_X\eth &= -i\omega_X\gamma^\nu\partial_\nu, \\ \eth\omega_X &= -i\gamma^\nu\partial_\nu\omega_X \\ &= -i\gamma^\nu\omega_X\partial_\nu - i\gamma^\nu\big(\partial_\nu\omega_X\big) \\ &= -i\gamma^\nu\omega_X\partial_\nu + \big(\eth\omega_X\big), \end{aligned} \tag{5.25}$$

Substituting (5.24) and (5.25) into (5.23), the result follows:

$$\eth_X^2 = -g^{\mu\nu}\partial_\mu\partial_\nu - \big(i\big(\eth\gamma^\nu\big) + i\omega_X\gamma^\nu + i\gamma^\nu\omega_X\big)\,\partial_\nu + \big(\eth\omega_X\big) + \omega_X^2. \qquad \blacksquare$$

In accordance with the notation (5.17–5.20), we define the covariant derivatives associated with the adjoint action of the corresponding connections as

$$\mathfrak{D}^{X}_{\mu} := \partial_{\mu} + [\omega^{X}_{\mu}, \cdot] \quad \text{and} \quad \mathfrak{D}^{\mathcal{S}}_{\mu} := \partial_{\mu} + [\omega^{\mathcal{S}}_{\mu}, \cdot], \tag{5.26}$$

using which we now give two lemmata that will be useful for the subsequent proofs.

**Lemma 5.6.** *One has*

$$\gamma^{\mu}\mathfrak{D}^{X}_{\mu}\gamma^{\nu} = -\Gamma^{\nu}\mathbb{I} - 2\gamma^{\mu}\gamma^{\nu}X_{\mu}. \tag{5.27}$$

**Proof.** Since the commutator $[\partial_{\mu}, \gamma^{\lambda}]$ acts on the Hilbert space $L^2(\mathcal{M}, \mathcal{S})$ as the bounded operator $(\partial_{\mu}\gamma^{\lambda})$, Def. (5.26) gives

$$\mathfrak{D}^{\mathcal{S}}_{\mu}\gamma^{\lambda} = (\partial_{\mu}\gamma^{\lambda}) + [\omega^{\mathcal{S}}_{\mu}, \gamma^{\lambda}] = [\nabla^{\mathcal{S}}_{\mu}, \gamma^{\lambda}]. \tag{5.28}$$

If $c$ denotes the Clifford action: $c(dx^{\lambda}) = \gamma^{\lambda}$, then by definition of the spin connection,

$$[\nabla^{\mathcal{S}}_{\mu}, c(dx^{\lambda})] = c(\nabla^{LV}_{\mu} dx^{\lambda}) = c(-\Gamma^{\lambda}_{\mu\nu} dx^{\nu}) = -\Gamma^{\lambda}_{\mu\nu} c(dx^{\nu}), \tag{5.29}$$

where $\nabla^{LV}$ denotes the covariant derivative (on the cotangent bundle $\mathcal{T}^*\mathcal{M}$) associated to the Levi-Civita connection, and we used the linearity of the Clifford action. (5.29) in terms of $\gamma$-matrices gives, for any $\mu, \nu$:

$$\mathfrak{D}^{\mathcal{S}}_{\mu}\gamma^{\lambda} = -\Gamma^{\lambda}_{\mu\nu}\gamma^{\lambda}. \tag{5.30}$$

From (5.19) and (5.26), we have that $\mathfrak{D}^{X}_{\mu} := \mathfrak{D}^{\mathcal{S}}_{\mu} + [X_{\mu}, \cdot]$, which acting on $\gamma^{\nu}$ becomes

$$\mathfrak{D}^{X}_{\mu}\gamma^{\nu} = \mathfrak{D}^{\mathcal{S}}_{\mu}\gamma^{\nu} + [X_{\mu}, \gamma^{\nu}] = -\Gamma^{\nu}_{\mu\kappa}\gamma^{\kappa} - 2\gamma^{\nu}X_{\mu}, \tag{5.31}$$

where

$$\left[X_{\mu}, \gamma^{\nu}\right] = X_{\mu}\gamma^{\nu} - \gamma^{\nu}X_{\mu} = \gamma^{\nu}\rho(X_{\mu}) - \gamma^{\nu}X_{\mu} = -2\gamma^{\nu}X_{\mu},$$

since $X_{\mu}\gamma^{\nu} = \gamma^{\nu}\rho(X_{\mu})$ with $X_{\mu} = f_{\mu}\gamma^5$, so $\rho(X_{\mu}) = -X_{\mu}$. Further, multiplying (5.31) by $\gamma^{\mu}$, we get

$$\gamma^{\mu}\mathfrak{D}^{X}_{\mu}\gamma^{\nu} = -\Gamma^{\nu}_{\mu\kappa}\gamma^{\mu}\gamma^{\kappa} - 2\gamma^{\mu}\gamma^{\nu}X_{\mu}. \tag{5.32}$$

Using the identity $\gamma^{\mu}\gamma^{\kappa} = g^{\mu\kappa}\mathbb{I} + \frac{1}{2}\left[\gamma^{\mu}, \gamma^{\kappa}\right]$ and the symmetry $\Gamma^{\nu}_{\mu\kappa} = \Gamma^{\nu}_{\kappa\mu}$,

$$\begin{aligned} \Gamma^{\nu}_{\mu\kappa}\gamma^{\mu}\gamma^{\kappa} &= \Gamma^{\nu}_{\mu\kappa}g^{\mu\kappa}\mathbb{I} + \frac{1}{2}\Gamma^{\nu}_{\mu\kappa}\left[\gamma^{\mu}, \gamma^{\kappa}\right] \\ &= \Gamma^{\nu}\mathbb{I} + \frac{1}{2}\left[\Gamma^{\nu}_{\mu\kappa}, \Gamma^{\nu}_{\kappa\mu}\right]\gamma^{\mu}\gamma^{\kappa} = \Gamma^{\nu}\mathbb{I}, \end{aligned} \tag{5.33}$$

substituting which in (5.32) the result follows. ■

**Corollary 5.6.1.** *One has*

$$\mathfrak{D}^X_\mu(\gamma^\lambda\gamma^\mu) = -\Gamma^\lambda\mathbb{I} - \gamma^\lambda\gamma^\mu\Gamma^\mu_{\mu\nu}. \tag{5.34}$$

**Proof.** Using the Leibniz rule and (5.31), it follows that

$$\begin{aligned}\mathfrak{D}^X_\mu(\gamma^\lambda\gamma^\mu) &= \mathfrak{D}^X_\mu(\gamma^\lambda)\gamma^\mu + \gamma^\lambda\mathfrak{D}^X_\mu(\gamma^\mu)\\ &= \big(-\Gamma^\lambda_{\mu\nu}\gamma^\nu - 2\gamma^\lambda X_\mu\big)\gamma^\mu + \gamma^\lambda\big(-\Gamma^\mu_{\mu\nu}\gamma^\nu - 2\gamma^\mu X_\mu\big)\\ &= -\gamma^\nu\gamma^\mu\Gamma^\lambda_{\mu\nu} + 2\gamma^\lambda\gamma^\mu X_\mu - \gamma^\lambda\gamma^\nu\Gamma^\mu_{\mu\nu} - 2\gamma^\lambda\gamma^\mu X_\mu\\ &= -\gamma^\nu\gamma^\mu\Gamma^\lambda_{\mu\nu} - \gamma^\lambda\gamma^\nu\Gamma^\mu_{\mu\nu}\\ &= -\Gamma^\lambda_{\mu\nu}\mathbb{I} - \gamma^\lambda\gamma^\nu\Gamma^\mu_{\mu\nu},\end{aligned}$$

where we used the fact that $X_\mu = f_\mu\gamma^5$ anticommutes with any $\gamma$-matrix and the last equality follows from (5.33). ■

**Lemma 5.7.** *One has*

$$g^{\mu\nu}(\mathfrak{D}^X_\mu g_{\nu\kappa}\mathbb{I}) = (\Gamma_\kappa + \Gamma^\mu_{\mu\kappa})\mathbb{I}. \tag{5.35}$$

**Proof.** Since $g_{\nu\kappa}\mathbb{I}$ is a multiple of the identity matrix; for any $\mu, \nu, \kappa$ one has

$$\mathfrak{D}^X_\mu g_{\nu\kappa}\mathbb{I} = \mathfrak{D}^S_\mu g_{\nu\kappa}\mathbb{I}.$$

By Leibniz rule, one has

$$\mathfrak{D}^S_\mu(g^{\mu\nu}g_{\nu\kappa}\mathbb{I}) = \begin{cases}\mathfrak{D}^S_\mu(\delta^\mu_\kappa\mathbb{I}) = 0\\ g^{\mu\nu}\mathfrak{D}^X_\mu(g_{\nu\kappa}\mathbb{I}) + \mathfrak{D}^S_\mu(g^{\mu\nu}\mathbb{I})g_{\nu\kappa}\end{cases}$$

Hence,

$$g^{\mu\nu}\mathfrak{D}^X_\mu(g_{\nu\kappa}\mathbb{I}) = -g_{\nu\kappa}\mathfrak{D}^S_\mu(g^{\mu\nu}\mathbb{I}) = -\frac{1}{2}g_{\nu\kappa}\mathfrak{D}^S_\mu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu). \tag{5.36}$$

By (5.30), we have

$$\begin{aligned}\mathfrak{D}^S_\mu(\gamma^\mu\gamma^\nu) &= -\gamma^\mu\Gamma^\nu_{\mu\lambda}\gamma^\lambda - \Gamma^\mu_{\mu\lambda}\gamma^\lambda\gamma^\nu = -\Gamma^\nu_{\mu\lambda}\gamma^\mu\gamma^\lambda - \Gamma^\mu_{\mu\lambda}\gamma^\lambda\gamma^\nu,\\ \mathfrak{D}^S_\mu(\gamma^\nu\gamma^\mu) &= -\gamma^\nu\Gamma^\mu_{\mu\lambda}\gamma^\lambda - \Gamma^\nu_{\mu\lambda}\gamma^\lambda\gamma^\mu = -\Gamma^\mu_{\mu\lambda}\gamma^\nu\gamma^\lambda - \Gamma^\nu_{\mu\lambda}\gamma^\lambda\gamma^\mu,\end{aligned}$$

so that

$$\begin{aligned}\mathfrak{D}^S_\mu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) &= \mathfrak{D}^S_\mu(\gamma^\mu\gamma^\nu) + \mathfrak{D}^S_\mu(\gamma^\nu\gamma^\mu)\\ &= -2\big(\Gamma^\mu_{\mu\lambda}g^{\mu\lambda} + \Gamma^\mu_{\mu\lambda}g^{\nu\lambda}\big)\mathbb{I} = -2\big(\Gamma^\nu + g^{\nu\lambda}\Gamma^\mu_{\mu\lambda}\big)\mathbb{I},\end{aligned}$$

and, therefore, (5.36) give

$$g^{\mu\nu}\mathfrak{D}^X_\mu(g_{\nu\kappa}\mathbb{I}) = g_{\nu\kappa}\big(\Gamma^\nu + g^{\nu\lambda}\Gamma^\mu_{\mu\lambda}\big)\mathbb{I} = (\Gamma_\kappa + \Gamma^\mu_{\mu\kappa})\mathbb{I}. \quad ■$$

Moving forward, we make use of the above Lem. 5.6 to obtain explicit expressions for the endomorphism terms $\alpha^\nu$ and $\beta$ in (5.22).

**Proposition 5.8.** *One has that*

$$\begin{aligned}\alpha^\nu &= 2g^{\mu\nu}\omega^X_\mu - 2\gamma^\mu\gamma^\nu X_\mu - \Gamma^\nu\mathbb{I},\\ \beta &= \gamma^\mu\gamma^\nu\big(\nabla^{\mathcal{S}}_\mu - X_\mu\big)\omega^X_\nu - \Gamma^\nu\omega^X_\nu.\end{aligned} \tag{5.37}$$

**Proof.** From (5.22), writing $\alpha^\nu$ as

$$\begin{aligned}\alpha^\nu &= \{\gamma^\nu, \gamma^\mu\omega^X_\mu\} + \gamma^\mu(\partial_\mu\gamma^\nu)\\ &= \gamma^\nu\gamma^\mu\omega^X_\mu + \gamma^\mu\omega^X_\mu\gamma^\nu + \gamma^\mu(\partial_\mu\gamma^\nu)\\ &= \gamma^\nu\gamma^\mu\omega^X_\mu + \gamma^\mu\gamma^\nu\omega^X_\mu + \gamma^\mu[\omega^X_\mu, \gamma^\nu] + \gamma^\mu(\partial_\mu\gamma^\nu)\\ &= \{\gamma^\mu, \gamma^\nu\}\omega^X_\mu + \gamma^\mu\mathfrak{D}^X_\mu\gamma^\nu\\ &= 2g^{\mu\nu}\omega^X_\mu - \Gamma^\nu\mathbb{I} - 2\gamma^\mu\gamma^\nu X_\mu,\end{aligned}$$

and the first result follows, using Lem. 5.6 and the identity $\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + g^{\mu\nu}\mathbb{I}$.

Next, $\beta$ in (5.22) can be expanded as

$$\begin{aligned}\beta &= \gamma^\mu\omega^X_\mu\gamma^\nu\omega^X_\nu + \gamma^\mu(\partial_\mu\gamma^\nu\omega^X_\nu)\\ &= \gamma^\mu[\omega^X_\mu, \gamma^\nu]\omega^X_\nu + \gamma^\mu\gamma^\nu\omega^X_\mu\omega^X_\nu + \gamma^\mu(\partial_\mu\gamma^\nu)\omega^X_\nu + \gamma^\mu\gamma^\nu(\partial_\mu\omega^X_\nu)\\ &= \gamma^\mu\gamma^\nu(\partial_\mu + \omega^X_\mu)\omega^X_\nu + \gamma^\mu\left((\partial_\mu\gamma^\nu) + [\omega^X_\mu, \gamma^\nu]\right)\omega^X_\nu\\ &= \gamma^\mu\gamma^\nu(\partial_\mu + \omega^{\mathcal{S}}_\mu + X_\mu)\omega^X_\nu + \gamma^\mu(\mathfrak{D}^X_\mu\gamma^\nu)\omega^X_\nu\\ &= \gamma^\mu\gamma^\nu(\nabla^{\mathcal{S}}_\mu + X_\mu)\omega^X_\nu - \Gamma^\nu\omega^X_\nu - 2\gamma^\mu\gamma^\nu X_\mu\omega^X_\nu\end{aligned}$$

and the second result follows. ■

For the connection $\omega$ defined in (5.15), we denote

$$\boldsymbol{\omega} := -i\gamma^\mu\omega_\mu, \tag{5.38}$$

recalling the notation $\mathbf{X} := -i\gamma^\mu X_\mu$, and obtain the following relation between them.

**Lemma 5.9.** *One has*

$$\omega_\mu = \omega^X_\mu - \chi_\mu \quad \textit{and} \quad \boldsymbol{\omega} = \boldsymbol{\omega}_X + 2\mathbf{X}, \tag{5.39}$$

*where $\omega^X_\mu$ and $\boldsymbol{\omega}_X$ are as in (5.19), and we have defined*

$$\chi_\mu := \gamma^\nu\gamma_\mu X_\nu. \tag{5.40}$$

**Proof.** Substituting $\alpha^\nu$, from Prop. 5.8, in $\omega_\mu$ given by (5.15), we have

$$
\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + \Gamma^\nu \cdot \mathrm{id}\right) \\
&= \frac{1}{2} g_{\mu\nu}\left(2g^{\lambda\nu}\omega^X_\lambda - 2\gamma^\lambda\gamma^\nu X_\lambda\right) \\
&= g_{\mu\nu}g^{\lambda\nu}\omega^X_\lambda - g_{\mu\nu}\gamma^\lambda\gamma^\nu X_\lambda \\
&= \omega^X_\mu - \gamma^\lambda\gamma_\mu X_\lambda,
\end{aligned}
$$

and the first result follows identifying $\gamma^\lambda\gamma_\mu X_\lambda =: \chi_\mu$. Further, multiplying the first result by $-i\gamma^\mu$, the second result is obtained

$$
\begin{aligned}
-i\gamma^\mu\omega_\mu &= -i\gamma^\mu\omega^X_\mu + i\gamma^\mu\gamma^\lambda\gamma_\mu X_\lambda, \\
\text{i.e.} \qquad \boldsymbol{\omega} &= \boldsymbol{\omega}_X - 2i\gamma^\lambda X_\lambda \\
&= \boldsymbol{\omega}_X + 2\mathbf{X},
\end{aligned}
$$

where we used the identity $\gamma^\mu\gamma^\lambda\gamma_\mu = -2\gamma^\lambda$. ■

We now give another lemma and use to compute the endomorphism $E$ that gives the potential terms in the spectral action of the minimally twisted manifold.

**Lemma 5.10.** *One has the following relations*

$$
g^{\mu\nu}\mathfrak{D}^X_\mu\chi_\nu = \gamma^\lambda\gamma^\mu\mathfrak{D}^X_\mu X_\lambda + \Gamma_\kappa\gamma^\lambda\gamma^\kappa X_\lambda - \Gamma^\lambda X_\lambda, \tag{5.41}
$$

$$
\chi\cdot\chi = -2\gamma^\lambda\gamma^\kappa X_\lambda X_\kappa, \qquad \Gamma^\mu\chi_\mu = \Gamma_\kappa\gamma^\lambda\gamma^\kappa X_\lambda. \tag{5.42}
$$

**Proof.** Using (5.40), the Leibniz rule for $\mathfrak{D}^X_\mu$ as in Lem. 5.6 and Cor. 5.6.1, we expand as following

$$
g^{\mu\nu}\mathfrak{D}^X_\mu\chi_\nu = g^{\mu\nu}\left(\mathfrak{D}^X_\mu\gamma^\lambda\gamma_\nu X_\lambda\right) = g^{\mu\nu}\gamma^\lambda\gamma_\nu\mathfrak{D}^X_\mu X_\lambda + g^{\mu\nu}(\mathfrak{D}^X_\mu\gamma^\lambda\gamma_\nu)X_\lambda
$$

where the first term is $\gamma^\lambda\gamma^\mu\mathfrak{D}^X_\mu X_\lambda$ and the second term becomes

$$
\begin{aligned}
g^{\mu\nu}(\mathfrak{D}^X_\mu\gamma^\lambda\gamma_\nu)X_\lambda &= g^{\mu\nu}(\mathfrak{D}^X_\mu g_{\nu\kappa}\gamma^\lambda\gamma^\kappa)X_\lambda \\
&= g^{\mu\nu}(\mathfrak{D}^X_\mu g_{\nu\kappa})\gamma^\lambda\gamma^\kappa X_\lambda + g^{\mu\nu}g_{\nu\kappa}\mathfrak{D}^X_\mu(\gamma^\lambda\gamma^\kappa)X_\lambda \\
&= (\Gamma_\kappa + \Gamma^\mu_{\mu\kappa})\gamma^\lambda\gamma^\kappa X_\lambda + \mathfrak{D}^X_\mu(\gamma^\lambda\gamma^\mu)X_\lambda \\
&= (\Gamma_\kappa + \Gamma^\mu_{\mu\kappa})\gamma^\lambda\gamma^\kappa X_\lambda - (\Gamma^\lambda\mathbb{I} + \gamma^\lambda\gamma^\mu\Gamma^\mu_{\mu\nu})X_\lambda \\
&= \Gamma_\kappa\gamma^\lambda\gamma^\kappa X_\lambda - \Gamma^\lambda X_\lambda,
\end{aligned}
$$

and so the first result follows. In the same manner, we obtain

$$\Gamma^\nu \chi_\nu = \Gamma^\mu \gamma^\lambda \gamma_\mu X_\lambda = \Gamma_\kappa g^{\mu\kappa} \gamma^\lambda \gamma_\mu X_\lambda = \Gamma_\kappa \gamma^\lambda \gamma^\kappa X_\lambda,$$

and

$$\begin{aligned}\chi \cdot \chi = g^{\mu\nu} \chi_\mu \chi_\nu &= g^{\mu\nu} (\gamma^\lambda \gamma_\mu X_\lambda)(\gamma^\kappa \gamma_\nu X_\kappa) \\ &= g^{\mu\nu} \gamma^\lambda \gamma_\mu \gamma^\kappa \gamma_\nu X_\lambda X_\kappa \\ &= g_{\mu\nu} \gamma^\lambda \gamma^\nu \gamma^\kappa \gamma^\mu X_\lambda X_\kappa = -2\gamma^\lambda \gamma^\kappa X_\lambda X_\kappa. \qquad \blacksquare\end{aligned}$$

.

**Proposition 5.11.** *The endomorphism term* (5.16) *for the Lichnerowicz formula* (5.14) *of the twisted-covariant Dirac operator* $\eth_X$ *is*

$$E = \frac{1}{2}\gamma^\mu \gamma^\nu \big( F^X_{\mu\nu} + 2\,\mathfrak{D}^X_\nu X_\mu + 4\,X_\mu X_\nu \big) - \Gamma^\mu X_\mu, \tag{5.43}$$

*where*

$$F^X_{\mu\nu} := \nabla^X_\mu \omega^X_\nu - \nabla^X_\nu \omega^X_\mu \tag{5.44}$$

*is the field strength of the connection* $\omega^X_\mu$ *and* $\mathfrak{D}^X_\mu$ *is the covariant derivative* (5.26) *of its adjoint action.*

**Proof.** Substituting $\beta$ from Prop. 5.8 into $E$ (5.16) and using $\omega_\mu = \omega^X_\mu - \chi_\mu$ of Lem. 5.9 in the form $\nabla_\mu = \nabla^X_\mu - \chi_\mu$, one has

$$\begin{aligned}E &= \gamma^\mu \gamma^\nu \big( \nabla^S_\mu - X_\mu \big) \omega^X_\nu - g^{\mu\nu} (\nabla^X_\mu - \chi_\mu) \omega_\nu - \Gamma^\nu (\omega^X_\mu - \omega_\mu) \\ &= \gamma^\mu \gamma^\nu \big( \nabla^S_\mu - X_\mu \big) \omega^X_\nu - g^{\mu\nu} \nabla^X_\mu \omega^X_\nu + g^{\mu\nu} \nabla^X_\mu \chi_\nu + g^{\mu\nu} \chi_\nu \omega_\mu - \Gamma^\nu \chi_\nu \\ &= \gamma^\mu \gamma^\nu \big( \nabla^S_\mu - X_\mu \big) \omega^X_\nu - \gamma^\mu \gamma^\nu \nabla^X_\mu \omega^X_\nu + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \nabla^X_\mu \omega^X_\nu \\ &\qquad + g^{\mu\nu} (\mathfrak{D}^X_\mu \chi_\nu + 2\chi_\nu \omega^X_\mu + \chi_\nu \omega_\mu) - \Gamma^\nu \chi_\nu \qquad (5.45) \\ &= \gamma^\mu \gamma^\nu \big( \nabla^S_\mu - X_\mu - \nabla^X_\mu \big) \omega^X_\nu + \frac{1}{2} \gamma^\mu \gamma^\nu (\nabla^X_\mu \omega^X_\nu - \nabla^X_\nu \omega^X_\mu) \\ &\qquad + g^{\mu\nu} \mathfrak{D}^X_\mu \chi_\nu + 2 g^{\mu\nu} \chi_\nu \omega^X_\mu - g^{\mu\nu} \chi_\nu \chi_\mu - \Gamma^\nu \chi_\nu \\ &= -2\gamma^\mu \gamma^\nu X_\mu \omega^X_\nu + \frac{1}{2} \gamma^\mu \gamma^\nu F^X_{\mu\nu} \\ &\qquad + g^{\mu\nu} \mathfrak{D}^X_\mu \chi_\nu + 2\gamma^\mu \gamma^\nu X_\mu \omega^X_\nu - \chi \cdot \chi - \Gamma^\nu \chi_\nu \qquad (5.46) \\ &= \frac{1}{2} \gamma^\mu \gamma^\nu F^X_{\mu\nu} + g^{\mu\nu} \mathfrak{D}^X_\mu \chi_\nu - \chi \cdot \chi - \Gamma^\nu \chi_\nu, \qquad (5.47)\end{aligned}$$

where in (5.45) we used the identity $g^{\mu\nu}\mathbb{I} = \gamma^\mu\gamma^\nu - \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and the relation from (5.19)

$$\mathfrak{D}^X_\mu \chi_\nu = \partial_\mu \chi_\nu + [\omega^X_\mu, \chi_\nu] = \nabla^X_\mu \chi_\nu - \chi_\nu \omega^X_\mu; \tag{5.48}$$

and in (5.46) we used (5.20) and (5.40). Finally, substituting in (5.47) the expressions from the previous Lem. 5.10 directly yields the result. ■

Prop. 5.11, of course, reduces to the correct expression in the flat case [DM, Prop. 5.3]. The only difference is that of the last term $\Gamma^\mu X_\mu$, which asserts a coupling between the $X_\mu$ field and the curvature. It is tempting to speculate that $X_\mu$ has something to do with torsion due to its form $X_\mu = -i\gamma^\mu f_\mu \gamma^5$ (4.14), which also appears as a modification of the spin connection (with curvature) in the above analysis. However, this is yet to be confirmed by fully computing the spectral action.

A related result in this context is [HPS], where the spectral action for pure gravity with torsion is calculated. The (skew-symmetric) torsion is incorporated into the *twisted* Dirac operators, which are twisted in a different sense than what we mean in our context. A comparative study might shed some light on the geometric understanding of the $X_\mu$ field.

# Conclusions

Here we conclude the thesis highlighting the key results with some passing comments.

The fermionic actions associated to the minimal twist of the spectral triples of a $U(1)$ gauge theory and electrodynamics respectively, yield the Weyl and the Dirac equations in lorentzian signature, although one starts with the euclidean signature (Prop. 4.10 and 4.19). That a similar transition of metric signature (from the riemannian to the pseudo-riemannian) at the level of fermionic action happens also for the minimal twist of the Standard Model should be checked. This will be the subject-matter of future works.

At any rate, these results we put forward here strengthen the suggestion of twisting non-commutative geometries as an alternative way to approach the problem of extending the theory of spectral triples to lorentzian manifolds. The fact that the twist does not satisfactorily implement the Wick rotation – it does so only for the Hilbert space – is not so relevant after all. What is far more important and interesting from a physical point of view than giving a purely spectral characterization of pseudo-riemannian manifolds is to arrive at an action that is meaningful in a lorentzian context. This thesis makes the case that it occurs for minimally twisted spectral triples, at least at the level of the fermionic action.

This reminds us of the results of [Ba] where, by dissociating the $KO$-dimension from the metric dimension, one imposes the lorentzian signature for the internal spectral triple, and thus obtains a fermionic action allowing right-handed neutrinos.

Indeed, the question of a lorentzian spectral action or the spectral action associated to twisted spectral triples remains wide open. The interpretation of the zeroth component of the real field $X_\mu$ as an energy (cf. discussions right after Prop. 4.10 and 4.19) should nevertheless play a role for the spectral action, where this field also appears (Prop. 5.11). As shown in [DM] for the twisted Standard Model that the contribution of this real field $X_\mu$ to the spectral action is minimized when $X_\mu$ vanishes, i.e. the case when no twisting occurs. Based on that and the results presented here regarding the Wick rotation of the fermionic action, one might wonder if the lorentzian (twisted) geometry is a vacuum excitation of the (non-twisted) riemannian geometry or, in other words, the twist is indeed a spontaneous breaking of the symmetry from a riemannian geometry to a pseudo-riemannian one.

The regularity condition imposed in [CMo, (3.4)] (see also Rem. 3) has its origin in Tomita-Takesaki theory (App. C). Particularly, the automorphism $\rho$ defining a twisted spectral triple should be seen as the evaluation of a one-parameter modular group $\{\rho_t\}$ of automorphisms at some specific value $t$. For the minimal twist of spectral triples, the flip (3.30) turned out to be the only possible automorphism that makes the twisted commutator bounded [LM1, Prop. 4.2]. It is not yet determined what the modular group of automorphisms corresponding to this flip would be. Should it exist, this will indicate that the time evolution in the Standard Model has its origin in such a modular group. This is precisely the essence of the 'thermal time hypothesis' proposed in [CR]. So far, this hypothesis has been applied to algebraic quantum field theory [Ma, MR], and for general considerations in quantum gravity [RS]. Its application to the Standard Model would be a novelty.

# Appendix A

# Gel'fand Duality

Gel'fand duality is an algebraic characterization of topological spaces, providing one-to-one correspondence between compact Hausdorff topological spaces and commutative $C^*$-algebras.

The following definitions are from [Su, §2.1, §4.3].

**Algebras.** An $\mathbb{F}$-*algebra* $A$ is a vector space over the field $\mathbb{F}$ with a bilinear associative product:

$$A \times A \to A, \quad (a, b) \mapsto ab, \qquad \forall a, b \in A. \tag{A.1}$$

$A$ is said to be *unital* if there exists a unit $1 \in A$ satisfying $1a = a1 = a$ $(\forall a \in A)$.

***-algebras and their representations.** An algebra $\mathcal{A}$ is called a $*$-*algebra* (or, an involutive algebra), if there exists an involution (that is, a conjugate linear map) $* : \mathcal{A} \to \mathcal{A}$ such that

$$(ab)^* = b^* a^*, \quad (a^*)^* = a, \qquad \forall a, b \in \mathcal{A}. \tag{A.2}$$

A *representation* $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is given by a $*$-algebra map

$$\pi : \mathcal{A} \to L(\mathcal{H}), \tag{A.3}$$

where $L(\mathcal{H})$ denotes the $*$-algebra of operators on $\mathcal{H}$ with the product given by composition and the involution given by hermitian conjugation.

**$C^*$-algebras and their representations.** A $C^*$-*algebra* $\mathfrak{A}$ is a complex norm-complete $*$-algebra that satisfies the $C^*$-property:

$$\|a^* a\| = \|a\|^2, \qquad \forall a \in \mathfrak{A}. \tag{A.4}$$

A *representation* $(\mathcal{H}, \pi)$ of a $C^*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is given by a $*$-algebra map

$$\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}), \tag{A.5}$$

where $\mathcal{B}(\mathcal{H})$ denotes the $*$-algebra of bounded operators on $\mathcal{H}$.

A representation $(\mathcal{H}, \pi)$ is called *irreducible* if $\mathcal{H} \neq 0$ and the only closed subspaces in $\mathcal{H}$ that are invariant under the action of $\mathfrak{A}$ are $\{0\}$ and $\mathcal{H}$ itself.

Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathfrak{A}$ are called *unitarily equivalent* if there exists a unitary map $\mathcal{U} : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_1(a) = \mathcal{U}^* \pi_2(a) \mathcal{U}, \qquad \forall a \in \mathfrak{A}. \tag{A.6}$$

Define the *structure space* $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ as the set of all unitary equivalence classes of the irreducible representations of $\mathfrak{A}$. And, let $C(\mathcal{X})$ denote the algebra of continuous $\mathbb{C}$-valued functions on a compact Hausdorff topological space $\mathcal{X}$. Then, Gel'fand duality asserts that

1. The structure space $\widehat{\mathfrak{A}}$ of a commutative (non-)unital $C^*$-algebra $\mathfrak{A}$ is a (locally) compact Hausdorff topological space, and $\mathfrak{A} \simeq C(\widehat{\mathfrak{A}})$ via the *Gel'fand transform*:

$$\mathfrak{A} \ni a \mapsto \widehat{a} \in \widehat{\mathfrak{A}}, \qquad \widehat{a}(\pi) = \pi(a). \tag{A.7}$$

2. For any compact Hausdorff topological space $\mathcal{X}$, we have $\widehat{C(\mathcal{X})} \simeq \mathcal{X}$.

# Appendix B

# Clifford Algebras

The definitions and the notations here are primarily taken from [Su, §4.1].

A **quadratic form** $\mathcal{Q}$ on a finite-dimensional $\mathbb{F}$-vector space $\mathcal{V}$ is a map $\mathcal{Q} : \mathcal{V} \to \mathbb{F}$ such that $\mathcal{Q}(\lambda v) = \lambda^2 \mathcal{Q}(v)$ for all $\lambda \in \mathcal{Q}, v \in \mathcal{V}$ and the function $\mathcal{Q}(v + w) - \mathcal{Q}(v) - \mathcal{Q}(w)$ is bilinear for all $v, w \in \mathcal{V}$.

Given a quadratic form $\mathcal{Q}$ on $\mathcal{V}$, the **Clifford algebra** $\mathrm{Cl}(\mathcal{V}, \mathcal{Q})$ is a unital associative algebra generated (over $\mathbb{F}$) by $\mathcal{V}$ satisfying $v^2 = \mathcal{Q}(v)1$ for all $v \in \mathcal{V}$.

**Property 1.** Clifford algebras are $\mathbb{Z}_2$-**graded algebras**, with **grading** $\chi$ given by

$$\chi(v_1 \cdots v_k) = (-1)^k (v_1 \cdots v_k), \tag{B.1}$$

and, thus, can be decomposed into even and odd parts, respectively, as follows:

$$\mathrm{Cl}(\mathcal{V}, \mathcal{Q}) = \mathrm{Cl}^0(\mathcal{V}, \mathcal{Q}) \oplus \mathrm{Cl}^1(\mathcal{V}, \mathcal{Q}). \tag{B.2}$$

**Property 2.** For all $v, w \in \mathcal{V}$, one has $vw + wv = 2g_{\mathcal{Q}}(v, w)$, where the symmetric bilinear form $g_{\mathcal{Q}} : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ associated to $\mathcal{Q}$ is given by

$$g_{\mathcal{Q}}(v, w) = \frac{1}{2}\big(\mathcal{Q}(v + w) - \mathcal{Q}(v) - \mathcal{Q}(w)\big). \tag{B.3}$$

For the Clifford algebras generated by the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, we fix the following notation

$$\mathrm{Cl}_n^{\pm} := \mathrm{Cl}(\mathbb{R}^n, \pm\mathcal{Q}_n), \qquad \mathbb{C}\mathrm{l}_n := \mathrm{Cl}(\mathbb{C}^n, \pm\mathcal{Q}_n), \tag{B.4}$$

with the standard quadratic form $\mathcal{Q}_n(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$. The algebras $\mathrm{Cl}_n^{\pm}$ are generated over $\mathbb{R}$ by $\{e_1, \ldots, e_n\}$ subject to

$$e_i e_j + e_j e_i = \pm 2\delta_{ij}, \qquad \forall i, j \in \{1, \ldots, n\}. \tag{B.5}$$

The even part $(Cl_n^{\pm})^0$ and the odd part $(Cl_n^{\pm})^1$ of $Cl_n^{\pm}$ consists of products, respectively, of an even and an odd number of $e_i$'s. $\mathbb{C}l_n$ is the complexification of the algebras $Cl_n^{\pm}$ and is, therefore, generated over $\mathbb{C}$ by the same $\{e_1, \ldots, e_n\}$ respecting (B.5).

Further, one checks that

$$\dim_{\mathbb{R}}(Cl_n^{\pm}) = \dim_{\mathbb{C}}(\mathbb{C}l_n) = 2^n. \tag{B.6}$$

The lower dimensional Clifford algebras (for $n = 1, 2$) are obtained explicitly as

$$Cl_1^+ \simeq \mathbb{R} \oplus \mathbb{R}, \quad Cl_1^- \simeq \mathbb{C}; \qquad Cl_2^+ \simeq M_2(\mathbb{R}), \quad Cl_2^- \simeq \mathbb{H}. \tag{B.7}$$

The map $\Phi(e_i) = e_{n+1}e_i$ on the generators extends to the following isomorphisms:

$$Cl_n^- \simeq (Cl_{n+1}^{\pm})^0. \tag{B.8}$$

Similarly, the map defined by

$$\Psi(e_i) = \begin{cases} 1 \otimes e_1, & i = 1, 2 \\ e_{i-2} \otimes e_1 e_2, & i = 3, \ldots, n \end{cases} \tag{B.9}$$

extends to

$$Cl_k^{\pm} \otimes_{\mathbb{R}} Cl_2^{\mp} \simeq Cl_{k+2}^{\mp}, \qquad \forall k \geqslant 1, \tag{B.10}$$

which, along with its base cases (B.7), recursively generates the Table B.1.

| $n$ | $Cl_n^+$ | $Cl_n^-$ | $\mathbb{C}l_n$ |
|---|---|---|---|
| 1 | $\mathbb{R} \oplus \mathbb{R}$ | $\mathbb{C}$ | $\mathbb{C} \oplus \mathbb{C}$ |
| 2 | $M_2(\mathbb{R})$ | $\mathbb{H}$ | $M_2(\mathbb{C})$ |
| 3 | $M_2(\mathbb{C})$ | $\mathbb{H} \oplus \mathbb{H}$ | $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ |
| 4 | $M_2(\mathbb{H})$ | $M_2(\mathbb{H})$ | $M_4(\mathbb{C})$ |
| 5 | $M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$ | $M_4(\mathbb{C})$ | $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ |
| 6 | $M_4(\mathbb{H})$ | $M_8(\mathbb{R})$ | $M_8(\mathbb{C})$ |
| 7 | $M_8(\mathbb{C})$ | $M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$ | $M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$ |
| 8 | $M_{16}(\mathbb{R})$ | $M_{16}(\mathbb{R})$ | $M_{16}(\mathbb{C})$ |

Table B.1: Clifford algebras $Cl_n^{\pm}$ and their complexifications $\mathbb{C}l_n$ for $n = 1, \ldots, 8$.

For $k = n + 2$, (B.10) gives

$$\begin{aligned} Cl_{n+4}^{\pm} &\simeq Cl_{n+2}^{\mp} \otimes_{\mathbb{R}} Cl_2^{\pm}, \\ &\simeq Cl_n^{\pm} \otimes_{\mathbb{R}} Cl_2^{\mp} \otimes_{\mathbb{R}} Cl_2^{\pm}, \\ &\simeq Cl_n^{\pm} \otimes_{\mathbb{R}} M_2(\mathbb{H}), \end{aligned} \tag{B.11}$$

where in the second step we used (B.10) for $k = n$ and in the third step $\mathbb{H} \otimes_{\mathbb{R}} M_2(\mathbb{R}) \simeq$

$M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \simeq M_2(\mathbb{H})$. Further, substituting $n \to n+4$ in (B.11), we have

$$\begin{aligned} Cl^{\pm}_{n+8} &\simeq Cl^{\pm}_{n+4} \otimes_{\mathbb{R}} M_2(\mathbb{H}), \\ &\simeq Cl^{\pm}_{n} \otimes_{\mathbb{R}} M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}), \\ &\simeq Cl^{\pm}_{n} \otimes_{\mathbb{R}} M_{16}(\mathbb{R}), \end{aligned} \tag{B.12}$$

where the second step uses (B.11) and the third $M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}) \simeq M_{16}(\mathbb{R})$.

Thus, with (B.12), one concludes that $Cl^{\pm}_{n+8}$ is Morita equivalent to $Cl^{\pm}_{n}$ and, therefore, one determines $Cl^{\pm}_{n}$ for all $n$.

In this sense, for the real Clifford algebras, Table B.1 has the periodicity of eight. Similarly, the complex Clifford algebras have periodicity two:

$$\mathbb{C}l_{n+2} \simeq \mathbb{C}l_n \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{B.13}$$

and, thus, $\mathbb{C}l_{n+2}$ is Morita equivalent to $\mathbb{C}l_n$.

### Clifford bundles

The Clifford algebraic structure can be naturally imported to riemannian manifolds, thanks to the metric structure on them.

A **riemannian metric** $g$ on a manifold $\mathcal{M}$ is a symmetric bilinear form $g : \Gamma(\mathcal{TM}) \times \Gamma(\mathcal{TM}) \to C(\mathcal{M})$ satisfying the following conditions:

(i) $g(X, Y)$ is a real function if $X$ and $Y$ are real vector fields;

(ii) $g$ is $C(\mathcal{M})$-bilinear, that is

$$g(fX, Y) = g(X, fY) = fg(X, Y), \qquad \forall f \in C(\mathcal{M}),\ \forall X, Y \in \Gamma(\mathcal{TM});$$

(iii) $g(X, X) \geqslant 0$ for all real vector fields $X$ and equality holds iff $X = 0$.

On the fibers $\mathcal{T}_x\mathcal{M}$ of the tangent bundle $\mathcal{TM}$ over a riemannian manifold $(\mathcal{M}, g)$, the inner product defined by the metric:

$$g_x(X_x, Y_x) := g(X, Y)|_x \tag{B.14}$$

associates the following quadratic form on the tangent space $\mathcal{T}_x\mathcal{M}$:

$$\mathcal{Q}_g(X_x) = g_x(X_x, X_x). \tag{B.15}$$

Then, at every $x \in \mathcal{M}$, one has the Clifford algebra $Cl(\mathcal{T}_x\mathcal{M}, \mathcal{Q}_g)$ and its corresponding complexification $\mathbb{C}l(\mathcal{T}_x\mathcal{M}, \mathcal{Q}_g)$.

The **Clifford bundle** $\mathcal{C}l^{\pm}(\mathcal{T}\mathcal{M})$ is the bundle of algebras $\mathcal{C}l(\mathcal{T}_x\mathcal{M}, \pm\mathcal{Q}_g)$ along with the transition functions $\tau$ inherited from the tangent bundle $\mathcal{T}\mathcal{M}$ (i.e. for open sets $U, V \subset \mathcal{M}$, $\tau_{UV} : U \cap V \to SO(n)$, where $n = \dim(\mathcal{M})$) and their action on each fiber $\mathcal{T}_x\mathcal{M}$ extended to $\mathcal{C}l(\mathcal{T}_x\mathcal{M}, \pm\mathcal{Q}_g)$ by

$$v_1 v_2 \cdots v_k \mapsto \tau_{UV}(v_1) \cdots \tau_{UV}(v_k), \qquad v_1, \ldots, v_k \in \mathcal{T}_x\mathcal{M}. \tag{B.16}$$

Similarly, complexified algebras $\mathbb{C}l(\mathcal{T}_x\mathcal{M}, \pm\mathcal{Q}_g)$ define the Clifford bundle $\mathbb{C}l(\mathcal{T}\mathcal{M})$.

The algebra of continuous real-valued sections of $\mathcal{C}l^{\pm}(\mathcal{T}\mathcal{M})$ is denoted by

$$\mathrm{Cliff}^{\pm}(\mathcal{M}) := \Gamma(\mathcal{C}l^{\pm}(\mathcal{T}\mathcal{M})), \tag{B.17}$$

and the algebra of continuous sections of $\mathbb{C}l(\mathcal{T}\mathcal{M})$ by

$$\mathbb{C}\mathrm{liff}(\mathcal{M}) := \mathrm{Cliff}^{\pm}(\mathcal{M}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{B.18}$$

# Appendix C

# Tomita-Takesaki Modular Theory

Modular theory first appeared in two unpublished lecture notes of Minoru Tomita [To1, To2] and a more accessible version was later presented by Masamichi Takesaki [Ta1]. It provides a way to construct 'modular automorphisms' of von Neumann algebras via polar decomposition of an involution. For a more involved account, see [Ta2].

A $C^*$-algebra (A.4) is a $*$-algebra of bounded operators on a Hilbert space $\mathcal{H}$ that is closed in the *operator norm topology*. In particular, a **von Neumann algebra** $\mathfrak{M}$ is a unital $C^*$-algebra closed in the *weak operator topology*. The **commutant** $\mathfrak{M}'$ of $\mathfrak{M}$ is defined as

$$\mathfrak{M}' := \{m' \in \mathfrak{M} : \ m'm = mm', \forall m \in \mathfrak{M}\}. \tag{C.1}$$

For a von Neumann algebra $\mathfrak{M}$, let a unit vector $\varpi \in \mathcal{H}$ be **separating** (that is, the map $\mathfrak{M} \to \mathfrak{M}\varpi$ is injective) and **cyclic** (that is, $\mathfrak{M}\varpi$ is dense in $\mathcal{H}$). Then, there exist two unique canonical operators, namely the **modular operator** $\Delta$ and the **modular conjugation** or **modular involution** $J$, such that

- $\Delta^* = \Delta$ is positive and invertible (but not bounded),
- the set $\{\Delta^{it} : t \in \mathbb{R}\}$ of unitaries induces a strongly continuous one-parameter group $\{\alpha_t\}$ of **modular automorphisms** $\alpha_t : \mathfrak{M} \to \mathfrak{M}$ (with respect to $\varpi$) defined by

$$\alpha_t(m) = \mathsf{Ad}(\Delta^{it})m = \Delta^{it} m \Delta^{-it}, \qquad \forall m \in \mathfrak{M},\ \forall t \in \mathbb{R}, \tag{C.2}$$

- $J = J^* = J^{-1}$ is **antilinear** (i.e. $\langle J\psi, J\phi\rangle = \overline{\langle \psi, \phi\rangle} = \langle \phi, \psi\rangle$, $\forall \psi, \phi \in \mathcal{H}$) and it commutes with $\Delta^{it}$, implying

$$\mathsf{Ad}(J)\Delta := J\Delta J^{-1} = \Delta^{-1}, \tag{C.3}$$

- $J : \mathfrak{M} \to \mathfrak{M}'$, defined by $J\mathfrak{M}J = \mathfrak{M}'$. Thus, $\mathfrak{M}$ is anti-isomorphic to its commutant

$\mathfrak{M}'$ and the anti-isomorphism is given by the $\mathbb{C}$-linear map

$$\mathfrak{M} \ni m \mapsto Jm^*J^{-1} \in \mathfrak{M}', \tag{C.4}$$

- $\varpi$ is a $+1$-eigenvector of both the operators, that is,

$$\Delta\varpi = \varpi = J\varpi, \tag{C.5}$$

- The unbounded antilinear operators $S_0$ and $F_0$ defined on $\mathcal{H}$ with domains $\mathfrak{M}\varpi$ and $\mathfrak{M}'\varpi$, respectively, by setting

$$\begin{aligned} S_0(m\varpi) &:= m^*\varpi, \qquad \forall m \in \mathfrak{M} \\ F_0(m'\varpi) &:= m'^*\varpi, \qquad \forall m' \in \mathfrak{M}'; \end{aligned} \tag{C.6}$$

extend to their respective closures – antilinear operators $S$ and $F = S^*$, defined on a dense subset of $\mathcal{H}$ – which have the following **polar decomposition**:

$$\begin{aligned} S &= J|S| = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J \\ F &= J|F| = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J, \end{aligned} \tag{C.7}$$

implying

$$\begin{aligned} \Delta &= S^*S = FS \\ \Delta^{-1} &= SF = SS^*. \end{aligned} \tag{C.8}$$

# Appendix D

# The Dirac Equation

## D.1 $\gamma$-matrices in chiral representation

In four-dimensional euclidean space, the gamma matrices are

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \tilde{\sigma}^{\mu} & 0 \end{pmatrix}, \qquad \gamma^{5} := \gamma^{1}\gamma^{2}\gamma^{3}\gamma^{0} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{D.1}$$

where, for $\mu = 0, j$, we define

$$\sigma^{\mu} := \{\mathbb{I}_2, -i\sigma_j\}, \qquad \tilde{\sigma}^{\mu} := \{\mathbb{I}_2, i\sigma_j\}, \tag{D.2}$$

with $\sigma_j$, for $j = 1, 2, 3$, being the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D.3}$$

In $(3+1)$-dimensional minkowski spacetime, the gamma matrices are

$$\gamma^{\mu}_{M} = \begin{pmatrix} 0 & \sigma^{\mu}_{M} \\ \bar{\sigma}^{\mu}_{M} & 0 \end{pmatrix}, \qquad \gamma^{5}_{M} := \gamma^{1}_{M}\gamma^{2}_{M}\gamma^{3}_{M}\gamma^{0}_{M} = -i\gamma^{5}, \tag{D.4}$$

where, for $\mu = 0, j$, we define

$$\sigma^{\mu}_{M} := \{\mathbb{I}_2, \sigma_j\}, \qquad \bar{\sigma}^{\mu}_{M} := \{\mathbb{I}_2, -\sigma_j\}, \tag{D.5}$$

with $\sigma_j$, for $j = 1, 2, 3$, being the Pauli matrices (D.3).

## D.2 Dirac lagrangian and Weyl equations

The Dirac lagrangian in euclidean space and minkowski spacetime, respectively, is

$$\begin{aligned} \mathcal{L} &:= \chi^\dagger(\eth + m)\psi & \eth &:= -i\gamma^\mu \partial_\mu \\ \mathcal{L}_M &:= -\bar{\Psi}(\eth_M + m)\Psi & \eth_M &:= -i\gamma^\mu_M \partial_\mu \end{aligned} \tag{D.6}$$

where $\chi, \psi$ are independent Dirac spinors, while the Dirac spinors $\Psi, \bar{\Psi}$ are related by: $\bar{\Psi} := \Psi^\dagger \gamma^0$. And $\gamma^\mu, \gamma^\mu_M$ are, respectively, the euclidean (D.1) and minkowskian (D.4) gamma matrices.

The Dirac spinor (or, the spin-$\frac{1}{2}$) representation of (the double cover of) the Lorentz group $SL(2,\mathbb{C})$ is reducible into two irreducible representations: $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$, which act only on the two-component *Weyl spinors* $\Psi_l$ and $\Psi_r$ of a Dirac spinor $\Psi$, defined, in the chiral representation (see §D), by

$$\Psi = \begin{pmatrix} \Psi_l \\ \Psi_r \end{pmatrix} \in L^2(\mathcal{M}, \mathcal{S}), \qquad \begin{matrix} \Psi_l \in L^2(\mathcal{M}, \mathcal{S})_+ \\ \Psi_r \in L^2(\mathcal{M}, \mathcal{S})_- \end{matrix} . \tag{D.7}$$

Under such decomposition into Weyl spinors, the Dirac lagrangian $\mathcal{L}_M$ becomes

$$\begin{aligned} \mathcal{L}_M &= \begin{pmatrix} \Psi_l^\dagger & \Psi_r^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \left[ \begin{pmatrix} 0 & i\sigma^\mu_M \partial_\mu \\ i\tilde{\sigma}^\mu_M \partial_\mu & 0 \end{pmatrix} - m \right] \begin{pmatrix} \Psi_l \\ \Psi_r \end{pmatrix} \\ &= i\Psi_l^\dagger \tilde{\sigma}^\mu_M \partial_\mu \Psi_l + i\Psi_r^\dagger \sigma^\mu_M \partial_\mu \Psi_r - m\left(\Psi_l^\dagger \Psi_r + \Psi_r^\dagger \Psi_l\right), \end{aligned} \tag{D.8}$$

which, for $m = 0$, describes the *Weyl fermions* (massless spin-$\frac{1}{2}$ particles) in quantum field theory. The corresponding *Weyl equations* of motion:

$$\begin{aligned} \mathcal{L}^l_M &:= i\Psi_l^\dagger \tilde{\sigma}^\mu_M \partial_\mu \Psi_l & \longrightarrow \quad \tilde{\sigma}^\mu_M \partial_\mu \Psi_l &= \left(\mathbb{I}_2 \partial_0 - \sigma_j \partial_j\right)\Psi_l &= 0, \\ \mathcal{L}^r_M &:= i\Psi_r^\dagger \sigma^\mu_M \partial_\mu \Psi_r & \longrightarrow \quad \sigma^\mu_M \partial_\mu \Psi_r &= \left(\mathbb{I}_2 \partial_0 + \sigma_j \partial_j\right)\Psi_r &= 0, \end{aligned} \tag{D.9}$$

are derived from the relevant lagrangian density, by treating the Weyl spinor $\Psi_{l/r}$ and its Hermitian conjugate $\Psi^\dagger_{l/r}$ as independent variables in the Euler-Lagrange equation:

$$\begin{aligned} \mathcal{L}^l_M &:= i\Psi_l^\dagger \tilde{\sigma}^\mu_M \partial_\mu \Psi_l & \longrightarrow \quad \tilde{\sigma}^\mu_M \partial_\mu \Psi_l &= (\mathbb{I}_2 \partial_0 - \textstyle\sum_{j=1}^3 \sigma_j \partial_j)\Psi_l &= 0, \\ \mathcal{L}^r_M &:= i\Psi_r^\dagger \sigma^\mu_M \partial_\mu \Psi_r & \longrightarrow \quad \sigma^\mu_M \partial_\mu \Psi_r &= (\mathbb{I}_2 \partial_0 + \textstyle\sum_{j=1}^3 \sigma_j \partial_j)\Psi_r &= 0, \end{aligned} \tag{D.10}$$

The plane-wave solutions of these equations, with $x^0$ identified to the time $t$ and $x^{j=1,2,3}$ the space coordinates, are

$$\Psi_{l/r}(x^0, x^j) = \Psi_0 e^{-i(p_j x^j - Et)} \tag{D.11}$$

where $(E, p_j)$ is the energy momentum 4-vector and $\Psi_0$ is a constant spinor, solution of

$$(E\mathbb{I}_2 - \tilde{\sigma}^j p_j)\Psi_0 = 0, \text{ for the left handed solution } \Psi_l, \tag{D.12}$$

$$(E\mathbb{I}_2 - \sigma^j p_j)\Psi_0 = 0, \text{ for the right handed solution } \Psi_r. \tag{D.13}$$

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