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REVIEW ARTICLE

Thermo-electric transport in gauge/gravity models

Andrea Amoretti[a], Alessandro Braggio[b,c], Nicola Maggiore[c,d] and Nicodemo Magnoli[c,d]

[a]Physique Théorique et Mathématique Université Libre de Bruxelles, Brussels, Belgium; [b]NEST, Istituto di Nanoscienze-CNR, Pisa, Italy; [c]I.N.F.N. – Sezione di Genova, Italy; [d]Dipartimento di Fisica, Universitá di Genova, Genova, Italy

ABSTRACT

In this review, we summarize recent results in the study of the thermo-electric transport properties of holographic models exhibiting mechanism of momentum dissipation. These models are of particular interests if applied to understand the transport mechanisms of strongly coupled condensed matter systems such as the high-temperature superconductors. After a brief introduction in which we point out the discrepancies between the experimentally measured transport properties of these materials and the prediction of the weakly coupled theory of Fermi Liquid, we will review the basic aspects of AdS/CFT correspondence and how gravitational models could help in understanding the peculiar properties of strongly coupled condensed matter systems.

1. Introduction

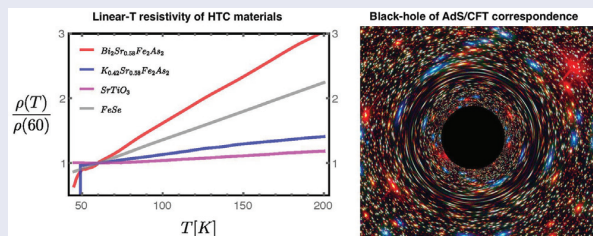The physics of the last century has been dominated by the study of weakly coupled systems. The effective field theories which usually describe these systems can be understood in terms of weakly interacting particles. Regarding condensed matter systems, one of the milestones in this respect is the so-called Landau Fermi liquid model (see e.g. [1]), which describes the vast majority of metals and insulators existing in nature.

The basic assumption from which the phenomenological Landau theory starts is that the qualitative picture for non-interacting Fermi gas in fact persists for a generic interacting fermionic system, also when the interactions between fermions are strong. Specifically, the basic Landau's starting assumptions are:

- There exists a Fermi surface which characterizes the ground state of a generic interacting fermionic system. In momentum space, this surface lies at
$\vec{k} = \vec{k}_F$, where $\vec{k}_F$ is called the Fermi momentum.
- The low energy excitations around the Fermi surface are weakly interacting particles, called quasi-particles, despite the (possibly strong) interactions between the fundamental fermions. The quasi-particles are characterized by the same charge and statistics of the underlying fundamental fermions.

The assumptions above imply that, near the Fermi surface, the retarded Green's function has the form ($\Sigma(\omega, k)$ is the self-energy):

$$G_R(\omega, \vec{k}) = \frac{1}{\omega - \epsilon_k + \Sigma(\omega, k)} \simeq \frac{Z}{\omega - v_F(k - k_F) + i\gamma_k(\omega)}, \qquad (1)$$

where $Z < 1$ is the quasi-particle weight, which measures the strength of the interaction between quasi-particles, v_F is the Fermi velocity and $\gamma_k(\omega)$ is the damping rate which is proportional to the imaginary part of the self-energy. Near the Fermi surface the self-energy must have the following frequency dependence:

$$\Re\Sigma(\omega, k_F) \sim \omega, \qquad \Im\Sigma(\omega, k_F) \sim \omega^2, \qquad (2)$$

so in the low-energy limit the quasiparticles are well-defined quasistationary excitations. The concept of quasi-particle is extremely powerful, and makes it possible to develop a general low energy theory, independently of the precise microscopic details of the system.

Despite the great success of the Landau Fermi liquid theory in describing condensed matter systems, in the 80s physicists start to realize that not all the existing materials can be described by this theory. The discovery of metals with no Fermi surface or no well-defined quasi-particles forced scientists to wonder if the strong interactions between the fundamental degrees of freedom of these systems could nullify the basic Fermi liquid phenomenological assumptions.

Probably, the most famous example of this kind of materials is that of High-T_c superconductors (HTc) [2]. Whereas 'ordinary' or metallic superconductors usually have transition temperatures T_c below 30 K (which is the maximum critical temperature predicted by BCS theory, see [3]), HTc have been observed with transition temperatures as high as 138 K. Moreover, in these peculiar materials both the transport properties of the non-superconducting phase and the superconducting pairing mechanism differ significantly from those predicted by the Fermi liquid and BCS theory.[1]
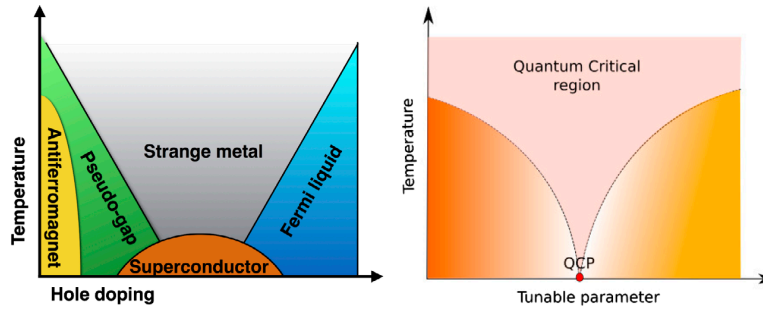
Figure 1. *Left:* Typical phase diagram of a High-T_c superconductor as a function of temperature and hole doping (density of carriers). *Right:* Phase diagram of a system in presence of a Quantum Critical Point.
Notes: The dashed lines mark a qualitative change in the physical properties of the system, even though a proper phase transition occurs only at zero temperature.

The phase diagram of a typical HTc (see Figure 1) is rather complicated and characterized by many different phases (see e.g. [4] for a review). The most surprising of these phases is for sure the strange metal phase, which is the region of the phase diagram on the top of the point in which the superconductive transition temperature is the highest (optimal doping point). This phase is characterized by an extremely stable linear in temperature resistivity, which seems to be insensitive to almost every scale of the system (such as Debye temperature, etc.). This abruptly deviates from the Fermi liquid prediction, which implies a T^2 scaling for the resistivity due to collisions between electrons in case of clean metals below the Debye temperature.

One of the first successful theoretical attempt to justify this exotic behavior was that of the Marginal Fermi Liquid (MFL) developed by Varma and collaborators (see [5] and references therein). We outlined before that such systems cannot be described by well-defined low-energy quasi-particles and do not have a Fermi surface. Then, the MFL is a theory that yields a Fermi surface in the weakest possible sense of the definition but otherwise does not make the same predictions as Fermi liquid theory. The basic assumption of the MFL is to take into account for unknown phenomenological polarization processes which modify the behavior of the self-energy (2) so that:

$$\Sigma_{\mathrm{MFL}}(\omega, k) \sim \omega \log \frac{\omega}{\omega_c} - i\omega, \qquad (3)$$

where ω_c is an high energy cut-off. A self-energy of this kind means that the Fermi surface still exists but is marginally defined, namely the quasi-particle weight Z vanishes logarithmically at the fermi surface. Moreover an imaginary part of Σ linear in ω means that quasi-particles are not any more stable. This phenomenologically inspired self-energy is compatible with a linear in temperature resistivity even though, in order to take into account for other exotic transport

properties of the strange metals, such as the Hall angle, other phenomenological inputs have to be included (see e.g. [5]).

Although a tremendous effort has been made to understand the strange metals better and beyond the phenomenology of the MFL, a complete theory which describes all the features of these systems is still lacking.

In recent times the idea that the phenomenon of *Quantum Criticality* could be the responsible of the Fermi liquid break down in the strange metals has assumed growing importance. This is the critical universal behavior that occurs in the vicinity of a Quantum Phase transition (see [6] for a review). A quantum phase transition is defined as a transition which occurs at zero temperature, due to the variation of some control parameter, such as a chemical potential or pressure. Such a phase transition cannot be generated by the competition between energy and entropy, like its finite-temperature counterpart, since classically the entropy has to vanish due to the third law of thermodynamics. Instead, it is originated by the competition between different terms in the Hamiltonian that describes the system. Now, the relevant aspect is that, if this phase transition is second order, the absence of a scale at the critical point means that the quantum field theory describing this point must be a scale-invariant field theory, having in mind that in the relativistic case, up to some subtleties, scale-invariance implies conformal invariance. The particular aspect of a quantum critical theory compared to a classical critical theory is that if one moves away from the quantum critical point by heating up the system, he lands in a Conformal field theory at finite temperature, whose dynamic is still controlled by the $T = 0$ conformal symmetry. Then, the only difference is that all dimension-full quantities are now expressed in terms of the only scale T.

Due to conformal invariance, it is possible to show that the transport coefficients, and the relaxation time to local equilibrium can be expressed by means of the fundamental quantities of nature and the temperature using dimensional analysis. This is in contradiction with Boltzmann theory of transport in Fermi liquid, which states that the transport coefficients are proportional to the mean free scattering time between quasi-particles. Rather, as conjectured in [6], at the quantum critical point the system behaves like a perfect fluid in which the relaxation time is as short as possible, and is determined universally by the absolute temperature by means of the indetermination principle. Away from the quantum critical point, the dissipation rate is much larger and in general it satisfies the inequality

$$\tau \geq \frac{\hbar}{k_B T} C, \tag{4}$$

where C is a dimensionless constant of order unity which could depend on the details of the systems. The transport coefficients are affected by this scaling argument as well, and we will come back to this point later in Section 6.

Comparing the typical Quantum Critical point phase diagram with that of the cuprates in Figure 1, it is very tempting to assume that the optimal doping

region at $T = 0$ is associated with a quantum phase transition. Although the idea that the physics underlying the strange metal behavior is a finite-temperature conformal field theory is fascinating, the details are not so simple. In particular scale-invariance is only observed in terms of energy–temperature scaling. In spatial directions, ARPES experiments [7] suggest that a distinct Fermi surface still exists. However, the idea that one can capture the basic ingredients of the physics of the strange metals by studying some sort of deformed strongly coupled conformal field theory has became one of the leading direction of research in this field.

From the theoretical point of view, there exist very few tools which allow us to analyze the properties of these complicated theories. However, in the last decade new techniques developed in the context of string theory have acquired greater and greater relevance in the study of strongly coupled systems. These techniques include the so-called AdS/CFT (holographic) correspondence [8], and the main goal of this review is to analyze what can be said on the thermo-electric transport properties of strongly coupled materials, such as the HTc, by means of holographic techniques.

2. Gauge-gravity duality

In this section we introduce the gauge/gravity duality using simple intuitive ideas. The AdS-CFT correspondence, where AdS stands for Anti de Sitter, and CFT for Conformal Field Theory, was originally formulated in the context of String Theory [8,9]. Excellent reviews on the role played by the gauge/gravity duality in condensed matter are [10–13]. While for CFT we refer the reader to the standard reference [14], we spend a few words on AdS (see [15] for details).

AdS_{d+1} is the maximally symmetric metric space in d + 1 dimension with negative curvature, where maximally symmetric means that it admits the maximum number of independent killing vectors. A convenient parametrization of the AdS_{d+1} metric is the Poincaré one:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta^{\mu\nu}dx_\mu dx_\nu), \tag{5}$$

where $\eta^{\mu\nu}$ is the diagonal flat metric with Minkoskian signature $(1, d - 1)$, L is a constant parameter called AdS radius, and the d + 1 coordinates are (t, x_i, z). In what follows the greek indices run other the boundary coordinates $\{t, x_i\}$. In its weakest form, the so-called bottom-up formulation, the AdS/CFT duality can be expressed in the following way:

d + 1-dimensional classical gravity theories on AdS_{d+1} vacuum are equivalent to the large N (degrees of freedom per site) limit of strongly coupled d-dimensional CFTs in flat space.

There are three issues which we have to clarify in order to understand the previous statement, namely:

- Why conformal field theories?
- How can we match the degrees of freedom of a d + 1-dimensional theory with those of a d-dimensional one?
- Why the dual CFTs in the large N limit are strongly coupled?

The answer to the first question is very simple if one has in mind the basic properties of CFTs and AdS spaces. In fact, the isometry group $SO(2,d)$ of AdS_{d+1} is exactly the symmetry group of a d dimensional CFT (see e.g. [16]). Then, if there is some region of AdS in which a quantum field theory lives, it is natural to assume that it has to be invariant under the same symmetry of AdS_{d+1}.

To get intuition on the second issue, we need the help of the holographic principle [17]. This principle states that a theory of gravity in d + 1 space time dimensions, in a local region of space has a number of degrees of freedom which scales like that of a quantum field theory in the boundary of that region.

To understand this basic principle we need to use the celebrated Bekenstein–Hawking area law [18,19] for the entropy of a black hole. According to [18,19], in fact, black holes are thermodynamical object and have an entropy which is proportional to the area A of their horizon, namely:

$$S_{BH} = \frac{A}{4G_{d+1}}, \tag{6}$$

where G_{d+1} is the Newton's constant in Planck units. Now, since we are considering a black hole, its entropy has to be the maximal entropy of anything else in the same volume. Consequently, each region of space has a maximum entropy scaling with the area of the boundary and not with the enclosed volume, as one may think. This is much smaller than the entropy of a local quantum field theory in the same space, which would have a number of states $N \sim e^V$, and the maximum entropy $S \sim \log N$ would have been proportional to the volume V. The maximum entropy in a region of space can instead be related to the number of degrees of freedom N_d of a local quantum field theory living in d dimensions.

The AdS/CFT correspondence is a particular realization of this principle, where the gravity theory lives in an AdS_{d+1} vacuum, and its degrees of freedom are encoded on the conformal boundary. To clarify this point, let us compute the area of the conformal boundary of AdS_{d+1}. Using the metric (5) embedded in a hyper-surface of constant radius z and time t, we obtain:

$$A = \int_{z \to 0} d^{d-1}x \sqrt{g_{d-1}} = \int_{z \to 0} d^{d-1}x \frac{L^{d-1}}{z^{d-1}}, \tag{7}$$

where g_{d-1} is the determinant of the embedded metric, and we have taken the limit $z \to 0$ since this is the locus where the conformal boundary is located. The integral (7) suffers from divergences coming both from the $z \to 0$ limit and from the $d^{d-1}x$ integration measure, and need to be regularized. In order to do this, we will perform the integral (7) up to a small value $z = \epsilon$ and we enclose the space in a closed space volume V_{d-1}, namely:

$$A = \frac{L^{d-1}}{\epsilon^{d-1}} V_{d-1}. \tag{8}$$

The maximum entropy in the bulk of AdS_{d+1} is, introducing the Planck scale length l_p,

$$N_{AdS} \sim \frac{A}{4G_{d+1}} \sim \frac{L^{d-1}}{\epsilon^{d-1}} \frac{V_{d-1}}{l_p^{d-1}}. \tag{9}$$

The dual quantum field theory in d dimensions is also UV and IR divergent and needs to be regularized in the same way, by introducing a box of volume V_{d-1}, and a short distance cut-off ϵ. The total number of degrees of freedom N_d of a quantum field theory in d dimensions is given by the number of spatial cells V_{d-1}/ϵ^{d-1} times the number of degrees of freedom per lattice site N. As an example, a quantum field theory with matrix fields ϕ_{ab} in the adjoint representation of the symmetry group $U(N)$ has a number of degrees of freedom per point equal to N^2. Thus

$$N_d \sim \frac{V_{d-1} N^2}{\epsilon^{d-1}}. \tag{10}$$

Finally, we have estimated the relation between the bulk gravitational degrees of freedom and those of the dual CFT and, using the holographic principle we have understood why the CFT lives in one less dimension. However, the estimation (10) allows us also to understand the first part of the third issue, namely why the number of degrees of freedom per site in the dual CFT has to be large. This is related to the fact that we are considering classical gravity. In order to do this, it is necessary that the typical excitation length of the gravity theory is much larger than the Planck's length l_p. In fact considering AdS, its typical length scale is given by its radius L, then, matching the relations (9) and (10), we obtain:
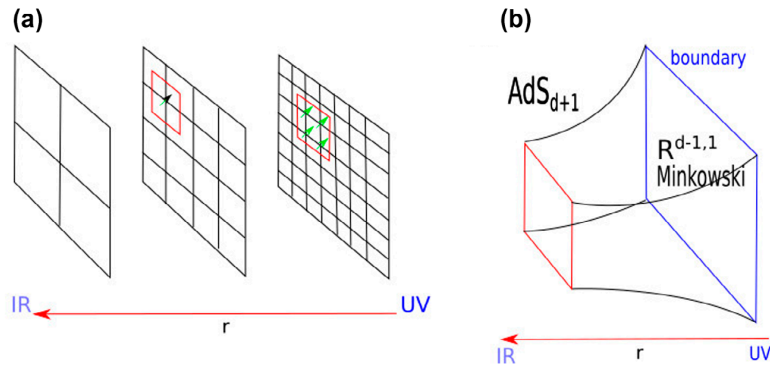
$$\frac{L^{d-1}}{l_p^{d-1}} \sim N^2 \gg 1, \tag{11}$$

which proves the assertion.

We need now only to understand the last part of the third issue, namely why the CFT is strongly coupled. This is related to the problem of giving physical interpretation of the extra radial gravitational coordinate z at the dual level. The cut-off R of this coordinates can be identified with the UV cut-off of the dual quantum field theory. Then we can already argue that the radial coordinate has to be related to the renormalization group flow in some way. It is tempting to identify this extra scale dimension with the radial dimension on the gravity side. In order to understand how this is possible, let us recall that the AdS_{d+1} metric can be cast, using the transformation $z = \frac{L^2}{r}$, in the following form:

$$ds^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2, \tag{12}$$

(a)

IR ←———————————— UV
r

(b)

AdS$_{d+1}$

boundary

R$^{d-1,1}$
Minkowski

IR   r   UV

Figure 2. Connection between the radial coordinates in AdS and the renormalization group flow in the dual quantum field theory.

where the conformal boundary is now located at $r \to \infty$. This parametrization makes clear that the AdS$_{d+1}$ geometry can be viewed as a family of copies of Minkowski spaces parametrized by the radial coordinate r, whose size is seen to shrink when r decreases from the conformal boundary $r \to \infty$ to the AdS horizon at $r \to 0$. This clarifies the UV/IR connection between gravity and the dual field theory and explains why the field theory living on the boundary is strongly coupled. In fact, from the view point of the gravity theory, physics near the conformal boundary $r \to \infty$ is the large volume physics, i.e. IR physics. Near the horizon $r = 0$ is instead the short distance UV physics. In contrast, from the view point of the quantum field theory, physics at large r corresponds to short distance UV physics and vice versa (see Figure 2).

We have given arguments to justify, at least at the conceptual level, the duality between a strongly coupled conformal field theory in the large N limit and a classical gravitational theory in one more space–time dimension. Since the gravitational theory is classical in principle, using the duality we can easily compute observables in the strongly coupled CFT. To do this, however, we need a prescription to relate observables of the gravitational theory to observables in the dual strongly coupled field theory. In particular, the fundamental objects of CFTs are the primary fields. Then in order to compute observables in the CFT we need a prescription to relate the fields in the gravity sector to the primary fields of the CFT. Let us consider a conformal field theory Lagrangian $\mathcal{L}_{\mathrm{CFT}}$. It can be perturbed by adding arbitrary functions, namely sources $h^A(x)$ of local operators $\mathcal{O}_A(x)$:

$$\mathcal{L}_{\mathrm{CFT}} \to \mathcal{L}_{\mathrm{CFT}} + \sum_A h^A(x)\mathcal{O}_A(x), \tag{13}$$

This is a UV perturbation because it is a perturbation of the bare Lagrangian by local operators. According to the general connection between the radial AdS coordinate and the renormalization group, it corresponds to a perturbation near the boundary $r \to \infty$ in AdS space. Thus the perturbation by a source $h(x)$

of the CFT will be encoded in the boundary condition on the bulk fields. Take now the source and extend it to the bulk side $h(x) \rightarrow h(x_\mu, r)$ with the extra coordinate r being the radial dimension of AdS_{d+1} (see the metric (12)). Fields in the boundary will be denoted with coordinates x, and bulk fields will be dependent on the coordinates (x_μ, r). Suppose $h(x_\mu, r)$ to be the solution of the equations of motion in the bulk with boundary condition

$$\lim_{r \to \infty} r^{-\bar{\alpha}} h(x_\mu, r) = h(x), \tag{14}$$

where $r^{\bar{\alpha}}$ is the leading r-dependent behaviur of the field at $r \to \infty$,[2] and another suitable boundary condition at the horizon to fix $h(x_\mu, r)$ uniquely. As a result we have a one to one map between bulk fields and boundary fields [9,20]. In fact, to each local operator $\mathcal{O}(x)$ corresponds a source $h(x)$, which is the boundary value in AdS of a bulk field $h(x_\mu, z)$. In order to deduce which field should be related to a given operator, symmetries come in help, because there is no completely general recipe. As a rule of the thumb, since internal symmetries of field in the gravitational sector are preserved in the dual field theory, in general, we can say that the spin of the bulk fields correspond to the spin of the dual operators in the boundary field theory. To make a quantitative example, let us analyze how a very fundamental quantity of a quantum field theory, the stress–energy tensor $T_{\mu\nu}$, is encoded in the dual gravitational sector using the previous prescription. In particular, the source of $T_{\mu\nu}$ should be a tensor $g_{\mu\nu}$. To have a gauge invariant coupling

$$\int \mathrm{d}^d x\, T_{\mu\nu}(x) g^{\mu\nu}(x), \tag{15}$$

$g_{\mu\nu}(x)$ should be the boundary value of a gauge field corresponding to the local translational invariance. The field we are talking about is of course the metric tensor $g_{ab}(x_\mu, r)$ with boundary value

$$\lim_{r \to \infty} g_{ab}(x_\mu, r) = g_{\mu\nu}(x), \tag{16}$$

where the latin indeces indicate the bulk coordinates. The right-hand side of the previous equation is to be intended as the embedding of the bulk metric on the hyper-surface $r = const$. The previous example allows us to make an important observation. In fact, we have just explained that the metric tensor $g_{\mu\nu}$, which encodes local diffeomorphisms invariance in the gravitational bulk, sources the stress–energy tensor $T_{\mu\nu}$ of the dual field theory, which is a global conserved current ($\partial_\mu T^{\mu\nu} = 0$) due to the global translational and rotational invariance of the dual field theory. In this sense we can affirm that, on the gravity side, the global symmetries arise as large gauge transformations, namely there is a correspondence between global symmetries in the gauge theory and gauge symmetries in the dual gravity theory. This connection between fields and operators allows us to express the duality as an equality between partition

functions. Namely, the proposal of Gubser, Klebanov, Polyakov and Witten (GKPW) [9,20], which, as everything in this framework is still a conjecture, affirms that the partition function of the CFT, $Z_{\text{CFT}}[\{h(x)\}]$, is equal to the partition function of the dual gravitational theory $Z_{\text{AdS}}[\{h(x_\mu, z)\}]$:

$$Z_{\text{CFT}}[\{h(x)\}] = Z_{\text{AdS}}[\{h(x_\mu, r)\}], \tag{17}$$

where $\{h(x)\}$ is the collection of all the sources associated to each local operator in the field theory side, and $\{h(x_\mu, r)\}$ is the collection of the bulk fields. However, we do not have a very useful idea of what is the right hand side of this equation, except in the large N limits, where this gravity theory becomes classical. In these limits we can do the path integral by a saddle point approximation, and the statement of the duality (17) becomes

$$Z_{\text{CFT}}[\{h(x)\}] = e^{i S_{\text{AdS}_{d+1}}[\{h(x_\mu, r)\}]}\Big|_{r \to \infty}, \tag{18}$$

where $S_{\text{AdS}_{d+1}}[\{h(x_\mu, r)\}]\big|_{r \to \infty}$ is the classical gravitational action evaluated on a solution of the equations of motion. Finally, we are able to formulate the first operative rule of the AdS/CFT correspondence, namely:

The gauge/gravity duality is a duality between partition functions which relate the partition function of a CFT in d dimension to the on-shell action of a gravitational theory in AdS_{d+1}, namely:

$$Z_{\text{CFT}}[\{h(x)\}] \leftrightarrow e^{i S_{\text{AdS}_{d+1}}[\{h(x_\mu, r)\}]}\Big|_{r \to \infty}.$$

The operators of the CFT are related to the fields in the bulk according to the following prescription:

field in AdS_{d+1} $\leftrightarrow$ local operators of CFT_d

spin of the gravitational fields $\leftrightarrow$ spin of the local CFT operators.

The sources for the operators are encoded in the boundary behavior of the fields in the gravitational side.

3. Temperature and chemical potential

Having set the basic principles of the AdS/CFT correspondence, we now need to discuss its applications to real world systems. Restricting to possible condensed matter applications, it is mandatory to face the problem of how we can introduce the concept of temperature and chemical potential in the holographic framework previously discussed.

Specifically we need a way to introduce the concept of temperature in gravity, and a natural object which comes to mind is the black hole. According to [19] in fact, we know that black holes are thermal objects which posses their own temperature and that respect the thermodynamic laws. It turns out (see e.g. [10] for details) that the gravity dual of an asymptotic anti-de Sitter black hole inherits

all the thermodynamical properties of the bulk black hole. To be more specific, the most simple asymptotically AdS black hole solution is the AdS-Schwarzschild solution of Einstein equations, given by the metric

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx_i dx^i\right), \qquad f(z) = 1 - \frac{z^d}{z_h^d}. \qquad (19)$$

At $z = z_h$, g_{tt} vanishes, and it can be proven that this is a real black hole horizon. The temperature of the black hole is exactly the temperature of the dual field theory, which is given by:

$$T = \frac{d}{4\pi z_h}. \qquad (20)$$

In the same spirit, the entropy of the dual field theory is exactly the entropy of the black hole (19), which can be computed by means of the area law introduced in (8), namely:

$$\mathcal{S} = \frac{S}{V_{d-1}} = \frac{L^{d-1}}{4G_{d+1}z_h^{d-1}}, \qquad (21)$$

where G_{d+1} is the d + 1-dimensional Newton constant and V_{d-1} is the d − 1-dimensional spatial volume of the dual field theory.

Having defined the temperature and the entropy, all the other thermodynamical quantities follow from basic thermodynamic relations. In this way we can express the thermodynamics of the dual field theory in terms of the horizon radius and the other parameters of the classical gravitational bulk theory.

Since most of the condensed matter typical setups are at finite charge density, we need also to understand what is the gravity dual of a theory at finite chemical potential, namely we want to find the gravitational analogous of a system with a $U(1)$ conserved symmetry. The solution to this issue allows us to clarify an important aspect of gauge/gravity duality, namely the correspondence between local and global symmetries. We have seen in the previous Section that the *global* conformal symmetry in the dual strongly coupled theory corresponds to the *local* diffeomorphism invariance in the bulk gravitational theory. This observation suggests the general correspondence:

Local gauge symmetries in the gravitational theory corresponds to global symmetries in the dual field theory.

To describe the physics of the global $U(1)$ symmetry we should therefore add a Maxwell field to our bulk space–time. The minimal bulk action is thus Einstein–Maxwell theory. The Einstein's equations of motion are:

$$R_{ab} - \frac{1}{2}g_{ab}\mathcal{R} - \frac{d(d-1)}{2L^2}g_{ab} = -\frac{\kappa_{d+1}^2}{2q^2}T_{ab}, \qquad (22)$$

where T_{ab} is the stress–energy tensor $T_{ab} = \frac{1}{4}g_{ab}F_{cd}F^{cd} - F_{ac}F_b^c$ and κ_{d+1} and q are the gravitational and Maxwell coupling constants. The equations of motion

for the electromagnetic field strength $F \equiv \partial_{[a}A_{b]}$ are $\nabla_a F^{ab} = 0$, where ∇_a is the usual covariant derivative.

Let us clarify what we need to obtain in the holographic framework. Our main purpose is to study a strongly coupled field theory at finite density and finite temperature. Regarding the finite temperature, we have learned that this is achieved by studying a black hole solution of the gravitational theory. Concerning the finite density issue, we have just learned that the gauge field in the gravitational side sources a conserved current density J_a in the dual field theory. If we want a finite density ρ we need to switch on the gauge field in the bulk so that the time component of J_μ, $\langle J_t \rangle = \rho$, has a non-zero expectation value. But, according to the standard holographic dictionary, the value of the field at the conformal boundary acts as the sources for the dual operator. Putting all together, and keeping in mind that at the dual level the source of the charge density ρ is the chemical potential μ, in order to find the gravitational dual of a finite charge density system we need to impose:

$$\lim_{z \to 0} A_t = \mu. \tag{23}$$

This is the first basic condition. The second one is that we want to recover scale-invariance at energy scales much greater than the chemical potential μ, namely we want the space–time to be asymptotically AdS.

We are now ready to find the gravity dual of a field theory at finite density and temperature. The transport properties and their relations with thermodynamics will be treated carefully in the next section after introducing disorder in our field theory.

4. Momentum dissipation in holography

In real materials, translational invariance is broken by the presence of a lattice and by various types of impurities. In order to simulate this situation in the holographic context various methods have been proposed. The first method [21–23] considers a small number of charged degrees of freedom in a bath of neutral degrees of freedom which can absorb momentum. The second method introduces the lattice as an infrared irrelevant operator [24], a spatially periodic bulk background solution [25] or a relevant scalar operator coupled with impurities [26].

An alternative to these approaches is to break the diffeomorphism invariance in the bulk introducing a mass for the graviton, in such a way that the dual boundary field theory has momentum dissipation. These models [27] have been studied as a possible modification of general relativity and go under the name of massive gravity. In what follows, for its simplicity, we will describe with some details the last approach.[3]
Specifically, we are interested in analyzing the holographic transport properties of a planar system, and consequently we need to consider the $3 + 1$-dimensional massive gravity action, which is given by:

$$S = \int d^4x \, \sqrt{-g} \left[\frac{1}{2\kappa_4^2} \left(R + \frac{6}{L^2} + \beta \left([\mathcal{K}]^2 - [\mathcal{K}^2] \right) \right) - \frac{1}{4q^2} F_{ab} F^{ab} \right], \quad (24)$$

where β is an arbitrary parameter having the dimension of a mass squared and the small square brackets denote a trace operation. There is also a boundary term, that we do not specify here, which is necessary in order to have a well-defined variational problem. The matrix $(\mathcal{K}^2)_b^a$ is defined in terms of the dynamical metric g_{ab} and a fiducial fixed metric f_{ab} in the following way

$$(\mathcal{K}^2)_b^a \equiv g^{ac} f_{cb}, \qquad \mathcal{K} \equiv \left(\sqrt{\mathcal{K}^2} \right)_b^a. \quad (25)$$

The fixed metric f_{ab} is the responsible of the breaking of diffeomorphisms invariance. As in [31], we consider the following form for f_{ab}:

$$f_{ab} = \mathrm{diag}(0, 0, 1, 1), \quad (26)$$

which corresponds to break diffeomorphisms in the (x, y) plane, but not in the (z, t) plane. At the dual level this means that the theory has conserved energy but no conserved momentum.

4.1. The dyonic solution: external magnetic field

At this point we need to include in our analysis an additional ingredient. Specifically, we want to discuss the effects due to the presence of an external magnetic field B orthogonal to the plane xy, with particular interest on its consequences on the thermo-electric transport coefficients in the holographic system at non-zero chemical potential μ. To include the constant magnetic field B we adopt the following ansatz for the background metric $g_{\mu\nu}$ and the gauge field A_μ

$$ds^2 = \frac{L^2}{z^2} \left[-f(z) dt^2 + dx^2 + dy^2 + \frac{1}{f(z)} dz^2 \right],$$
$$A = \phi(z) \, dt + B x \, dy. \quad (27)$$

Substituting this ansatz within the equations of motion derived from (24), we obtain the following black-brane solution

$$\phi(z) = \mu - q^2 \rho z = \mu \left(1 - \frac{z}{z_h} \right), \qquad \rho \equiv \frac{\mu}{q^2 z_h},$$
$$f(z) = 1 - \frac{z^3}{z_h^3} + \beta \left(z^2 - \frac{z^3}{z_h} \right) - \frac{z^3}{z_h} \left(1 - \frac{z}{z_h} \right) \frac{\kappa^2 \left(B^2 z_h^2 + \mu^2 \right)}{2 L^2 q^2}, \quad (28)$$

where we have denoted with z_h the horizon location defined by the vanishing of the emblackening factor, namely $f(z_h) = 0$. The definition of ρ is actually substantiated by the explicit analysis of the thermodynamics that we perform in Section 4.1.1.

4.1.1. Thermodynamics

As discussed in the previous section, the black brane solution (28) corresponds to a planar dyonic black hole having both electric and magnetic charges. From the boundary theory standpoint, B represents a magnetic field perpendicular to the spatial manifold xy which enters the boundary thermodynamical quantities; as usual in gauge/gravity, these are derived from the bulk on-shell action. The temperature T and the entropy density $\mathcal{S}$ are the easiest thermodynamical quantities to compute since they are determined from the horizon data, namely

$$T = -\frac{f'(z_h)}{4\pi} = -\frac{\kappa_4^2 z_h^2 \left(B^2 z_h^2 + \mu^2\right) - 2L^2 q^2 \left(\beta z_h^2 + 3\right)}{8\pi L^2 q^2 z_h}, \quad \mathcal{S} = \frac{2\pi L^2}{\kappa_4^2 z_h^2}. \quad (29)$$

In order to compute the energy density $\mathcal{E}$, the pressure P, the charge density ρ and the magnetization M, we need to evaluate explicitly the Landau potential Ω which, according to the holographic dictionary, is identified with the on-shell bulk action. Once the Landau potential is known, the other thermodynamical quantities follow easily by means of standard thermodynamical relations. We explicitly obtain

$$P = -\frac{\Omega}{V} = -\frac{3B^2 z_h}{4q^2} + \frac{L^2}{2\kappa_4^2 z_h^3} - \frac{\beta L^2}{2\kappa_4^2 z_h} + \frac{\mu^2}{4q^2 z_h}, \quad (30)$$

$$\mathcal{E} = -P + \mathcal{S}T + \mu\rho = \frac{B^2 z_h}{2q^2} + \frac{L^2}{\kappa_4^2 z_h^3} + \frac{\beta L^2}{\kappa_4^2 z_h} + \frac{\mu^2}{2q^2 z_h}, \quad (31)$$

$$\rho = \frac{\partial \mathcal{E}}{\partial \mu} = \frac{\mu}{q^2 z_h}, \qquad M = -\frac{\partial \mathcal{E}}{\partial B} = -\frac{B z_h}{q^2}. \quad (32)$$

4.2. Relation between massive gravity and momentum dissipation mechanisms

In the previous Section, we have analyzed the basic properties of the background black-brane solution in the presence of a mass term for the graviton. We need now to understand how a mass potential for the graviton is related to momentum dissipation in the dual strongly coupled field theory.

The basic idea illustrated in [31] is that the massive gravity potential breaks the diffeomorphism invariance in the bulk. Since we have learned in the previous part that diffeomorphism invariance in the bulk is related to the conservation of the stress–energy tensor in the dual field theory, a gravitational theory with a mass potential would correspond to a dual theory where

$$\partial_\mu T^{\mu\nu} \neq 0. \quad (33)$$

Specifically, since considering the fiducial metric (26) corresponds to breaking the diffeomorphism invariance along the x and y directions one would expect that the model we are considering corresponds to a theory in which momentum is not conserved in some way.

A more precise statement was provided in [32] where, by analyzing the poles of the correlation functions in the hydrodynamic limit (namely at sufficiently low momentum dissipation rate τ^{-1}, where momentum is an almost conserved quantity), it was realized that massive gravity is the dual gravitational realization of a system in which the conservation law for the stress–energy tensor is modified as follows:

$$\partial_t T^{tt} = 0, \qquad \partial_t T^{ti} = -\tau^{-1} T^{ti}, \tag{34}$$

where τ^{-1} is the momentum dissipation rate. At order $\mathcal{O}(\beta)$ the scattering rate is expressed in terms of the thermodynamical quantities (30) and of the graviton mass β in the following way[4]:

$$\tau^{-1} = -\frac{\mathcal{S}\beta}{2\pi(\mathcal{E} + P)}. \tag{35}$$

A further evidence of the analogy between massive gravity and momentum dissipation was provided in [37], where it was proven that the holographic lattice [25] gives an effective mass term for the graviton.

As a final comment it is important to note that the explicit form of the scattering rate (35) constrains the possible values of the mass parameter β. In particular, since τ^{-1} has to be positive, β must assume negative values.

5. Thermo-electric transport coefficients

We are now ready to analyze the transport properties of the holographic model previously described. Restricting to linear response theory, in the presence of an external magnetic field the transport coefficients relate the charge density $\vec{J}$ and the heat current $\vec{Q}$ to the external electric field $\vec{E}$ and thermal gradient $\vec{\nabla} T$ in the following way (see e.g. [38]):

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} T \\ \hat{\alpha} T & \hat{\bar{\kappa}} T \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T/T \end{pmatrix}, \tag{36}$$

where the electric conductivity $\hat{\sigma}$ respects the following condition:

$$\sigma_{ij} = \sigma_{xx}\delta_{ij} + \sigma_{xy}\epsilon_{ij}, \qquad \epsilon_{ij} = -\epsilon_{ji}, \tag{37}$$

and there are analogous relations for the thermo-electric conductivity $\hat{\alpha}$ and the thermal conductivity $\hat{\bar{\kappa}}$.

The DC transport coefficients can be expressed in terms of the Kubo formulæ in the following way [39]:

$$\sigma_{ij} = \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{i\omega} \left(G^R_{J_i J_j}(\omega, k) - G^R_{J_i J_j}(0, k) \right), \tag{38}$$

$$\alpha_{ij} = \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{i\omega T} \left(G^R_{Q_i J_j}(\omega, k) - G^R_{Q_i J_j}(0, k) \right), \tag{39}$$

$$\bar{\kappa}_{ij} = \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{i\omega T} \left(G^R_{Q_i Q_j}(\omega, k) - G^R_{Q_i Q_j}(0, k) \right), \qquad (40)$$

where the functions G^R represent the retarded Green functions of the charge current J_μ and the energy momentum tensor $T_{\mu\nu}$. To compute the previous quantities in a strongly coupled field theory using standard QFT techniques is in general extremely difficult. Fortunately, holography provides us with a relatively simple method to compute them. The details of the computation in the case of the massive gravity model previously described are rather technical and go beyond the purposes of this review. We refer the interested reader to [40–46] for a detailed analysis. The final outcome is:

$$\sigma_{xx} = \frac{\mathcal{E} + P}{\tau} \frac{\rho^2 + \sigma_Q \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \qquad \sigma_{xy} = \rho B \frac{\rho^2 + \sigma_Q \left(B^2 \sigma_Q + 2\frac{\mathcal{E}+P}{\tau} \right)}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \quad (41)$$

$$\alpha_{xx} = \rho \, \mathcal{S} \frac{\mathcal{E} + P}{\tau} \frac{1}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \qquad \alpha_{xy} = \mathcal{S} B \frac{\rho^2 + \sigma_Q \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \quad (42)$$

$$\bar{\kappa}_{xx} = \frac{\mathcal{S}^2 T \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \qquad \bar{\kappa}_{xy} = \frac{B\rho \, \mathcal{S}^2 T}{B^2 \rho^2 + \left(B^2 \sigma_Q + \frac{\mathcal{E}+P}{\tau} \right)^2}, \quad (43)$$

where σ_Q is a characteristic (quantum critical) conductivity at zero charge density, $\mathcal{E}$, P, ρ and $\mathcal{S}$ are the thermodynamical quantities defined in Section 4.1.1 and τ is the dissipation time defined in (35).

6. Holographic inspired phenomenology

The holographic result obtained in the previous Section is potentially amenable of direct experimental confirmation for an isotropic strongly coupled system in two spatial dimensions. At the phenomenological level it is easy to see that the six transport coefficients (41)–(43) depend only on four parameters: two thermodynamical variable $\mathcal{S}$ and ρ and two dynamical parameters σ_Q and $\frac{\tau}{\mathcal{E}+P}$. Consequently, if the holographic picture is generically valid in a two dimensional strongly correlated material, just four phenomenological entries are needed to fully determine the transport properties of the system. This theoretical result is demanding for an experimental testing.

One of the major problems in performing this kind of measurements in interesting strongly coupled materials, such as the HTc, is that typically certain transport coefficients are dominated by the effects of phonons, while we are interested in extracting just the electrons response. This is actually not the case for the electric conductivity, where the phonons are typically suppressed. In this case we get some phenomenological insight using the holographic result (41). It was noted in [47], indeed, that holography naturally solves the puzzle of the linear in temperature behavior of the resistivity and the concomitant T^2 scaling

of the Hall angle in the strange metal phase of the cuprates (see e.g. [4]). In fact, expanding σ_{xx} and σ_{xy} at low magnetic field, the electric conductivity follows an inverse Matthiessen's rule, namely

$$\sigma_{xx} = \sigma_Q + \sigma_D, \quad \text{with} \quad \sigma_D = \frac{\rho^2 \tau}{\mathcal{E} + P}, \tag{44}$$

while the Hall angle $\tan \theta_H$ does not depend on σ_Q

$$\tan \theta_H = \frac{\sigma_{xy}}{\sigma_{xx}} \sim \frac{B}{\rho} \sigma_D. \tag{45}$$

Assuming that near the quantum critical region σ_Q is greater that the Drude contribution σ_D, and imposing that $\sigma_Q \sim 1/T$ and $\sigma_D \sim 1/T^2$, we find the following scaling for the resistivity and the Hall angle:

$$\rho_{xx} \sim T, \qquad \tan \theta_H \sim \frac{1}{T^2}, \tag{46}$$

which are precisely the same scalings experimentally measured in the cuprates [4]. Another reasonable assumption is that the charge density ρ is temperature independent, a condition that can be easily achieved in standard experimental setups. In order to determine the fourth parameter, a promising quantity to be measured is the hall thermal conductivity $\bar{\kappa}_{xy}$, which, being a transverse transport coefficient is almost unaffected by phonons. However, this quantity is very difficult to measure and currently there are very few measurements for the cuprates [48,49]. As noted in [44] (see also [50–52]), imposing the scaling measured in [48,49] in the holographic theoretical prediction (43) leads to nontrivial agreement for some of the other transport coefficients of the cuprates, such as the magneto-resistance and the Hall Lorentz ratio [4], even though a more precise experimental characterization of the whole set of transport coefficients is needed in order to be conclusive on the agreement between holographic prediction and the scaling behavior of these quantities in the cuprates.

7. Conclusions and outlook

In this review, we have outlined some possible applications of AdS/CFT techniques to the analysis of strongly coupled condensed matter systems. Specifically, we have focused our attention on holographic models exhibiting mechanisms of momentum dissipation concentrating on their thermo-electric transport properties. The most relevant result is that holography seems to insist that just four phenomenological entries are needed in order to fully determine the six independent transport coefficients in a strongly correlated plasma. In the last Section we have shown how, just imposing reasonable scaling behavior for some thermodynamical quantities, holography incorporates the nontrivial scalings for the resistivity, and the Hall angle measured in the HTc. However, more work

has to be done both in the theoretical and in the experimental direction. From the theoretical point of view, it is still an open question to find a stable black hole solution that naturally includes all the scalings of the cuprates (see e.g. [43]). From the experimental point of view, a more careful analysis of the thermal transport coefficients in the strange metal phase of the HTc is needed in order to make sensible comparisons with the holographic theoretical predictions. This actually set the basements of an extremely intriguing scenario where, probably for the first time, string theorists could work side-by-side with condensed matter experimentalists!

Notes

1. See [3] for a theoretical review on BCS and [4] and references therein for a review on transport properties in HTc.
2. The specific value of $\bar{\alpha}$ depends both on the kind of fields under consideration and on the dimensionality of the spacetime.
3. See also [28–30] for further developments and discussions about holographic condensed matter applications of massive gravity.
4. For a more precise definition of the momentum dissipation rate see [33–36].

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Disclosure statement

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