Topologically Protected Duality on The Boundary of Maxwell-BF Theory

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Abstract: The Maxwell-BF theory with a single-sided planar boundary is considered in Euclidean four-dimensional spacetime. The presence of a boundary breaks the Ward identities, which describe the gauge symmetries of the theory, and, using standard methods of quantum field theory, the most general boundary conditions and a nontrivial current algebra on the boundary are derived. The electromagnetic structure, which characterizes the boundary, is used to identify the three-dimensional degrees of freedom, which turn out to be formed by a scalar field and a vector field, related by a duality relation. The induced three-dimensional theory shows a strong–weak coupling duality, which separates different regimes described by different covariant actions. The role of the Maxwell term in the bulk action is discussed, together with the relevance of the topological nature of the bulk action for the boundary physics.

Keywords: quantum field theory; topological quantum field theory; duality in gauge field theories; boundary quantum field theory

PACS: 03.70.+k Theory of quantized fields; 11.10.-z Field theory

1. Introduction

Topological field theories have been the subject of a thorough investigation in theoretical physics [1–3]. The initial aim was to unveil to what extent they could give hints for better understanding gravity without matter [4], but it was soon recognized that they also had and still have a different role if one adds a boundary [5,6]. Indeed, it is the boundary which plays a physical role and it is on the boundary that the local observables of a new and different physics live. The introduction of a boundary in a field theory was first proposed by Symanzik who introduced a separability ansatz in order to study the Casimir effect of two parallel plates [7]. The method proposed by Symanzik concerns a space divided into a left and a right hand side, and it has been applied to topological field theories of different types [8], obtaining results particularly relevant for the theory of the fractional quantum Hall effect [9] and of the topological insulators in three and four spacetime dimensions [10]. Later on, field theories with a single-sided boundary have also been considered, which also lead to interesting results. The first and most studied model is the Chern–Simons theory in three dimensions [11,12]. Soon after appeared the BF models [13], the generalizations of the Chern–Simons model which can only live in an odd spacetime [14,15] . The topological nature of the Chern–Simons action is that it does not depend on the metric tensor and hence the energy momentum tensor vanishes. Later on, Witten proposed another kind of topological theory where the action is not in the cohomology of the BRS operator and hence the theory has no physical observables [16]. The common denominator to all these models is that they are gauge field theories with the gauge field A_μ as the main actor and where the Maxwell term does not appear since it breaks the topological nature of the model, which seems to be at the

basis of duality relations characterizing the boundary degrees of freedom of these models, whose bulk theories are purely topological [17]. Duality relations of this type are known [18,19] to allow to extract fermionic degrees of freedom out of bosonic ones, in a way compatible with the existence of Hall or quantum spin states on the edge of higher-dimensional bulk theories. It is natural to ask the question to what extent this is true, investigating whether fermionizing duality relations hold also on the boundary of non-topological field theories, which would broaden the possible candidates for the theories of fractional quantum Hall effect and of topological insulators. Moreover, the Maxwell coupling is expected to be quite relevant in whatever physics may arise on the boundary and that is why we are studying models where the Maxwell term is included in the bulk action. This has been done for Chern–Simons theory with both double- [20] and single-sided [21] boundary, with significantly different results. We note also that the introduction of a Maxwell term in the Chern–Simons action gives rise to topologically massive theories [22,23] by means of a mechanism which cannot to be replicated in spacetime dimensions other than three. The question also arises in what sense a topological theory with a Maxwell coupling is still topological. We propose here the Maxwell-BF model as a new kind of topological theory in four spacetime dimensions. The model does not fit into the known "topological" classes of quantum field theories since it does depend explicitly on the metric and the Maxwell term makes it cohomologically nontrivial. Nevertheless, the bulk theory has no local observables due to the equations of motion, which enforce the field strength $F_{\mu\nu}$ to vanish, and hence the gauge field A_μ is pure gauge. This model has never been considered before with a boundary and the question is not irrelevant. The fact that the physics of the model live on the boundary is neither intuitive nor immediately deducible as a simple exercise.

The paper is organized as follows. In Section 2, the model with planar boundary is introduced. The boundary conditions and the Ward identities, broken by the presence of the boundary, are derived. A kind of electromagnetic structure is found on the boundary, with Maxwell equations solved by potentials, which will play the role of degrees of freedom for the 3D theory. The identification of electric and magnetic fields makes possible a physical interpretation of the role of the Maxwell term, as deformation of the magnetic field. In Section 3, following standard methods of quantum field theory, the algebra formed by the conserved electromagnetic currents is found, which heavily depends on the Maxwell term in the bulk action. The algebra, written in terms of the 3D potentials, allows for the construction, in Section 4, of the 3D theory, whose symmetries are identified. The holographic contact is realized by means of the equations of motion of the 3D theory, which are required to be compatible with the boundary conditions of the bulk 4D theory. The resulting equation is recognized to be the duality relation which characterizes the existence of fermionic degrees of freedom on the boundary and therefore turns out not to be peculiar of purely topological bulk field theories only. In other words, the fact that the physical properties are the same in the holographic theory, whether the Maxwell term is present or not, clarifies the meaning of topological quantum field theories when a boundary is introduced.

2. The Model: Action, Boundary Conditions and Ward Identities

The action of the 4D Maxwell-BF theory in Euclidean spacetime is

$$S_{bulk} = \int d^4x \, \theta(x_3) \left(k_1 \, \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} B_{\rho\sigma} + k_2 \, F_{\mu\nu} F_{\mu\nu} \right) , \tag{1}$$

where the presence of the step-function $\theta(x_3)$ restricts the model on the half-space $x_3 \geq 0$, with planar boundary at $x_3 = 0$, $F_{\mu\nu}(x)$ is the electromagnetic tensor for the gauge field $A_\mu(x)$ and $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$ is the rank-2 antisymmetric tensor of the 4D topological BF theory [24,25]. The canonical mass dimensions of the quantum fields are

$$[A] = 1 \; ; \; [B] = 2 . \tag{2}$$

Finally, k_1 and k_2 are constants that could be reabsorbed by a redefinition of the fields, but we prefer to keep to be able to trace the contributions of the topological BF and Maxwell F^2 term, respectively.

In absence of boundary, i.e., without the θ-function in Equation (1), the action of the Maxwell-BF theory is invariant under the following two symmetries:

$$
\begin{aligned}
\delta^{(1)} A_\mu &= \partial_\mu \Lambda \\
\delta^{(1)} B_{\mu\nu} &= 0
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\delta^{(2)} A_\mu &= 0 \\
\delta^{(2)} B_{\mu\nu} &= \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu \,,
\end{aligned}
\tag{4}
$$

where $\Lambda(x)$ and $\zeta_\mu(x)$ are gauge parameters. The presence of the boundary in S_{bulk} breaks the $\delta^{(2)}$-invariance, preserving the usual gauge symmetry $\delta^{(1)}$:

$$
\delta^{(1)} S_{bulk} = 0 \;;\;\; \delta^{(2)} S_{bulk} = -4k_2 \int d^4x \, \delta(x_3) \zeta_i \epsilon_{ijk} \partial_j A_k \,,
\tag{5}
$$

where Latin indices run from 0 to 2: $i, j, \dots = \{0, 1, 2\}$.

The total action S_{tot} is composed by four terms

$$
S_{tot} = S_{bulk} + S_{gf} + S_{ext} + S_{bd} \,,
\tag{6}
$$

where S_{bulk} is given by Equation (1), S_{gf} is the gauge fixing term

$$
S_{gf} = \int d^4x \, \theta(x_3) \left(b A_3 + d_i B_{3i} \right) \,,
\tag{7}
$$

and $b(x)$ and $d_i(x)$ are Lagrange multipliers implementing the gauge conditions

$$
A_3 = B_{3i} = 0 \,.
\tag{8}
$$

In S_{ext}, external sources $J_i(x)$ and $J_{ij}(x)$ are introduced

$$
S_{ext} = \int d^4x \, \theta(x_3) \left(J_i A_i + \frac{1}{2} J_{ij} B_{ij} \right) \,,
\tag{9}
$$

by means of which the quantum fields surviving the gauge conditions in Equation (8) can be defined. Finally, S_{bd} is the most general boundary term defined on $x_3 = 0$ compatible with power counting

$$
S_{bd} = \int d^4x \, \delta(x_3) \left(a_1 \epsilon_{ijk} A_i \partial_j A_k + a_2 A_i \partial_3 A_i + a_3 \epsilon_{ijk} A_i B_{jk} + \frac{m}{2} A_i A_i \right) \,,
\tag{10}
$$

where a_i and m are constant parameters and the canonical mass assignments in Equation (2) and $[\delta] = 1$ have been used. Notice that

$$
\begin{aligned}
\int d^4x \, \theta(x_3) \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} &= 2 \int d^4x \, \theta(x_3) \partial_\mu (\epsilon_{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}) \\
&= -4 \int d^4x \, \delta(x_3) \epsilon_{ijk} A_i \partial_j A_k \,,
\end{aligned}
\tag{11}
$$

which justifies the fact that we did not introduce in S_{bulk} (Equation (1)) a term $F\tilde{F}$ in favor of its Chern-Simons-like boundary counterpart, identified by the constant a_1 in Equation (10). In addition,

we chose to keep 3D covariance on the boundary. A more general, non-covariant boundary term could have been written [26–28].

The equations of motion of the fields A_i and B_{ij} are

$$
\begin{aligned}
\frac{\delta S_{tot}}{\delta A_i} &= \theta(x_3)[-2k_1\epsilon_{ijk}\partial_3 B_{jk} - 4k_2\partial_3^2 A_i - 4k_2\partial_j^2 A_i + 4k_2\partial_i\partial_j A_j + J_i] \\
&\quad + \delta(x_3)[(a_3 - 2k_1)\epsilon_{ijk}B_{jk} + (a_2 - 4k_2)\partial_3 A_i + 2a_1\epsilon_{ijk}\partial_j A_k + m A_i] = 0
\end{aligned}
\tag{12}
$$

and

$$
\frac{\delta S_{tot}}{\delta B_{ij}} = \theta(x_3)[4k_1\epsilon_{ijk}\partial_3 A_k + J_{ij}] + \delta(x_3)[2a_3\epsilon_{ijk}A_k] = 0 .
\tag{13}
$$

From the equations of motion (Equations (12) and (13)) and performing $\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}dx_3$, we get the boundary conditions

$$
(a_3 - 2k_1)\epsilon_{ijk}B_{jk} + (a_2 - 4k_2)\partial_3 A_i + 2a_1\epsilon_{ijk}\partial_j A_k + m A_i\Big|_{x_3=0} = 0
\tag{14}
$$

$$
a_3 A_i\big|_{x_3=0} = 0 .
\tag{15}
$$

The equations of motion lead also to the following Ward identities, broken due to the presence of the boundary

$$
\int_0^\infty dx_3\,\partial_i J_i = -\partial_i\left(2k_1\epsilon_{ijk}B_{jk} + 4k_2\partial_3 A_i\right)\Big|_{x_3=0}
\tag{16}
$$

$$
\int_0^\infty dx_3\,\partial_j J_{ij} = (4k_1 - 2a_3)\,\epsilon_{ijk}\partial_j A_k\Big|_{x_3=0} ,
\tag{17}
$$

where, to obtain Equation (16), the boundary conditions in Equation (14) have been used. On the boundary $x_3 = 0$ and at vanishing external sources, i.e., on the mass shell, the above Ward identities imply

$$
\partial_i\left(2k_1\epsilon_{ijk}B_{jk} + 4k_2\partial_3 A_i\right)\Big|_{x_3=0} = 0
\tag{18}
$$

and

$$
\epsilon_{ijk}\partial_j A_k\Big|_{x_3=0} = 0 .
\tag{19}
$$

Notice that, due to Equation (19), the a_1-term in the boundary action S_{bd} in Equation (10) vanishes. Therefore, without loss of generality, we may rule it out

$$
a_1 = 0 .
\tag{20}
$$

Equations (18) and (19) reveal an electromagnetic structure on the boundary $x_3 = 0$, since they suggest to define an "electric" and a "magnetic" field:

$$
\begin{aligned}
\mathcal{E}_i &\equiv A_i \tag{21}\\
\mathcal{H}_i &\equiv k_1\epsilon_{ijk}B_{jk} + 2k_2\partial_3 A_i , \tag{22}
\end{aligned}
$$

which allow identifying the degrees of freedom on the boundary $x_3 = 0$ as the corresponding electromagnetic potentials. Indeed, Equations (18) and (19) are solved by introducing a 3D vector field $\xi_i(X)$ and a 3D scalar field $\Phi(X)$, respectively:

$$
\sqrt{M}\epsilon_{ijk}\partial_j\xi_k(X) \equiv 2k_1\epsilon_{ijk}B_{jk}(x) + 4k_2\partial_3 A_i(x)\Big|_{x_3=0}
\tag{23}
$$

$$
\frac{1}{\sqrt{M}}\partial_i\Phi(X) \equiv A_i(x)\big|_{x_3=0} ,
\tag{24}
$$

where a massive scaling parameter M has been introduced in order to make compatible the mass dimensions in Equation (2) of the 4D fields A_μ and $B_{\mu\nu}$ with those of their 3D boundary counterparts Φ and ξ_i [29]. In 3D spacetime dimensions, in fact, vector fields and scalar fields should have the following mass dimensions:

$$[\xi] = [\Phi] = \frac{1}{2} . \tag{25}$$

A comment is in order: It turns out that a particular solution of the general boundary conditions in Equations (14) and (15) exists, which makes the physics independent of the Maxwell term. In fact, the choice

$$a_2 \neq 4k_2 \; ; a_3 = 2k_1 \; ; m = \text{any} \tag{26}$$

implies, from Equations (14) and (15), (Dirichlet and) Neumann boundary conditions for the gauge field A_i on $x_3 = 0$

$$\partial_3 A_i = 0|_{x_3=0} . \tag{27}$$

Consequently, the Ward identities do not depend on the coupling k_2 of the Maxwell term in the action in Equation (1), and the theory is indistinguishable, under any respect, from the pure topological BF theory with planar boundary. The fact that the Neumann condition for the gauge field A_i, which is a solution of the general boundary conditions in Equations (14) and (15), makes the Maxwell term transparent and the non-topological theory equivalent to a topological one, is the first nontrivial result of this paper. Since the aim of this paper is to study if and how the non-topological Maxwell term has an impact on the physics on the boundary, we proceed from now disregarding the solution in Equation (26). In particular, the boundary condition in Equation (15) is solved by

$$a_3 = 0 . \tag{28}$$

The k_2-Maxwell term manifests itself on the r.h.s. of the Ward identity in Equation (16) by means of $\partial_3 A_i|_{x_3=0}$. We observe that, on the boundary $x_3 = 0$, the fields $A_i|_{x_3=0}$ and $\partial_3 A_i|_{x_3=0}$ must be treated as independent dynamical fields [30]. Consequently, we need to couple, on the boundary $x_3 = 0$, an external source $\widehat{J}_i$ to $\partial_3 A_i|_{x_3=0}$, as done in [31], where the 3D Maxwell theory with boundary has been studied:

$$S_{ext} \rightarrow \hat{S}_{ext} = \int d^4x \left[\theta(x_3) \left(J_i A_i + \frac{1}{2} J_{ij} B_{ij} \right) + \delta(x_3) \widehat{J}_i \partial_3 A_i \right] . \tag{29}$$

3. The Boundary Algebra

Differentiating the two Ward identities in Equations (16) and (17) with respect to the external sources $J_i(x)$, $J_{ij}(x)$ and $\hat{J}_i(x)$ and then going at $J = 0$ lead to six algebraic relations. We consider the subalgebra obtained as follows.

Differentiating the Ward identity in Equation (16) with respect to $J_m(x')$, and then going at vanishing external source, we get

$$\partial_m^X \delta^{(3)}(X - X') = \partial_i^X \left(-2k_1 \epsilon_{ijk} \Delta_{A_m B_{jk}}(X', X) - 4k_2 \Delta_{A_m \partial_3 A_i}(X', X) \right) . \tag{30}$$

In Equation (30), $\partial_i^X \equiv \frac{\partial}{\partial X_i}$, and the time-ordered propagator between two generic fields $\Phi(X)$ and $\Phi(X)$ is defined on the generating functional of the connected Green functions $Z_c[J]$, as usual, as

$$\Delta_{\Phi\Psi}(X, X') \equiv \left. \frac{\delta^{(2)} Z_c}{\delta J_\Phi(X) \delta J_\Psi(X')} \right|_{J_\Phi = J_\Psi = 0} \tag{31}$$
$$= \theta(x_0 - x_0') \langle \Phi(X) \Psi(X') \rangle + \theta(x_0' - x_0) \langle \Psi(X') \Phi(X) \rangle.$$

Hence, from Equation (30), we have

$$
\begin{aligned}
\partial_m^X \delta^{(3)}(X - X') = {}& \delta(x_0 - x_0') \langle [A_m(X'), 2k_1 \widetilde{B}(X) + 4k_2 \partial_3 A_0(X)] \rangle \\
& - 2\theta(x_0 - x_0') \langle \left[\partial_i^X \left(k_1 \epsilon_{ijk} B_{jk} + 2k_2 \partial_3 A_i \right)(X) A_m(X') \right] \rangle \\
& - 2\theta(x_0' - x_0) \langle \left[A_m(X') \partial_i^X \left(k_1 \epsilon_{ijk} B_{jk} + 2k_2 \partial_3 A_i \right)(X) \right] \rangle,
\end{aligned}
\tag{32}
$$

where we define

$$
\widetilde{B}(X) \equiv \epsilon_{\alpha\beta} B_{\alpha\beta}(X) .
\tag{33}
$$

The last two terms on the r.h.s. of Equation (32) vanish on-shell due to Equation (18), so that, at vanishing external sources, we get

$$
\partial_m^X \delta^{(3)}(X - X') = \delta(x_0 - x_0') \langle [A_m(X'), 2k_1 \widetilde{B}(X) + 4k_2 \partial_3 A_0(X)] \rangle .
\tag{34}
$$

Following the same steps, differentiating the Ward identity in Equation (16) with respect to the external sources $J_{mn}(x')$ and $\widehat{J}_m(x')$, we get, respectively,

$$
\delta(x_0 - x_0') \langle [B_{mn}(X'), k_1 \widetilde{B}(X) + 2k_2 \partial_3 A_0(X)] \rangle = 0
\tag{35}
$$

and

$$
\delta(x_0 - x_0') \langle [k_1 \widetilde{B}(X) + 2k_2 \partial_3 A_0(X), \partial_3 A_m(X')] \rangle = 0 .
\tag{36}
$$

On the other hand, deriving the Ward identity in Equation (17) with respect to $J_{mn}(x')$, we obtain

$$
\delta(x_0 - x_0') \langle [B_{mn}(X'), k_1 \widetilde{B}(X) + 2k_2 \partial_3 A_0(X)] \rangle = 0 .
\tag{37}
$$

From the above algebraic relations, we get the following subalgebra formed by equal-time commutators:

$$
\langle [A_\alpha(X), 2k_1 \widetilde{B}(X') + 4k_2 \partial_3 A_0(X')] \rangle_{x_0 = x_0'} = \partial_\alpha^X \delta^{(2)}(X - X')
\tag{38}
$$

$$
\langle [A_\alpha(X), A_\beta(X')] \rangle_{x_0 = x_0'} = 0
\tag{39}
$$

$$
\left\langle \left[2k_1 \widetilde{B}(X) + 4k_2 \partial_3 A_0(X), 2k_1 \widetilde{B}(X') + 4k_2 \partial_3 A_0(X') \right] \right\rangle_{x_0 = x_0'} = 0 ,
\tag{40}
$$

which, written in terms of the 3D boundary fields $\xi_i(X)$ (Equation (23)) and $\Phi(X)$ (Equation (24)), implies

$$
\langle [\Phi(X), \epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X')] \rangle_{x_0 = x_0'} = \delta^{(2)}(X - X')
\tag{41}
$$

$$
\langle [\Phi(X), \Phi(X')] \rangle_{x_0 = x_0'} = 0
\tag{42}
$$

$$
\langle [\epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X), \epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X')] \rangle_{x_0 = x_0'} = 0 .
\tag{43}
$$

4. The Action Induced on the 3D Boundary

The commutators in Equations (41)–(43) can be interpreted as equal-time canonical commutation relations for the 3D canonically conjugate variables

$$
\begin{aligned}
q(X) &\equiv \frac{1}{\sqrt{M}} \Phi(X) \\
p(X) &\equiv \sqrt{M} \epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X) ,
\end{aligned}
\tag{44}
$$

and this allows us to identify the most general action $S_{3D}[\Phi, \xi]$ induced on the planar boundary $x_3 = 0$ of the 4D Maxwell-BF theory, which must display the following features:

1. The 3D action $S_{3D}[\Phi, \xi]$ must be a local integrated functional of the 3D fields in Equations (23) and (24), with canonical mass dimension equal to three.
2. The 3D Lagrangian $\mathcal{L}_{3D}[\Phi, \xi]$ must be such that the relation between the canonically conjugate variables in Equation (44) holds true:

$$p(X) = \frac{\partial \mathcal{L}_{3D}}{\partial \dot{q}(X)} \, , \tag{45}$$

which implies that the Lagrangian $\mathcal{L}_{3D}[\Phi, \xi]$ must contain time derivatives only in the term $p\dot{q}$.

3. The action $S_{3D}[\Phi, \xi]$ must display the two symmetries, which leave invariant the definitions in Equation (23) and (24):

    (a) gauge symmetry

    $$\delta_{gauge}\xi_i = \partial_i \Lambda \tag{46}$$

    (b) shift symmetry

    $$\delta_{shift}\Phi = \text{constant} \, . \tag{47}$$

The most general action satisfying the above requests is

$$S_{3D}[\Phi, \xi] = \int d^3 X \left[c_1(\epsilon_{\alpha\beta}F_{\alpha\beta})(\partial_0\Phi) + c_2 F_{\alpha\beta}F_{\alpha\beta} + c_3 \partial_\alpha \Phi \partial_\alpha \Phi \right] , \tag{48}$$

where $F_{\alpha\beta} = \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha$ and c_i, $i = 1, 2, 3$ are constants. From the action $S_{3D}[\Phi, \xi]$, we get the equations of motion

$$\frac{\delta S_{3D}}{\delta \Phi} = -2\partial_\alpha(c_1\epsilon_{\alpha\beta}\partial_0\xi_\beta + c_3\partial_\alpha\Phi) = 0 \tag{49}$$

$$\frac{\delta S_{3D}}{\delta \xi_\alpha} = 2\partial_\beta(c_1\epsilon_{\alpha\beta}\partial_0\Phi + 2c_2 F_{\alpha\beta}) = 0 \, . \tag{50}$$

5. Holographic Constraint and Duality

The equations of motion (Equations (49) and (50)) of the scalar-vector 3D action $S_{3D}[\Phi, \xi]$ (Equation (48)) must be compatible with the boundary conditions in Equations (14) and (15) of the 4D Maxwell-BF theory on the planar boundary $x_3 = 0$. To make this holographic contact [32], the boundary condition in Equation (14) written in terms of the boundary degrees of freedom ξ_i (Equation (23)) and Φ (Equation (24)) is

$$\epsilon_{ijk}\partial_j\xi_k - \kappa\partial_i\Phi = 0 \, , \tag{51}$$

where, besides Equations (20) and (28), we cho0se

$$a_2 = 0 \, , \tag{52}$$

and we define the dimensionless normalized mass parameter

$$\kappa \equiv \frac{m}{M} \, . \tag{53}$$

We recognize in Equation (51) the duality relation found in [33], which extracts fermionic degrees of freedom from bosonic ones [18,19,34]. We come back to this point below. Here, it appears as the unique boundary condition that relates the 4D Maxwell-BF theory with boundary and its holographic 3D counterpart(s), as explicitly shown below.

Notice that the three components $i = \{0, \alpha\}$ of Equation (51) are

$$\epsilon_{\alpha\beta} F_{\alpha\beta} - 2\kappa\partial_0 \Phi \;=\; 0 \tag{54}$$

$$\epsilon_{\alpha\beta}\partial_0\xi_\beta - \epsilon_{\alpha\beta}\partial_\beta\xi_0 + \kappa\partial_\alpha\Phi \;=\; 0\,, \tag{55}$$

which are compatible with Equations (49) and (50) if

$$c_2 = -\frac{1}{2\kappa}c_1 \;;c_3 = \kappa c_1 \tag{56}$$

and if the temporal gauge choice for the 3D gauge field ξ_i is imposed

$$\xi_0 = 0\,. \tag{57}$$

6. Summary of Results and Discussion

When a boundary is introduced in a quantum field theory, a crucial role is played by the boundary term, which in the case studied in this paper is represented by S_{bd} (Equation (10)), which depends by a number of constant parameters which need to be fine tuned in order to determine the holographic theory induced on the boundary. For the 4D Maxwell-BF theory, the boundary term finally reduces to

$$S_{bd} = \frac{m}{2}\int d^4x\,\delta(x_3)A_i A_i\,, \tag{58}$$

which depends on one massive parameter m only.

In presence of a boundary, the question naturally arises of which boundary conditions should be imposed. The procedure described in this paper leads to the following boundary condition compatible with the holographic construction:

$$2k_1\epsilon_{ijk}B_{jk} + 4k_2\partial_3 A_i - mA_i\Big|_{x_3=0} = 0\,, \tag{59}$$

which involves both the k_1-BF and the k_2-Maxwell terms. Notice that it depends on one parameter only (m). It is of a nonstandard type, since it does not fall into the usual Dirichlet, Neumann or Robin boundary conditions on each field, separately. On the contrary, it mixes both the fields, and the effect of the Maxwell term is to introduce a dependence on the transverse component of the gauge field with respect to the planar boundary, which is independent of the longitudinal components, and consequently must be treated as an independent dynamical variable on the boundary. As we remarked, an unexpected consequence of the boundary conditions in Equations (14) and (15) is that they can be solved by the set of parameters in Equation (26), which implies, in particular, Neumann boundary condition for the gauge field A_μ. This corresponds to eliminating any dependence from k_2, i.e., from the Maxwell term, in the physics on the boundary, represented by the current algebra as well as the boundary conditions themselves.

The boundary breaks all the invariances of the unbounded theory: translations, parity and gauge symmetries. The first consequence concerns the choice of the gauge conditions, implemented by the gauge fixing term S_{gf} in Equation (7), which does not need to be covariant. With deeper consequences, the Ward identities describing the Ward identities are broken by boundary terms:

$$\int_0^\infty dx_3\,\partial_i J_i = -\partial_i\left(2k_1\epsilon_{ijk}B_{jk} + 4k_2\partial_3 A_i\right)\Big|_{x_3=0} \tag{60}$$

$$\int_0^\infty dx_3\,\partial_j J_{ij} = 4k_1\,\epsilon_{ijk}\partial_j A_k\Big|_{x_3=0}\,. \tag{61}$$

At $J = 0$, i.e., going on-shell, the vanishing of the boundary breakings lead to define on $x_3 = 0$ an "electric" field and a "magnetic" field, and the corresponding potentials:

$$\mathcal{E}_i \equiv A_i \propto \partial_i \Phi \tag{62}$$

$$\mathcal{H}_i \equiv k_1 \epsilon_{ijk} B_{jk} + 2k_2 \partial_3 A_i \propto \epsilon_{ijk} \partial_j \xi_k , \tag{63}$$

which allow physically interpreting the contribution of the Maxwell term as a deformation of the magnetic field on the boundary. Without Maxwell term, i.e., at $k_2 = 0$, the fields A_i and B_{ij} are interpreted as electric and magnetic fields on the boundary, respectively. Their transverse component do not enter in the game and could safely be eliminated by Neumann boundary conditions. In the presence of the Maxwell term, this is no longer true for the gauge field A, for which both the longitudinal and transverse components are physically important: the former as electric field, and the latter as deformation of the main magnetic field represented by the dual of the B-field. The potentials corresponding to the boundary electric and magnetic fields (Equations (62) and (63)) are the degrees of freedom by means of which the holographic 3D theory is constructed: a vector field ξ_i and a scalar field Φ.

The algebra obtained from the broken Ward identities in Equations (60) and (61) by differentiating them with respect to the external sources J, written in terms of the boundary degrees of freedom is

$$\langle [\Phi(X), \epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X')] \rangle_{x_0 = x_0'} = \delta^{(2)}(X - X') \tag{64}$$

$$\langle [\Phi(X), \Phi(X')] \rangle_{x_0 = x_0'} = 0 \tag{65}$$

$$\langle [\epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X), \epsilon_{\alpha\beta} \partial_\alpha \xi_\beta(X')] \rangle_{x_0 = x_0'} = 0 , \tag{66}$$

which can be seen as equal-time canonical commutation relations between canonical variables $q(X)$ and $p(X)$.

Once the canonical variables are identified, the corresponding action is found to be

$$S_{3D}[\Phi, \xi] = c_1 \int d^3 X \left[(\epsilon_{\alpha\beta} F_{\alpha\beta})(\partial_0 \Phi) - \frac{1}{2\kappa} F_{\alpha\beta} F_{\alpha\beta} + \kappa \partial_\alpha \Phi \partial_\alpha \Phi \right] , \tag{67}$$

which is the most general 3D local integrated functional built with a scalar and a vector field, respecting power counting and invariant under the gauge (Equation (46)) and shift (Equation (47)) symmetries, and, most importantly, whose equations of motion are compatible with the boundary conditions in Equation (59):

$$\epsilon_{ijk} F_{jk} - 2\kappa \partial_i \Phi = 0 . \tag{68}$$

Equation (68) coincides with the duality relation between a scalar and a vector field which has been invoked in [18,19,33] as the main tool for the mechanism of fermionization of bosonic degrees of freedom. In other words, the effective dynamical variables living on the boundary of the 4D Maxwell-BF are fermionic. This feature is crucial for the interpretation of the boundary degrees of freedom as the edge states of the 3D topological insulators [35,36]. The new fact that we are recovering here is that this property, which has been related to the topological character of the bulk theory [17], indeed also holds for a non-topological theory such as Maxwell-BF theory. We comment in more detail on this point below. The duality relation in Equation (68) has also more field theoretical consequences. Thanks to Equation (68), indeed, the action in Equation (67) is covariant, despite the appearances. In fact, the scalar field Φ can be eliminated from the action through Equation (68), and we find the 3D Maxwell theory

$$S_{Max}[\xi] = \frac{c_1}{2\kappa} \int d^3 X \, F_{ij} F_{ij} . \tag{69}$$

Alternatively, the duality relation in Equation (68) allows us to trade the vector for the scalar field

$$S_{KK}[\Phi] = \kappa c_1 \int d^3 X \, \partial_i \Phi \partial_i \Phi \,. \tag{70}$$

The scalar–vector 3D theory described by Equation (67), the Maxwell action in Equation (69) and the massless Klein–Gordon action in Equation (70) are all holographic counterparts of the 4D Maxwell-BF theory with planar boundary. The two actions in Equations (69) and (70) are related by the duality relation in Equation (51), which may be written in a way to emphasize its strong–weak coupling aspect:

$$\partial_i \Phi \leftrightarrow \epsilon_{ijk} F_{jk} \quad \cup \quad \kappa \leftrightarrow \frac{1}{\kappa} \,. \tag{71}$$

In this form, it is apparent that the coupling κ in Equation (53) governs the regimes where one type of action dominates with respect to the other: at strong coupling (very large κ), the dominating term is the Maxwell one, while, at weak coupling (very small κ), it is the massless scalar action which dominates. At intermediate regimes, both the degrees of freedom are present, and the relevant action is Equation (67), whose degrees of freedom are fermionic, due to Equation (68), as remarked above. It is this intermediate regime which is relevant for the topological insulators, for which the same action in Equation (67) has been proposed in [35]. Surprisingly, the order parameter κ, which distinguishes the various regimes, is directly related through Equation (53) to the only effective parameter m, which survives in the boundary term in Equation (58), and hence plays a much more crucial role than one might expect at first sight.

We conclude this paper with a remark concerning the effect of the presence of the Maxwell term in the bulk action in Equation (1) together with a general consideration on the topological nature of quantum field theories. The main feature of topological quantum field theories is the lack of local observables, the only observables being global geometrical properties of the manifolds on which they are built. Technically, this means that the local cohomology of the BRS operator is empty. Examples of topological quantum field theories are the 3D Chern–Simons theory and BF theories, which may be defined in any spacetime dimensions. The 4D Maxwell-BF theory studied in this paper is not topological, since its action depends on the metric through the Maxwell term, and its cohomological structure is nontrivial. Nevertheless, the bulk action has no local observables since the equations of motion enforce the vanishing of the $F_{\mu\nu}$ tensor; thus, we are looking at a different class of "topological" theories where the boundary is expected to carry all the physical information. If a boundary is introduced, as above, this nontriviality reflects in the algebra derived from the Ward identities in Equations (60) and (61), which is rather complicated. We write down the four relations (out of six) as Equations (34)–(37), where the dependence on the Maxwell term is highlighted by the coefficient k_2. In addition, the boundary condition in Equation (59) explicitly depends on the Maxwell term, which physically results in a perturbation of the magnetic field $\mathcal{H}$ (Equation (63)). On the other hand, when constructing the holographic scalar-vector 3D theory, the presence of the Maxwell term is buried in the definition of the vector potential ξ_i in Equation (63), and both 3D actions (in their duality-related representations, Equations (67), (69) and (70), and the duality relation, Equation (68)) are the same as in the case of pure BF with boundary. In other words, from the holographic point of view, Maxwell-BF theory with boundary is indistinguishable from the pure topological BF theory, as if holography would protect the topological character of bulk theories.

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References

1. Witten, E. Supersymmetry and Morse theory. *J. Diff. Geom.* **1982**, *17*, 661. [CrossRef]
2. Witten, E. Topological Quantum Field Theory. *Commun. Math. Phys.* **1988**, *117*, 353. [CrossRef]
3. Atiyah, M. Topological quantum field theories. *Inst. Hautes Etudes Sci. Publ. Math.* **1989**, *68*, 175. [CrossRef]
4. Witten, E. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nuclear Phys. B* **1988**, *311*, 46. [CrossRef]
5. Moore, G.W.; Seiberg, N. Taming the Conformal Zoo. *Phys. Lett. B* **1989**, *220*, 422. [CrossRef]
6. Cattaneo, A.S.; Mnev, P.; Reshetikhin, N. Classical and quantum Lagrangian field theories with boundary. *PoS CORFU* **2011**, *2011*, 044. [CrossRef]
7. Symanzik, K. Schrodinger Representation and Casimir Effect in Renormalizable Quantum Field Theory. *Nuclear Phys. B* **1981**, *190*, 1. [CrossRef]
8. Amoretti, A.; Blasi, A.; Maggiore, N.; Magnoli, N. Three-dimensional dynamics of four-dimensional topological BF theory with boundary. *New J. Phys.* **2012**, *14*, 113014. [CrossRef]
9. Blasi, A.; Ferraro, D.; Maggiore, N.; Magnoli, N.; Sassetti, M.; Symanzik's Method Applied To The Fractional Quantum Hall Edge States. *Ann. Phys.* **2008**, *17*, 885. [CrossRef]
10. Amoretti, A.; Braggio, A.; Caruso, G.; Maggiore, N.; Magnoli, N. Introduction of a boundary in topological field theories. *Phys. Rev. D* **2014**, *90*, 125006. [CrossRef]
11. Chern, S.S.; Simons, J. Characteristic forms and geometric invariants. *Ann. Math.* **1974**, *99*, 48. [CrossRef]
12. Blasi, A.; Collina, R. The Chern-Simons model with boundary: A Cohomological approach. *Int. J. Mod. Phys. A* **1992**, *7*, 3083. [CrossRef]
13. Horowitz, G.T. Exactly Soluble Diffeomorphism Invariant Theories. *Commun. Math. Phys.* **1989**, *125*, 417. [CrossRef]
14. Karlhede, A.; Rocek, M. Topological Quantum Field Theories in Arbitrary Dimensions. *Phys. Lett. B* **1989**, *224*, 58. [CrossRef]
15. Blasi, A.; Maggiore, N.; Montobbio, M. Noncommutative two dimensional BF model. *Nuclear Phys. B* **2006**, *740*, 281. [CrossRef]
16. Birmingham, D.; Blau, M.; Rakowski, M.; Thompson, G. Topological field theory. *Phys. Rept.* **1991**, *209*, 129. [CrossRef]
17. Amoretti, A.; Braggio, A.; Caruso, G.; Maggiore, N.; Magnoli, N. Holography in flat spacetime: 4D theories and electromagnetic duality on the border. *J. High Energy Phys.* **2014**, *1404*, 142. [CrossRef]
18. Aratyn, H. A Bose Representation For The Massless Dirac Field In Four-dimensions. *Nuclear Phys. B* **1983**, *227*, 172. [CrossRef]
19. Aratyn, H. Fermions From Bosons In (2+1)-dimensions. *Phys. Rev. D* **1983**, *28*, 2016. [CrossRef]
20. Blasi, A.; Maggiore, N.; Magnoli, N.; Storace, S. Maxwell-Chern-Simons Theory with Boundary. *Class. Quant. Grav.* **2010**, *27*, 165018. [CrossRef]
21. Maggiore, N. Holographic reduction of Maxwell-Chern-Simons theory. *Eur. Phys. J. Plus* **2018**, *133*, 281. [CrossRef]
22. Deser, S.; Jackiw, R.; Templeton, S. Topologically Massive Gauge Theories. *Ann. Phys.* **1982**, *140*, 372–411. Erratum in: **2000**, *281*, 409. Erratum in: **1988**, *185*, 406. [CrossRef]
23. Deser, S.; Jackiw, R.; Templeton, S. Three-Dimensional Massive Gauge Theories. *Phys. Rev. Lett.* **1982**, *48*, 975. [CrossRef]
24. Blau, M.; Thompson, G. A New Class of Topological Field Theories and the Ray-singer Torsion. *Phys. Lett. B* **1989**, *228*, 64. [CrossRef]
25. Blau, M.; Thompson, G. Topological Gauge Theories of Antisymmetric Tensor Fields. *Ann. Phys.* **1991**, *205*, 130 [CrossRef]
26. Maggiore, N. From Chern–Simons to Tomonaga–Luttinger. *Int. J. Mod. Phys. A* **2018**, *33*, 1850013. [CrossRef]
27. Blasi, A.; Maggiore, N. Massive deformations of rank-2 symmetric tensor theory (a.k.a. BRS characterization of Fierz–Pauli massive gravity). *Class. Quant. Grav.* **2017**, *34*, 015005. [CrossRef]
28. Blasi, A.; Maggiore, N. Massive gravity and Fierz-Pauli theory. *Eur. Phys. J. C* **2017**, *77*, 614. [CrossRef]
29. Blasi, A.; Maggiore, N.; Montobbio, M. Instabilities of noncommutative two dimensional bf model. *Mod. Phys. Lett. A* **2005**, *20*, 2119. [CrossRef]
30. Karabali, D.; Nair, V.P. Boundary Conditions as Dynamical Fields. *Phys. Rev. D* **2015**, *92*, 125003. [CrossRef]

31. Maggiore, N. Conserved chiral currents on the boundary of 3D Maxwell theory. *J. Phys. A* **2019**, *52*, 115401. [CrossRef]

32. Amoretti, A.; Braggio, A.; Maggiore, N.; Magnoli, N. Thermo-electric transport in gauge/gravity models. *Adv. Phys. X* **2017**, *2*, 409. [CrossRef]

33. Amoretti, A.; Blasi, A.; Caruso, G.; Maggiore, N.; Magnoli, N. Duality and Dimensional Reduction of 5D BF Theory. *Eur. Phys. J. C* **2013**, *73*, 2461. [CrossRef]

34. Amoretti, A.; Braggio, A.; Caruso, G.; Maggiore, N.; Magnoli, N. 3+1D Massless Weyl spinors from bosonic scalar-tensor duality. *Adv. High Energy Phys.* **2014**, *2014*, 635286. [CrossRef]

35. Cho, G.Y.; Moore, J.E. Topological BF field theory description of topological insulators. *Ann. Phys.* **2011**, *326*, 1515. [CrossRef]

36. Blasi, A.; Braggio, A.; Carrega, M.; Ferraro, D.; Maggiore, N.; Magnoli, N. Non-Abelian BF theory for 2+1 dimensional topological states of matter. *New J. Phys.* **2012**, *14*, 013060. [CrossRef]