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A New Variational Approach to Linearization of Traction Problems in Elasticity

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Abstract In a recent paper, we deduced a new energy functional for pure traction problems in elasticity, as the variational limit of nonlinear elastic energy functional for a material body subject to an equilibrated force field: a kind of Gamma limit with respect to the weak convergence of strains, when

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a suitable small parameter tends to zero. This functional exhibits a gap, that makes it different from the classical linear elasticity functional. Nevertheless, a suitable compatibility condition on the force field ensures coincidence of related minima and minimizers. Here, we show some relevant properties of the new functional and prove stronger convergence of minimizing sequences for suitable choices of nonlinear elastic energies.

Keywords Calculus of Variations, Pure Traction problems, Linear Elasticity, Nonlinear Elasticity, Finite Elasticity, Critical points, Gamma-convergence, Asymptotic analysis, nonlinear Neumann problems.

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Dedicated to Alexander Ioffe on the occasion of his 80^{th} Birthday

1 Introduction

In the article [1] we studied the variational deduction of pure traction problem in linear elasticity starting from general theory of finite elasticity and provided a rigorous deduction of the limit energy functional by a kind of Gamma convergence approach. Quite surprisingly and in contrast with the case of Dirichlet boundary condition, in the case of pure traction the limit functional deduced in [1] is different from the classical energy of linear elasticity: however such new functional achieves the same minimum and has the same set of minimizing displacements, provided an additional compatibility condition is fulfilled. In the present article we show additional structural properties of this new limit functional in the most general setting, together with more refined convergence results in the case of Saint Venant–Kirchhoff energy density.

2 Preliminaries

This paper is focused on the properties of the functional

$$\mathcal{F}(\mathbf{v}) \;:=\; \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \int_{\Omega} \mathcal{V}_0\left(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\right)\, d\mathbf{x} \;-\; \mathcal{L}(\mathbf{v})\ .$$

Here and in the sequel, we set: $N = 2,3$, $\mathcal{M}^{N\times N}_{skew}$ denotes the set of skew-symmetric $N\times N$ real matrices, $\Omega \subset \mathbb{R}^N$ is a Lipschitz open set representing the reference configuration of an hyperelastic material body undergoing pure traction, $\mathcal{V}_0(\mathbf{x},\cdot)$ are uniformly positive definite quadratic forms on square matrices, the vector field $\mathbf{v}$ in $H^1(\Omega,\mathbb{R}^N)$ denotes a displacement and $\mathbb{E}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$ denotes the related linearized strain, while $\mathcal{L}(\mathbf{v})$ represents the work done by the load for displacement $\mathbf{v}$,

$$\mathcal{L}(\mathbf{v}) \;:=\; \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}\, d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x}\,, \qquad \mathbf{f}\in L^2(\partial\Omega;\mathbb{R}^N),\ \mathbf{g}\in L^2(\Omega)\,,$$

here $\mathbf{f}$ and $\mathbf{g}$ are, respectively, the prescribed boundary and body force fields, moreover we assume that the total load is equilibrated, say

$$\mathcal{L}(\mathbf{z}) \;=\; 0 \qquad \forall\,\mathbf{z}\,:\ \mathbb{E}(\mathbf{z}) \equiv \mathbf{0}\ .$$

Motivations for studying functional $\mathcal{F}$ and its minimization over $\mathbf{v}$ in $H^1(\Omega,\mathbb{R}^N)$ rely on the variational asymptotic analysis developed in [1], where we proved that for pure traction problems in elasticity a gap arises between the classical linearized elasticity functional $\mathcal{E}$,

$$\mathcal{E}(\mathbf{v}) := \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}))\, d\mathbf{x} - \mathcal{L}(\mathbf{v})\,,$$

and the rigorous variational limit of nonlinear elastic energy of a material body subject to an equilibrated force field. Actually such limit is the functional $\mathcal{F}$, provided the load fulfils a suitable compatibility condition: see (8) and Theorem 5.1 below.

The inequality $\mathcal{F}(\mathbf{v}) \leq \mathcal{E}(\mathbf{v})$ for every $\mathbf{v}$ is straightforward. Nevertheless the two functionals cannot coincide: indeed $\mathcal{F}(\mathbf{v}) = -\mathcal{L}(\mathbf{v}) < \mathcal{E}(\mathbf{v})$ whenever $\mathbf{v}(\mathbf{x}) = \frac{1}{2}\mathbf{W}^2\mathbf{x}$ with $\mathbf{W} \neq \mathbf{0}$ skew symmetric matrix.

Notwithstanding this gap, in [1] we showed that the two functionals $\mathcal{F}$ and $\mathcal{E}$ have the same minimum and same set of minimizers, when the load is equilibrated and compatible (see Theorem (5.1) below).

In the case $N = 2$, the gap between the two functionals can be better clarified as follows (see Remark 2.6 in [1] for more details):

$$\mathcal{F}(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v}) - \frac{1}{4}\left(\int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbf{I}) d\mathbf{x}\right)^{-1} \left[\left(\int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v})\, d\mathbf{x}\right)^{-}\right]^2,$$

where $\alpha^- = \max(-\alpha, 0)$, thus

$$\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v}) \qquad \text{if } N = 2 \quad \text{and} \quad \int_{\Omega} D\mathcal{V}_0(x, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v})\, dx \geq 0\,.$$

Even more explicitly, in a particular case, when $N = 2$ and $\mathcal{V}_0(\mathbf{x},\mathbf{B}) = 4\mu|\mathbf{B}|^2 + 2\lambda|\mathrm{Tr}\mathbf{B}|^2$ with $\lambda,\, \mu > 0$, we get

$$\mathcal{F}(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v}) \,-\, \frac{1}{4}\,|\,\Omega\,|^{-1}\left[\left(\int_\Omega \operatorname{div}\mathbf{v}\; d\mathbf{x}\right)^{-}\right]^2,$$

such evaluation approximately means that for every displacement $\mathbf{v}$ such that the associated deformed configuration $(\mathbf{I}+\mathbf{v})(\Omega)$ of a 2D homogeneous material has greater area than reference configuration Ω, the global energy $\mathcal{F}(\mathbf{v})$ provided by new functional $\mathcal{F}$ evaluated at $\mathbf{v}$ is the same as the one provided by classical linearized elasticity, say $\mathcal{E}(\mathbf{v})$.

The rigorous derivation of the variational theory of linear elasticity [2] from the theory of finite elasticity [3,4] was achieved in [5] through arguments based on De Giorgi theory of Gamma convergence, thus providing a mathematical justification of the classical elasticity in small deformations regime, at least for *Dirichlet or mixed boundary value problem*.

In a more recent paper [1], we have focussed the analysis on the analogous variational question related to Neumann type condition, say the *pure traction problem in elasticity*: the case where the elastic body is subject to a system of equilibrated forces and no Dirichlet condition is assigned on the boundary.

In the present paper, we prove some relevant properties concerning the structure of the new functional and improve its variational connection for a large class of nonlinear energies.

In Section 4, we prove that $\mathcal{F}$ is sequentially lower semicontinuous weak respect to the natural but very weak notion of convergence, namely weak L^2

convergence of linearized strains (see Proposition (4.3)), nevertheless $\mathcal{F}$ exhibits a kind of ”nonlocal” behavior (see Remark 3.2).

In the 2D case, we can prove that $\mathcal{F}$ is a convex functional for every choice of the positive definite quadratic form $\mathcal{V}_0$ or, equivalently, for the variational limit of every nonlinear stored energy density $\mathcal{W}$ fulfilling structural assumptions of general kind in the theory of elasticity: this is shown by making explicit its first variation and showing that the second variation cannot be negative (see (23) and Proposition 4.1).

On the other hand, in the 3D case the functional $\mathcal{F}$ cannot be convex for whatever choice of the positive definite quadratic form $\mathcal{V}_0$ or, equivalently for every nonlinear stored energy density $\mathcal{W}$ fulfilling the standard structural assumptions: see Proposition 4.2 and the general counterexample to convexity therein.

The dichotomy above relies on the fact that there exist pairs of skew-symmetric matrices $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{M}^{3\times 3}_{skew}$ such that $\mathbf{W}_1^2 + \mathbf{W}_2^2$ is not the square of any skew-symmetric matrix: e.g. see (18); while in the 2D case the matrix $\mathbf{W}^2$ is a nonpositive multiple of the identity for every skew-symmetric matrix $\mathbf{W}$.

Notice that $\mathcal{F}$ is not subadditive: indeed even in dimension $N = 2$ formulæ (15) and (19) in Section 4 show that functional $\mathcal{F}$ cannot be subadditive on disjoint sets.

In Section 5, for reader's convenience we summarize and comment preliminary main results of [1] about the variational convergence of pure traction problems, namely functional $\mathcal{F}$ deduced as a weak Gamma limit of functionals $\mathcal{F}_h$ related to general nonlinear elastic energies.

Eventually, in Section 6 we refine the convergence properties for *minimizing sequences of the sequence of functionals* $\mathcal{F}_h$ (e.g $\mathcal{F}(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$): if $\mathcal{W}$ is the *Saint Venant–Kirchhoff energy density* (24) then we show by Theorem 6.1 that there exist subsequences of functionals $\mathcal{F}_h$ and of related minimizing sequence $\mathbf{v}_h$, such that (without relabeling) $\mathbf{v}_h - \mathbb{P}\mathbf{v}_h$ converges weakly in $H^1(\Omega;\mathbb{R}^N)$ and strongly in $W^{1,q}(\Omega,\mathbb{R}^N)$ $(1 \le q < 2)$ to a minimizer of $\mathcal{F}$, provided both (7) and (8) hold true; here and in the sequel $\mathbb{P}$ denotes the orthogonal projection on infinitesimal rigid displacements.

On the other hand, if the strict inequality in compatibility condition (8) is replaced by weak inequality, still over the collection of skew symmetric matrices, then $\operatorname{argmin} \mathcal{F}$ still contains $\operatorname{argmin} \mathcal{E}$ and $\min \mathcal{F} = \min \mathcal{E}$ holds true, but $\mathcal{F}$ may have infinitely many minimizing critical points which are not minimizers of $\mathcal{E}$.

Therefore, only two cases are allowed: either $\min \mathcal{F} = \min \mathcal{E}$ or $\inf \mathcal{F} = -\infty$; actually the second case arises in presence of compressive surface load.

We mention several contributions facing issues in elasticity, which are strictly connected with the context of present paper: [6−21].

3 Asymptotic Analysis of Pure Traction Problem

Referring to the open set $\Omega \subset \mathbb{R}^N,\ N = 2, 3$, as the reference configuration of an hyperelastic material body, the stored energy due to a deformation $\mathbf{y}$ can be expressed as a functional of the deformation gradient $\nabla \mathbf{y}$ as follows

$$\int_{\Omega} \mathcal{W}(\mathbf{x}, \nabla \mathbf{y})\, d\mathbf{x}\,,$$

where $\mathcal{W} : \Omega \times \mathcal{M}^{N\times N} \to [0, +\infty]$ is a coercive frame indifferent function, $\mathcal{M}^{N\times N}$ is the set of real $N \times N$ matrices and $\mathcal{W}(\mathbf{x}, \mathbf{F}) < +\infty$ if and only if $\det \mathbf{F} > 0$.

Then due to frame indifference there exists a function $\mathcal{V}$ such that

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}))\,, \qquad \forall\, \mathbf{F} \in \mathcal{M}^{N\times N}, \ \text{ a.e. } \mathbf{x} \in \Omega.$$

We set $\mathbf{F} = \mathbf{I} + h\mathbf{B}$, where $h > 0$ is an adimensional small parameter and

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) := h^{-2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}).$$

We assume that the reference configuration has zero energy and is stress free, i.e.

$$\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \quad D\mathcal{W}(\mathbf{x}, \mathbf{I}) = \mathbf{0} \quad \text{for a.e. } \mathbf{x} \in \Omega\,,$$

and that $\mathcal{W}$ is regular enough in the second variable, then Taylor's formula entails

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) = \mathcal{V}_0(\mathbf{x}, \operatorname{sym}\mathbf{B}) + o(1) \qquad \text{as } h \to 0_+$$

where $\operatorname{sym}\mathbf{B} := \frac{1}{2}(\mathbf{B}^T + \mathbf{B})$ and

$$\mathcal{V}_0(\mathbf{x}, \operatorname{sym}\mathbf{B}) := \frac{1}{2} \operatorname{sym}\mathbf{B}\, D^2\mathcal{V}(\mathbf{x}, \mathbf{0}) \operatorname{sym}\mathbf{B}.$$

If the deformation $\mathbf{y}$ is close to the identity up to a small displacement, say $\mathbf{y}(\mathbf{x}) = \mathbf{x} + h\mathbf{v}(\mathbf{x})$ with bounded $\nabla \mathbf{v}$, then, by setting $\mathbb{E}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v}^T + \nabla \mathbf{v})$, one easily obtains

$$\lim_{h\to 0}\int_{\Omega}\mathcal{V}_h(\mathbf{x},\nabla\mathbf{v})\,d\mathbf{x}=\int_{\Omega}\mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v}))\,d\mathbf{x}\,. \tag{1}$$

Right-hand side in (1) represents the classical linear elastic deformation energy and such a limit was retained to establish a reasonable justification of linearized elasticity. Moreover in [5] it is proved by Γ-convergence techniques that, under standard structural conditions on $\mathcal{W}$, actually the linear elastic problem is achieved in the limit by exploiting the weak convergence of $H^1(\Omega,\mathbb{R}^N)$, in case of Dirichlet or mixed boundary condition.

The variational limit is different when no Dirichlet boundary condition is present, as we outline briefly here.

In [1], we studied the case of Neumann boundary conditions (pure traction problem in elasticity) assuming that $\mathbf{f}\in L^2(\partial\Omega;\mathbb{R}^N)$, $\mathbf{g}\in L^2(\Omega;\mathbb{R}^N)$ are, respectively, the prescribed boundary and body force fields, and the whole system of forces is equilibrated, namely it fulfils the condition of equilibrated load

$$\mathcal{L}(\mathbf{z})\;=\;0 \qquad \forall\mathbf{z}\;:\;\mathbb{E}(\mathbf{z})\equiv\mathbf{0}\,, \tag{2}$$

which is a standard necessary condition for pure traction in linear elasticity, where

$$\mathcal{L}(\mathbf{v})\;:=\;\int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^{N-1}+\int_{\Omega}\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}\,.$$

We considered the sequence of energy functionals

$$\mathcal{F}_h(\mathbf{v})\;=\;\int_{\Omega}\mathcal{V}_h(\mathbf{x},\nabla\mathbf{v})d\mathbf{x}-\mathcal{L}(\mathbf{v})\,,\qquad \mathbf{v}\in H^1(\Omega,\mathbb{R}^N)\,, \tag{3}$$

and we inquired whether the asymptotic relationship $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$ as $h \downarrow 0$ implies, up to subsequences, some kind of weak convergence of $\mathbf{v}_h$ to a minimizer $\mathbf{v}_0$ of a suitable limit functional in $H^1(\Omega; \mathbb{R}^N)$; to this aim, next example is highly explicative: assume

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \begin{cases} |\mathbf{F}^T\mathbf{F} - \mathbf{I}|^2, & \text{if } \det \mathbf{F} > 0, \\ \\ +\infty, & \text{otherwise,} \end{cases} \tag{4}$$

$\mathbf{g} \equiv \mathbf{f} \equiv \mathbf{0}$, hence $\inf \mathcal{F}_h = 0$ for every $h > 0$, then by choosing a fixed nontrivial $N \times N$ skew-symmetric matrix $\mathbf{W}$, a real number $0 < 2\alpha < 1$ and setting

$$\mathbf{z}_h := h^{-\alpha}\, \mathbf{W}\, \mathbf{x}\,, \tag{5}$$

we get $\mathcal{F}_h(\mathbf{z}_h) = \inf \mathcal{F}_h + o(1)$, though $\mathbf{z}_h$ has no subsequence weakly converging in $H^1(\Omega; \mathbb{R}^N)$.

Therefore, in contrast to [5], one cannot expect weak $H^1(\Omega; \mathbb{R}^N)$ compactness of minimizing sequences for pure traction problem, not even in the simplest case of null external forces: we emphasize that in general nonlinear elasticity setting this difficulty cannot be easily circumvented by standard translations, since $\mathcal{F}_h(\mathbf{v}_h) \neq \mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h)$. Nevertheless, we will show in Theorem 6.1 below that, at least for some special $\mathcal{W}$, if $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$ then up to subsequences $\mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$.

For this reason, we exploited a much weaker topology: in order to have in general some kind of precompactness for sequences $\mathbf{v}_h$ fulfilling $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$, the key idea in our approach consists in working with a very weak notion, say weak $L^2(\Omega;\mathbb{R}^N)$ convergence of linear strains $\mathbb{E}(\mathbf{v}_h)$. Since such convergence does not imply an analogous convergence of the skew symmetric part of the gradient of displacements, one may expect that the Γ limit functional is different from the point-wise limit of $\mathcal{F}_h$, as actually is the case. Under some natural assumptions on $\mathcal{W}$, a careful application of the Rigidity Lemma of [28] together with a suitable tuning of asymptotic analysis with Euler-Rodrigues formula for rotations show that, if $\mathbb{E}(\mathbf{v}_h)$ are bounded in L^2, then up to subsequences $\sqrt{h}\,\nabla\mathbf{v}_h$ converges strongly in L^2 to a constant skew symmetric matrix, while the variational limit of the sequence $\mathcal{F}_h$ with respect to the w-L^2 convergence of linear strains turns out to be the functional $\mathcal{F}$, defined by

$$\mathcal{F}(\mathbf{v}) \;:=\; \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \int_\Omega \mathcal{V}_0\left(\mathbf{x},\,\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2\right)d\mathbf{x} \;-\; \mathcal{L}(\mathbf{v})\,. \qquad (6)$$

In [1], we proved that, if loads are equilibrated

$$\mathcal{L}(\mathbf{z}) \;=\; 0 \qquad \forall\,\mathbf{z}\,:\;\mathbb{E}(\mathbf{z})\equiv\mathbf{0}\,, \qquad (7)$$

and fulfil the compatibility condition

$$\int_{\partial\Omega}\mathbf{f}\cdot\mathbf{W}^2\mathbf{x}\,d\mathcal{H}^{N-1}+\int_\Omega\mathbf{g}\cdot\mathbf{W}^2\mathbf{x}\,d\mathbf{x}\;<\;0 \qquad \forall\,\text{skew symmetric matrix } \mathbf{W}\neq\mathbf{0}, \qquad (8)$$

then the pure traction problem in linear elasticity is rigorously deduced via Γ-convergence from the corresponding pure traction problem formulated in nonlinear elasticity, referring to weak L^2 convergence of the linear strains; moreover minimizers of $\mathcal{F}$ coincide with the ones of linearized elasticity functional $\mathcal{E}$

$$\mathcal{E}(\mathbf{v}) := \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}))\, d\mathbf{x} - \mathcal{L}(\mathbf{v})\,, \tag{9}$$

thus providing a complete variational justification of pure traction problems in linear elasticity at least if (8) is satisfied. In particular, as it is shown in Remark 2.8, this is true when $\mathbf{g} \equiv 0$, $\mathbf{f} = f\mathbf{n}$ with $f > 0$ and $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$, that is when we are in presence of tension-like surface forces.

4 Structural Properties of Functional $\mathcal{F}$

In this section, we develop further the analysis of structural properties of functional $\mathcal{F}$ defined by (6), focussing mainly on convexity and semicontinuity issues.

All along the paper we assume that the reference configuration of the elastic body is a

$$\text{bounded, connected open set } \Omega \subset \mathbb{R}^N \text{ with Lipschitz boundary, } N = 2, 3, \tag{10}$$

and set these notations: the generic point $\mathbf{x} \in \Omega$ has components x_j referring to the standard basis vectors $\mathbf{e}_j$ in $\mathbb{R}^N$; $\mathcal{L}^N$ and $\mathcal{B}^N$ denote respectively the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbb{R}^N$.

The notation for vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^N$ and $N \times N$ real matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ are as follows:

$\mathbf{a} \cdot \mathbf{b} = \sum_j \mathbf{a}_j \mathbf{b}_j$; $\mathbf{A} \cdot \mathbf{B} = \sum_{i,j} \mathbf{A}_{i,j} \mathbf{B}_{i,j}$; $[\mathbf{AB}]_{i,j} = \sum_k \mathbf{A}_{i,k} \mathbf{B}_{k,j}$; $|\mathbf{F}|^2 =$

$\mathrm{Tr}(\mathbf{F}^T\mathbf{F}) = \sum_{i,j} F_{i,j}^2$ denotes the squared Euclidean norm of $\mathbf{F}$ in the space $\mathcal{M}^{N\times N}$ of $N\times N$ real matrices; $\mathbf{I} \in \mathcal{M}^{N\times N}$ denotes the identity matrix, $SO(N)$ denotes the group of rotation matrices, $\mathcal{M}^{N\times N}_{sym}$ and $\mathcal{M}^{N\times N}_{skew}$ denote respectively the sets of symmetric and skew-symmetric matrices. For every $\mathbf{B} \in \mathcal{M}^{N\times N}$ we define $\mathrm{sym}\,\mathbf{B} := \frac{1}{2}(\mathbf{B}+\mathbf{B}^T)$ and $\mathrm{skew}\,\mathbf{B} := \frac{1}{2}(\mathbf{B}-\mathbf{B}^T)$.

First we recall that the minimum at right-hand side in definition (6) of $\mathcal{F}$ exists for every $\mathbf{v}$ in $H^1(\Omega,\mathbb{R}^N)$, so that $\mathcal{F}(\mathbf{v})$ is well defined: precisely the finite dimensional minimization problem has exactly two solutions, which differs only by the sign, since strict convexity of the positive definite quadratic form $\mathcal{V}_0(\mathbf{x},\cdot)$ entails

$$\lim_{|\mathbf{W}|\to+\infty,\,\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \int_\Omega \mathcal{V}_0\left(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2\right)\,d\mathbf{x} \;=\; +\infty \tag{11}$$

and hence the existence of a unique minimizing argument $\mathbf{W}^2$.

We also highlight a straightforward consequence of (6), which proves useful in the sequel:

$$\mathcal{F}(\mathbf{v}) \;=\; -\mathcal{L}(\mathbf{v}) \qquad \text{for every } \mathbf{v}(\mathbf{x}) = \mathbf{W}^2\mathbf{x}\,,\ \text{with } \mathbf{W}\in\mathcal{M}^{N\times N}_{skew}\,. \tag{12}$$

Proposition 4.1 *If $N=2$, then functional $\mathcal{F}$ is convex for every choice of the positive definite quadratic form $\mathcal{V}_0$.*

Proof For every $\varepsilon>0$ we define $\varphi_\varepsilon \in C^2(\mathbb{R})$ as

$$\varphi_\varepsilon(t) = \begin{cases} t^2-\varepsilon t+\frac{\varepsilon^2}{3}\,, & \text{if } t\le 0\,,\\ (3\varepsilon)^{-1}(\varepsilon-t)^3\,, & \text{if } 0\le t\le\varepsilon\,,\\ 0\,, & \text{otherwise,} \end{cases} \tag{13}$$

and introduce the C^2 functionals $\mathcal{F}_\varepsilon$ by setting

$$\mathcal{F}_\varepsilon(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v}) - \frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})d\mathbf{x}\right)^{-1}\varphi_\varepsilon\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x}\right) \quad \forall\,\mathbf{v}\in H^1(\Omega,\mathbb{R}^N)\,. \tag{14}$$

Then, by (13), (14) and representation

$$\mathcal{F}(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v}) - \frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})d\mathbf{x}\right)^{-1}\left[\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x}\right)^{-}\right]^2, \tag{15}$$

we get $\varphi_\varepsilon(t) \geq (t^-)^2$, hence

$$\mathcal{F}_\varepsilon \leq \mathcal{F}\;, \qquad \mathcal{F} = \sup_{\varepsilon>0}\mathcal{F}_\varepsilon\;.$$

Moreover, we claim that $\mathcal{F}_\varepsilon$ is convex for every $\varepsilon > 0$ and this property entails the convexity of $\mathcal{F}$ since $\mathcal{F}$ is the supremum of a family of convex functions. Indeed $\mathcal{F}_\varepsilon$ is a C^2 functional on the whole space $H^1(\Omega,\mathbb{R}^N)$; therefore, its second variation, for every $\mathbf{u},\mathbf{v}\in H^1(\Omega,\mathbb{R}^N)$, is

$$\begin{aligned}
&\mathbf{v}^T\delta^2\mathcal{F}_\varepsilon(\mathbf{u})\mathbf{v} = \mathbf{v}^T\delta^2\mathcal{E}(\mathbf{u})\mathbf{v}\\
&-\frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})d\mathbf{x}\right)^{-1}\varphi''_\varepsilon\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{u})\,d\mathbf{x}\right)\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x}\right)^2 =\\
&= 2\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v}))\,d\mathbf{x} -\\
&-\frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})d\mathbf{x}\right)^{-1}\varphi''_\varepsilon\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{u})\,d\mathbf{x}\right)\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x}\right)^2.
\end{aligned} \tag{16}$$

By taking into account that $0 \leq \varphi''_\varepsilon \leq 2$, we get

$$\mathbf{v}^T\delta^2\mathcal{F}_\varepsilon(\mathbf{u})\mathbf{v} \geq 2\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v}))\,d\mathbf{x} - \frac{1}{2}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})d\mathbf{x}\right)^{-1}\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x}\right)^2. \tag{17}$$

Then, since $\mathcal{V}_0$ is a positive definite quadratic form, by representation (15) we obtain that the right hand side of (17) is equal to $2\big(\mathcal{F}(\mathbf{v})+\mathcal{L}(\mathbf{v})\big)$ if

$$\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,d\mathbf{x} \ \geq \ 0\,;$$

else it is equal to $\mathcal{F}(-\mathbf{v})+\mathcal{L}(-\mathbf{v})$.

In both cases (6) entails $\mathbf{v}^T\delta^2\mathcal{F}_\varepsilon(\mathbf{u})\mathbf{v}\geq 0$ for every $\mathbf{u}$, $\mathbf{v}\in H^1(\Omega,\mathbb{R}^N)$.

Therefore $\mathcal{F}_\varepsilon$ is convex and claim is proved. □

Proposition 4.2 *If $N=3$, then functional $\mathcal{F}$ is nonconvex for every choice of the positive definite quadratic form $\mathcal{V}_0$.*

Proof Set

$$\mathbf{W}_1=\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1,\ \ \mathbf{W}_2=\mathbf{e}_2\otimes\mathbf{e}_3-\mathbf{e}_3\otimes\mathbf{e}_2. \tag{18}$$

Then

$$\frac{1}{2}(\mathbf{W}_1^2+\mathbf{W}_2^2)=-\frac{1}{2}(\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_3)-\mathbf{e}_2\otimes\mathbf{e}_2:=\mathbf{A}$$

and by choosing $\mathbf{v}(\mathbf{x}):=\mathbf{A}\mathbf{x}$ we get $\mathbb{E}(\mathbf{v})=\mathbf{A}\notin\{\mathbf{W}^2:\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}\}$.

Hence, $\mathcal{F}(\mathbf{v})>-\mathcal{L}(\mathbf{v})$ for every possible choice of the positive definite quadratic form $\mathcal{V}_0$. Whereas, by setting

$$\mathbf{v}_1(\mathbf{x}):=\mathbf{W}_1^2\mathbf{x}\,,\quad \mathbf{v}_2(\mathbf{x}):=\mathbf{W}_2^2\mathbf{x}\,,$$

due to (12), we get $\mathcal{F}(\mathbf{v}_1)=-\mathcal{L}(\mathbf{v}_1)$, $\mathcal{F}(\mathbf{v}_2)=-\mathcal{L}(\mathbf{v}_2)$. Hence

$$\mathcal{F}(\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2))=\mathcal{F}(\mathbf{v})>-\mathcal{L}(\mathbf{v})=-\tfrac{1}{2}(\mathcal{L}(\mathbf{v}_1)+\mathcal{L}(\mathbf{v}_2))=\tfrac{1}{2}(\mathcal{F}(\mathbf{v}_1)+\mathcal{F}(\mathbf{v}_2)) \tag{19}$$

thus proving that $\mathcal{F}$ is not convex in the 3D case for every choice of $\mathcal{V}_0$. □

Although existence of minimizers of $\mathcal{F}$ is already a direct consequence of convergence results in [23], in the next Proposition we provide a direct proof of sequential lower semicontinuity of $\mathcal{F}$ with respect to the natural, very weak convergence, for both cases of dimension 2 and 3.

Proposition 4.3 *Assume that the standard structural conditions and* (7) *holds true.*

Then, for every $\mathbf{v}_n, \mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ *such that* $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbb{E}(\mathbf{v})$ *in* $L^2(\Omega; \mathcal{M}^{N\times N})$ *we have*

$$\liminf_{n\to+\infty} \mathcal{F}(\mathbf{v}_n) \;\geq\; \mathcal{F}(\mathbf{v})$$

Proof Let $\mathbf{v}_n, \mathbf{v}$ belong to $H^1(\Omega; \mathbb{R}^N)$ and fulfil $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbb{E}(\mathbf{v})$ in $L^2(\Omega; \mathcal{M}^{N\times N})$. Then $\mathbb{E}(\mathbf{v}_n)$ is bounded in $L^2(\Omega; \mathcal{M}^{N\times N})$. If $\liminf_{n\to+\infty} \mathcal{F}(\mathbf{v}_n) = +\infty$, then the claim is trivial, so we may also assume without restriction that $\mathcal{F}(\mathbf{v}_n) \leq C$. Assumption (7) of equilibrated load entails $\mathcal{F}(\mathbf{v}_n) = \mathcal{F}(\mathbf{v}_n - \mathbb{P}\mathbf{v}_n)$, so may suppose that $\mathbb{P}\mathbf{v}_n \equiv \mathbf{0}$. We choose

$$\mathbf{W}_n \in \operatorname{argmin}\left\{ \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\, \mathbb{E}(\mathbf{v}_n) - \tfrac{1}{2}\mathbf{W}^2\Big)\, d\mathbf{x} :\; \mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \right\}. \tag{20}$$

Hence, if C_K the Korn-Poincaré inequality in Ω and $\alpha > 0$ is the uniform coercivity constant of $\mathcal{V}_0$, say $\mathcal{V}_0(\mathbf{x}, \mathbf{M}) \geq \alpha |\mathbf{M}|^2$, we get

$$\begin{aligned} \alpha \int_\Omega |\mathbb{E}(\mathbf{v}_n) - \tfrac{1}{2}\mathbf{W}_n^2|^2\, d\mathbf{x} \leq C + \mathcal{L}(\mathbf{v}_n) \;=\; C + \mathcal{L}(\mathbf{v}_n - \mathbb{P}\mathbf{v}_n) \;\leq \\ \leq C + C_K(\|\mathbf{f}\|_{L^2(\partial\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)})\|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega;\mathcal{M}^{N\times N})}\,, \end{aligned} \tag{21}$$

Therefore $|\mathbf{W}_n^2|$ is bounded and since $\mathbf{W}_n$ is real skew-symmetric we obtain that $|\mathbf{W}_n|$ is bounded too. So we may suppose that, up to subsequences,

$\mathbf{W}_n \to \mathbf{W}$ in $\mathcal{M}^{N\times N}_{skew}$. By taking into account that $\mathbb{P}\mathbf{v}_n \equiv \mathbf{0}$ we get $\mathbf{v}_n \rightharpoonup \mathbf{v}$ in $H^1(\Omega, \mathbb{R}^N)$ hence by recalling that $\mathcal{V}_0(\mathbf{x},\cdot)$ is a convex quadratic form

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{F}(\mathbf{v}_n) = & \ \liminf_{n\to+\infty} \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\, \mathbb{E}(\mathbf{v}_n) - \tfrac{1}{2}\mathbf{W}_n^2\Big)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}_n) \geq \\ & \geq \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\Big)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}) \geq \mathcal{F}(\mathbf{v}),\end{aligned} \tag{22}$$

which proves the claimed lower semicontinuity inequality. □

Remark 3.1 The first variation of $\mathcal{F}$ can be explicitly evaluated in the 2D case, thanks to representation (15), as follows

$$\begin{aligned}\delta\mathcal{F}(\mathbf{v})[\varphi] = & \int_\Omega D\mathcal{V}_0\big(\mathbf{x}, \mathbb{E}(\mathbf{v})\big)\cdot \mathbb{E}(\varphi)\, d\mathbf{x} \; - \; \mathcal{L}(\varphi) \\ & + \frac{1}{2}\left(\int_\Omega \mathcal{V}_0(\mathbf{I}) d\mathbf{x}\right)^{-1} \left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot \mathbb{E}(\mathbf{v})\, d\mathbf{x}\right)^{-} \int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot \mathbb{E}(\varphi)\, d\mathbf{x} \; = \\ = \; \delta\mathcal{E}(\mathbf{v})[\varphi] \; + \; & \frac{1}{2}\left(\int_\Omega \mathcal{V}_0(\mathbf{I}) d\mathbf{x}\right)^{-1} \left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot \mathbb{E}(\mathbf{v})\, d\mathbf{x}\right)^{-} \int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot \mathbb{E}(\varphi)\, d\mathbf{x}\end{aligned} \tag{23}$$

for every $\mathbf{v}$, $\varphi \in H^1(\Omega; \mathbb{R}^N)$.

Remark 3.2 Functional $\mathcal{F}$ exhibits a nonlocal behavior: precisely in 2D, due to the representations (15) and (23) respectively of the functional and its first variation, $\mathcal{F}(\mathbf{v})$ is the sum of a contribution $\mathcal{E}(\mathbf{v})$ due to local functional $\mathcal{E}$ related to linear elasticity plus a possibly vanishing correction with global dependance on $\mathbf{v}$ explicitly evaluated by

$$-\frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I}) d\mathbf{x}\right)^{-1} \left[\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot \mathbb{E}(\mathbf{v})\,;\, d\mathbf{x}\right)^{-}\right]^2 .$$

In the case of Saint Venant–Kirchhoff energy density

$$\mathcal{W}(\mathbf{x},\mathbf{F}) = \begin{cases} \mu|\mathbf{F}^T\mathbf{F} - \mathbf{I}|^2 + \frac{\lambda}{2}|\ \mathrm{Tr}\ (\mathbf{F}^T\mathbf{F} - \mathbf{I})|^2, & \text{if } \det \mathbf{F} > 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{24}$$

corresponding to the limit quadratic form $\mathcal{V}_0(\mathbf{x},\mathbf{B}) = 4\mu|\mathbf{B}|^2 + 2\lambda|\mathrm{Tr}\mathbf{B}|^2$ with $\lambda,\, \mu > 0$, the correction simplifies as follows:

$$-\frac{1}{4\,|\Omega|}\left(\int_\Omega \mathrm{div}\,\mathbf{v}\, d\mathbf{x}\right)^- \ .$$

Moreover, the nonlocal coefficient $\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\, d\mathbf{x}\right)^-$ appears in Euler equations too.

5 Variational Convergence Results

In this section, we recall the main results of [1] about the variational convergence of pure traction problems. To this aim, basic notation and assumptions for general nonlinear energies is introduced first.

Still we assume that the reference configuration of the elastic body is a

$$\text{bounded, connected open set } \Omega \subset \mathbb{R}^N \text{ with Lipschitz boundary, } N = 2, 3, \tag{25}$$

and set these notations: the generic point $\mathbf{x} \in \Omega$ has components x_j referring to the standard basis vectors $\mathbf{e}_j$ in $\mathbb{R}^N$; $\mathcal{L}^N$ and $\mathcal{B}^N$ denote respectively the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbb{R}^N$.

For every $\mathcal{U} : \Omega \times \mathcal{M}^{N\times N} \to \mathbb{R}$, with $\mathcal{U}(\mathbf{x},\cdot) \in C^2$ a.e. $\mathbf{x} \in \Omega$, we denote the gradient and the hessian of g with respect to the second variable by $D\mathcal{U}(\mathbf{x},\cdot)$ and $D^2\mathcal{U}(\mathbf{x},\cdot)$ respectively.

For every displacements field $\mathbf{v} \in H^1(\Omega;\mathbb{R}^N)$, $\mathbb{E}(\mathbf{v}) := \mathrm{sym}\,\nabla\mathbf{v}$ denotes the infinitesimal strain tensor field, $\mathcal{R} := \{\mathbf{v} \in H^1(\Omega;\mathbb{R}^N) : \mathbb{E}(\mathbf{v}) = \mathbf{0}\}$ the set of infinitesimal rigid displacements and $\mathbb{P}\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto $\mathcal{R}$.

We consider a body made of an hyperelastic material, say there exists a $\mathcal{L}^N\times\mathcal{B}^{N^2}$measurable $\mathcal{W}:\Omega\times\mathcal{M}^{N\times N}\to[0,+\infty]$ such that, for a.e. $\mathbf{x}\in\Omega$, $\mathcal{W}(\mathbf{x},\nabla\mathbf{y}(\mathbf{x}))$ represents the stored energy density, when $\mathbf{y}(x)$ is the deformation and $\nabla\mathbf{y}(\mathbf{x})$ is the deformation gradient.

Moreover we assume that for a.e. $\mathbf{x}\in\Omega$

$$\mathcal{W}(\mathbf{x},\mathbf{F})=+\infty \qquad \text{if } \det\mathbf{F}\le 0 \quad \text{(orientation preserving condition)}\,, \tag{26}$$

$$\mathcal{W}(\mathbf{x},\mathbf{RF})=\mathcal{W}(\mathbf{x},\mathbf{F}) \qquad \forall\,\mathbf{R}\in SO(N) \quad \forall\,\mathbf{F}\in\mathcal{M}^{N\times N} \qquad \text{(frame indifference)}\,, \tag{27}$$

$$\exists \text{ a neighborhood } \mathcal{A} \text{ of } SO(N) \text{ s.t.} \quad \mathcal{W}(\mathbf{x},\cdot)\in C^2(\mathcal{A})\,, \tag{28}$$

$$\exists\, C>0 \text{ independent of } \mathbf{x}:\ \mathcal{W}(\mathbf{x},\mathbf{F})\ge C\,\mathrm{dist}^2\big(\mathbf{F},SO(N)\big) \tag{29}$$

$$\forall\,\mathbf{F}\in\mathcal{M}^{N\times N} \text{ (coerciveness)},$$

$$\mathcal{W}(\mathbf{x},\mathbf{I})=0\,, \quad D\mathcal{W}(\mathbf{x},\mathbf{I})=0\,, \qquad \text{for a.e. } \mathbf{x}\in\Omega\,, \tag{30}$$

that is the reference configuration has zero energy and is stress free, so by (27) we get also

$$\mathcal{W}(\mathbf{x},\mathbf{R})=0, \quad D\mathcal{W}(\mathbf{x},\mathbf{R})=0 \qquad \forall\,\mathbf{R}\in SO(N)\,.$$

By frame indifference there exists a $\mathcal{L}^N\times\mathcal{B}^N$-measurable function $\mathcal{V}:\Omega\times\mathcal{M}^{N\times N}\to[0,+\infty]$ such that for every $\mathbf{F}\in\mathcal{M}^{N\times N}$

$$\mathcal{W}(\mathbf{x},\mathbf{F})=\mathcal{V}(\mathbf{x},\tfrac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I})) \tag{31}$$

and by (28)

$$\exists \text{ a neighborhood } \mathcal{O} \text{ of } \mathbf{0} \text{ such that } \mathcal{V}(\mathbf{x},\cdot) \in C^2(\mathcal{O}),\ \text{a.e. } x \in \Omega\,. \qquad (32)$$

In addition we assume that there exists $\gamma > 0$ independent of $\mathbf{x}$ such that

$$\left|\,\mathbf{B}^T\, D^2\mathcal{V}(\mathbf{x},\mathbf{D})\,\mathbf{B}\,\right| \;\leq\; 2\,\gamma\,|\mathbf{B}|^2 \quad \forall\,\mathbf{D}\!\in\!\mathcal{O},\ \forall\,\mathbf{B}\!\in\!\mathcal{M}^{N\times N}. \qquad (33)$$

By (30) and Taylor expansion with Lagrange reminder we get, for a.e. $\mathbf{x} \in \Omega$ and suitable $t \in (0,1)$ depending on $\mathbf{x}$ and on $\mathbf{B}$:

$$\mathcal{V}(\mathbf{x},\mathbf{B}) \;=\; \frac{1}{2}\,\mathbf{B}^T D^2\mathcal{V}(\mathbf{x},t\mathbf{B})\,\mathbf{B}\,. \qquad (34)$$

Hence by (33)

$$\mathcal{V}(\mathbf{x},\mathbf{B}) \;\leq\; \gamma\,|\mathbf{B}|^2 \qquad \forall\;\mathbf{B} \in \mathcal{M}^{N\times N} \cap \mathcal{O}\,. \qquad (35)$$

According to (31) for a.e. $\mathbf{x}\!\in\!\Omega,\ h\!>\!0$ and every $\mathbf{B} \in \mathcal{M}^{N\times N}$ we set

$$\mathcal{V}_h(\mathbf{x},\mathbf{B}) := \frac{1}{h^2}\,\mathcal{W}(\mathbf{x},\mathbf{I}+h\mathbf{B}) = \frac{1}{h^2}\,\mathcal{V}(\mathbf{x},h\,\mathrm{sym}\,\mathbf{B} + \tfrac{1}{2}h^2\mathbf{B}^T\mathbf{B})\;. \qquad (36)$$

Taylor's formula with (30),(36) entails

$$\mathcal{V}_h(\mathbf{x},\mathbf{B}) = \frac{1}{2}\,(\mathrm{sym}\,\mathbf{B})\,D^2\mathcal{V}(\mathbf{x},\mathbf{0})\,(\mathrm{sym}\,\mathbf{B}) + o(1),$$

so

$$\mathcal{V}_h(\mathbf{x},\mathbf{B}) \;\to\; \mathcal{V}_0(\mathbf{x},\mathrm{sym}\ \mathbf{B}) \text{ as } h \to 0_+\,, \qquad (37)$$

where the point-wise limit of integrands is the quadratic form $\mathcal{V}_0$ defined by

$$\mathcal{V}_0(\mathbf{x},\mathbf{B}) \;:=\; \frac{1}{2}\,\mathbf{B}^T D^2\mathcal{V}(\mathbf{x},\mathbf{0})\,\mathbf{B} \quad \text{a.e. } \mathbf{x} \in \Omega,\ \mathbf{B} \in \mathcal{M}^{N\times N}\,. \qquad (38)$$

The symmetric fourth order tensor $D^2\mathcal{V}(\mathbf{x},\mathbf{0})$ in (38) plays the role of the classical elasticity tensor.

By (29) we get

$$\mathcal{V}_h(x,\mathbf{B}) \;=\; \frac{1}{h^2}\,\mathcal{W}(x,\mathbf{I}+h\mathbf{B}) \;\geq\; C\,|\,2\,\mathrm{sym}\mathbf{B} \,+\, h\,\mathbf{B}^T\mathbf{B}\,|^2 \tag{39}$$

so that (38) and (39) imply the ellipticity of $\mathcal{V}_0$:

$$\mathcal{V}_0(\mathbf{x},\mathrm{sym}\,\mathbf{B}) \;\geq\; 4\,C\,|\mathrm{sym}\,\mathbf{B}|^2 \qquad \text{a.e. } \mathbf{x}\in\Omega,\;\; \mathbf{B}\in\mathcal{M}^{N\times N}\,. \tag{40}$$

For a suitable choice of the adimensional parameter $h>0$, the functional representing the total energy is labeled by $\mathcal{F}_h : H^1(\Omega;\mathbb{R}^N)\to\mathbb{R}\cup\{+\infty\}$ and defined as follows

$$\mathcal{F}_h(\mathbf{v}) \;:=\; \int_\Omega \mathcal{V}_h(\mathbf{x},\nabla\mathbf{v})\,d\mathbf{x} \,-\, \mathcal{L}(\mathbf{v})\,, \tag{41}$$

where $\mathcal{L}$ is defined by (2).

In order to describe the asymptotic behavior as $h\downarrow 0$ of functionals $\mathcal{F}_h$, we refer to the limit energy functional $\mathcal{F} : H^1(\Omega;\mathbb{R}^N)\to\mathbb{R}$ defined by (6).

In this section, we assume (25) together with the *standard structural conditions* (26)-(30),(33) as usual in scientific literature concerning elasticity theory and we refer to the notations (31),(36),(38),(41).

Definition 5.1 Given an infinitesimal sequence h_j of positive real numbers, we say that $\mathbf{v}_j\in H^1(\Omega;\mathbb{R}^N)$ is a *minimizing sequence* of the sequence of functionals $\mathcal{F}_{h_j}$ if

$$(\mathcal{F}_{h_j}(\mathbf{v}_j) - \inf\mathcal{F}_{h_j}) \to 0 \quad \text{ as } \quad h_j\downarrow 0\,.$$

We proved that for every given infinitesimal sequence h_j actually the minimizing sequences of the sequence of functionals $\mathcal{F}_{h_j}$ exists. For reader's convenience we recall here the main results of [1]: see Lemma 3.1, Theorem 2.2, Remark 2.5, Theorem 4.1 and Corollary 4.2 therein.

Lemma 5.1 *Assume the standard structural conditions together with* (7) *and* (8).

Then, there is a constant K, dependent only on Ω and the coercivity constant of of the stored energy density appearing in (29)*, such that*

$$\inf_{h>0} \inf_{\mathbf{v}\in H^1} \mathcal{F}_h(\mathbf{v}) \ \geq \ -K\left(\|\mathbf{f}\|^2_{L^2} + \|\mathbf{g}\|^2_{L^2}\right). \tag{42}$$

Theorem 5.1 *Assume that the standard structural conditions and* (7),(8) *hold true. Then:*

$$\min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \ = \ \min_{\mathbf{w}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{E}(\mathbf{w})\,; \tag{43}$$

$$\mathrm{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\,\mathcal{F} \ = \ \mathrm{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\,\mathcal{E}\,; \tag{44}$$

for every sequence of strictly positive real numbers $h_j \downarrow 0$ there are minimizing sequences of the sequence of functionals $\mathcal{F}_{h_j}$;

for every minimizing sequence $\mathbf{v}_j \in H^1(\Omega;\mathbb{R}^N)$ of $\mathcal{F}_{h_j}$ there exist a subsequence and a displacement $\mathbf{v}_0 \in H^1(\Omega;\mathbb{R}^N)$ such that, without relabeling,

$$\mathbb{E}(\mathbf{v}_j) \ \rightharpoonup \ \mathbb{E}(\mathbf{v}_0) \qquad \textit{weakly in } L^2(\Omega;\mathcal{M}^{N\times N})\,, \tag{45}$$

$$\sqrt{h_j}\ \nabla\mathbf{v}_j \ \to \ \mathbf{0} \qquad \textit{strongly in } L^2(\Omega;\mathcal{M}^{N\times N})\,, \tag{46}$$

$$\lim_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \ = \ \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \ = \ \mathcal{F}(\mathbf{v}_0) \ = \ \mathcal{E}(\mathbf{v}_0)\,. \tag{47}$$

If strong inequality in the compatibility condition (8) is replaced by a weak inequality, then the uniform estimate (42) still hold true and also minimizing sequences of the sequence of functionals $\mathcal{F}_{h_j}$ exist for every infinitesimal sequence h_j, but the minimizers coincidence (44) for $\mathcal{F}$ and $\mathcal{E}$ cannot hold anymore. Nevertheless the following general result holds true.

Proposition 5.1 *If the structural assumptions together with* (7) *are fulfilled, but* (8) *is replaced by*

$$\mathcal{L}(\mathbf{W}^2\mathbf{x}) \leq 0 \qquad \forall\, \mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \tag{48}$$

then $\operatorname{argmin} \mathcal{F}$ *is still nonempty and*

$$\min \mathcal{F} = \min \mathcal{E}\,, \tag{49}$$

but the coincidence of minimizers sets is replaced by the inclusion

$$\operatorname{argmin} \mathcal{E} \subset \operatorname{argmin} \mathcal{F}\,. \tag{50}$$

If (48) *holds true and there exists* $\mathbf{U} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{U} \neq \mathbf{0}$ *such that* $\mathcal{L}(\mathbf{U}^2\mathbf{x}) = 0$, *then* $\mathcal{F}$ *admits infinitely many minimizers which are not minimizers of* $\mathcal{E}$, *precisely*

$$\operatorname{argmin} \mathcal{E} \;\underset{\neq}{\subset}\; \operatorname{argmin} \mathcal{E} + \left\{ \mathbf{U}^2\mathbf{x} \,:\; \mathbf{U} \in \mathcal{M}^{N\times N}_{skew}\,,\; \mathcal{L}(\mathbf{U}^2\mathbf{x}) = 0 \right\} \;\subset\; \operatorname{argmin} \mathcal{F}\,, \tag{51}$$

where the last inclusion is an equality in 2D:

$$\operatorname{argmin} \mathcal{E} \;\underset{\neq}{\subset}\; \operatorname{argmin} \mathcal{E} + \{ -t\,\mathbf{x} \,:\; t \geq 0 \} \;=\; \operatorname{argmin} \mathcal{F}\,, \qquad \text{if } N = 2\,. \tag{52}$$

Remark 4.1 The compatibility condition (8) cannot be dropped in Theorem 5.1 even if the (necessary) condition (7) holds true. Moreover, plain substitution of strong inequality in (8) with weak inequality leads to a lack of compactness for minimizing sequences.

Indeed, if $\mathbf{n}$ denotes the outer unit normal vector to $\partial\Omega$ and we choose $\mathbf{f} = f\mathbf{n}$ with $f < 0$, $\mathbf{g} \equiv 0$, then

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2\, \mathbf{x}\, d\mathcal{H}^{N-1} = 2f(\mathrm{Tr}\, \mathbf{W}^2)|\Omega| > 0 \tag{53}$$

and the strict inequality in (8) is reversed in a strong sense by any $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \setminus \{\mathbf{0}\}$;

fix a sequence of positive real numbers such that $h_j \downarrow 0$, $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{W} \not\equiv \mathbf{0}$, and set $\mathbf{v}_j = {h_j}^{-1}(\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W})\,\mathbf{x}$; then $\mathbf{I} + \left(\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W}\right) \in SO(N)$ and

$$\mathcal{F}_{h_j}(\mathbf{v}_j) = -\frac{f}{2h_j}\int_{\partial\Omega} \mathbf{W}^2\mathbf{x} \cdot \mathbf{n}\, d\mathcal{H}^{n-1} = -\frac{f}{2h_j}(\mathrm{Tr}\, \mathbf{W}^2)|\Omega| \to -\infty\,. \tag{54}$$

On the other hand, assume (25), $\mathcal{W}$ as in (4) and $\mathbf{f} = \mathbf{g} \equiv \mathbf{0}$, so that the compatibility inequality is susbstituted by the weak inequality; if $\mathbf{v}_j$ are defined as above then, hence by frame indifference,

$$\mathcal{F}_{h_j}(\mathbf{v}_j) = 0 = \inf \mathcal{F}_{h_j} \tag{55}$$

but $\mathbb{E}(v_j)$ has no weakly convergent subsequences in $L^2(\Omega; \mathcal{M}^{N\times N})$.

Remark 4.2 It is worth noticing that the compatibility condition (8) holds true when $\mathbf{g} \equiv 0$, $\mathbf{f} = f\mathbf{n}$ with $f > 0$ and $\mathbf{n}$ the outer unit normal vector to $\partial\Omega$.

Indeed let $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}, \mathbf{W} \not\equiv \mathbf{0}$: hence by (7) and the Divergence Theorem we get

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\,\mathbf{x}\,d\mathcal{H}^{N-1} \;=\; 2f(\mathrm{Tr}\,\mathbf{W}^2)|\Omega| \;<\; 0\,, \tag{56}$$

thus proving (8) in this case. This means that in presence of tension-like surface forces and of null body forces the compatibility condition holds true.

It is quite natural to ask whether condition (8), which was essential in the proof of Theorem 5.1, may be dropped in order to obtain at least existence of $\min\mathcal{F}$: the answer is negative.

Indeed the next remark shows that, when compatibility inequality in (8) is reversed for at least one choice of the skew-symmetric matrix $\mathbf{W}$, then $\mathcal{F}$ is unbounded from below.

Remark 4.3 If

$$\exists\,\mathbf{W}_* \in \mathcal{M}^{N\times N}_{skew}\ : \qquad \mathcal{L}(\mathbf{z}_{\mathbf{W}_*}) > 0\,, \quad \text{where } \mathbf{z}_{\mathbf{W}_*} = \frac{1}{2}\mathbf{W}_*^2\mathbf{x}\,, \tag{57}$$

then

$$\inf_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \;=\; -\infty. \tag{58}$$

Indeed we get

$$\inf_{H^1(\Omega;\mathbb{R}^N)} \mathcal{F} \;=\; \min_{H^1(\Omega;\mathbb{R}^N)} \mathcal{E} \;-\; \sup_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \mathcal{L}(\mathbf{z}_{\mathbf{W}}) \qquad \text{where } \ \mathbf{z}_{\mathbf{W}} = \frac{1}{2}\mathbf{W}^2\mathbf{x}\,. \tag{59}$$

Hence

$$\inf_{H^1(\Omega;\mathbb{R}^N)} \mathcal{F} \;\le\; \min_{H^1(\Omega;\mathbb{R}^N)} \mathcal{E} \;-\; \tau\mathcal{L}(\,\mathbf{z}_{\mathbf{W}_*}) \qquad \forall\,\tau > 0,$$

which entails (58).

Next example shows that, in case of uniform compression on the whole boundary, the functional $\mathcal{F}$ is unbounded from below, regardless of convexity or non-convexity of Ω and $\mathcal{F}$.

Example 4.1 Assume $\Omega \subset \mathbb{R}^N$ is a Lipschitz, connected open set, $N = 2, 3$, $\mathbf{g} \equiv \mathbf{0}$, $\mathbf{f} = -\mathbf{n}$, where $\mathbf{n}$ denotes the outer unit normal vector to $\partial\Omega$. (examples of 2D domains under equilibrated, but not compatible, compressive load are shown in Fig.1).

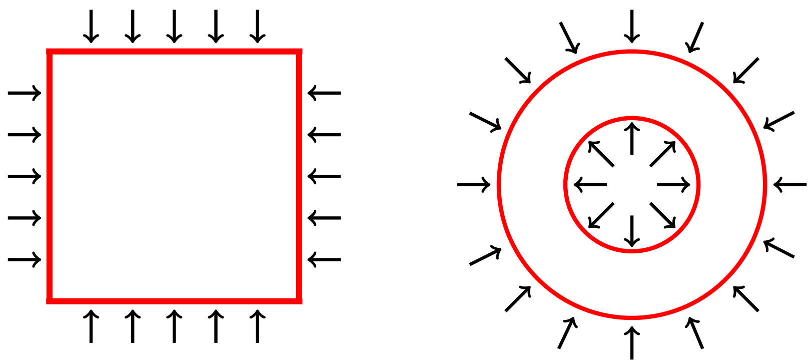

Fig. 1 Equilibrated but not compatible compressive load (Example 4.1).

Then (57) holds true hence, by Remark 4.3, $\inf_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = -\infty$.

Indeed, for every $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ such that $|\mathbf{W}|^2 = 2$ we obtain

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} = -\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} = -\int_{\Omega} \operatorname{div}(\mathbf{W}^2\mathbf{x})\, d\mathbf{x} =$$
$$= -\,|\Omega|\,\operatorname{Tr}\mathbf{W}^2 = 2\,|\Omega| > 0\,.$$

Summarizing, only two cases are allowed: either $\min \mathcal{F} = \min \mathcal{E}$ or $\inf \mathcal{F} = -\infty$: the second case actually arises in presence of compressive surface load.

The new functional $\mathcal{F}$ somehow preserves memory of instabilities which are typical of finite elasticity, while they disappear in the linearized model described by $\mathcal{E}$. In the light of Theorem 5.1, as far as pure traction problems are considered, it seems reasonable that the range of validity of linear elasticity should be restricted to a certain class of external loads, explicitly those verifying (8): a remarkable example in such class is a uniform normal tension load at the boundary as in Remark 4.2 while in the other cases equilibria of a linearly elastic body could be better described through critical points of $\mathcal{F}$, whose existence in general seems to be an interesting and open problem.

6 Strong Convergence of Minimizing Sequences of $\mathcal{F}_h$

In this section, we prove that for the special class of Saint Venant–Kirchhoff energy density it is possible to choose a subsequence of functionals $\mathcal{F}_h$ defined by (41) and a corresponding minimizing sequence, according to Definition 5.1, which is weakly converging in $H^1(\Omega;\mathbb{R}^N)$ to a minimizer of functional $\mathcal{F}$ defined by (6). Moreover, thanks to a result of [5], this convergence entails strong convergence in $W^{1,q}(\Omega;\mathbb{R}^N)$ for $1 \le q < 2$.

Before stating the main result of this section we notice that, by frame indifference (27) and equilibrated load condition (7), without loss of of generality we can assume

$$\int_\Omega x_i \, d\mathbf{x} \; = \; 0 \quad \forall\, i = 1 \dots N\,, \qquad \int_\Omega x_i \, x_j \, d\mathbf{x} \; = \; 0 \quad \forall\, i,j = 1 \dots N,\; i \neq j. \tag{60}$$

Therefore, if I_k denotes the moment of inertia of Ω with respect to the k-th axis, by (60) we get

$$\mathbb{P}\,\mathbf{u}(\mathbf{x}) \;=\; \mathbf{a}\times\mathbf{x}, \quad \mathbf{a}_k \;=\; I_k^{-1}\int_\Omega (\mathbf{x}\times\mathbf{v})_k\, d\mathbf{x} \tag{61}$$

so

$$(\nabla\,\mathbb{P}\,\mathbf{u}(\mathbf{x}))_k \;=\; \mathbf{a}\times\mathbf{e}_k. \tag{62}$$

Theorem 6.1 *Let* $\mu>0,\,\lambda>0$ *be the Lamé constants and*

$$\mathcal{W}(\mathbf{x},\mathbf{F}) \;=\; \mathcal{W}(\mathbf{F}) \;:=\; \begin{cases} \mu|\mathbf{F}^T\mathbf{F}-\mathbf{I}|^2+\frac{\lambda}{2}|\;\mathrm{Tr}\;(\mathbf{F}^T\mathbf{F}-\mathbf{I})|^2\,, & \textit{if}\ \det\mathbf{F}>0,\\ \\ +\infty\,, & \textit{else,} \end{cases} \tag{63}$$

the stored energy density, assume (7), (8) *and let let* h_j *be a sequence of strictly positive real numbers with* $h_j\to 0$.

Then, there exists a (not relabeled) subsequence of functionals $\mathcal{F}_{h_j}$ *and a minimizing sequence* $\mathbf{w}_j$ *weakly converging in* $H^1(\Omega;\mathbb{R}^N)$ *and strongly converging in* $W^{1,q}(\Omega,\mathbb{R}^N)$ *to* $\mathbf{w}_0$ *in* $\operatorname{argmin}\mathcal{E}$*, for* $1\le q<2$.

Proof First of all we notice that (63) entails (29), hence Theorem 5.1 applies to the present situation. By recalling Proposition 5.3 of [5] it will be enough to show that there exists a minimizing sequence $\mathbf{w}_j$ for functionals $\mathcal{F}_{h_j}$ (say $\mathcal{F}_{h_j}(\mathbf{w}_j)=\inf\mathcal{F}_{h_j}+o(1)$) weakly converging in $H^1(\Omega;\mathbb{R}^N)$ to $\mathbf{w}_0\in\operatorname{argmin}\mathcal{F}$ and

$$\lim_{h_j\to 0}\mathcal{F}_{h_j}(\mathbf{v}_j) = \int_\Omega \mathcal{V}_0\big(\mathbb{E}(\mathbf{v}_0)\big)\,d\mathbf{x}-\mathcal{L}(\mathbf{v}_0) = \mathcal{E}(\mathbf{v}_0). \tag{64}$$

It is worth noticing that according to (63)

$$\mathcal{V}_0(\mathbf{x},\mathbf{B}) \equiv \mathcal{V}_0(\mathbf{B}) = 4\mu|\mathbf{B}|^2 + 2\lambda|\ \mathrm{Tr}\ \mathbf{B}|^2. \tag{65}$$

To this aim let $\mathbf{v}_j$ be a minimizing sequence for functionals $\mathcal{F}_{h_j}$: by Theorem 5.1 there exist a (not relabeled) subsequence h_j and $\mathbf{v}_j,\ \mathbf{v}_0 \in H^1(\Omega;\mathbb{R}^N)$ such that

$$\mathbb{E}(\mathbf{v}_j) \ \rightharpoonup\ \mathbb{E}(\mathbf{v}_0) \ \text{ in } \ L^2(\Omega;\mathcal{M}^{N\times N}), \tag{66}$$

$$\mathcal{F}(\mathbf{v}_0) = \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = \lim_{h_j\to 0} \mathcal{F}_{h_j}(\mathbf{v}_j) \ = \mathcal{E}(\mathbf{v}_0) \ = \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{E}(\mathbf{v})\,, \tag{67}$$

$$\sqrt{h_j}\,\nabla\mathbf{v}_j \ \to\ \mathbf{0} \quad \text{in} \quad L^2(\Omega;\mathcal{M}^{N\times N}) \tag{68}$$

and by (67), (68) and convexity of $\mathcal{V}_0$

$$\mathcal{E}(\mathbf{v}_0) \ = \ \mathcal{F}(\mathbf{v}_0) \ = \ \lim_{h_j\to 0} \mathcal{F}_{h_j}(\mathbf{v}_j) \ =$$

$$\lim_{h_j\to 0} \int_\Omega \mathcal{V}_0\big(\mathbb{E}(\mathbf{v}_j) + \tfrac{1}{2}h_j \nabla\mathbf{v}_j^T\nabla\mathbf{v}_j\big)\,d\mathbf{x} - \mathcal{L}(\mathbf{v}_j) \ \geq \tag{69}$$

$$\int_\Omega \mathcal{V}_0\big(\mathbb{E}(\mathbf{v}_0)\big)\,d\mathbf{x} - \mathcal{L}(\mathbf{v}_0) \ = \ \mathcal{E}(\mathbf{v}_0)\,.$$

Thanks to (60), (61) and (62) we get

$$\int_\Omega (\mathbf{x}\times\mathbf{v}_{h_j})\,d\mathbf{x} = \int_\Omega \mathbf{x}\times(\mathbf{v}_{h_j} - |\Omega|^{-1}\int_\Omega \mathbf{v}_{h_j}\,d\mathbf{x})\,d\mathbf{x}$$

which, thanks to (68), implies

$$\sqrt{h_j}\,\nabla(\mathbb{P}\mathbf{v}_j) \to \mathbf{0} \tag{70}$$

so that

$$\mathbf{B}_j := \frac{h}{2}\left\{\nabla(\mathbb{P}\mathbf{v}_j)^T\nabla(\mathbb{P}\mathbf{v}_j) + \nabla\mathbf{v}_j^T\nabla(\mathbb{P}\mathbf{v}_j) - \nabla(\mathbb{P}\mathbf{v}_j)^T\nabla\mathbf{v}_j\right\} \to \mathbf{0} \tag{71}$$

strongly in $L^2(\Omega;\mathcal{M}^{N\times N})$.

Since $\mathbf{v}_j$ is a minimizing sequence, (63) and Poincaré-Korn inequality yield

$$\begin{gathered}\int_\Omega |\mathbb{E}(\mathbf{v}_j)+\tfrac{1}{2}h_j\nabla\mathbf{v}_j^T\nabla\mathbf{v}_j|^2\,d\mathbf{x}\le C+\mathcal{L}(\mathbf{v}_j)= \\ C+\mathcal{L}(\mathbf{v}_j-\mathbb{P}\mathbf{v}_j)\le C+C'\left(\int_\Omega|\mathbb{E}(\mathbf{v}_j)|^2\,d\mathbf{x}\right)^{\frac{1}{2}},\end{gathered}\tag{72}$$

hence $\mathbf{D}_j:=\mathbb{E}(\mathbf{v}_j)+\frac{1}{2}h_j\nabla\mathbf{v}_j^T\nabla\mathbf{v}_j$ are equibounded in $L^2(\Omega;\mathcal{M}^{N\times N})$ and by setting $\mathbf{w}_j:=\mathbf{v}_j-\mathbb{P}\mathbf{v}_j$, by recalling that $\mathbf{B}\to\mathcal{V}_0(\mathbf{B})$ is convex we have

$$\mathcal{F}_{h_j}(\mathbf{v}_j)-\mathcal{F}_{h_j}(\mathbf{w}_j)\ge\int_\Omega\mathbf{B}_{h_j}\cdot\mathcal{V}_0'(\mathbf{D}_j+\mathbf{B}_j)\,d\mathbf{x}.\tag{73}$$

Since $|\mathcal{V}_0'(\mathbf{B})|\le C|\mathbf{B}|$ for some $C>0$, by (71) and (72) we get

$$\left|\int_\Omega\mathbf{B}_j\cdot\mathcal{V}_0'(\mathbf{D}_j+\mathbf{B}_j)\,d\mathbf{x}\right|\le C\int_\Omega\left(|\mathbf{B}_j|^2+\left|\mathbf{B}_{h_j}\right||\mathbf{D}_j|\right)\,d\mathbf{x}\to 0\tag{74}$$

that is

$$\mathcal{F}_{h_j}(\mathbf{v}_j)\ge\mathcal{F}_{h_j}(\mathbf{w}_j)+o(1)\tag{75}$$

which proves that $\mathbf{w}_j$ is a minimizing sequence too. It is now readily seen that $\mathbf{w}_j$ are equibounded in $H^1(\Omega;\mathbb{R}^N)$ and (64) follows from (69) so the claim is proven. □

Remark 5.1 By inspection of the proof, Theorem 6.1 holds true also for more general energies: e.g., if $\mathcal{W}$ is a convex function of $\mathbf{F}^T\mathbf{F}-\mathbf{I}$ with quadratic growth and if $\mathcal{W}$ is finite if and only if $\det\mathbf{F}>0$.

7 Conclusions

The validation analysis of linearized elasticity performed in [1] concerning pure traction problems through Γ-convergence, quite surprisingly highlighted a new kind of limit energy functional $\mathcal{F}$.

Properties of this new functional and its relationship with the classical energy of linear elasticity have been investigated in the present paper, delivering fine differences among the dimensions $N = 2$ and $N = 3$.

The appearance, under suitable condition on the load, of infinitely many minimizers of the functional $\mathcal{F}$, which are not minimizers of the classical elasticity energy, requires further analysis and suggests that $\mathcal{F}$ could be more appropriate approximate energy, keeping memory of large instabilities affecting the nonlinear theory.

Indeed, due to our analysis, the classical linearized model of elasticity proves inadequate for a body uniformly compressed along its whole boundary in the direction of inward normal.

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