ON MULTIPLE FREQUENCY POWER DENSITY MEASUREMENTS II. THE FULL MAXWELL'S EQUATIONS

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Abstract. We shall give conditions on the illuminations φ_i such that the solutions to Maxwell's equations

$$\begin{cases} \mathrm{curl} E^i = \mathbf{i}\omega\mu H^i & \text{in } \Omega, \\ \mathrm{curl} H^i = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E^i & \text{in } \Omega, \\ E^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega, \end{cases}$$

satisfy certain non-zero qualitative properties inside the domain Ω, provided that a finite number of frequencies ω are chosen in a fixed range. The illuminations are explicitly constructed. This theory finds applications in several hybrid imaging problems, where unknown parameters have to be imaged from internal measurements. Some of these examples are discussed. This paper naturally extends a previous work of the author [Inverse Problems 29 (2013) 115007], where the Helmholtz equation was studied.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain and consider the Dirichlet boundary value problem for Maxwell's equations

$$(1) \qquad \begin{cases} \mathrm{curl} E_\omega^\varphi = \mathbf{i}\omega\mu H_\omega^\varphi & \text{in } \Omega, \\ \mathrm{curl} H_\omega^\varphi = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E_\omega^\varphi & \text{in } \Omega, \\ E_\omega^\varphi \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases}$$

where μ, ε and σ are uniformly positive definite real tensors representing the magnetic permeability, the electric permittivity and the conductivity, respectively, φ is a given illumination and $\omega > 0$. It is well known that the above problem is well-posed and admits a unique solution $(E_\omega^\varphi, H_\omega^\varphi)$ [28].

We want to find suitable illuminations φ_i such that the corresponding solutions to (1) satisfy certain non-zero conditions in Ω. For example, we may look for three illuminations φ_1, φ_2 and φ_3 such that

$$(2) \qquad \left|\det \begin{bmatrix} E_\omega^{\varphi_1} & E_\omega^{\varphi_2} & E_\omega^{\varphi_3} \end{bmatrix}(x)\right| > 0,$$

or, more generally, for b illuminations $\varphi_1, \dots, \varphi_b$ such that the corresponding solutions verify r conditions given by

$$(3) \qquad \left|\zeta_l\big((E_\omega^{\varphi_1}, H_\omega^{\varphi_1}), \dots, (E_\omega^{\varphi_b}, H_\omega^{\varphi_b})\big)(x)\right| > 0, \qquad l = 1, \dots, r,$$

where the maps ζ_l depend on $(E_\omega^{\varphi_i}, H_\omega^{\varphi_i})$ and their derivatives.

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This problem is generally studied for a fixed frequency $\omega > 0$. A possible approach to find suitable illuminations uses complex geometric optics (CGO) solutions for Maxwell's equations. They were introduced by Colton and Päivärinta [23], and generalize the CGO solutions originally introduced for the Calderón problem [26] by Sylvester and Uhlmann [32]. These are highly oscillatory solutions of (1) and can be chosen to satisfy (3) (see [22]). Unfortunately, CGO solutions have some drawbacks. First, their construction depends on the knowledge of the parameters μ, ε and σ, which in inverse problems are usually unknown. Second, they may be numerically unstable to implement due to their high oscillations. Third, the parameters have to be very smooth. In the elliptic case, another construction method for the illuminations uses the Runge approximation, which ensures that locally the solutions of the general problem behave like the solutions to the constant coefficient case. This approach was first introduced by Bal and Uhlmann in the case of elliptic equations [19]. As in the case of CGO solutions, the suitable illuminations are not explicitly constructed and may depend on the unknown parameters μ, ε and σ.

In this paper we propose an alternative strategy to this issue based on the use of multiple frequencies in a fixed range $K_{ad} = [K_{\min}, K_{\max}]$, for some $0 < K_{\min} < K_{\max}$. Namely, given the maps ζ_l, we shall give conditions on the illuminations such that the corresponding solutions satisfy the required properties, provided that a finite number of frequencies are used in the range K_{ad}. These conditions may depend on some (or none) of the parameters. More precisely, there exist illuminations φ_i and a finite number of frequencies $K \subseteq K_{ad}$ such that (3) is satisfied for every $x \in \Omega$ and for some $\omega \in K$ depending on x. For example, we shall show that the choice $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is sufficient for condition (2), provided that σ is close to a constant. The main idea behind this method is simple: if the illuminations are suitably chosen then the zero level sets of functionals depending on E^{φ}_{ω} and H^{φ}_{ω} *move* when the frequency changes. The proof is based on the regularity theory for Maxwell's equations, on the analyticity of the map $\omega \mapsto (E^{\varphi}_{\omega}, H^{\varphi}_{\omega})$ for complex frequencies ω and on the fact that the required conditions are satisfied in $\omega = 0$.

This work is the natural completion of a previous paper [1], where the same problem was studied for the Helmholtz equation. The generalization to Maxwell's equations is not straightforward. Regularity theory for Maxwell's equations is less developed than elliptic regularity theory, and a recent work [4] has been used here to have minimal regularity assumptions on the coefficients. Well-posedness for (1) with $\omega \in \mathbb{C} \setminus \Sigma$ follows by a standard argument, but is not as classical as with the Helmholtz equation, since we consider a complex frequency-dependent refractive index $\varepsilon + \mathbf{i}\omega^{-1}\sigma$.

This theory finds applications in the reconstruction procedures of several hybrid problems. In hybrid imaging techniques different types of waves are combined simultaneously to obtain high-resolution and high-contrast images (see [12, 27, 13, 5]). In a first step, internal functionals, or *power densities*, are measured inside the domain. In a second step, the parameters have to be reconstructed from the knowledge of these power densities. In most situations, the problem is modeled by the Helmholtz equation [34, 9, 17, 11, 19, 18, 10, 7] or by the full Maxwell's equations [29, 25, 20, 22], and the second step usually requires the availability of solutions satisfying certain qualitative properties inside the domain, such as (2) or, more generally, (3), for some maps ζ_l depending on the particular problem under consideration. These conditions have been shown to be satisfied by suitable CGO solutions or by means of the Runge approximation in the elliptic case. The multi-frequency approach described in this work is an alternative. Since K is finite, the dependence of the frequency on x does not constitute a source of instability in the reconstruction (see Section 4).

In [1] the microwave imaging by ultrasound deformation technique modeled by the Helmholtz equation was studied as an application. In this paper we provide three different examples of applications. Two are related to the problem of reconstructing the electromagnetic parameters from the knowledge of the magnetic field inside the domain. The third one deals with an inverse problem for electro-seismic conversion. In the literature regarding hybrid inverse problems, the Helmholtz equation approximation, rather than the full Maxwell's equations, has often been considered. It is likely that as this domain develops, other reconstruction algorithms requiring (3) to be satisfied for some maps ζ_l will appear, thereby providing possible new applications for this theory.

The multi-frequency approach described in this work provides a good alternative to the use of CGO or of the Runge approximation to find solutions satisfying qualitative properties inside the domain. However, a few points remain unsolved, and the same aspects are unsolved also in the case of the Helmholtz equation studied in [1]. First, is it possible to find a lower bound in (3)? Second, can we quantify the number of needed frequencies? In the case of the Helmholtz equation, numerical simulations were performed and suggest that in two dimensions three frequencies are sufficient. Third, can we remove the assumption on σ being close to a constant (when (2) is considered)? A possible way to tackle the last point consists in the use of generic illuminations, that in some situations may be sufficient for a general σ (see Remark 7). Some of these issues are addressed in [2, 6].

This paper is organized as follows. In Section 2 we precisely describe the setting and state the main theoretical result. Then, we apply this to three particular hybrid problems and study the reconstruction procedures. In Section 3 we prove the main result, and more details on the multi-frequency approach are given. In Section 4 the examples of hybrid problems are discussed in more detail. Section 5 is devoted to the proof of some results regarding well-posedness, regularity and analyticity properties of Maxwell's equations which are stated in Section 3.

2. Main results

Let $\Omega \subseteq \mathbb{R}^3$ be a $C^{\kappa+1,1}$ bounded and simply connected domain for some $\kappa \in \mathbb{N}^*$ with a simply connected boundary $\partial\Omega$. Let $\mu, \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ be real tensors satisfying the ellipticity conditions

$$\lambda |\xi|^2 \le \xi\cdot\mu\xi \le \Lambda |\xi|^2, \quad \lambda |\xi|^2 \le \xi\cdot\varepsilon\xi \le \Lambda |\xi|^2, \quad \lambda |\xi|^2 \le \xi\cdot\sigma\xi \le \Lambda |\xi|^2, \qquad \xi \in \mathbb{R}^3, \tag{4}$$

for some $\lambda, \Lambda > 0$. Moreover, as far as the regularity of these coefficients is concerned, we assume that

$$\mu, \varepsilon, \sigma \in W^{\kappa,p}(\Omega; \mathbb{R}^{3\times 3}) \tag{5}$$

for some $p > 3$ and $\kappa \in \mathbb{N}^*$. We shall study problem (1) with illumination satisfying

$$\varphi \in W^{\kappa,p}(\Omega; \mathbb{C}^3), \qquad \mathrm{curl}\varphi \cdot \nu = 0 \text{ on } \partial\Omega. \tag{6}$$

The second of these conditions is required to make the problem well-posed if $\omega = 0$ (see Section 5). The natural functional space associated to (1) is

$$H(\mathrm{curl}, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) : \mathrm{curl} u \in L^2(\Omega; \mathbb{C}^3)\}.$$

2.1. **A multi-frequency approach to the boundary control of Maxwell's equations.** Let $K_{ad} = [K_{\min}, K_{\max}]$ be the range of frequencies we have access to, for some $0 < K_{\min} < K_{\max}$.

Definition 1. Given a finite set $K \subseteq K_{ad}$ and illuminations $\varphi_1, \dots, \varphi_b$ satisfying (6), we say that $K \times \{\varphi_1, \dots, \varphi_b\}$ is *a set of measurements*.

Let $K \times \{\varphi_1, \dots, \varphi_b\}$ be a set of measurements. For any $(\omega, \varphi_i) \in K \times \{\varphi_1, \dots, \varphi_b\}$ denote the unique solution to (1) in $H(\mathrm{curl}, \Omega)^2$ by (E^i_ω, H^i_ω), namely

$$\begin{cases} \mathrm{curl} E^i_\omega = \mathbf{i}\omega\mu H^i_\omega & \text{in } \Omega, \\ \mathrm{curl} H^i_\omega = -\mathbf{i} q_\omega E^i_\omega & \text{in } \Omega, \\ E^i_\omega \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where

$$q_\omega = \omega\varepsilon + \mathbf{i}\sigma. \tag{8}$$

In Proposition 15 we shall show that $(E^i_\omega, H^i_\omega) \in C^{\kappa-1}(\overline{\Omega}; \mathbb{C}^6)$.

The reconstruction procedures in hybrid problems often requires the availability of a certain number of illuminations such that the corresponding solutions to (7) and their derivatives up to the $(\kappa - 1)$-th order satisfy some non-zero conditions inside the domain. Let $b \in \mathbb{N}^*$ be the number of illuminations and $r \in \mathbb{N}^*$ be the number of non-zero conditions. These conditions depend on the particular problem under consideration, but it turns out that most of them can be written in the following form.

Definition 2. Let $b, r \in \mathbb{N}^*$ be two positive integers, $s > 0$ and let

$$\zeta = (\zeta_1, \dots, \zeta_r) \colon C^{\kappa-1}(\overline{\Omega}; \mathbb{C}^6)^b \longrightarrow C(\overline{\Omega}; \mathbb{C})^r \quad \text{be analytic.} \tag{9}$$

A set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$ is ζ*-complete* if for every $x \in \overline{\Omega}$ there exists $\omega = \omega(x) \in K$ such that

$$\begin{aligned} &1.\ \left|\zeta_1\big((E^1_\omega, H^1_\omega), \dots, (E^b_\omega, H^b_\omega)\big)(x)\right| \geq s, \\ &\ \vdots \qquad\qquad\qquad \vdots \\ &\mathrm{r.}\ \left|\zeta_r\big((E^1_\omega, H^1_\omega), \dots, (E^b_\omega, H^b_\omega)\big)(x)\right| \geq s. \end{aligned} \tag{10}$$

The notion of analyticity for maps between complex Banach spaces will be given in Section 3. This notion generalizes the usual one for maps of complex variable, and for now we anticipate some basic facts (detailed in Lemma 14): multilinear bounded functions are analytic, the composition of analytic functions is analytic, and ζ is analytic if and only if ζ_l is analytic for every l.

The analyticity assumption on ζ is not particularly restrictive, as it will be clear from the following three examples of maps ζ's . In the next subsection, we shall see that these examples are motivated by some hybrid problems.

Example 3. Take $b = 3$, $r = 1$, $\kappa = 1$ and $\zeta^{(1)}$ defined by

$$\zeta^{(1)}((u_1, v_1), (u_2, v_2), (u_3, v_3)) = \det \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}, \qquad (u_i, v_i) \in C(\overline{\Omega}; \mathbb{C}^6).$$

The map $\zeta^{(1)}$ is multilinear and bounded, whence analytic. In this case, the condition characterizing $\zeta^{(1)}$-complete sets of measurements is

$$\left|\det\left[E_\omega^1 \quad E_\omega^2 \quad E_\omega^3\right](x)\right| \geq s. \tag{11}$$

In other words, (11) signals the availability, in every point, of three independent electric fields and, in particular, of one non-vanishing electric field.

Example 4. Take $b = 6$, $r = 1$ and $\kappa = 2$. Consider the function $\eta\colon C^1(\overline{\Omega};\mathbb{C}^3)^2 \longrightarrow C(\overline{\Omega};\mathbb{C}^3)$ given by

$$\eta(u_1,u_2) = (\nabla u_1)u_2-(\nabla u_2)u_1+\mathrm{div}u_1u_2-\mathrm{div}u_2u_1-2\,{}^t(\nabla u_1)u_2+2\,{}^t(\nabla u_2)u_1. \tag{12}$$

Define now $\zeta^{(2)}\colon C^1(\overline{\Omega};\mathbb{C}^6)^6 \longrightarrow C(\overline{\Omega};\mathbb{C})$ by

$$\zeta^{(2)}((u_1,v_1),\dots,(u_6,v_6)) = \det\left[\eta(u_1,u_2) \quad \eta(u_3,u_4) \quad \eta(u_5,u_6)\right].$$

As before, the map $\zeta^{(2)}$ is multilinear and bounded, whence analytic. The interpretation of the corresponding condition (10) is not immediate, and the reader is referred to the following subsection for an application.

Example 5. Take $b = 6$, $r = 2$ and $\kappa = 2$. As $r = 2$, the map $\zeta^{(3)}$ considered here involves two conditions. The first one is given by $\zeta^{(2)}$, and the second one refers to the availability of one non-vanishing electric field. More precisely, we define $\zeta^{(3)}\colon C^1(\overline{\Omega};\mathbb{C}^6)^6 \longrightarrow C(\overline{\Omega};\mathbb{C})^2$ by

$$\zeta^{(3)}((u_1,v_1),\dots,(u_6,v_6)) = \left(\zeta^{(2)}((u_1,v_1),\dots,(u_6,v_6))\,,\,(u_1)_2\right).$$

The two components of this map are analytic, and so $\zeta^{(3)}$ is analytic.

Before studying the construction of ζ-complete sets of measurements, let us make a comment on (1). As already stated in the introduction, our strategy to make use of several frequencies starts with a study of (1) when $\omega = 0$. For $\omega > 0$, the well-posedness for this problem is classical [28]. However, in order to make the problem well-posed in the case $\omega = 0$, we need to add some constraints on H_ω, as it will be shown in Section 5. In particular, we look for solutions $(E_\omega, H_\omega) \in H(\mathrm{curl},\Omega) \times H^\mu(\mathrm{curl},\Omega)$, where

$$H^\mu(\mathrm{curl},\Omega) = \{v \in H(\mathrm{curl},\Omega) : \mathrm{div}(\mu v) = 0 \text{ in } \Omega,\ \mu v\cdot\nu = 0 \text{ on } \partial\Omega\}.$$

We shall see that problem (1) is well-posed in $H(\mathrm{curl},\Omega)\times H^\mu(\mathrm{curl},\Omega)$ for all $\omega \geq 0$.

The main result of this work deals with the construction of ζ-complete sets of measurements.

Theorem 6. *Assume that* (4) *and* (5) *hold. Let $\hat\sigma \in W^{\kappa,p}(\Omega;\mathbb{R}^{3\times3})$ be an arbitrary matrix-valued function satisfying* (4), *$\varphi_1,\dots,\varphi_b$ satisfy* (6) *and ζ be as in* (9). *Suppose that*

$$\zeta_l\big((\hat E_0^1,\hat H_0^1),\dots,(\hat E_0^b,\hat H_0^b)\big)(x) \neq 0, \qquad x \in \overline{\Omega},\ l = 1,\dots,r, \tag{13}$$

where $(\hat E_0^i,\hat H_0^i) \in H(\mathrm{curl},\Omega) \times H^\mu(\mathrm{curl},\Omega)$ is the solution to (7) *with $\hat\sigma$ in lieu of σ and $\omega = 0$, namely*

$$\begin{cases} \mathrm{curl}\hat E_0^i = 0 & \text{in } \Omega,\\ \mathrm{div}(\hat\sigma\hat E_0^i) = 0 & \text{in } \Omega,\\ \hat E_0^i\times\nu = \varphi_i\times\nu & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \mathrm{curl}\hat H_0^i = \hat\sigma\hat E_0^i & \text{in } \Omega,\\ \mathrm{div}(\mu\hat H_0^i) = 0 & \text{in } \Omega,\\ \mu\hat H_0^i\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

There exists $\delta > 0$ *such that if* $\|\sigma - \hat{\sigma}\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ *then we can choose a finite* $K \subseteq K_{ad}$ *such that*

$$K \times \{\varphi_1, \dots, \varphi_b\}$$

is a ζ*-complete set of measurements.*

In the following remark we discuss assumption (13) and the dependence of the construction of the illuminations on the electromagnetic parameters.

Remark 7. Suppose that we are in the simpler case $\hat{\sigma} = \sigma$. This result states that we can construct a ζ-complete set of measurements, if the illuminations $\varphi_1, \dots, \varphi_b$ are chosen in such a way that (13) holds true. It is in general much easier to satisfy (13) than (10), as $\omega = 0$ makes problem (7) simpler (see next subsection). Note that (14) does not depend on ε, so that the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is independent of ε. For the same reason, their construction may depend on σ and μ. However, in the cases where the maps ζ_l involve only the electric field E, it depends on σ, and not on ε and μ (see Corollary 10).

A typical application of the theorem is in the case where σ is a small perturbation of a known constant tensor $\hat{\sigma}$. Then, the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is independent of σ. A similar argument would work if μ is a small perturbation of a constant tensor $\hat{\mu}$. We have decided to omit it for simplicity, since in the applications we have in mind the maps ζ_l do not depend on the magnetic field H, thereby making assumption (13) independent of μ.

In [1], we showed that in the case of the Helmholtz equation there exist *occulting* illuminations, that is, boundary conditions for which a finite number of frequencies is not sufficient, and so the assumption corresponding to (13) cannot be completely removed. Yet, it is very likely that this assumption can be considerably weakened. Further investigations in this direction may lead to a construction totally independent of the electromagnetic parameters. For example, it is natural to wonder whether generic illuminations are non-occulting. In the case of the Helmholtz equation, we can prove that this is the case if we consider the map $\zeta(u) = \nabla u$ [3].

In the following remark we discuss the construction of the finite set of frequencies K.

Remark 8. The proof of the theorem is constructive as far as the choice of the set K is concerned. Given a converging sequence in K_{ad}, a suitable finite subsequence will be sufficient. More details are given in Lemma 19. An upper bound for the number of needed frequencies is still unavailable. Numerical simulations for the Helmholtz model were performed in [1], which suggest that three frequencies are sufficient in two dimensions. We believe that four frequencies should be sufficient in the case of Maxwell's equations in dimension three.

Finally, we compare this construction with the CGO approach.

Remark 9. The regularity of the coefficients required for this approach is much lower than the regularity required if CGO solutions are used. Indeed, if the conditions depend on the derivatives up to the $(\kappa - 1)$-th order, with CGO we need at least $W^{\kappa+2,p}$ [22], while with this approach we require the parameters to be in $W^{\kappa,p}$ for some $p > 3$. Note that these regularity assumptions are in some sense minimal: in order for (10) to be meaningful pointwise, we need $E^i_\omega, H^i_\omega \in C^{\kappa-1}$.

Moreover, as discussed in Remark 7 and in § 2.2 below, by using this approach the construction is usually explicit and does not require highly oscillatory illuminations.

In the case where the conditions given by the map ζ are independent of the magnetic field H, Theorem 6 can be rewritten in the following form.

Corollary 10. *Assume that* (4) *and* (5) *hold. Let* $\hat{\sigma} \in W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})$ *satisfy* (4) *and* ζ *be as in* (9) *and independent of* H*. Take* $\psi_1,\dots,\psi_b \in W^{\kappa+1,p}(\Omega;\mathbb{C})$*. Suppose that*

$$\zeta_l\big(\nabla w^1,\dots,\nabla w^b\big)(x) \neq 0, \qquad x \in \overline{\Omega},\, l = 1,\dots,r, \tag{15}$$

where $w^i \in H^1(\Omega;\mathbb{C})$ *is the solution to*

$$\begin{cases} \operatorname{div}(\hat{\sigma}\nabla w^i) = 0 & \text{in } \Omega,\\ w^i = \psi_i & \text{on } \partial\Omega. \end{cases}$$

There exists $\delta > 0$ *such that if* $\|\sigma - \hat{\sigma}\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ *then we can choose a finite* $K \subseteq K_{ad}$ *such that*

$$K \times \{\nabla\psi_1,\dots,\nabla\psi_b\}$$

is a ζ*-complete set of measurements.*

In other words, if the required qualitative properties do not depend on H, then the problem of finding ζ-complete sets is completely reduced to satisfying the same conditions for the gradients of solutions to the conductivity equation. This last problem has received a considerable attention in the literature (see e.g. [8, 21, 36, 19, 15, 14, 16]).

2.2. **Applications to hybrid problems.** In this subsection we apply the theory introduced so far to three reconstruction procedures arising in hybrid imaging. In each case, the reconstruction is possible if we have a ζ-complete sets of measurements, where ζ is one of the maps discussed in Examples 3, 4 and 5. Only the main points are discussed here, and full details are given in Section 4.

2.2.1. *Reconstruction of* ε *and* σ *from knowledge of internal magnetic fields - first method.* Combining boundary measurements with an MRI scanner, we can measure the internal magnetic fields H^i_ω [29, 25]. Assuming $\mu = 1$, the electromagnetic parameters to image are ε and σ, and both are assumed isotropic. We shall consider two different reconstruction algorithms. For the first method, the relevant map ζ is the determinant $\zeta^{(1)}$ introduced in Example 3.

Let $K \times \{\varphi_1,\varphi_2,\varphi_3\}$ be a set of measurements and consider problem (7). We shall show that q_ω given by (8) satisfies a first order partial differential equation in Ω, with coefficients depending on the magnetic fields, thereby known. This equation is of the form

$$\nabla q_\omega M^{(1)}_\omega = F^{(1)}(\omega, q_\omega, H^i_\omega, \Delta H^i_\omega) \qquad \text{in } \Omega,$$

where $M^{(1)}_\omega$ is the 3×6 matrix-valued function given by

$$M^{(1)}_\omega = \begin{bmatrix} \operatorname{curl} H^1_\omega \times \mathbf{e}_1 & \operatorname{curl} H^1_\omega \times \mathbf{e}_2 & \cdots & \operatorname{curl} H^3_\omega \times \mathbf{e}_1 & \operatorname{curl} H^3_\omega \times \mathbf{e}_2 \end{bmatrix},$$

and $F^{(1)}$ is a given vector-valued function. If

$$\left|\det\begin{bmatrix} E^1_\omega & E^2_\omega & E^3_\omega \end{bmatrix}(x)\right| > 0, \tag{16}$$

then $M^{(1)}_\omega(x)$ admits a right inverse $(M^{(1)}_\omega)^{-1}(x)$. The equation for q_ω becomes

$$\nabla q_\omega(x) = F^{(1)}(\omega, q_\omega, H^i_\omega, \Delta H^i_\omega)(M^{(1)}_\omega)^{-1}(x). \tag{17}$$

Suppose now that $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ is det-complete. This implies that (16) is satisfied everywhere in Ω for some $\omega = \omega(x)$. We shall see that in this case it is possible to integrate (17) and reconstruct uniquely q_ω, provided that q_ω is known at one point of $\overline{\Omega}$.

In this example, we have seen that $\zeta^{(1)}$-complete sets are sufficient to be able to image the electromagnetic parameters. Moreover, $\zeta^{(1)}$-complete sets can be explicitly constructed by using Corollary 10.

Proposition 11. *Assume that* (4) *and* (5) *hold with* $\kappa = 1$ *and let* $\hat{\sigma} \in \mathbb{R}^{3\times 3}$ *be positive definite. There exists* $\delta > 0$ *such that if* $\|\sigma - \hat{\sigma}\|_{W^{1,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ *then we can choose a finite* $K \subseteq K_{ad}$ *such that*

$$K \times \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

is a $\zeta^{(1)}$*-complete set of measurements.*

Proof. We want to apply Corollary 10 with $\zeta = \zeta^{(1)}$ and $\psi_i = x_i$ for $i = 1, 2, 3$. We only need to show that (15) holds. Since $w^i = x_i$, for every $x \in \overline{\Omega}$ there holds

$$\zeta\big(\nabla w^1, \nabla w^2, \nabla w^3\big)(x) = \det \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{bmatrix} = 1 \neq 0,$$

as desired. □

2.2.2. *Reconstruction of* ε *and* σ *from knowledge of internal magnetic fields - second method.* We consider the reconstruction procedure discussed in [25] for the same hybrid problem studied before. As we shall see, in this case the relevant map ζ is $\zeta^{(2)}$, introduced in Example 4.

Let $K \times \{\varphi_1, \ldots, \varphi_6\}$ be a set of measurements and consider problem (7). We shall prove that q_ω satisfies a first order partial differential equation in Ω, with coefficients depending on the magnetic fields, thereby known. This equation is of the form

$$\nabla q_\omega M_\omega^{(2)} = q_\omega F^{(2)}(\mathrm{curl} H_\omega^i) \qquad \text{in } \Omega, \tag{18}$$

where $M_\omega^{(2)}$ is the 3×3 matrix-valued function given by

$$M_\omega^{(2)} = \begin{bmatrix} \eta(\mathrm{curl}H_\omega^1, \mathrm{curl}H_\omega^2) & \eta(\mathrm{curl}H_\omega^3, \mathrm{curl}H_\omega^4) & \eta(\mathrm{curl}H_\omega^5, \mathrm{curl}H_\omega^6) \end{bmatrix},$$

with η given by (12), and $F^{(2)}$ is a vector-valued function. It turns out that $\eta(\mathrm{curl}H_\omega^i, \mathrm{curl}H_\omega^j) = -q_\omega^2 \eta(E_\omega^i, E_\omega^j)$, whence

$$M_\omega^{(2)} = -q_\omega^2 \begin{bmatrix} \eta(E_\omega^1, E_\omega^2) & \eta(E_\omega^3, E_\omega^4) & \eta(E_\omega^5, E_\omega^6) \end{bmatrix}.$$

Therefore, if

$$\left|\det \begin{bmatrix}\eta(E_\omega^1, E_\omega^2) & \eta(E_\omega^3, E_\omega^4) & \eta(E_\omega^5, E_\omega^6)\end{bmatrix}(x)\right| > 0, \tag{19}$$

then the equation for q_ω becomes

$$\nabla q_\omega(x) = q_\omega F^{(2)}(\mathrm{curl} H_\omega^i)(M_\omega^{(2)})^{-1}(x). \tag{20}$$

Suppose now that $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ is $\zeta^{(2)}$-complete. This implies that (19) is satisfied everywhere in Ω for some $\omega = \omega(x)$. As in the previous case, it is possible to integrate (20) and reconstruct q_ω uniquely, provided that q_ω is known at one point of $\overline{\Omega}$.

In this example, we have seen that $\zeta^{(2)}$-complete sets are sufficient to be able to reconstruct the electromagnetic parameters. As in the previous case, $\zeta^{(2)}$-complete sets can be explicitly constructed using Corollary 10. Let I denote the 3×3 identity matrix.

Proposition 12. *Assume that* (4) *and* (5) *hold with* $\kappa = 2$ *and take* $\hat{\sigma} > 0$. *There exists* $\delta > 0$ *such that if* $\|\sigma - \hat{\sigma} I\|_{W^{2,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ *then we can choose a finite* $K \subseteq K_{ad}$ *such that*

$$K \times \{\mathbf{e}_2, \nabla(x_1x_2), \mathbf{e}_3, \nabla(x_2x_3), \mathbf{e}_1, \nabla(x_1x_3)\}$$

is a $\zeta^{(2)}$*-complete set of measurements.*

Proof. We want to apply Corollary 10 with $\zeta = \zeta^{(2)}$ and $\psi_1 = x_2$, $\psi_2 = x_1x_2$, $\psi_3 = x_3$, $\psi_4 = x_2x_3$, $\psi_5 = x_1$ and $\psi_6 = x_1x_3$. We only need to show that (15) holds. Since $w^i = \psi_i$, a trivial calculation shows that for every $x \in \overline{\Omega}$

$$\begin{aligned}\zeta^{(2)}\big(\nabla w^1, \dots, \nabla w^6\big)(x) &= \det\big[\eta(\mathbf{e}_2, \nabla(x_1x_2)) \quad \eta(\mathbf{e}_3, \nabla(x_2x_3)) \quad \eta(\mathbf{e}_1, \nabla(x_1x_3))\big] \\ &= 1,\end{aligned}$$

as desired. □

In [25], complex geometric optics solutions are used to make problem (18) solvable. With this approach, the six illuminations are given explicitly and do not depend on the coefficients. However, it has to be noted that the assumption $\|\sigma - \hat{\sigma}\|_{W^{2,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ can be restrictive (see Remark 7).

2.2.3. *Inverse problem of electro-seismic conversion.* Electro-seismic conversion is the generation of a seismic wave in a fluid-saturated porous material when an electric field is applied [37]. The problem is modeled by the coupling of Maxwell's equations and Biot's equations. We consider the hybrid inverse problem introduced in [22]. In the first step, by inverting Biot's equation, the quantity $D_\omega^i = LE_\omega^i$ is recovered in Ω, where $L > 0$ is a possibly varying coefficient representing the coupling between electromagnetic and mechanic effects. In a second step, the electromagnetic parameters ε and σ have to be imaged from the knowledge of D_ω^i. As in the previous cases, we assume that $\mu = 1$ and that ε and σ are isotropic.

Let $K \times \{\varphi_1, \dots, \varphi_6\}$ be a set of measurements. Since $\mathrm{curl} H_\omega^i = -\mathbf{i} q_\omega E_\omega^i$, this problem is very similar to the one discussed in 2.2.2, and L plays the role of $-\mathbf{i}q_\omega$. Therefore, if $K \times \{\varphi_1, \dots, \varphi_6\}$ is $\zeta^{(2)}$-complete, it is possible to reconstruct the coefficient L. Once L is known, the electric field can be easily obtained as $E_\omega^i = L^{-1}D_\omega^i$. Finally, ε and σ can be reconstructed via

$$\omega q_\omega E_\omega^1(x) = \mathrm{curl}\mathrm{curl} E_\omega^1(x),$$

provided that E_ω^1 is non-vanishing. In particular, this is true if $\left|(E_\omega^1)_2\right|(x) > 0$. This is the second condition in the definition of $\zeta^{(3)}$ in Example 5. Therefore, if $K \times \{\varphi_1, \dots, \varphi_6\}$ is $\zeta^{(3)}$-complete then it is possible to uniquely reconstruct σ and ε if these are known at one point in $\overline{\Omega}$.

The construction of $\zeta^{(3)}$-complete sets of measurements is analogous to the construction of $\zeta^{(2)}$-complete sets.

Proposition 13. *Assume that* (4) *and* (5) *hold with* $\kappa = 2$ *and take* $\hat{\sigma} > 0$. *There exists* $\delta > 0$ *such that if* $\|\sigma - \hat{\sigma} I\|_{W^{2,p}(\Omega;\mathbb{R}^{3\times 3})} \leq \delta$ *then we can choose a finite* $K \subseteq K_{ad}$ *such that*

$$K \times \{\mathbf{e}_2, \nabla(x_1x_2), \mathbf{e}_3, \nabla(x_2x_3), \mathbf{e}_1, \nabla(x_1x_3)\}$$

is a $\zeta^{(3)}$*-complete set of measurements.*

The same comment given after Proposition 12 is relevant here: in [22], complex geometric optics solutions are used to make this problem solvable.

3. A multi-frequency approach to the boundary control of Maxwell's equations

In this section we prove Theorem 6. In subsection 3.1 we study some basic properties of analytic functions in Banach spaces. In subsection 3.2 we state the main results regarding well-posedness, regularity and analyticity properties for problem (1). Finally, in subsection 3.3, the proof of Theorem 6 is given.

3.1. Analytic functions.

Analytic functions in a Banach space setting were studied in [33]. Let E and E' be complex Banach spaces, $D \subseteq E$ be an open set and take $f\colon D \to E'$. We say that f admits a Gateaux differential in $x_0 \in D$ with respect to the direction $y \in E$ if the limit

$$\lim_{\tau\to 0} \frac{f(x_0+\tau y)-f(x_0)}{\tau}$$

exists in E'. We say that f is analytic in x_0 if it is continuous in x_0 and admits a Gateaux differential in x_0 with respect to every direction $y \in E$. We say that f is analytic in D (or simply analytic) if it is analytic in every point of D. With this definition, it is clear that this notion extends the classical notion of analyticity for functions of complex variable.

The following lemma summarizes some of the basic properties of analytic functions that are of interest to us.

Lemma 14. *Let $E_1,\dots,E_r$, E and E' be complex Banach spaces. Let $D \subseteq E$ be an open set.*

1. *If $f\colon E_1\times\cdots\times E_r \to E'$ is multilinear and bounded then f is analytic.*
2. *If $f\colon D\to E_1$ and $g\colon E_1 \to E'$ are analytic then $g\circ f\colon D\to E'$ is analytic.*
3. *Take $f=(f_1,\dots f_r)\colon D\to E_1\times\cdots\times E_r$. Then f is analytic if and only if f_l is analytic for every $l=1,\dots,r$.*

Proof. Parts (1) and (3) trivially follow from the definition. Part (2) is shown in [35]. □

3.2. Maxwell's equations.

The proofs of the results stated in this subsection are given in Section 5.

We first study well-posedness for the problem at hand. As it will be clear from the proof of Theorem 6, we need to study problem (1) in the more general case of complex frequency. The case $\omega \neq 0$ is classical [31], and the problem is well-posed except for a discrete set of complex resonances. Well-posedness in the case $\omega = 0$ will follow from a standard argument involving the Helmholtz decomposition.

Proposition 15. *Assume that* (4) *and* (6) *hold for some $p>3$ and $\kappa = 1$. There exists a discrete set $\Sigma \subseteq \mathbb{C}\setminus\{0\}$ such that for all $\omega \in \mathbb{C}\setminus\Sigma$ the problem*

$$\begin{cases} \mathrm{curl} E_\omega = \mathbf{i}\omega\mu H_\omega & \text{in } \Omega,\\ \mathrm{curl} H_\omega = -\mathbf{i}q_\omega E_\omega & \text{in } \Omega,\\ \mathrm{div}(\mu H_\omega) = 0 & \text{in } \Omega,\\ E_\omega\times\nu = \varphi\times\nu & \text{on } \partial\Omega,\\ \mu H_\omega\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

admits a unique solution $(E_\omega, H_\omega) \in H(\mathrm{curl}, \Omega) \times H^\mu(\mathrm{curl}, \Omega)$ *and*

$$\|(E_\omega, H_\omega)\|_{H(\mathrm{curl},\Omega)^2} \leq C \|\varphi\|_{H(\mathrm{curl},\Omega)},$$

for some $C > 0$ *independent of* φ.

We next state a regularity result for the solution (E_ω, H_ω), which follows from the regularity theorems in [4].

Proposition 16. *Assume that* (4), (5) *and* (6) *hold for some* $p > 3$ *and* $\kappa \in \mathbb{N}^*$. *For* $\omega \in \mathbb{C} \setminus \Sigma$ *let* $(E_\omega, H_\omega) \in H(\mathrm{curl}, \Omega)$ *be the unique solution to* (21). *Then* $(E_\omega, H_\omega) \in C^{\kappa-1}(\overline{\Omega}; \mathbb{C}^6)$ *and*

$$\|(E_\omega, H_\omega)\|_{C^{\kappa-1}(\overline{\Omega};\mathbb{C}^6)} \leq C \|\varphi\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)},$$

for some $C > 0$ *independent of* φ. *Moreover, if* $\omega = 0$ *there holds*

$$\|(E_0, H_0)\|_{W^{\kappa,p}(\Omega;\mathbb{C}^6)} \leq C' \|\varphi\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)},$$

for some $C' > 0$ *depending on* Ω, λ, Λ, κ, p *and* $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times3})^3}$ *only.*

We finally state that (E_ω, H_ω) depends analytically on ω. This is the main result of Section 5, and represents one of the main ingredients of the proof of Theorem 6,

Proposition 17. *Under the hypotheses of Proposition 16 the map*

$$S_\varphi \colon \mathbb{C} \setminus \Sigma \longrightarrow C^{\kappa-1}(\overline{\Omega}; \mathbb{C}^6), \qquad \omega \longmapsto (E_\omega, H_\omega)$$

is analytic.

3.3. **Proof of Theorem 6.** The rest of this section is devoted to the proof of Theorem 6. We first need two preliminary lemmata.

One of the main tools of the multi-frequency approach is the study of (10) in the case $\omega = 0$. The following lemma shows that assumption (13) implies that (10) is satisfied in $\omega = 0$ if the conductivity σ is a perturbation of $\hat{\sigma}$, namely

$$\sigma = \hat{\sigma} + t, \qquad \|t\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times3})} \leq \delta,$$

for some sufficiently small $\delta > 0$. Denote the solution to (7) corresponding to $\sigma = \hat{\sigma} + t$ and $\omega = 0$ by $(E_0^i(t), H_0^i(t))$.

Lemma 18. *Assume that the hypotheses of Theorem 6 hold true and write* $\sigma = \hat{\sigma} + t$. *There exists* $\delta > 0$ *such that if* $\|t\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times3})} \leq \delta$ *then*

$$\left|\zeta_l\big((E_0^1(t), H_0^1(t)), \ldots, (E_0^b(t), H_0^b(t))\big)(x)\right| > 0, \qquad l = 1, \ldots, r,\ x \in \overline{\Omega}. \tag{22}$$

Proof. Assume $\|t\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times3})} \leq \eta$, for some $\eta > 0$ sufficiently small so that $\hat{\sigma} + t$ always satisfies the ellipticity condition (4). We use the decomposition $E_0^i(t) = \tilde{E}_0^i(t) + \varphi_i$, where

$$\tilde{E}_0^i(t) \in H_0(\mathrm{curl}, \Omega) = \{u \in H(\mathrm{curl}, \Omega) : u \times \nu = 0 \text{ on } \partial\Omega\}.$$

By construction, $(\tilde{E}_0^i(t), H_0^i(t))$ is a solution to

$$\begin{cases} \mathrm{curl} H_0^i(t) - (\hat{\sigma} + t)\tilde{E}_0^i(t) = (\hat{\sigma} + t)\varphi_i & \text{in } \Omega, \\ \mathrm{curl} \tilde{E}_0^i(t) = -\mathrm{curl}\varphi & \text{in } \Omega, \end{cases}$$

together with $\mathrm{div}(\mu H_0^i(t)) = 0$ in Ω and $\mu H_0^i(t) \cdot \nu = 0$ on $\partial\Omega$. Writing $u^i(t) = E_0^i(t) - E_0^i(0) = \tilde{E}_0^i(t) - \tilde{E}_0^i(0)$ and $v^i(t) = H_0^i(t) - H_0^i(0)$, an easy calculation shows that

$$\begin{cases} \mathrm{curl} v^i(t) - \hat{\sigma} u^i(t) = t E_0^i(t) & \text{in } \Omega, \\ \mathrm{curl} u^i(t) = 0 & \text{in } \Omega. \end{cases}$$

We can now apply Proposition 25 (see Section 5) to obtain that

$$\left\|(u^i(t),v^i(t))\right\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2} \le c_1 \left\|tE_0^i(t)\right\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)},$$

for some $c_1 > 0$ independent of t (but depending on η). Moreover, by Proposition 16 we have $\left\|E_0^i(t)\right\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} \le c_2$ for some c_2 independent of t. Thus, the Sobolev Embedding Theorem yields

$$\left\|\big(E_0^i(t)-E_0^i(0),H_0^i(t)-H_0^i(0)\big)\right\|_{C^{\kappa-1}(\overline{\Omega};\mathbb{C}^6)} \le c\left\|t\right\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})}, \tag{23}$$

for some $c > 0$ independent of t.

Note now that (13) yields

$$\left|\zeta_l\big((E_0^1(0),H_0^1(0)),\dots,(E_0^b(0),H_0^b(0))\big)(x)\right| > 0, \qquad l=1,\dots,r,\ x\in\overline{\Omega}.$$

Therefore, in view of (23) and the continuity of ζ we obtain the result. □

In order to prove Theorem 6, we still need to *transfer* property (22) to any range of frequencies. This is the content of Lemma 19, that generalizes [1, Lemma 4.1] to Maxwell's equations.

Lemma 19. *Assume that* (4) *and* (5) *hold. Let* $\varphi_1,\dots,\varphi_b$ *satisfy* (4) *and* ζ *be as in* (9). *Assume that for* $\omega = 0$

$$\zeta_l\big((E_0^1,H_0^1),\dots,(E_0^b,H_0^b)\big)(x) \ne 0, \qquad x\in\overline{\Omega},\ l=1,\dots,r.$$

Take $\omega_n,\omega\in K_{ad}$ *with* $\omega_n\to\omega$ *and* $\omega_n\ne\omega$. *Then there exists a finite* $N\subseteq\mathbb{N}$ *such that*

$$\sum_{n\in N}\left|\zeta_l\big((E_{\omega_n}^1,H_{\omega_n}^1),\dots,(E_{\omega_n}^b,H_{\omega_n}^b)\big)(x)\right| > 0, \qquad x\in\overline{\Omega},\ l=1,\dots,r.$$

In particular, we can choose a finite $K\subseteq K_{ad}$ *such that*

$$\sum_{\omega\in K}\left|\zeta_l\big((E_{\omega}^1,H_{\omega}^1),\dots,(E_{\omega}^b,H_{\omega}^b)\big)(x)\right| > 0, \qquad x\in\overline{\Omega},\ l=1,\dots,r.$$

Proof. Take $x\in\overline{\Omega}$. Consider the map $g_x\colon\mathbb{C}\setminus\Sigma\to\mathbb{C}$ defined by

$$g_x(\omega)=\prod_{l=1}^{r}\zeta_l\big((E_{\omega}^1,H_{\omega}^1),\dots,(E_{\omega}^b,H_{\omega}^b)\big)(x), \qquad \omega\in\mathbb{C}\setminus\Sigma.$$

Proposition 17 states that the map $\omega\in\mathbb{C}\setminus\Sigma\mapsto(E_\omega^i,H_\omega^i)\in C^{\kappa-1}(\overline{\Omega},\mathbb{C}^6)$ is analytic. Hence, in view of (9), Lemma 14 yields that g_x is analytic. Thus, as $g_x(0)\ne 0$, the set $\{\omega'\in\mathbb{C}\setminus\Sigma : g_x(\omega')=0\}$ has no accumulation points in $\mathbb{C}\setminus\Sigma$ by the analytic continuation theorem. Since by assumption ω is an accumulation point for the sequence (ω_n), this implies that there exists $n_x\in\mathbb{N}$ such that $g_x(\omega_{n_x})\ne 0$. As a consequence, in view of the continuity of the map $\prod_{l=1}^{r}\zeta_l\big((E_{\omega_{n_x}}^1,H_{\omega_{n_x}}^1),\dots,(E_{\omega_{n_x}}^b,H_{\omega_{n_x}}^b)\big)$, we can find $r_x>0$ such that

$$g_y(\omega_{n_x})\ne 0, \qquad y\in B(x,r_x)\cap\overline{\Omega}. \tag{24}$$

Since $\overline{\Omega}=\bigcup_{x\in\overline{\Omega}}\big(B(x,r_x)\cap\overline{\Omega}\big)$, there exist $x_1,\dots,x_M\in\overline{\Omega}$ satisfying

$$\overline{\Omega}=\bigcup_{m=1}^{M}\big(B(x_m,r_{x_m})\cap\overline{\Omega}\big). \tag{25}$$

Defining $N=\{n_{x_m} : m=1:,\dots,M\}$, by (24) and (25) we obtain the result. □

We are now in a position to prove Theorem 6

Proof of Theorem 6. Take δ as in Lemma 18 and suppose $\|\sigma-\hat\sigma\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})}\le\delta$. By Lemma 18 we have

$$\zeta_l\big((E_0^1,H_0^1),\dots,(E_0^b,H_0^b)\big)(x)\neq 0,\qquad l=1,\dots,r,\ x\in\overline{\Omega}.$$

As a result, in view of Lemma 19, we can choose a finite $K\subseteq K_{ad}$ such that

$$\sum_{\omega\in K}\big|\zeta_l\big((E_\omega^1,H_\omega^1),\dots,(E_\omega^b,H_\omega^b)\big)(x)\big|>0,\qquad x\in\overline{\Omega},\ l=1,\dots,r.$$

Finally, as $\overline{\Omega}$ is compact and $\zeta_l\big((E_\omega^1,H_\omega^1),\dots,(E_\omega^b,H_\omega^b)\big)$ are continuous maps, we obtain that $K\times\{\varphi_1,\dots,\varphi_b\}$ is a ζ-complete set of measurements for some $s>0$. □

4. Applications to hybrid problems

In this section we analyze the hybrid problems introduced in Subsection 2.2 in more detail. The inverse problem for electro-seismic conversion was fully studied in 2.2.3 and no further details will be given here.

4.1. Reconstruction of ε and σ from knowledge of internal magnetic fields - first method.

Let us recall the problem discussed in 2.2.1. Combining boundary measurements with an MRI scanner, we are able to measure the magnetic fields. From the knowledge of H, the electromagnetic parameters have to be imaged. Consider problem (7)

$$\begin{cases}\mathrm{curl}E_\omega^i=\mathbf{i}\omega H_\omega^i & \text{in }\Omega,\\ \mathrm{curl}H_\omega^i=-\mathbf{i}q_\omega E_\omega^i & \text{in }\Omega,\\ E_\omega^i\times\nu=\varphi_i\times\nu & \text{on }\partial\Omega.\end{cases}\tag{26}$$

Recall that $q_\omega=\omega\varepsilon+\mathbf{i}\sigma$. We assume $\mu=1$ and $\varepsilon,\sigma>0$, namely we study the isotropic case. Given a set of measurements $K\times\{\varphi_1,\dots,\varphi_b\}$, we measure H_ω^i in Ω and want to reconstruct ε and σ.

Two interesting issues of practical importance are not considered in this work. First, the case where one or two components of the magnetic fields are measured. In such a case, the rotation of the MRI scanner is avoided. The reader is referred to [30], where a low frequency approximation is considered. Second, it is possible to consider anisotropic coefficients, which in some cases are a better model for human tissues [25].

We now describe the first method to reconstruct σ and ε. Let $K\times\{\varphi_1,\varphi_2,\varphi_3\}$ be a $\zeta^{(1)}$-complete set of measurement (see Proposition 11). Namely, for every $x\in\overline{\Omega}$ there exists $\omega(x)\in K$ such that

$$\big|\det\big[\ E^1_{\omega(x)}\quad E^2_{\omega(x)}\quad E^3_{\omega(x)}\ \big](x)\big|\ge s',$$

for some $s'>0$ independent of x. Thus, (26) implies

$$\begin{aligned}&\big|\det\big[\ \mathrm{curl}H^1_{\omega(x)}\quad \mathrm{curl}H^2_{\omega(x)}\quad \mathrm{curl}H^3_{\omega(x)}\ \big](x)\big|\\ &\qquad=\big|q^3_{\omega(x)}\det\big[\ E^1_{\omega(x)}\quad E^2_{\omega(x)}\quad E^3_{\omega(x)}\ \big](x)\big|\ge s,\end{aligned}\tag{27}$$

for some $s>0$ independent of x. This inequality will be necessary in the following.

We now proceed to eliminate the unknown electric field from system (26), in order to obtain an equation with only ε and σ as unknowns and the magnetic field as a known datum. An immediate calculation shows that for any $\omega\in K$ and $i=1,2,3$ there holds $\mathrm{curl}(\mathbf{i}q_\omega^{-1}\mathrm{curl}H_\omega^i)=\mathbf{i}\omega H_\omega^i$ in Ω, whence

$$\nabla q_\omega\times\mathrm{curl}H_\omega^i=q_\omega\mathrm{curl}\mathrm{curl}H_\omega^i-q_\omega^2\omega H_\omega^i=-q_\omega\Delta H_\omega^i-q_\omega^2\omega H_\omega^i\qquad\text{in }\Omega,$$

where the last identity is a consequence of the fact that H_ω^i is divergence free, since $\mu = 1$. Taking now scalar product with $\mathbf{e}_j$ for $j = 1, 2$ we have

$$\nabla q_\omega \cdot \left(\mathrm{curl} H_\omega^i \times \mathbf{e}_j\right) = -q_\omega \Delta (H_\omega^i)_j - q_\omega^2 \omega (H_\omega^i)_j \qquad \text{in } \Omega.$$

We can now write these 6 equations in a more compact form. By introducing the 3×6 matrix

$$M_\omega^{(1)} = \begin{bmatrix} \mathrm{curl} H_\omega^1 \times \mathbf{e}_1 & \mathrm{curl} H_\omega^1 \times \mathbf{e}_2 & \cdots & \mathrm{curl} H_\omega^3 \times \mathbf{e}_1 & \mathrm{curl} H_\omega^3 \times \mathbf{e}_2 \end{bmatrix}$$

and the six-dimensional horizontal vector

$$v_\omega = \left((H_\omega^1)_1, (H_\omega^1)_2, \ldots, (H_\omega^3)_1, (H_\omega^3)_2\right)$$

we obtain

$$\nabla q_\omega M_\omega^{(1)} = -q_\omega \Delta v_\omega - q_\omega^2 \omega v_\omega \qquad \text{in } \Omega. \tag{28}$$

We now want to right invert the matrix $M_\omega^{(1)}$ to obtain a well-posed first order PDE for q_ω. The following lemma gives a sufficient condition for the matrix $M_\omega^{(1)}$ to admit a right inverse.

Lemma 20. *Let $G_1, G_2, G_3 \in \mathbb{C}^3$ be linearly independent. Then the 3×6 matrix*

$$\begin{bmatrix} G_1 \times \mathbf{e}_1 & G_1 \times \mathbf{e}_2 & \cdots & G_3 \times \mathbf{e}_1 & G_3 \times \mathbf{e}_2 \end{bmatrix}$$

has rank three.

Proof. Take $u \in \mathbb{C}^3$ such that $G_i \times \mathbf{e}_j \cdot u = 0$ for every $i = 1, 2, 3$ and $j = 1, 2$. We need to prove that $u = 0$. Since $\mathbf{e}_j \times u \cdot G_i = 0$ for all i and j and $\{G_i : i = 1, 2, 3\}$ is a basis of $\mathbb{C}^3$ by assumption, we obtain $\mathbf{e}_j \times u = 0$ for $j = 1, 2$, namely $u = 0$. □

Define now for any $\omega \in K$ the set

$$\Omega_\omega = \{x \in \overline{\Omega} : \left|\det \left[\mathrm{curl} H_\omega^1 \quad \mathrm{curl} H_\omega^2 \quad \mathrm{curl} H_\omega^3\right](x)\right| > \frac{s}{2}\}.$$

Since $\frac{s}{2} < s$, in view of (27) we obtain the cover

$$\overline{\Omega} = \bigcup_{\omega \in K} \Omega_\omega.$$

As the sets Ω_ω are relatively open in $\overline{\Omega}$, they must overlap, and this will be exploited below in the reconstruction. For any $\omega \in K$ and $x \in \Omega_\omega$, in view of Lemma 20 the matrix $M_\omega^{(1)}(x)$ admits a right inverse, which with an abuse of notation we denote by $(M_\omega^{(1)})^{-1}(x)$. Therefore, problem (28) becomes

$$\nabla q_\omega = -q_\omega \Delta v_\omega (M_\omega^{(1)})^{-1} - q_\omega^2 \omega v_\omega (M_\omega^{(1)})^{-1} \qquad \text{in } \Omega_\omega. \tag{29}$$

It is now possible to integrate this PDE and reconstruct ε and σ in every $x \in \Omega$ if these are known for one value $x_0 \in \overline{\Omega}$. It is the nature of the multi-frequency approach that the relevant conditions are satisfied only locally for a fixed value of the frequency $\omega \in K$. In other words, (29) is not satisfied everywhere but only in Ω_ω. As a consequence, it is not possible to reconstruct ε and σ in x after one simple integration of (29). The process is more involved, and is similar to the algorithm described in [14].

Suppose now that $q_\omega(x_0)$ is known for some $x_0 \in \overline{\Omega}$ and take $x \in \overline{\Omega}$. Let Ω_ω^j for $j \in J_\omega$ be the connected components of Ω_ω, for a suitable set J_ω. Since $\overline{\Omega}$ is compact, from the cover $\overline{\Omega} = \bigcup_{\omega\in K}\bigcup_{j\in J_\omega}\Omega_\omega^j$, we can extract a finite subcover

$$\overline{\Omega} = \bigcup_{\omega\in K}\bigcup_{j\in J'_\omega}\Omega_\omega^j,$$

where $J'_\omega \subseteq J_\omega$ is finite. Hence, as $\overline{\Omega}$ is connected and Ω_ω^j are relatively open in $\overline{\Omega}$ and connected, we can find a smooth path $\gamma\colon [0,1] \to \overline{\Omega}$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and

$$\gamma([0,1]) = \bigcup_{m=0}^{M-1}\gamma([t_m, t_{m+1}]),$$

for some $M \in \mathbb{N}^*$, where $t_0 = 0$, $t_M = 1$ and $\gamma([t_m, t_{m+1}]) \subseteq \Omega_{\omega_m}^{j_m}$ for some $\omega_m \in K$ and $j_m \in J'_{\omega_m}$. Starting with $m = 0$, we integrate (29) with $\omega = \omega_m$ along $\gamma([t_m, t_{m+1}])$ and obtain q_{ω_m} in $\gamma(t_{m+1})$. Thus, we can reconstruct σ and ε in $\gamma(t_{m+1})$ and so $q_{\omega_{m+1}}(\gamma(t_{m+1}))$. Repeating this process $M-1$ times we obtain $\varepsilon(x)$ and $\sigma(x)$, as desired.

4.2. **Reconstruction of ε and σ from knowledge of internal magnetic fields - second method.** We consider here the problem of Subsection 4.1 and give details of the reconstruction algorithm summarized in 2.2.2.

Let $K\times\{\varphi_1,\dots,\varphi_6\}$ be a $\zeta^{(2)}$-complete set of measurement (see Proposition 12). Namely, for every $x \in \overline{\Omega}$ there exists $\omega(x) \in K$ such that

$$\left|\det\left[\ \eta(E^1_{\omega(x)}, E^2_{\omega(x)}) \quad \eta(E^3_{\omega(x)}, E^4_{\omega(x)}) \quad \eta(E^5_{\omega(x)}, E^6_{\omega(x)})\ \right](x)\right| \geq s, \tag{30}$$

for some $s > 0$ independent of x, where η is given by (12).

In view of (26), for $\omega \in K$ there holds

$$\mathrm{curlcurl}E^1_\omega \cdot E^2_\omega - \mathrm{curlcurl}E^2_\omega \cdot E^1_\omega = 0 \qquad \text{in } \Omega.$$

An easy calculation shows that substituting $E^i_\omega = \mathbf{i}q_\omega^{-1}\mathrm{curl}H^i_\omega$ in this identity we obtain

$$\nabla q_\omega \cdot \eta(\mathrm{curl}H^1_\omega, \mathrm{curl}H^2_\omega) = q_\omega\,\gamma(\mathrm{curl}H^1_\omega, \mathrm{curl}H^2_\omega) \qquad \text{in } \Omega, \tag{31}$$

where $\gamma\colon C^1(\overline{\Omega};\mathbb{C}^3)^2 \to C(\overline{\Omega};\mathbb{C})$ is defined by

$$\gamma(u_1,u_2) = (\nabla\mathrm{div}u_1)\cdot u_2 - (\nabla\mathrm{div}u_2)\cdot u_1 - \Delta u_1\cdot u_2 + \Delta u_2\cdot u_1.$$

Repeating the same argument with the other illuminations and combining the resulting equations we have

$$\nabla q_\omega M^{(2)}_\omega = q_\omega(\gamma(\mathrm{curl}H^1_\omega,\mathrm{curl}H^2_\omega),\gamma(\mathrm{curl}H^3_\omega,\mathrm{curl}H^4_\omega),\gamma(\mathrm{curl}H^5_\omega,\mathrm{curl}H^6_\omega)) \qquad \text{in } \Omega, \tag{32}$$

where $M^{(2)}_\omega$ is the 3×3 matrix-valued function given by

$$M^{(2)}_\omega = \left[\ \eta(\mathrm{curl}H^1_\omega,\mathrm{curl}H^2_\omega) \quad \eta(\mathrm{curl}H^3_\omega,\mathrm{curl}H^4_\omega) \quad \eta(\mathrm{curl}H^5_\omega,\mathrm{curl}H^6_\omega)\ \right]$$

By definition of η and since $\mathrm{curl}H^i_\omega = -\mathbf{i}q_\omega E^i_\omega$ we have $\eta(\mathrm{curl}H^i_\omega,\mathrm{curl}H^j_\omega) = -q_\omega^2\eta(E^i_\omega,E^j_\omega)$, whence

$$M^{(2)}_\omega = -q_\omega^2\left[\ \eta(E^1_\omega,E^2_\omega) \quad \eta(E^3_\omega,E^4_\omega) \quad \eta(E^5_\omega,E^6_\omega)\ \right].$$

Therefore, inverting $M^{(2)}_\omega$ in (32) we obtain

$$\nabla q_\omega = q_\omega(\gamma(\mathrm{curl}H_\omega^1, \mathrm{curl}H_\omega^2), \dots, \gamma(\mathrm{curl}H_\omega^5, \mathrm{curl}H_\omega^6))(M_\omega^{(2)})^{-1} \qquad \text{in } \Omega_\omega,$$

where

$$\Omega_\omega = \{x \in \overline{\Omega} : \left|\det \begin{bmatrix} \eta(E_\omega^1, E_\omega^2) & \eta(E_\omega^3, E_\omega^4) & \eta(E_\omega^5, E_\omega^6) \end{bmatrix}(x)\right| > \frac{s}{2}\}.$$

In view of (30) we have $\overline{\Omega} = \bigcup_{\omega\in K} \Omega_\omega$, and so q_ω can be reconstructed everywhere in Ω following the algorithm discussed in the previous subsection, provided that q_ω is known at one point in $\overline{\Omega}$.

5. Maxwell's equations

In this section we prove the results stated in Subsection 3.2. Propositions 15 and 16 are standard results, and their proofs will be detailed for completeness. On the other hand, Proposition 17 is less standard and requires the careful analysis of Maxwell's equations given in this section.

Let us recall problem (1)

$$\begin{cases} \mathrm{curl}E_\omega = \mathbf{i}\omega\mu H_\omega & \text{in } \Omega, & (33a)\\ \mathrm{curl}H_\omega = -\mathbf{i}q_\omega E_\omega & \text{in } \Omega, & (33b)\\ E_\omega \times \nu = \varphi \times \nu & \text{on } \partial\Omega. & (33c) \end{cases}$$

We now justify the introduction of the additional constraint $H_\omega \in H^\mu(\mathrm{curl}, \Omega)$ in order to make (33) well-posed in the case $\omega = 0$. For any $p \in H^1(\Omega;\mathbb{C})$ the quantity $(0, \nabla p)$ is a solution to the homogeneous equation. From (33a) it follows that $\omega\mathrm{div}(\mu H_\omega) = 0$ in Ω. Thus, it is natural to impose in general that

$$\mathrm{div}(\mu H_\omega) = 0 \qquad \text{in } \Omega,$$

which is meaningful also when $\omega = 0$. Taking in the above example $p \in H^1(\Omega;\mathbb{C}) \setminus \{0\}$ such that $\mathrm{div}(\mu\nabla p) = 0$ in Ω, we have that $(0, \nabla p)$ is still a solution to the homogeneous equation. This shows that the above condition is not sufficient to guarantee well-posedness. In this example, we would need a boundary condition on p. In view of [28, (3.52)] we have

$$\mathrm{curl}w \cdot \nu = \mathrm{div}_{\partial\Omega}(w \times \nu), \qquad w \in H(\mathrm{curl}, \Omega), \tag{34}$$

whence from (33a) we obtain

$$\mathbf{i}\omega\mu H_\omega \cdot \nu = \mathrm{curl}E_\omega \cdot \nu = \mathrm{div}_{\partial\Omega}(E_\omega \times \nu) = \mathrm{div}_{\partial\Omega}(\varphi \times \nu) = \mathrm{curl}\varphi \cdot \nu \qquad \text{on } \partial\Omega.$$

This suggests to assume (6) and impose $\mu H_\omega \cdot \nu = 0$ on $\partial\Omega$. This argument justifies assumption (6) on φ and the constraint $H_\omega \in H^\mu(\mathrm{curl}, \Omega)$.

To simplify our study of (33), we first do a lifting of the boundary condition φ. Namely, write

$$E_\omega = \tilde{E}_\omega + \varphi, \tag{35}$$

where $\tilde{E}_\omega \in H_0(\mathrm{curl}, \Omega)$, and obtain

$$\begin{cases} \mathrm{curl}\tilde{E}_\omega = \mathbf{i}\omega\mu H_\omega - \mathrm{curl}\varphi & \text{in } \Omega,\\ \mathrm{curl}H_\omega = -\mathbf{i}q_\omega \tilde{E}_\omega - \mathbf{i}q_\omega\varphi & \text{in } \Omega. \end{cases} \tag{36}$$

5.1. **Well-posedness.** This subsection is devoted to the proof of Proposition 15. As far as the case $\omega \neq 0$ is concerned, we follow [31]. The case $\omega = 0$ is still standard, but requires additional care.

Introduce the space

$$X = L^2(\Omega;\mathbb{C}^3) \times \{v \in L^2(\Omega;\mathbb{C}^3) : \mathrm{div}(\mu v) = 0 \text{ in } \Omega,\ \mu v \cdot \nu = 0 \text{ on } \partial\Omega\},$$

equipped with the norm $\|(u,v)\|_X^2 = \|u\|^2_{L^2(\Omega;\mathbb{C}^3)} + \|v\|^2_{L^2(\Omega;\mathbb{C}^3)}$. Consider its subspace $\mathcal{D}(T) = H_0(\mathrm{curl},\Omega) \times H^\mu(\mathrm{curl},\Omega)$ and the operator

$$T\colon \mathcal{D}(T) \longrightarrow X, \qquad T(u,v) = \mathbf{i}\left(\varepsilon^{-1}(\mathrm{curl} v - \sigma u), -\mu^{-1}\mathrm{curl} u\right).$$

In view of (34) we have $\mathrm{curl} u \cdot \nu = 0$ for all $u \in H_0(\mathrm{curl},\Omega)$. Therefore $T(u,v) \in X$ for all $(u,v) \in \mathcal{D}(T)$, and so T is well-defined.

The following lemma states that (36) can be recast as a Fredholm-type equation involving the operator T.

Lemma 21. *Assume that* (4) *and* (6) *hold and take* $\omega \in \mathbb{C}$ *and* $(\tilde{E}_\omega, H_\omega) \in \mathcal{D}(T)$. *Then* $(\tilde{E}_\omega, H_\omega)$ *is solution to* (36) *if and only if*

$$(T - \omega)(\tilde{E}_\omega, H_\omega) = (\varepsilon^{-1} q_\omega \varphi, \mathbf{i}\mu^{-1}\mathrm{curl}\varphi). \tag{37}$$

Proof. Showing this equivalence is just a matter of writing down the relevant identities, and the details are left to the reader. □

The previous lemma states the equivalence between our original problem (36) and the Fredholm-type equation (37). The first natural step towards the study of the latter is the characterization of the spectrum of T, which we will denote by $\Sigma = \sigma(T)$.

The spectrum of an extension $\tilde{T}$ of T was studied in [31]. Consider the space $\tilde{X} = L^2(\Omega;\mathbb{C}^3) \times L^2(\Omega;\mathbb{C}^3)$ equipped with the norm $\|(u,v)\|^2_{\tilde{X}} = \|u\|^2_{L^2(\Omega;\mathbb{C}^3)} + \|v\|^2_{L^2(\Omega;\mathbb{C}^3)}$, its subspace $\mathcal{D}(\tilde{T}) = H_0(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)$ and the operator

$$\tilde{T}\colon \mathcal{D}(\tilde{T}) \longrightarrow \tilde{X}, \qquad \tilde{T}(u,v) = \mathbf{i}\left(\varepsilon^{-1}(\mathrm{curl} v - \sigma u), -\mu^{-1}\mathrm{curl} u\right).$$

Lemma 22 ([31, Proposition 3.1])**.** *Assume that* (4) *holds. The spectrum of* $\tilde{T}$ *is discrete.*

As it has already been pointed out, $\tilde{T}(0,\nabla p) = 0$ for every $p \in H^1(\Omega;\mathbb{C})$, namely $\tilde{T}$ is not injective. Therefore $0 \in \sigma(\tilde{T})$. The restriction we set in this work to the domain and the codomain of the operator T are motivated by the need of studying (36), whence (37), also in the case $\omega = 0$. Thus, we shall now prove that $0 \notin \Sigma$.

Lemma 23. *Assume that* (4) *holds. The operator* T *is invertible and* $T^{-1}\colon X \to \mathcal{D}(T)$ *is continuous, namely*

$$\left\|(u,v)\right\|_X \le C\,\|T(u,v)\|_X\,, \qquad (u,v) \in X,$$

for some $C > 0$ *depending on* Ω, λ *and* Λ *only.*

Proof. Let $(F,G) \in X$. We need to show that there exists a unique $(u,v) \in \mathcal{D}(T)$ such that $T(u,v) = (F,G)$ and that $\left\|(u,v)\right\|_X \le C\,\|(F,G)\|_X$, for some $C > 0$ depending on Ω, λ and Λ only. In the following we shall denote different such constants with the same letter c.

Let us rewrite $T(u,v)=(F,G)$ as

$$\begin{cases} \sigma u - \mathrm{curl} v = \mathbf{i}\varepsilon F & \text{in } \Omega, \\ \mathrm{curl} u = \mathbf{i}\mu G & \text{in } \Omega. \end{cases} \tag{38}$$

In view of the Helmholtz decomposition [24, Chapter I, Corollary 3.4], we can write $u = \nabla p + \mathrm{curl}\Phi$ for some $p \in H^1(\Omega;\mathbb{C})$ and $\Phi \in H^1(\Omega;\mathbb{C}^3)$ such that $\mathrm{div}\,\Phi = 0$ in Ω and $\Phi \times \nu = 0$ on $\partial\Omega$. Since $\mathrm{curl}(\mathrm{curl}\Phi) = \nabla(\mathrm{div}\Phi) - \Delta\Phi = -\Delta\Phi$, the second equation of (38) yields

$$\begin{cases} -\Delta\Phi = \mathbf{i}\mu G & \text{in } \Omega, \\ \mathrm{div}\,\Phi = 0 & \text{in } \Omega, \\ \Phi \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus Φ is uniquely determined by G and in view of [24, Chapter I, Theorem 3.8] there holds

$$\|\mathrm{curl}\Phi\|_{H^1(\Omega;\mathbb{C}^3)} \le c\big(\|\mathrm{curlcurl}\Phi\|_{L^2(\Omega;\mathbb{C}^3)} + \|\mathrm{divcurl}\Phi\|_{L^2(\Omega;\mathbb{C})}\big) \le c\,\|G\|_{L^2(\Omega;\mathbb{C}^3)}\,. \tag{39}$$

We now want to find suitable boundary conditions satisfied by p. We denote the surface gradient by $\nabla_{\partial\Omega}$, the surface divergence by $\mathrm{div}_{\partial\Omega}$ and the surface scalar curl by $\mathrm{curl}_{\partial\Omega}$ [28, Section 3.4]. By (34) and [28, (3.15)] we have

$$0 = \mathbf{i}\mu G\cdot\nu = \mathrm{curlcurl}\Phi\cdot\nu = \mathrm{div}_{\partial\Omega}(\mathrm{curl}\Phi\times\nu) = \mathrm{curl}_{\partial\Omega}\mathrm{curl}\Phi \qquad \text{on } \partial\Omega.$$

(Note that $\mathrm{curl}\Phi\cdot\nu = \mathrm{div}_{\partial\Omega}(\Phi\times\nu) = 0$ on $\partial\Omega$, so that $\mathrm{curl}\Phi$ is a tangential vector field, and so we can apply to it the surface scalar curl.) As a result, since $\partial\Omega$ is simply connected, there exists a unique $r \in H^1(\partial\Omega;\mathbb{C})$ such that $\mathrm{curl}\Phi = -\nabla_{\partial\Omega} r$ on $\partial\Omega$ and $\int_{\partial\Omega} r\,ds = 0$. Poincaré inequality gives

$$\|r\|_{H^1(\partial\Omega;\mathbb{C})} \le c\,\|\nabla_{\partial\Omega} r\|_{L^2_t(\partial\Omega,\mathbb{C}^3)} = c\,\|\mathrm{curl}\Phi\|_{L^2_t(\partial\Omega,\mathbb{C}^3)}\,, \tag{40}$$

where $L^2_t(\partial\Omega,\mathbb{C}^3)$ denotes the space of tangential vector fields in $L^2(\partial\Omega,\mathbb{C}^3)$. As $u\times\nu = 0$ on $\partial\Omega$ we have $\nabla_{\partial\Omega} r\times\nu = -\mathrm{curl}\Phi\times\nu = \nabla p\times\nu = \nabla_{\partial\Omega} p\times\nu$ on $\partial\Omega$, whence $\nabla_{\partial\Omega} p = \nabla_{\partial\Omega} r$ on $\partial\Omega$. Since p is defined up to a constant, we can set $p = r$ on $\partial\Omega$. Thus, in view of the first equation of (38), we must look for a solution to

$$\begin{cases} -\mathrm{div}(\sigma\nabla p) = \mathrm{div}(\sigma\mathrm{curl}\Phi - \mathbf{i}\varepsilon F) & \text{in } \Omega, \\ p = r & \text{on } \partial\Omega. \end{cases}$$

Therefore, p is uniquely determined by Φ, F and r and the following estimate holds

$$\begin{aligned} \|p\|_{H^1(\Omega;\mathbb{C})} &\le c\left(\|\mathrm{curl}\Phi\|_{L^2(\Omega;\mathbb{C}^3)} + \|F\|_{L^2(\Omega;\mathbb{C}^3)} + \|r\|_{H^1(\partial\Omega;\mathbb{C})}\right) \\ &\le c\left(\|\mathrm{curl}\Phi\|_{H^1(\Omega;\mathbb{C}^3)} + \|F\|_{L^2(\Omega;\mathbb{C}^3)}\right) \\ &\le c\left(\|G\|_{L^2(\Omega;\mathbb{C}^3)} + \|F\|_{L^2(\Omega;\mathbb{C}^3)}\right), \end{aligned} \tag{41}$$

where the second inequality is a consequence of (40) and the third one follows from (39). We have proven that u is uniquely determined by F and G and combining (39) and (41) gives the estimate

$$\|u\|_{L^2(\Omega;\mathbb{C}^3)} \le c\,\|(F,G)\|_X\,.$$

The well-posedness for v follows from [24, Chapter I, Theorem 3.5] applied to the first equation of (38). Indeed, $\mathrm{div}(\sigma u - \mathbf{i}\varepsilon F) = 0$ in Ω and so there exists a unique $v \in H(\mathrm{curl}, \Omega)$ such that

$$\begin{cases} \mathrm{curl} v = \sigma u - \mathbf{i}\varepsilon F & \text{in } \Omega, \\ \mathrm{div}(\mu v) = 0 & \text{in } \Omega, \\ \mu v \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

and the norm estimate $\|v\|_{L^2(\Omega;\mathbb{C}^3)} \leq c(\|u\|_{L^2(\Omega;\mathbb{C}^3)} + \|F\|_{L^2(\Omega;\mathbb{C}^3)})$ holds.

Finally, we have proven that there exists a unique $(u, v) \in \mathcal{D}(T)$ such that (38) holds true. Moreover, thanks to the estimates on the norms of u and v, T^{-1} is continuous. □

Combining Lemmata 22 and 23 and using the fact that $\Sigma \subseteq \sigma(\tilde{T})$ we obtain the following characterization of the spectrum of T.

Proposition 24. *Assume that* (4) *holds. The spectrum* Σ *of* T *is discrete. Moreover,* $0 \notin \Sigma$*. In particular, for all* $w \in \mathbb{C} \setminus \Sigma$ *and* $(F, G) \in X$ *the equation*

$$(T - \omega)(u, v) = (F, G) \tag{42}$$

has a unique solution $(u, v) \in \mathcal{D}(T)$ *such that*

$$\|(u, v)\|_X \leq C \|(F, G)\|_X,$$

for some $C > 0$ *independent of* F *and* G.

We are now in a position to prove Proposition 15.

Proof of Proposition 15. For $\omega \in \mathbb{C}\setminus\Sigma$, set $F = \varepsilon^{-1} q_\omega \varphi$ and $G = \mathbf{i}\mu^{-1}\mathrm{curl}\varphi$ in (42). In view of Proposition 24, equation (37) has a unique solution $(\tilde{E}_\omega, H_\omega) \in \mathcal{D}(T)$ satisfying the norm estimate

$$\|(\tilde{E}_\omega, H_\omega)\|_X \leq C \|\varphi\|_{H(\mathrm{curl},\Omega)},$$

for some C independent of φ. After setting $E_\omega = \tilde{E}_\omega + \varphi$ as in (35), Lemma 21 implies that (E_ω, H_ω) is the unique solution to (21). Moreover, the relevant norm estimate holds. This completes the proof. □

5.2. **Regularity properties.** The focus of this subsection is the proof of Proposition 16. First, we study the regularity of the solutions to (42) by applying the regularity theory for Maxwell's equations discussed in [4].

Proposition 25. *Assume that* (4) *and* (5) *hold. Take* $\omega \in \mathbb{C} \setminus \Sigma$ *and* $(F, G) \in X$ *such that* $F \in W^{\kappa,p}(\Omega;\mathbb{C}^3)$ *and* $G \in W^{\kappa-1,p}(\Omega;\mathbb{C}^3)$*. Let* $(u, v) \in \mathcal{D}(T)$ *be a solution to* (42)*. Then* $u, v \in W^{\kappa,p}(\Omega;\mathbb{C}^3)$ *and*

$$\|(u, v)\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2} \leq C \left(\|(u, v)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|F\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} + \|G\|_{W^{\kappa-1,p}(\Omega;\mathbb{C}^3)} \right), \tag{43}$$

for some $C > 0$ *depending on* Ω, ω, λ, Λ, κ, p *and* $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})^3}$ *only. Moreover, if* $\omega = 0$ *we have*

$$\|(u, v)\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2} \leq C' \left(\|F\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} + \|G\|_{W^{\kappa-1,p}(\Omega;\mathbb{C}^3)} \right),$$

for some $C' > 0$ *depending on* Ω, λ, Λ, κ, p *and* $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})^3}$ *only.*

Proof. By definition of T, (42) can be rewritten as

$$\begin{cases} -q_\omega u + \mathbf{i}\mathrm{curl} v = \varepsilon F & \text{in } \Omega, \\ -\mathbf{i}\mu^{-1}\mathrm{curl} u - \omega v = G & \text{in } \Omega. \end{cases}$$

Substituting $v \to -v$, this problem can be recast as

$$\begin{cases} \mathrm{curl} v = \mathbf{i}\omega'\varepsilon' u + J_e & \text{in } \Omega, \\ \mathrm{curl} u = -\mathbf{i}\omega'\mu' v + J_m & \text{in } \Omega, \end{cases}$$

where $\omega' = 1$, $\varepsilon' = \omega\varepsilon + \mathbf{i}\sigma$, $J_e = \mathbf{i}\varepsilon F$, $\mu' = \omega\mu$ and $J_m = \mathbf{i}\mu G$. This system's form was considered in [4]. In view of [4, Proposition 5] we obtain that (u_l, v_l) satisfies for any $l = 1, 2, 3$ and all $g \in H^2(\Omega; \mathbb{C})$

$$\int_\Omega u_l \mathrm{div}({}^t\varepsilon' \nabla \bar g)\, dx = \int_{\partial\Omega} (\partial_l \bar g)\varepsilon' u \cdot \nu \, ds + \int_\Omega (\partial_l \varepsilon' u - \varepsilon'(\mathbf{e}_l \times (\mathbf{i}\mu G - \mathbf{i}\omega\mu v)) - \mathbf{i}\mathbf{e}_l \mathrm{div}(\mathbf{i}\varepsilon F)) \cdot \nabla \bar g \, dx,$$

and

$$\int_\Omega v_l \mathrm{div}({}^t(\omega\mu)\nabla \bar g)\, dx = -\int_{\partial\Omega} (\mathbf{e}_l \times (v \times \nu)) \cdot ({}^t(\omega\mu)\nabla \bar g)\, ds + \int_\Omega (\omega(\partial_l \mu) v - \omega\mu(\mathbf{e}_l \times (i\varepsilon F + \mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)u))) \cdot \nabla \bar g \, dx.$$

Multiplying the first equation by $-\mathbf{i}$ we obtain

$$\int_\Omega u_l \mathrm{div}({}^t(\sigma - \mathbf{i}\omega\varepsilon)\nabla \bar g)\, dx = \int_{\partial\Omega} (\partial_l \bar g)(\sigma - \mathbf{i}\omega\varepsilon) u \cdot \nu \, ds + \int_\Omega (\partial_l(\sigma - \mathbf{i}\omega\varepsilon) u - (\sigma - \mathbf{i}\omega\varepsilon)(\mathbf{e}_l \times (\mathbf{i}\mu G - \mathbf{i}\omega\mu v)) - \mathbf{i}\mathbf{e}_l \mathrm{div}(\varepsilon F)) \cdot \nabla \bar g \, dx. \tag{44}$$

Similarly, we can multiply the second equation by ω^{-1} and obtain

$$\int_\Omega v_l \mathrm{div}({}^t\mu \nabla \bar g)\, dx = -\int_{\partial\Omega} (\mathbf{e}_l \times (v \times \nu)) \cdot ({}^t\mu \nabla \bar g)\, ds + \int_\Omega ((\partial_l \mu) v - \mu(\mathbf{e}_l \times (i\varepsilon F + \mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)u))) \cdot \nabla \bar g \, dx. \tag{45}$$

Note that this last operation is allowed if $\omega \neq 0$. If $\omega = 0$, we can argue as follows. Take a sequence $\omega_n \in \mathbb{C} \setminus (\Sigma \cup \{0\})$ such that $\omega_n \to 0$. Since the resolvent operator $(T - \omega)^{-1}$ is analytic in ω, then the map $\omega \in \mathbb{C} \setminus \Sigma \mapsto (T - \omega)^{-1}(F, G) \in X$ is continuous. Therefore, as (45) is satisfied for all ω_n, it has to hold also for $\omega = 0$.

A careful analysis shows that the proof of [4, Theorem 9] only relies on the system (44)-(45) and not on the original form of Maxwell's equations considered in [4] (see also [3]). Therefore, it is possible to apply this result to the previous system (44)-(45) and obtain that $u, v \in W^{\kappa,p}(\Omega; \mathbb{C}^3)$ and

$$\|(u, v)\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2} \leq c \left(\|(u, v)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|\varepsilon F\|_{W^{\kappa,p}(\mathrm{div},\Omega)} + \|\mu G\|_{W^{\kappa,p}(\mathrm{div},\Omega)} \right),$$

for some $c > 0$ depending on Ω, ω, λ, Λ, κ, p and $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})^3}$ only, where

$$W^{\kappa,p}(\mathrm{div}, \Omega) = \{u \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3) : \mathrm{div} u \in W^{\kappa-1,p}(\Omega; \mathbb{C})\}.$$

Estimate (43) follows from $\|\varepsilon F\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} \leq c\|F\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)}$, which is an easy consequence of the Sobolev Embedding Theorem.

Finally, the last estimate follows from (43) and Lemma 23. □

We are now ready to prove Proposition 16.

Proof of Proposition 16. We have already seen (see the proof of Proposition 15) that setting $E_\omega = \tilde{E}_\omega + \varphi$, $(\tilde{E}_\omega, H_\omega)$ is a solution to (42) with $F = \varepsilon^{-1} q_\omega \varphi$ and $G = \mathbf{i}\mu^{-1}\mathrm{curl}\varphi$. Thus, in view of Proposition 25, $\tilde{E}_\omega, H_\omega \in W^{\kappa,p}(\Omega;\mathbb{C}^3)$ and

$$\left\|(\tilde{E}_\omega, H_\omega)\right\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2} \le C\Big(\left\|(\tilde{E}_\omega, H_\omega)\right\|_{L^2(\Omega;\mathbb{C}^3)^2} + \left\|\varepsilon^{-1} q_\omega \varphi\right\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} + \left\|\mu^{-1}\mathrm{curl}\varphi\right\|_{W^{\kappa-1,p}(\Omega;\mathbb{C}^3)}\Big),$$

for some $C > 0$ depending on Ω, ω, λ, Λ, κ, p and $\|(\mu,\varepsilon,\sigma)\|_{W^{\kappa,p}(\Omega;\mathbb{R}^{3\times 3})^3}$ only. Combining this inequality with the estimate for $\left\|(E_\omega, H_\omega)\right\|_{H(\mathrm{curl},\Omega)^2}$ given in Proposition 15 we obtain the first part of the result, as $W^{\kappa,p}(\Omega;\mathbb{C}^3)^2$ is continuously embedded into $C^{\kappa-1}(\overline{\Omega};\mathbb{C}^6)$. Finally, the estimate on $\|(E_0,H_0)\|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2}$ follows from the second part of Proposition 25. □

5.3. **Analyticity properties.** We end this section by proving Proposition 17. This will be a consequence of the results discussed so far in this section and of the following lemma, which generalizes [1, Proposition 3.5].

Lemma 26. *Let Y be a Banach space. Take an operator $T\colon \mathcal{D}(T) \subseteq Y \to Y$ and denote its resolvent set by $\rho(T)$. Take $Y_1 \subseteq Y$ and $Y_2 \subseteq \mathcal{D}(T) \cap Y_1$ with norms $\|\ \|_{Y_1}$ and $\|\ \|_{Y_2}$, respectively. Assume that the inclusion $i\colon Y_2 \to Y_1$ is continuous and that for all $\omega \in \rho(T)$ the operator $(T-\omega)^{-1}\colon Y_1 \to Y_2$ is well-defined and bounded. Take $N \in Y_2$ and let $g\colon \rho(T) \to Y_1$ be such that $g(\omega) - g(\omega_0) = (\omega - \omega_0)N$ for all $\omega, \omega_o \in \rho(T)$. Then the map*

$$\omega \in \rho(T) \longmapsto (T-\omega)^{-1} g(\omega) \in Y_2$$

is analytic.

Proof. Denote the map $\omega \in \rho(T) \longmapsto (T-\omega)^{-1} g(\omega) \in Y_2$ by f. Take $\omega_0 \in \rho(T)$: we shall prove that f is analytic in ω_0. Denote the operator $(T-\omega_0)^{-1}\colon Y_1 \to Y_2$ by B. Let $r > 0$ be such that $B(\omega_0, r) \subseteq \rho(T)$, take $\omega \in B(\omega_0, r)$ and set $h = \omega - \omega_0$. Introduce the operator $C_h\colon Y_2 \to Y_1$ defined by $y \mapsto hy$, whose norm satisfies $\|C_h\| \le \|i\|\, r$. A straightforward calculation shows that

$$(T-\omega_0)(f(\omega) - f(\omega_0)) - h(f(\omega) - f(\omega_0)) = h(N + f(\omega_0)),$$

where the equality makes sense in Y_1. We can now apply the operator B to both sides of this equation and get

$$(I - BC_h)(f(\omega) - f(\omega_0)) = BC_h(N + f(\omega_0)).$$

If $r < (\|i\|\,\|B\|)^{-1}$ we have that $\|BC_h\| < 1$ and so

$$f(\omega) - f(\omega_0) = \sum_{n=1}^{\infty} (BC_h)^n (N + f(\omega_0)), \qquad \omega \in B(\omega_0, r),$$

whence the result. □

We are now in a position to prove Proposition 17.

Proof of Proposition 17. Since $E_\omega = \tilde{E}_\omega + \varphi$, it is enough to show the analyticity of the map $\omega \mapsto (\tilde{E}_\omega, H_\omega)$. We want to apply Lemma 26 with $Y = X$, $Y_1 = \{(F,G) \in X : F \in W^{\kappa,p}(\Omega;\mathbb{C}^3),\ G \in W^{\kappa-1,p}(\Omega;\mathbb{C}^3)\}$ equipped with the norm $\|\ \|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)} \times \|\ \|_{W^{\kappa-1,p}(\Omega;\mathbb{C}^3)}$, $Y_2 = \mathcal{D}(T) \cap W^{\kappa,p}(\Omega;\mathbb{C}^3)^2$ equipped with the norm $\|\ \|_{W^{\kappa,p}(\Omega;\mathbb{C}^3)^2}$, $N = (0,\varphi)$ and $g(\omega) = (\varepsilon^{-1} q_\omega \varphi, \mathbf{i}\mu^{-1}\mathrm{curl}\varphi)$. Let us now check that the assumptions of the lemma are verified. The continuity of the inclusion $Y_2 \subseteq Y_1$ is trivial. The continuity of $(T-\omega)^{-1} \colon Y_1 \to Y_2$ follows from Propositions 24 and 25. Finally, a direct computation shows that $g(\omega) - g(\omega_0) = (\omega - \omega_0)N$. Therefore the result follows by Lemma 26, as $W^{\kappa,p}(\Omega;\mathbb{C}^3)^2$ is continuously embedded into $C^{\kappa-1}(\overline{\Omega};\mathbb{C}^6)$. ◻

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References

[1] G. S. Alberti. On multiple frequency power density measurements. *Inverse Problems*, 29(11):115007, 2013. doi:10.1088/0266-5611/29/11/115007.

[2] G. S. Alberti. Enforcing local non-zero constraints in PDEs and applications to hybrid imaging problems. *ArXiv e-prints*, June 2014. arXiv:1406.3248.

[3] G. S. Alberti. *On local constraints and regularity of PDE in electromagnetics. Applications to hybrid imaging inverse problems.* PhD thesis, University of Oxford, 2014. URL: http://ora.ox.ac.uk/objects/uuid:1b30b3b7-29b1-410d-ae30-bd0a87c9720b.

[4] G. S. Alberti and Y. Capdeboscq. Elliptic Regularity Theory Applied to Time Harmonic Anisotropic Maxwell's Equations with Less than Lipschitz Complex Coefficients. *SIAM J. Math. Anal.*, 46(1):998–1016, 2014. doi:10.1137/130929539.

[5] G. S. Alberti and Y. Capdeboscq. A propos de certains problèmes inverses hybrides. In *Seminaire: Equations aux Dérivées Partielles. 2013–2014*, Sémin. Équ. Dériv. Partielles, page Exp. No. II. École Polytech., Palaiseau, 2014. URL: http://slsedp.cedram.org/cedram-bin/article/SLSEDP_2013-2014____A2_0.pdf.

[6] G. S. Alberti and Y. Capdeboscq. On local non-zero constraints in PDE with analytic coefficients. *ArXiv e-prints*, 2015. arXiv:1501.01449.

[7] G. Alessandrini. Global stability for a coupled physics inverse problem. *Inverse Problems*, 30(7):075008, 2014. doi:10.1088/0266-5611/30/7/075008.

[8] G. Alessandrini and R. Magnanini. Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions. *SIAM J. Math. Anal.*, 25(5):1259–1268, 1994. doi:10.1137/S0036141093249080.

[9] H. Ammari, Y. Capdeboscq, F. de Gournay, A. Rozanova-Pierrat, and F. Triki. Microwave imaging by elastic deformation. *SIAM J. Appl. Math.*, 71(6):2112–2130, 2011. doi:10.1137/110828241.

[10] H. Ammari, P. Garapon, H. Kang, and H. Lee. A method of biological tissues elasticity reconstruction using magnetic resonance elastography measurements. *Quart. Appl. Math.*, 66(1):139–175, 2008.

[11] H. Ammari, J. Garnier, W. Jing, and L. H. Nguyen. Quantitative thermo-acoustic imaging: An exact reconstruction formula. *J. Differential Equations*, 254(3):1375–1395, 2013. doi:10.1016/j.jde.2012.10.019.

[12] H. Ammari and H. Kang. Expansion methods. In *Handbook of Mathematical Methods in Imaging*, pages 447–499. Springer, New York, 2011.

[13] G. Bal. Hybrid inverse problems and internal functionals. In G. Uhlmann, editor, *Inverse Problems and Applications: Inside Out II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 325–368. Cambridge Univ. Press, Cambridge, 2013.

[14] G. Bal, E. Bonnetier, F. Monard, and F. Triki. Inverse diffusion from knowledge of power densities. *Inverse Probl. Imaging*, 7(2):353–375, 2013. doi:10.3934/ipi.2013.7.353.
[15] G. Bal and M. Courdurier. Boundary control of elliptic solutions to enforce local constraints. *J. Differential Equations*, 255(6):1357–1381, 2013. doi:10.1016/j.jde.2013.04.039.
[16] G. Bal, C. Guo, and F. Monard. Inverse anisotropic conductivity from internal current densities. *Inverse Problems*, 30(2):025001, 21, 2014. doi:10.1088/0266-5611.
[17] G. Bal, K. Ren, G. Uhlmann, and T. Zhou. Quantitative thermo-acoustics and related problems. *Inverse Problems*, 27(5):055007, 15, 2011. doi:10.1088/0266-5611/27/5/055007.
[18] G. Bal and G. Uhlmann. Inverse diffusion theory of photoacoustics. *Inverse Problems*, 26(8):085010, 20, 2010. doi:10.1088/0266-5611/26/8/085010.
[19] G. Bal and G. Uhlmann. Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions. *Communications on Pure and Applied Mathematics*, 66(10):1629–1652, 2013. doi:10.1002/cpa.21453.
[20] G. Bal and T. Zhou. Hybrid inverse problems for a system of Maxwell's equations. *Inverse Problems*, 30(5):055013, 2014. doi:10.1088/0266-5611/30/5/055013.
[21] Y. Capdeboscq, J. Fehrenbach, F. de Gournay, and O. Kavian. Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements. *SIAM J. Imaging Sci.*, 2(4):1003–1030, 2009. doi:10.1137/080723521.
[22] J. Chen and Y. Yang. Inverse problem of electro-seismic conversion. *Inverse Problems*, 29(11):115006, 2013.
[23] D. Colton and L. Päivärinta. The uniqueness of a solution to an inverse scattering problem for electromagnetic waves. *Archive for Rational Mechanics and Analysis*, 119(1):59–70, 1992. doi:10.1007/BF00376010.
[24] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.
[25] C. Guo and G. Bal. Reconstruction of complex-valued tensors in the maxwell system from knowledge of internal magnetic fields. *Inverse Problems and Imaging*, 8(4):1033–1051, 2014. doi:10.3934/ipi.2014.8.1033.
[26] O. Kavian. Lectures on parameter identification. In *Three courses on partial differential equations*, volume 4 of *IRMA Lect. Math. Theor. Phys.*, pages 125–162. de Gruyter, Berlin, 2003.
[27] P. Kuchment. Mathematics of hybrid imaging: A brief review. In I. Sabadini and D. C. Struppa, editors, *The Mathematical Legacy of Leon Ehrenpreis*, volume 16 of *Springer Proceedings in Mathematics*, pages 183–208. Springer Milan, 2012. doi:10.1007/978-88-470-1947-8_12.
[28] P. Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.
[29] J. K. Seo, D. Kim, J. Lee, O. I. Kwon, S. Z. K. Sajib, and E. J. Woo. Electrical tissue property imaging using MRI at dc and Larmor frequency. *Inverse Problems*, 28(8):084002, 26, 2012. doi:10.1088/0266-5611/28/8/084002.
[30] J. K. Seo, J. Yoon, E. J. Woo, and O. Kwon. Reconstruction of conductivity and current density images using only one component of magnetic field measurements. *Biomedical Engineering, IEEE Transactions on*, 50(9):1121–1124, 2003. doi:10.1109/TBME.2003.816080.
[31] E. Somersalo, D. Isaacson, and M. Cheney. A linearized inverse boundary value problem for Maxwell's equations. *J. Comput. Appl. Math.*, 42(1):123–136, 1992. doi:10.1016/0377-0427(92)90167-V.
[32] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987. doi:10.2307/1971291.
[33] A. E. Taylor. Analytic functions in general analysis. *Ann. Sc. Norm. Super. Pisa, II. Ser.*, 6:277–292, 1937.
[34] F. Triki. Uniqueness and stability for the inverse medium problem with internal data. *Inverse Problems*, 26(9):095014, 11, 2010. doi:10.1088/0266-5611/26/9/095014.
[35] E. F. Whittlesey. Analytic functions in Banach spaces. *Proc. Amer. Math. Soc.*, 16:1077–1083, 1965.
[36] T. Widlak and O. Scherzer. Hybrid tomography for conductivity imaging. *Inverse Problems*, 28(8):084008, 28, 2012. doi:10.1088/0266-5611/28/8/084008.

[37] Z. Zhu, M. W. Haartsen, and M. N. Toksöz. Experimental studies of seismoelectric conversions in fluid-saturated porous media. *Journal of Geophysical Research: Solid Earth*, 105(B12):28055–28064, 2000. doi:10.1029/2000JB900341.

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