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# Nested sequences of stars and starshaped sets

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In this paper we consider nested (decreasing and increasing) sequences of stars or starshaped sets in Banach spaces. The intersection, if decreasing, and the closure of the union, if increasing, are studied with regards to the preservation of these properties. Among other results we show that the closure of an increasing sequence of stars is a star if and only if the sequence of their centers is weakly convergent. Similar results, for starshaped sets, are true exactly in reflexive spaces.


## 1. Introduction

Let $X$ be a real Banach space. We will denote by $B(X)$ its closed unit ball. Here $c_0$, $l_1$ and $c$ stand for the classical sequence spaces with their natural norm and $\{e_n\}$ stands for their standard basis. The closure, the convex hull and the closed convex hull of $A$ will be denoted, respectively, by $\overline{A}$, $co(A)$, and $\overline{co}(A)$. A nonempty set $S \subset X$ is said to be *balanced* if $tS \subset S$ for every $t \in [-1, 1]$. A nonempty, closed and bounded set $S \subset X$ is said to be a *star* if there exists a $c \in S$, called *center* of $S$, such that $S - c$ is balanced. A nonempty, closed and bounded set $C \subset X$ is called *starshaped*, and $c$ is a center, if $C = \bigcup_{x \in C}[c, x]$ where $[c, x]$ denotes the closed segment (whereas $(c, x)$ is the open segment). For a starshaped set $C$, its *kernel*, that we shall denote by $ker(C)$, is the set of centers of $C$. Few basic properties concerning the sets we are considering are straightforward: stars sets are starshaped and the kernel of a starshaped set is always convex. It is also simple to see that the center of a bounded star is unique. Contrary to convex sets, closed stars and starshaped sets are not, in general, weakly closed: consider, for example in the Banach space $l_1$, the set $S = \bigcup_n [-e_n, e_n]$.

Properties of decreasing sequences of sets have been considered in many cases: for example in every Banach space, every decreasing sequence of balls has nonempty intersection. Another well known property

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involving sequences of convex sets is the following: a convex, closed set $C$ is weakly compact if and only if every decreasing sequence of closed convex sets in $C$ has nonempty intersection.

In this paper we consider "monotone" (increasing or decreasing) sequences of stars, or starshaped sets. We completely characterize what happens for their union or intersection: namely, we indicate in which spaces the union or the intersection of sets in one of these classes have the corresponding property.

## 2. Increasing sequences of starshaped sets

In this section we assume that $\{C_n\}$ is a uniformly bounded (UB for short) sequence of starshaped sets, that is $\overline{\bigcup C_n}$ is a bounded set. If we neglect the (UB) property the next simple example shows that also in the plane the set $\overline{\bigcup C_n}$ need not to be starshaped.

**Example 2.1.** Consider $T_k = \left\{(x,y) \in \mathbb{R}^2 : \max(0, k-x) \le y \le \min(k, k+1-x)\right\}$ and $C_n = \bigcup_{k=1}^{n} T_k$. Then $C_n$ are starshaped with:

$$ker(C_n) = \left\{(x,y) \in \mathbb{R}^2 : \max(0, n-x) \le y \le \min(1, n+1-x)\right\}.$$

It is easy to check that $\overline{\bigcup C_n}$ is not starshaped.

The next two examples show that also with the (UB) condition the closure of an increasing sequence of starshaped sets could loose this property.

**Example 2.2.** Consider the increasing, (UB) sequence of starshaped sets:

$$C_n = \{x \in c_0 : \ 0 \le x_k \le 1 \text{ for } 1 \le k \le n \text{ with } \max x_k = 1, x_k = 0 \text{ for } k \ge n+1\}.$$

Then $ker(C_n) = \{(1, 1, \cdots, 1, 0, 0, \cdots)\}$ and $C = \overline{\bigcup C_n}$ is the subset of $c_0$ of all sequences with all components in $[0, 1]$ and at least one component equal to 1. The set C is not starshaped: by contradiction, let be a sequence $c = \{c_n\} \in c_0$ in $ker(C)$ and suppose $c_{n_0} < 1$. The elements of the standard vector basis $\{e_n\}$ are in $C$ but the segment $(c, e_{n_0})$ is not contained in $C$ since no coordinate is equal to 1.

**Example 2.3.** Consider the following subsets $C_n$ of $B(l_1)$: $x = \{x_k\} \in C_n$ if the following conditions are satisfied:

1) $x_k \ge 0$ for every $k$.
2) $\sum_{k=1}^{n} x_k = 1$.
3) either $x_k \le \frac{1}{k}$ for $k = 1, 2, \cdots n$, or if $N = \max\left\{j \in \{1, \cdots, n\} : x_j > \frac{1}{j}\right\}$ we have $x_1 = x_2 = \cdots = x_{N-1}$.

Then it is easy to verify that $C_n$ is an increasing (UB) sequence of starshaped sets with $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, 0, \cdots) \in ker(C_n)$. The set $C = \overline{\bigcup C_n}$ is not starshaped: by contradiction, suppose that $c = \{c_n\}$ is a sequence in $ker(C)$. We can suppose that $c_n < 1$ and since $\lambda c + (1-\lambda)e_n$ is in $C$, for $\lambda$ small enough we have that $\lambda c_n + 1 - \lambda > \frac{1}{n}$, and this implies that $c_1 = c_2 = \cdots = c_{n-1}$. Since $n$ is arbitrary we obtain that $c = 0$: an absurdity.

The last example can be generalized to any arbitrary nonreflexive space:

**Proposition 2.4.** *In every nonreflexive space there exists an increasing (UB) sequence of starshaped sets such that* $\overline{\bigcup C_n}$ *is not starshaped.*

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**Proof.** By James' theorem (see for instance [6, Th. 10.3]) if $X$ is a nonreflexive space there exist $\epsilon > 0$, $\{x_n\} \subset B(X)$ and $\{x_n^*\} \subset B(X^*)$ such that $x_k^*(x_i) \geq \varepsilon$ for $k \leq i$ and $x_k^*(x_i) = 0$ for $k > i$. Similarly to the last example we consider the sets $C_n$; a vector $x \in X$ belongs to $C_n$ if

1. $x = \sum_{k=1}^{n} \lambda_k x_k$ with $0 \leq \lambda_k$ and $\sum_{k=1}^{n} \lambda_k = 1$.
2. either $\lambda_k \leq \frac{1}{k}$ for $k = 1, 2, \cdots n$ or if $N = \max\left\{j \in \{1, \cdots, n\} : \lambda_j > \frac{1}{j}\right\}$ we have $\lambda_1 = \lambda_2 = \cdots = \lambda_{N-1}$.

Again $\{C_n\}$ is an increasing (UB) sequence of starshaped sets but $\overline{\bigcup C_n}$ is not a starshaped set. □

In both Examples 2.2 and 2.3 it is easy to verify that no sequence of centers is weakly convergent. The next theorem shows that this is a sufficient condition for an increasing nested sequence of (UB) starshaped sets to have the closure of the union starshaped. We start with a simple lemma:

**Lemma 2.1.** *Let $\{C_n\}$ be an increasing (UB) sequence of starshaped sets and $n_1 < n_2 < \cdots < n_k$. Then, if $y \in C_{n_1}$ and $c_{n_j} \in ker(C_{n_j})$ (for $j = 1, 2, \cdots, k$), we have $co(y, c_{n_1}, c_{n_2}, \cdots, c_{n_{k-1}}, c_{n_k}) \subset C_{n_k}$.*

**Proof.** Let $y \in C_{n_1}$ and $c_{n_1} \in ker(C_{n_1})$, then $[y, c_{n_1}] \subset C_{n_1}$. Let $c_{n_2} \in C_{n_2}$. Since for a convex combination, we have

$$\lambda_1 y + \lambda_2 c_{n_1} + \lambda_3 c_{n_2} = (1 - \lambda_3)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2} y + \frac{\lambda_2}{\lambda_1 + \lambda_2} c_{n_1}\right) + \lambda_3 c_{n_2}$$

it follows that $co(y, c_{n_1}, c_{n_2}) \subset C_{n_2}$. The lemma follows easily by induction. □

**Theorem 2.1.** *Let $\{C_n\}$ be an increasing (UB) sequence of starshaped sets with $c_n \in ker(C_n)$. Then, if $\{c_n\}$ has a subsequence weakly convergent to $c$, then the set $\overline{\bigcup C_n}$ is starshaped and $c \in ker(\overline{\bigcup C_n})$.*

**Proof.** Since $\bigcup C_n = \bigcup C_{n_k}$ we can suppose that $c_n$ is weakly convergent to $c$. Moreover by Mazur lemma, we have that for every $n$, $c \in \overline{co}(c_n, c_{n+1}, \cdots)$ and by the previous lemma, this implies that $c \in \overline{\bigcup C_n}$. Finally suppose $y \in \overline{\bigcup C_n}$ and let $z \in [c, y]$, say $z = \lambda y + (1-\lambda)c$. There exists $\bar{y} \in C_N$ such that $\|\bar{y} - y\| < \varepsilon$ and similarly there exists $\bar{c} \in co(c_N, c_{N+1}, \cdots, c_m) \subset C_m$ such that $\|\bar{c} - c\| < \varepsilon$. Define $\bar{z} = \lambda\bar{y} + (1 - \lambda)\bar{c}$. Then by the Lemma 2.1 $\bar{z} \in C_m$ and

$$\|\bar{z} - z\| \leq \lambda\|\bar{y} - y\| + (1 - \lambda)\|\bar{c} - c\| < \varepsilon$$

So $[y, c] \subset \overline{\bigcup C_n}$. □

Since for (UB) increasing sequence $\{C_n\}$ every sequence of elements in the kernels is bounded, we have the following:

**Corollary 2.5.** *Let be $X$ a reflexive space and $\{C_n\}$ an increasing (UB) sequence of starshaped sets. Then $\overline{\bigcup C_n}$ is a starshaped set.*

Next result shows that, in nonreflexive spaces, if we add the hypothesis that $\overline{\bigcup C_n}$ is starshaped then not necessarily a sequence of points in their kernels is weakly convergent.

**Proposition 2.6.** *In every nonreflexive space $X$ there exists an increasing (UB) sequence $\{C_n\}$ of starshaped sets such that $\overline{\bigcup C_n}$ is starshaped but no sequence in the kernels of $\{C_n\}$ is weakly convergent.*

**Proof.** Let $X$ be a nonreflexive space and $C_n = co(x_1, x_2, \cdots, x_n) \bigcup [x_n, x_{n+1}]$, where $\{x_n\}$ is a sequence in $B(X)$ of linearly independent vectors without weakly convergent subsequences. Every $C_n$ is starshaped with $ker(C_n) = \{x_n\}$ and $\overline{\bigcup C_n} = \overline{co}(x_1, x_2, \cdots)$ is clearly a starshaped set. □

We conclude this section by indicating the following nice characterization of reflexive spaces: it springs immediately from the last results.

**Theorem 2.2.** *A Banach space is reflexive if and only if for any increasing (UB) sequence $\{C_n\}$ of starshaped sets, the closure of their union is starshaped if and only if the kernels of $\{C_n\}$ have a weakly convergent subsequence.*

## 3. Decreasing sequences of starshaped sets

Decreasing sequences of starshaped sets and their intersections are already considered in the literature. In [1, (2.5)] the authors proved the following result:

**Theorem 3.1.** *If $X$ is a nonreflexive Banach space then each nonempty bounded weakly compact subset of $X$ is the intersection of a decreasing sequence of weakly closed starshaped sets.*

We can use the same method of the previous section to obtain a similar description for the intersection of starshaped sequences.

**Lemma 3.1.** *Let $\{C_n\}$ be a decreasing sequence of starshaped sets and $n_1 < n_2 < \cdots < n_k$. Then, if $c_{n_j} \in ker(C_{n_j})$ for $j = 1, 2, \cdots, k$ and $y \in C_{n_k}$ we have $co(y, c_{n_1}, c_{n_2}, \cdots, c_{n_{k-1}}, c_{n_k}) \subset C_{n_1}$.*

**Proof.** Suppose that $c_{n_k} \in ker(C_{n_k})$. Then $[y, c_{n_k}] \subset C_{n_k}$ an so $co(y, c_{n_k}, c_{n_{k-1}}) \subset C_{n_{k-1}}$. The claim follows easily by induction. □

The following result may be proved in much the same way as Theorem 2.1 simply replacing Lemma 2.1 with Lemma 3.1.

**Theorem 3.2.** *Assume that $\{C_n\}$ is a decreasing sequence of starshaped sets and consider $c_n \in ker(C_n)$. If $\{c_n\}$ has a subsequence weakly convergent to $c$, then $\bigcap C_n$ is a nonempty starshaped set and $c \in ker(\bigcap C_n)$.*

As a consequence of this theorem and Theorem 3.1 we obtain the following result, which slightly generalizes [3, (4.2)].

**Corollary 3.1.** *A Banach space $X$ is a reflexive space if and only if every decreasing intersection of starshaped sets is a nonempty starshaped set.*

## 4. Increasing sequences of stars

In this Section we consider (UB) increasing sequences $\{S_n\}$ of stars and we denote by $\{s_n\}$ the sequence of their (unique) centers. The first result is the analogue of Theorem 2.1 (or Theorem 3.2). We omit its proof since it can be obtained simply adapting the technique used in Theorem 2.1.

**Theorem 4.1.** *Consider an increasing, (UB) sequence of stars $\{S_n\}$, and let $\{s_n\}$ the sequence of their centers. If $s_{n_k} \to s$ weakly then $\overline{\bigcup S_n}$ is a star and $s$ is its center.*

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A simple example shows that, in general, the closure of the union of an increasing (UB) sequence of stars need not be a star:

**Example 4.1.** Consider the sets: $S_n = \{x \in c_0 : 0 \leq x_k \leq 1 \text{ for } 1 \leq k \leq n \,, x_k = 0 \text{ for } k > n\}$. Then they are stars with centers $s_n = (1/2, 1/2, \cdots, 1/2, 0, 0 \cdots)$. Clearly $\overline{\bigcup S_n}$ is a convex set but not a star.

Next result shows that the previous example is, in some sense, the only possible example. To prove this we recall two fundamental results of Rosenthal (see [5] and [4]):

**Theorem 4.2** *(First Rosenthal's Theorem). Every bounded sequence in a Banach space has either a weak-Cauchy subsequence, or a subsequence equivalent to the standard $l_1 -$ basis.*

**Theorem 4.3** *(Second Rosenthal's Theorem). Every non-trivial weak-Cauchy sequence in a Banach space has either a strongly summing subsequence, or a convex block basis equivalent to the summing basis.*

We recall that a sequence $\{x_n\}$ in a Banach space is called *strongly summing* if $\{x_n\}$ is weak-Cauchy so that whenever $\{\alpha_n\}$ is a sequence of scalars satisfying $\sup_n \| \sum_{j=1}^{n} \alpha_j x_j \| < \infty$, then $\sum \alpha_n$ converges.

**Theorem 4.4.** *Suppose that $\{S_n\}$ a sequence in a Banach space $X$ of increasing (UB) stars such that $\overline{\bigcup S_n}$ is not a star. Then $X$ contains $c_0$.*

**Proof.** Consider the sequence $\{s_n\}$ of centers of $\{S_n\}$ and assume that $n_1 < n_2 < \cdots < n_k$. Then $s_{n_1} \in S_{n_1} \subset S_{n_2}$ and this implies that $2s_{n_2} - s_{n_1} \in S_{n_2} \subset S_{n_3}$. So $2s_{n_3} - 2s_{n_2} + s_{n_1} \in S_{n_3}$. By a simple iteration we obtain that

$$2s_{n_k} - 2s_{n_{k-1}} + 2s_{n_{k-2}} - \cdots \pm 2s_{n_2} \mp s_{n_1} \in S_{n_k}.$$

Moreover since $\{S_n\}$ is uniformly bounded there exists a constant $M$ such that

$$\left\| 2s_{n_k} - 2s_{n_{k-1}} + 2s_{n_{k-2}} - \cdots \pm 2s_{n_2} \mp s_{n_1} \right\| < M \tag{4.1}$$

By Theorem 4.1 $\{s_n\}$ has no weakly convergent subsequence and so by the first Rosenthal's theorem $\{s_n\}$ has a subsequence equivalent to the standard basis of $l_1$ or has a weak-Cauchy subsequence. Clearly, by (4.1) no subsequence of $\{s_n\}$ is equivalent to the standard basis of $l_1$. So we can suppose that $\{s_n\}$ has a non trivial weak-Cauchy subsequence. By the second Rosenthal's theorem this subsequence has a "strongly summing subsequence" or a "convex block basis" equivalent to the summing basis. Again, by (4.1), no subsequence of $\{s_n\}$ can be *strongly summing*. This implies that a convex block of $\{s_n\}$ spans a subspace isomorphic to $c_0$. □

Since a convex block of centers belongs to suitable $S_n$ we can state the following:

**Corollary 4.2.** *Suppose that $\{S_n\}$ is an increasing, (UB) sequence of stars such that $\overline{\bigcup S_n}$ is not a star. Then there exists a sequence $\{x_n\}$, with $x_n \in S_n$, equivalent to the summing basis of $c_0$.*

Proposition 2.6 shows that in every nonreflexive space $X$ there exists an increasing (UB) sequence $\{C_n\}$ of starshaped sets such that $\overline{\bigcup C_n}$ is starshaped but no sequence with elements in their kernels is weakly convergent. We show that for increasing sequences of stars the situation is different.

**Theorem 4.5.** *Suppose that $\{S_n\}$ is an increasing (UB) sequence of stars, with centers $s_n \in S_n$, bounded by a constant $M$, and such that $\overline{\bigcup S_n}$ is a star. Then there exists a subsequence of $\{s_n\}$ which is Cesaro-summable.*

**Proof.** Let be $s$ the center of $S = \overline{\bigcup S_n}$. Then there exists $y_1 \in S_{n_1}$ such that $\|s - y_1\| \leq 1/2$. Since $2s_{n_1} - y_1 \in S_{n_1} \subset S$ we have: $2s - (2s_{n_1} - y_1) \in S$. So there exists $y_2 \in S_{n_2}$ (with $n_1 < n_2$) such that $\| [2s - (2s_{n_1} - y_1)] - y_2\| \leq 1/4$. Again, since $2s_{n_2} - y_2 \in S_{n_2} \subset S$ we have: $2s - (2s_{n_2} - y_2) \in S$. So there exists $y_3 \in S_{n_3}$ (with $n_2 < n_3$) such that $\| [2s - (2s_{n_2} - y_2)] - y_3\| \leq 1/8$. Notice that:

$$\begin{aligned}\|5s - 2s_{n_1} - 2s_{n_2}\| &= \|(2s - 2s_{n_2} + y_2 - y_3) + (2s - 2s_{n_1} + y_1 - y_2) + (s - y_1) + y_3\| \leq \\ &\leq \|2s - 2s_{n_2} + y_2 - y_3\| + \|2s - 2s_{n_1} + y_1 - y_2\| + \|s - y_1\| + \|y_3\| \leq \\ &\leq 1/8 + 1/4 + 1/2 + \|y_3\| < 1 + M.\end{aligned}$$

Clearly by induction we obtain:

$$\|(2k+1)s - 2(s_{n_1} + s_{n_2} + \cdots + s_{n_k})\| < 1 + \|y_{n_k}\| < 1 + M. \tag{4.2}$$

From this we obtain $s = \lim_{k\to\infty} \frac{2}{2k+1} \sum_{j=1}^{k} s_{n_j}$. □

**Theorem 4.6.** *If $\overline{\bigcup S_n}$ is a star then there exists a subsequence of $\{s_n\}$ weakly convergent to the center $s$ of $\overline{\bigcup S_n}$.*

**Proof.** By Theorem 4.4 and Theorem 4.5 we can suppose that there exists a subsequence of $\{s_n\}$ which is weak-Cauchy and such that $s = \lim_{n\to\infty} \frac{2}{2n+1} \sum_{k=1}^{n} s_k$. Let $x^* \in X^*$ and $N$ such that, for every $n \geq N$ we have $| x^*(s_n) - x^*(s_N) | < \varepsilon$. So:

$$\begin{aligned}| x^*(s) - x^*(s_N) | &= \left| \lim_{n\to\infty} \left( \frac{2}{2n+1} \left( \sum_{k=1}^{n} (x^*(s_k) - x^*(s_N)) \right) - \frac{x^*(s_N)}{2n+1} \right) \right| \leq \\ &\leq 2 \left| \lim_{n\to\infty} \frac{\sum_{k=1}^{N} (x^*(s_k) - x^*(s_N)) + \sum_{k=N+1}^{n} (x^*(s_k) - x^*(s_N))}{2n+1} \right| + \lim_{n\to\infty} \left| \frac{x^*(s_N)}{2n+1} \right| \leq \\ &\leq 2 \lim_{n\to\infty} \left| \frac{\sum_{k=1}^{N} (x^*(s_k) - x^*(s_N))}{2n+1} \right| + 2 \lim_{n\to\infty} \left| \frac{\sum_{k=N+1}^{n} (x^*(s_k) - x^*(s_N))}{2n+1} \right| \leq \\ &\leq 2 \lim_{n\to\infty} \frac{\sum_{k=N+1}^{n} |x^*(s_k) - x^*(s_N)|}{2n+1} \leq \lim_{n\to\infty} \frac{2(n-N)}{2n+1} \varepsilon = \varepsilon\end{aligned}$$

So $\{s_n\}$ is a weakly convergent sequence and from Theorem 4.1 we can assert that the sequence weakly converges to the center of $\overline{\bigcup S_n}$. □

## 5. Decreasing sequences of stars

Some simple examples show that we can have different behavior when we consider the intersection of a decreasing sequence of stars.

**Example 5.1.** Consider the stars $S_n = \{x \in c_0 : x_1 = x_2 = \cdots = x_n = 1, -1 \leq x_k \leq 1, k = n+1, \cdots\}$. Then $\bigcap S_n$ is the empty set. Moreover the sequence of the centers $s_n = (1, 1, \cdots, 1, 0, \cdots)$ is not weakly convergent.

**Example 5.2.** Consider now $S_n = \{x \in c_0 : 0 \leq x_k \leq 1, k = 1, 2, \cdots, n; -1 \leq x_k \leq 1, k \geq n+1\}$; then the intersection is nonempty but $\bigcap S_n = \{x \in c_0 : 0 \leq x_k \leq 1\}$ is not a star. Again the sequence of centers $s_n = (1/2, \cdots, 1/2, 0, 0, \cdots)$ is not weakly convergent.

**Example 5.3.** Finally if we consider $S_n = \{x \in c : x_1 = x_2 = \cdots = x_n = 1, 0 \leq x_k \leq 1, k = n+1, \cdots\}$ then $\bigcap S_n = \{(1, 1, 1, \cdots)\}$ is a star but again the sequence of centers $s_n = (1, \cdots, 1, 1/2, 1/2 \cdots)$ is not weakly convergent.

By a similar argument used in Theorem 2.1 we can prove:

**Theorem 5.1.** *Let $\{S_n\}$ be a decreasing sequence of stars with $\{s_n\}$ the sequence of their centers. If $\{s_n\}$ has a subsequence weakly convergent to $s$, then $\bigcap S_n$ is a nonempty star and $s$ is its center.*

In [2] the author obtained the following result:

**Theorem 5.2.** *Le $X$ be a Banach space. The following are equivalent:*

(1) *Every decreasing sequence of stars has nonempty intersection.*
(2) *$X$ does not contain $c_0$.*

We remark that it is not difficult to fit the proof of Theorem 4.4 to the decreasing case. So we obtain, with a different proof, the result of the above theorem. More precisely we can prove the following extension of Theorem 5.2:

**Theorem 5.3.** *Suppose that $\{S_n\}$ is a decreasing sequence of stars such that $\bigcap S_n$ is not a star. Then there exists a sequence $\{x_n\} \in S_n$ equivalent to the summing basis of $c_0$.*

As a final remark we observe that we cannot extend Theorem 4.5 to decreasing sequences of stars. Indeed consider Example 5.3, then the sequence of centers does not contain Cesaro-summable subsequences.

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