Globalizing F-invariants

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Abstract

In this paper we define and study the global Hilbert-Kunz multiplicity and the global F-signature of prime characteristic rings which are not necessarily local. Our techniques are made meaningful by extending many known theorems about Hilbert-Kunz multiplicity and F-signature to the non-local case.

Keywords: Hilbert-Kunz multiplicity, F-signature, globalizing

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1. Introduction

Throughout, R will be a commutative Noetherian ring with identity. Unless otherwise stated, R is of prime characteristic p. Let $F^e : R \to R$ be the eth iterate of the Frobenius endomorphism, that is $F^e(r) = r^{p^e}$. Kunz's work in [25] equates flatness of F^e with the property that R is regular, a foundational result indicating asymptotic measurements of the Frobenius endomorphism can be used to measure the severity of the singularities of R. We will focus on the numerical invariants Hilbert-Kunz multiplicity and F-signature.

For the sake of simplicity in introducing Hilbert-Kunz multiplicity and F-signature, assume that $(R, \mathfrak{m}, k)$ is a complete local domain, with unique maximal ideal $\mathfrak{m}$, dimension d, residue field k, and $k^{1/p}$ is finite as a k-vector space. If $I \subseteq R$ is an ideal, $I^{[p^e]} = (i^{p^e} \mid i \in I)$ is the expansion of I along F^e. If M is a finite length R-module let $\lambda(M)$ denote the length of M. If I is an $\mathfrak{m}$-primary ideal, so is $I^{[p^e]}$ for each $e \in \mathbb{N}$.

Definition 1.1. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p and I an $\mathfrak{m}$-primary ideal. The *Hilbert-Kunz multiplicity of I* is

$$\mathrm{e}_{\mathrm{HK}}(I) = \lim_{e \to \infty} \lambda(R/I^{[p^e]})/p^{e \dim(R)}.$$

Monsky proved the existence of the limit $\mathrm{e}_{\mathrm{HK}}(I)$ in [29]. The Hilbert-Kunz multiplicity of R is defined to be $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$. It is well known that $\mathrm{e}_{\mathrm{HK}}(R) \geqslant 1$ with equality if and only if R is regular, [42].

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More generally, it is known that sufficiently small values of Hilbert-Kunz multiplicity imply the properties of being Gorenstein and strongly F-regular, [6, 4].

Denote by F^e_*R the R-module obtained via restriction of scalars via F^e. Our hypotheses imply that R is F-finite, that is, F^e_*R is a finitely generated R-module for each $e \in \mathbb{N}$. Moreover, we have that $\lambda(R/\mathfrak{m}^{[p^e]}R)/p^{ed} = \mu(F^e_*R)/\operatorname{rank}(F^e_*R)$, where $\mu(\_\_)$ denotes the minimal number of generators of a finitely generated R-module. In particular,

$$\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \mu(F^e_*R)/\operatorname{rank}(F^e_*R).$$

Thus, the Hilbert-Kunz multiplicity of R is the asymptotic growth rate of the minimal number of generators of F^e_*R compared to its rank, a measurement that can also be discussed for rings that are not necessarily local.

Now let R be an F-finite domain, not necessarily local. With the above observation, we define the *global Hilbert-Kunz multiplicity* of R, still denoted $\mathrm{e}_{\mathrm{HK}}(R)$, as

$$\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \mu(F^e_*R)/\operatorname{rank}(F^e_*R),$$

provided the limit exists. Our first main result is the existence of the corresponding limit for any F-finite ring. In addition, we relate $\mathrm{e}_{\mathrm{HK}}(R)$ with the Hilbert-Kunz multiplicities $\mathrm{e}_{\mathrm{HK}}(R_P)$ of the localizations at primes $P \in \operatorname{Spec}(R)$, showing that such an invariant, even though it is defined globally, captures the local properties of the ring. Finally, as for the Hilbert-Kunz multiplicity of a local ring, we show that small values of $\mathrm{e}_{\mathrm{HK}}(R)$ imply that R has mild singularities. We summarize all these results in the following theorem. We point out that our results hold in a more general setup than the one in which we state them here, as we will show in Section 3.

Theorem A. *Let R and T be F-finite domains, not necessarily local.*

1. *(Theorem 3.8) The limit $\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \mu(F^e_*R)/\operatorname{rank}(F^e_*R)$ exists.*
2. *(Theorem 3.16) We have $\mathrm{e}_{\mathrm{HK}}(R) = \max\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in \operatorname{Spec}(R)\}$.*
3. *(Theorem 3.20) The ring R is regular if and only if $\mathrm{e}_{\mathrm{HK}}(R) = 1$.*
4. *(Theorem 3.20) Let $e = \max\{e(R_P) \mid P \in \operatorname{Spec}(R)\}$ where $e(R_P)$ is the Hilbert-Samuel multiplicity of the local ring R_P. If $\mathrm{e}_{\mathrm{HK}}(R) \leqslant 1 + \max\{1/\dim(R)!, 1/e\}$, then R is strongly F-regular and Gorenstein.*
5. *(Theorem 5.7) If $R \to T$ is faithfully flat, then $\mathrm{e}_{\mathrm{HK}}(R) \leqslant \mathrm{e}_{\mathrm{HK}}(T)$.*

We now turn our attention to the F-signature. To introduce it, we return to the assumptions that $(R, \mathfrak{m}, k)$ is a complete local domain of dimension d, and $k^{1/p}$ is a finite k-vector space. As noted before, these assumptions guarantee that R is F-finite. We denote by $\operatorname{frk}(F^e_*R)$ the maximal number of free R-summands appearing in all direct sum decompositions, equivalently the maximal number of free R-summands appearing in a single direct sum decomposition, of F^e_*R into indecomposable modules.

Definition 1.2. Let $(R, \mathfrak{m}, k)$ be local domain of prime characteristic p and assume that R is F-finite. The *F-signature of* R is

$$\mathrm{s}(R) = \lim_{e \to \infty} \mathrm{frk}(F_*^e R) / \mathrm{rank}(F_*^e R).$$

Tucker proved the existence of $\mathrm{s}(R)$ in [41]. Before Tucker's proof of the existence of $\mathrm{s}(R)$, the study of the asymptotic growth rate of the number of free summands of $F_*^e R$ originated in [37]. Huneke and Leuschke coined the term F-signature in [22] and were able to show it exists under the additional assumption that R is Gorenstein. There were a number of papers written which established the existence of the F-signature for certain classes of rings, see [22], [35], [3], [44], and [1]. As remarked by third author in [44], the study of F-signature is closely related to relative Hilbert-Kunz multiplicity studied by Watanabe and Yoshida in [43]. Similar to Hilbert-Kunz multiplicity, particular values of $\mathrm{s}(R)$ determine the severity of the singularity of R. Most notably, $\mathrm{s}(R) = 1$ if and only if R is regular, as shown by Huneke and Leuschke in [22], and $\mathrm{s}(R) > 0$ if and only if R is strongly F-regular by work of Aberbach and Leuschke in [5].

In order to globalize F-signature, note that the numbers $\mathrm{frk}(F_*^e R)$ make sense also for F-finite rings which are not necessarily local. Unlike the local case, one has to consider all direct sum decompositions of $F_*^e R$ to determine $\mathrm{frk}(F_*^e R)$ and not just a single direct sum decomposition. Nevertheless, it is possible to study the sequence of measurements $\mathrm{frk}(F_*^e R)/\mathrm{rank}(F_*^e R)$ for any F-finite ring. We prove that the limit $\mathrm{s}(R)$ of such a sequence exists, and we call it the global F-signature of R. As with global Hilbert-Kunz multiplicity, we relate $\mathrm{s}(R)$ with the local F-signatures $\mathrm{s}(R_P)$, for $P \in \mathrm{Spec}(R)$. In addition, we show that special values of $\mathrm{s}(R)$ detect the singularities of the ring R, as in the case of local rings. Our main results about F-signature, here stated for simplicity in a more restrictive setup than the one in which they actually hold, are summarized in the following theorem.

Theorem B. *Let R and T be F-finite domains, not necessarily local.*

1. *(Theorem 4.7) The limit* $\mathrm{s}(R) = \lim_{e \to \infty} \mathrm{frk}(F_*^e R)/\mathrm{rank}(F_*^e R)$ *exists.*
2. *(Theorem 4.13) We have* $\mathrm{s}(R) = \min\{\mathrm{s}(R_P) \mid P \in \mathrm{Spec}(R)\}$.
3. *(Theorem 4.15) The ring R is regular if and only if* $\mathrm{s}(R) = 1$.
4. *(Theorem 4.15) The ring R is strongly F-regular if and only if* $\mathrm{s}(R) > 0$.
5. *(Theorem 5.1) If $R \to T$ is faithfully flat, then* $\mathrm{s}(R) \geqslant \mathrm{s}(T)$.

This article is organized as follows. Section 2 is used to set up notation and recall previously known results. Section 3 develops the theory of global Hilbert-Kunz multiplicity of finitely generated modules over non-local F-finite rings. We then introduce the theory of global F-signature of finitely generated R-modules over non-local F-finite rings in Section 4. The global F-signature of a pair $(R, \mathscr{D})$ where R is an F-finite ring and $\mathscr{D}$ is a Cartier subalgebra is also introduced. In section 5 we study global Hilbert-Kunz multiplicity and global F-signature of faithfully flat extensions. Besides showing similarities between the local and global theory of Hilbert-Kunz multiplicity and F-signature, we also provide examples in Sections 3, 4, and 5 which illustrate their differences.

This article globally defines and analyzes two important F-invariants which have historically been studied locally. In [12], the authors of this paper establish results similar to those of this paper for other numerical invariants, including Frobenius Betti numbers. In doing so, we globalize more F-invariants of interest.

2. Background

If R is a domain and M a finitely generated R-module, the rank of M is defined as $\operatorname{rank}_R(M) = \dim_K(M \otimes_R K)$, where K is the fraction field of R. When R is not a domain, the notion of rank is not necessarily uniquely defined. In particular, in this article we will need to use two different definitions. Given a finitely generated R-module M, we define the rank of M as $\operatorname{rank}_R(M) = \max\{\operatorname{rank}_{R/Q}(M/QM) \mid Q \in \operatorname{Min}(R)\}$, and we define the min-rank of M as $\text{min-rank}_R(M) = \min\{\operatorname{rank}_{R/Q}(M/QM) \mid Q \in \operatorname{Min}(R)\}$. The reason for giving the name of rank to the maximum of the ranks modulo minimal primes is that this is the definition that we will mostly use in this article. Clearly, the two notions agree when R is a domain. As discussed in the introduction, we use $\lambda_R(\_\_)$ and $\mu_R(\_\_)$ to denote the length of a finite length R-module and the minimal number of generators of a finitely generated R-module respectively. If confusion is not likely to arise, we commonly omit subscripts from these notations.

2.1. F-finite rings

As discussed in the introduction, R is F-finite if for some, equivalently for all positive integers $e \in \mathbb{N}$, F^e_*R is a finitely generated R-module. Every F-finite ring is excellent, [26, Theorem 2.5]. If R is F-finite and M a finitely generated R-module, then F^e_*M is a finitely generated R-module for each $e \in \mathbb{N}$. Once again, F^e_*M is the R-module M obtained via restriction of scalars by F^e. If R is a domain, then F^e_*R is naturally isomorphic with R^{1/p^e}, the ring of p^eth roots of R, as R lies in an algebraic closure of its fraction field. However, we will refrain from using this notation and henceforth use $F^e_*\_\_$.

Let R be an F-finite ring. Given $P \in \operatorname{Spec}(R)$ let $\kappa(P) = R_P/PR_P$ be the residue field of R_P and let $\alpha(P) = \log_{p^e}[F^e_*\kappa(P) : \kappa(P)]$, which is independent of the choice of $e > 0$. Let $\gamma(R) = \max\{\alpha(Q) \mid Q \in \min(R)\}$. It is easily verified that, if R is a domain, then $\operatorname{rank}(F^e_*R) = p^{e\gamma(R)}$ for each $e \in \mathbb{N}$. If M is a finitely generated R-module, $I = \operatorname{Ann}_R(M)$, we define $\gamma(M)$ as $\gamma(R/I)$.

2.2. Hilbert-Kunz multiplicity

Suppose $R = (R, \mathfrak{m}, k)$ is a local ring of prime characteristic p, of dimension d, and M a finitely generated R-module. Let $I \subseteq R$ be an ideal. If $I = (i_1, \ldots, i_s)$, then one easily checks that $I^{[p^e]} = (i_1^{p^e}, \ldots, i_s^{p^e})$. So for each $e \in \mathbb{N}$ there are inclusions of ideals $I^{sp^e} \subseteq I^{[p^e]} \subseteq I^{p^e}$. Therefore if I is $\mathfrak{m}$-primary, so is each $I^{[p^e]}$, and we have the set of inequalities

$$\lambda(M/I^{p^e}M) \leqslant \lambda(M/I^{[p^e]}M) \leqslant \lambda(M/I^{sp^e}M).$$

So as a function, $\lambda(M/I^{[p^e]}M) = O(p^{e\dim(M)})$. Monsky proved that the following limit exists:

$$\lim_{e\to\infty} \frac{1}{p^{ed}}\lambda(M/I^{[p^e]}M).$$

Its limit is denoted $\mathrm{e_{HK}}(I, M)$, and is called the Hilbert-Kunz multiplicity of I with respect to M. Moreover, $\lambda(M/I^{[p^e]}M) = \mathrm{e_{HK}}(I, M)p^{ed} + O(p^{e(d-1)})$.

We let $\mathrm{e_{HK}}(M) = \mathrm{e_{HK}}(\mathfrak{m}, M)$ and call this number the Hilbert-Kunz multiplicity of M. Hilbert-Kunz multiplicity is additive on short exact sequences. So if I is an $\mathfrak{m}$-primary ideal and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of finitely generated R-modules, then $\mathrm{e_{HK}}(I, M) = \mathrm{e_{HK}}(I, M') + \mathrm{e_{HK}}(I, M'')$, see [29, Theorem 1.8]. Because of this, study of the Hilbert-Kunz multiplicity of a finitely generated R-module can typically be reduced to the scenario that R is a domain and $M = R$.

There are theorems which relate values of $\lambda(R/\mathfrak{m}^{[p^e]}R)$ and $\mathrm{e_{HK}}(R)$ with the severity of the singularity of $(R, \mathfrak{m}, k)$, the first of which is Kunz's Theorem: If d is the Krull dimension of R and $e \in \mathbb{N}$ then $\lambda(R/\mathfrak{m}^{[p^e]}R) \geqslant p^{ed}$ with equality if and only if R is a regular local ring, [25, Theorem 3.3]. In particular, if R is regular then $\mathrm{e_{HK}}(R) = 1$. Kunz's Theorem motivates the philosophy that particular values of $\mathrm{e_{HK}}(R)$ controls the severity of the singularities of R. This philosophy has been significantly developed since Kunz's work.

Theorem 2.1. *Let $(R, \mathfrak{m}, k)$ be a formally unmixed local ring of characteristic p.*

1. [42, Theorem 1.5] *Then R is regular if and only if $\mathrm{e_{HK}}(R) = 1$.*[1]
2. [4, 6] *Let e be the Hilbert-Samuel multiplicity of R. If $\mathrm{e_{HK}}(R) \leqslant 1 + \max\{1/\dim(R)!, 1/e\}$, then R is strongly F-regular and Gorenstein.*
3. [4, 6, 10] *There is a number $\delta > 0$ depending only on the dimension of R such that if $\mathrm{e_{HK}}(R) \leqslant 1 + \delta$, then R is regular.*

Suppose that $(R, \mathfrak{m}) \to (T, \mathfrak{n})$ is a flat local homomorphism of characteristic p local rings, M a finitely generated R-module, and $M_T = M \otimes_R T$. Then Kunz's methods in the proof of [26, Theorem 3.6, Proposition 3.9] can be used to show $\lambda_R(M/\mathfrak{m}^{[p^e]}M)/p^{e\dim(R)} \leqslant \lambda_T(M_T/\mathfrak{n}^{[p^e]}M_T)/p^{e\dim(T)}$ for all $e \in \mathbb{N}$. Furthermore, equality holds if the closed fiber of $R \to T$ is regular. We formally state this result in the module case for future reference and attribute it to Kunz.

[1] See [24] for a simpler proof.

Theorem 2.2. *Let $(R, \mathfrak{m}, k) \to (T, \mathfrak{n}, l)$ be a flat local ring homomorphism of local rings of prime characteristic p, and M a finitely generated R-module. Denote $M_T = M \otimes_R T$. Then for each $e \in \mathbb{N}$ we have $\lambda_R(M/\mathfrak{m}^{[p^e]}M)/p^{e\dim(R)} \leqslant \lambda_T(M_T/\mathfrak{n}^{[p^e]}M_T)/p^{e\dim(T)}$, hence $\mathrm{e}_{\mathrm{HK}}(M) \leqslant \mathrm{e}_{\mathrm{HK}}(M_T)$. Moreover, if $T/\mathfrak{m}T$ is regular, then for each $e \in \mathbb{N}$ we have that $\lambda_R(M/\mathfrak{m}^{[p^e]}M)/p^{e\dim(R)} = \lambda_T(M_T/\mathfrak{n}^{[p^e]}M_T)/p^{e\dim(T)}$, hence $\mathrm{e}_{\mathrm{HK}}(M) = \mathrm{e}_{\mathrm{HK}}(M_T)$.*

Kunz asked if R is an excellent equidimensional ring of prime characteristic p, for each $e \in \mathbb{N}$ is the function $\lambda_e : \mathrm{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \lambda(R_P/P^{[p^e]}R_P)/p^{e\,\mathrm{ht}(P)}$ upper semi-continuous, see [26, Problem page 1006]. Shepherd-Barron provides a counter-example to Kunz's problem and showed the answer to the question is yes under the stronger assumption R is locally equidimensional, see [34]. As with Theorem 2.2, Shepherd-Barron's solution to Kunz's question is easily generalized to the module case.

Theorem 2.3. *Let R be a locally equidimensional excellent ring of prime characteristic p and let M be a finitely generated R-module. Then for each $e \in \mathbb{N}$ the function $\lambda_e : \mathrm{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \lambda(M_P/P^{[p^e]}M_P)/p^{e\,\mathrm{ht}(P)}$ is upper semi-continuous.*

Remark 2.4. The proof of Theorem 2.3 shows that the function λ_e is dense upper semi-continuous. That is for each $P \in \mathrm{Spec}(R)$ there exists a dense open set U containing P such that $\lambda_e(Q) \leqslant \lambda_e(P)$ for each $Q \in U$.

2.3. F-signature

Suppose that $(R, \mathfrak{m}, k)$ is an F-finite local ring of dimension d and prime characteristic p. For each $e \in \mathbb{N}$, let $a_e(R)$ be the maximal number of free R-summands appearing in various direct sum decompositions of F_*^eR, and call $a_e(R)$ the eth Frobenius splitting number of R. Suppose that $F_*^eR \cong R^{\oplus n} \oplus M_e$ where M_e does not contain a free summand. Consider the sets $I_e = \{r \in R \mid \varphi(F_*^e r) \in \mathfrak{m}, \forall \varphi \in \mathrm{Hom}_R(F_*^eR, R)\}$, introduced by Aberbach and Enescu [2]. Then, one can easily verify that I_e is an ideal of R, and that $F_*^eI_e \cong \mathfrak{m}R^{\oplus n} \oplus M_e$. Therefore $a_e(R)$ is the maximal number of free R-summands appearing in any direct sum decomposition of F_*^eR. Motivated by Hilbert-Kunz multiplicity, it is natural to study the sequence of numbers $\lambda(R/I_e)/p^{ed} = a_e(R)/p^{e\gamma(R)}$ for the purpose of studying the severity of the singularities of R.

Theorem 2.5. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of prime characteristic p.*

1. [41, Main Result] *The following limit exists,*

$$\lim_{e \to \infty} \frac{a_e(R)}{p^{e\gamma(R)}}.$$

Its limit is denoted $\mathrm{s}(R)$, and is called the F-signature of R.

2. *As a function in e, $a_e(R) = \mathrm{s}(R)p^{\gamma(R)} + O(p^{e(\gamma(R)-1)})$.* [2]

3. [22, Corollary 16] *We have $\mathrm{s}(R) = 1$ if and only if R is regular.*

4. [5, Main Result] *We have $\mathrm{s}(R) > 0$ if and only if R is strongly F-regular.*

Any strongly F-regular local ring is a domain. So the study of the F-signature is typically of interest when R is a domain.

Remark 2.6. Define the F-signature of a finitely generated R-module M as follows. Let $a_e(M)$ be the largest rank of a free module appearing in various or, equivalently, in a single direct sum decomposition of $F^e_* M$. The F-signature of M is defined to be $\mathrm{s}(M) = \lim_{e\to\infty} a_e(M)/p^{e\gamma(R)}$, which exists by some simple reductions to the scenario that $M = R$. Moreover, positivity of $\mathrm{s}(M)$ implies positivity of $\mathrm{s}(R)$. To see this one only needs to observe $a_e(M) \leqslant a_e(R^{\oplus \mu(M)}) = \mu(M)a_e(R)$. It is also the case $\mathrm{s}(M) = \mathrm{rank}_R(M)\,\mathrm{s}(R)$, see [41, Theorem 4.11].

The third author naturally extends the notion of F-signature to all local rings which are not assumed to be F-finite in [44]. Let $E_R(k)$ denote the injective hull of k and let $u \in E_R(k)$ generate the socle. Let $I_e = \{r \in R \mid u \otimes F^e_* r = 0 \in E_R(k) \otimes F^e_* R\}$. If $(R, \mathfrak{m}, k)$ is F-finite and of dimension d, then $\lambda(R/I_e)/p^{ed} = a_e(R)/p^{e\gamma(R)}$. If $(R, \mathfrak{m}, k)$ is not necessarily F-finite, the F-signature of R is defined to be $\lim_{e\to\infty} \lambda(R/I_e)/p^{ed}$. The third author's observations in [44] and Tucker's work in [41] provide the existence of the F-signature of a non-F-finite local ring. Moreover, parts 3 and 4 of Theorem 2.5 remain valid without the F-finite assumption.

The third author has shown that if $(R, \mathfrak{m}, k)$ is a non-regular local of of prime characteristic p and dimension d, then $\mathrm{s}(R) < 1 - \frac{1}{d!p^d}$, see [44, Theorem 3.1]. We show how part 3 of Theorem 2.1 provides the existence of a constant $\delta > 0$, depending only on the dimension of a local ring, such that if $(R, \mathfrak{m}, k)$ is local of dimension d, of any prime characteristic, and non-regular, then $\mathrm{s}(R) < 1 - \delta$.

Theorem 2.7. *Fix $d \in \mathbb{N}$. There is a number $\delta > 0$ such that if $(R, \mathfrak{m}, k)$ is an F-finite of dimension d, of any prime characteristic, and such that $\mathrm{s}(R) \geqslant 1 - \delta$, then R is regular.*

Proof. Assume that R is non-regular and let δ be as in part 3 of Theorem 2.1. If R is not strongly F-regular, then $\mathrm{s}(R) = 0$ by part 4 of Theorem 2.5. Thus we may assume R is strongly F-regular, in particular R is a domain and $\mathrm{e}_{\mathrm{HK}}(R) > 1+\delta$ by part 3 of Theorem 2.1. By [22, Proposition 14] $(e(R)-1)(1-\mathrm{s}(R)) \geqslant \mathrm{e}_{\mathrm{HK}}(R)-1$ where $e(R)$ is the Hilbert-Samuel multiplicity of R. As $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{m}^{p^e}$, $\lambda(R/\mathfrak{m}^{[p^e]}) \geqslant \lambda(R/\mathfrak{m}^{p^e})$. Dividing by p^{ed} and letting $e \to \infty$ shows $\mathrm{e}_{\mathrm{HK}}(R) \geqslant \frac{e(R)}{d!}$. Simple manipulations of the these two inequalities provide the

[2]This result can be pieced together from results in [41] and [8]. See [31, Theorem 3.6] for a direct proof.

following:

$$\begin{aligned} \mathrm{s}(R) &\leqslant 1 - \frac{\mathrm{e_{HK}}(R) - 1}{d!\,\mathrm{e_{HK}}(R) - 1} = \frac{d!\,\mathrm{e_{HK}}(R) - \mathrm{e_{HK}}(R)}{d!\,\mathrm{e_{HK}}(R) - 1} \\ &= \frac{d! - 1}{d! - \frac{1}{\mathrm{e_{HK}}(R)}} < \frac{d! - 1}{d! - \frac{1}{\delta+1}} = 1 - \frac{\delta}{d!(\delta+1) - 1}. \end{aligned}$$

□

Given a local ring $(R, \mathfrak{m}, k)$ of prime characteristic p and of dimension d, let $s_e(R) = \lambda(R/I_e)/p^{ed}$ and call this number the eth normalized Frobenius splitting number of R. In [44], the third author proves if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat, then for each $e \in \mathbb{N}$, $s_e(R) \geqslant s_e(T)$. In other words, the normalized Frobenius splitting numbers can only decrease after flat extensions. We formally state this theorem for future reference.

Theorem 2.8 ([44, Theorem 5.4 (3), Theorem 5.5]). *Let $(R, \mathfrak{m}, k) \to (T, \mathfrak{n}, l)$ be a flat local ring homomorphism of local rings of prime characteristic p and let M be a finitely generated R-module. Then $s_e(M) \geqslant s_e(M \otimes_R T)$ for each $e \in \mathbb{N}$, hence $\mathrm{s}(M) \geqslant \mathrm{s}(M \otimes_R T)$. Moreover, if $T/\mathfrak{m}T$ is regular, then $s_e(M) = s_e(M \otimes_R T)$ for each $e \in \mathbb{N}$, hence $\mathrm{s}(M) = \mathrm{s}(M \otimes_R T)$.*

2.4. Cartier subalgebras and the F-signature

In [8, 9] Blickle, Schwede, and Tucker use the language of Cartier subalgebras to greatly generalize the notion of the F-signature. Their generalization of the F-signature provide the correct framework to answer a question of Aberbach and Enescu, see [2, Question 4.9] and [8, Remark 4.6].

We make the assumption that R is an F-finite ring, not necessarily local. One can make $\mathscr{C}^R = \oplus_{e \in \mathbb{N}} \operatorname{Hom}_R(F^e_* R, R)$ a graded $\mathbb{F}_p$-algebra in a natural way. The 0th graded piece of $\mathscr{C}^R$ is $\operatorname{Hom}_R(R, R) \cong R$. If $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ and $\psi \in \operatorname{Hom}_R(F^{e'}_* R, R)$, then we let $\varphi \bullet \psi = \varphi \circ F^e_* \psi \in \operatorname{Hom}_R(F^{e+e'}_* R, R)$. One should observe that $\mathscr{C}^R$ is non-commutative and that $R \cong \operatorname{Hom}_R(R, R)$ is not central in $\mathscr{C}^R$. If $r \in R$, $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$, and $F^e_* s \in F^e_* R$, then $r \bullet \varphi(F^e_* s) = r\varphi(F^e_* s) = \varphi(rF^e_* s) = \varphi(F^e_* r^{p^e} s) \neq \varphi(F^e_* rs) = \varphi(F^e_* s) \bullet r$.

A Cartier subalgebra $\mathscr{D}$ is a graded $\mathbb{F}_p$-subalgebra of $\mathscr{C}^R$ such that the 0th graded piece of $\mathscr{D}$ is $\operatorname{Hom}_R(R, R)$, which is all of the 0th graded piece of $\mathscr{C}^R$. Let $\mathscr{D}_e$ denote the eth graded piece of $\mathscr{D}$. We refer the reader to [7] for a more thorough introduction to Cartier subalgebras.

Given a Cartier subalgebra $\mathscr{D}$ we call a summand M of $F^e_* R$ a $\mathscr{D}$-summand if $M \cong R^{\oplus n}$ is free and the map $F^e_* R \to M \cong R^{\oplus n}$ is a direct sum of elements of $\mathscr{D}_e$. The assumption that $\mathscr{D}_0 = \operatorname{Hom}_R(R, R)$ implies that the chosen isomorphism of $M \cong R^{\oplus n}$ does not affect whether M is a $\mathscr{D}$-summand or not. If $R = (R, \mathfrak{m}, k)$ is local, then the eth Frobenius splitting number of $(R, \mathscr{D})$ is defined to be the maximal rank of a free $\mathscr{D}$-summand appearing in various direct sum decompositions of $F^e_* R$ and is denoted $a_e(R, \mathscr{D})$. As with the usual Frobenius splitting numbers, one only needs to look at a single direct sum decomposition of

F^e_*R to determine $a_e(R,\mathscr{D})$, see [8, Proposition 3.5]. Observe that if $\mathscr{D}=\mathscr{C}^R$ then $a_e(R,\mathscr{D})=a_e(R)$ is the usual eth Frobenius splitting number of R. To ease notation, we will typically write $\mathrm{s}_e(R,\mathscr{D})$ to represent $a_e(R,\mathscr{D})/p^{e\gamma(R)}$.

Suppose R is an F-finite domain. We define two classes of Cartier subalgebras which arise from geometric considerations, see [18, 17, 40]. Let $0\neq\mathfrak{a}\subseteq R$ be an ideal. For $t\in\mathbb{R}_{\geqslant 0}$, define $\mathscr{C}^{\mathfrak{a}^t}=\bigoplus_{e\geqslant 0}\mathscr{C}_e^{\mathfrak{a}^t}$, where

$$\begin{aligned}\mathscr{C}_e^{\mathfrak{a}^t} &= F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil}\operatorname{Hom}_R(F^e_*R,R)\\ &= \{\phi(F^e_*x\cdot \_\_)\mid F^e_*x\in F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil} \text{ and } \phi\in\operatorname{Hom}_R(F^e_*R,R)\}.\end{aligned}$$

Suppose R is an F-finite normal domain and Δ is an effective $\mathbb{Q}$-divisor on $\operatorname{Spec}(R)$, define $\mathscr{C}^{(R,\Delta)}=\bigoplus_{e\geqslant 0}\mathscr{C}_e^{(R,\Delta)}$, where

$$\begin{aligned}\mathscr{C}_e^{(R,\Delta)} &= \{\phi\in\operatorname{Hom}_R(F^e_*R,R)\mid \Delta_\phi\geqslant\Delta\}\\ &= \operatorname{im}\left(\operatorname{Hom}_R(F^e_*R(\lceil(p^e-1)\Delta\rceil),R)\to\operatorname{Hom}_R(F^e_*R,R)\right).\end{aligned}$$

Given a Cartier subalgebra $\mathscr{D}$, let $\Gamma_{\mathscr{D}}=\{e\in\mathbb{N}\mid\mathscr{D}_e\neq 0\}$. One can easily check that $\Gamma_{\mathscr{D}}$ is a subsemigroup of $\mathbb{N}$. A Cartier subalgebra $\mathscr{D}$, or the pair $(R,\mathscr{D})$, is called strongly F-regular if for every $r\in R$ there is an $e\in\Gamma_{\mathscr{D}}$ and $\varphi\in\mathscr{D}_e$ such that $\varphi(F^e_*r)=1$. Blickle, Schwede, and Tucker prove the following:

Theorem 2.9. *Let $(R,\mathfrak{m},k)$ be an F-finite local domain and let $\mathscr{D}$ be a Cartier subalgebra.*

1. [8, Theorem 3.11] *The following limit exists,*

$$\lim_{e\in\Gamma_{\mathscr{D}}\to\infty}\mathrm{s}_e(R,\mathscr{D}).$$

   Its limit is denoted $\mathrm{s}(R,\mathscr{D})$ and is called the F-signature of $(R,\mathscr{D})$.

2. [8, Theorem 3.18] *We have $\mathrm{s}(R,\mathscr{D})>0$ if and only if $(R,\mathscr{D})$ is strongly F-regular.*

2.5. Basic element results

Unless otherwise stated, the results which we recall in this subsection are characteristic independent. We only require that R is Noetherian of finite Krull dimension d. The first result we recall is a weakening of the Forster-Swan Theorem. We refer the reader to [16] and [39] for the original statements. We also recommend reading [14] and the material surrounding [28, Theorem 5.8] for a historical discussion of the Theorem.

Theorem 2.10 (Forster-Swan Theorem). *If A is a Noetherian ring of finite Krull dimension, e.g., A is of prime characteristic and F-finite, and N is a finitely generated R-module, then $\mu_A(N)\leqslant\max\{\mu_{A_P}(N_P)\mid P\in\operatorname{Supp}(N)\}+\dim(A)$.*

Another result which can be obtained by basic element techniques is Stafford's generalization of Serre's Splitting Theorem, [33, Theorem 1]

Theorem 2.11 ([38]). [3] *Let R be a Noetherian ring of finite Krull dimension d, e.g., R is an F-finite ring of prime characteristic p. Suppose that M is a finitely generated R-module and that for each $P \in \operatorname{Spec}(R)$, M_P contains a free R_P-summand of rank at least $d+1$, then M contains a free summand.*

We now introduce some terminology in order to recall another theorem from [11]. Let R be a commutative Noetherian ring, M a finitely generated R-module. Let $\mathscr{E}$ be a submodule of $\operatorname{Hom}_R(M,R)$. We say that a summand N of M is a free $\mathscr{E}$-summand if $N \cong R^{\oplus n}$ is free, and the projection $\varphi : M \to N \cong R^{\oplus n}$ is a direct sum of elements of $\mathscr{E}$. Observe that that choice of an isomorphism $N \cong R^{\oplus n}$ does not affect whether or not N is an $\mathscr{E}$-summand.

Theorem 2.12 ([11, Theorem C]). *Let R be a commutative Noetherian ring of dimension d, let M a finitely generated R-module, and let $\mathscr{E}$ be an R-submodule of $\operatorname{Hom}_R(M,R)$. Assume that, for each $P \in \operatorname{Spec}(R)$, M_P contains a free $\mathscr{E}_P$-summand of rank at least $d+1$. Then M contains a free $\mathscr{E}$-summand.*

Theorem 2.12 applies to Cartier algebras. The assumption that $\mathscr{D}$ is a Cartier algebra implies that $\mathscr{D}_e \subseteq \operatorname{Hom}_R(F^e_*R, R)$ is an R-submodule, and Theorem 2.12 yields the following.

Theorem 2.13. *Let R be an F-finite domain, $\mathscr{D}$ be a Cartier algebra, and $e \in \Gamma_{\mathscr{D}}$. Suppose that, for all $P \in \operatorname{Spec}(R)$, we have $a_e(R_P, \mathscr{D}_P) \geqslant \dim(R) + m$, where m is a fixed positive integer. Then $a_e(R,\mathscr{D}) \geqslant m$.*

3. Global Hilbert-Kunz multiplicity

Recall that if R is an F-finite ring then we let $\gamma(R) = \max\{\alpha(Q) \mid Q \in \operatorname{Min}(R)\}$. If R is F-finite and M is a finitely generated R-module, define

$$\mathrm{e}_{\mathrm{HK}}(M) = \lim_{e\to\infty} \frac{\mu(F^e_*M)}{p^{e\gamma(R)}}.$$

Theorem 3.8 shows that the limit exists, and we call it the *global Hilbert-Kunz multiplicity of M*. We observe that $\mathrm{e}_{\mathrm{HK}}(M)$ agrees with the usual Hilbert-Kunz multiplicity of M if R is local, see Remark 3.13.

Before continuing, observe that $\mu(F^e_*M)$ can be reinterpreted as the smallest rank of a free module F mapping onto F^e_*M. If R is local then the set of finitely generated free modules agrees with the set of finitely generated projective modules. Thus it is natural to set $\tilde{\mu}(F^e_*M)$ to be the smallest rank of a projective module mapping onto F^e_*M and studying the limit $\lim_{e\to\infty} \tilde{\mu}(F^e_*M)/p^{e\gamma(R)}$. We refer the reader to the proof of Theorem 3.16 to see that Theorem 2.10 shows

$$\mu(F^e_*M) - \dim(R) \leqslant \tilde{\mu}(F^e_*M) \leqslant \mu(F^e_*M)$$

[3] The authors of this paper have recently written a paper providing alternative proofs of Theorem 2.11 in the commutative case, see [11]. Moreover, the results of [11] allows us to establish the existence of a global F-signature of Cartier subalgebra in Theorem 4.19.

and therefore the sequences $\mu(F^e_* M)/p^{e\gamma(R)}$ and $\tilde{\mu}(F^e_* M)/p^{e\gamma(R)}$ converge to the same limit. But first, we wish to establish the existence of global Hilbert-Kunz multiplicity without referencing any of the basic element results found in Section 2.5.

The following Lemma is a global version of an observation made by Dutta in [13]. Lemma 3.1 has shown itself to be useful in positive characteristic commutative algebra. Huneke's survey paper [21] uses a local version of Lemma 3.1 to prove the existence of Hilbert-Kunz multiplicity and the F-signature. Lemma 3.1 is used by the second author in [32] to establish the presence of strong uniform bounds found in all F-finite rings.

Lemma 3.1 ([32, Lemma 2.2]). *Let R be an F-finite domain. Then there exists a finite set of nonzero primes $\mathcal{S}(R)$, and a constant C, such that for every $e \in \mathbb{N}$,*

1. *there is a containment of R-modules $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$,*
2. *which has a prime filtration with prime factors isomorphic to R/Q, where $Q \in \mathcal{S}(R)$,*
3. *and for each $Q \in \mathcal{S}(R)$, the prime factor R/Q appears no more than $Cp^{e\gamma(R)}$ times in the chosen prime filtration of $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$.*

Corollary 3.2. *Let R be an F-finite ring and M a finitely generated R-module. Then $\mu(F^e_* M) = O(p^{e\gamma(M)})$.*

Proof. Counting minimal number of generators is sub-additive on short exact sequences and restricting scalars is exact. Thus by considering a prime filtration of M, we are reduced to showing that if $M = R$ is an F-finite domain, then there is a constant C such that for every $e > 0$, $\mu(F^e_* R) \leqslant Cp^{e\gamma(R)}$.

Suppose that R is an F-finite domain and let $\mathcal{S}(R)$ and C be as in Lemma 3.1. If $\mathcal{S}(R)$ is empty, i.e., for each e we can take the inclusions $R^{\oplus p^{e\gamma(R)}} \subseteq F^e_* R$ to be the surjective as well, then there is nothing to show. For each $e > 0$ let $T_e = F^e_* R/R^{\oplus p^{e\gamma(R)}}$. Then we can find a prime filtration of T_e, whose prime factors are isomorphic to R/Q, where $Q \in \mathcal{S}(R)$, and such a prime factor appears no more than $Cp^{e\gamma(R)}$ times. In particular, T_e has a prime filtration with no more than $C|\mathcal{S}(R)|p^{e\gamma(R)}$ prime factors. By considering the short exact sequence $0 \to R^{\oplus p^{e\gamma(R)}} \to F^e_* R \to T_e \to 0$ and the prime filtration of T_e, we have that $\mu(F^e_* R) \leqslant \mu(R^{\oplus p^{e\gamma(R)}}) + \mu(T_e) = p^{e\gamma(R)} + \mu(T_e) \leqslant (1 + C|\mathcal{S}(R)|)p^{e\gamma(R)}$. □

Lemma 3.3. *Let R be an F-finite ring and let M, N be finitely generated R-modules which are isomorphic at minimal primes of R. Then $\mu(F^e_* M) = \mu(F^e_* N) + O(p^{e(\gamma(R)-1)})$.*

Proof. As M and N are assumed to be isomorphic at minimal primes, we can find right exact sequences $M \to N \to T_1 \to 0$ and $N \to M \to T_2 \to 0$ such that T_1 and T_2 are not supported at any minimal prime of R. Hence for each $e \in \mathbb{N}$ we have right exact sequences $F^e_* M \to F^e_* N \to F^e_* T_1 \to 0$ and $F^e_* N \to F^e_* M \to F^e_* T_2 \to 0$. It follows that

$$|\mu(F^e_* M) - \mu(F^e_* N)| \leqslant \max\{\mu(F^e_* T_1), \mu(F^e_* T_2)\}.$$

Therefore by Corollary 3.2,

$$|\mu(F^e_* M) - \mu(F^e_* N)| = O\left(p^{e(\max\{\gamma(T_1),\gamma(T_2)\})}\right).$$

For $i = 1, 2$, let $I_i = \mathrm{Ann}_R(T_i)$. Then there is a $P_i \in \mathrm{Spec}(R)$ such that $\gamma(T_i) = \alpha(P_i/I_i) + \mathrm{ht}(P_i/I_i)$. Observe that $\alpha(P_i/I_i) + \mathrm{ht}(P_i/I_i) = \alpha(P_i) + \mathrm{ht}(P_i/I_i) < \alpha(P_i) + \mathrm{ht}(P_i) \leqslant \gamma(R)$, which completes the proof of the Lemma. ☐

Remark 3.4. The method of Lemma 3.3 shows something a bit stronger. If we set $\mathrm{Assh}(R) = \{P \in \mathrm{Min}(R) \mid \alpha(P) = \gamma(R)\}$ and assume that M, N are finitely generated R-modules which are isomorphic at the minimal primes in $\mathrm{Assh}(R)$, then $\mu(F^e_* M) = \mu(F^e_* N) + O(p^{e(\gamma(R)-1)})$. Recall that in Section 2 we used $\mathrm{Assh}(R)$ to denote the set of minimal primes Q of a local ring $(R, \mathfrak{m}, k)$ such that $\dim(R/Q) = \dim(R)$. The following Lemma justifies our use of $\mathrm{Assh}(R)$.

Lemma 3.5. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and let P be a minimal prime of R. Then $\alpha(P) = \gamma(R)$ if and only if $\dim(R/P) = \dim(R)$.*

Proof. Observe that if $P \in \mathrm{Min}(R)$, then $\alpha(P) = \alpha(P/P)$ in the local domain R/P. By [26, Proposition 2.3], $\alpha(P) = \dim(R/P) + \alpha(\mathfrak{m}/P) = \dim(R/P) + \alpha(\mathfrak{m})$. This shows a minimal prime $P \in \mathrm{Min}(R)$ in a local ring satisfies $\dim(R) = \dim(R/P)$ if and only if $\alpha(P) = \gamma(R)$. ☐

The following is a Corollary to Lemma 3.3.

Corollary 3.6. *Let R be an F-finite ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of finitely generated R-modules. Then $\mu(F^e_* M) = \mu(F^e_* M' \oplus F^e_* M'') + O(p^{e(\gamma(R)-1)})$.*

Proof. First suppose that R is a reduced ring. Then M is isomorphic to $M' \oplus M''$ at minimal primes of R. Hence by Lemma 3.3, $\mu(F^e_* M) = \mu(F^e_* M' \oplus F^e_* M'') + O(p^{e(\gamma(R)-1)})$.

Now suppose that R is not necessarily reduced and choose e_0 large enough such that $\sqrt{0}^{[p^{e_0}]} = 0$. Denote by S the image of R under the e_0th iterate of Frobenius, $F^{e_0} : R \to R$. Then S is a reduced ring which R is module finite over. Suppose that N is a finitely generated R-module. Observe that the elements $n_1, \ldots, n_\ell$ are a generating set for N as an S-module if and only if $F^{e_0}_* n_1, \ldots, F^{e_0}_* n_\ell$ are a generating set for $F^{e_0}_* N$ as an R-module. It follows that for each $e \in \mathbb{N}$ that $\mu_R(F^{e+e_0}_* N) = \mu_S(F^e_* N)$, which reduces the proof of the Corollary to the reduced case. ☐

One should observe that $\mu(F^e_* M' \oplus F^e_* M'') \leqslant \mu(F^e_* M') + \mu(F^e_* M'')$, but equality does not necessarily hold since R is not assumed to be local. If fact, one can not even hope to prove $\mu(F^e_* M) = \mu(F^e_* M') + \mu(F^e_* M'') + O(p^{e(\gamma(R)-1)})$. If such an inequality held, then one could establish that global Hilbert-Kunz multiplicity was additive on short exact sequences.

Example 3.7. Global Hilbert-Kunz multiplicity is not additive on direct summands, hence not additive on short exact sequences. Let $R = \mathbb{F}_p \times \mathbb{F}_p$. For each $e \in \mathbb{N}$, $F^e_* R \cong R$, hence $\mu(F^e_* R) = 1$ for each $e \in \mathbb{N}$ and $\mathrm{e_{HK}}(R) = 1$. Let $M_1 = \mathbb{F}_p \times 0$ and $M_2 = 0 \times \mathbb{F}_p$, the two direct summands of $\mathbb{F}_p$ of R. Then for each $e \in \mathbb{N}$, $F^e_* M_1 \cong M_1$ and $F^e_* M_2 \cong M_2$. Hence $\mathrm{e_{HK}}(M_1) = 1$ and $\mathrm{e_{HK}}(M_2) = 1$, but $\mathrm{e_{HK}}(M_1 \oplus M_2) \neq 2$. Nevertheless, Corollary 3.17 below shows that global Hilbert-Kunz multiplicity is additive if R is assumed to be a domain.

We now prove the existence of global Hilbert-Kunz multiplicity.

Theorem 3.8. *Let R be an F-finite ring and M a finitely generated R-module. Then the limit $\mathrm{e_{HK}}(M) = \lim\limits_{e\to\infty} \mu(F^e_* M)/p^{e\gamma(R)}$ exists. Moreover, there is a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$, $\mathrm{e_{HK}}(M) \leqslant \dfrac{\mu(F^e_* M)}{p^{e\gamma(R)}} + \dfrac{C}{p^e}$.*

Proof. Suppose that $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$ is a prime filtration of M with $M_i/M_{i-1} \cong R/Q_i$. Repeated use of Corollary 3.6 allows us to reduce proving the existence of global Hilbert-Kunz multiplicity to the scenario that $M \cong R/Q_1 \oplus R/Q_2 \oplus \cdots \oplus R/Q_\ell$, a direct sum of modules of the form R/Q_i where $Q_i \in \mathrm{Spec}(R)$.

Suppose that $\mathrm{Assh}(R)$ is as in Remark 3.4. By rearranging and relabeling as necessary, we may assume that $Q_1, \ldots, Q_i \in \mathrm{Assh}(R)$ and $Q_{i+1}, \ldots, Q_\ell \notin \mathrm{Assh}(R)$. Hence M and $R/Q_1 \oplus R/Q_2 \oplus \cdots \oplus R/Q_i$ are isomorphic when localized at each $Q \in \mathrm{Assh}(R)$. Thus by Remark 3.4, we are further reduced to the scenario $M \cong R/Q_1 \oplus R/Q_2 \oplus \cdots \oplus R/Q_\ell$ where each $Q_i \in \mathrm{Assh}(R)$.

A prime Q is an element of $\mathrm{Assh}(R)$ if and only if $F_*(R/Q)$ has rank $p^{\gamma(R)}$ as an R/Q-module. It follows that there is a right exact sequence

$$M^{\oplus p^{\gamma(R)}} \to F_* M \to T \to 0$$

such that $T_Q = 0$ for each $Q \in \mathrm{Assh}(R)$. As restricting scalars is exact, for each $e \in \mathbb{N}$ there is a right exact sequence

$$F^e_* M^{\oplus p^{\gamma(R)}} \to F^{e+1}_* M \to F^e_* T \to 0.$$

For each $e \in \mathbb{N}$,

$$\mu(F^{e+1}_* M) \leqslant \mu(F^e_* M^{\oplus p^{\gamma(R)}}) + \mu(F^e_* T) \leqslant p^{\gamma(R)} \mu(F^e_* M) + \mu(F^e_* T).$$

As T is not supported at any prime of $\mathrm{Assh}(R)$, by Corollary 3.2 there is a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$, after dividing by $p^{(e+1)\gamma(R)}$,

$$\frac{\mu(F^{e+1}_* M)}{p^{(e+1)\gamma(R)}} \leqslant \frac{\mu(F^e_* M)}{p^{e\gamma(R)}} + \frac{C}{p^e}.$$

The theorem follows by [31, Lemma 3.5] □

Corollary 3.9 (Associativity Formula). *Let R be an F-finite ring and M a finitely generated R-module. Then*

$$\mu(F^e_*M) = \mu\left(\bigoplus_{Q\in \mathrm{Assh}(R)} \bigoplus_{i=1}^{\lambda_{R_Q}(M_Q)} F^e_*(R/Q)\right) + O(p^{e(\gamma(R)-1)}).$$

In particular,

$$\mathrm{e}_{\mathrm{HK}}(M) = \mathrm{e}_{\mathrm{HK}}\left(\bigoplus_{Q\in \mathrm{Assh}(R)} \bigoplus_{i=1}^{\lambda(M_Q)} F^e_*(R/Q)\right).$$

Proof. In the proof of Theorem 3.8 it was observed that

$$\mu(F^e_*M) = \mu(F^e_*(R/Q_1 \oplus R/Q_2 \cdots \oplus R/Q_\ell)) + O(p^{e(\gamma(R)-1)}),$$

where R/Q_i are the various prime factors appearing in a prime filtration of M with $Q_i \in \mathrm{Assh}(R)$. Given a prime $Q \in \mathrm{Assh}(R)$, the number of times R/Q appears as a prime factor in a prime filtration of M is precisely $\lambda_{R_Q}(M_Q)$. $\square$

If $(R, \mathfrak{m}, k)$ is a local F-finite ring and M a finitely generated R-module, then Monsky's original proof of the existence of Hilbert-Kunz multiplicity in [29] showed $\mu(F^e_*M) = \mathrm{e}_{\mathrm{HK}}(M)p^{e\gamma(R)} + O(p^{e(\gamma(R)-1)})$. Equivalently, there is a constant $C \in \mathbb{R}$ such that $|\mu(F^e_*M) - \mathrm{e}_{\mathrm{HK}}(M)p^{e\gamma(R)}| \leqslant Cp^{e(\gamma(R)-1)}$. To extend Monsky's original result to the global case, we first record the following application of Theorem 2.10.

Lemma 3.10. *Let R be a Noetherian ring of finite Krull dimension. Suppose that M is a finitely generated R-module. Then for each $n \in \mathbb{N}$,*

$$|\mu(M^{\oplus n}) - n\mu(M)| \leqslant n \dim(R).$$

Proof. It is easy to see that $\mu(M^{\oplus n}) \leqslant n\mu(M)$. By Theorem 2.10 there is a $P \in \mathrm{Spec}(R)$ such that $\mu(M) \leqslant \mu(M_P)+\dim(R)$. Hence $n\mu_R(M) \leqslant n\mu_{R_P}(M_P)+n\dim(R) = \mu_{R_P}(M_P^{\oplus n})+n\dim(R) \leqslant \mu_R(M^{\oplus n})+n\dim(R)$. $\square$

Theorem 3.11. *Let R be an F-finite ring and M a finitely generated R-module. Then $\mu(F^e_*M) = \mathrm{e}_{\mathrm{HK}}(M)p^{e\gamma(R)}+ O(p^{e(\gamma(R)-1)})$.*

Proof. As in the proof of Theorem 3.8, one can reduce all considerations to the scenario $M \cong R/Q_1 \oplus \cdots R/Q_\ell$ where each $Q_i \in \mathrm{Assh}(R)$. Hence there is a right exact sequence of the form

$$F_*M \to M^{\oplus p^{\gamma(R)}} \to T \to 0$$

such that $T_Q = 0$ for each $Q \in \mathrm{Assh}(R)$. For each $e \in \mathbb{N}$,

$$\mu(F^e_*M^{\oplus p^{\gamma(R)}}) \leqslant \mu(F^{e+1}_*M) + \mu(F^e_*T).$$

By Lemma 3.10,

$$p^{\gamma(R)}\mu(F_*^e M) \leqslant \mu(F_*^{e+1}M) + \mu(F_*^e T) + p^{\gamma(R)}\dim(R).$$

By Corollary 3.2 there is a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$, $\mu(F_*^e T) \leqslant Cp^{e(\gamma(R)-1)}$. Dividing by $p^{(e+1)\gamma(R)}$ and applying a crude estimate shows

$$\frac{\mu(F_*^e M)}{p^{e\gamma(R)}} \leqslant \frac{\mu(F_*^{e+1}M)}{p^{(e+1)\gamma(R)}} + \frac{C+\dim(R)}{p^e}.$$

The theorem follows from Theorem 3.8 and [31, Lemma 3.5]. $\square$

Corollary 3.12. *Let R be an F-finite ring and M a finitely generated R-module. Then the limit $\overline{\mathrm{e}_{\mathrm{HK}}}(M) = \lim\limits_{e\to\infty} \mu(F_*^e M)/p^{e\gamma(M)}$ exists and $\overline{\mathrm{e}_{\mathrm{HK}}}(M) \geqslant 1$. Moreover, $\mu(F_*^e M) = \overline{\mathrm{e}_{\mathrm{HK}}}(M)p^{e\gamma(M)} + O(p^{e(\gamma(M)-1)})$.*

Proof. For existence, apply Theorems 3.8 and 3.11 to the module M, but viewed as an $R/\operatorname{Ann}_R(M)$-module. To see that $\overline{\mathrm{e}_{\mathrm{HK}}}(M) \geqslant 1$, one may assume that $\gamma(M) = \gamma(R)$ and show $\mathrm{e}_{\mathrm{HK}}(M) \geqslant 1$. The assumption $\gamma(M) = \gamma(R)$ is equivalent to $M_Q \neq 0$ for some $Q \in \operatorname{Assh}(R)$. Then $\mu_R(F_*^e M) \geqslant \mu_{R_Q}(F_*^e M_Q) = \lambda_{R_Q}(M_Q/Q^{[p^e]}M_Q)p^{e\alpha(Q)} \geqslant \lambda_{R_Q}(M_Q)p^{e\gamma(R)}$. Divide by $p^{e\gamma(R)}$ and let $e \to \infty$ to see $\mathrm{e}_{\mathrm{HK}}(M) \geqslant \lambda(M_Q) \geqslant 1$. $\square$

Remark 3.13. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring and let M be a finitely generated R-module. Then for each $e \in \mathbb{N}$, $\mu(F_*^e M)/p^{e\gamma(R)} = \lambda(M/\mathfrak{m}^{[p^e]}M)/\mathfrak{p}^{e\dim(R)}$, see Lemma 3.5. In particular, the global Hilbert-Kunz multiplicity of M is the same as the one defined in the local setting.

Suppose that R is an F-finite domain. Then for each $P \in \operatorname{Spec}(R)$, $\operatorname{rank}_R(F_*^e R) = \operatorname{rank}_{R_P}(F_*^e R_P)$. It follows that $\mu_R(F_*^e R)/\operatorname{rank}_R(F_*^e R) \geqslant \mu_{R_P}(F_*^e R_P)/\operatorname{rank}_{R_P}(F_*^e R_P)$ and therefore $\mathrm{e}_{\mathrm{HK}}(R) \geqslant \mathrm{e}_{\mathrm{HK}}(R_P)$. Theorem 3.16 below shows that under such hypotheses, $\mathrm{e}_{\mathrm{HK}}(R) = \max\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in \operatorname{Spec}(R)\}$. It will not always be the case that global Hilbert-Kunz multiplicity is an upper bound of $\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in \operatorname{Spec}(R)\}$, see Example 3.18 below. To better describe the scenario in all F-finite rings, let

$$Z_R = \{P \in \operatorname{Spec}(R) \mid \operatorname{ht}(P) + \alpha(P) = \gamma(R)\}.$$

Observe that if R is an F-finite domain, then $Z_R = \operatorname{Spec}(R)$. More generally, $P \in Z_R$ if and only if there is some $Q \in \operatorname{Min}(R)$ such that $\gamma(R_Q) = \gamma(R)$ and $Q \subseteq P$ if and only if there is some $Q \in \operatorname{Assh}(R)$ such that $Q \subseteq P$. Therefore $Z_R = \cup_{Q\in\operatorname{Assh}(R)} V(Q)$ is a closed set.

The following theorem is a generalization of Smirnov's theorem that Hilbert-Kunz multiplicity is upper semi-continuous on the spectrum of rings which are locally equidimensional, [36, Main Result].

Theorem 3.14. *Let R be an F-finite ring and M a finitely generated R-modules. For each $e \in \mathbb{N}$ the function $\mu_e : Z_R \to \mathbb{R}$ sending $P \mapsto \mu_{R_P}(F_*^e M_P)/p^{e\gamma(R_P)}$ is upper semi-continuous. Moreover, the functions μ_e converge uniformly to their limit, namely $\mathrm{e}_{\mathrm{HK}} : Z_R \to \mathbb{R}$ sending $P \mapsto \mathrm{e}_{\mathrm{HK}}(M_P)$. In particular, the function $\mathrm{e}_{\mathrm{HK}} : Z_R \to \mathbb{R}$ is upper semi-continuous and $\sup\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\} = \max\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\}$.*

Proof. For each $e \in \mathbb{N}$, the function $\tilde{\mu}_e : \mathrm{Spec}(R) \to \mathbb{N}$ sending $P \mapsto \mu_{R_P}(F^e_* M_P)$ is easily seen to be upper semi-continuous on all of $\mathrm{Spec}(R)$. For each $P \in Z_R$, $\gamma(R_P) = \gamma(R)$. Hence μ_e is upper semi-continuous on Z_R. As $\mu_{R_P}(F^e_* M_P)/p^{e\gamma(R_P)} = \lambda_{R_P}(M_P/P^{[p^e]}M_P)/p^{e\,\mathrm{ht}(P)}$, the uniform convergence of μ_e follows from [32, Theorem 5.1] ☐

Our next theorem relates the global Hilbert-Kunz multiplicity of an F-finite ring with the Hilbert-Kunz multiplicities of various localizations of R. We first need a lemma.

Lemma 3.15. *Let R be an F-finite ring. Suppose M is a finitely generated R-module such that $\gamma(M) = \gamma(R)$. There exists $e_0 \in \mathbb{N}$ such that for all $e \geqslant e_0$, $\emptyset \neq \{P \in \mathrm{Spec}(R) \mid \mu(F^e_* M) \leqslant \mu(F^e_* M_P) + \dim(R)\} \subseteq Z_R$. In particular, $\{P \mid \mu(F^e_* M_P) = \max\{\mu(F^e_* M_Q)\}\} \subseteq Z_R$ for all $e \geqslant e_0$.*

Proof. Suppose that M is a finitely generated R-module such that $\gamma(M) = \gamma(R)$. Then $\mathrm{e_{HK}}(M) \geqslant 1$ by Corollary 3.12. By Theorem 2.10, for each $e \in \mathbb{N}$ there exists $P_e \in \mathrm{Spec}(R)$ such that $\mu_R(F^e_* M) \leqslant \mu_{R_{P_e}}(F^e_* M_{P_e}) + \dim(R)$. By [32, Proposition 3.3] there is a constant $C \in \mathbb{R}$ such that for each $P \in \mathrm{Spec}(R)$, $\lambda_{R_P}(M_P/P^{[p^e]}M_P) \leqslant Cp^{e\,\mathrm{ht}(P)}$. Equivalently, there is a constant $C \in \mathbb{R}$ such that for each $P \in \mathrm{Spec}(R)$, $\mu_{R_P}(F^e_* M_P) \leqslant Cp^{e(\mathrm{ht}(P)+\alpha(P))}$. Suppose there existed an infinite subset $\Gamma \subseteq \mathbb{N}$ such that for each $e \in \Gamma$ the prime P_e could be chosen such that $P_e \notin Z_R$. Then for each $e \in \Gamma$,

$$\mu_R(F^e_* M) \leqslant \mu_{R_{P_e}}(F^e_* M_{P_e}) + \dim(R) \leqslant Cp^{e(\mathrm{ht}(P)+\alpha(P))} + \dim(R) \leqslant Cp^{e(\gamma(R)-1)} + \dim(R).$$

Dividing by $p^{e\gamma(R)}$ and letting $e \in \Gamma \to \infty$ shows $\mathrm{e_{HK}}(M) = 0$. ☐

Theorem 3.16. *Let R be an F-finite ring and let M be a finitely generated R-module. Then the following limits exist.*

1. $\mathrm{e_{HK}}(M) = \lim\limits_{e\to\infty} \mu_R(F^e_* M)/p^{e\gamma(R)}$,
2. $\lim\limits_{e\to\infty} \tilde{\mu}(F^e_* M)/p^{e\gamma(R)}$,
3. $\lim\limits_{e\to\infty} \lambda_{R_{Q_e}}(M_{Q_e}/Q_e^{[p^e]}M_{Q_e})/p^{e\,\mathrm{ht}(Q_e)}$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\mu_{R_{Q_e}}(F^e_* M_{Q_e}) = \max\{\mu_{R_P}(F^e_* M_P) \mid P \in \mathrm{Spec}(R)\},$$
4. $\lim\limits_{e\to\infty} \mathrm{e_{HK}}(M_{Q_e})$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\mu(F^e_* M_{Q_e}) = \max\{\mu_{R_P}(F^e_* M_P) \mid P \in \mathrm{Spec}(R)\}.$$

The above limits agree, with the common value being $\max\{\mathrm{e_{HK}}(M_P) \mid P \in Z_R\}$.

Proof. We begin by showing that the limits in 1, 3, and 4 agree. It is clear that $\mu_R(F^e_* M) \geqslant \mu_{R_P}(F^e_* M_P)$ for all $P \in \mathrm{Spec}(R)$. So for every $P \in Z_R$,

$$\frac{\mu(F^e_* M)}{p^{e\gamma(R)}} \geqslant \frac{\mu_{R_P}(F^e_* M_P)}{p^{e\gamma(R)}} = \frac{\lambda(M_P/P^{[p^e]}M_P)}{p^{e(\gamma(R)-\alpha(P))}} = \frac{\lambda(M_P/P^{[p^e]}M_P)}{p^{e\,\mathrm{ht}(P)}}.$$

Letting $e \to \infty$ we see that $\mathrm{e}_{\mathrm{HK}}(M) \geqslant \mathrm{e}_{\mathrm{HK}}(M_P)$ for every $P \in Z_R$. This shows that $\mathrm{e}_{\mathrm{HK}}(M) \geqslant \sup\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\}$ which is equal to $\max\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\}$ by Theorem 3.14.

By [32, Theorem 5.1] and Theorem 3.8, if $\epsilon > 0$ then for $e \gg 0$,

1. $\left|\dfrac{\lambda_{R_P}(M_P/P^{[p^e]}M_P)}{p^{e\,\mathrm{ht}(P)}} - \mathrm{e}_{\mathrm{HK}}(M_P)\right| < \epsilon/3$ for all $P \in \mathrm{Spec}(R)$,
2. $\left|\dfrac{\mu_R(F^e_*M)}{p^{e\gamma(R)}} - \mathrm{e}_{\mathrm{HK}}(M)\right| < \epsilon/3$, and
3. $\dfrac{\dim(R)}{p^{e\gamma(R)}} < \epsilon/3$.

For each $e > 0$ let $Q_e \in \mathrm{Spec}(R)$ be such that $\max\{\mu_{R_P}(F^e_*M_P) \mid P \in \mathrm{Spec}(R)\} = \mu_{R_{Q_e}}(F^e_*M_{Q_e})$. By Theorem 2.10 $\mu_R(F^e_*M) \leqslant \mu_{R_{Q_e}}(F^e_*M_{Q_e}) + \dim(R)$ and by Lemma 3.15, the prime $Q_e \in Z_R$ for all $e \gg 0$.

Therefore if e is suitably large,

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}(M_{Q_e}) \leqslant \mathrm{e}_{\mathrm{HK}}(M) &< \frac{\mu_R(F^e_*M)}{p^{e\gamma(R)}} + \epsilon/3 \leqslant \frac{\mu_{R_{Q_e}}(F^e_*M_{Q_e})}{p^{e\gamma(R)}} + \frac{\dim(R)}{p^{e\gamma(R)}} + \epsilon/3 \\ &< \frac{\mu_{R_{Q_e}}(F^e_*M_{Q_e})}{p^{e\gamma(R)}} + \epsilon/3 + \epsilon/3 = \frac{\lambda_{R_{Q_e}}(M_{Q_e}/Q_e^{[p^e]}M_{Q_e})}{p^{e\,\mathrm{ht}(P)}} + \epsilon/3 + \epsilon/3 \\ &< \mathrm{e}_{\mathrm{HK}}(M_{Q_e}) + \epsilon/3 + \epsilon/3 + \epsilon/3 = \mathrm{e}_{\mathrm{HK}}(M_{Q_e}) + \epsilon.\end{aligned}$$

Thus $\mathrm{e}_{\mathrm{HK}}(M) \leqslant \max\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\}$ and we must have equality. Furthermore, the above chain of inequalities shows that the limits in 3 and 4 of the statement of the theorem exist and both converge to $\mathrm{e}_{\mathrm{HK}}(M)$.

It remains to show the limits in 1 and 2 agree. Recall that $\tilde{\mu}(F^e_*M)$ is the smallest rank of a projective module mapping onto F^e_*M. Since every free module is projective, $\tilde{\mu}(F^e_*M) \leqslant \mu(F^e_*M)$. By Theorem 2.10 there is $Q \in \mathrm{Spec}(R)$ such that $F^e_*M_Q$ is generated by at most $\mu(F^e_*M) - \dim(R)$ elements. If P is a projective module mapping onto F^e_*M then $\mathrm{rank}(P) \geqslant \mathrm{rank}(P_Q) \geqslant \mu(F^e_*M_Q) \geqslant \mu(F^e_*M) - \dim(R)$. Therefore $\mu(F^e_*M) - \dim(R) \leqslant \tilde{\mu}(F^e_*M) \leqslant \mu(F^e_*M)$ and the limits in 1 and 2 agree. □

Corollary 3.17. *Let R be an F-finite ring such that $Z_R = V(Q)$ for some prime ideal Q, e.g., R is a domain. Then global Hilbert-Kunz multiplicity is additive on short exact sequences. Furthermore, if M is any finitely generated R-module, then $\mathrm{e}_{\mathrm{HK}}(M) = \lambda_{R_Q}(M_Q)\,\mathrm{e}_{\mathrm{HK}}(R/Q)$.*

Proof. Let $\ell = \lambda_{R_Q}(M_Q)$. It is enough to show that $\mathrm{e}_{\mathrm{HK}}(M) = \ell\,\mathrm{e}_{\mathrm{HK}}(R/Q)$. Corollary 3.9 shows that $\mathrm{e}_{\mathrm{HK}}(M) = \mathrm{e}_{\mathrm{HK}}(\bigoplus^{\ell} R/Q)$. We can now use Theorem 3.16 to conclude that

$$\mathrm{e}_{\mathrm{HK}}(M) = \max\{\mathrm{e}_{\mathrm{HK}}(M_P) \mid P \in Z_R\} = \ell \max\{\mathrm{e}_{\mathrm{HK}}(R_P/QR_P) \mid P \in Z_R\} = \ell\,\mathrm{e}_{\mathrm{HK}}(R/Q).$$

The proof is complete. □

Example 3.18. If $Z_R \neq \mathrm{Spec}(R)$, then global Hilbert-Kunz multiplicity is not an upper bound of $\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in \mathrm{Spec}(R)\}$. Let K be an F-finite field and $(T, \mathfrak{m})$ a local F-finite domain such that $\gamma(K) > \gamma(T)$ and

$\mathrm{e_{HK}}(T) > 1$ and let $R = K \times T$. Then Z_R consists of the single prime $0 \times T$, hence by Theorem 3.16 $\mathrm{e_{HK}}(R) = 1 < \mathrm{e_{HK}}(R_{K\times\mathfrak{m}}) = \mathrm{e_{HK}}(T)$.

We now provide the global analogue of Theorem 2.1. We remark that F-finite domains satisfy the hypotheses Lemma 3.19 and Theorem 3.20.

Lemma 3.19. *Let R be an F-finite ring such that $Z_R = \operatorname{Spec}(R)$ and such that every associated prime of R is minimal. Then for each $P \in \operatorname{Spec}(R)$, R_P is formally unmixed.*

Proof. The assumption that $Z_R = \operatorname{Spec}(R)$ implies $Z_{R_P} = \operatorname{Spec}(R_P)$ for each $P \in \operatorname{Spec}(R)$. Hence R is locally equidimensional by Lemma 3.5. By Ratliff, the completion of an excellent equidimensional local ring is equidimensional, see [23, Corollary B.4.3 and Theorem B.5.1]. As R is excellent, $R_P \to \widehat{R_P}$ has regular fibers by [27, Section 33, Lemma 4]. In particular, all associated primes of $\widehat{R_P}$ are minimal, completing the proof of the Lemma. □

Theorem 3.20. *Let R be an F-finite ring such that $Z_R = \operatorname{Spec}(R)$ and such that every associated prime of R is minimal.*

1. *Then R is regular if and only if $\mathrm{e_{HK}}(R) = 1$.*
2. *Let $e = \max\{e(R_P) \mid P \in \operatorname{Spec}(R)\}$, where $e(R_P)$ is the Hilbert-Samuel multiplicity of the local ring R_P. If $\mathrm{e_{HK}}(R) \leqslant 1 + \max\{1/\dim(R)!, 1/e\}$, then R is strongly F-regular and Gorenstein.*
3. *There is a number $\delta > 0$, depending only on the dimension of R such that if $\mathrm{e_{HK}}(R) \leqslant 1 + \delta$, then R is regular.*

Proof. By Theorem 3.16, $\mathrm{e_{HK}}(R) = \max\{\mathrm{e_{HK}}(R_P) \mid P \in \operatorname{Spec}(R)\}$. Hence $\mathrm{e_{HK}}(R) = 1$ if and only $\mathrm{e_{HK}}(R_P) = 1$ for each $P \in \operatorname{Spec}(R)$ if and only if R_P is a regular local ring for each $P \in \operatorname{Spec}(R)$ by Lemma 3.19 and part 1 of Theorem 2.1 if and only if R is regular, this proves 1.

For 2, apply Lemma 3.19 and part 2 of Theorem 2.1 to know that for each $P \in \operatorname{Spec}(R)$ that R_P is strongly F-regular and Gorenstein, hence R is strongly F-regular and Gorenstein. The proof of 3 is parallel to that 2. One only needs to reference part 3 of Theorem 2.1 instead of part 2 of Theorem 2.1. Let $\delta(i)$ be a number as in part 3 of Theorem 2.1, that works for rings of dimension i, and let $\delta = \min\{\delta(i) \mid i \leqslant d\}$. □

4. Global F-signature

Let R be an F-finite ring and M a finitely generated R-module. Consider the following sequences of numbers.

1. Let $a_e(M) = \operatorname{frk}(F^e_* M)$ be the largest rank of a free summand appearing in the various direct sum decompositions of $F^e_* M$. Then $a_e(M) \leqslant \operatorname{rank}(F^e_* M) \leqslant \operatorname{rank}(M) p^{e\gamma(R)} = O(p^{e\gamma(R)})$.

2. Let $\tilde{a}_e(M)$ be the largest min-rank of a projective summand appearing in various direct sum decompositions of F^e_*M. Then $a_e(M) \leqslant \tilde{a}_e(M) \leqslant \operatorname{rank}(F^e_*M) \leqslant \operatorname{rank}(M)p^{e\gamma(R)} = O(p^{e\gamma(R)})$.

Remark 4.1. If R is local, then $a_e(M) = \tilde{a}_e(M)$ and $\mathrm{s}(M) = \lim_{e\to\infty} a_e(M)/p^{e\gamma(R)}$. If R is non-local, then Serre's Splitting Theorem, Theorem 2.11, shows that for each $e > 0$ we have $a_e(M) \leqslant \tilde{a}_e(M) \leqslant a_e(M) + d$. Hence the limit $\lim_{e\to\infty} a_e(M)/p^{e\gamma(R)}$ exists if and only if the limit $\lim_{e\to\infty} \tilde{a}_e(M)/p^{e\gamma(R)}$ exists. Moreover, if the limits do exist then their limits are equal.

We also remark that $a_e(M)$ can be equivalently defined as the largest rank of free module F for which there exists an onto map $M \to F$. This definition has the benefit that it does not require one to consider all direct sum decompositions of F^e_*M to compute $a_e(M)$ and therefore $\mathrm{s}(M)$. However, the validity of considering various direct sum decompositions is established in Theorem 4.13.

If R is F-finite, not necessarily local, and M is a finitely generated R-module, we define

$$\mathrm{s}(M) = \lim_{e\to\infty} \frac{a_e(M)}{p^{e\gamma(R)}}.$$

We show in Theorem 4.7 that the limit $\mathrm{s}(M)$ exists, and we call it the *global F-signature of* M. Note that, when R is local, this is the usual definition of F-signature of a module M.

Remark 4.2. Suppose that R is an F-finite ring. The existence of a finitely generated R-module M and $e > 0$ such that $a_e(M) > 0$ implies $a_e(R) > 0$. In particular, R is reduced. Recall the notation $Z_R = \{P \in \operatorname{Spec}(R) \mid \alpha(P) + \operatorname{ht}(P) = \gamma(R)\}$ from Section 3. Observe that, if $Z_R \neq \operatorname{Spec}(R)$, then for any finitely generated R-module M and any $P \notin Z_R$ we have $a_e(M) \leqslant a_e(M_P) \leqslant O(p^{e(\gamma(R)-1)})$. It follows that, in this case, $\mathrm{s}(M) = 0$ for any finitely generated R-module M. These observations allow us to reduce our considerations to the scenario that R is reduced and $Z_R = \operatorname{Spec}(R)$. In particular, $\operatorname{Assh}(R) = \operatorname{Min}(R)$.

Suppose that R is an F-finite reduced ring, and let M be a finitely generated R-module. For each $P \in \operatorname{Spec}(R)$, we see $a_e(M) \leqslant a_e(M_P)$. Now further assume that $Z_R = \operatorname{Spec}(R)$. Then if there exists $P \in \operatorname{Spec}(R)$ such that $\mathrm{s}(M_P) = 0$ then $\mathrm{s}(M)$ exists and is equal to 0. If this does not hold, then R is strongly F-regular and hence a direct product of integral domains by part 4 of Theorem 2.5. We therefore reduce many considerations in this section to the case that R is a domain.

Lemma 4.3. *Let R be a Noetherian ring of finite Krull dimension. Let $M' \to M \to M'' \to 0$ be a right exact sequence of finitely generated R-modules. Then* $\operatorname{frk}_R(M) \leqslant \operatorname{frk}_R(M') + \mu_R(M'') + \dim(R)$.

Proof. For each $P \in \operatorname{Spec}(R)$, $\operatorname{frk}_{R_P}(M_P) \leqslant \operatorname{frk}_{R_P}(M'_P) + \mu_{R_P}(M''_P)$, see [31, Lemma 2.1]. In particular, $\operatorname{frk}_R(M) \leqslant \operatorname{frk}_{R_P}(M'_P) + \mu_R(M'')$ for each $P \in \operatorname{Spec}(R)$. By Theorem 2.11 there is a prime $P \in \operatorname{Spec}(R)$ such that $\operatorname{frk}_{R_P}(M'_P) \leqslant \operatorname{frk}_R(M') + \dim(R)$. Therefore $\operatorname{frk}_R(M) \leqslant \operatorname{frk}_R(M') + \mu_R(M'') + \dim(R)$. ☐

Lemma 4.4. *Let R be an F-finite ring and let M, N be two finitely generated R-modules isomorphic at each prime $P \in \operatorname{Assh}(R)$. Then $a_e(M) = a_e(N) + O(p^{e(\gamma(R)-1)})$.*

Proof. There are two right exact sequences $M \to N \to T_1 \to 0$ and $N \to M \to T_2 \to 0$ such that $(T_1)_P = (T_2)_P = 0$ for each $P \in \operatorname{Assh}(R)$. By Lemma 4.3

$$|a_e(M) - a_e(N)| \leqslant \max\{\mu(F^e_* T_1) + \dim(R), \mu(F^e_* T_2) + \dim(R)\}.$$

The result follows from Lemma 3.2. □

Corollary 4.5. *Let R be an F-finite ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of finitely generated R-modules. Then $a_e(M) = a_e(M' \oplus M'') + O(p^{e(\gamma(R)-1)})$.*

Proof. Without loss of generality, one may assume that R is reduced. In particular, M is isomorphic to $M' \oplus M''$ at all $P \in \operatorname{Assh}(R)$, and the result follows by Lemma 4.4. □

Example 4.6. As with global Hilbert-Kunz, one cannot expect $\mathrm{s}(M_1 \oplus M_2) = \mathrm{s}(M_1) + \mathrm{s}(M_2)$. Let $R = \mathbb{F}_p \times \mathbb{F}_p$, $M_1 = \mathbb{F}_p \times 0$, and $M_2 = 0 \times \mathbb{F}_p$. Observe that $\gamma(R) = 0$. Hence $\mathrm{s}(M_1) = \mathrm{s}(M_2) = 0$ whereas $\mathrm{s}(R) = 1$.

Theorem 4.7. *Let R be an F-finite ring and M a finitely generated R-module. Then the limit $\mathrm{s}(M) = \lim_{e\to\infty} a_e(M)/p^{e\gamma(R)}$ exists. Moreover, there exists a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$, $a_e(M) \leqslant \mathrm{s}(M)p^{e\gamma(R)} + Cp^{e(\gamma(R)-1)}$.*

Proof. Without loss of generality, one may assume that R is reduced and $\alpha(P) + \operatorname{ht}(P) = \gamma(R)$ for each $P \in \operatorname{Spec}(R)$. By considering a prime filtration of M, repeated use of Corollary 4.5 allows one to reduce all considerations to the scenario that $M \cong R/Q_1 \oplus \cdots \oplus R/Q_\ell$ where $Q_i \in \operatorname{Min}(R)$. As R is a reduced and $\alpha(P) + \operatorname{ht}(P) = \gamma(R)$ for each $P \in \operatorname{Spec}(R)$, there is a short exact sequence

$$0 \to F_* M \to M^{\oplus p^{\gamma(R)}} \to T \to 0$$

so that $T_Q = 0$ for each $Q \in \operatorname{Min}(R)$. For each $e \in \mathbb{N}$

$$a_e(M^{\oplus p^{\gamma(R)}}) \leqslant a_{e+1}(M) + \mu(F^e_* T) + \dim(R)$$

by Lemma 4.3. By Corollary 3.2 there is a constant $C \in \mathbb{R}$ such that $\mu(F^e_* T) \leqslant Cp^{e(\gamma(R)-1)}$. Note $p^{\gamma(R)} a_e(M) \leqslant a_e(M^{\oplus p^{\gamma(R)}})$, dividing the above inequality by $p^{(e+1)\gamma(R)}$ and applying crude estimates,

$$\frac{a_e(M)}{p^{e\gamma(R)}} \leqslant \frac{a_{e+1}(M)}{p^{(e+1)\gamma(R)}} + \frac{C + \dim(R)}{p^e}.$$

The theorem follows from [31, Lemma 3.5]. □

Lemma 4.8. *Let R be a Noetherian ring of finite Krull dimension, of any characteristic. Suppose that M is a finitely generated R-module. Then for each $n \in \mathbb{N}$,*

$$|\operatorname{frk}(M^{\oplus n}) - n\operatorname{frk}(M)| \leqslant n\dim(R).$$

Proof. It is clear that $n\operatorname{frk}(M) \leqslant \operatorname{frk}(M^{\oplus n})$. By Theorem 2.11 there exists a $P \in \operatorname{Spec}(R)$ such that $\operatorname{frk}(M_P) \leqslant \operatorname{frk}(M) + \dim(R)$. Hence $\operatorname{frk}(M^{\oplus n}) \leqslant \operatorname{frk}_{R_P}(M_P^{\oplus n}) = n\operatorname{frk}(M_P) \leqslant n\operatorname{frk}(M) + n\dim(R)$. □

Theorem 4.9. *Let R be an F-finite ring and M a finitely generated R-module. Then $a_e(M) = \mathrm{s}(M)p^{e\gamma(R)} + O(p^{e(\gamma(R)-1)})$.*

Proof. Without loss of generality, we may assume that R is reduced and $\alpha(P) + \operatorname{ht}(P) = \gamma(R)$ for each $P \in \operatorname{Spec}(R)$. As in the proof of Theorem 4.7, we may assume $M \cong R/Q_1 \oplus \cdots \oplus R/Q_\ell$ where $Q_i \in \operatorname{Min}(R)$. In this case, there is a short exact sequence

$$0 \to M^{\oplus p^{\gamma(R)}} \to F_*M \to T \to 0$$

so that $T_P = 0$ for each $P \in \operatorname{Min}(R)$. For each $e \in \mathbb{N}$

$$a_{e+1}(M) \leqslant a_e(M^{\oplus p^{\gamma(R)}}) + \mu(F_*^e T) + \dim(R)$$

by Lemma 4.3. By Corollary 3.2 there is a constant $C \in \mathbb{R}$ such that for each $e \in \mathbb{N}$ $\mu(F_*^e T) \leqslant Cp^{e(\gamma(R)-1)}$. Hence by Lemma 4.8,

$$a_{e+1}(M) \leqslant p^{\gamma(R)}a_e(M) + Cp^{e(\gamma(R)-1)} + p^{\gamma(R)}\dim(R) + \dim(R).$$

Dividing by $p^{(e+1)\gamma(R)}$ and applying a crude estimate shows

$$\frac{a_{e+1}(M)}{p^{(e+1)\gamma(R)}} \leqslant \frac{a_e(M)}{p^{e\gamma(R)}} + \frac{C + 2\dim(R)}{p^e}.$$

The theorem follows from Theorem 4.7 and [31, Lemma 3.5]. □

Lemma 4.10. *Let R be an F-finite ring of dimension d and M a finitely generated R-module. For each $e \in \mathbb{N}$ choose a decomposition $F_*^e M \cong R^{\oplus n_e} \oplus M_e$ such that M_e does not have a free summand. There exists $Q \in \operatorname{Spec}(R)$ such that $\operatorname{frk}(F_*^e M_Q) \leqslant n_e + d$.*

Proof. By Theorem 2.11 there is a $Q \in \operatorname{Spec}(R)$ such that $\operatorname{frk}((M_e)_Q) \leqslant d$. The desired result now follows since $\operatorname{frk}(F_*^e M_Q) = n_e + \operatorname{frk}((M_e)_Q)$. □

Lemma 4.11. *Let R be an F-finite ring of dimension d and let M be a finitely generated R-module. For each $e \in \mathbb{N}$ choose a decomposition $F_*^e M \cong \Omega_e \oplus M_e$ such that Ω_e is projective of min-rank m_e and M_e does not have a projective summand. Then there exists $Q \in \operatorname{Spec}(R)$ such that $\operatorname{frk}(F_*^e M_Q) \leqslant m_e + \dim(R)$.*

Proof. By Theorem 2.11 there is a $Q \in \operatorname{Spec}(R)$ such that $\operatorname{frk}((M_e)_Q) \leqslant d$, else M_e has a free, and hence projective, summand. The desired result now follows as in Lemma 4.10. □

Lemma 4.12. *Let R be an F-finite ring such that $Z_R = \operatorname{Spec}(R)$ and M a finitely generated R-module. For each $e > 0$, let $Q_e \in \operatorname{Spec}(R)$ be such that $a_e(M_{Q_e}) = \min\{a_e(M_P) \mid P \in \operatorname{Spec}(R)\}$. Then both $\frac{a_e(M_{Q_e})}{p^{e\gamma(R)}}$ and $\mathrm{s}(M_{Q_e})$ converge to $\min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$ as $e \to \infty$.*

Proof. Since $Z_R = \operatorname{Spec}(R)$, the F-signature function $\operatorname{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \mathrm{s}(R_P)$ is lower semi-continuous by [32, Theorem 6.4]. Therefore there exists $Q \in \operatorname{Spec}(R)$ such that $\mathrm{s}(M_Q) = \inf\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$. By [32, Theorem 5.6] the functions $\mathrm{s}_e : \operatorname{Spec}(R) \to \mathbb{R}$ sending a prime $P \mapsto \mathrm{s}_e(M_P) = a_e(M_P)/p^{e\gamma(R)}$ converge uniformly to their limit, namely $\mathrm{s} : \operatorname{Spec}(R) \to \mathbb{R}$ sending a prime $P \mapsto \mathrm{s}(M_P)$, the F-signature of M_P. Let $\epsilon > 0$ and $e_0 \gg 0$ such that for $e \geqslant e_0$, $|\mathrm{s}_e(M_P) - \mathrm{s}(M_P)| < \epsilon/2$ for every $P \in \operatorname{Spec}(R)$. Then for $e \geqslant e_0$

$$\mathrm{s}(M_Q) \leqslant \mathrm{s}(M_{Q_e}) < \mathrm{s}_e(M_{Q_e}) + \epsilon/2 \leqslant \mathrm{s}_e(M_Q) + \epsilon/2 < \mathrm{s}(M_Q) + \epsilon/2 + \epsilon/2 = \mathrm{s}(M_Q) + \epsilon.$$

The lemma now follows. □

Theorem 4.13. *Let R be an F-finite ring such that $Z_R = \operatorname{Spec}(R)$, and M a finitely generated R-module. Then the following limits exist:*

1. $\mathrm{s}(M) = \lim_{e\to\infty} \dfrac{a_e(M)}{p^{e\gamma(R)}}$,
2. $\mathrm{s}(M) = \lim_{e\to\infty} \dfrac{\tilde{a}_e(M)}{p^{e\gamma(R)}}$, *where $\tilde{a}_e(M)$ is the largest min-rank of a projective summand appearing in various direct sum decompositions of $F^e_* M$,*
3. $\lim_{e\to\infty} \dfrac{n_e}{p^{e\gamma(R)}}$, *where n_e is the rank of a free direct summand of $F^e_* M$ appearing in a choice of decomposition $F^e_* M \cong R^{\oplus n_e} \oplus M_e$, where M_e has no free summand,*
4. $\lim_{e\to\infty} \dfrac{m_e}{\operatorname{rank}(F^e_* R)}$, *where m_e is the min-rank of a project summand Ω_e of $F^e_* M$ appearing in a choice of decomposition $F^e_* M \cong \Omega_e \oplus M_e$, where M_e has no projective summand,*
5. $\lim_{e\to\infty} \dfrac{a_e(M_{Q_e})}{p^{e\gamma(R)}}$, *where $Q_e \in \operatorname{Spec}(R)$ is chosen such that*
$$a_e(M_{Q_e}) = \min\{a_e(M_P) \mid P \in \operatorname{Spec}(R)\},$$
6. $\lim_{e\to\infty} \mathrm{s}(M_{Q_e})$, *where $Q_e \in \operatorname{Spec}(R)$ is chosen such that*
$$a_e(M_{Q_e}) = \min\{a_e(M_P) \mid P \in \operatorname{Spec}(R)\}.$$

Moreover, all of the above limits agree, with common value being $\min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$.

Proof. The existence and agreements of the limits in (1) and (2) is the content of Theorem 4.7 and Remark 4.1. The existence and agreements of the limits in (5) and (6) is the content of Lemma 4.12.

The proof of the theorem is easily reduced to showing the convergence of the sequences in (3) and (4) to $\min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$. Let $Q_e \in \operatorname{Spec}(R)$ be as in Lemma 4.12. Then by Lemmas 4.10 and 4.11, $a_e(M_{Q_e}) \leqslant n_e + d$ and $a_e(M_{Q_e}) \leqslant m_e + d$. Observe that $m_e, n_e \leqslant a_e(M_{Q_e})$. Therefore $\frac{a_e(M_{Q_e})-d}{p^{e\gamma(R)}} \leqslant \frac{n_e}{p^{e\gamma(R)}} \leqslant \frac{a_e(M_{Q_e})}{p^{e\gamma(R)}}$ and $\frac{a_e(M_{Q_e})-d}{p^{e\gamma(R)}} \leqslant \frac{m_e}{p^{e\gamma(R)}} \leqslant \frac{a_e(M_{Q_e})}{p^{e\gamma(R)}}$. By Lemma 4.12, $\frac{n_e}{p^{e\gamma(R)}}$ and $\frac{m_e}{p^{e\gamma(R)}}$ must converge to $\min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$. □

Corollary 4.14. *Let R be an F-finite ring such that $Z_R = \operatorname{Spec}(R)$, and let M be a finitely generated R-module. Then $\mathrm{s}(M) = \min\{\operatorname{rank}_{R_P}(M_P)\,\mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\}$. In addition, if R is a domain, then $\mathrm{s}(M) = \operatorname{rank}_R(M)\,\mathrm{s}(R)$.*

Proof. By Theorem 4.13, $\mathrm{s}(M) = \min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\}$. For each $P \in \operatorname{Spec}(R)$, $\mathrm{s}(M_P) = \operatorname{rank}_{R_P}(M_P)\,\mathrm{s}(R_P)$, and the first claim follows. If R is a domain, we have that $\operatorname{rank}_{R_P}(M_P) = \operatorname{rank}_R(M)$ for any $P \in \operatorname{Spec}(R)$. Thus, in this case, we have $\mathrm{s}(M) = \operatorname{rank}_R(M) \min\{\mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\}$, which is $\operatorname{rank}(M)\,\mathrm{s}(R)$ by a repeated application of Theorem 4.13. □

Theorem 4.15. *Let R be an F-finite ring.*

1. *If $Z_R = \operatorname{Spec}(R)$ then $\mathrm{s}(R) = 1$ if and only if R is regular.*
2. *If $Z_R = \operatorname{Spec}(R)$ then $\mathrm{s}(R) > 0$ if and only if R is strongly F-regular.*
3. *There is a number $\delta > 0$, depending only on the dimension of R, such that, if R is an F-finite ring of dimension $\dim(R) \leqslant d$ and $\mathrm{s}(R) \geqslant 1-\delta$, then R is regular.*

Proof. The ring R is regular if and only if for each $Q \in \operatorname{Spec}(R)$ the local ring R_Q is a regular local ring. The local ring R_Q is regular if and only if $\mathrm{s}(R_Q) = 1$ by part 3 of Theorem 2.5. By Theorem 4.13 this will happen if and only if $\mathrm{s}(R) = 1$ under the assumption $Z_R = \operatorname{Spec}(R)$.

For 2, an F-finite ring is strongly F-regular if and only if each localization of R at a prime ideal is strongly F-regular. This is equivalent to $\mathrm{s}(R_Q) > 0$ for each $Q \in \operatorname{Spec}(R)$ by part 4 of Theorem 2.5. This is equivalent to $\mathrm{s}(R) = \min\{\mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\} > 0$

To prove 3 let $\delta(i)$ be a number as in Theorem 2.7, that works for rings of dimension i, and let $\delta = \min\{\delta(i) \mid i \leqslant d\}$. Without loss of generality, we may assume $\mathrm{s}(R) > 0$, thus we may assume that $Z_R = \operatorname{Spec}(R)$. If $\mathrm{s}(R) \geqslant 1-\delta$, then $\mathrm{s}(R_P) \geqslant 1-\delta$ for each $P \in \operatorname{Spec}(R)$. It follows that R_P is regular for each $P \in \operatorname{Spec}(R)$, that is, R is regular. □

Example 4.16. If $Z_R \neq \operatorname{Spec}(R)$, i.e., if there exists $P \in \operatorname{Spec}(R)$ such that $\alpha(P) + \operatorname{ht}(P) \neq \gamma(R)$, then $\mathrm{s}(R) = 1$ is not equivalent to R being regular and $\mathrm{s}(R) > 0$ is not equivalent to R being strongly F-regular. Let $R = \mathbb{F}_p \times \mathbb{F}_p(t)$. Then R is regular, hence strongly F-regular. But $\alpha(P) + \operatorname{ht}(P)$ varies at the two different prime ideals of R, hence $\mathrm{s}(R) = 0$ by Remark 4.2.

4.1. Global F-signature of a Cartier subalgebra

In what follows, R is an F-finite ring and $\mathscr{D}$ is a Cartier subalgebra. Given a choice of direct summand M of F^e_*R, with splitting $M \subseteq F^e_*R \to M$, we say that a summand N of M is a $\mathscr{D}$-summand if $N \cong R^{\oplus n}$ is free and the natural projection map $F^e_*R \to M \to N$ is a direct sum of elements of $\mathscr{D}_e$. The choice of isomorphism $N \cong R^{\oplus n}$ does not change whether or not N is a $\mathscr{D}$-summand. We denote by $a(M, \mathscr{D})$ the largest rank of a $\mathscr{D}$-summand appearing in various direct sum decompositions of M. Recall that $a(F^e_*R, \mathscr{D}) = a^{\mathscr{D}}_e(R)$ is the usual eth Frobenius splitting number of the pair $(R, \mathscr{D})$, see Section 2.

Lemma 4.17. *Let R be an F-finite ring and M be a direct summand of F^e_*R. Suppose that $x \in M$ and that $(Rx)_Q \subseteq M_Q$ is a $\mathscr{D}_Q$-summand for each $Q \in \operatorname{Spec}(R)$. Then $Rx \subseteq M$ is a $\mathscr{D}$-summand.*

Proof. Our assumptions allow us to find $s_1, \ldots, s_n \in R$ such that $(s_1, \ldots, s_n) = R$ and such that $(Rx)_{s_i} \subseteq M_{s_i}$ is a $\mathscr{D}_{s_i}$-summand. After replacing s_i by powers of themselves, we can find $\varphi_1, \ldots, \varphi_n \in \mathscr{D}_e$ such that $\varphi_i(x) = s_i$. There are elements $r_1, \ldots, r_n \in R$ such that $r_1 s_1 + \cdots + r_n s_n = 1$. Let $\varphi = r_1\varphi_1 + \cdots + r_n\varphi_n \in \mathscr{D}_e$, then $\varphi(x) = 1$. $\square$

Lemma 4.18. *Let R be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. For each $e \in \Gamma_{\mathscr{D}}$, let $Q_e \in \operatorname{Spec}(R)$ be such that $a_e(R_{Q_e}, \mathscr{D}_{Q_e}) = \min\{a_e(R_P, \mathscr{D}_P) \mid P \in \operatorname{Spec}(R)\}$. Then the sequence $\mathrm{s}_e(R_{Q_e}, \mathscr{D}_{Q_e})$ converges to a limit as $e \in \Gamma_{\mathscr{D}} \to \infty$. Moreover, if $(R, \mathscr{D})$ satisfies condition (†), then the limit converges to $\min\{\mathrm{s}(R_P, \mathscr{D}_P) \mid P \in \operatorname{Spec}(R)\}$.*

Proof. By [32, Theorem 6.4], there is a constant $C \in \mathbb{R}$ such that for each $e, e' \in \Gamma_{\mathscr{D}}$ and each $P \in \operatorname{Spec}(R)$,

$$\mathrm{s}_e(R_P, \mathscr{D}_P) - \mathrm{s}_{e+e'}(R_P, \mathscr{D}_P) < \frac{C}{p^e}.$$

It follows that, for each $e, e' \in \Gamma_{\mathscr{D}}$, we have

$$\mathrm{s}_e(R_{Q_e}, \mathscr{D}_{Q_e}) \leqslant \mathrm{s}_e(R_{Q_{e+e'}}, \mathscr{D}_{Q_{e+e'}}) \leqslant \mathrm{s}_{e+e'}(R_{Q_{e+e'}}, \mathscr{D}_{Q_{e+e'}}) + \frac{C}{p^e},$$

and we conclude that the limit $\lim_{e\to\infty} \mathrm{s}_e(R_{Q_e}, \mathscr{D}_{Q_e})$ exists by [31, Lemma 3.5].

Now assume that $(R, \mathscr{D})$ satisfies (†). Then the functions $\mathrm{s}_e : \operatorname{Spec}(R) \to \mathbb{R}$, defined as $Q \mapsto \mathrm{s}_e(R_Q, \mathscr{D}_Q)$, converge uniformly to their limit, namely $\mathrm{s} : \operatorname{Spec}(R) \to \mathbb{R}$ sending a prime Q to the the F-signature $\mathrm{s}(R_Q, \mathscr{D}_Q)$ of the pair $(R_Q, \mathscr{D}_Q)$. This allows one to proceed as in the proof of Lemma 4.12. $\square$

We say that a projective summand Ω of F^e_*R is a $\mathscr{D}$-summand if $a(\Omega_Q, \mathscr{D}_Q) = \operatorname{rank}(\Omega_Q)$ for each $Q \in \operatorname{Spec}(R)$. We call a projective summand Ω of F^e_*R a free $\mathscr{D}$-summand if Ω is free and a $\mathscr{D}$-summand. Let $a_e(R, \mathscr{D})$ be the largest rank of a free $\mathscr{D}$-summand appearing in various direct sum decompositions of F^e_*R, and denote by $\tilde{a}_e(R, \mathscr{D})$ the largest min-rank of a projective $\mathscr{D}$-summand appearing in various direct sum decompositions of F^e_*R. We define the *global F-signature of the pair* $(R, \mathscr{D})$ as

$$\mathrm{s}(R, \mathscr{D}) = \lim_{e \in \Gamma_{\mathscr{D}} \to \infty} \frac{a_e(R, \mathscr{D})}{p^{e\gamma(R)}}.$$

We show the existence of this limit in the following theorem, and we relate it with other limits as in Theorem 4.13.

Theorem 4.19. *Let R be an F-finite domain of dimension d and let $\mathscr{D}$ be a Cartier subalgebra. Then the following limits exist:*

1. $\mathrm{s}(R,\mathscr{D}) = \lim_{e\in\Gamma_{\mathscr{D}}\to\infty} \frac{a_e(R,\mathscr{D})}{p^{e\gamma(R)}}$,
2. $\lim_{e\in\Gamma_{\mathscr{D}}\to\infty} \frac{\tilde{a}_e(R,\mathscr{D})}{p^{e\gamma(R)}}$,
3. $\lim_{e\in\Gamma_{\mathscr{D}}\to\infty} \frac{n_e}{p^{e\gamma(R)}}$, *where n_e is the rank of a free $\mathscr{D}$-summand of F^e_*R appearing in a choice of decomposition $F^e_*R \cong R^{n_e} \oplus M_e$ where M_e has no free $\mathscr{D}$-summand,*
4. $\lim_{e\in\Gamma_{\mathscr{D}}\to\infty} \frac{m_e}{p^{e\gamma(R)}}$, *where m_e is the min-rank of a project $\mathscr{D}$-summand Ω_e of F^e_*R appearing in a choice of decomposition $F^e_*R \cong \Omega_e \oplus M_e$ where M_e has no projective $\mathscr{D}$-summand,*
5. $\lim_{e\in\Gamma_{\mathscr{D}}\to\infty} \frac{a_e(R_{Q_e},\mathscr{D}_{Q_e})}{p^{e\gamma(R)}}$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*

$$a_e(R_{Q_e},\mathscr{D}_{Q_e}) = \min\{a_e(R_P,\mathscr{D}_P) \mid P \in \mathrm{Spec}(R)\}.$$

Moreover, all of the above limits agree. If $(R,\mathscr{D})$ satisfies condition (†), then all the above limits equal $\min\{\mathrm{s}(R_P,\mathscr{D}_P) \mid P \in \mathrm{Spec}(R)\}$.

Proof. The convergence of the limit in (5) is the content of Lemma 4.18. Suppose that n_e and m_e are as in (3) and (4). Then Theorem 2.13 easily implies that $m_e \leqslant a_e(R_{Q_e},\mathscr{D}_{Q_e}) \leqslant m_e + d$ and $n_e \leqslant a_e(R_{Q_e},\mathscr{D}_{Q_e}) \leqslant n_e + d$. It follows that the limits in $(1)-(4)$ all exist and are equal to the limit in (5). If we assume that $(R,\mathscr{D})$ satisfies (†), then Lemma 4.18 implies that the common limit value is indeed $\min\{\mathrm{s}(R_P,\mathscr{D}_P) \mid P \in \mathrm{Spec}(R)\}$. □

Corollary 4.20. *Let R be an F-finite domain and let $\mathscr{D}$ be a Cartier algebra satisfying condition (†). Then $\mathrm{s}(R,\mathscr{D}) > 0$ if and only if $(R,\mathscr{D})$ is strongly F-regular.*

Proof. A pair $(R,\mathscr{D})$ is strongly F-regular if and only if for each $P \in \mathrm{Spec}(R)$ the pair $(R_P,\mathscr{D}_P)$ is strongly F-regular. Positivity of $\mathrm{s}(R_P,\mathscr{D}_P)$ is equivalent to strong F-regularity of $(R_P,\mathscr{D}_P)$ by part 2 of Theorem 2.9. By [32, Theorem 6.4] and Theorem 4.19 there is a $Q \in \mathrm{Spec}(R)$ such that $\mathrm{s}(R,\mathscr{D}) = \mathrm{s}(R_Q,\mathscr{D}_Q)$. □

Corollary 4.20 brings up the following natural question.

Question 4.21. Let R be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. Is positivity of $\mathrm{s}(R,\mathscr{D})$ equivalent to strong F-regularity of $\mathscr{D}$?

Suppose that R is an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. Suppose that one could show that the functions $\mathrm{s}_e : \mathrm{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \mathrm{s}_e(R_P,\mathscr{D}_P)$ converge uniformly to their limit function, namely

$\mathrm{s} : \mathrm{Spec}(R) \to \mathbb{R}$ which sends $P \mapsto \mathrm{s}(R_P, \mathscr{D}_P)$. Then one can follow the methods of Theorem 4.13 to establish $\mathrm{s}(R, \mathscr{D}) = \min\{\mathrm{s}(R_P, \mathscr{D}_P) \mid P \in \mathrm{Spec}(R)\}$. Such a result would establish a positive answer to Question 4.21. We therefore ask the following more specific question.

Question 4.22. Suppose that R is an F-finite domain and $\mathscr{D}$ a Cartier subalgebra. Do the functions $\mathrm{s}_e : \mathrm{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \mathrm{s}_e(R_P, \mathscr{D}_P)$ converge uniformly to their limit as $e \in \Gamma_{\mathscr{D}} \to \infty$?

5. Global F-invariants under faithfully flat extensions

5.1. *Global F-signature*

We now study the behavior of global F-signature under faithfully flat extensions. Recall that if R is an F-finite ring, then we let $Z_R = \{P \in \mathrm{Spec}(R) \mid \alpha(P) + \mathrm{ht}(P) = \gamma(R)\}$. Let M be a finitely generated R-module. Remark 4.2 and Theorem 4.13 combined state that $\mathrm{s}(M) = 0$ if $Z_R \neq \mathrm{Spec}(R)$, and that $\mathrm{s}(M) = \min\{\mathrm{s}(M_P) \mid P \in \mathrm{Spec}(R)\}$ if $Z_R = \mathrm{Spec}(R)$.

Theorem 5.1. *Let $R \to T$ be a faithfully flat map of F-finite rings such that $Z_R = \mathrm{Spec}(R)$ and $Z_T = \mathrm{Spec}(T)$, and M a finitely generated R-module. Then $\mathrm{s}(M) \geqslant \mathrm{s}(M \otimes_R T)$. If moreover the closed fibers of $R \to T$ are regular, then $\mathrm{s}(M) = \mathrm{s}(M \otimes_R T)$.*

Proof. By Theorem 4.13, $\mathrm{s}(M) = \min\{\mathrm{s}(M_P) \mid P \in \mathrm{Spec}(R)\}$ and $\mathrm{s}(M \otimes_R T) = \min\{\mathrm{s}(M \otimes_R T_Q) \mid Q \in \mathrm{Spec}(T)\}$. Let $P \in \mathrm{Spec}(R)$ be such that $\mathrm{s}(M) = \mathrm{s}(M_P)$ and let $Q \in \mathrm{Spec}(T)$ be such that $Q \cap R = P$. By Theorem 2.8, $\mathrm{s}(M_P) \geqslant \mathrm{s}(M \otimes_R T_Q)$, hence $\mathrm{s}(M) \geqslant \mathrm{s}(M \otimes_R T)$.

Suppose that $R \to T$ has regular closed fibers and let $Q \in \mathrm{Spec}(T)$ be such that $\mathrm{s}(M \otimes_R T) = \mathrm{s}(M \otimes_R T_Q)$. If $\mathfrak{m}$ is a maximal ideal of T containing Q, then $\mathrm{s}(M \otimes_R T_{\mathfrak{m}}) \leqslant \mathrm{s}(M \otimes_R T_Q)$. Thus without loss of generality we may assume that Q is maximal, thus $P = R \cap Q$ is maximal in R. By Theorem 2.8, $\mathrm{s}(M_P) = \mathrm{s}(M \otimes_R T_Q)$, it follows that $\mathrm{s}(M) = \mathrm{s}(M \otimes_R T)$. ☐

Suppose that $R \to T$ is a faithfully flat extension of F-finite rings satisfying the hypotheses of Theorem 5.1. Example 5.3 below shows that it need not be the case that $a_e(R)/p^{e\gamma(R)} \geqslant a_e(T)/p^{e\gamma(T)}$, even though the inequality holds after taking limits. One should compare this to the local situation in Theorem 2.8. Before providing such an example, we first discuss the existence of an F-finite regular ring R such that $F^e_* R$ is not free. The class of examples we discuss were already known to exist by experts.[4]

Example 5.2. If R is a regular F-finite domain, then $F^e_* R$ need not be free as an R-module. Let k be an algebraic closed field of characteristic p, X an elliptic curve over k, as in [19, Chapter 4.4], $x_0 \in X$ be

[4]The class of examples we discuss in Example 5.2 were communicated to us by Florian Enescu. Florian Enescu learned of such examples from Mohan Kumar.

a chosen point for the group law on X, and let $K(X)$ be the function field of X. Assume X is ordinary, that is the Frobenius morphism $F : X \to X$ induces an injective map of 1-dimensional vector spaces $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$. The assumption that X is ordinary guarantees that the map of structure sheaves $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ is split. Denote by $\mathcal{E}$ the cokernel of $\mathcal{O}_X \to F^e_* \mathcal{O}_X$. Then $\mathcal{E} \cong \mathcal{O}_X(x_1 - x_0) \oplus \cdots \oplus \mathcal{O}_X(x_{p^e-1} - x_0)$ where $x_0, x_1, \ldots, x_{p^e-1}$ are the p^e distinct p^e-torsion points of X, see [30, Example 2.18, Exercise 2.19] for further details.

If $\operatorname{char} k \neq 2$ or if $e > 0$ let $U = X - \{x_1, \ldots, x_{p^e-2}\}$. If $\operatorname{char} k = 2$ and $e = 1$ let $U = X - \{x_2\}$ for some point x_2 which is not a 2-torsion point of X. As X is a non-singular projective curve, U is an open affine set. Let $R = \Gamma(U, \mathcal{O}_X)$ and $M = \Gamma(U, \mathcal{O}_X(x_{p^e-1} - x_0))$, then $F^e_* R \cong R^{\oplus p^e - 1} \oplus M$ is projective of rank p^e. By examining the p^eth exterior product of $R^{\oplus p^e-1} \oplus M$, one sees that $F^e_* R$ is a free R-module of rank p^e if and only if M is a free module of rank 1. We claim that M is not free. Else, M is identified with $R \cdot f$ for some $f \in K(X)$. Equivalently, the divisor $x_{p^e-1} - x_0$ is linearly equivalent to 0 on U. As $x_0, x_{p^e-1} \notin U$, this will imply $x_{p^e-1} - x_0$ is linearly equivalent to 0 on X, contradicting that x_0, x_{p^e-1} are distinct points.

Example 5.3. Suppose that $R \to T$ is a faithfully flat map of F-finite domains. Then it does not necessarily follow that $a_e(R)/p^{e\gamma(R)} \geqslant a_e(T)/p^{e\gamma(T)}$ for each $e \in \mathbb{N}$, even though the inequality holds after taking limits. Let R be a Dedekind domain affine over the algebraically closed field k of characteristic p. Then $F^e_* R$ is projective of rank p^e. By [33, Theorem 1], $p^e - 1 \leqslant a_e(R) \leqslant p^e$ with $a_e(R) = p^e$ if and only if $F^e_* R$ is free. Let R be as in Example 5.2, so that $F^e_* R$ is not free. Consider the faithfully flat extension $R \to R[t] \to T = R[t]_W$ where W is the multiplicative set $R[t] - \cup_{\mathfrak{m} \in \operatorname{Max}(R)} \mathfrak{m} R[t]$. Observe that T is a Dedekind domain and $F^e_* T$ is projective of rank p^{2e}. By [33, Theorem 1], $a_e(T)$ is either $p^{2e} - 1$ or p^{2e}. Then $a_e(R)/p^{e\gamma(R)} = \frac{p^e-1}{p^e} < \frac{p^{2e}-1}{p^{2e}} \leqslant a_e(T)/p^{e\gamma(T)}$.

We now discuss the behavior of global F-signature of F-finite faithfully flat extensions of rings which are either F-finite or essentially of finite type over an excellent local ring. Recall that, by [44], given any d-dimensional local ring $(R, \mathfrak{m}, k)$ of prime characteristic and a finitely generated R-module M, we can define a sequence $\#(F^e_* M)/p^{ed}$ that agrees with $a_e(M)/p^{e\gamma(R)}$ when R is F-finite. We still denote an element of this sequence by $\mathrm{s}_e(M)$, even when R is not F-finite. Let R be either F-finite or essentially of finite type over an excellent local ring and let M a finitely generated R-module. We define the *local-minimal F-signature of M* as

$$\min\{\mathrm{s}(M_P) \mid P \in \operatorname{Spec}(R)\} = \min\{\operatorname{rank}_{R_P}(M_P)\,\mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\},$$

and we denote it by $\mathrm{s}_{\mathrm{loc}}(M)$. We note that such a minimum exists, since in our assumptions the F-signature function $\mathrm{s} : \operatorname{Spec}(R) \to \mathbb{R}$, sending $P \mapsto \mathrm{s}(R_P)$, is lower semi-continuous by [32, Theorem 5.7]. In particular, R is strongly F-regular if and only if $\mathrm{s}_{\mathrm{loc}}(R) > 0$ by part 4 of Theorem 2.5. Observe that, when R is F-finite and $Z_R = \operatorname{Spec}(R)$, $\mathrm{s}_{\mathrm{loc}}(M)$ coincides with the global F-signature $\mathrm{s}(M)$ defined in Section 4. See Theorem 4.13.

In Theorem 5.6, we show equality between $\mathrm{s}_{\mathrm{loc}}(M)$,

$$\sup\{\mathrm{s}_{\mathrm{loc}}(T \otimes_R M) \mid R \to T \text{ is faithfully flat and } T \text{ is F-finite}\},$$

and

$$\sup\{\mathrm{s}(T \otimes_R M) \mid R \to T \text{ is faithfully flat and } T \text{ is F-finite}\}.$$

We begin with a lemma.

Lemma 5.4. *Let R be an F-finite locally equidimensional ring. Then there is a faithfully flat extension $R \to T$ with regular fibers such that T is F-finite, $\gamma(T) = \gamma(R)$, and $Z_T = \operatorname{Spec}(T)$.*

Proof. By [26, Proposition 2.3], $R \cong T_1 \times \cdots \times T_n$ is a direct product of F-finite rings such that $Z_{T_i} = \operatorname{Spec}(T_i)$. For each $1 \leqslant i \leqslant n$ let $E_i = T_i[x_1, \cdots, x_{\gamma(R)-\gamma(T_i)}]$. Observe that $T_i \to E_i$ is a faithfully flat map of F-finite rings such that $Z_{E_i} = \operatorname{Spec}(E_i)$, with regular fibers, and $\gamma(E_i) = \gamma(R)$. Let $T = E_1 \times \cdots \times E_n$ and $R \to T$ be the natural map. It is easily verified that $Z_T = \operatorname{Spec}(T)$. $\square$

We will use Hochster's and Huneke's gamma constructions to prove Theorem 5.6 below. We briefly recall some basic properties of gamma constructions, all of which can be found in [20, Section 6]. Suppose that R is essentially of finite type over a complete local ring $(A, \mathfrak{m}, k)$. Let Λ be a p-base for k. For each cofinite subset of $\Gamma \subseteq \Lambda$, there is an associated F-finite ring R^Γ and faithfully flat purely inseparable ring homomorphism $R \to R^\Gamma$. It follows that $\operatorname{Spec}(R^\Gamma) \to \operatorname{Spec}(R)$ is a homeomorphism with inverse map $P \mapsto P_\Gamma = \sqrt{PR^\Gamma}$.

For every given $P \in \operatorname{Spec}(T)$ there exists a cofinite subset $\Gamma_0 \subseteq \Lambda$ such that $PR^\Gamma = P_\Gamma$ for all cofinite subsets $\Gamma \subseteq \Gamma_0$. Therefore, for every given P and cofinite $\Gamma_1 \subseteq \Lambda$, there exists a cofinite $\Gamma_2 \subseteq \Gamma_1$ such that $PR^\Gamma = P_\Gamma$ for all cofinite subsets $\Gamma \subseteq \Gamma_2$.

Suppose that R is essentially of finite type over a complete local ring $(A, \mathfrak{m}, k)$. Let Λ be a p-base for k and let $\Gamma \subseteq \Lambda$ be a cofinite subset. Then for each $P \in \operatorname{Spec}(R)$ we have flat map of local rings $R_P \to (R^\Gamma)_{P_\Gamma} =: R^\Gamma_{P_\Gamma}$. Then $\mathrm{s}(M_P) \geqslant \mathrm{s}(M \otimes_R R^\Gamma_{P_\Gamma})$, with equality if $PR^\Gamma_{P_\Gamma}$ is prime, see Theorem 2.8. We remark that it is not necessarily the case that there exists $\Gamma \subseteq \Lambda$ cofinite such that PR_Γ is prime for every $P \in \operatorname{Spec}(R)$. Hence one cannot necessarily expect to find $\Gamma \subseteq \Lambda$ such that $\mathrm{s}(M_P) = \mathrm{s}(M \otimes_R R^\Gamma_{P_\Gamma})$ for all $P \in \operatorname{Spec}(R)$. However, we show in Theorem 5.6 below that one can find $\Gamma \subseteq \Lambda$ such that $\mathrm{s}(M_P)$ and $\mathrm{s}(M \otimes_S R^\Gamma_{P_\Gamma})$ are arbitrarily close for all $P \in \operatorname{Spec} R$.

Remark 5.5. Let R be either F-finite or essentially of finite type over an excellent local ring and let M be a finitely generated R-module. Assume that $R \to T$ is faithfully flat and T is F-finite. If $\mathrm{s}_{\mathrm{loc}}(M) = 0$ then it easily follows by Theorem 2.8 that $\mathrm{s}_{\mathrm{loc}}(M \otimes_R T) = 0$ and therefore $\mathrm{s}(M \otimes_R T) = 0$. If $\mathrm{s}_{\mathrm{loc}}(M) > 0$ then $\mathrm{s}_{\mathrm{loc}}(R) > 0$ and R is strongly F-regular by part 4 of Theorem 2.5. In particular, R is locally equidimensional and if R is F-finite, the functions $\mathrm{s}_e : \operatorname{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \mathrm{s}_e(M_P)$ are lower semi-continuous by [15, Corollary 2.5 and Remark 5.5]. If R is essentially of finite type over an excellent local ring $(A, \mathfrak{m}, k)$, then $R \to \hat{A} \otimes_A R$ is faithfully flat with regular fibers, [27, Section 33, Lemma 4]. It follows by Theorem 2.8 that $\mathrm{s}_{\mathrm{loc}}(R) = \mathrm{s}_{\mathrm{loc}}(\hat{A} \otimes_A R)$. In particular, $\hat{A} \otimes_A R$ remains strongly F-regular and therefore locally equidimensional. Hence $\mathrm{s}_e : \mathrm{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \mathrm{s}_e(M_P)$ is lower semi-continuous by [15, Theorem 5.1 and Remark 5.5]. It follows that if R is strongly F-regular, then for each $e \in \mathbb{N}$ there exists $Q_e \in \mathrm{Spec}(R)$ such that $\mathrm{s}_e(M_{Q_e}) = \min\{\mathrm{s}_e(M_P) \mid P \in \mathrm{Spec}(R)\}$.

Theorem 5.6. *Let R be either F-finite or essentially of finite type over an excellent local ring and M a finitely generated R-module. If R is not strongly F-regular then $\mathrm{s}_{\mathrm{loc}}(M) = \mathrm{s}_{\mathrm{loc}}(M \otimes_R T) = \mathrm{s}(M \otimes_R T) = 0$ for every faithfully flat F-finite extension $R \to T$. If R is strongly F-regular then the following limits exist:*

1. $\lim\limits_{e\to\infty} s_e(M_{Q_e})$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\mathrm{s}_e(M_{Q_e}) = \min\{\mathrm{s}_e(M_P) \mid P \in \mathrm{Spec}(R)\},$$
2. $\lim\limits_{e\to\infty} \mathrm{s}(M_{Q_e})$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\mathrm{s}_e(M_{Q_e}) = \min\{\mathrm{s}_e(M_P) \mid P \in \mathrm{Spec}(R)\},$$

and they agree with the local-minimal F-signature $\mathrm{s}_{\mathrm{loc}}(M)$. Moreover,

$$\begin{aligned}\mathrm{s}_{\mathrm{loc}}(M) &= \sup\{\mathrm{s}_{\mathrm{loc}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}\\ &= \sup\{\mathrm{s}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}.\end{aligned}$$

Under the assumption that R is F-finite,

$$\begin{aligned}\mathrm{s}_{\mathrm{loc}}(M) &= \max\{\mathrm{s}_{\mathrm{loc}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}\\ &= \max\{\mathrm{s}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}.\end{aligned}$$

Proof. By Remark 5.5 we may assume that R is strongly F-regular. For each $e \in \mathbb{N}$ let $\mathrm{s}_e : \mathrm{Spec}(R) \to \mathbb{R}$ be the function sending $P \mapsto \mathrm{s}_e(M_P)$ and let $\mathrm{s} : \mathrm{Spec}(R) \to \mathbb{R}$ be the function mapping $P \mapsto \mathrm{s}(M_P)$. The functions s_e converge uniformly to s by [32, Theorem 5.6]. It follows that the limits in (1) and (2) exist and are equal to $\mathrm{s}_{\mathrm{loc}}(M)$. See Lemma 4.12 for a similar argument.

Let $R \to T$ be faithfully flat, with T an F-finite ring. Let $P \in \mathrm{Spec}(R)$ be chosen such that $\mathrm{s}_{\mathrm{loc}}(M) = \mathrm{s}(M_P)$. As $R \to T$ is faithfully flat there exists $Q \in T$ such that $Q \cap R = P$. By Theorem 2.8 $\mathrm{s}(M_P) \geqslant \mathrm{s}(M \otimes_R T_Q)$, hence $\mathrm{s}_{\mathrm{loc}}(M) \geqslant \mathrm{s}_{\mathrm{loc}}(M \otimes_R T)$. If $Z_T \neq \mathrm{Spec}(T)$, then $\mathrm{s}(M \otimes_R T) = 0$, see Remark 4.2. Else $Z_T = \mathrm{Spec}(T)$ and $\mathrm{s}(M \otimes_R T) = \mathrm{s}_{\mathrm{loc}}(M \otimes_R T)$ by Theorem 4.13. This shows

$$\begin{aligned}\mathrm{s}_{\mathrm{loc}}(M) &\geqslant \sup\{\mathrm{s}_{\mathrm{loc}}(M \otimes_R T) \mid R \to T \text{ is faithfully flat and } T \text{ is F-finite}\}\\ &\geqslant \sup\{\mathrm{s}(M \otimes_R T) \mid R \to T \text{ is faithfully flat and } T \text{ is F-finite}\}.\end{aligned}$$

Suppose that R is F-finite. We show the existence of a faithfully flat F-finite extension $R \to T$ such that $\mathrm{s}_{\mathrm{loc}}(M) = \mathrm{s}(M \otimes_R T)$. Since R is strongly F-regular, we have $R \cong D_1 \times \cdots \times D_n$ is a product of F-finite domains D_i by part 4 of Theorem 2.5. By Lemma 5.4 there exists a faithfully flat extension $R \to T$ with regular fibers, T is F-finite, and $Z_T = \mathrm{Spec}(T)$. In particular, $\mathrm{s}_{\mathrm{loc}}(M) = \mathrm{s}_{\mathrm{loc}}(M \otimes_R T)$ by Theorem 2.8. As $Z_T = \mathrm{Spec}(T)$ we see that $\mathrm{s}(M \otimes_R T) = \mathrm{s}_{\mathrm{loc}}(M \otimes_R T)$ by Theorem 4.13.

Now suppose that R is essentially of finite type over an excellent local ring $(A, \mathfrak{m}, k)$. Let $\epsilon > 0$. We are going to show the existence of a faithfully flat extension $R \to T$ such that T is F-finite and $\mathrm{s}(M \otimes_R T) > \mathrm{s}_{\mathrm{loc}}(M) - \epsilon$, which will complete the proof of the theorem. Denote by $\hat{A}$ the completion of A with respect to its maximal ideal. Then $R \to \hat{A} \otimes_A R$ is faithfully flat with regular fibers, [27, Section 33, Lemma 4] and, by Theorem 2.8, we have that $\mathrm{s}_{\mathrm{loc}}(M) = \mathrm{s}_{\mathrm{loc}}(\hat{A} \otimes_A M)$. Thus we may replace R with $\hat{A} \otimes_A R$ and assume that R is essentially of finite type over a complete local ring.

Abusing notation, we let $(A, \mathfrak{m}, k)$ be a complete local ring which R is essentially of finite type over. Without loss of generality, assume that $\epsilon < \mathrm{s}_{\mathrm{loc}}(M)$. Let Λ be a p-base for a coefficient field $k \subseteq A$. For each cofinite subset $\Gamma \subseteq \Lambda$ let

$$U_\Gamma = \{P \in \mathrm{Spec}(R) \mid \mathrm{s}(M \otimes_R R^\Gamma_{P_\Gamma}) > \mathrm{s}_{\mathrm{loc}}(M) - \epsilon\}.$$

For each Γ, the induced map of spectra $\mathrm{Spec}(R^\Gamma) \to \mathrm{Spec}(R)$ is a homeomorphism, hence by [32, Theorem 5.7] the sets U_Γ are open. Moreover, if $\Gamma' \subseteq \Gamma$, then Theorem 2.8 shows that $U_{\Gamma'} \supseteq U_\Gamma$. As $\mathrm{Spec}(R)$ is Noetherian, there exists some cofinite subset $\Gamma \subseteq \Lambda$ such that U_Γ is maximal. We claim that $U_\Gamma = \mathrm{Spec}(R)$. Else, there exists $P \in \mathrm{Spec}(R) - U_\Gamma$. There exists some cofinite subset $\Gamma' \subseteq \Gamma$ such that $PR^{\Gamma'} = P_{\Gamma'}$, i.e., $PR^{\Gamma'}$ is prime. In which case, $R_P \to R^{\Gamma'}_{P_{\Gamma'}}$ is a faithfully flat local homomorphism whose closed fiber is a field. By Theorem 2.8, $\mathrm{s}(M_P) = \mathrm{s}(M \otimes_R R^{\Gamma'}_{P_{\Gamma'}})$. Therefore $P \in U_{\Gamma'}$, and then $P \in U_\Gamma$ by maximality. This contradicts the choice of P. Thus we have $\mathrm{s}(M \otimes_R R^\Gamma_{P_\Gamma}) > \mathrm{s}_{\mathrm{loc}}(M) - \epsilon > 0$ for all $P_\Gamma \in \mathrm{Spec}(R^\Gamma)$, which implies $\mathrm{s}_{\mathrm{loc}}(M \otimes_R R^\Gamma) > \mathrm{s}_{\mathrm{loc}}(M) - \epsilon$. In particular, R^Γ is strongly F-regular and is a direct product of F-finite domains. By Lemma 5.4 there exists faithfully flat F-finite extension $R^\Gamma \to T$ with regular fibers and such that $Z_T = \mathrm{Spec}(T)$. Hence $\mathrm{s}_{\mathrm{loc}}(M \otimes_R R^\Gamma) = \mathrm{s}_{\mathrm{loc}}(M \otimes_R T)$ by Theorem 2.8 and $\mathrm{s}_{\mathrm{loc}}(M \otimes_R T) = \mathrm{s}(M \otimes_R T)$ by Theorem 4.13. Therefore $\mathrm{s}(M \otimes_R T) > \mathrm{s}_{\mathrm{loc}}(M) - \epsilon$, which completes the proof. $\square$

5.2. *Global Hilbert-Kunz multiplicity*

We now discuss the behavior of global Hilbert-Kunz multiplicity under faithfully flat extensions. Recall that if R is F-finite and M a finitely generated R-module then $\mathrm{e}_{\mathrm{HK}}(R) = \max\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in Z_R\}$ by Theorem 3.16.

Theorem 5.7. *Let $R \to T$ be a faithfully flat extension of F-finite rings and let M be a finitely generated R-module. If each $P \in Z_R$ is a contraction of a prime $Q \in Z_T$, then $\mathrm{e}_{\mathrm{HK}}(M) \leqslant \mathrm{e}_{\mathrm{HK}}(M \otimes_R T)$. In particular,* *if R and T are domains, or more generally if R and T are such that $Z_R = \operatorname{Spec}(R)$ and $Z_T = \operatorname{Spec}(T)$, then $\mathrm{e_{HK}}(M) \leqslant \mathrm{e_{HK}}(M \otimes_R T)$ with equality if the closed fibers of $R \to T$ are regular.*

PROOF. By Theorem 3.16, $\mathrm{e_{HK}}(M) = \max\{\mathrm{e_{HK}}(M_P) \mid P \in Z_R\}$ and $\mathrm{e_{HK}}(M \otimes_R T) = \max\{\mathrm{e_{HK}}(M \otimes_R T_Q) \mid Q \in Z_T\}$. Let $P \in Z_R$ be such that $\mathrm{e_{HK}}(M) = \mathrm{e_{HK}}(M_P)$. By assumption, there exists $Q \in Z_T$ such that $Q \cap R = P$. By Theorem 2.2 we obtain that $\mathrm{e_{HK}}(M) = \mathrm{e_{HK}}(M_P) \leqslant \mathrm{e_{HK}}(M \otimes_R T_Q) \leqslant \mathrm{e_{HK}}(M \otimes_R T)$. Now suppose $Z_R = \operatorname{Spec}(R)$, $Z_T = \operatorname{Spec}(T)$, and the closed fibers of $R \to T$ are regular. Then there exists $\mathfrak{n} \in \operatorname{Max}(T)$ such that $\mathrm{e_{HK}}(M \otimes_R T) = \mathrm{e_{HK}}(M \otimes_R T_{\mathfrak{n}})$. Let $\mathfrak{m}$ be the contraction of $\mathfrak{n}$ in R, then $R_{\mathfrak{m}} \to T_{\mathfrak{n}}$ is flat with regular fiber. By Theorem 2.2, $\mathrm{e_{HK}}(M_{\mathfrak{m}}) = \mathrm{e_{HK}}(M \otimes_R T_{\mathfrak{n}}) = \mathrm{e_{HK}}(M \otimes_R T)$. The theorem follows since $\mathrm{e_{HK}}(M) \geqslant \mathrm{e_{HK}}(M_{\mathfrak{m}})$. $\square$

Example 5.8. For an arbitrary faithfully flat extension $R \to T$ of F-finite rings, it need not be the case that $\mathrm{e_{HK}}(R) \leqslant \mathrm{e_{HK}}(T)$. Suppose that R is an F-finite domain such that $\mathrm{e_{HK}}(R) > 1$. Let $S = K[x]$ where K is the fraction field of R. Take T to be the direct product of rings $R \times S$. Then the natural map $R \to T$ is faithfully flat and $\mathrm{e_{HK}}(T) = 1 < \mathrm{e_{HK}}(R)$.

Example 5.9. If $R \to T$ is a faithfully flat map of F-finite domains, then it need not be the case that $\mu(F^e_* R)/p^{e\gamma(R)} \leqslant \mu(F^e_* T)/p^{e\gamma(T)}$, even though the inequality holds after taking limits. One should compare this to the local situation in Theorem 2.2. In fact, the same example used in Example 5.3 demonstrates such phenomena. Suppose that R is a Dedekind domain affine over an algebraically closed field k of characteristic p. Then $F^e_* R$ is projective of rank p^e. Hence by Theorem 2.10, $\mu(F_* R)$ is either p^e or $p^e + 1$. The case that $\mu(F_* R) = p^e$ corresponds to the case that $F^e_* R$ is free and $\mu(F^e_* R) = p^e + 1$ corresponds to the case that $F^e_* R$ is not free. Suppose that R is as in Example 5.2, that is $F^e_* R$ is not free. Consider the faithfully flat extension $R \to R[t] \to T = R[t]_W$ where W is the multiplicative set $R[t] - \cup_{\mathfrak{m} \in \operatorname{Max}(R)} \mathfrak{m} R[t]$. Then T is a Dedekind domain and $F^e_* T$ is a projective T-module of rank p^{2e}. By Theorem 2.10, $\mu(F^e_* T)$ is either p^{2e} or $p^{2e} + 1$. But $\mu(F^e_* R)/p^{e\gamma(R)} = \frac{p^e+1}{p^e} > \frac{p^{2e}+1}{p^{2e}} \geqslant \mu(F^e_* T)/p^{e\gamma(T)}$.

Suppose that R is either F-finite or essentially of finite type over an excellent local ring, and M is a finitely generated R-module. We defined $\mathrm{s_{loc}}(M)$ and showed in Theorem 5.6 that if $R \to T$ is faithfully flat and T is F-finite, then $\mathrm{s_{loc}}(M) \geqslant \mathrm{s_{loc}}(M \otimes_R T) \geqslant \mathrm{s}(M \otimes_R T)$. Moreover, for $\epsilon > 0$, there exists $R \to T$ faithfully flat and F-finite such that $\mathrm{s_{loc}}(M) < \mathrm{s}(M \otimes_R T) + \epsilon$. We now develop an analogous theory for Hilbert-Kunz multiplicity.

Define the *local-maximal Hilbert-Kunz multiplicity of M* to be

$$\mathrm{e^{loc}_{HK}}(M) = \sup\{\mathrm{e_{HK}}(M_P) \mid P \in \operatorname{Spec}(R)\}.$$

As the Hilbert-Kunz multiplicity function is not upper semi-continuous without the locally equidimensional hypothesis, there may not be a prime $P \in \operatorname{Spec}(R)$ such that $\mathrm{e^{loc}_{HK}}(M) = \mathrm{e_{HK}}(M_P)$. Suppose that $R \to T$ is faithfully flat and T is F-finite. It easily follows by Theorem 2.2 that $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) \leqslant \mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T)$. However, it may be the case that $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T) > \mathrm{e}_{\mathrm{HK}}(M \otimes_R T)$ or it may be the case that there is faithfully flat $T \to T'$ such that T' is F-finite and $\mathrm{e}_{\mathrm{HK}}(M \otimes_R T) > \mathrm{e}_{\mathrm{HK}}(M \otimes_R T')$, see Example 5.8. Nevertheless, we can still develop an analogue of Theorem 5.6 for Hilbert-Kunz multiplicity, but under appropriate hypotheses.

Theorem 5.10. *Let R be a locally equidimensional ring which is either F-finite or essentially of finite type over an excellent local ring $(A, \mathfrak{m}, k)$. Let M be a finitely generated R-module. Then the following limits exist:*

1. $\lim\limits_{e\to\infty} \lambda(M_{Q_e}/Q_e^{[p^e]}M_{Q_e})/p^{e\,\mathrm{ht}(Q_e)}$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\lambda(M_{Q_e}/Q_e^{[p^e]}M_{Q_e})/p^{e\,\mathrm{ht}(Q_e)} = \max\{\lambda(M_P/P^{[p^e]}M_P)/p^{e\,\mathrm{ht}(P)} \mid P \in \mathrm{Spec}(R)\},$$
2. $\lim\limits_{e\to\infty} \mathrm{e}_{\mathrm{HK}}(M_{Q_e})$, *where $Q_e \in \mathrm{Spec}(R)$ is chosen such that*
$$\lambda(M_{Q_e}/Q_e^{[p^e]}M_{Q_e})/p^{e\,\mathrm{ht}(Q_e)} = \max\{\lambda(M_P/P^{[p^e]}M_P)/p^{e\,\mathrm{ht}(P)} \mid P \in \mathrm{Spec}(R)\}.$$

All of the above limits agree, with common value being the local-maximal Hilbert-Kunz multiplicity $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M)$. Under the assumption that R is F-finite,

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) &= \min\{\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}\\ &= \min\{\mathrm{e}_{\mathrm{HK}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat, } T \textit{ is F-finite, and } Z_T = \mathrm{Spec}(T)\}.\end{aligned}$$

In the case that R is essentially of finite type over an excellent local ring $(A, \mathfrak{m}, k)$ such that $\hat{A} \otimes_A R$ is locally equidimensional,

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) &= \inf\{\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat and } T \textit{ is F-finite}\}\\ &= \inf\{\mathrm{e}_{\mathrm{HK}}(M \otimes_R T) \mid R \to T \textit{ is faithfully flat, } T \textit{ is F-finite, and } Z_T = \mathrm{Spec}(T)\}.\end{aligned}$$

Proof. Similar to the proof of Theorem 5.6, the existence of the limits in (1) and (2) and their convergence to $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M)$ is a statement about the uniform limit of semi-continuous functions defined on a quasi-compact topological space. See [32, Theorem 5.6] for necessary details.

Suppose that $R \to T$ is faithfully flat and T is F-finite. It easily follows by Theorem 2.2 that $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) \leqslant \mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T)$. Moreover, if $Z_T = \mathrm{Spec}(T)$ then $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T) = \mathrm{e}_{\mathrm{HK}}(M \otimes_R T)$ by Theorem 3.16. Therefore

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) &\leqslant \inf\{\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M \otimes_R T) \mid R \to T \text{ is faithfully flat and } T \text{ is F-finite}\}\\ &\leqslant \inf\{\mathrm{e}_{\mathrm{HK}}(M \otimes_R T) \mid R \to T \text{ is faithfully flat, } T \text{ is F-finite, and } Z_T = \mathrm{Spec}(T)\}.\end{aligned}$$

Suppose that R is F-finite, we show the existence of a faithfully flat extension $R \to T$ such that $\mathrm{e}_{\mathrm{HK}}^{\mathrm{loc}}(M) = \mathrm{e}_{\mathrm{HK}}(M \otimes_R T)$. We are assuming that R is locally equidimensional. Let T be as in Lemma 5.4, that is $R \to T$

is faithfully flat, with regular fibers, T is F-finite, and $Z_T = \operatorname{Spec}(T)$. By Theorem 2.2 and Theorem 3.16, $\mathrm{e}^{\mathrm{loc}}_{\mathrm{HK}}(M) = \mathrm{e}^{\mathrm{loc}}_{\mathrm{HK}}(M \otimes_R T) = \mathrm{e}_{\mathrm{HK}}(M \otimes_R T)$.

The corresponding statement in the case that R is essentially of finite type over an excellent local ring $(A, \mathfrak{m}, k)$ such that $\hat{A} \otimes_A R$ is locally equidimensional follows by standard uses of the gamma construction, [20, Section 6]. $\square$

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