Hadamard states for quantum Abelian duality

Marco Benini[1,a], **Matteo Capoferri**[2,b], **Claudio Dappiaggi**[3,4,c]

[1] Institut für Mathematik, Universität Potsdam, Karl-Liebknecht-Str. 24-25, 14476 Potsdam, Germany.

[2] Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK.

[3] Dipartimento di Fisica, Università degli Studi di Pavia, Via Bassi 6, I-27100 Pavia, Italy.

[4] Istituto Nazionale di Fisica Nucleare - Sezione di Pavia, Via Bassi 6, I-27100 Pavia, Italy.

[a] mbenini87@gmail.com, [b] matteo.capoferri@gmail.com, [c] claudio.dappiaggi@unipv.it

Version of May 12, 2017

Abstract. Abelian duality is realized naturally by combining differential cohomology and locally covariant quantum field theory. This leads to a C*-algebra of observables, which encompasses the simultaneous discretization of both magnetic and electric fluxes. We discuss the assignment of physically well-behaved states on this algebra and the properties of the associated GNS triple. We show that the algebra of observables factorizes as a suitable tensor product of three C*-algebras: the first factor encodes dynamical information, while the other two capture topological data corresponding to electric and magnetic fluxes. On the former factor and in the case of ultrastatic globally hyperbolic spacetimes with compact Cauchy surfaces, we exhibit a state whose two-point correlation function has the same singular structure of a Hadamard state. Specifying suitable counterparts also on the topological factors we obtain a state for the full theory, ultimately implementing Abelian duality transformations as Hilbert space isomorphisms.

Keywords: Abelian gauge theory, differential cohomology, algebraic quantum field theory, Hadamard states

MSC2010: 81T13, 81T05

1 Introduction

The implementation of the principle of local gauge invariance in the framework of algebraic quantum field theory has been a topic of considerable interest in the past years. Besides the obvious connections to models of major physical interest, a notable feature, which has emerged from the earliest investigations, is the violation of the principle of general local covariance [BFV03] in the prime example of an Abelian gauge theory, i.e. electromagnetism [DL12, BDS13, BDHS14, SDH14]: the assignment of an algebra of observables which encodes the canonical commutation relations as well as information on the dynamics and on gauge invariance violates the isotony axiom, i.e. the morphism between observable algebras induced by a spacetime isometric embedding fails in general to be injective on account of topological obstructions.

In addition, gauge theories are of paramount interest to various areas of mathematical physics since they are the natural playground for the appearance of dualities, the most famous example being the electric/magnetic duality in source-free Maxwell theory. In this context, the discrete nature of magnetic fluxes comes from refining the Faraday tensor as the curvature of a circle bundle connection, whereas discrete electric fluxes stem from the so-called Dirac charge quantization. The natural generalization of this mechanism goes under the name of *Abelian duality*.

In this paper we focus on this specific aspect of Abelian gauge theories, further elaborating on the results of [BBSS15, BBSS16]. The starting point of these papers is the reformulation in the framework of locally covariant quantum field theory [BSS14] of the well-established implementation of Abelian gauge theories by means of differential cohomology [Fre00, Sza12]. This viewpoint has been taken a step further in [BBSS16] by implementing Abelian duality naturally on globally hyperbolic spacetimes, resulting in a functorial assignment of self-dual gauge fields, which encompasses both the causality and the time-slice axiom. This novel approach has at least two net advantages. On the one hand it yields a simultaneous discretization of electric and magnetic fluxes, which is manifest at the level of the algebra of observables. On the other hand, it enhances the Hamiltonian description by [FMS07a, FMS07b] in a spacetime covariant fashion: Abelian duality transformations become natural isomorphisms between suitable quantum field theory functors.

The results obtained in [BBSS16] do not achieve a full-fledged description of Abelian duality yet, namely the assignment of a suitable quantum state that recovers the usual interpretation of quantum theories is missing. Addressing this aspect is the main goal of our paper. While the existence of states is not a matter of debate, not all of them should be considered physically acceptable. Already for the scalar field, in order to guarantee finite quantum fluctuations for all observables as well as the existence of a covariant construction of an algebra of Wick polynomials [KM15], one needs to focus on the restricted class of *Hadamard states*. These are characterized by a specific singular structure of the underlying two-point correlation function [Rad96]. Existence, properties and construction schemes for these states have been thoroughly studied in the past twenty years, including a class of examples of Abelian gauge theories [DS13, FP03, BDM14, GW14]. In the context of Abelian duality, similar results cannot be obtained blindly due to the non-trivial (and fascinating) entanglement of dynamical and topological degrees of freedom [BBSS16]: finding a physically relevant state implementing Abelian duality transformations as Hilbert space isomorphisms is the challenging task addressed here.

In this paper we focus on globally hyperbolic spacetimes of arbitrary dimension, though with compact Cauchy surfaces. For the symplectic Abelian group of observables associated to Abelian duality [BBSS16] we provide a (non-canonical) decomposition into three symplectic Abelian subgroups. The first captures dynamical information, while the other two encode the topological degrees of freedom associated to magnetic and electric fluxes. This decomposition carries over upon Weyl quantization: the full C^*-algebra of observables is isomorphic to an appropriate tensor product of three C^*-algebras, each associated to one of the symplectic Abelian subgroups aforementioned. Therefore, a state for the full C^*-algebra is tantamount to one for each tensor factor. Furthermore, having disentangled the topological and the dynamical degrees of freedom, for the latter we can investigate the existence of states with two-point correlation function fulfilling the microlocal spectrum condition [SV01]. With a slight abuse of terminology we will still call them *Hadamard states*, although the underlying algebra does not fulfil the standard CCR relations of a scalar field that lie at the core of [Rad96]. First we assign states to the C^*-algebras encompassing the topological degrees of freedom and then we focus on the dynamical degrees of freedom. To make our analysis of the dynamical sector more concrete, we will construct these Hadamard states explicitly by focusing on ultra-static backgrounds. Although other situations could be treated as well (however only as a case-by-case analysis), our purpose is to show that Abelian duality offers an elegant interpretation of topological quantities, as well as effective tools to cope with them. Such features are not sensitive to the geometry of the underlying spacetime (only by its topology). As such, non-ultra-static backgrounds would only result in a more involved analysis, without providing any additional insight. As a by-product of our construction, we prove that Abelian duality transformations are implemented at the level of the GNS triple as Hilbert space isomorphisms, thus closing the gap with the existing literature based on a direct Hilbert space description [FMS07a, FMS07b].

As for the structure of the paper, in Section 2 we introduce our notation and conventions, recollecting in particular the most important results from [BSS14, BBSS16, BBSS15]. Most notably, we discuss the symplectic Abelian group of observables. In Section 3, we introduce three auxiliary symplectic Abelian groups, corresponding to the dynamical sector, to the torsion-free topological sector and to the torsion topological sector of our model. These are then used in Section 3.4 to present the symplectic Abelian group of observables as a direct sum of the mentioned sectors. In Section 4, firstly we recall that we can associate a C*-algebra of Weyl type to each symplectic Abelian group. Secondly we prove that the splitting of the preceding section entails a factorization of the full C*-algebra of observables into a suitable tensor product of C*-algebras, each factor being associated to one of the sectors. Focusing on the case of ultra-static, globally hyperbolic spacetimes with compact Cauchy surface, in Section 4.2 we construct the ground state for the C*-algebra associated to the dynamical sector, proving that its two-point correlation function fulfils the microlocal spectrum condition. Then we exhibit states also for the remaining two sectors, see Sections 4.3 and 4.4, thus showing that Abelian duality transformations are implemented as isomorphisms between the relevant GNS triples. Section 4.5 concludes the paper discussing the example of the Lorentz cylinder, providing a Fourier expansion of the two-point correlation function for the dynamical sector of the theory: remarkably, Abelian duality naturally circumvents the infrared obstructions to the existence of ground states in 1+1 dimensions.

2 Preliminaries

In the present section we quickly recapitulate the background material for the rest of the paper. We also take the chance to introduce our notation and conventions. Let us remark once and for all that, unless otherwise stated, for any $m \in \mathbb{Z}_{\geq 0}$, the term *(m-dimensional) manifold* refers to a connected, second-countable, Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^m$ and that comes equipped with a smooth structure (an atlas with smooth transition maps). We will only consider manifolds of *finite type*, namely those admitting a finite good cover.[1] For Abelian groups we will always adopt the additive notation, therefore we define the circle group $\mathbb{T}$ additively as the quotient $\mathbb{R}/\mathbb{Z}$ of Abelian groups (rather than multiplicatively as the group of unit complex numbers). The multiplicative notation is reserved to ring multiplication, e.g. $\wedge$ for the graded algebra of differential forms, $\smile$ for the graded ring structure on cohomology, $\cdot$ for the graded differential cohomology ring and, more generally, juxtaposition for associative algebras.

2.1 Differential cohomology

The most efficient mathematical tool to describe configurations for the field theory that will be introduced in Section 2.2 is provided by differential cohomology. For a full presentation of the subject we refer the reader to the existing literature, e.g. [BB14]. Here we will just recall few basic facts, mainly in order to introduce our notation. Differential cohomology for a manifold M is a graded ring $\hat{\mathrm{H}}^\bullet(M;\mathbb{Z})$ that arises as a suitable refinement of the cohomology ring $\mathrm{H}^\bullet(M;\mathbb{Z})$ with $\mathbb{Z}$-coefficients by the graded algebra $\Omega^\bullet(M)$ of differential forms. More precisely, for $k \in \mathbb{Z}_{\geq 0}$, k-differential cohomology is defined (up to natural isomorphism) as the unique contravariant functor $\hat{\mathrm{H}}^k(\,\cdot\,;\mathbb{Z})$ from a suitable category of smooth spaces (including manifolds) to Abelian groups, equipped with four natural transformations[2] ι, κ, curv, char (relating it to differential forms and cohomology

[1] The finite-type requirement ensures that the cohomology groups considered in the following are finitely generated as Abelian groups. This is a rather mild restriction, e.g. Cartesian spaces, compact manifolds, as well as all globally hyperbolic Lorentzian manifolds with compact Cauchy surface (which are the backgrounds of main interest for the core of the paper), are of finite type. Counterexamples can be produced by removing from a manifold of finite type a closed subset containing infinitely many disconnected components, e.g. a Cartesian space without the lattice of points with integer coordinates.

[2] Note that in the following we will often refer to the components of a natural transformation implicitly. In particular, with abuse of notation, we will denote any natural transformation and its components by the same symbol.

classes) forming the commutative diagram (2.1), subject to the condition that all rows and columns are short exact sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathrm{H}^{k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \xrightarrow{\nu} & \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathbb{Z}}(M)} & \xrightarrow{\mathrm{d}} & \mathrm{d}\,\Omega^{k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1}(M;\mathbb{T}) & \xrightarrow{\kappa} & \hat{\mathrm{H}}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}} & \Omega^k_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow{\scriptstyle [\,\cdot\,]} & & \\
0 & \longrightarrow & \mathrm{H}^k_{\mathrm{tor}}(M;\mathbb{Z}) & \xrightarrow[\mathrm{j}]{} & \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow[\mathrm{q}]{} & \mathrm{H}^k_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.1}
$$

Several equivalent explicit realizations of differential cohomology have been developed in the literature, e.g. Cheeger-Simons differential characters [CS85], de Rham-Federer characters [HLZ03], Hopkins-Singer cocycles [HS05], isomorphism classes of higher circle bundles equipped with connection.

Remark 2.1. For our purposes there is no practical convenience in choosing any of these concrete models, as the only information we will need is commutativity of diagram (2.1) and exactness of its rows and columns. Nonetheless, it is worth looking at the model provided by higher circle bundles and connections to give a geometric interpretation of the objects and of the morphisms in diagram (2.1). In fact, this point of view justifies the origin of the names usually attributed to the natural transformations ι, κ, curv, char:

- The *characteristic class map* $\mathrm{char} : \hat{\mathrm{H}}^k(M;\mathbb{Z}) \to \mathrm{H}^k(M;\mathbb{Z})$ assigns to each isomorphism class of circle $(k-1)$-bundles with connection the characteristic class of the underlying bundle, the so-called Chern class, which is an element of the k-th cohomology group with $\mathbb{Z}$-coefficients (as a concrete example, the reader can consider singular cohomology).

- The *curvature map* $\mathrm{curv} : \hat{\mathrm{H}}^k(M;\mathbb{Z}) \to \Omega^k_{\mathbb{Z}}(M)$ assigns the curvature of the connection, which is an element of the Abelian group $\Omega^k_{\mathbb{Z}}(M)$ of differential k-forms having integral periods, i.e. taking integer values when evaluated on any k-cycle.

- The *inclusion of flat connections* $\kappa : \mathrm{H}^{k-1}(M;\mathbb{T}) \to \hat{\mathrm{H}}^k(M;\mathbb{Z})$ provides (isomorphism classes of) circle $(k-1)$-bundles equipped with a flat connection, which are classified by $\mathrm{H}^{k-1}(M;\mathbb{T})$, i.e. $(k-1)$-cohomology with $\mathbb{T}$-coefficients.

- The *inclusion of trivial bundles* $\iota : \Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M) \to \hat{\mathrm{H}}^k(M;\mathbb{Z})$ equips the trivial circle $(k-1)$-bundle with the connection defined by a $(k-1)$-form, known to physicists as a (higher analogue of the) vector potential, and takes the corresponding isomorphism class, which is reflected by the quotient by $(k-1)$-forms with integral periods.

The interpretation illustrated above is perhaps most familiar in degree $k=2$ as circle 1-bundles with connection are the usual principal circle bundles equipped with a connection in the ordinary sense; in degree $k=1$, instead, differential cohomology reduces to smooth $\mathbb{T}$-valued functions (the corresponding characteristic class is known as the winding number). For $k \geq 3$, one encounters higher analogues of principal circle bundles (also known as gerbes), together with the appropriate notion of a connection. The remaining morphisms displayed in diagram (2.1) are defined as follows:

- $\nu : \mathrm{H}^{k-1}(M;\mathbb{R})/\mathrm{H}^{k-1}_{\mathrm{free}}(M;\mathbb{Z}) \to \Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M)$ exploits de Rham theorem to assign to singular cohomology with $\mathbb{R}$-coefficients the corresponding de Rham cohomology class. More in detail, de Rham theorem provides the natural isomorphism $\mathrm{H}^{k-1}(M;\mathbb{R}) \simeq \Omega^{k-1}_{\mathrm{d}}(M)/\,\mathrm{d}\,\Omega^{k-2}(M)$ between singular cohomology and de Rham cohomology, which restricts to the free part of singular cohomology as $\mathrm{H}^{k-1}_{\mathrm{free}}(M;\mathbb{Z}) \simeq \Omega^{k-1}_{\mathbb{Z}}(M)/\,\mathrm{d}\,\Omega^{k-2}(M)$ (recall that differential forms with integral periods are automatically closed). ν is obtained composing the quotient between these isomorphisms with the inclusion $\Omega^{k-1}_{\mathrm{d}}(M)/\,\mathrm{d}\,\Omega^{k-2}(M) \subseteq \Omega^{k-1}(M)/\,\mathrm{d}\,\Omega^{k-2}(M)$.

- $\mathrm{d} : \Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M) \to \mathrm{d}\,\Omega^{k-1}(M)$ is simply defined by the standard differential on forms.

- $\subseteq : \mathrm{d}\,\Omega^{k-1}(M) \to \Omega^{k}_{\mathbb{Z}}(M)$ is the inclusion.

- $[\,\cdot\,] : \Omega^{k}_{\mathbb{Z}}(M) \to \mathrm{H}^{k}_{\mathrm{free}}(M;\mathbb{Z})$ acts upon the de Rham cohomology class of $\omega \in \Omega^{k}_{\mathbb{Z}}(M)$ assigning to it the corresponding free $\mathbb{Z}$-valued singular cohomology class.

- Lastly, the left column and bottom row of diagram (2.1) are obtained by decomposing into short exact sequences the long exact singular cohomology sequence associated to the short exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$. More explicitly, $\mu : \mathrm{H}^{k-1}(M;\mathbb{R})/\mathrm{H}^{k-1}_{\mathrm{free}}(M;\mathbb{Z}) \to \mathrm{H}^{k-1}(M;\mathbb{T})$ is induced by the homomorphism $\mathbb{R} \to \mathbb{T}$ of coefficients, $\beta : \mathrm{H}^{k-1}(M;\mathbb{T}) \to \mathrm{H}^{k}_{\mathrm{tor}}(M;\mathbb{Z})$ and $\mathrm{j} : \mathrm{H}^{k}_{\mathrm{tor}}(M;\mathbb{Z}) \to \mathrm{H}^{k}(M;\mathbb{Z})$ provide the epi-mono factorization of the Bockstein homomorphism through the torsion part $\mathrm{H}^{k}_{\mathrm{tor}}(M;\mathbb{Z})$ of the cohomology group $\mathrm{H}^{k}(M;\mathbb{Z})$ and $\mathrm{q} : \mathrm{H}^{k}(M;\mathbb{Z}) \to \mathrm{H}^{k}_{\mathrm{free}}(M;\mathbb{Z})$ projects onto the free part $\mathrm{H}^{k}_{\mathrm{free}}(M;\mathbb{Z})$ of $\mathrm{H}^{k}(M;\mathbb{Z})$, namely its quotient by the torsion subgroup.

In analogy with cohomology and differential forms, differential cohomology can be equipped with a natural graded ring structure, denoted by $\,\cdot\,$. This is uniquely specified by naturality, by the condition that both char and curv become ring homomorphisms and by the following compatibility condition for ι and κ:

$$\iota[A] \cdot h = \iota[A \wedge \mathrm{curv}\, h], \qquad\qquad \kappa\, u \cdot h = \kappa(u \smile \mathrm{char}\, h), \tag{2.2}$$

for all $h \in \hat{\mathrm{H}}^{k}(M;\mathbb{Z})$, $[A] \in \Omega^{\ell}(M)/\Omega^{\ell}_{\mathbb{Z}}(M)$ and $u \in \mathrm{H}^{\ell}(M;\mathbb{T})$ (actually, the last compatibility condition is redundant). Further details about differential cohomology and its graded ring structure can be found in [BB14, SS08]. Let us also mention that relative versions of differential cohomology exist (see [BB14] for a comparison among different approaches) and these can be used to realize differential cohomology with restricted support. For example, models with compact support have been considered in [HLZ03, BBSS15].

2.2 Configurations

By means of differential cohomology in degree k on an m-dimensional globally hyperbolic spacetime[3] M we can introduce the model under investigation, see also [FMS07a, FMS07b, BBSS16]. To provide some intuition, let us focus for the moment on the case $k = 2$ and $m = 4$, which is perhaps the most familiar one, postponing a more formal and exhaustive presentation. The model we consider aims to reconcile the duality transformation between "electric and magnetic degrees of

[3] Here and in the following, the term *spacetime* refers to an oriented and time-oriented Lorentzian manifold.

freedom" characterising the source-free vacuum Maxwell theory with a more modern perspective on electromagnetism based on circle bundles and connections, in the spirit of Yang-Mills theory. This approach succeeds in encompassing the Aharonov-Bohm effect by means of non-trivial flat connections and, at the same time, possesses the interesting feature of naturally encoding discrete magnetic fluxes into the theory. In order to do so, we describe our gauge fields as being formed by a pair of principal circle bundles P and $\tilde{P}$ over M, equipped with connections A and $\tilde{A}$ respectively. A gauge transformation is a pair of the usual connection-preserving principal bundle isomorphisms, one for the (P, A)-component and the other for the $(\tilde{P}, \tilde{A})$-component. This is not a mere duplication of the degrees of freedom as one can infer from the equation

$$F_A = *F_{\tilde{A}}, \tag{2.3}$$

that is, the curvature F_A of A matches $*F_{\tilde{A}}$, the Hodge dual of the curvature of $\tilde{A}$. While this condition implies that both A and $\tilde{A}$ are Maxwell connections (recall that both F_A and $F_{\tilde{A}}$ are automatically closed), the converse is not true. In fact, the above equation imposes a strong constraint relating A to $\tilde{A}$, thus allowing to recover the electric/magnetic duality transformation of ordinary source-free vacuum Maxwell theory: given a solution $F \in \Omega^2(M)$ of the source-free vacuum Maxwell equations, by taking $- * F$ one obtains another such solution where the "electric and magnetic degrees of freedom" are interchanged. Here the analogue of this duality transformation is obtained by swapping the components (P, A) and $(\tilde{P}, \tilde{A})$, up to a sign (cf. (2.6)). To summarize, by imposing a suitable field equation on an enlarged configuration space, we are able to reconcile the electric/magnetic duality transformation of Maxwell theory with the point of view on electromagnetism motivated by Yang-Mills theory. As a by-product, along with the discretization of magnetic fluxes, we also obtain the electric counterparts to be discretized. In fact, while A is responsible for the (discrete) magnetic flux as usual, the equation of motion entails that $\tilde{A}$ becomes responsible for the electric flux, which as a consequence is now a discrete quantity too.[4]

In the following we provide a more formal definition of the relevant configuration space in arbitrary degree k and dimension m, we rewrite the equation of motion and we solve the associated Cauchy problem. Although these aspects have been developed in full detail in [BBSS16], for ease of reference we briefly recapitulate here the main results. Using $*$ to denote the Hodge dual induced by the metric and by the orientation of M and introducing the convenient notation

$$A^{p,q} \doteq A^p \times A^q \tag{2.4}$$

for any $p, q \in \mathbb{Z}_{\geq 0}$ and any graded Abelian group $A^\bullet$, we specify the Abelian group

$$\mathfrak{C}^k(M; \mathbb{Z}) \doteq \{(h, \tilde{h}) \in \hat{\mathrm{H}}^{k,m-k}(M; \mathbb{Z}) : \ \mathrm{curv}\, h = * \,\mathrm{curv}\, \tilde{h}\}, \tag{2.5}$$

encompassing the configurations $(h, \tilde{h})$ for the field theory of interest. As explained in more detail above, in degree $k = 2$ and dimension $m = 4$, (2.5) provides a *semiclassical* refinement of Maxwell theory in the vacuum and without external sources, capable of encoding the discretization of both electric and magnetic fluxes that originate from topological features of the underlying spacetime. This refinement has further pleasant features: it encodes the Aharonov-Bohm effect (in the form of non-trivial flat connections) and, at the same time, it allows us to define a counterpart of the duality transformation between electric and magnetic degrees of freedom typical of the source-free

[4] Notice that in this context both electric and magnetic fluxes are not related to external sources, rather they arise on account of topological features of the background spacetime M.

vacuum Maxwell theory, see [FMS07a, FMS07b, BBSS16]. In fact, recalling that $** = (-1)^{k(m-k)+1}$ on k-forms over M, one obtains a natural isomorphism

$$\zeta : \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^{m-k}(M;\mathbb{Z}), \qquad (h,\tilde{h}) \longmapsto (\tilde{h}, (-1)^{k(m-k)+1} h), \tag{2.6}$$

that interchanges precisely the (discrete) magnetic and electric fluxes, which are carried by h and $\tilde{h}$ respectively and which are detected by the associated characteristic classes via char. In particular, for $m = 2k$ this duality transformation becomes a natural automorphism that can be used to identify self-dual configurations [BBSS16, Sect. 7].

Any configuration in $\mathfrak{C}^k(M;\mathbb{Z})$ is fully determined by initial data in $\hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ on a spacelike Cauchy surface Σ of M. More precisely, the following Cauchy problem for $(h,\tilde{h}) \in \hat{\mathrm{H}}^{k,m-k}(M;\mathbb{Z})$ with initial data $(h_\Sigma, \tilde{h}_\Sigma) \in \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ is well-posed:

$$\mathrm{curv}\, h = *\, \mathrm{curv}\, \tilde{h}, \tag{2.7a}$$

$$i_\Sigma^*\, h = h_\Sigma, \tag{2.7b}$$

$$i_\Sigma^*\, \tilde{h} = \tilde{h}_\Sigma, \tag{2.7c}$$

where $i_\Sigma : \Sigma \to M$ is the embedding of the spacelike Cauchy surface into M and $i_\Sigma^* \doteq \hat{\mathrm{H}}^p(i_\Sigma;\mathbb{Z}) : \hat{\mathrm{H}}^p(M;\mathbb{Z}) \to \hat{\mathrm{H}}^p(\Sigma;\mathbb{Z})$ denotes the differential cohomology pullback. Well-posedness is equivalent to the Abelian group homomorphism

$$i_\Sigma^* : \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}), \qquad (h,\tilde{h}) \longmapsto (i_\Sigma^*\, h, i_\Sigma^*\, \tilde{h}) \tag{2.8}$$

being an isomorphism, which is shown in full detail in [BBSS16, Sect. 2]. Here we just sketch the basic idea behind the proof. In order to do so, however, we need to introduce first a counterpart of (2.1) where $\hat{H}^k(M,\mathbb{Z})$ is replaced by $\mathfrak{C}^k(M;\mathbb{Z})$, in such a way that the diagram is commutative and its rows and columns still form exact sequences. Defining topologically trivial configurations as

$$\mathfrak{T}^k(M;\mathbb{Z}) \doteq \big\{([A],[\tilde{A}]) \in \Omega^{k-1,m-k-1}(M)/\Omega^{k-1,m-k-1}_{\mathbb{Z}}(M) :\ \mathrm{d}\, A = *\, \mathrm{d}\, \tilde{A}\big\}, \tag{2.9}$$

the above-mentioned diagram reads

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \xrightarrow{\nu\times\nu} & \mathfrak{T}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{d}_1} & \mathrm{d}\,\Omega^{k-1}\cap *\,\mathrm{d}\,\Omega^{m-k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu\times\mu} & & \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}}\cap *\,\Omega^{m-k}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta\times\beta} & & \downarrow{\scriptstyle \mathrm{char}\times\mathrm{char}} & & \downarrow{\scriptstyle ([\,\cdot\,],[*^{-1}\,\cdot\,])} & & \\
0 & \longrightarrow & \mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z}) & \xrightarrow[\mathrm{j}\times\mathrm{j}]{} & \mathrm{H}^{k,m-k}(M;\mathbb{Z}) & \xrightarrow[\mathrm{q}\times\mathrm{q}]{} & \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.10}$$

where the subscript $_1$ denotes the precomposition with the projection on the first factor. By pullback along the embedding $i_\Sigma : \Sigma \to M$, diagram (2.10) maps to a similar diagram that is obtained by "doubling" (2.1) on Σ:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} & \xrightarrow{\nu\times\nu} & \frac{\Omega^{k-1,m-k-1}(\Sigma)}{\Omega^{k-1,m-k-1}_{\mathbb{Z}}(\Sigma)} & \xrightarrow{\mathrm{d}\times\mathrm{d}} & \mathrm{d}\,\Omega^{k-1,m-k-1}(\Sigma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu\times\mu} & & \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) & \xrightarrow{\mathrm{curv}\times\mathrm{curv}} & \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta\times\beta} & & \downarrow{\scriptstyle \mathrm{char}\times\mathrm{char}} & & \downarrow{\scriptstyle ([\cdot],[\cdot])} & & \\
0 & \longrightarrow & \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) & \xrightarrow[\mathrm{j}\times\mathrm{j}]{} & \mathrm{H}^{k,m-k}(\Sigma;\mathbb{Z}) & \xrightarrow[\mathrm{q}\times\mathrm{q}]{} & \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.11}
$$

Evidently, the embedding $i_\Sigma : \Sigma \to M$ induces a morphism of diagrams with source given by (2.10) and target given by (2.11). Notice that, indeed, the isomorphism (2.8) eventually extends to an isomorphism between the full diagrams (2.10) and (2.11). Therefore, besides providing an equivalent way to describe the configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ in terms of initial data, the assignment of a spacelike Cauchy surface allows us to consistently relate all sorts of information encoded by (2.10) with their counterparts in (2.11). We will often switch between these two viewpoints in the following, especially in Section 3.

We illustrate how the above mentioned isomorphism between the diagrams (2.10) and (2.11) arises, while we recall the argument used in [BBSS16, Sect. 2] to prove well-posedness of the Cauchy problem (2.7). To start with, focus on the middle horizontal part of the morphism of diagrams induced by $i_\Sigma : \Sigma \to M$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}} \cap * \,\Omega^{m-k}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^*_\Sigma} & & \downarrow{\scriptstyle i^*_\Sigma} & & \downarrow{\scriptstyle (i^*_\Sigma,\, i^*_\Sigma \circ *^{-1})} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) & \xrightarrow{\mathrm{curv}\times\mathrm{curv}} & \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma) & \longrightarrow & 0
\end{array}
\tag{2.12}
$$

Note that the central morphism in (2.12) is precisely (2.8). Recall that, being a globally hyperbolic spacetime, M is homotopic to its Cauchy surface Σ. Therefore, it follows by homotopy invariance of cohomology groups that we have an isomorphism of Abelian groups on the left of (2.12). To prove that the right morphism is an isomorphism too, we exploit well-posedness of the Cauchy problem for the Maxwell equation [DL12]: notice that $F \in \Omega^k_{\mathbb{Z}} \cap * \,\Omega^{m-k}_{\mathbb{Z}}(M)$ is a solution of the Maxwell equation because forms with integral periods are in particular closed. Then the right morphism assigns precisely the initial datum for F, whose components have integral periods by homotopy invariance of cohomology groups. Conversely, homotopy invariance of homology groups entails that an initial datum with integral periods leads to a solution F of the Maxwell Cauchy problem lying in $\Omega^k_{\mathbb{Z}} \cap * \,\Omega^{m-k}_{\mathbb{Z}}(M)$: to check this, recall that each cycle on M can be presented as the sum between a cycle on Σ and the boundary of a chain, which is irrelevant when evaluated on a closed form, such as F and $*^{-1}F$; therefore, the periods of both F and $*^{-1}F$ are determined by the periods of the initial datum. Since (2.12) is a commutative diagram with short exact rows and we have isomorphisms on the left and on the right, the five lemma entails that the central morphism, i.e. (2.8), is an isomorphism too.

Remark 2.2. Similar conclusions hold true for a modified version $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of the configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ with support restricted to spacelike compact regions, see [BBSS16, Sect. 3]. In particular, the embedding $i_\Sigma : \Sigma \to M$ of a spacelike Cauchy surface into the globally hyperbolic spacetime M induces an isomorphism similar to (2.8):

$$i_\Sigma^* : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z}), \tag{2.13}$$

where $\hat{\mathrm{H}}^p_{\mathrm{c}}(\Sigma;\mathbb{Z})$ is the Abelian group of p-differential characters with compact support on Σ (cf. [BBSS15, HLZ03]). The Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of spacelike compact configurations plays an important role because it can be used to introduce a class of well-behaved functionals on $\mathfrak{C}^{k,m-k}(M;\mathbb{Z})$ that can be regarded as *observables* for this theory, [BBSS16, Sect. 4.2]. Notice that diagrams (2.10) and (2.11) have counterparts with spacelike compact and, respectively, compact support (see [BBSS15, BBSS16]) that provide crucial information about the observables of the model under consideration. Since in the rest of the paper we will be dealing with globally hyperbolic spacetimes admitting *compact* Cauchy surfaces, there will be no distinction between $\mathfrak{C}^k(M;\mathbb{Z})$ and $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$, thus allowing for a remarkable simplification: we will be in a position to introduce observables in Section 2.3 without mentioning any restriction on their support. Nonetheless, it should be observed that in the generic situation (i.e. *arbitrary* Cauchy surface) the support restriction is crucial to define observables, which otherwise would be *ill-defined*, cf. [BBSS16, Sect. 4]. This inevitable support restriction is also the origin of the potential *degeneracies* in the presymplectic structure associated to the model considered here, which of course can only arise in the case of a *non-compact* Cauchy surface, cf. [BBSS16, Prop. 4.5]. Working with globally hyperbolic spacetimes that admit compact Cauchy surfaces will therefore enable us to always deal with *symplectic* structures (more generally, with weakly non-degenerate pairings), a feature that plays a major role in many of the constructions in Section 3.

2.3 Observables

In this section we recall the notion of observable considered in [BBSS16, Sect. 4]. From now on and unless otherwise stated we will consider globally hyperbolic spacetimes M admitting a *compact* Cauchy surface, thus avoiding many of the technical complications that arise in the generic situation, as already mentioned in Remark 2.2 above.

In view of the aforementioned assumption, we introduce a non-degenerate pairing $\sigma : \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k(M;\mathbb{Z}) \to \mathbb{T}$ and we use it to interpret $\mathfrak{C}^k(M;\mathbb{Z})$ as the Abelian group labelling a well-behaved class of observables on $\mathfrak{C}^k(M;\mathbb{Z})$ of the form $\sigma(\,\cdot\,,(h,\tilde{h}))$, $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$. At the same time, this pairing will also provide a symplectic structure on $\mathfrak{C}^k(M;\mathbb{Z})$. The procedure to define σ involves the isomorphism (2.8) between configurations and initial data on a spacelike Cauchy surface Σ of M and the definition of a suitable pairing σ^Σ on initial data $\hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$. To introduce $\sigma^\Sigma : \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \to \mathbb{T}$ we proceed as follows:

1. Recalling the ring structure for differential cohomology on Σ, we introduce a bi-homomorphism of Abelian groups

$$\begin{aligned} \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) &\longrightarrow \hat{\mathrm{H}}^m(\Sigma;\mathbb{Z}), \\ ((h_\Sigma,\tilde{h}_\Sigma),(h'_\Sigma,\tilde{h}'_\Sigma)) &\longmapsto \tilde{h}_\Sigma \cdot h'_\Sigma - \tilde{h}'_\Sigma \cdot h_\Sigma; \end{aligned} \tag{2.14}$$

2. Due to $\dim \Sigma = m-1$, it follows that $\hat{\mathrm{H}}^m(\Sigma;\mathbb{Z})$ is isomorphic to $\mathrm{H}^{m-1}(\Sigma;\mathbb{R})/\mathrm{H}^{m-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})$ in a natural way;

3. Since Σ is compact and oriented (its orientation is induced by orientation and time-orientation of M), we can consider its fundamental class $[\Sigma] \in H_{m-1}(\Sigma)$;

4. Σ is also connected, hence the canonical evaluation of cohomology classes on $[\Sigma]$ yields an isomorphism $\mathrm{H}^{m-1}(\Sigma;\mathbb{R})/\mathrm{H}^{m-1}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \to \mathbb{T}$.

These considerations lead to the definition of σ^Σ given below:

$$\sigma^\Sigma : \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{T}, \quad ((h_\Sigma, \tilde{h}_\Sigma),(h'_\Sigma,\tilde{h}'_\Sigma)) \longmapsto (\tilde{h}_\Sigma \cdot h'_\Sigma - \tilde{h}'_\Sigma \cdot h_\Sigma)[\Sigma]. \tag{2.15}$$

Notice that σ^Σ is a weakly non-degenerate pairing by [BBSS15, Prop. 5.6]; moreover, it is antisymmetric. Pulling back σ^Σ along the isomorphism $i_\Sigma^* : \mathfrak{C}^k(M;\mathbb{Z}) \to \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ in (2.8) we obtain

$$\sigma \doteq \sigma^\Sigma \circ (i_\Sigma^* \times i_\Sigma^*) : \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \mathbb{T}. \tag{2.16}$$

An argument based on Stokes' theorem shows that σ does not depend on the chosen (compact) spacelike Cauchy surface Σ, cf. [BBSS16, Lem. 8.4]. Clearly, σ inherits the properties of σ^Σ, in particular it is a weakly non-degenerate antisymmetric pairing.

For $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$, $\sigma(\,\cdot\,,(h,\tilde{h}))$ defines a functional on $\mathfrak{C}^k(M;\mathbb{Z})$. [BBSS16, BBSS15] show that these functionals are distinguished, to the extent that they provide smooth characters on the configuration space. We interpret these functionals as the observables for the model under consideration. Since σ is weakly non-degenerate, functionals of this type form an Abelian group isomorphic to $\mathfrak{C}^k(M;\mathbb{Z})$. Furthermore, σ is antisymmetric, therefore we regard

$$(\mathfrak{C}^k(M;\mathbb{Z}),\sigma) \tag{2.17}$$

as the symplectic Abelian group of observables for our theory. Recalling (2.15) and (2.16) and on account of graded commutativity, it follows that the duality isomorphism $\zeta : \mathfrak{C}^k(M;\mathbb{Z}) \to \mathfrak{C}^{m-k}(M;\mathbb{Z})$ defined in (2.6) preserves σ, hence the Abelian duality transformation ζ is implemented symplectically at the level of observables.

3 Symplectically orthogonal decomposition

The goal of this section is to establish a decomposition of the Abelian group of observables $\mathfrak{C}^k(M;\mathbb{Z})$ that is compatible with its symplectic structure σ. As we will see later, at the level of quantum algebras this decomposition corresponds to a convenient factorization that will enable us to introduce a state on the full algebra by looking at each factor separately. Once again, we consider a globally hyperbolic spacetimes M admitting a compact spacelike Cauchy surface Σ. The peculiarity of the compact Cauchy surface case is that all pairings we are going to consider are weakly non-degenerate, while they happen to have degeneracies in general. As above, we will denote the embedding of Σ into M by i_Σ and we will often implicitly consider the isomorphism $i_\Sigma^* : \mathfrak{C}^k(M;\mathbb{Z}) \to \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ in (2.8). In particular, recall that i_Σ^* extends to an isomorphism between the diagrams (2.10) and (2.11) and that it relates the symplectic structure σ on $\mathfrak{C}^k(M;\mathbb{Z})$, cf. (2.16), to the symplectic structure σ^Σ on $\hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$, cf. (2.15). Our task is twofold: first, we aim at introducing suitable symplectic structures on certain auxiliary Abelian groups arising from the diagrams (2.10) and (2.11); second, we want to provide a symplectically orthogonal decomposition of the symplectic Abelian group $(\mathfrak{C}^k(M;\mathbb{Z}),\sigma)$ in terms of the above-mentioned auxiliary symplectic Abelian groups.

3.1 Dynamical sector

With *dynamical sector* we refer to the top-right corner of diagram (2.10), i.e. the vector space

$$\mathsf{Dyn}^k(M) \doteq \mathrm{d}\,\Omega^{k-1}(M) \cap * \,\mathrm{d}\,\Omega^{m-k-1}(M). \tag{3.1}$$

The reader should keep in mind that, as a special case of [BBSS16, Th. 2.5], one obtains an isomorphism induced by the restriction to the spacelike Cauchy surface Σ of M:

$$i_\Sigma^* : \mathsf{Dyn}^k(M) \xrightarrow{\simeq} \mathrm{d}_\Sigma\,\Omega^{k-1,m-k-1}(\Sigma), \qquad \mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A} \longmapsto (i_\Sigma^*\,\mathrm{d}\,A, i_\Sigma^*\,\mathrm{d}\,\tilde{A}). \tag{3.2}$$

In the following we will often omit to spell out both ways to express an element of $\mathsf{Dyn}(M)$, that is to say that we will only present it as $\mathrm{d}\,A \in \mathsf{Dyn}^k(M)$, although by definition there exists $\tilde{A} \in \Omega^{m-k-1}(M)$ such that $\mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A}$.

We can endow $\mathsf{Dyn}(M)$ with a pairing, namely

$$\sigma_{\mathsf{Dyn}} : \mathsf{Dyn}^k(M) \times \mathsf{Dyn}^k(M) \longrightarrow \mathbb{R}, \qquad (\mathrm{d}\,A, \mathrm{d}\,A') \longmapsto \int_\Sigma i_\Sigma^*(\tilde{A} \wedge \mathrm{d}\,A' - \tilde{A}' \wedge \mathrm{d}\,A). \tag{3.3}$$

Stokes theorem implies that σ_{Dyn} is well-defined and that its definition does not depend on the choice of Σ. Clearly, there is an equivalent pairing $\sigma^\Sigma_{\mathsf{Dyn}}$ on the isomorphic vector space $\mathrm{d}_\Sigma\,\Omega^{k-1,m-k-1}(\Sigma)$ and explicitly defined using the same formula (note the subscript $_\Sigma$ to distinguish the differential on Σ from the one on M):

$$\sigma^\Sigma_{\mathsf{Dyn}}((\mathrm{d}_\Sigma A_\Sigma, \mathrm{d}_\Sigma A'_\Sigma), (\mathrm{d}_\Sigma \tilde{A}_\Sigma, \mathrm{d}_\Sigma \tilde{A}'_\Sigma)) = \int_\Sigma \tilde{A}_\Sigma \wedge \mathrm{d}_\Sigma A'_\Sigma - \tilde{A}'_\Sigma \wedge \mathrm{d}_\Sigma A_\Sigma, \tag{3.4}$$

for $(\mathrm{d}_\Sigma A_\Sigma, \mathrm{d}_\Sigma A'_\Sigma), (\mathrm{d}_\Sigma \tilde{A}_\Sigma, \mathrm{d}_\Sigma \tilde{A}'_\Sigma) \in \mathrm{d}_\Sigma\,\Omega^{k-1,m-k-1}(\Sigma)$. From its definition and using Stokes theorem, one can conclude that $\sigma^\Sigma_{\mathsf{Dyn}}$ is both weakly non-degenerate and anti-symmetric. Therefore, $(\mathrm{d}_\Sigma\,\Omega^{k-1,m-k-1}(\Sigma), \sigma^\Sigma_{\mathsf{Dyn}})$ and $(\mathsf{Dyn}(M), \sigma_{\mathsf{Dyn}})$ are isomorphic symplectic vector spaces.

The dynamical sector carries a natural duality isomorphism, the counterpart of (2.6), which evidently preserves the symplectic structure σ_{Dyn}:

$$\zeta_{\mathsf{Dyn}} : \mathsf{Dyn}^k(M) \longrightarrow \mathsf{Dyn}^{m-k}(M), \qquad \mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A} \longmapsto \mathrm{d}\,\tilde{A} = *\,\mathrm{d}(-1)^{k(m-k)+1}A. \tag{3.5}$$

Remark 3.1. We just established a symplectic structure σ_{Dyn} for the vector space $\mathsf{Dyn}^k(M)$, the top-right corner of diagram (2.10); moreover, we introduced an isomorphic symplectic vector space $(\mathrm{d}_\Sigma\,\Omega^{k-1,m-k-1}(\Sigma), \sigma^\Sigma_{\mathsf{Dyn}})$ corresponding to initial data on a (compact) spacelike Cauchy surface Σ. Recall that $(\mathfrak{C}^k(M;\mathbb{Z}), \sigma)$ is just a symplectic Abelian group, rather than a symplectic vector space; therefore, in order to relate $(\mathsf{Dyn}^k(M), \sigma_{\mathsf{Dyn}})$ to $(\mathfrak{C}^k(M;\mathbb{Z}), \sigma)$, we will have to forget the multiplication by scalars in $\mathsf{Dyn}^k(M)$ and turn σ_{Dyn} into a $\mathbb{T}$-valued bi-homomorphism, i.e. post-compose it with the quotient $\mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. We will do so in Section 3.4 in order to identify $(\mathsf{Dyn}^k(M), \sigma_{\mathsf{Dyn}})$ as a direct summand of $(\mathfrak{C}^k(M;\mathbb{Z}), \sigma)$.

3.2 Torsion-free topological sector

The second contribution we consider for our decomposition of $\mathfrak{C}^k(M;\mathbb{Z})$ contains topological information only (refer to Remark 3.2 for a physical interpretation). We quotient out the part related to torsion subgroups as those have to be treated separately (and have their own interpretation, cf.

[FMS07b]). This leads to the *torsion-free topological sector*, namely the Abelian group arising from the direct sum between the top-left and the bottom-right corners of diagram (2.10):

$$\mathsf{Top}^k_{\mathrm{free}}(M) \doteq \frac{\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z})} \oplus \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z}). \tag{3.6}$$

Also $\mathsf{Top}^k_{\mathrm{free}}(M)$ has an equivalent description in terms of data specified on a (compact) spacelike Cauchy surface Σ:

$$i_\Sigma^* : \mathsf{Top}^k_{\mathrm{free}}(M) \xrightarrow{\simeq} \frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \oplus \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z}). \tag{3.7}$$

This isomorphism, obtained by pulling cohomology classes back along $i_\Sigma : \Sigma \to M$, is just an instance of homotopy invariance of cohomology groups. Further information can be found in [BBSS16, Lem. 8.2].

We equip $\mathsf{Top}^k_{\mathrm{free}}(M)$ with a pairing σ_{free} induced by the cup product $\smile$ between cohomology groups. The procedure to define σ_{free} is similar to the one adopted in Section 2.3, namely we introduce a pairing $\sigma^\Sigma_{\mathrm{free}}$ on the right-hand side of (3.7) and then we induce σ_{free} on $\mathsf{Top}^k_{\mathrm{free}}(M)$ via the isomorphism (3.7). As a first step, following [BBSS15, (5.22)], we observe that the cohomological cup product provides a bi-homomorphism of Abelian groups:

$$\mathrm{H}^p(\Sigma;\mathbb{T}) \times \mathrm{H}^{m-p-1}(\Sigma;\mathbb{Z}) \longrightarrow \mathrm{H}^{m-1}(\Sigma;\mathbb{T}), \qquad (f_\Sigma,\zeta_\Sigma) \longmapsto f_\Sigma \smile \zeta_\Sigma. \tag{3.8}$$

Note that for dimensional reasons $\mathrm{H}^{m-1}(\Sigma;\mathbb{T})$ is naturally isomorphic to the quotient of $\mathrm{H}^{m-1}(\Sigma;\mathbb{R})$ by its subgroup $\mathrm{H}^{m-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})$, cf. (2.1). Therefore, repeating arguments 3. and 4. of Section 2.3 and with the help of diagram (2.1) (recall the morphisms μ and q in particular), we can introduce the non-degenerate pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{free}} : \frac{\mathrm{H}^p(\Sigma;\mathbb{R})}{\mathrm{H}^p_{\mathrm{free}}(\Sigma;\mathbb{Z})} \times \mathrm{H}^{m-p-1}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{T}, \qquad (u_\Sigma,z_\Sigma) \longmapsto (\mu\, u_\Sigma \smile \zeta_\Sigma)[\Sigma], \tag{3.9}$$

where ζ_Σ is any element of $\mathrm{H}^{m-p-1}(\Sigma;\mathbb{Z})$ such that $\mathrm{q}\,\zeta_\Sigma = z_\Sigma$. This definition is well-posed on account of the properties of the cup product. A suitable combination of the pairings $\langle\cdot,\cdot\rangle_{\mathrm{free}}$ for different degrees provides

$$\sigma^\Sigma_{\mathrm{free}} : \left(\frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \oplus \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})\right) \times \left(\frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \oplus \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})\right) \longrightarrow \mathbb{T}, \tag{3.10a}$$

where

$$\begin{aligned} &\sigma^\Sigma_{\mathrm{free}}((u,\tilde{u},z,\tilde{z}),(u',\tilde{u}',z',\tilde{z}')) \\ &\quad \doteq \langle \tilde{u}_\Sigma, z'_\Sigma\rangle_{\mathrm{free}} - (-1)^{k(m-k)}\langle u_\Sigma, \tilde{z}'_\Sigma\rangle_{\mathrm{free}} - \langle \tilde{u}'_\Sigma, z_\Sigma\rangle_{\mathrm{free}} + (-1)^{k(m-k)}\langle u'_\Sigma, \tilde{z}_\Sigma\rangle_{\mathrm{free}}. \end{aligned} \tag{3.10b}$$

$\sigma^\Sigma_{\mathrm{free}}$ is clearly antisymmetric and inherits non-degeneracy from $\langle\cdot,\cdot\rangle_{\mathrm{free}}$. These properties are directly transferred to the pairing σ_{free}, defined on $\mathsf{Top}^k_{\mathrm{free}}(M)$ as the pullback of $\sigma^\Sigma_{\mathrm{free}}$ along (3.7):

$$\sigma_{\mathrm{free}} \doteq \sigma^\Sigma_{\mathrm{free}} \circ (i_\Sigma^* \times i_\Sigma^*) : \mathsf{Top}^k_{\mathrm{free}}(M) \times \mathsf{Top}^k_{\mathrm{free}}(M) \longrightarrow \mathbb{T}. \tag{3.11}$$

Notice that σ_{free} is actually independent of the choice of Σ. In fact, for any choice of Cauchy surface Σ, $i_{\Sigma\,*}[\Sigma]$ is the unique generator of $\mathrm{H}_{m-1}(M) \simeq \mathbb{Z}$. Summing up, the right-hand side of (3.7) equipped with $\sigma^{\Sigma}_{\mathrm{free}}$ and $(\mathsf{Top}^k_{\mathrm{free}}(M),\sigma_{\mathrm{free}})$ are isomorphic symplectic Abelian groups. In Section 3.4 we will identify $(\mathsf{Top}^k_{\mathrm{free}}(M),\sigma_{\mathrm{free}})$ as a direct summand of the symplectic Abelian group $(\mathfrak{C}^k(M;\mathbb{Z}),\sigma)$.

As the dynamical sector, also $(\mathsf{Top}^k_{\mathrm{free}}(M),\sigma_{\mathrm{free}})$ carries a counterpart of the duality isomorphism (2.6):

$$\zeta_{\mathrm{free}}:\mathsf{Top}^k_{\mathrm{free}}(M)\longrightarrow\mathsf{Top}^{m-k}_{\mathrm{free}}(M),\quad (u,\tilde{u},z,\tilde{z})\longmapsto(\tilde{u},(-1)^{k(m-k)+1}u,\tilde{z},(-1)^{k(m-k)+1}z). \tag{3.12}$$

Note that the one above is a natural isomorphism preserving σ_{free}, hence the duality transformation is symplectically implemented also on the torsion-free topological sector.

3.3 Torsion topological sector

This is the last contribution we have to consider in order to decompose $\mathfrak{C}^k(M;\mathbb{Z})$. Again it contains information of purely topological nature, but it is quite special in that it relates to the torsion part of certain cohomology groups. An interpretation of these quantities in terms of non-commutativity between electric and magnetic fluxes can be found in [FMS07a, FMS07b], see also Remark 3.2. As in the previous sections, we will provide two equivalent ways to describe the object of interest, related by an isomorphism induced by the embedding $i_\Sigma:\Sigma\to M$ of a (compact) spacelike Cauchy surface Σ into the globally hyperbolic spacetime M. Subsequently, we will introduce a suitable symplectic structure. We will refer to the Abelian group

$$\mathsf{Top}^k_{\mathrm{tor}}(M)\doteq\mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z}) \tag{3.13}$$

as the *torsion topological sector*. Since the embedding $i_\Sigma:\Sigma\to M$ is a retraction, homotopy invariance of cohomology implies that the restriction along i_Σ induces an equivalent description of $\mathsf{Top}^k_{\mathrm{tor}}(M)$ in terms of cohomology groups of the Cauchy surface Σ:

$$i^*_\Sigma:\mathsf{Top}^k_{\mathrm{tor}}(M)\xrightarrow{\simeq}\mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}). \tag{3.14}$$

Notice that this is an isomorphism of Abelian groups in contrast to (3.2), which is an isomorphism of vector spaces.

Adopting the Cauchy surface point of view on $\mathsf{Top}^k_{\mathrm{tor}}(M)$, it is easy to construct a symplectic structure, that essentially arises from the torsion linking form. We introduce a non-degenerate $\mathbb{T}$-valued pairing between $\mathrm{H}^p_{\mathrm{tor}}(\Sigma;\mathbb{Z})$ and $\mathrm{H}^{m-p}_{\mathrm{tor}}(\Sigma;\mathbb{Z})$ using (3.8) and (2.1) (recall the morphisms j and β in particular):

$$\langle\cdot,\cdot\rangle_{\mathrm{tor}}:\mathrm{H}^p_{\mathrm{tor}}(\Sigma;\mathbb{Z})\times\mathrm{H}^{m-p}_{\mathrm{tor}}(\Sigma;\mathbb{Z})\to\mathbb{T},\qquad (t_\Sigma,t'_\Sigma)\longmapsto(u_\Sigma\smile \mathrm{j}\,t'_\Sigma)[\Sigma], \tag{3.15}$$

for any $u_\Sigma\in\mathrm{H}^{p-1}(\Sigma;\mathbb{T})$ such that $\beta\,u_\Sigma=t_\Sigma$. Notice that this definition is well-posed on account of the properties of the cup product. Using $\langle\cdot,\cdot\rangle_{\mathrm{tor}}$, we can introduce the bi-homomorphism:

$$\sigma^\Sigma_{\mathrm{tor}}:\mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z})\times\mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z})\longrightarrow\mathbb{T},\quad ((t_\Sigma,\tilde{t}_\Sigma),(t'_\Sigma,\tilde{t}'_\Sigma))\longrightarrow\langle\tilde{t}_\Sigma,t'_\Sigma\rangle_{\mathrm{tor}}-\langle\tilde{t}'_\Sigma,t_\Sigma\rangle_{\mathrm{tor}}. \tag{3.16}$$

Since the pairing $\langle\cdot,\cdot\rangle_{\mathrm{tor}}$ is non-degenerate, also $\sigma^\Sigma_{\mathrm{tor}}$ is such. Furthermore, it is clearly antisymmetric, hence $(\mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}),\sigma^\Sigma_{\mathrm{tor}})$ is a symplectic Abelian group. The isomorphism allows us to transfer $\sigma^\Sigma_{\mathrm{tor}}$ to $\mathsf{Top}^k_{\mathrm{tor}}(M)$ by setting

$$\sigma_{\mathrm{tor}}\doteq\sigma^\Sigma_{\mathrm{tor}}\circ(i^*_\Sigma\times i^*_\Sigma):\mathsf{Top}^k_{\mathrm{tor}}(M)\times\mathsf{Top}^k_{\mathrm{tor}}(M)\longrightarrow\mathbb{T}. \tag{3.17}$$

The argument that makes the definition in (3.11) independent of the choice of Cauchy surface can be applied here too to show that also σ_{tor} does not depend on such choice. With the last equation, we have endowed $\mathsf{Top}^k_{\mathrm{tor}}(M)$ with a natural symplectic structure, so that $(\mathsf{Top}^k_{\mathrm{tor}}(M), \sigma_{\mathrm{tor}})$ is a symplectic Abelian group.

Similarly to the dynamical and the torsion-free topological sectors, also the torsion topological sector carries a natural duality isomorphism, compatible with the symplectic structure σ_{tor}:

$$\zeta_{\mathrm{tor}} : \mathsf{Top}^k_{\mathrm{tor}}(M) \longrightarrow \mathsf{Top}^{m-k}_{\mathrm{tor}}(M), \qquad (t, \tilde{t}) \longmapsto (\tilde{t}, (-1)^{k(m-k)+1} t). \tag{3.18}$$

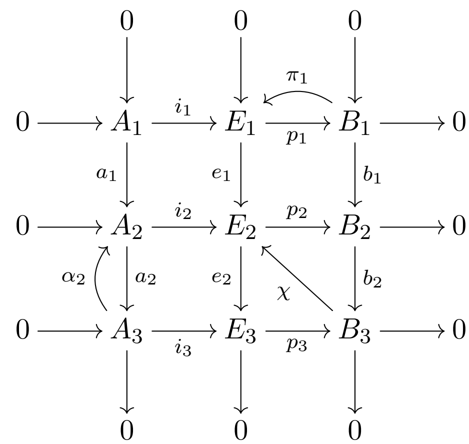
Remark 3.2. To illustrate the physical content of the topological sectors of the configuration space $\mathfrak{C}^k(M;\mathbb{Z})$, it is perhaps most effective to look first at the various components without separating out the torsion sector from the torsion-free sector. Looking back at diagram (2.10), one realizes that the relevant topological information is stored in the Abelian group $\mathrm{H}^{k,m-k}(M;\mathbb{Z})$, corresponding to admissible characteristic classes, and in the Abelian group $\mathrm{H}^{k-1,m-k-1}(M;\mathbb{T})$, classifying all flat fields. (As a side remark, note that there is some common information which is encoded by both groups, namely the torsion classes described by $\mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z})$.) Recalling Section 2.2, one realizes that $\mathrm{H}^{k,m-k}(M;\mathbb{Z})$ is responsible for the *discretized magnetic and electric fluxes*. In fact, given a configuration $(h, \tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$, in full analogy with the usual interpretation of the Chern class of a principal circle bundle, one realizes that $\mathrm{char}\, h \in \mathrm{H}^k(M;\mathbb{Z})$ carries the magnetic flux. On the other end, the condition $\mathrm{curv}\, h = * \,\mathrm{curv}\, \tilde{h}$ indicates that the curvature of $\tilde{h}$ corresponds (up to appropriate sign) to the Hodge dual of the curvature of h, whose restriction to the initial data surface corresponds to the electric field (as in ordinary Maxwell theory). Therefore one is naturally led to interpret the characteristic class $\mathrm{char}\, \tilde{h} \in \mathrm{H}^{m-k}(M;\mathbb{Z})$ as witnessing the discretization of the electric flux. As far as the Abelian group $\mathrm{H}^{k-1,m-k-1}(M;\mathbb{Z})$ of flat fields is concerned, this has the usual interpretation in terms of the Aharonov-Bohm effect: while this feature is not present in ordinary Maxwell theory (as it is only concerned with curvatures), the modern approach to electromagnetism as a gauge theory of bundles and connections regards (non-trivial) flat connections as the witnesses of the Aharonov-Bohm effect. In the same spirit, $\mathrm{H}^{k-1,m-k-1}(M;\mathbb{Z})$, which rightfully corresponds to flat fields (being the kernel of the curvature map curv), is interpreted as classifying higher analogues of the Aharonov-Bohm configurations.

3.4 Symplectically orthogonal decomposition

Recall that M is an m-dimensional globally hyperbolic spacetime admitting a compact spacelike Cauchy surface Σ. So far the assumption of a compact Cauchy surface was just meant to simplify our presentation. In Theorem 3.5 this assumption will be used in a crucial way.

We will now present a procedure to decompose orthogonally (however not canonically) the symplectic Abelian group $(\mathfrak{C}^k(M;\mathbb{Z}), \sigma)$ into the three symplectic Abelian groups $(\mathsf{Dyn}^k(M), \sigma_{\mathsf{Dyn}})$, $(\mathsf{Top}^k_{\mathrm{free}}(M), \sigma_{\mathrm{free}})$ and $(\mathsf{Top}^k_{\mathrm{tor}}(M), \sigma_{\mathrm{tor}})$ introduced in the previous sections. To achieve this result, we need to choose (non-canonical) splittings for the short exact sequences in (2.10). Before we prove that splittings of the desired type actually exist, let us illustrate the assumptions that ensure the compatibility of our decomposition with the relevant symplectic structures.

Lemma 3.3. *Consider a commutative diagram*



(3.19)

of Abelian groups whose rows and columns are short exact sequences and assume that π_1, α_2, χ split the relevant short exact sequences, i.e.

$$p_1 \circ \pi_1 = \mathrm{id}_{B_1}, \qquad a_2 \circ \alpha_2 = \mathrm{id}_{A_3}, \qquad b_2 \circ p_2 \circ \chi = \mathrm{id}_{B_3}\,. \tag{3.20}$$

Then all sequences in the diagram split and the map

$$I \doteq e_1 \circ \pi_1 + e_1 \circ i_1 + \chi + i_2 \circ \alpha_2 : B_1 \oplus A_1 \oplus B_3 \oplus A_3 \longrightarrow E_2 \tag{3.21}$$

provides an isomorphism of Abelian groups.

Proof. Commutativity of the bottom right square and the last identity of (3.20) entail that $\beta_2 \doteq p_2 \circ \chi : B_3 \to B_2$ splits the right column and that $\pi_3 \doteq e_2 \circ \chi : B_3 \to E_3$ splits the bottom row. Therefore, by exactness there exist unique homomorphisms $\beta_1 : B_2 \to B_1, \iota_3 : E_3 \to A_3$ such that

$$\beta_2 \circ b_2 + b_1 \circ \beta_1 = \mathrm{id}_{B_2}, \qquad \pi_3 \circ p_3 + i_3 \circ \iota_3 = \mathrm{id}_{E_3}\,. \tag{3.22}$$

Similarly, the first two identities of (3.20) entail that there exist unique homomorphisms $\iota_1 : E_1 \to A_1$ and $\alpha_1 : A_2 \to A_1$ such that

$$\pi_1 \circ p_1 + i_1 \circ \iota_1 = \mathrm{id}_{E_1}, \qquad \alpha_2 \circ a_2 + a_1 \circ \alpha_1 = \mathrm{id}_{A_2}\,. \tag{3.23}$$

Combining (3.22) and (3.23), we can also split the central column and row. In fact, introducing

$$\epsilon_2 \doteq \chi \circ p_3 + i_2 \circ \alpha_2 \circ \iota_3 : E_3 \longrightarrow E_2, \qquad \pi_2 \doteq \chi \circ b_2 + e_1 \circ \pi_1 \circ \beta_1 : B_2 \longrightarrow E_2, \tag{3.24}$$

it is easy to confirm that $e_2 \circ \epsilon_2 = \mathrm{id}_{E_3}$ and $p_2 \circ \pi_2 = \mathrm{id}_{B_2}$. In particular, by exactness there exist unique homomorphisms $\epsilon_1 : E_2 \to E_1, \iota_2 : E_2 \to A_2$ such that

$$\pi_2 \circ p_2 + i_2 \circ \iota_2 = \mathrm{id}_{E_2}, \qquad \epsilon_2 \circ e_2 + e_1 \circ \epsilon_1 = \mathrm{id}_{E_2}\,. \tag{3.25}$$

With these preparations, we obtain a candidate for the inverse of I:

$$J \doteq \big((p_1, \iota_1) \oplus (p_3, \iota_3)\big) \circ (\epsilon_1, e_2) : E_2 \longrightarrow B_1 \oplus A_1 \oplus B_3 \oplus A_3. \tag{3.26}$$

To confirm that J is the inverse of I, observe that (3.24) entails the identities $\epsilon_2 \circ \pi_3 = \chi$ and $\epsilon_2 \circ i_3 = i_2 \circ \alpha_2$, hence

$$I = (e_1 + \epsilon_2) \circ \big((\pi_1 + i_1) \oplus (\pi_3 + i_3)\big) : B_1 \oplus A_1 \oplus B_3 \oplus A_3 \longrightarrow E_2. \tag{3.27}$$

Using the splitting identities, we conclude that $I \circ J$ and $J \circ I$ are the appropriate identity morphisms. □

In the specific case of (2.10), the splittings we are interested in are depicted in the diagram below:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & \overset{\eta}{\curvearrowleft} & \downarrow & & \\
0 & \longrightarrow & \dfrac{\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \xrightarrow{\nu\times\nu} & \mathfrak{T}^k(M;\mathbb{Z}) & \xrightarrow[\mathrm{d}_1]{} & \mathsf{Dyn}^k(M) & \longrightarrow & 0 \\
 & & {\scriptstyle \mu\times\mu}\downarrow & & {\scriptstyle \iota\times\iota}\downarrow & & \downarrow{\scriptstyle\subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}}\cap *\,\Omega^{m-k}_{\mathbb{Z}}(M) & \longrightarrow & 0\\
 & & {\scriptstyle\xi}\uparrow\ \downarrow{\scriptstyle\beta\times\beta} & & {\scriptstyle \mathrm{char}\times\mathrm{char}}\downarrow & \overset{\chi}{\nwarrow} & \downarrow{\scriptstyle([\,\cdot\,],[*^{-1}\,\cdot\,])} & & \\
0 & \longrightarrow & \mathsf{Top}^k_{\mathrm{tor}}(M) & \xrightarrow[\mathrm{j}\times\mathrm{j}]{} & \mathrm{H}^{k,m-k}(M;\mathbb{Z}) & \xrightarrow[\mathrm{q}\times\mathrm{q}]{} & \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{3.28}
$$

More explicitly, the splitting conditions read

$$
(\mathrm{q}\times\mathrm{q})\circ(\mathrm{char}\times\mathrm{char})\circ\chi=\mathrm{id}_{\mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})},\quad (\beta\times\beta)\circ\xi=\mathrm{id}_{\mathsf{Top}^k_{\mathrm{tor}}(M)},\quad \mathrm{d}_1\circ\eta=\mathrm{id}_{\mathsf{Dyn}^k(M)}\,. \tag{3.29}
$$

Lemma 3.4. *Let M be an m-dimensional globally hyperbolic spacetime admitting a compact space-like Cauchy surface Σ. Assuming splittings $\chi:\mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})\to\mathfrak{C}^k(M;\mathbb{Z})$, $\eta:\mathsf{Dyn}^k(M)\to\mathfrak{T}^k(M;\mathbb{Z})$ and $\xi:\mathsf{Top}^k_{\mathrm{tor}}(M)\to\mathrm{H}^{k-1,m-k-1}(M;\mathbb{T})$ according to (3.28) and (3.29), the following identities are fulfilled:*

$$
\sigma\circ\Big(\big((\iota\times\iota)\circ\eta\big)\times\big((\iota\times\iota)\circ\eta\big)\Big)=\sigma_{\mathsf{Dyn}}, \tag{3.30a}
$$

$$
\sigma\circ\Big(\big((\kappa\times\kappa)\circ\xi\big)\times\big((\kappa\times\kappa)\circ\xi\big)\Big)=\sigma_{\mathrm{tor}}, \tag{3.30b}
$$

$$
\sigma\circ\big((\iota\times\iota)\times(\kappa\times\kappa)\big)=0. \tag{3.30c}
$$

Furthermore, for each $(u,\tilde{u},z,\tilde{z}),(u',\tilde{u}',z',\tilde{z}')\in\mathsf{Top}^k_{\mathrm{free}}(M)$, one has

$$
\sigma\big((\iota\times\iota)(\nu\times\nu)(u,\tilde{u}),\chi(z',\tilde{z}')\big)+\sigma\big(\chi(z,\tilde{z}),(\iota\times\iota)(\nu\times\nu)(u',\tilde{u}')\big)=\sigma_{\mathrm{free}}\big((u,\tilde{u},z,\tilde{z}),(u',\tilde{u}',z',\tilde{z}')\big). \tag{3.30d}
$$

Proof. It is easier to show the desired identities using the equivalent definition of the relevant symplectic structures in terms of data on the Cauchy surface Σ. The proof uses (2.2) extensively. Consider $\mathrm{d}\,A,\mathrm{d}\,A'\in\mathsf{Dyn}^k(M)$ and denote the images of $(\iota\times\iota)\,\eta\,\mathrm{d}\,A$ and $(\iota\times\iota)\,\eta\,\mathrm{d}\,A'$ along the isomorphism (2.8) by $(\iota[A_\Sigma],\iota[\tilde{A}_\Sigma])\in\hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ and respectively by $(\iota[A'_\Sigma],\iota[\tilde{A}'_\Sigma])\in\hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$. According to Section 2.3, to determine $\sigma\big((\iota\times\iota)\,\eta\,\mathrm{d}\,A,(\iota\times\iota)\,\eta\,\mathrm{d}\,A'\big)\in\mathbb{T}$ we have to consider

$$
\begin{aligned}
\iota[\tilde{A}_\Sigma]\cdot\iota[A'_\Sigma]-\iota[\tilde{A}'_\Sigma]\cdot\iota[A_\Sigma]&=\iota\,[\tilde{A}_\Sigma\wedge\mathrm{curv}\,\iota[A'_\Sigma]]-\iota\,[\tilde{A}'_\Sigma\wedge\mathrm{curv}\,\iota[A_\Sigma]]\\
&=\iota[\tilde{A}_\Sigma\wedge\mathrm{d}_\Sigma\,A'_\Sigma]-\iota[\tilde{A}'_\Sigma\wedge\mathrm{d}_\Sigma\,A_\Sigma].
\end{aligned} \tag{3.31}
$$

Notice the use of (2.2) to establish the first equality. Evaluation on the fundamental class $[\Sigma]\in\mathrm{H}_{m-1}(\Sigma)$ of Σ concludes the proof of the first identity of (3.30) once one recalls also Section 3.1 and (3.29), which entail

$$
i_\Sigma^*\,\mathrm{d}\,A=i_\Sigma^*\,\mathrm{d}_1\,\eta\,\mathrm{d}\,A=\mathrm{d}_1([A'_\Sigma],[\tilde{A}'_\Sigma]). \tag{3.32}
$$

A similar argument proves the second identity of (3.30) too.

To prove the third identity, consider $([A_\Sigma],[\tilde{A}_\Sigma]) \in \Omega^{k-1,m-k-1}(\Sigma)/\Omega_{\mathbb{Z}}^{k-1,m-k-1}(\Sigma)$ and $(u_\Sigma,\tilde{u}_\Sigma) \in \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T})$ and evaluate $\sigma^\Sigma\big((\iota[A_\Sigma],\iota[\tilde{A}_\Sigma]),(\kappa\, u_\Sigma,\kappa\,\tilde{u}_\Sigma)\big)$. According to Section 2.3, this involves the following computation:

$$\begin{aligned}\iota[\tilde{A}_\Sigma]\cdot\kappa\, u_\Sigma - \kappa\,\tilde{u}_\Sigma\cdot\iota[A_\Sigma] &= \iota[\tilde{A}_\Sigma\wedge \operatorname{curv}\kappa\, u_\Sigma] - \kappa(\tilde{u}_\Sigma \smile \operatorname{char}\iota[A_\Sigma])\\ &= \iota[\tilde{A}_\Sigma\wedge 0] - \kappa(\tilde{u}_\Sigma\smile 0) = 0.\end{aligned} \tag{3.33}$$

Notice that we are again using (2.2).

For the last identity of (3.30), take $(u_\Sigma,\tilde{u}_\Sigma),(u'_\Sigma,\tilde{u}'_\Sigma) \in \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})/\mathrm{H}_{\mathrm{free}}^{k-1,m-k-1}(\Sigma;\mathbb{Z})$ and $(z,\tilde{z}),(z',\tilde{z}') \in \mathrm{H}_{\mathrm{free}}^{k,m-k}(M;\mathbb{Z})$ and introduce

$$(h_\Sigma,\tilde{h}_\Sigma) \doteq i_\Sigma^*\,\chi(z,\tilde{z}) \in \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}), \qquad (h'_\Sigma,\tilde{h}'_\Sigma) \doteq i_\Sigma^*\,\chi(z',\tilde{z}') \in \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}), \tag{3.34}$$

where (2.8) has been used. In view of Section 2.3 computing $\sigma^\Sigma\big((\iota\,\nu\, u_\Sigma,\iota\,\nu\,\tilde{u}_\Sigma),(h'_\Sigma,\tilde{h}'_\Sigma)\big)$ involves the following calculation, based on (2.11) and (2.2):

$$\begin{aligned}\iota\,\nu\,\tilde{u}_\Sigma\cdot h'_\Sigma - \tilde{h}'_\Sigma\cdot\iota\,\nu\, u_\Sigma &= \kappa\,\mu\,\tilde{u}_\Sigma\cdot h'_\Sigma - \tilde{h}'_\Sigma\cdot\kappa\,\mu\, u_\Sigma\\ &= \kappa(\mu\,\tilde{u}_\Sigma\smile\operatorname{char} h'_\Sigma) - (-1)^{k(m-k)}\,\kappa(\mu\, u_\Sigma\smile\operatorname{char}\tilde{h}'_\Sigma).\end{aligned} \tag{3.35}$$

On account of (3.29), we observe that the element $(\mathrm{q}\times\mathrm{q})(\operatorname{char}\times\operatorname{char})(h'_\Sigma,\tilde{h}'_\Sigma)$ is precisely the restriction $(z'_\Sigma,\tilde{z}'_\Sigma)$ to Σ of $(z',\tilde{z}')$. Therefore, evaluating (3.35) on the fundamental class $[\Sigma] \in \mathrm{H}_{m-1}(\Sigma)$ of Σ and recalling (3.9), we obtain

$$\sigma^\Sigma\big((\iota\,\nu\, u_\Sigma,\iota\,\nu\,\tilde{u}_\Sigma),(h'_\Sigma,\tilde{h}'_\Sigma)\big) = \langle\tilde{u}_\Sigma,z'_\Sigma\rangle_{\mathrm{free}} - (-1)^{k(m-k)}\langle u_\Sigma,\tilde{z}'_\Sigma\rangle_{\mathrm{free}}. \tag{3.36}$$

A similar argument shows that

$$\sigma^\Sigma\big((h_\Sigma,\tilde{h}_\Sigma)),(\iota\,\nu\, u'_\Sigma,\iota\,\nu\,\tilde{u}'_\Sigma)\big) = -\langle\tilde{u}'_\Sigma,z_\Sigma\rangle_{\mathrm{free}} + (-1)^{k(m-k)}\langle u'_\Sigma,\tilde{z}_\Sigma\rangle_{\mathrm{free}}. \tag{3.37}$$

We conclude combining the last two equations and recalling Section 3.2. □

Lemma 3.3 entails that the one defined below is an isomorphism of Abelian groups:

$$\begin{aligned}&\mathsf{Dyn}^k(M)\oplus\mathsf{Top}^k_{\mathrm{free}}(M)\oplus\mathsf{Top}^k_{\mathrm{tor}}(M) \xrightarrow{\simeq} \mathfrak{C}^k(M;\mathbb{Z}),\\ &\big(\mathrm{d}A,(u,\tilde{u},z,\tilde{z}),(t,\tilde{t})\big) \longmapsto (\iota\times\iota)\big(\eta\,\mathrm{d}A + (\nu\times\nu)(u,\tilde{u})\big) + \chi(z,\tilde{z}) + (\kappa\times\kappa)\,\xi(t,\tilde{t}).\end{aligned} \tag{3.38}$$

Furthermore, if we assume that the splittings fulfil the compatibility conditions

$$\sigma\circ(\chi\times\chi) = 0, \qquad \sigma\circ\Big(\big((\kappa\times\kappa)\circ\xi\,\big)\times\chi\Big) = 0, \qquad \sigma\circ\Big(\big((\iota\times\iota)\circ\eta\,\big)\times\chi\Big) = 0 \tag{3.39}$$

with respect to the symplectic structure σ on $\mathfrak{C}^k(M;\mathbb{Z})$, recalling Lemma 3.4, we conclude that (3.38) is also an isomorphism of symplectic Abelian groups. The symplectic structure on the source is obtained combining those of each summand, cf. (3.3), (3.11) and (3.17). In fact, Lemma 3.4 and (3.39) entail the following identity:

$$\sigma\begin{pmatrix}(\iota\times\iota)\,\eta\,\mathrm{d}A + (\iota\times\iota)(\nu\times\nu)(u,\tilde{u}) + \chi(z,\tilde{z}) + (\kappa\times\kappa)\,\xi(t,\tilde{t}),\\ (\iota\times\iota)\,\eta\,\mathrm{d}A' + (\iota\times\iota)(\nu\times\nu)(u',\tilde{u}') + \chi(z',\tilde{z}') + (\kappa\times\kappa)\,\xi(t',\tilde{t}')\end{pmatrix} \tag{3.40}$$

$$= \sigma_{\mathsf{Dyn}}(\mathrm{d}A,\mathrm{d}A') + \sigma_{\mathrm{free}}\big((u,\tilde{u},z,\tilde{z}),(u',\tilde{u}',z',\tilde{z}')\big) + \sigma_{\mathrm{tor}}\big((t,\tilde{t}),(t',\tilde{t}')\big). \tag{3.41}$$

Theorem 3.5. *Let M be an m-dimensional globally hyperbolic spacetime admitting a compact spacelike Cauchy surface Σ. Then there exist splittings as per* (3.28) *and* (3.29) *fulfilling* (3.39) *and compatible with the duality isomorphisms, cf.* (2.6)*,* (3.12)*,* (3.18)*.*

Proof. In practice, it is easier to work using the equivalent description of $\mathfrak{C}^k(M;\mathbb{Z})$ in terms of initial data provided by the restriction along the embedding $i_\Sigma : \Sigma \to M$ of the spacelike Cauchy surface Σ into the spacetime M. In particular, we will construct the splittings with reference to (2.11). The actual statement of the theorem is then obtained via the isomorphism relating (2.10) to (2.11), see (2.8). Notice that the assumption of Σ being compact is crucial for this proof. Without this assumption we would be forced to introduce suitable support restrictions in order to make the various pairings considered below well-defined, cf. Remark 2.2. This merely technical complication comes together with a deeper obstruction to our argument, namely the fact that similar pairings in the support-restricted setting would be degenerate. In practice, to the best of our knowledge, the scenarios with non-compact Cauchy surfaces can only be dealt with as a case-by-case analysis, see also the explicit examples analyzed in [Cap16].

We start constructing $\chi : \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \to \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ such that $(\mathrm{q}\times\mathrm{q})\circ(\mathrm{char}\times\mathrm{char})\circ\chi = \mathrm{id}_{\mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})}$ and $\sigma^\Sigma \circ (\chi\times\chi) = 0$. By definition $\mathrm{H}^p_{\mathrm{free}}(\Sigma;\mathbb{Z})$ is a free Abelian group. In particular, we can choose bases

$$\{z_i : i = 1,\dots,n\} \subseteq \mathrm{H}^k_{\mathrm{free}}(\Sigma;\mathbb{Z}), \qquad \{\tilde{z}_{\tilde{i}} : \tilde{i} = 1,\dots,\tilde{n}\} \subseteq \mathrm{H}^{m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z}). \tag{3.42}$$

Since both $\mathrm{q} : \mathrm{H}^p(\Sigma;\mathbb{Z}) \to \mathrm{H}^p_{\mathrm{free}}(\Sigma;\mathbb{Z})$ and $\mathrm{char} : \hat{\mathrm{H}}^p(\Sigma;\mathbb{Z}) \to \mathrm{H}^p(\Sigma;\mathbb{Z})$ are surjective, we can choose

$$\{h_i : i = 1,\dots,n\} \subseteq \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z}), \qquad \{\tilde{h}'_{\tilde{i}} : \tilde{i} = 1,\dots,\tilde{n}\} \subseteq \hat{\mathrm{H}}^{m-k}(\Sigma;\mathbb{Z}) \tag{3.43a}$$

such that

$$\mathrm{q}\,\mathrm{char}\, h_i = z_i, \qquad \mathrm{q}\,\mathrm{char}\, \tilde{h}'_{\tilde{i}} = \tilde{z}_{\tilde{i}}. \tag{3.43b}$$

By evaluation on the fundamental class $[\Sigma]$ of Σ, we introduce a set of real numbers

$$\{c_{\tilde{i}i} : \tilde{i} = 1,\dots,\tilde{n},\, i = 1,\dots,n\} \subseteq \mathbb{R} : \qquad c_{\tilde{i}i} \mod \mathbb{Z} = (\tilde{h}'_{\tilde{i}} \cdot h_i)[\Sigma]. \tag{3.44}$$

Consider the non-degenerate pairing

$$\mathrm{H}^{m-k-1}(\Sigma;\mathbb{R}) \times \mathrm{H}^k(\Sigma;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (\tilde{r}, r) \longmapsto (\tilde{r} \smile r)[\Sigma] \tag{3.45}$$

for cohomology with real coefficients on Σ. Since $\mathrm{H}^k_{\mathrm{free}}(\Sigma;\mathbb{Z})$ is a lattice in $\mathrm{H}^k(\Sigma;\mathbb{R})$, $\{z_i\}$ is a basis of $\mathrm{H}^k(\Sigma;\mathbb{R})$ too. Then we can select its dual basis via the non-degenerate pairing displayed above:

$$\{\tilde{r}_i : i = 1,\dots,n\} \subseteq \mathrm{H}^{m-k-1}(\Sigma;\mathbb{R}) : \qquad (\tilde{r}_i \smile z_j)[\Sigma] = \delta_{ij}. \tag{3.46}$$

We define

$$\{\tilde{s}_{\tilde{i}} : \tilde{i} = 1,\dots,\tilde{n}\} \subseteq \mathrm{H}^{m-k-1}(\Sigma;\mathbb{R}), \qquad \{\tilde{u}_{\tilde{i}} : \tilde{i} = 1,\dots,\tilde{n}\} \subseteq \frac{\mathrm{H}^{m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \tag{3.47a}$$

according to

$$\tilde{s}_{\tilde{i}} \doteq \sum_{i=1}^{n} c_{\tilde{i}i}\, \tilde{r}_i, \qquad \tilde{u}_{\tilde{i}} \doteq \tilde{s}_{\tilde{i}} \mod \mathrm{H}^{m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z}). \tag{3.47b}$$

By construction, one finds

$$(\tilde{s}_{\tilde{i}} \smile z_i)[\Sigma] = \sum_{j=1}^{n} c_{\tilde{i}j} (\tilde{r}_j \smile z_i)[\Sigma] = c_{\tilde{i}i}. \tag{3.48}$$

Therefore, setting also

$$\{\tilde{h}_{\tilde{i}} \doteq \tilde{h}'_{\tilde{i}} - \kappa\, \mu\, \tilde{u}_{\tilde{i}} : \tilde{i} = 1, \dots, \tilde{n}\} \subseteq \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}), \tag{3.49}$$

we get

$$\begin{aligned}(\tilde{h}_{\tilde{i}} \cdot h_i)[\Sigma] &= (\tilde{h}'_{\tilde{i}} \cdot h_i)[\Sigma] - (\mu\, \tilde{u}_{\tilde{i}} \smile \operatorname{char} h_i)[\Sigma] \\ &= c_{\tilde{i}i} - (\tilde{s}_{\tilde{i}} \smile \mathrm{q}\operatorname{char} h_i)[\Sigma] \mod \mathbb{Z} \\ &= c_{\tilde{i}i} - (\tilde{s}_{\tilde{i}} \smile z_i)[\Sigma] \mod \mathbb{Z} = 0.\end{aligned} \tag{3.50}$$

Hence, the formula

$$\chi : \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{k,m-k}(\Sigma : \mathbb{Z}), \qquad (z_i, 0) \longmapsto (h_i, 0), \qquad (0, \tilde{z}_{\tilde{i}}) \longmapsto (0, \tilde{h}_{\tilde{i}}) \tag{3.51}$$

uniquely specifies the sought homomorphism on the basis $\{(z_i,0),(0,\tilde{z}_{\tilde{i}}) : i = 1,\dots,n, \tilde{i} = 1,\dots,\tilde{n}\}$ of $\mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})$. The splitting χ can be made compatible with the duality isomorphisms ζ_{free} in (3.12) and ζ in (2.6). In fact, it is sufficient to consider also

$$\tilde{\chi} : \mathrm{H}^{m-k,k}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{m-k,k}(\Sigma : \mathbb{Z}), \qquad (\tilde{z}_{\tilde{i}}, 0) \longmapsto (\tilde{h}_{\tilde{i}}, 0), \qquad (0, z_i) \longmapsto (0, h_i) \tag{3.52}$$

to conclude that $\zeta \circ ((\iota\,\nu \times \iota\,\nu) \times \chi) = ((\iota\,\nu \times \iota\,\nu) \times \tilde{\chi}) \circ \zeta_{\mathrm{free}}$.

As a second step, we focus on the construction of

$$\eta : \mathrm{d}_\Sigma\, \Omega^{k-1,m-k-1}(\Sigma) \longmapsto \frac{\Omega^{k-1,m-k-1}(\Sigma)}{\Omega^{k-1,m-k-1}_{\mathbb{Z}}(\Sigma)} \tag{3.53}$$

such that $(\mathrm{d}_\Sigma \times \mathrm{d}_\Sigma) \circ \eta = \mathrm{id}_{\mathrm{d}_\Sigma\, \Omega^{k-1,m-k-1}(\Sigma)}$ and $\sigma^\Sigma \circ \Big(((\iota \times \iota) \circ \eta\,) \times \chi \Big) = 0$. First of all, we observe that a splitting $\eta' = \eta'_1 \times \eta'_2$ exists. In fact, on account of [BSS14, Section A.1], we obtain η'_1 and η'_2 such that $\mathrm{d}_\Sigma \circ \eta'_1 = \mathrm{id}_{\mathrm{d}_\Sigma\, \Omega^{k-1}(\Sigma)}$ and $\mathrm{d}_\Sigma \circ \eta'_2 = \mathrm{id}_{\mathrm{d}_\Sigma\, \Omega^{m-k-1}(\Sigma)}$. Our goal is to define η as a suitable modification of η'. For this purpose, we observe that $\mathrm{H}^p(\Sigma;\mathbb{R})/\mathrm{H}^p_{\mathrm{free}}(\Sigma;\mathbb{Z})$ is the Pontryagin dual of $\mathrm{H}^{m-p-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})$, cf. [BBSS15, Rem. 5.7]. In particular, recalling the definition of $\sigma^\Sigma_{\mathrm{free}}$ in (3.11), we observe that

$$\frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \longrightarrow \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})^\star, \qquad (u, \tilde{u}) \longmapsto \sigma^\Sigma_{\mathrm{free}} \longmapsto \sigma^\Sigma_{\mathrm{free}}((u,\tilde{u},0,0),\cdot) \tag{3.54}$$

provides the Pontryagin duality isomorphism. This observation allows us to define

$$\Delta_\eta : \mathrm{d}_\Sigma\, \Omega^{k-1,m-k-1}(\Sigma) \longmapsto \frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \tag{3.55a}$$

by setting

$$\sigma^\Sigma_{\mathrm{free}}\Big((\Delta_\eta(\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A}), 0, 0), (0,0,z,\tilde{z}) \Big) \doteq \sigma^\Sigma((\iota \times \iota)\, \eta'(\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A}), \chi(z,\tilde{z})) \tag{3.55b}$$

for each $(\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A}) \in \mathrm{d}_\Sigma \Omega^{k-1,m-k-1}(\Sigma)$ and each $(z,\tilde{z}) \in \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})$. In fact, for each $(\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A})$, the right-hand side yields a group character on $\mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})$; hence, $\Delta_\eta(\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A})$ fulfilling the defining condition displayed above exists and is unique. Notice that Δ_η is actually the Cartesian product of two homomorphisms exactly as η' due to the fact that its defining equation (3.55) does not mix components. Introducing:

$$\eta \doteq \eta' - (\nu \times \nu) \circ \Delta_\eta : \mathrm{d}_\Sigma \Omega^{k-1,m-k-1}(\Sigma) \longrightarrow \frac{\Omega^{k-1,m-k-1}(\Sigma)}{\Omega^{k-1,m-k-1}_{\mathbb{Z}}(\Sigma)} \tag{3.56}$$

and recalling also the last equation of Lemma 3.4, we find that η fulfils the desired requirement:

$$\begin{aligned}
&\sigma^\Sigma\Big((\iota\times\iota)\,\eta(\mathrm{d}\,A, \mathrm{d}\,\tilde{A}), \chi(z,\tilde{z})\Big) \\
&\quad = \sigma^\Sigma\Big((\iota\times\iota)\,\eta'(\mathrm{d}\,A, \mathrm{d}\,\tilde{A}), \chi(z,\tilde{z})\Big) - \sigma^\Sigma\Big((\iota\times\iota)(\nu\times\nu)\Delta_\eta(\mathrm{d}\,A, \mathrm{d}\,\tilde{A}), \chi(z,\tilde{z})\Big) \\
&\quad = \sigma^\Sigma_{\mathrm{free}}\Big((\Delta_\eta(\mathrm{d}\,A, \mathrm{d}\,\tilde{A}),0,0),(0,0,z,\tilde{z})\Big) - \sigma^\Sigma_{\mathrm{free}}\Big((\Delta_\eta(\mathrm{d}\,A, \mathrm{d}\,\tilde{A}),0,0),(0,0,z,\tilde{z})\Big) = 0.
\end{aligned} \tag{3.57}$$

Once again η is the Cartesian product of two morphisms, namely its first component η_1, which does not depend on the second argument of η, and its second component η_2, which is instead independent of the first argument. It is now easy to confirm the compatibility with the duality isomorphisms ζ_{Dyn} of (3.5) and ζ of (2.6) by introducing a second splitting whose components are obtained flipping the components of the splitting constructed above:

$$\tilde{\eta} : \mathrm{d}_\Sigma \Omega^{m-k-1,k-1}(\Sigma) \longrightarrow \frac{\Omega^{m-k-1,k-1}(\Sigma)}{\Omega^{m-k-1,k-1}_{\mathbb{Z}}(\Sigma)} \qquad (\mathrm{d}_\Sigma A, \mathrm{d}_\Sigma \tilde{A}) \longmapsto (\eta_2(\mathrm{d}_\Sigma A), \eta_1(\mathrm{d}_\Sigma \tilde{A})). \tag{3.58}$$

As a consequence, we find $\zeta \circ (\iota\times\iota) \circ \eta = (\iota\times\iota)\circ\tilde{\eta}\circ\zeta_{\mathsf{Dyn}}$, which is the desired compatibility.

The last step consists in providing $\xi : \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) \to \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T})$ such that $(\beta\times\beta)\circ\xi = \mathrm{id}_{\mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z})}$ and $\sigma^\Sigma \circ \Big(\big((\kappa\times\kappa)\circ\xi\,\big)\times\chi\Big) = 0$. To start with, note that $\mathrm{H}^{p-1}(\Sigma;\mathbb{R})/\mathrm{H}^{p-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})$ is a divisible group, so that there exists a splitting $\xi' = \xi'_1 \times \xi'_2 : \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) \to \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T})$. It only remains to modify ξ' in order to obtain the desired ξ. An argument similar to the one we used to define Δ_η, cf. (3.55), allows us to introduce

$$\Delta_\xi : \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) \longrightarrow \frac{\mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(\Sigma;\mathbb{Z})} \tag{3.59a}$$

by setting

$$\sigma^\Sigma_{\mathrm{free}}\Big((\Delta_\xi(t,\tilde{t}),0,0),(0,0,z,\tilde{z})\Big) \doteq \sigma^\Sigma((\kappa\times\kappa)\,\xi'(t,\tilde{t}),\chi(z,\tilde{z})) \tag{3.59b}$$

for each $(t,\tilde{t}) \in \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z})$ and each $(z,\tilde{z}) \in \mathrm{H}^{k,m-k}_{\mathrm{free}}(\Sigma;\mathbb{Z})$. Now consider

$$\xi \doteq \xi' - (\mu\times\mu)\circ\Delta_\xi : \mathrm{H}^{k,m-k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) \longrightarrow \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T}). \tag{3.60}$$

Repeating the calculation in (3.57), one can confirm that ξ fulfils the desired property. By the same argument valid for Δ_η, Δ_ξ is a Cartesian product of two morphisms and so is ξ. Introducing a new splitting with flipped components

$$\tilde{\xi} : \mathrm{H}^{m-k,k}_{\mathrm{tor}}(\Sigma;\mathbb{Z}) \longrightarrow \mathrm{H}^{m-k-1,k-1}(\Sigma;\mathbb{T}), \qquad (t,\tilde{t}) \longmapsto (\xi_2\,t, \xi_1\,\tilde{t}), \tag{3.61}$$

one can confirm that $\zeta\circ(\iota\times\iota)\circ\xi = (\iota\times\iota)\circ\tilde{\xi}\circ\zeta_{\mathrm{tor}}$, which expresses the compatibility between the splittings ξ and $\tilde{\xi}$ and the duality isomorphisms ζ of (2.6) and ζ_{tor} of (3.18). □

Remark 3.6. While for generic (m,k) the question does not make sense, for $m = 2k$ one would like to find a splitting that is self-compatible with the duality isomorphisms in (2.6), (3.12), (3.18) (Theorem 3.5 only guarantees the existence of a second splitting that is consistent with the first one via the duality isomorphisms). This question can be answered by finding splittings whose components are two copies of the same morphism. To illustrate how to achieve this result, let us find a suitable splitting of the form

$$\chi = \chi_1 \times \chi_2 : \mathrm{H}^{k,k}_{\mathrm{free}}(\Sigma;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{k,k}(\Sigma;\mathbb{Z}) \tag{3.62}$$

with $\chi_1 = \chi_2$. Adopting the notation of the proof of Theorem 3.5, we consider dual bases

$$\{z_i : i = 1, \dots, n\} \subseteq \mathrm{H}^k_{\mathrm{free}}(\Sigma;\mathbb{Z}), \qquad \{\tilde{r}_i : i = 1, \dots, n\} \subseteq \mathrm{H}^{k-1}(\Sigma;\mathbb{R}) \tag{3.63}$$

and we choose arbitrarily

$$\{h'_i : i = 1, \dots, n\} \subseteq \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z}) \qquad \text{such that} \qquad \mathrm{q}\,\mathrm{char}\, h'_i = z_i. \tag{3.64}$$

Selecting a collection of real numbers

$$\{c_{ij} \in [0,1) : i,j = 1, \dots, n\} \qquad \text{such that} \qquad c_{ij} \mod \mathbb{Z} = (h'_i \cdot h'_j)[\Sigma], \tag{3.65}$$

one concludes that $c_{ij} = (-1)^{k^2} c_{ji}$, hence the set

$$\Big\{ h_i \doteq h'_i - \kappa\,\mu \sum_{j=1}^{n} \tfrac{1}{2}\, c_{ij}\, \tilde{r}_j : i = 1, \dots, n \Big\} \tag{3.66}$$

satisfies the condition $(h_i \cdot h_j)[\Sigma] = 0$. Therefore we obtain the desired splitting specifying

$$\chi_1 = \chi_2 : \mathrm{H}^k_{\mathrm{free}}(\Sigma;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z}), \qquad z_i \longmapsto h_i. \tag{3.67}$$

Similar conclusions follow for the other relevant splittings. In particular, one obtains a symplectically orthogonal decomposition also for the self-dual theory, which has been investigated in [BBSS16, Sect. 7].

4 Quantization and states

The goal of the present section is to construct quantized C*-algebras of observables for the symplectic Abelian group $\mathfrak{C}^k(M)$, the dynamical sector $\mathsf{Dyn}^k(M)$, the torsion-free topological sector $\mathsf{Top}^k_{\mathrm{free}}(M)$ and the torsion topological sector $\mathsf{Top}^k_{\mathrm{tor}}(M)$. With Corollary 4.3 we will show that the decomposition of Theorem 3.5 has a quantum counterpart in terms of an appropriate tensor product of C*-algebras. This allows us to define a state on the quantized C*-algebra associated to $\mathfrak{C}^k(M)$ by defining states on the C*-algebras associated to each sector, cf. Proposition 4.7, Proposition 4.9 and Proposition 4.12. In particular, for the dynamical sector we construct a Hadamard state (this feature is not of interest for the other sectors). In terms of induced GNS representations, one obtains Hilbert spaces equipped with isomorphisms implementing the duality isomorphism $\zeta : \mathfrak{C}^k(M;\mathbb{Z}) \to \mathfrak{C}^{m-k}(M;\mathbb{Z})$ of (2.6). In particular, for $m = 2k$, these isomorphisms are in fact automorphisms, i.e. unitary operators. In Section 4.3 we will study in detail the GNS representation induced by the state on the torsion-free topological sector $\mathsf{Top}^k_{\mathrm{free}}(M)$ and we will make further comments on it in Remark 4.11. The main results of this section are summarized in Theorem 4.14.

4.1 Quantization

We quantize the symplectic Abelian group $\mathfrak{C}^k(M;\mathbb{Z})$ implementing canonical commutation relations of Weyl type, thus obtaining a C*-algebra of observables. In view of the symplectically orthogonal decomposition constructed in the previous section, we obtain an analogous factorization at the level of C*-algebras. The analysis that we are going to present applies to any symplectically orthogonal decomposition. Although we are interested in applying it to $\mathfrak{C}^k(M;\mathbb{Z})$, $\mathsf{Dyn}^k(M)$, $\mathsf{Top}^k_{\mathrm{free}}(M)$ and $\mathsf{Top}^k_{\mathrm{tor}}(M)$, it is more convenient to work in the general setting, applying it to the case in hand only at the end. Let G_1, G_2 be Abelian groups equipped with a presymplectic form

$$\sigma_i : G_i \times G_i \longrightarrow \mathbb{T}, \qquad i = 1,2, \tag{4.1}$$

i.e. an antisymmetric bi-homomorphism. For $i = 1,2$ one forms the unital $*$-algebra $\mathcal{A}(G_i)$ generated by the symbols $\{W(g_i),\, g_i \in G_i\}$ and subject to the defining relations

$$W(g_i)^* = W(-g_i), \qquad W(g_i)\,W(g_i') = \exp(2\pi i\,\sigma_i(g_i, g_i'))\,W(g_i + g_i'). \tag{4.2}$$

Each of these algebras can be equipped with the following norm, defined in [MSTV73]:

$$\|\cdot\|_1 : \mathcal{A}(G_i) \longrightarrow \mathbb{R}, \qquad \Big\|\sum_{j=1}^N \alpha_j W(g_j)\Big\|_1 \doteq \sum_{j=1}^N |\alpha_j|, \tag{4.3}$$

where we consider arbitrary (but finite) linear combinations of the generators of $\mathcal{A}(G_i)$. Upon completion, we obtain Banach $*$-algebras

$$\mathcal{B}(G_i) \doteq \overline{(\mathcal{A}(G_i), \|\cdot\|_1)}, \qquad i = 1,2. \tag{4.4}$$

On the other hand, taking $(G_1 \oplus G_2, \sigma)$ as our starting point, where we define

$$\sigma((g_1,g_2),(g_1',g_2')) \doteq \sigma_1(g_1,g_1') + \sigma_2(g_2,g_2'), \tag{4.5}$$

for all $(g_1,g_2),(g_1',g_2') \in G_1 \oplus G_2$, the same procedure can be repeated, resulting first in the $*$-algebra $\mathcal{A}(G_1 \oplus G_2)$ and then in the Banach $*$-algebra $\mathcal{B}(G_1 \oplus G_2)$. The latter comes together with two canonical homomorphisms of Banach $*$-algebras

$$\iota_i : \mathcal{B}(G_i) \longrightarrow \mathcal{B}(G_1 \oplus G_2), \qquad i = 1,2, \tag{4.6}$$

which are completely specified by their action on generators, namely $\iota_1(W(g_1)) \doteq W(g_1, 0)$ for all $g_1 \in G_1$ and similarly for $\iota_2(W(g_2)) \doteq W(0, g_2)$ for all $g_2 \in G_2$. Furthermore, we can consider the algebraic tensor product $\mathcal{B}(G_1) \otimes \mathcal{B}(G_2)$. This is a $*$-algebra with respect to the product and the involution that are defined componentwise by the counterparts on each factor. We equip $\mathcal{B}(G_1) \otimes \mathcal{B}(G_2)$ with the norm

$$\|a\|_{\hat{\otimes}} = \inf\left(\sum_{k=1}^N \|a_{k,1}\|_1\,\|a_{k,2}\|_1\right), \qquad a \in \mathcal{B}(G_1) \otimes \mathcal{B}(G_2). \tag{4.7a}$$

The infimum is taken over all possible presentations of a as

$$a = \sum_{k=1}^N a_{1,k} \otimes a_{2,k}, \tag{4.7b}$$

with $a_{i,k} \in \mathcal{B}(G_i)$. The completion of $\mathcal{B}(G_1) \otimes \mathcal{B}(G_2)$ with respect to (4.7) leads to a Banach $*$-algebra [Gui65] denoted by

$$\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2). \tag{4.8}$$

Since (4.2) and (4.5) entail that $\iota_1(a_1)\,\iota_2(a_2) = \iota_2(a_2)\,\iota_1(a_1)$ for each $a_1 \in \mathcal{B}(G_1), a_2 \in \mathcal{B}(G_2)$, recalling the universal property of $\hat{\otimes}$, cf. [Gui65], we obtain a Banach $*$-algebra morphism

$$I : \mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2) \longrightarrow \mathcal{B}(G_1 \oplus G_2), \tag{4.9a}$$

uniquely specified by

$$I(a_1 \otimes a_2) = \iota_1(a_1)\,\iota_2(a_2) \tag{4.9b}$$

for $a_i \in \mathcal{B}(G_i)$. Our goal is to show that I is an isomorphism of Banach $*$-algebras. It suffices to exhibit its inverse. In fact, consider the $*$-homomorphism

$$J : \mathcal{A}(G_1 \oplus G_2) \longrightarrow \mathcal{B}(G_1) \otimes \mathcal{B}(G_2), \tag{4.10a}$$

defined on generators by

$$J(W(g_1, g_2)) = W(g_1) \otimes W(g_2), \tag{4.10b}$$

for $g_i \in G_i$. From (4.3) and (4.7), one obtains the inequality

$$\|J(a)\|_{\hat{\otimes}} \leq \|a\|_1 \tag{4.11}$$

for all $a \in \mathcal{A}(G_1 \oplus G_2)$, which entails that J can be uniquely extended to a Banach $*$-algebra morphism after taking the completions on codomain and domain. With a slight abuse of notation, we denote this extension by

$$J : \mathcal{B}(G_1 \oplus G_2) \longrightarrow \mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2). \tag{4.12}$$

A direct inspection of the definitions of I and J unveils that

$$I \circ J = \mathrm{id}_{\mathcal{B}(G_1 \oplus G_2)}, \qquad J \circ I = \mathrm{id}_{\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2)}, \tag{4.13}$$

which is tantamount to saying that I is an isomorphism of Banach $*$-algebras. In other words, $\mathcal{B}(G_1 \otimes G_2)$ is isomorphic to $\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2)$. To conclude our analysis we need to move to the level of C*-algebras. To this end we recall that, to each unital Banach $*$-algebra, one can assign functorially its canonical enveloping C*-algebra, see [Dix77, Sect. 2.7]. This functor, which will be denoted by $\mathcal{C}^*$, is the left-adjoint of the forgetful functor from unital C*-algebras to unital Banach $*$-algebras. We consider the C*-algebras $\mathcal{C}^*(\mathcal{B}(G_i))$, $i = 1, 2$, $\mathcal{C}^*(\mathcal{B}(G_1 \oplus G_2))$ and $\mathcal{C}^*(\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2))$. Having already established that $\mathcal{B}(G_1 \oplus G_2)$ is isomorphic via (4.10) to $\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2)$, by functoriality it follows that

$$\mathcal{C}^*(\mathcal{B}(G_1 \oplus G_2)) \simeq \mathcal{C}^*(\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2)). \tag{4.14}$$

Following [Gui65], we introduce a C*-norm $\|\cdot\|_{\check{\otimes}}$ on the algebraic tensor product $C_1 \otimes C_2$ of two C*-algebras C_1, C_2 as the least upper bound of all C*-subcross seminorms. The C*-algebra $C_1\check{\otimes}C_2$ obtained by completion with respect to $\|\cdot\|_{\check{\otimes}}$ is characterized by the following universal property: If $\psi_i : C_1 \to C_3$ $(i = 1, 2)$ are two commuting morphisms of unital C*-algebras from C_i into C_3, then there exists a unique C*-algebra morphism $\Psi : C_1\check{\otimes}C_2 \to C_3$ such that $\Psi(c_1 \otimes c_2) = \psi_1(c_1)\,\psi_2(c_2)$ for all $c_1 \in C_1$ and for all $c_2 \in C_2$. The following property relates $\hat{\otimes}$ and $\check{\otimes}$ via $\mathcal{C}^*$ [Gui65]: given two unital Banach $*$-algebras, the enveloping C*-algebra of their $\hat{\otimes}$-tensor product is naturally isomorphic to the $\check{\otimes}$-tensor product of their enveloping C*-algebras. Therefore we obtain

$$\mathcal{C}^*(\mathcal{B}(G_1)\hat{\otimes}\mathcal{B}(G_2)) \simeq \mathcal{C}^*(\mathcal{B}(G_1))\check{\otimes}\mathcal{C}^*(\mathcal{B}(G_2)). \tag{4.15}$$

Summing up, we have the following:

Proposition 4.1. *Let (G_1, σ_1) and (G_2, σ_2) be two presymplectic Abelian groups. Then there is a canonical isomorphism of C*-algebras:*

$$\mathcal{C}^*(\mathcal{B}(G_1 \oplus G_2)) \simeq \mathcal{C}^*(\mathcal{B}(G_1)) \check{\otimes} \mathcal{C}^*(\mathcal{B}(G_2)). \tag{4.16}$$

Remark 4.2. We observe that, as a consequence of [Dix77, Prop. 2.7.1], for a (pre)symplectic Abelian group G, the C*-enveloping algebra associated to $\mathcal{B}(G)$ is isomorphic to the C*-algebra associated to G defined in [MSTV73], which encodes the Weyl canonical commutation relations. For this reason, we will denote $\mathcal{C}^*(\mathcal{B}(G))$ with the symbol $\mathcal{W}(G)$. In particular, this observation entails that the quantization prescription considered in [BDHS14, BBSS16] is equivalent to the one adopted here.

The preceding analysis can be applied to the scenario of interest to us. Recalling the symplectically orthogonal decomposition in (3.38), as well as Proposition 4.1, we conclude that the C*-algebra of observables $\mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z}))$ can be factorized as a $\check{\otimes}$-tensor product of three contributions.

Corollary 4.3. *The following is an isomorphism of C*-algebras:*

$$\mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z})) \simeq \mathcal{W}(\mathsf{Dyn}^k(M)) \,\check{\otimes}\, \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \,\check{\otimes}\, \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M)). \tag{4.17}$$

Remark 4.4. The duality transformations ζ, ζ_{Dyn}, ζ_{free}, ζ_{tor}, cf. (2.6), (3.5), (3.12), (3.18), have quantized counterparts

$$\mathcal{W}(\zeta) : \mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z})) \longrightarrow \mathcal{W}(\mathfrak{C}^{m-k}(M;\mathbb{Z})), \qquad W(h,\tilde{h}) \longmapsto W(\zeta(h,\tilde{h})), \tag{4.18a}$$

$$\mathcal{W}(\zeta_{\mathsf{Dyn}}) : \mathcal{W}(\mathsf{Dyn}^k(M;\mathbb{Z})) \longrightarrow \mathcal{W}(\mathsf{Dyn}^{m-k}(M;\mathbb{Z})), \qquad W(\mathrm{d}\,A) \longmapsto W(\zeta_{\mathsf{Dyn}}(\mathrm{d}\,A)), \tag{4.18b}$$

$$\mathcal{W}(\zeta_{\mathrm{free}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M;\mathbb{Z})) \longrightarrow \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{free}}(M;\mathbb{Z})), \quad W(u,\tilde{u},z,\tilde{z}) \longmapsto W(\zeta_{\mathrm{free}}(u,\tilde{u},z,\tilde{z})), \tag{4.18c}$$

$$\mathcal{W}(\zeta_{\mathrm{tor}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M;\mathbb{Z})) \longrightarrow \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{tor}}(M;\mathbb{Z})), \qquad W(t,\tilde{t}) \longmapsto W(\zeta_{\mathrm{tor}}(t,\tilde{t})) \tag{4.18d}$$

at the C* algebra level defined as the unique extensions of the obvious formulas given on generators. The compatibility between splittings and duality transformations stated in Theorem 3.5 and in Remark 3.6 induces a similar property between the factorization of Corollary 4.3 and the quantum duality transformations of (4.18).

Remark 4.5. Now that our quantization prescription has been set up, following [FMS07a, FMS07b], we would like to pinpoint an interesting feature related to the torsion topological sector of our model: $\mathsf{Top}^k_{\mathrm{tor}}(M)$ is responsible for the non-commutativity between certain basic observables measuring magnetic and electric fluxes respectively, a feature that has no counterpart in ordinary Maxwell theory. This can be realized by looking at the C*-algebra $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$: consider $t \in \mathrm{H}^k_{\mathrm{tor}}(M;\mathbb{Z})$ and $\tilde{t} \in \mathrm{H}^{m-k}_{\mathrm{tor}}(M;\mathbb{Z})$ and take the basic observables $W(0,\tilde{t}), W(t,0) \in \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$. Recalling Remark 3.2, these observables are only sensitive to (the torsion part of) either the magnetic or the electric flux. An elementary calculation using the relations (4.2) and (3.17) shows that

$$W(0,\tilde{t})\, W(t,0) = \exp(4\pi i \langle i^*_\Sigma\, \tilde{t}, i^*_\Sigma\, t\rangle_{\mathrm{tor}})\, W(t,0)\, W(0,\tilde{t}). \tag{4.19}$$

Provided that the topology of the background spacetime supports the relevant torsion classes, e.g. $\mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z}) \simeq (\mathbb{Z}_q)^2$ with $q \geq 3$ (examples can be constructed using lens spaces, cf. [Hat02, Ex. 2.43] and [BBSS16, Ex. 7.3]), it follows from the above computation that $W(0,\tilde{t})$ and $W(t,0)$ do *not* commute.

4.2 States for the dynamical sector

With Corollary 4.3 we have established a factorization of the algebra of observables induced by the symplectically orthogonal splitting in (3.38). This allows us to define states on $\mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z}))$ by assigning a state on each of the $\check{\otimes}$-tensor factors. We start from the dynamical sector, cf. Section 3.1, i.e. we look for a Hadamard state on $\mathcal{W}(\mathsf{Dyn}^k(M))$. Our approach is motivated by the following proposition, where the requirement of a compact Cauchy surface is inessential and it can be easily removed by introducing differential forms with timelike compact support, see e.g. [Ben16] for analogous results in this more general case (we refrain from this level of generality here):

Proposition 4.6. *Let M be an m-dimensional globally hyperbolic spacetime (admitting a compact Cauchy surface) and consider the causal propagator $G : \Omega^p_{\mathrm{c}}(M) \to \Omega^p(M)$ for the normally hyperbolic differential operator $\Box \doteq \delta\,\mathrm{d} + \mathrm{d}\,\delta$ defined on p-forms (see [BGP07, Bär15]). Then the following is an isomorphism of vector spaces:*

$$L : \frac{\Omega^k_{\mathrm{c}}(M)}{\Omega^k_{\mathrm{c,d}}(M) \oplus \Omega^k_{\mathrm{c},\delta}(M)} \longrightarrow \mathsf{Dyn}^k(M), \qquad [\rho] \longmapsto \mathrm{d}(G\,\delta\,\rho) = *\,\mathrm{d}\left((-1)^{mk+1} G * \mathrm{d}\,\rho\right), \tag{4.20}$$

where the subscripts $_{\mathrm{d}}$ and $_{\delta}$ denote the kernels of $\mathrm{d} : \Omega^k_{\mathrm{c}}(M) \to \Omega^{k+1}_{\mathrm{c}}(M)$ and $\delta : \Omega^k_{\mathrm{c}}(M) \to \Omega^{k-1}_{\mathrm{c}}(M)$, respectively.

Proof. First of all, notice that L is well-defined. In fact, this follows from the fact that G is the causal propagator for $\Box$ on p-forms and that both d and $*$ intertwine $\Box$ (defined on forms of suitable degrees). In particular, one obtains $\mathrm{d}\,G\,\delta = G(\Box - \delta\,\mathrm{d}) = -\,\delta\,G\,\mathrm{d}$ on k-forms with compact support. This confirms that L is well-defined and that the equality displayed in its definition holds true.

To confirm injectivity, let us consider $\rho \in \Omega^k_{\mathrm{c}}(M)$ such that $G\,\mathrm{d}\,\delta\,\rho = 0$. Then by the properties of the causal propagator, see e.g. [BGP07], there exists $\alpha \in \Omega^k_{\mathrm{c}}(M)$ such that $\Box\alpha = \mathrm{d}\,\delta\,\rho$. In particular, $\mathrm{d}\,\alpha = 0$ and $\delta\,\rho = \delta\,\alpha$. Since also $G\,\delta\,\mathrm{d}\,\rho = 0$, a similar argument allows us to find $\tilde{\alpha} \in \Omega^k_{\mathrm{c}}(M)$ such that $\delta\,\tilde{\alpha} = 0$ and $\mathrm{d}\,\rho = \mathrm{d}\,\tilde{\alpha}$. Combining these results, one has the identity

$$\Box\rho = \delta\,\mathrm{d}\,\alpha + \mathrm{d}\,\delta\,\tilde{\alpha} = \Box(\alpha + \tilde{\alpha}), \tag{4.21}$$

therefore $\rho = \alpha + \tilde{\alpha} \in \Omega^k_{\mathrm{c,d}}(M) \oplus \Omega^k_{\mathrm{c},\delta}(M)$.

To show that L is also surjective, consider $\mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A} \in \mathsf{Dyn}^k(M)$. By the standard Lorenz gauge fixing we are allowed to assume without loss of generality that $\delta\,A = 0$ and $\delta\,\tilde{A} = 0$. In fact, it is enough to observe that $\delta\,\mathrm{d}\,A = 0$ and $\delta\,\mathrm{d}\,\tilde{A} = 0$ and use [Ben16, Lem. 7.2]. With this further assumption, $\mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A}$ entails that $\Box A = 0$ and $\Box\tilde{A} = 0$. Therefore we find $\alpha \in \Omega^{k-1}_{\mathrm{c}}(M)$ and $\tilde{\alpha} \in \Omega^{m-k-1}_{\mathrm{c}}(M)$ such that $G\alpha = A$ and $G\tilde{\alpha} = \tilde{A}$. From $\mathrm{d}\,A = *\,\mathrm{d}\,\tilde{A}$ it follows that there exists $\rho \in \Omega^k_{\mathrm{c}}(M)$ such that $\mathrm{d}\,\alpha - *\,\mathrm{d}\,\tilde{\alpha} = \Box\rho$. Evaluating the left and the right-hand side on $\mathrm{d}\,\delta$, one obtains $\Box\,\mathrm{d}\,\alpha = \mathrm{d}\,\delta\,\mathrm{d}\,\alpha = \Box\,\mathrm{d}\,\delta\,\rho$, hence $\mathrm{d}\,\alpha = \mathrm{d}\,\delta\,\rho$. This allows us to conclude that $\mathrm{d}\,A = \mathrm{d}\,G\,\delta\,\rho$. Since L is clearly linear, we conclude that L is an isomorphism of vector spaces as claimed. □

The isomorphism in Proposition 4.6 can be promoted to one of symplectic vector spaces. In fact, we can equip the vector space

$$\Omega^k_{\mathrm{c}}(M)_{\mathsf{Dyn}} \doteq \frac{\Omega^k_{\mathrm{c}}(M)}{\Omega^k_{\mathrm{c,d}}(M) \oplus \Omega^k_{\mathrm{c},\delta}(M)} \tag{4.22}$$

with a symplectic structure as follows:[5]

$$\tau_{\mathsf{Dyn}} : \Omega^k_{\mathrm{c}}(M)_{\mathsf{Dyn}} \times \Omega^k_{\mathrm{c}}(M)_{\mathsf{Dyn}} \longrightarrow \mathbb{R}, \qquad ([\rho],[\rho']) \longmapsto \int_M \rho \wedge *G\,\mathrm{d}\,\delta\,\rho'. \tag{4.23}$$

Notice that there are other equivalent formulas defining τ_{Dyn}:

$$-\int_M \mathrm{d}\,\rho \wedge *G\,\mathrm{d}\,\rho' = \int_M \rho \wedge *G\,\mathrm{d}\,\delta\,\rho' = \int_M \delta\,\rho \wedge *G\,\delta\,\rho'. \tag{4.24}$$

These identities, which follow from the fact that d and its formal adjoint δ intertwine the causal propagators, show that τ_{Dyn} is well-defined. To confirm that τ_{Dyn} is antisymmetric recall that $\Box$ is formally self-adjoint, hence the causal propagator G is formally anti-selfadjoint. Being also non-degenerate (to prove it one argues as for injectivity in Proposition 4.6), τ_{Dyn} is indeed a symplectic form. With a quite standard, although lengthy, computation, one checks that the isomorphism L is compatible with the symplectic structures τ_{Dyn} and σ_{Dyn}, respectively defined on the source and on the target. This calculation is based on Stokes theorem and on the properties of the retarded/advanced Green operators $G^{\pm} : \Omega^p_{\mathrm{c}} \to \Omega^p(M)$ for $\Box : \Omega^p(M) \to \Omega^p(M)$:

$$\begin{aligned}\tau_{\mathsf{Dyn}}([\rho],[\rho']) &= \int_{J^+_M(\Sigma)} \Box G^-\rho \wedge *\,\mathrm{d}\,G\,\delta\,\rho' + \int_{J^-_M(\Sigma)} \Box G^+\rho \wedge *\,\mathrm{d}\,G\,\delta\,\rho' \\ &= \int_\Sigma \delta\,G\rho \wedge *\,\mathrm{d}\,G\,\delta\,\rho' - \int_\Sigma \mathrm{d}\,G\,\delta\,\rho' \wedge *\,\mathrm{d}\,G\rho \\ &= \sigma_{\mathsf{Dyn}}(\mathrm{d}(G\,\delta\,\rho), \mathrm{d}(G\,\delta\,\rho')),\end{aligned} \tag{4.25}$$

where Σ is a Cauchy surface of M while $J^{\pm}_M(\Sigma)$ denotes its causal future/past. For later reference, let us observe how ζ_{Dyn} of (3.5) looks like from this point of view:

$$\zeta_{\mathsf{Dyn}} : \Omega^k_{\mathrm{c}}(M)_{\mathsf{Dyn}} \longrightarrow \Omega^{m-k}_{\mathrm{c}}(M)_{\mathsf{Dyn}}, \qquad [\rho] \longmapsto [(-1)^{k(m-k)} * \rho]. \tag{4.26}$$

This alternative, yet equivalent, perspective on the symplectic vector space $\mathsf{Dyn}^k(M)$ suggests us how to introduce a two-point function that will be later used to define a Hadamard state on $\mathcal{W}(\mathsf{Dyn}^k(M))$. In fact, due to [SV01], one always obtains a Hadamard two-point function $\mathfrak{W}_k \in \Omega^{2k}_{\mathrm{c}}(M \times M)'$ associated to $\Box : \Omega^k(M) \to \Omega^k(M)$. For example, when dealing with ultra-static spacetimes, one way to achieve this result is to adopt the so-called *positive frequencies* prescription, which leads to the ground state (see e.g. [Wal94]). Then, mimicking the formula for the symplectic form τ_{Dyn}, one is induced to regard $\mathfrak{W}_k \circ (\mathrm{id} \otimes \mathrm{d}\,\delta)$ as a natural candidate for the two-point function of the quantum field theory corresponding to $\mathsf{Dyn}^k(M)$.

For the sake of concreteness, let us focus on the case of an ultra-static globally hyperbolic spacetime M admitting a compact Cauchy surface Σ. This means that we can present M as

$$M \simeq \mathbb{R} \times \Sigma, \qquad g = -\,\mathrm{d}\,t \otimes \mathrm{d}\,t + h, \tag{4.27}$$

where h is a Riemannian metric on Σ (constant in $t \in \mathbb{R}$).[6] This allows us to decompose differential forms on M in terms of sections of the pullbacks along the projection $\pi_2 : M \to \Sigma$ of the bundles

[5] Very similar symplectic structures appear in the literature, e.g. in [Dim92, FP03, DL12, FL16], as well as others. Nonetheless, it should be mentioned that there are quite subtle, yet crucial differences between these approaches and ours. These reside essentially in the domain of the symplectic structure and they are mainly related to the fact that the topological degrees of freedom in the present context are isolated from the dynamical ones and treated separately.

[6] Although non-ultra-static spacetimes could be considered as well, we refrain from doing so. In fact, the more generic situation would require a case-by-case study leading to analytical complications that would not result in further insights on the nature of the main problem under study.

$\bigwedge^p T^*\Sigma$ of skew-symmetric p-cotensors over Σ. Specifically, one has:

$$\Omega^k(M) = \Gamma\left(M, \pi_2^* \bigwedge^k T^*\Sigma\right) \oplus dt \wedge \Gamma\left(M, \pi_2^* \bigwedge^{k-1} T^*\Sigma\right). \tag{4.28}$$

With respect to this decomposition, $\Box$ takes the form

$$\Box(\omega_\Sigma + \mathrm{d}\,t \wedge \omega_t) = (\partial_t^2 \omega_\Sigma + \triangle\omega_\Sigma) + \mathrm{d}\,t \wedge (\partial_t^2 \omega_t + \triangle\omega_t). \tag{4.29}$$

This allows us to use the spectral calculus associated to the Hodge-de Rham Laplacian $\triangle = \delta_\Sigma\, \mathrm{d}_\Sigma + \mathrm{d}_\Sigma\, \delta_\Sigma$ on Σ (note that differential, codifferential and Hodge dual are indicated with a subscript $_\Sigma$ whenever they refer to the geometry of the Cauchy surface Σ, instead of that of the whole spacetime M). In particular, for p-forms on Σ we have the Hodge decomposition into harmonic, exact and coexact contributions:

$$\Omega^p(\Sigma) = \mathcal{H}^p(\Sigma) \oplus \mathrm{d}_\Sigma\, \Omega^{p-1}(\Sigma) \oplus \delta_\Sigma\, \Omega^{p+1}(\Sigma). \tag{4.30}$$

We denote the projections on the harmonic part and the projection on its orthogonal complement by:

$$\pi^p_{\mathcal{H}} : \Omega^p(\Sigma) \longrightarrow \mathcal{H}^p(\Sigma), \qquad \pi^p_\perp : \Omega^p(\Sigma) \longrightarrow \mathrm{d}_\Sigma\, \Omega^{p-1}(\Sigma) \oplus \delta_\Sigma\, \Omega^{p+1}(\Sigma). \tag{4.31}$$

With these preparations, we can write down a quite explicit formula for the causal propagator G associated to $\Box$ acting on $\Omega^k(M)$. Regarding ρ_Σ and ρ_t as smoothly $\mathbb{R}$-parametrized differential forms on Σ, i.e. $t \in \mathbb{R} \mapsto \rho_\Sigma(t,\cdot) \in \Omega^k(\Sigma)$ and $t \in \mathbb{R} \mapsto \rho_t(t,\cdot) \in \Omega^{k-1}(\Sigma)$, we obtain

$$\begin{aligned} G : \Omega^k_{\mathrm{c}}(M) &\longrightarrow \Omega^k(M), \\ \rho_\Sigma + \mathrm{d}\,t \wedge \rho_t &\longmapsto G_{\mathcal{H}}\, \pi^k_{\mathcal{H}}\, \rho_\Sigma + G_\perp\, \pi^k_\perp\, \rho_\Sigma + \mathrm{d}\,t \wedge (G_{\mathcal{H}}\, \pi^{k-1}_{\mathcal{H}}\, \rho_t + G_\perp\, \pi^{k-1}_\perp\, \rho_t), \end{aligned} \tag{4.32a}$$

where

$$(G_{\mathcal{H}}\, \alpha_{\mathcal{H}})(t,\cdot) = \int_{\mathbb{R}} (t - t')\, \alpha_{\mathcal{H}}(t',\cdot)\, \mathrm{d}\,t', \tag{4.32b}$$

$$(G_\perp\, \alpha_\perp)(t,\cdot) = \int_{\mathbb{R}} \triangle^{-\frac{1}{2}} \sin(\triangle^{\frac{1}{2}}(t - t'))\, \alpha_\perp(t',\cdot)\, \mathrm{d}\,t', \tag{4.32c}$$

for $\alpha_{\mathcal{H}}, \alpha_\perp \in \Gamma_{\mathrm{c}}(M, \pi_2^* \bigwedge^p T^*\Sigma)$ such that, for each $t \in \mathbb{R}$, $\alpha_{\mathcal{H}}(t,\cdot) \in \mathcal{H}^p(\Sigma)$ and $\alpha_\perp(t,\cdot) \in \mathrm{d}_\Sigma\, \Omega^{p-1}(\Sigma) \oplus \delta_\Sigma\, \Omega^{p+1}(\Sigma)$.

Following an approach inspired by [FP03], we introduce a bidistribution $\mathfrak{W}_k \in \Omega^{2k}_{\mathrm{c}}(M \times M)'$ where only the part orthogonal to the harmonic one contributes:

$$\begin{aligned} \mathfrak{W}_k : \Omega^k_{\mathrm{c}}(M) \otimes \Omega^k_{\mathrm{c}}(M) &\longrightarrow \mathbb{C} \\ (\rho_\Sigma + \mathrm{d}\,t \wedge \rho_t) \otimes (\rho'_\Sigma + \mathrm{d}\,t \wedge \rho'_t) &\longmapsto \mathfrak{W}_\perp(\pi^k_\perp\, \rho_\Sigma \otimes \pi^k_\perp\, \rho'_\Sigma) - \mathfrak{W}_\perp(\pi^{k-1}_\perp\, \rho_t \otimes \pi^{k-1}_\perp\, \rho'_t), \end{aligned} \tag{4.33a}$$

where

$$\mathfrak{W}_\perp(\alpha_\perp \otimes \alpha'_\perp) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle \alpha_\perp(t,\cdot),\, \tfrac{1}{2} \triangle^{-\frac{1}{2}} \exp(-i\, \triangle^{\frac{1}{2}}\, (t - t'))\, \alpha'_\perp(t',\cdot) \right\rangle \mathrm{d}\,t\ \mathrm{d}\,t', \tag{4.33b}$$

for $\alpha_\perp, \alpha'_\perp \in \Gamma_{\mathrm{c}}(M, \pi_2^* \bigwedge^p T^*\Sigma)$ $(p = k-1, k)$ such that, for each $t \in \mathbb{R}$, $\alpha_\perp(t,\cdot), \alpha'_\perp(t,\cdot) \in \mathrm{d}_\Sigma\, \Omega^{p-1}(\Sigma) \oplus \delta_\Sigma\, \Omega^{p+1}(\Sigma)$, and where $\langle\cdot,\cdot\rangle$ denotes the L^2-scalar product on $\Omega^p(\Sigma)$. A straightforward computation allows us to confirm that $\mathfrak{W}_k$ is a bisolution of $\Box$. In fact, for $\alpha_\perp, \alpha'_\perp$, as above we have

$$\mathfrak{W}_\perp((\partial_t^2 \alpha_\perp + \triangle\alpha_\perp) \otimes \alpha'_\perp) = \mathfrak{W}_\perp(\alpha_\perp \otimes (\partial_t^2 \alpha'_\perp + \triangle\alpha'_\perp)) = 0. \tag{4.34}$$

An argument similar to the one illustrated in [FP03, Appendix B][7] allows us to conclude that $\mathfrak{W}_k$ fulfils the microlocal spectrum condition (recall that $\mathfrak{W}_k$ is a $\Box$-bisolution whose antisymmetric part differs from $-i\,G$ only for the harmonic contribution, which is smooth). Using $\mathfrak{W}_k$ we introduce[8]

$$\omega_2 \doteq (\mathrm{id} \otimes \mathrm{d}\,\delta)\,\mathfrak{W}_k = \mathfrak{W}_k \circ (\mathrm{id} \otimes \mathrm{d}\,\delta) \in \Omega_{\mathrm{c}}^{2(m-k)}(M \times M)'. \tag{4.35}$$

Recalling that for each $\omega \in \Omega^k(M)$ one has

$$\mathrm{d}\,\delta(\omega_\Sigma + \mathrm{d}\,t \wedge \omega_t) = \mathrm{d}_\Sigma\,\delta_\Sigma\,\omega_\Sigma + \mathrm{d}_\Sigma\,\partial_t\omega_t + \mathrm{d}\,t \wedge (\partial_t^2\omega_t + \mathrm{d}_\Sigma\,\delta_\Sigma\,\omega_t + \delta_\Sigma\,\partial_t\omega_\Sigma), \tag{4.36a}$$

$$\delta\,\mathrm{d}(\omega_\Sigma + \mathrm{d}\,t \wedge \omega_t) = \partial_t^2\omega_\Sigma + \delta_\Sigma\,\mathrm{d}_\Sigma\,\omega_\Sigma - \mathrm{d}_\Sigma\,\partial_t\omega_t + \mathrm{d}\,t \wedge (\delta_\Sigma\,\mathrm{d}_\Sigma\,\omega_t - \delta_\Sigma\,\partial_t\omega_\Sigma), \tag{4.36b}$$

one can confirm that

$$\omega_2 = (\mathrm{d}\,\delta \otimes \mathrm{id})\,\mathfrak{W}_k. \tag{4.37}$$

Furthermore, since $\mathfrak{W}_k$ is a bisolution of $\Box$, it follows that

$$\omega_2 = -(\mathrm{id} \otimes \delta\,\mathrm{d})\,\mathfrak{W}_k = -(\delta\,\mathrm{d} \otimes \mathrm{id})\,\mathfrak{W}_k. \tag{4.38}$$

Similarly, recalling that for each $\omega \in \Omega^p(M)$ one has

$$\mathrm{d}(\omega_\Sigma + \mathrm{d}\,t \wedge \omega_t) = \mathrm{d}_\Sigma\,\omega_\Sigma + \mathrm{d}\,t \wedge (\partial_t\omega_\Sigma - \mathrm{d}_\Sigma\,\omega_t), \tag{4.39a}$$

$$\delta(\omega_\Sigma + \mathrm{d}\,t \wedge \omega_t) = \delta_\Sigma\,\omega_\Sigma + \partial_t\omega_t - \mathrm{d}\,t \wedge \delta_\Sigma\,\omega_t, \tag{4.39b}$$

one shows also that

$$\omega_2 = (\delta \otimes \delta)\,\mathfrak{W}_{k-1} = -(\mathrm{d} \otimes \mathrm{d})\,\mathfrak{W}_{k+1}. \tag{4.40}$$

These observations entail that ω_2 vanishes both on closed and on coclosed forms, thus yielding

$$\omega_2 : \Omega_{\mathrm{c}}^k(M)_{\mathsf{Dyn}} \otimes \Omega_{\mathrm{c}}^k(M)_{\mathsf{Dyn}} \longrightarrow \mathbb{C}, \qquad [\rho] \otimes [\rho'] \longmapsto \omega_2(\rho \otimes \rho'). \tag{4.41}$$

Notice that the antisymmetric part of ω_2 agrees with τ_{Dyn}:

$$\omega_2([\rho] \otimes [\rho'] - [\rho'] \otimes [\rho]) = -i\,\tau_{\mathsf{Dyn}}([\rho], [\rho']) \tag{4.42}$$

for all $[\rho], [\rho'] \in \Omega^k(M)_{\mathsf{Dyn}}$. Eq. (4.40) is also crucial to confirm that ω_2 inherits the microlocal spectrum condition from $\mathfrak{W}_{k+1}$. In fact, we are going to show that

$$\mathrm{WF}(\omega_2) = \mathrm{WF}(\mathfrak{W}_{k+1}). \tag{4.43}$$

First of all, notice that the principal symbol of $\mathrm{d} \otimes \mathrm{d}$ is the homomorphism of vector bundles over $M \times M$:

$$\begin{aligned} \sigma_{\mathrm{d} \otimes \mathrm{d}} : T^*(M \times M) \otimes \Big(\bigwedge^k T^*M \boxtimes \bigwedge^k T^*M\Big) &\longrightarrow \bigwedge^{k+1} T^*M \boxtimes \bigwedge^{k+1} T^*M) \\ (p, p') \otimes (\omega \otimes \omega') &\longmapsto (p \wedge \omega) \otimes (p' \wedge \omega'). \end{aligned} \tag{4.44}$$

[7] For the sake of completeness, let us stress that our model has the remarkable feature of admitting a splitting into dynamical and topological sectors. In this way one circumvents the obstructions that arise in [FP03] with certain non-trivial homology groups. Therefore, as opposed to [FP03], we do not need to assume these homology groups to be trivial. It is sufficient to separate the dynamical degrees of freedom from the topological ones.

[8] Here and in the following the symbols denoting two-point functions, as well as states, (typically ω, possibly with a subscript) will not contain any reference to the degree k of the theory as this can be easily inferred from the context.

In particular, it follows that $(p, p') \in \mathrm{Char}(\mathrm{d}\otimes\mathrm{d}) \subseteq T^*(M\times M)\setminus\{0\}$ if and only if precisely one of p and p' vanishes (the case of $p=0$ and $p'=0$ is excluded per definition). Then the microlocal spectrum condition for $\mathfrak{W}_{k+1}$ entails that $\mathrm{WF}(\mathfrak{W}_{k+1})\cap\mathrm{Char}(\mathrm{d}\otimes\mathrm{d})=\emptyset$. Taking into account also [Hör03, Ch. 8], we have the chain of inclusions $\mathrm{WF}(\omega_2)\subseteq\mathrm{WF}(\mathfrak{W}_{k+1})\subseteq\mathrm{WF}(\omega_2)\cup\mathrm{Char}(\mathrm{d}\otimes\mathrm{d})$. Therefore (4.43) follows, showing that ω_2 inherits the microlocal spectrum condition from $\mathfrak{W}_{k+1}$.

To summarize, we constructed a bidistribution ω_2 that fulfils the microlocal spectrum condition, that descends to the quotient in $\Omega^k_{\mathsf{c}}(M)_{\mathsf{Dyn}}$ and that is compatible with the canonical commutation relations encoded in $\mathcal{W}(\mathsf{Dyn}^k(M))$. A straightforward computation conducted expanding (4.40) allows us to confirm that ω_2 is also non-negative, cf. [FP03] for a similar argument:

$$\omega_2([\rho]\otimes[\rho])\geq 0 \tag{4.45}$$

for all $[\rho]\in\Omega^k_{\mathsf{c}}(M)_{\mathsf{Dyn}}$. In fact, introducing also the projections

$$\pi^p_{\mathrm{d}}:\Omega^p(\Sigma)\longrightarrow \mathrm{d}_\Sigma\,\Omega^{p-1}(\Sigma),\qquad \pi^p_{\delta}:\Omega^p(\Sigma)\longrightarrow \delta_\Sigma\,\Omega^{p+1}(\Sigma), \tag{4.46}$$

that decompose $\pi^p_\perp$ as $\pi^p_\perp=(\pi^p_{\mathrm{d}},\pi^p_\delta)$ and recalling (4.39), for all $\rho,\rho'\in\Omega^k_{\mathsf{c}}(M)$ one obtains

$$\begin{aligned}\omega_2(\rho\otimes\rho') &= \mathfrak{W}_{k-1}(\delta\,\rho\otimes\delta\,\rho')\\ &= \mathfrak{W}_\perp\Big(\pi^{k-1}_\delta(\delta_\Sigma\,\rho_\Sigma+\partial_t\rho_t)\otimes\pi^{k-1}_\delta(\delta_\Sigma\,\rho'_\Sigma+\partial_t\rho'_t)\Big)\\ &\quad+\mathfrak{W}_\perp\Big(\pi^{k-1}_{\mathrm{d}}(\partial_t\rho_t)\otimes\pi^{k-1}_{\mathrm{d}}(\partial_t\rho'_t)\Big)-\mathfrak{W}_\perp\Big(\pi^{k-2}_\delta(\delta_\Sigma\,\rho_t)\otimes\pi^{k-2}_\delta(\delta_\Sigma\,\rho'_t)\Big)\\ &= \mathfrak{W}_\perp\Big(\pi^{k-1}_\delta(\delta_\Sigma\,\rho_\Sigma+\partial_t\rho_t)\otimes\pi^{k-1}_\delta(\delta_\Sigma\,\rho'_\Sigma+\partial_t\rho'_t)\Big)\geq 0,\end{aligned} \tag{4.47}$$

where the two contributions appearing in the third line cancel out due to

$$\begin{aligned}\mathfrak{W}_\perp\Big(\pi^{k-2}_\delta(\delta_\Sigma\,\rho_t)\otimes\pi^{k-2}_\delta(\delta_\Sigma\,\rho'_t)\Big) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\langle\pi^{k-1}_{\mathrm{d}}\rho_t,\tfrac{1}{2}\triangle^{\frac{1}{2}}\,\exp(-i\,\triangle^{\frac{1}{2}}(t-t'))\,\pi^{k-1}_{\mathrm{d}}\rho'_t\rangle\,\mathrm{d}\,t\,\mathrm{d}\,t'\\ &= \mathfrak{W}_\perp\Big(\pi^{k-1}_{\mathrm{d}}(\partial_t\rho_t)\otimes\pi^{k-1}_{\mathrm{d}}(\partial_t\rho'_t)\Big).\end{aligned} \tag{4.48}$$

We are now in a position to define the desired state on $\mathcal{W}(\mathsf{Dyn}^k(M))$:

Proposition 4.7. *Let M be an m-dimensional ultra-static globally hyperbolic spacetime with compact Cauchy surface and recall Proposition 4.6. Then*

$$\omega_{\mathsf{Dyn}}:\mathcal{W}(\mathsf{Dyn}^k(M))\longrightarrow\mathbb{C},\qquad W(L[\rho])\longmapsto\exp\big(-2\pi\,\omega_2([\rho]\otimes[\rho])\big). \tag{4.49}$$

is a state on the C^*-algebra $\mathcal{W}(\mathsf{Dyn}^k(M))$ that fulfils the microlocal spectrum condition. Furthermore, for the state on $\mathcal{W}(\mathsf{Dyn}^k(M))$ and its analogue on $\mathcal{W}(\mathsf{Dyn}^{m-k}(M))$, one has

$$\omega_{\mathsf{Dyn}}\circ\mathcal{W}(\zeta_{\mathsf{Dyn}})=\omega_{\mathsf{Dyn}}, \tag{4.50}$$

where $\mathcal{W}(\zeta_{\mathsf{Dyn}}):\mathcal{W}(\mathsf{Dyn}^k(M))\to\mathcal{W}(\mathsf{Dyn}^{m-k}(M))$ is the duality isomorphism introduced in Remark 4.4.

Proof. We have shown that ω_2 is a bidistribution that fulfils the microlocal spectrum condition, that descends to the quotient $\Omega^k(M)_{\mathsf{Dyn}}\simeq\mathsf{Dyn}^k(M)$, whose antisymmetric part coincides with $-i\,\tau_{\mathsf{Dyn}}$ (which is equivalent to $-i\,\sigma_{\mathsf{Dyn}}$ under the isomorphism of Proposition 4.6, cf. (4.25)) and that is non-negative. Therefore ω_{Dyn} will be a "Hadamard state" for the C*-algebra $\mathcal{W}(\mathsf{Dyn}^k(M))$ as soon as we confirm that it is sufficient to specify it on the generators of the $*$-algebra $\mathcal{A}(\mathsf{Dyn}^k(M))$, cf. Section 4.1. Note that it is positive and normalized on $\mathcal{A}(\mathsf{Dyn}^k(M))$. Furthermore, it is immediate to check continuity with respect to $\|\cdot\|_1$ because the exponential factor is bounded from above by 1. In particular, ω_{Dyn} can be extended to the $\|\cdot\|_1$-completion of $\mathcal{A}(\mathsf{Dyn}^k(M))$, i.e. the Banach $*$-algebra $\mathcal{B}(\mathsf{Dyn}^k(M))$. By a standard property of the enveloping C*-algebra [Dix77, Prop. 2.7.4], the representations of $\mathcal{W}(\mathsf{Dyn}^k(M)) = \mathcal{C}^*(\mathcal{B}(\mathsf{Dyn}^k(M)))$ are in bijective correspondence with those of $\mathcal{B}(\mathsf{Dyn}^k(M))$. Therefore it is sufficient to specify ω_{Dyn} on $\mathcal{A}(\mathsf{Dyn}^k(M))$ (as we did) in order to obtain a unique canonical extension to $\mathcal{W}(\mathsf{Dyn}^k(M))$.

To confirm that our prescription for the construction of ω_{Dyn} is compatible with $\mathcal{W}(\zeta_{\mathsf{Dyn}}) : \mathcal{W}(\mathsf{Dyn}^k(M)) \to \mathcal{W}(\mathsf{Dyn}^{m-k}(M))$, let us observe that, on account of a similar property of the L^2-scalar product on $\Omega^p(\Sigma)$, for all $\alpha_\perp \in \Gamma_{\mathrm{c}}(M, \pi_2^* \bigwedge^p T^*\Sigma)$ and $\beta_\perp \in \Gamma_{\mathrm{c}}(M, \pi_2^* \bigwedge^{m-p-1} T^*\Sigma)$ such that, for all $t \in \mathbb{R}$, $\alpha_\perp(t,\cdot) \in \mathrm{d}_\Sigma \Omega^{p-1}(\Sigma) \oplus \delta_\Sigma \Omega^{p+1}(\Sigma)$ and $\beta_\perp(t,\cdot) \in \mathrm{d}_\Sigma \Omega^{m-p-2}(\Sigma) \oplus \delta_\Sigma \Omega^{m-p}(\Sigma)$, one has

$$\mathfrak{W}_\perp(\alpha_\perp \otimes *_\Sigma \beta_\perp) = \mathfrak{W}_\perp((-1)^{mp} *_\Sigma \alpha_\perp \otimes \beta_\perp), \tag{4.51}$$

As a direct consequence, for all $[\rho], [\rho'] \in \Omega^k_{\mathrm{c}}(M)_{\mathsf{Dyn}}$ one finds

$$\omega_2(\zeta_{\mathsf{Dyn}}[\rho] \otimes \zeta_{\mathsf{Dyn}}[\rho']) = \mathfrak{W}_{m-k-1}(\delta *\rho \otimes \delta *\rho') = -\mathfrak{W}_{k+1}(\mathrm{d}\,\rho \otimes \mathrm{d}\,\rho') = \omega_2([\rho] \otimes [\rho']). \tag{4.52}$$

Notice that we used (4.26) and (4.40) for the first step, (4.51) for the second one and again (4.40) to conclude. In particular, this entails the desired claimed relation between the state on $\mathcal{W}(\mathsf{Dyn}^k(M))$ and the state on $\mathcal{W}(\mathsf{Dyn}^{m-k}(M))$. □

Remark 4.8. To conclude this section, we observe that ω_{Dyn} has been constructed so to be a *ground state*, as per [SV00, App. A].

4.3 States for the torsion-free topological sector

In this section we exhibit a state on $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ commenting, in particular, on its significance. Our approach is inspired by [AMS93] and is motivated by the observation in [FMS07a, FMS07b] that the quantum Hilbert space associated to the model considered here is expected to carry a grading according to the free part of the magnetic and electric fluxes. Recalling Section 4.1, we have that $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ is the enveloping C*-algebra associated to the Banach $*$-algebra $\mathcal{B}(\mathsf{Top}^k_{\mathrm{free}}(M))$ obtained as the $\|\cdot\|_1$-completion of the $*$-algebra $\mathcal{A}(\mathsf{Top}^k_{\mathrm{free}}(M))$. Recalling also Section 3.2, we denote the generators of $\mathcal{A}(\mathsf{Top}^k_{\mathrm{free}}(M))$ by $W(u,\tilde{u},z,\tilde{z})$ for $(u,\tilde{u},z,\tilde{z}) \in \mathsf{Top}^k_{\mathrm{free}}(M)$.

Proposition 4.9. *Let $\omega_{\mathrm{free}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathbb{C}$ be the linear functional specified by*

$$\omega_{\mathrm{free}}(W(u,\tilde{u},z,\tilde{z})) = \begin{cases} 1 & \text{if } z = 0, \tilde{z} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.53}$$

Then ω_{free} is a state on the C-algebra $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$. Furthermore the state on $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ and its analogue on $\mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{free}}(M))$ are compatible with the duality isomorphism $\mathcal{W}(\zeta_{\mathrm{free}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{free}}(M))$ introduced in Remark 4.4, i.e.,*

$$\omega_{\mathrm{free}} \circ \mathcal{W}(\zeta_{\mathrm{free}}) = \omega_{\mathrm{free}}. \tag{4.54}$$

Proof. The functional is normalized since the unit in $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ is the element $W(0,0,0,0)$ and, by definition, $\omega_{\mathrm{free}}(W(0,0,0,0)) = 1$. To prove positivity, let I be an index set of finite cardinality and let $a = \sum_{i\in I} \alpha_i\, W(u_i,\tilde{u}_i,z_i,\tilde{z}_i)$, where $\alpha_i \in \mathbb{C}$ and $(u_i,\tilde{u}_i,z_i,\tilde{z}_i) \in \mathsf{Top}^k_{\mathrm{free}}(M)$ for all $i \in I$.

Without loss of generality, we assume $(u_i, \tilde{u}_i, z_i, \tilde{z}_i) \neq (u_j, \tilde{u}_j, z_j, \tilde{z}_j)$ for all $i, j \in I$ such that $i \neq j$. Set $i \sim j$ if and only if $z_i = z_j$ and $\tilde{z}_i = \tilde{z}_j$. Clearly, $\sim$ is an equivalence relation. Let $\tilde{I} = I/\sim$ and let us indicate with $\tilde{i}$ the equivalence class of $i \in I$. Using (3.11) and (4.53), we obtain

$$\omega_{\mathrm{free}}(a^* a) = \sum_{\tilde{i} \in \tilde{I}} \left| \sum_{i \in \tilde{i}} \alpha_i \exp\Big(2\pi i\, \sigma\big((u_i, \tilde{u}_i, 0, 0), (0, 0, z_i, \tilde{z}_i)\big)\Big) \right|^2 \geq 0, \tag{4.55}$$

which guarantees the positivity of ω_{free}. Furthermore, ω_{free} is clearly continuous with respect to the norm $\| \cdot \|_1$, hence it induces a unique state on the Banach $*$-algebra $\mathcal{B}(\mathsf{Top}^k_{\mathrm{free}}(M))$. By [Dix77, Prop. 2.7.4] this provides a unique representation, hence a state, also on the enveloping C*-algebra $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) = \mathcal{C}^*(\mathcal{B}(\mathsf{Top}^k_{\mathrm{free}}(M)))$.

To confirm that our prescription is compatible with the relevant duality isomorphism note that the last two components of $\zeta_{\mathrm{free}}(u, \tilde{u}, z, \tilde{z}) \in \mathsf{Top}^{m-k}_{\mathrm{free}}(M)$ vanish if and only if the last two components of $(u, \tilde{u}, z, \tilde{z}) \in \mathsf{Top}^k_{\mathrm{free}}(M)$ vanish. Therefore $\omega_{\mathrm{free}}(\mathcal{W}(\zeta_{\mathrm{free}})\, W(u, \tilde{u}, z, \tilde{z})) = \omega_{\mathrm{free}}(W(u, \tilde{u}, z, \tilde{z}))$, leading to the conclusion. □

Observe that the state is not faithful: by direct inspection of (4.53) one finds $0 \neq a \in \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ such that $\omega_{\mathrm{free}}(a^* a) = 0$. For example, such an a is given by

$$a = W(0, 0, z, \tilde{z}) - \exp\Big(2\pi i\, \sigma_{\mathrm{free}}\big((0, 0, z, \tilde{z}), (u, \tilde{u}, z, \tilde{z})\big)\Big)\, W(u, \tilde{u}, z, \tilde{z}). \tag{4.56}$$

Remark 4.10. A faithful alternative to ω_{free} is the state $\widetilde{\omega}_{\mathrm{free}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathbb{C}$ defined by

$$\widetilde{\omega}_{\mathrm{free}}(W(u, \tilde{u}, z, \tilde{z})) = \begin{cases} 1 & \text{if } u = 0, \tilde{u} = 0, z = 0, \tilde{z} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.57}$$

Although the GNS representation induced by $\widetilde{\omega}_{\mathrm{free}}$ is faithful, ω_{free} leads to a more appealing interpretation, which is why we regard it as our prime example.

We regard (4.53) as our prime example, being motivated by the observation in [FMS07a, FMS07b] that the Hilbert space associated to the quantization of Abelian duality is graded according to the *free part* of the magnetic and electric fluxes. In fact, as it will be evident from the construction of the induced GNS representation, the resulting Hilbert space will reproduce the expected grading by $\mathrm{H}^{k,m-k}_{\mathrm{free}}(M; \mathbb{Z})$. To show this fact explicitly, we construct below the GNS representation associated to the state ω_{free}. The Gelfand ideal $\mathcal{I}^k_{\mathrm{free}} \subseteq \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ of ω_{free} is precisely generated by elements of $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ of the form (4.56). Hence the GNS Hilbert space is the completion

$$\mathscr{H}^k_{\mathrm{free}} \doteq \overline{\mathscr{D}^k_{\mathrm{free}}} \tag{4.58a}$$

of the pre-Hilbert space

$$\mathscr{D}^k_{\mathrm{free}} \doteq \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))/\mathcal{I}^k_{\mathrm{free}} = \bigoplus_{(z,\tilde{z}) \in \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})} \mathbb{C}\, |z, \tilde{z}\rangle \tag{4.58b}$$

equipped with the scalar product

$$\langle \cdot | \cdot \rangle : \mathscr{D}^k_{\mathrm{free}} \times \mathscr{D}^k_{\mathrm{free}} \longrightarrow \mathbb{C}, \qquad \langle z', \tilde{z}' | z, \tilde{z} \rangle \doteq \omega_{\mathrm{free}}\big(W(0, 0, z', \tilde{z}')^*\, W(0, 0, z, \tilde{z})\big) \tag{4.58c}$$

induced by ω_{free}, where for notational convenience we set

$$|z, \tilde{z}\rangle \doteq [W(0, 0, z, \tilde{z})] \in \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))/\mathcal{I}^k_{\mathrm{free}}. \tag{4.58d}$$

The expected grading by free magnetic and electric fluxes appears explicitly in the presentation of the pre-Hilbert $\mathscr{D}^k_{\mathrm{free}}$ as a direct sum of copies of $\mathbb{C}$ labelled by $\mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})$. This grading carries over to its completion, i.e. the Hilbert space $\mathscr{H}^k_{\mathrm{free}}$.

The GNS representation associated to ω_{free} is defined by

$$\pi^k_{\mathrm{free}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \longrightarrow \mathcal{BL}(\mathscr{H}^k_{\mathrm{free}}), \qquad W(u,\tilde{u},z,\tilde{z}) \longmapsto \pi^k_{\mathrm{free}}(W(u,\tilde{u},z,\tilde{z})), \tag{4.59a}$$

where $\pi^k_{\mathrm{free}}(W(u,\tilde{u},z,\tilde{z}))$ acts on $\mathscr{H}^k_{\mathrm{free}}$ according to

$$\begin{aligned} \pi^k_{\mathrm{free}}(W(u,\tilde{u},z,\tilde{z})) : \mathscr{H}^k_{\mathrm{free}} &\longrightarrow \mathscr{H}^k_{\mathrm{free}}, \\ |z',\tilde{z}'\rangle &\longmapsto \exp\Big(2\pi i\,\sigma_{\mathrm{free}}\big((u,\tilde{u},0,0),(0,0,z+2z',\tilde{z}+2\tilde{z}')\big)\Big)\,|z+z',\tilde{z}+\tilde{z}'\rangle. \end{aligned} \tag{4.59b}$$

As a by-product, the cyclic vector of the GNS representation is $|0,0\rangle$. Furthermore, one observes that generators of the form $W(u,\tilde{u},0,0)$ act on $\mathscr{H}^k_{\mathrm{free}}$ by multiplication with a phase that depends linearly on u and $\tilde{u}$, while those of the form $W(0,0,z,\tilde{z})$ act on $\mathscr{H}^k_{\mathrm{free}}$ by shift:

$$\Phi^k(u,\tilde{u}) \doteq \pi^k_{\mathrm{free}}(W(u,\tilde{u},0,0)), \qquad \Sigma^k(z,\tilde{z}) \doteq \pi^k_{\mathrm{free}}(W(0,0,z,\tilde{z})). \tag{4.60}$$

In particular, it holds that

$$\Phi^k : \mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})/\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z}) \longmapsto \mathcal{BL}(\mathscr{H}^k_{\mathrm{free}}), \qquad (u,\tilde{u}) \longmapsto \Phi^k(u,\tilde{u}) \tag{4.61}$$

is a strongly continuous family of unitary operators linearly parametrized by the quotient of $\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})$ by $\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z})$. In particular, for each $(r,\tilde{r}) \in \mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})$, Stone's theorem provides an unbounded densely defined self-adjoint operator

$$P^k(r,\tilde{r}) : \mathscr{D}^k_{\mathrm{free}} \to \mathscr{H}^k_{\mathrm{free}}, \qquad |z,\tilde{z}\rangle \longmapsto 2\,\widetilde{\sigma}_{\mathrm{free}}\big((r,\tilde{r},0,0),(0,0,z,\tilde{z})\big)\,|z,\tilde{z}\rangle \tag{4.62}$$

that generates $t \in \mathbb{R} \mapsto \Phi^k\left(t(r,\tilde{r})\right) = \exp\left(2\pi i t\, P^k(r,\tilde{r})\right)$. Here $\widetilde{\sigma}_{\mathrm{free}}$ is the lift of σ_{free} defined by

$$\begin{aligned} \widetilde{\sigma}_{\mathrm{free}} : \big(\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R}) \oplus \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})\big)^{\times 2} &\longrightarrow \mathbb{R}, \\ \big((r,\tilde{r},z,\tilde{z}),(r',\tilde{r}',z',\tilde{z}')\big) &\longmapsto (\iota^*_\Sigma\tilde{r} \smile \iota^*_\Sigma z')[\Sigma] - (-1)^{k(m-k)}(\iota^*_\Sigma r \smile \iota^*_\Sigma \tilde{z}')[\Sigma] \\ &\qquad - (\iota^*_\Sigma\tilde{r}' \smile \iota^*_\Sigma z)[\Sigma] + (-1)^{k(m-k)}(\iota^*_\Sigma r' \smile \iota^*_\Sigma \tilde{z})[\Sigma] \end{aligned} \tag{4.63}$$

in terms of the cohomological pairing on Σ introduced in (3.45). Notice that (the spectra of) the operators $P^k(r,\tilde{r})$ are directly related to the values of z and $\tilde{z}$, which correspond to magnetic and electric fluxes [BBSS16, FMS07a, FMS07b]. As such, we regard the operators $P^k(r,\tilde{r})$ as *flux observables* (for $r=0$ only the magnetic flux is tested, conversely for $\tilde{r}=0$ only the electric flux). The shift operators $\Sigma^k(z,\tilde{z})$ instead are precisely those modifying such fluxes: adding z to the magnetic flux and $\tilde{z}$ to the electric one, they map between different degrees of the $\mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})$-graded Hilbert space $\mathscr{H}^k_{\mathrm{free}}$. Because of this appealing interpretation, that resembles the quantum mechanical description of a system formed by point particles freely moving on the circle with momenta $(z,\tilde{z})$, we regard ω_{free} as our prime example of state for the torsion-free topological sector.

Remark 4.11. We already observed that the duality isomorphism $\mathcal{W}(\zeta_{\mathrm{free}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{free}}(M))$ preserves the states ω_{free} defined on the source and the target, cf. Proposition 4.9. As a consequence, we obtain the isomorphism

$$U^k_{\mathrm{free}} : \mathscr{H}^k_{\mathrm{free}} \longrightarrow \mathscr{H}^{m-k}_{\mathrm{free}}, \qquad |z,\tilde{z}\rangle \longmapsto |\tilde{z},(-1)^{k(m-k)+1}z\rangle \tag{4.64}$$

between the GNS Hilbert spaces implementing the duality isomorphism $\mathcal{W}(\zeta_{\mathrm{free}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{free}}(M))$ of Remark 4.4 at the level of GNS representations:

$$U^k_{\mathrm{free}}\, \pi_{\mathrm{free}}(\cdot)\, (U^k_{\mathrm{free}})^{-1} = \pi_{\mathrm{free}} \circ \mathcal{W}(\zeta_{\mathrm{free}}). \tag{4.65}$$

As a by-product, we obtain that the operators in (4.60) and (4.62) are intertwined by these isomorphisms. In particular, for $m = 2k$, U^k_{free} is the unitary operator (Hilbert space automorphism) on $\mathscr{H}^k_{\mathrm{free}}$ that interchanges magnetic and electric fluxes (with a sign that accounts for the appropriate degrees):

$$P^k(\tilde{r}, (-1)^{k^2+1} r)\, U^k_{\mathrm{free}} = U^k_{\mathrm{free}}\, P^k(r, \tilde{r}). \tag{4.66}$$

4.4 States for the torsion topological sector

On the torsion topological sector we introduce a state similar to the one of Remark 4.10. Examples of spacetimes for which this sector is non-trivial are illustrated in [BBSS16, FMS07a, FMS07b], see also Remark 4.5.

Proposition 4.12. *Let $\omega_{\mathrm{tor}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M)) \to \mathbb{C}$ be the linear functional specified by*

$$\omega_{\mathrm{tor}}(W(t, \tilde{t})) = \begin{cases} 1 & \text{if } t = 0, \tilde{t} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.67}$$

Then ω_{tor} is a faithful state on the C^*-algebra $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$. Furthermore, for the state on $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$ and its analogue on $\mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{tor}}(M))$, one has

$$\omega_{\mathrm{tor}} \circ \mathcal{W}(\zeta_{\mathrm{tor}}) = \omega_{\mathrm{tor}}, \tag{4.68}$$

where $\mathcal{W}(\zeta_{\mathrm{tor}}) : \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M)) \to \mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{tor}}(M))$ denotes the duality isomorphism introduced in Remark 4.4.

Proof. Normalization and continuity with respect to $\|\cdot\|_1$ are immediate. For positivity, consider $a = \sum_{i\in I} \alpha_i\, W(t_i, \tilde{t}_i)$, where I is a finite set that labels $(t_i, \tilde{t}_i) \in \mathsf{Top}^k_{\mathrm{tor}}(M)$ faithfully, meaning that $i \neq j$ implies $(t_i, \tilde{t}_i) \neq (t_j, \tilde{t}_j)$. Then one finds

$$\omega_{\mathrm{tor}}(a^*\, a) = \sum_{i\in I} |\alpha_i|^2 \geq 0. \tag{4.69}$$

Then by the same argument presented in the proof of Proposition 4.9, we obtain the state ω_{tor} on the C*-algebra $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$. The identity displayed above also shows that the state ω_{tor} is faithful. Furthermore, ω_{tor} on $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$ and its analogue on $\mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{tor}}(M))$ are clearly related by ζ_{tor}. In fact, $(t, \tilde{t}) = 0 \in \mathsf{Top}^k_{\mathrm{tor}}(M)$ if and only if $\zeta_{\mathrm{tor}}(t, \tilde{t}) = 0 \in \mathsf{Top}^{m-k}_{\mathrm{tor}}(M)$. □

Remark 4.13. Notice that, passing to the GNS representations associated to ω_{tor} on $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$ and on $\mathcal{W}(\mathsf{Top}^{m-k}_{\mathrm{tor}}(M))$, we would obtain a Hilbert space isomorphism implementing $\mathcal{W}(\zeta_{\mathrm{tor}})$. The procedure is identical to the one of Remark 4.11. In fact, whenever the states satisfy a relation such as the one in (4.68), the above mentioned Hilbert space isomorphism is just a by-product of the GNS construction. In particular, for $m = 2k$ this leads to the duality isomorphism being unitarily implemented at the GNS level (i.e. as an automorphism of the GNS Hilbert space).

We are now in a position to state and prove the main result of Section 4:

Theorem 4.14. *Let M be an m-dimensional ultra-static globally hyperbolic spacetime M with compact Cauchy surface. Then via the factorization of Corollary 4.3 we obtain a state*

$$\omega : \mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z})) \longmapsto \mathbb{C} \tag{4.70}$$

by tensoring the states

$$\omega_{\mathsf{Dyn}} : \mathcal{W}(\mathsf{Dyn}^k(M)) \to \mathbb{C}, \qquad \omega_{\mathrm{free}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \to \mathbb{C}, \qquad \omega_{\mathrm{tor}} : \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M)) \to \mathbb{C} \tag{4.71}$$

of Propositions 4.7, 4.9 and 4.12. In addition, this construction is compatible with the duality isomorphism $\mathcal{W}(\zeta) : \mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z})) \to \mathcal{W}(\mathfrak{C}^{m-k}(M;\mathbb{Z}))$ of Remark 4.4, namely the states on the source and on the target are related by

$$\omega \circ \mathcal{W}(\zeta) = \omega. \tag{4.72}$$

Furthermore, the state on the dynamical sector $\mathcal{W}(\mathsf{Dyn}^k(M))$ fulfils the microlocal spectrum condition.[9]

Proof. By Corollary 4.3 we obtain a state on $\mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z}))$ by assigning one on the $\check{\otimes}$-tensor product of the C*-algebras $\mathcal{W}(\mathsf{Dyn}^k(M))$, $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ and $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$, cf. [Gui65] and Section 4.1. By [Gui65] two commuting representations (one for each factor) on a common Hilbert space provide a unique representation of the $\check{\otimes}$-tensor product. Since it is always possible to merge via the tensor product the carrier Hilbert spaces associated to two representations into a single counterpart on which the original representations act on one component and trivially on the other (hence they commute), it is sufficient for us to provide a representation of each $\check{\otimes}$-tensor factor. Indeed, this amounts to assigning a state on each sector, namely on $\mathcal{W}(\mathsf{Dyn}^k(M))$, $\mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M))$ and $\mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$ respectively. This task is accomplished by Propositions 4.7 (M is ultra-static), 4.9 and 4.12. In particular, Proposition 4.7 provides a Hadamard state.

Concerning the behaviour with respect to the duality isomorphism $\mathcal{W}(\zeta) : \mathcal{W}(\mathfrak{C}^k(M;\mathbb{Z})) \to \mathcal{W}(\mathfrak{C}^{m-k}(M;\mathbb{Z}))$, we observe that Propositions 4.7, 4.9 and 4.12 provide relations similar to (4.72) for each $\check{\otimes}$-tensor factor. Furthermore, in Remark 4.4 we observed that the factorization of Corollary 4.3 intertwines the duality isomorphism $\mathcal{W}(\zeta)$ with the $\check{\otimes}$-tensor product of the duality isomorphisms $\mathcal{W}(\zeta_{\mathsf{Dyn}})$, $\mathcal{W}(\zeta_{\mathrm{free}})$ and $\mathcal{W}(\zeta_{\mathrm{tor}})$. Therefore, the claim follows from the definition of ω. □

Remark 4.15. Although we do not explicitly pursue this goal here, let us mention that, using also the self-consistent splitting established in Remark 3.6, our analysis can be straightforwardly adapted to the case of self-dual configurations. In particular, one obtains an analogue of Theorem 4.14. However, this requires some care in the presence of torsion. In fact, one should keep in mind that the symplectic structure in the self-dual subtheory is not only the restriction of the symplectic structure σ on $\mathfrak{C}^k(M;\mathbb{Z})$ defined in (2.16), but it has to be *rescaled* by a factor of 1/2. This has to be done in order to avoid artificial degeneracies in the torsion topological sector that would otherwise show up whenever a $\mathbb{Z}_2$-summand is present. Refer to [BBSS16, Sect. 7] for further information about self-dual Abelian gauge fields.

[9] Note that the microlocal spectrum condition, which guarantees the existence of Wick polynomials and the finiteness of quantum fluctuations, makes sense only on $\mathcal{W}(\mathsf{Dyn}^k(M))$. The other sectors only possess "finitely many" degrees of freedom. In fact, they correspond to group characters on a *finitely generated* Abelian group (the topological configuration space), which corresponds to flat fields, representing Aharonov-Bohm configurations, and characteristic classes, interpreted as magnetic and electric fluxes.

4.5 An example: the Lorentz cylinder

In the last section we discuss explicitly a simple but instructive example. Additional ones are present in [Cap16]. We consider the so-called Lorentz cylinder $M = \mathbb{R} \times \mathbb{S}^1$ (notice that our convention is to set the length, and not the radius, of the circle to 1). Introducing the standard coordinates (t, θ), we endow M with the ultra-static metric $g = -\,\mathrm{d}\,t \otimes \mathrm{d}\,t + d\theta \otimes d\theta$. In addition, we focus our attention on the degree $k = 1$, i.e. $\mathfrak{C}^1(M;\mathbb{Z})$. Notice that any one of its elements is nothing else than a pair $(h, \tilde{h}) \in C^\infty(M;\mathbb{T}) \times C^\infty(M;\mathbb{T})$ such that $\mathrm{d}\log h = *\,\mathrm{d}\log \tilde{h}$.

Since $H^0(\mathbb{S}^1;\mathbb{Z}) \simeq \mathbb{Z} \simeq H^1(\mathbb{S}^1;\mathbb{Z})$, it ensues that $\mathsf{Top}^1_{\mathrm{tor}}(M)$ is trivial, while $\mathsf{Top}^1_{\mathrm{free}}(M) \simeq \mathbb{T}^2 \oplus \mathbb{Z}^2$. Furthermore, $\mathsf{Dyn}^1(M) = \mathrm{d}\,C^\infty(M) \cap *\,\mathrm{d}\,C^\infty(M)$. Hence, as a consequence of Corollary 4.3, the C*-algebra of observables consists of two factors only:

$$\mathcal{W}(\mathfrak{C}^1(M;\mathbb{Z})) \simeq \mathcal{W}(\mathsf{Dyn}^1(M)) \check{\otimes} \mathcal{W}(\mathsf{Top}^1_{\mathrm{free}}(M)). \tag{4.73}$$

A state thereon is completely specified by assigning it independently on each factor of the tensor product. As for the state on $\mathcal{W}(\mathsf{Top}^1_{\mathrm{free}}(M))$, this is nothing but (4.53): the analysis of Section 4.3 applies slavishly to the topological torsion-free sector, eventually leading to a $\mathbb{Z}^2$-graded Hilbert space. The $\mathbb{Z}^2$-summand of $\mathsf{Top}^1_{\mathrm{free}}(M)$ corresponds to the magnetic and electric fluxes, here realised by the windings of h and $\tilde{h}$ respectively around $\mathbb{S}^1$, for $(h, \tilde{h}) \in \mathfrak{C}^1(M;\mathbb{Z})$. The $\mathbb{T}^2$-summand of $\mathsf{Top}^1_{\mathrm{free}}(M)$ corresponds instead to flat fields, which in this context amount to h and $\tilde{h}$ being *constant* $\mathbb{T}$-valued functions on M (these are precisely the 0-mode configurations).

On the dynamical sector $\mathcal{W}(\mathsf{Dyn}^1(M))$, one can specialize further the formula of Section 4.2 for the two-point function of the state, confirming the absence of the contribution from 0-modes, which are in fact already taken care of by the topological sector. Let us take $\rho, \rho' \in \Omega^1_c(M)$. Starting from (4.40), we consider $\mathfrak{W}_0$. (4.39) shows that $\delta\,\rho = \delta_{\mathbb{S}^1}\,\rho_{\mathbb{S}^1} + \partial_t \rho_t$, where we regard $t \mapsto \rho_{\mathbb{S}^1}(t,\cdot)$ and $t \mapsto \rho_t(t,\cdot)$ as smoothly $\mathbb{R}$-parametrized k-forms on $\mathbb{S}^1$, for $k = 1, 0$ respectively. In addition, recalling that $\pi^0_\perp$ is defined in (4.31) as the projection onto the orthogonal complement of the harmonic part of $\Omega^0(\Sigma)$, it holds that

$$\pi^0_\perp(\delta\,\rho) = \sum_{n>0} c_n(t)\, \cos(2\pi n\theta) + d_n(t)\, \sin(2\pi n\theta), \tag{4.74}$$

where

$$c_n(t) \doteq 2\int_0^1 \cos(2\pi n\theta')\; \delta\,\rho(t,\theta')\, d\theta', \qquad d_n(t) \doteq 2\int_0^1 \sin(2\pi n\theta')\; \delta\,\rho(t,\theta')\, d\theta'. \tag{4.75}$$

Notice that the contribution from the mode $n = 0$ is the only part of $\delta\,\rho$ that is missing from the right-hand side of (4.74): this is the crucial step that allows us to get rid of 0-modes (i.e. harmonic forms). Performing the $\pi^0_\perp$-projection is well-motivated in the present context, as the contribution coming from harmonic forms is already taken into account in the topological sector. By writing the same expression for ρ', we can now evaluate directly (4.33):

$$\begin{aligned}\omega_2([\rho] \otimes [\rho']) &= \mathfrak{W}_0(\delta\,\rho \otimes \delta\,\rho') = \mathfrak{W}_\perp(\pi^0_\perp(\delta\,\rho) \otimes \pi^0_\perp(\delta\,\rho')) \\ &= \sum_{n>0} \frac{1}{4\pi n}\Big(\widehat{\delta\,\rho}(n,n)\,\widehat{\delta\,\rho'}(-n,-n) + \widehat{\delta\,\rho}(n,-n)\,\widehat{\delta\,\rho'}(-n,n)\Big),\end{aligned} \tag{4.76}$$

$[\rho], [\rho'] \in \Omega^1_{\mathrm{c}}(M)_{\mathsf{Dyn}}$ and

$$\widehat{\delta\,\rho}(n,m) = \int_{\mathbb{R}} dt \int_0^1 d\theta\, e^{-2\pi i n t} e^{2\pi i m\theta}\, \delta\,\rho(t,\theta), \tag{4.77}$$

for all $m, n \in \mathbb{Z}$.

Remark 4.16. Observe that (4.76) and the ensuing ω_{Dyn} identify a ground state for the dynamical sector of our theory. At first glance, this might appear as a contradiction to the renowned no-go result for the existence of ground states for a massless scalar field on a two-dimensional globally hyperbolic spacetime, see for example [SCH13]. The origin of this obstruction lies in the presence of an infrared singularity, which is reflected in the contribution of the 0-mode to the Fourier expansion of the two-point function. It is noteworthy that our implementation of Abelian duality automatically removes this pernicious feature by the separation between dynamical and topological degrees of freedom, which in this specific case include the 0-modes: as one can infer by direct inspection of (4.76), there is no contribution to the sum coming from $n = 0$.

Acknowledgements

The authors are grateful to Nicolò Drago and Alexander Schenkel for stimulating discussions and valuable suggestions and would like to thank the anonymous referees for their encouragement towards improving the structure and clarity of the paper. M.C. and C.D. are grateful to the Institute of Mathematics of the University of Potsdam for the kind hospitality during the realization of part of this work. The work of M.B. has been supported by a research fellowship of the Alexander von Humboldt foundation. The work of M.C. has been partially supported by IUSS (Pavia). The work of C.D. has been supported by the University of Pavia.

References

[AMS93] F. Acerbi, G. Morchio, F. Strocchi, *Theta vacua, charge confinement and charged sectors from nonregular representations of CCR algebras*, Lett. Math. Phys. **27** (1993), 1–11.

[ARN16] I. Agullo, A. del Rio and J. Navarro-Salas, *Electromagnetic duality anomaly in curved spacetimes*, arXiv:1607.08879 [gr-qc].

[BB14] C. Bär and C. Becker. *Differential Characters.* Lect. Notes Math. 2112, Springer, 2014, 198p.

[BGP07] C. Bär, N. Ginoux, and F. Pfäffle. *Wave Equations on Lorentzian Manifolds and Quantization*, American Mathematical Society, 2007, 193p.

[Bär15] C. Bär. *Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes*, Commun. Math. Phys. **333** (2015), 1585–1615.

[BBSS15] C. Becker, M. Benini, A. Schenkel, and R. J. Szabo. *Cheeger-Simons differential characters with compact support and Pontryagin duality*, arXiv:1511.00324 [math.DG].

[BBSS16] C. Becker, M. Benini, A. Schenkel, and R. J. Szabo. *Abelian Duality on Globally Hyperbolic Spacetimes*, Commun. Math. Phys. **349** (2017), 361–392. doi: 10.1007/s00220-016-2669-9.

[BSS14] C. Becker, A. Schenkel, and R. J. Szabo. *Differential cohomology and locally covariant quantum field theory*, arXiv:1406.1514 [hep-th].

[Ben16] M. Benini, *Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies,* J. Math. Phys. **57** (2016), 053502.

[BDHS14] M. Benini, C. Dappiaggi, T.-P. Hack, and A. Schenkel. *A C^*-Algebra for Quantized Principal U(1)-Connections on Globally Hyperbolic Lorentzian Manifolds*, Commun. Math. Phys. **332** (2014), 477–504.

[BDM14] M. Benini, C. Dappiaggi, S. Murro, *Radiative observables for linearized gravity on asymptotically flat spacetimes and their boundary induced states*, J. Math. Phys. **55** (2014), 082301.

[BDS13] M. Benini, C. Dappiaggi and A. Schenkel, *Quantized Abelian principal connections on Lorentzian manifolds*, Commun. Math. Phys. **330** (2014), 123–152, arXiv:1303.2515 [math-ph].

[BS06] A. N. Bernal and M. Sánchez. *Further Results on the Smoothability of Cauchy Hypersurfaces and Cauchy Time Functions*, Lett. Math. Phys.**77** (2006), 183–197.

[BDFY15] R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, editors. *Advances in Algebraic Quantum Field Theory*, Springer International Publishing, 2015, 453p.

[BFV03] R. Brunetti, K. Fredenhagen, and R. Verch. *The Generally Covariant Locality Principle – A New Paradigm for Local Quantum Field Theory*, Commun. Math. Phys. **237** (2003), 31–68.

[Cap16] M. Capoferri, *Algebra of observables and states for quantum Abelian duality*, M.Sc. thesis, University of Pavia, 2016, arXiv:1611.09055 [math-ph].

[CS85] J. Cheeger and J. Simons, *Differential characters and geometric invariants.* Lect. Notes Math. 1167, Springer, 1985.

[DL12] C. Dappiaggi and B. Lang. *Quantization of Maxwell's Equations on Curved Backgrounds and General Local Covariance*, Lett. Math. Phys. **101** (2012), 265–287.

[DS13] C. Dappiaggi and D. Siemssen, *Hadamard States for the Vector Potential on Asymptotically Flat Spacetimes*, Rev. Math. Phys. **25** (2013), 1350002, arXiv:1106.5575 [gr-qc].

[Dim92] J. Dimock, *Quantized electromagnetic field on a manifold*, Rev. Math. Phys. **4** (1992), 223–233.

[Dix77] J. Dixmier. *C^*-algebras.* North Holland Publishing Company, 1977, 506p.

[FP03] C. J. Fewster, M. J. Pfenning, *A quantum weak energy inequality for spin-one fields in curved space-time,* J. Math. Phys. **44** (2003), 4480.

[FL16] C. J. Fewster and B. Lang. *Dynamical Locality of the Free Maxwell Field*, Ann. Henri Poinc. **17** (2016), 401–436.

[Fre00] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, In Cambridge 2000, Surveys in differential geometry, 129–194 [hep-th/0011220].

[FMS07a] D. S. Freed, G. W. Moore, and G. Segal. *Heisenberg groups and noncommutative fluxes*, Ann. Phys. **322** (2007), 236–285.

[FMS07b] D. S. Freed, G. W. Moore, and G. Segal. *The Uncertainty of Fluxes*, Commun. Math. Phys. **271** (2007) 247–274.

[Ful89] S. A. Fulling *Aspects of Quantum Field Theory in Curved Spacetime*, London Mathematical Society Student Texts 17, 1989, 328p.

[GW14] C. Gérard and M. Wrochna, *Hadamard States for the Linearized Yang-Mills Equation on Curved Spacetime*, Commun. Math. Phys. **337** (2015), no. 1, 253–320, arXiv:1403.7153 [math-ph].

[Gui65] A. Guichardet. *Tensor product of C^*-algebras*, Sov. Math., **6** (1965) 210–213, and Lect. Notes Series no. 12, Aarhus Universitet, 1969.

[HLZ03] F. R. Harvey, H. B. Lawson, Jr. and J. Zweck, *The de Rham-Federer theory of differential characters and character duality*, Amer. J. Math. **125** (2003), 791–847.

[Hat02] A. Hatcher, *Algebraic topology.* Cambridge University Press, 2002.

[HS05] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Diff. Geom. **70** (2005), 329–452.

[Hör03] L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Classics in Mathematics, Springer-Verlag, 2003, 440p.

[KM15] V. Moretti and I. Khavkine, *Algebraic QFT in curved spacetime and quasifree Hadamard states: An introduction*, in R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason (eds.), Advances in algebraic quantum field theory, Springer, 2015, arXiv:1412.5945 [math-ph].

[KW91] B. S. Kay and R. M. Wald. *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon*, Phys. Rep. **207** (1991), 49–136.

[MSTV73] J. Manuceau, M. Sirugue, D. Testard, and A. Verbeure. *The smallest C^*-algebra for canonical commutations relations*, Commun. Math. Phys. **32** (1973), 231–243.

[Rad96] M. J. Radzikowski. *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Commun. Math. Phys. **179** (1996), 529–553.

[SV00] H. Sahlmann and R. Verch. *Passivity and Microlocal Spectrum Condition*, Commun. Math. Phys. **214** (2000), 705–731.

[SV01] H. Sahlmann, R. Verch, *Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime*, Rev. Math. Phys. **13** (2001), 1203–1246.

[SDH14] K. Sanders, C. Dappiaggi and T. P. Hack, *Electromagnetism, Local Covariance, the Aharonov-Bohm Effect and Gauss' Law*, Commun. Math. Phys. **328** (2014), 625–667 arXiv:1211.6420 [math-ph].

[SCH13] S. Schubert. *Über die Charakterisierung von Zuständen hinsichtlich der Erwartungswerte quadratischer Operatoren.* M.Sc. thesis, Universität Hamburg, 2013.

[SS08] J. Simons and D. Sullivan. *Axiomatic characterization of ordinary differential cohomology*, J. Topol. **1** (2008), 45–56.

[Sza12] R. J. Szabo, *Quantization of Higher Abelian Gauge Theory in Generalized Differential Cohomology,* PoS ICMP **2012** (2012), 009, arXiv:1209.2530 [hep-th].

[Wal94] R. M. Wald. *Quantum Field Theory on Curved Spacetime and Black Hole Thermodynamics*, Chicago Lectures in Physics, University of Chicago Press, 1994, 220p.