A SUFFICIENT CONDITION FOR STRONG F-REGULARITY

ALESSANDRO DE STEFANI AND LUIS NÚÑEZ-BETANCOURT

Abstract. Let $(R, \mathfrak{m}, K)$ be an F-finite Noetherian local ring which has a canonical ideal $I \subsetneq R$. We prove that if R is S_2 and $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module, then R is a strongly F-regular ring. In particular, under these assumptions, R is a Cohen-Macaulay normal domain.

1. Introduction

Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of positive characteristic and let $x \in \mathfrak{m}$ be a nonzero divisor in R. A central question in the study of singularities is whether good properties of the ring R/xR imply good properties of R. This is related to whether a type of singularity deforms. It is known that F-purity and strong F-regularity, two important and well studied types of singularities in positive characteristic, do not deform. This was showed by an example of Fedder [Fed83] for F-purity, and by an example of Singh [Sin99], for strong F-regularity. However, if R is a Gorenstein ring, both F-purity and strong F-regularity do deform.

Enescu [Ene03] changed gears by looking at canonical ideals instead of ideals generated by nonzero divisors. Suppose that R has a canonical ideal, i.e. an ideal I such that $I \cong \omega_R$. Note that in a Gorenstein ring, an ideal is generated by a nonzero divisor if and only if it is a canonical ideal. Recently, Ma [Ma12] showed that, under mild assumptions on R, if R/I is F-pure, then R is also F-pure [Ma14, Theorem 3.4]. Inspired by his result, we investigate if R/I being strongly F-regular implies that R is strongly F-regular or, equivalently, if R/I being F-rational implies that R is strongly F-regular. Our main result is Theorem 3.8, which is a more general version of the following.

Theorem 1.1. Let $(R, \mathfrak{m}, K)$ be an excellent local ring of dimension d and characteristic $p > 0$. Suppose that R is S_2 and it has a canonical ideal $I \cong \omega_R$ such that R/I is F-rational. Then R is a strongly F-regular ring.

This theorem extends a result of Enescu [Ene03, Corollary 2.9] by dropping the hypotheses of R being a Cohen-Macaulay (normal) domain. We point out that these three conditions are implied by R being strongly F-regular. In his work, Enescu uses properties of pseudo-canonical covers, while we focus on an interplay between Frobenius actions and p^{-1}-linear maps combined with structural properties of local cohomology. As a consequence of Theorem 1.1, we extend a result of Goto, Hayasaka, and Iai [GHI03, Corollary 2.4] to rings which are not necessarily Cohen-Macaulay. Specifically, we show that, under the assumptions of Theorem 1.1, if R/I is regular, then R is also regular (see Corollary 3.9).

Throughout this article $(R, \mathfrak{m}, K)$ will denote a Noetherian local ring of Krull dimension d and characteristic $p > 0$. $(-)^\vee$ denotes the Matlis dual functor $\operatorname{Hom}_R(\ -\ , E_R(K))$. In addition, ω_R denotes a canonical module for R, which is a finitely generated R-module satisfying ${\omega_R}^\vee \cong H^d_{\mathfrak{m}}(R)$.

2. Preliminaries

2.1. **Canonical modules.** In this section we present several facts and properties regarding canonical modules over rings which are not necessarily Cohen-Macaulay. We refer to [Aoy83, HH94b, Ma14] for details.

We recall that not every ring has a canonical module; however, every complete ring has one. In fact, if R is a homomorphic image of a Gorenstein local ring $(S, \mathfrak{n}, L)$ of dimension n, then $\omega_R \cong \operatorname{Ext}^{n-d}_S(R, S)$.

Proposition 2.1 ([Aoy83, Corollary 4.3])**.** Let $(R, \mathfrak{m}, K)$ be a local ring with canonical module ω_R. If R is equidimensional, then for every prime ideal P, $(\omega_R)_P$ is a canonical module for R_P.

Proposition 2.2 ([Ma14, Proposition 2.4])**.** Let $(R, \mathfrak{m}, K)$ be a local ring with canonical module ω_R. If R is equidimensional and unmixed, then the following conditions are equivalent:

(1) ω_R is isomorphic to an ideal $I \subseteq R$.
(2) R is generically Gorenstein, i.e. if $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Min}(R)(= \operatorname{Ass}(R))$.
(3) ω_R has rank 1.

Moreover, when any of these equivalent conditions hold, I is a height one ideal containing a nonzero divisor of R, and R/I is equidimensional and unmixed [Ma14, Proposition 2.6]. If, in addition, R is Cohen-Macaulay, then R/I is Gorenstein.

Definition 2.3. Let k be a positive integer. Recall that a finitely generated R-module M is said to satisfy *Serre's condition* S_k (or simply M is S_k) if

$$\operatorname{depth}(M_\mathfrak{p}) \geqslant \min\{k, \operatorname{ht}(\mathfrak{p})\}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

R is S_k if it satisfies Serre's condition S_k as a module over itself. If R is S_2, then R is unmixed. Furthermore, when $(R, \mathfrak{m}, K)$ is local and catenary (e.g. when it is excellent), the S_2 condition also implies that R is equidimensional. If R is excellent and S_2, then its $\mathfrak{m}$-adic completion $\widehat{R}$ is also S_2, and if I is a canonical ideal of R, then $I\widehat{R} = \widehat{I}$ is a canonical ideal of $\widehat{R}$.

2.2. **Methods in positive characteristic.** We recall some of the definitions of singularities for rings of positive characteristic. We refer the interested reader to [Hun96, Smi01, ST12, BS13] for surveys and a book on these topics.

Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of characteristic $p > 0$, and let $F^e : R \to R$ be the e-th iteration of the Frobenius endomorphism on R, where e is a positive integer. Let M be an R-module. By $F^e_*(M)$ we denote M viewed as a module over R via the action of F^e. Specifically, for any $F^e_*(m_1), F^e_*(m_2) \in F^e_* M$ and for any $r \in R$ we have

$$F^e_*(m_1) + F^e_*(m_2) = F^e_*(m_1 + m_2) \quad \text{ and } \quad r \cdot F^e_*(m_1) = F^e_*(r^{p^e} m_1).$$

If $e = 1$, we omit e in the notation. When R is reduced, the endomorphism F^e can be identified with the inclusion of R into R^{1/p^e}, the ring of its p^e-th roots.

Definition 2.4. R is called *F-finite* if $F_*(R)$ is a finitely generated R-module.

A local ring $(R, \mathfrak{m}, K)$ is F-finite if and only if it is excellent and $[K : K^p] < \infty$ [Kun76].

Definition 2.5. R is *F-pure* if $F \otimes 1_M : R \otimes_R M \to R \otimes_R M$ is injective for all R-modules M. R is *F-split* if the map $R \to F_* R$ splits.

Remark 2.6. If R is an F-pure ring, F itself is injective and R must be a reduced ring. We have that R is F-split if and only if R is a direct summand of $F_* R$. If R is an F-finite ring, R is F-pure if and only R is F-split [HR74, Lemma 5.1]. As a consequence, if $(R, \mathfrak{m}, K)$ is F-finite, we have that R is F-pure if and only if $\widehat{R}$ is F-pure. If R is F-finite, we use the word F-pure to refer to both.

Definition 2.7 ([BB11])**.** We say that an additive map $\phi : M \to M$ is *p^{-e}-linear* if $\phi(r^{p^e} v) = r\phi(v)$ for every $r \in R, v \in M$.

There is a bijective correspondence between p^{-1}-linear maps on M and R-module homomorphisms $F_* M \to M$.

Definition 2.8. We define the ring $R\{F\}$ as $\frac{R\langle F \rangle}{R(r^p F - Fr \mid r \in R)}$, the non-commutative R-algebra generated by F with relations $r^p \cdot F = F \cdot r$, for $r \in R$.

Definition 2.9. We say that an R-module M has a *Frobenius action*, if there is an additive map $F : M \to M$ such that $F(ru) = r^p F(u)$ for $u \in M$ and $r \in R$.

There is a natural equivalence between $R\{F\}$-modules and R-modules with a Frobenius action. In addition, every Frobenius action of M corresponds to an R-module homomorphism $M \to F_* M$. If R is complete and F-finite, then $(F_* M)^\vee \cong F_*(M^\vee)$ [BB11, Lemma 5.1]. Then, there is an induced map

$$F_*(M^\vee) \cong (F_* M)^\vee \to (M)^\vee,$$

which gives a correspondence between Frobenius actions on M and p^{-1}-linear maps on $M^\vee$. In this case, we have that the Frobenius map $F : R \to R$ induces a Frobenius action on $H^d_{\mathfrak{m}}(R)$. Suppose R has a canonical module ω_R. Let $\Phi : \omega_R \to \omega_R$ be the p^{-1}-linear map corresponding to the Matlis dual of $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$. We will refer to Φ as the *trace map* of ω_R.

Definition 2.10. A local ring $(R, \mathfrak{m}, K)$ of dimension d is called *F-rational* if it is Cohen-Macaulay and $H^d_{\mathfrak{m}}(R)$ is simple as a $R\{F\}$-module.

We point out that this is not the original definition introduced by Hochster and Huneke [HH90], which is in terms of tight closure: R is F-rational if the ideals generated by parameters are tightly closed. However, both definitions are equivalent due to Smith [Smi97, Theorem 2.6]. F-rational local rings have nice singularties; for example, they are normal domains.

Definition 2.11 ([HH90])**.** An F-finite ring R is *strongly F-regular* if for all nonzero elements $c \in R$, the R-linear homomorphism $\varphi : R \to F^e_*(R)$ defined by $\varphi(1) = F^e_*(c)$ splits for $e \gg 0$.

Theorem 2.12 ([HH89, Theorem 3.1 c)])**.** Every F-finite regular ring is strongly F-regular.

It is a well-known fact that strong F-regularity implies F-rationality. This could be seen from the relation that these notions have with tight closure (see [HH90]).

Definition 2.13 ([Sch10])**.** Suppose that R is an integral domain. The *test ideal* $\tau(R) \subseteq R$ is defined as the smallest non-zero compatible ideal of R.

$\tau(R)$ is the big test ideal originally defined by Hochster and Huneke [HH90] in terms of tight closure. Schwede [Sch10, Theorem 6.3] proved that the definition above is equivalent. We have that R is strongly F-regular if and only if $\tau(R) = R$. Furthermore, $\tau(R) = \operatorname{ann}_{E_R(K)} 0^*_{E_R(K)}$, where $0^*_{E_R(K)}$ denotes the tight closure of 0 in the injective hull $E_R(K)$ of K. We have that R is strongly F-regular if and only if $0^*_{E_R(K)} = 0$.

Remark 2.14. One can define strongly F-regular rings for rings that are not F-finite by requiring that for all nonzero elements $c \in R$, the R-linear homomorphism $\varphi : R \to F^e_*(R)$ defined by $\varphi(1) = F^e_*(c)$ is pure for $e \gg 0$. If R is F-finite, this definition is equivalent to Definition 2.11. Furthermore, R is strongly F-regular if and only if $\tau(R) = \operatorname{ann}_{E_R(K)} 0^*_{E_R(K)} = R$, as in the F-finite case [Smi93, Theorem 7.1.2]. Regular rings are strongly F-regular also for non F-finite rings. This can be proven using the Gamma construction [HH94a, Discussion 6.11 & Lemma 6.13] and Aberbach and Enescu's results on base change for test ideals [AE03, Corollary 3.8].

Remark 2.15. Let $\phi : R \to R$ be a p^{-e}-linear map. For every ideal $J \subseteq R$ we have $\phi(J^{[p^e]}) \subseteq J$. If ϕ is surjective then equality holds. Furthermore, if ϕ is surjective then R is F-pure. In fact, there exists an element $r \in R$ such that $\phi(r) = 1$, and then the R-linear homomorphism $\varphi : F^e_* R \to R$ defined by $\varphi(F^e_* x) = \phi(rx)$ gives the desired splitting.

Definition 2.16 ([Sch10])**.** Let $\phi : R \to R$ be a p^{-e}-linear map and let $J \subseteq R$ be an ideal. J is ϕ-compatible if $\phi(J) \subseteq J$. An ideal J is said to be compatible if it is ϕ-compatible for all p^{-e}-linear maps $\phi : R \to R$ and all $e \in \mathbb{N}$.

Compatible ideals in the definition above were used previously in different contexts [MR85, Smi97, LS01, HT04].

Remark 2.17. If R is a Gorenstein F-finite ring, we have that $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ as $F^e_* R$-modules, and the isomorphism is given by precomposition with multiplication by elements in $F^e_* R$. Let Φ be the p^{-1}-map corresponding to a generator of $\operatorname{Hom}_R(F_* R, R) \cong F^e_* R$ as a $F_* R$-module. If an ideal J is Φ-compatible, then it is compatible [ST12, Theorem 3.7].

3. Canonical ideals and strong F-regularity

We start by proving preparation lemmas that imply that under the hypotheses of Theorem 1.1, R is F-pure.

Lemma 3.1. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of dimension d. Suppose that R has a canonical ideal $I \subsetneq R$ such that $H^{d-1}_\mathfrak{m}(R/I)$ is a simple $R\{F\}$-module. Then the Frobenius map $F : H^{d-1}_\mathfrak{m}(R/I) \to H^{d-1}_\mathfrak{m}(R/I)$ is injective.

Proof. Since the Frobenius action on $\operatorname{Ker}(F) \subseteq H^{d-1}_\mathfrak{m}(R/I)$ is trivial, we have that $\operatorname{Ker}(F)$ is an $R\{F\}$-submodule. Then either $\operatorname{Ker}(F) = 0$ or $\operatorname{Ker}(F) = H^{d-1}_\mathfrak{m}(R/I)$, because $H^{d-1}_\mathfrak{m}(R/I)$ is simple. If $\operatorname{Ker} F = H^{d-1}_\mathfrak{m}(R/I)$, then every R-submodule of $\operatorname{Ker} F$ is an $R\{F\}$-module. Since $H^{d-1}_\mathfrak{m}(R/I)$ is a simple $R\{F\}$-module, $\operatorname{Ker} F = H^{d-1}_\mathfrak{m}(R/I)$ must be a simple R-module, that is $H^{d-1}_\mathfrak{m}(R/I) \cong R/\mathfrak{m} = k$. Since $\dim(R/I) = d - 1$, this is only possible if $d = 1$ and R/I is zero-dimensional. However, if $\dim(R/I) = 0$, we have $H^{d-1}_\mathfrak{m}(R/I) = R/I = R/\mathfrak{m}$ and $\operatorname{Ker}(F) = 0$ in this case. ∎

Remark 3.2. Suppose that $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module. From the short exact sequence $0 \to I \to R \to R/I \to 0$, we obtain an exact sequence of $R\{F\}$-modules

$$H^{d-1}_{\mathfrak{m}}(R/I) \longrightarrow H^{d}_{\mathfrak{m}}(I) \longrightarrow H^{d}_{\mathfrak{m}}(R) \longrightarrow 0.$$

The map $H^{d}_{\mathfrak{m}}(I) \to H^{d}_{\mathfrak{m}}(R)$ is not injective by [Ma14, Lemma 3.3], and its kernel is a non-zero $R\{F\}$-submodule of $H^{d}_{\mathfrak{m}}(I)$. Since $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module, the first map in the sequence above must be injective. Hence, when $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module, we have a short exact sequence of $R\{F\}$-modules

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(R/I) \longrightarrow H^{d}_{\mathfrak{m}}(I) \longrightarrow H^{d}_{\mathfrak{m}}(R) \longrightarrow 0.$$

Remark 3.3. Note that $H^{d-1}_{\mathfrak{m}}(R/I)$ and $H^{d}_{\mathfrak{m}}(R)$ are simple $R\{F\}$-modules if and only if $H^{d-1}_{\widehat{\mathfrak{m}}}(\widehat{R}/\widehat{I})$ and $H^{d}_{\widehat{\mathfrak{m}}}(\widehat{R})$ are simple $\widehat{R}\{F\}$-modules.

Lemma 3.4. Let $(R, \mathfrak{m}, K)$ be an excellent local ring of dimension d. Suppose that R is equidimensional and unmixed, and it has a canonical ideal $I \subsetneq R$. If $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module, then $H^{d-1}_{\mathfrak{m}}(R/I) \cong (R/I)^{\vee}$.

Proof. By Remark 3.3 we can assume that R is complete. By Remark 3.2 we have a short exact sequence of $R\{F\}$-modules $0 \to H^{d-1}_{\mathfrak{m}}(R/I) \to H^{d}_{\mathfrak{m}}(I) \to H^{d}_{\mathfrak{m}}(R) \to 0$. Taking the Matlis dual we get an exact sequence of R-modules:

$$0 \longrightarrow H^{d}_{\mathfrak{m}}(R)^{\vee} \cong J \longrightarrow H^{d}_{\mathfrak{m}}(I)^{\vee} \cong R \longrightarrow H^{d-1}_{\mathfrak{m}}(R/I)^{\vee} \longrightarrow 0,$$

where $J \cong \omega_R$ is potentially another canonical ideal for R. We then get that $H^{d-1}_{\mathfrak{m}}(R/I)^{\vee} \cong R/J =: \omega_{R/I}$ is a canonical module for R/I, and we want to show that $J = I$. We have a homomorphism of R/I-modules:

$$R/I \longrightarrow \operatorname{Hom}_{R/I}(\omega_{R/I}, \omega_{R/I}) \cong \operatorname{Hom}_{R/I}(R/J, R/J) \cong R/J,$$

which is just the map induced by the inclusion $I \subseteq J$. Since R is equidimensional and unmixed, so is R/I, and the kernel of the above map is trivial [HH94b]. In addition, this kernel is J/I. Therefore, $J = I$ and $H^{d-1}_{\mathfrak{m}}(R/I) \cong (R/I)^{\vee}$. ☐

Proposition 3.5. Let $(R, \mathfrak{m}, K)$ be an F-finite Noetherian local ring of dimension d. Suppose that R is S_2 and it has a canonical ideal $I \subsetneq R$ such that $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module. Then, R/I is an F-pure ring. As a consequence, R is an F-pure ring.

Proof. By Remarks 3.3 and 2.6, we may assume that R is a complete ring. We have that the Frobenius action on $H^{d-1}_{\mathfrak{m}}(R/I)$ is injective by Lemma 3.1. This induces a surjective p^{-1}-linear map on $R/I = \left(H^{d-1}_{\mathfrak{m}}(R/I)\right)^{\vee}$. Then, R/I is F-pure by Remark 2.15. Therefore, R is also F-pure [Ma14, Theorem 3.4]. ☐

The simplicity of $H^{d-1}_{\mathfrak{m}}(R/I)$ forces R/I to have mild singularities, as we show in the following result. This result will be needed in the proof of Theorem 3.8.

Theorem 3.6. Let $(R, \mathfrak{m}, K)$ be a Noetherian local F-finite ring of dimension d. Suppose that R is equidimensional and unmixed, and that it has a canonical ideal $I \subsetneq R$ such that $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module. Then R/I is a strongly F-regular Gorenstein ring.

Proof. We note that R is strongly F-regular and Gorenstein if and only if its completion is also strongly F-regular and Gorenstein. We can assume that R is complete by Remark 3.3. By Lemma 3.4, it follows that $H^{d-1}_{\mathfrak{m}}(R/I)^{\vee} = \omega_{R/I} \cong R/I$. To prove that R/I is Gorenstein, it remains to show that it is Cohen-Macaulay. Then, it suffices to show that R/I is strongly F-regular. Let $\Phi : R/I \to R/I$ be the p^{-1}-linear map which is dual to the Frobenius action on $H^{d-1}_{\mathfrak{m}}(R/I)$. We note that Φ is surjective by Lemma 3.1. Let $c \in R/I$ be a nonzero element and set $J := \bigcup_{e\in\mathbb{N}} \Phi^e(c(R/I))$. We have that J is an ideal of R/I that contains c. In addition, $\Phi(J) = J$ by Remark 2.15 and Lemma 3.1. Since $c \neq 0$, J is a nonzero ideal compatible with Φ, and thus $J^{\vee}$ corresponds to a nonzero $R\{F\}$-submodule of $H^{d-1}_{\mathfrak{m}}(R/I)$. But the latter is a simple $R\{F\}$-module, therefore $J^{\vee} = H^{d-1}_{\mathfrak{m}}(R/I)$, and hence $J = R/I$. In particular, there exists an element $r \in R/I$ and an integer N such that $\Phi^N(rc) = 1$. Let $\varphi : F^N_*(R/I) \to R/I$ be the R/I-linear map defined by $\varphi(F^N_* x) = \Phi^N(rx)$ for all $x \in R/I$. We have that $\varphi(F^e_* c) = 1$. Since $0 \neq c \in R/I$ was chosen arbitrarily, we conclude that R/I is strongly F-regular. □

We recall a result of Goto, Hayasaka, and Iai [GHI03] that is needed to prove Theorem 1.1.

Proposition 3.7 ([GHI03, Corollary 2.4])**.** Let $(S, \mathfrak{m}, K)$ be a Cohen-Macaulay local ring which has a canonical ideal $I \subsetneq S$ such that S/I is a regular local ring. Then R is regular.

Now, we are ready to prove our main theorem.

Theorem 3.8. Let $(R, \mathfrak{m}, K)$ be a Noetherian local F-finite ring of dimension d and characteristic $p > 0$. Suppose that R is S_2 and it has a canonical ideal $I \subsetneq R$ such that $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module. Then R is a strongly F-regular ring.

Proof. Under our assumptions on R, we have that $\tau(\widehat{R}) \cap R = \tau(R)$ [LS01, Theorem 2.3] and $I\widehat{R}$ is a canonical ideal for $\widehat{R}$. Thus, it suffices to prove our claim assuming that R is complete by Remark 3.3. By Theorem 3.6, I is a height one prime ideal. Since R is equidimensional, we have that $(\omega_R)_I$ is a canonical module for R_I, and in particular IR_I is a canonical ideal for R_I. Since R_I is a one-dimensional Cohen-Macaulay local ring and R_I/IR_I is a field, we have that R_I must be regular by Proposition 3.7.

We finish proving the theorem by means of contradiction. Assume that R is not strongly F-regular. Let $\tau(R)$ be the test ideal of R. We claim that $\tau(R) \subseteq I$. Let $N = \operatorname{ann}_{E_R(K)} \tau(R)$, which is a submodule of $E_R(K) \cong H^d_{\mathfrak{m}}(I)$ compatible with every Frobenius action on $E_R(K)$ (see [LS01]). In particular, N is an $R\{F\}$-submodule of $H^d_{\mathfrak{m}}(I)$. As $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-submodule of $H^d_{\mathfrak{m}}(I)$, it must be contained in N. In fact, they cannot be disjoint because they both contain the socle of $H^d_{\mathfrak{m}}(I)$. Taking annihilators in R and applying Matlis duality, we get

$$\tau(R) = \operatorname{ann}_R(N) \subseteq \operatorname{ann}_R(H^{d-1}_{\mathfrak{m}}(R/I)) = I.$$

Since the test ideal defines the non-strongly F-regular locus [LS01, Theorem 7.1], R_I is not strongly F-regular. This is a contradiction because every regular ring is strongly F-regular by Theorem 2.12. □

We are now ready to prove Theorem 1.1. We proceed by reducing to the F-finite case via gamma construction. This method is well-know to the experts. We refer to [HH94a, Dicussion 6.11 & Lemma 6.13] for definitions and properties.

Proof of Theorem 1.1. Completion does not change the assumption that R/I is F-rational by Remark 3.3. In addition, if $\widehat{R}$ is strongly F-regular, then so is R, because $\tau(R)\widehat{R} = \tau(\widehat{R})$ [LS01, Theorem 2.3]. We now consider a p-base for $K^{1/p}$, Λ, and a cofinite set $\Gamma \subseteq \Lambda$. Consider the faithfully flat extension $R \to R^\Gamma$ given by the gamma construction. We have that R^Γ is a complete F-finite local ring with maximal ideal $\mathfrak{m}^\Gamma := \mathfrak{m}R^\Gamma$ and residue field $K^\Gamma \cong K \otimes_R R^\Gamma$. Since R is complete, there exists a Gorenstein local ring $(S, \mathfrak{n}, K)$, with $\dim(R) = \dim(S)$, such that R is a homomorphic image of S. By functoriality of the Gamma construction, we have that S^Γ maps homomorphically onto R^Γ. Furthermore, the map $S \to S^\Gamma$ is local and flat, with K^Γ as closed fiber. Since S is a Gorenstein ring, so is S^Γ. Because I is a canonical module of R, we have that $I \cong \operatorname{Hom}_S(R, S)$. Since $S \to S^\Gamma$ is faithfully flat, we have

$$I \otimes_R R^\Gamma \cong \operatorname{Hom}_S(R, S) \otimes_S S^\Gamma \cong \operatorname{Hom}_{S^\Gamma}(R \otimes_S S^\Gamma, S^\Gamma) \cong \operatorname{Hom}_{S^\Gamma}(R^\Gamma, S^\Gamma).$$

Therefore, $I \otimes_R R^\Gamma \cong \omega_{R^\Gamma}$ is a canonical module for R^Γ. In addition, $I \otimes_R R^\Gamma \cong IR^\Gamma$; therefore, $I^\Gamma := IR^\Gamma \subsetneq R^\Gamma$ is a canonical ideal of R^Γ. Now consider the flat local homomorphism $R/I \to R^\Gamma/I^\Gamma$ induced by the gamma construction above. We have that R/I is excellent because it is complete. In addition, R/I is a Cohen-Macaulay domain because it is F-rational. Furthermore, the closed fiber is K^Γ, and R/I is Gorenstein by Lemma 3.4. We have that R^Γ/I^Γ is F-rational because parameter ideals are tightly closed in R^Γ/I^Γ [Abe01, Proposition 3.2]. Since R^Γ is F-finite, we conclude that R^Γ is strongly F-regular by Theorem 3.8. In this case, $\tau(R^\Gamma) = R^\Gamma$, so that $0^*_{E_{R^\Gamma}(K^\Gamma)} = 0$. Now, we apply [AE03, Corollary 3.8] to get that $\tau(R)R^\Gamma = \tau(R^\Gamma) = R^\Gamma$. Therefore, $\tau(R) = R$, and so, R is strongly F-regular. ☐

As a corollary of this result, we weaken the Cohen-Macaulay assumption in Proposition 3.7 to S_2 for excellent rings of positive characteristic.

Corollary 3.9. Let $(R, \mathfrak{m}, K)$ be an excellent local ring of positive characteristic. Suppose that R is S_2, and it has a canonical ideal $I \subsetneq R$ such that R/I is regular. Then R is regular.

Proof. As R/I is regular, it is an F-rational ring. By Theorem 1.1 R is strongly F-regular; therefore, R is Cohen-Macaulay. We can now apply Proposition 3.7 to get the desired result. ☐

Finally, we give an example which shows that the sufficient condition for strong F-regularity in Theorem 3.8 is not necessary. That is, if R is strongly F-regular, we cannot always find a canonical ideal $I \subseteq R$ such that $H^{d-1}_{\mathfrak{m}}(R/I)$ is a simple $R\{F\}$-module, or equivalently such that R/I is strongly F-regular.

Example 3.10. Let K be a field with $\operatorname{char}(K) \neq 3$, and consider the two dimensional domain

$$R = K[\![s^3, s^2t, st^2, t^3]\!] \cong K[\![x, y, z, w]\!]/J,$$

where $J = (z^2 - yw, yz - xw, y^2 - xz)$. Then R has type two, and it is a direct summand of $k[\![s, t]\!]$; therefore, it is strongly F-regular [HH89, Theorem 3.1 (e)]. Any canonical ideal of R is two-generated, say $I = (f, g)$. Denote by $\mathfrak{m}$ the maximal ideal of R. If R/I was strongly F-regular, then it would be regular because it is a one-dimensional ring. This would be equivalent to $\dim_K(\mathfrak{m}/(\mathfrak{m}^2 + I + J)) = 1$. Since $J \subseteq \mathfrak{m}^2$, we have

$$\dim_K\left(\frac{\mathfrak{m}}{\mathfrak{m}^2 + I + J}\right) = \dim_K\left(\frac{\mathfrak{m}}{\mathfrak{m}^2 + (f, g)}\right) \geqslant 2.$$

Remark 3.11. If R is an F-finite normal domain, one can use a more geometric argument to show that if R/I is strongly F-regular, then so is R. This is done by choosing a canonical divisor on Spec(R) corresponding to the ideal $I \cong \omega_R$, and by using F-adjunction [Sch09].

Acknowledgments

We would like to thank Linquan Ma for valuable comments which greatly improved the exposition of the paper and for suggesting to us the one-dimensional version of Proposition 3.7. We thank Karl Schwede for pointing out Remark 3.11. We also thank Craig Huneke for useful discussions. We thank the referee for helpful comments. In particular, for suggesting to drop the assumption of F-finite in Theorem 1.1. The first author was partially supported by NSF Grant DMS-1259142. The second author thanks the National Council of Science and Technology of Mexico (CONACyT) for support through Grant #207063.

References

[Abe01] Ian M. Aberbach. Extension of weakly and strongly F-regular rings by flat maps. *J. Algebra*, 241(2):799–807, 2001. 7

[AE03] Ian M. Aberbach and Florian Enescu. Test ideals and base change problems in tight closure theory. *Trans. Amer. Math. Soc.*, 355(2):619–636, 2003. 4, 7

[Aoy83] Yôichi Aoyama. Some basic results on canonical modules. *J. Math. Kyoto Univ.*, 23(1):85–94, 1983. 2

[BB11] Manuel Blickle and Gebhard Böckle. Cartier modules: finiteness results. *J. Reine Angew. Math.*, 661:85–123, 2011. 3

[BS13] Manuel Blickle and Karl Schwede. p^{-1}-linear maps in algebra and geometry. In *Commutative algebra*, pages 123–205. Springer, New York, 2013. 2

[Ene03] Florian Enescu. Applications of pseudocanonical covers to tight closure problems. *J. Pure Appl. Algebra*, 178(2):159–167, 2003. 1

[Fed83] Richard Fedder. F-purity and rational singularity. *Trans. Amer. Math. Soc.*, 278(2):461–480, 1983. 1

[GHI03] Shiro Goto, Futoshi Hayasaka, and Shin-Ichiro Iai. The a-invariant and Gorensteinness of graded rings associated to filtrations of ideals in regular local rings. *Proc. Amer. Math. Soc.*, 131(1):87–94 (electronic), 2003. 1, 6

[HH89] Melvin Hochster and Craig Huneke. Tight closure and strong F-regularity. *Mém. Soc. Math. France (N.S.)*, (38):119–133, 1989. Colloque en l'honneur de Pierre Samuel (Orsay, 1987). 3, 7

[HH90] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990. 3, 4

[HH94a] Melvin Hochster and Craig Huneke. F-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346(1):1–62, 1994. 4, 6

[HH94b] Melvin Hochster and Craig Huneke. Indecomposable canonical modules and connectedness. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, volume 159 of *Contemp. Math.*, pages 197–208. Amer. Math. Soc., Providence, RI, 1994. 2, 5

[HR74] Melvin Hochster and Joel L. Roberts. Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. *Advances in Math.*, 13:115–175, 1974. 3

[HT04] Nobuo Hara and Shunsuke Takagi. On a generalization of test ideals. *Nagoya Math. J.*, 175:59–74, 2004. 4

[Hun96] Craig Huneke. *Tight closure and its applications*, volume 88 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With an appendix by Melvin Hochster. 2

[Kun76] Ernst Kunz. On Noetherian rings of characteristic p. *Amer. J. Math.*, 98(4):999–1013, 1976. 2

[LS01] Gennady Lyubeznik and Karen E. Smith. On the commutation of the test ideal with localization and completion. *Trans. Amer. Math. Soc.*, 353(8):3149–3180 (electronic), 2001. 4, 6, 7

[Ma12] Linquan Ma. Finiteness properties of local cohomology for F-pure local rings. *Preprint*, 2012. 1

[Ma14] Linquan Ma. A sufficient condition for F-purity. *J. Pure Appl. Algebra*, 218(7):1179–1183, 2014. 1, 2, 5

[MR85] V. B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties. *Ann. of Math. (2)*, 122(1):27–40, 1985. 4

[Sch09] Karl Schwede. F-adjunction. *Algebra Number Theory*, 3(8):907–950, 2009. 8

[Sch10] Karl Schwede. Centers of F-purity. *Math. Z.*, 265(3):687–714, 2010. 3, 4

[Sin99] Anurag K. Singh. F-regularity does not deform. *Amer. J. Math.*, 121(4):919–929, 1999. 1

[Smi93] Karen Ellen Smith. *Tight closure of parameter ideals and F-rationality.* ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of Michigan. 4

[Smi97] Karen E. Smith. F-rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997. 3, 4

[Smi01] Karen E. Smith. Tight closure and vanishing theorems. In *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, volume 6 of *ICTP Lect. Notes*, pages 149–213. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001. 2

[ST12] Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012. 2, 4

Department of Mathematics, University of Virginia, Charlottesville, VA 22903
Email address: `ad9fa@virginia.edu`

Department of Mathematics, University of Virginia, Charlottesville, VA 22903
Email address: `lcn8m@virginia.edu`