

A ZARISKI-NAGATA THEOREM FOR SMOOTH \mathbb{Z} -ALGEBRAS

ALESSANDRO DE STEFANI, ELOÍSA GRIFO, AND JACK JEFFRIES

ABSTRACT. In a polynomial ring over a perfect field, the symbolic powers of a prime ideal can be described via differential operators: a classical result by Zariski and Nagata says that the n -th symbolic power of a given prime ideal consists of the elements that vanish up to order n on the corresponding variety. However, this description fails in mixed characteristic. In this paper, we use p -derivations, a notion due to Buium and Joyal, to define a new kind of differential powers in mixed characteristic, and prove that this new object does coincide with the symbolic powers of prime ideals. This seems to be the first application of p -derivations to Commutative Algebra.

1. INTRODUCTION

The subject of symbolic powers is both a classical Commutative Algebra topic and an active area of current research. While there are many open problems in the setting of algebras containing a field, even the results that are well-understood for algebras over fields are mostly open for \mathbb{Z} -algebras and local rings of mixed characteristic. Thanks to the perfectoid spaces techniques of Schölze [?] as applied to Commutative Algebra by André and Bhatt, a major advance has happened recently [And16, Bha16]. Ma and Schwede have shown that a theorem of Ein-Lazarsfeld-Smith [ELS01] and Hochster-Huneke [HH02] on the uniform containment of symbolic and ordinary powers of ideals holds for regular rings of mixed characteristic [MS17].

In this paper, we are interested in generalizing another classical result on symbolic powers to the case of mixed characteristic and smooth \mathbb{Z} -algebras: the Zariski-Nagata Theorem, which establishes that symbolic powers of prime ideals can be described using differential operators.

Zariski's Main Lemma on Holomorphic Functions [Zar49], together with work by Nagata [Nag62, p. 143], states that if P is a prime ideal in a polynomial ring over a field, then

$$P^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq P \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}^n.$$

This result was later refined by Eisenbud and Hochster [EH79], and can be rephrased using differential powers of ideals, a fact which was well-known in characteristic 0 and extended to perfect fields in [DDSG⁺17]. More precisely, if R is a smooth algebra over a perfect field K , and Q is a prime ideal in R , the Zariski-Nagata Theorem states that the n -th symbolic power $Q^{(n)}$ of Q consists of the elements in R that are taken inside Q by every K -linear differential operator of order at most $n - 1$.

Rather than using perfectoid techniques, our generalization of Zariski-Nagata makes use of a different arithmetic notion of derivative, the notion of a p -derivation, defined by Joyal [Joy85] and Buium [Bui95] independently. From a Commutative Algebra point of view, p -derivations are rather exotic maps from a ring to itself — in particular, they are not even additive — but they do have many applications to Arithmetic Geometry, such as in [Bui05, Bui15, Bor09]. To the best of our knowledge, this is the first application of p -derivations to Commutative Algebra.

While our results cover a more general setting, let us describe the case where $R = A[x_1, \dots, x_n]$, where A denotes the integers \mathbb{Z} or the p -adic integers \mathbb{Z}_p . Given a prime ideal Q in R , we study two different types of differential powers associated to Q . The first one is defined just in terms of

The third author was partially supported by NSF Grant DMS #1606353.

differential operators, as in the statement of the Zariski-Nagata Theorem. More precisely, given an integer $n \geq 1$, the n -th (A -linear) differential power of Q is defined as

$$Q^{(n)A} = \{f \in R \mid \partial(f) \in Q \text{ for all } \partial \in D_{R|A}^{n-1}\},$$

where $D_{R|A}^{n-1}$ is the set of A -linear differential operators on R of order at most $n-1$ (see Definition 2.3). If Q does not contain any prime integer, then $Q^{(n)A}$ coincides with the n -th symbolic power of Q .

Theorem A. (see Theorem 3.6) Let $R = A[x_1, \dots, x_n]$, where $A = \mathbb{Z}$ or $A = \mathbb{Z}_p$, and Q be a prime ideal of R such that $Q \cap A = (0)$. Then $Q^{(n)} = Q^{(n)A}$ for all $n \geq 1$.

More generally, the previous result holds if R is an essentially smooth algebra over A , where A is either \mathbb{Z} or a DVR of mixed characteristic.

If the prime ideal Q contains a prime integer p , then differential powers are not sufficient to characterize symbolic powers, as one can see in Remark 3.8. To overcome this issue, we combine differential operators and p -derivations to define the mixed differential powers of an ideal. Given a fixed p -derivation δ , the n -th mixed differential power of Q is the ideal given by

$$Q^{(n)_{\text{mix}}} = \{f \in R \mid (\delta^a \circ \partial)(f) \in I \text{ for all } \partial \in D_{R|A}^b \text{ with } a + b \leq n - 1\}.$$

In principle, the mixed differential powers of an ideal depend on the choice of a p -derivation δ . However, in our setting $Q^{(n)_{\text{mix}}}$ is independent of the choice of the p -derivation (see Corollary 3.21). This new notion of mixed differential powers allows us to characterize symbolic powers of prime ideals that contain a given integer p .

Theorem B. (see Theorem 3.20) Let $R = A[x_1, \dots, x_n]$, where $A = \mathbb{Z}$ or $A = \mathbb{Z}_p$, and Q be a prime ideal of R such that $Q \cap A = (p)$, for a prime p . Then $Q^{(n)} = Q^{(n)_{\text{mix}}}$ for all $n \geq 1$.

More generally, we show this holds for R an essentially smooth algebra over A , where A is either \mathbb{Z} or a DVR with uniformizer p , as long as R has a p -derivation and A/pA satisfies some additional assumptions (e.g., A/pA is perfect).

2. BACKGROUND

2.1. Essentially smooth algebras. Throughout, we say that a ring S is *smooth* over a subring B if the inclusion map $B \rightarrow S$ is formally smooth and S is a finitely generated B -algebra. We say that S is *essentially smooth* over B if the inclusion map $B \rightarrow S$ is formally smooth and S is a localization of a finitely generated B -algebra (i.e., S is essentially of finite type over B). Polynomial extensions of rings are smooth. The following structure lemma asserts that in our setting, every smooth map is locally of this form.

Lemma 2.1. Let A be either

- (1) a field, or
- (2) \mathbb{Z} or a DVR with uniformizer $p \in \mathbb{Z}$.

Let (R, \mathfrak{m}) be a local ring that is essentially smooth over A , and suppose in Case (2) that $p \in \mathfrak{m}$. Then $R \cong A[x_1, \dots, x_d]_P$ for some nonnegative integer d and some prime ideal P inside $A[x_1, \dots, x_d]$, and in Case (2), $p \in P$.

Proof. By assumption, we can write R as T/I , where $T = A[x_1, \dots, x_d]_P$, for some prime ideal P of $A[x_1, \dots, x_d]$ and an ideal I of T . We prove the claim by induction on d .

Case (1): If $d = 0$, there is nothing to show, since our assumptions imply that $R \cong A$. We now assume $d \geq 1$. If $g_1, \dots, g_s \in T$ is a generating set of I , we may assume that every g_i belongs to P^2T , otherwise T/g_iT is isomorphic to the localization of a polynomial ring over A in one fewer

variable, and we are done by induction. Then, $\dim_A(P/P^2) = d$, but if $I \neq 0$, then $\dim(T/I) < d$, and R is not regular, so that R is not smooth over A . Thus, $I = 0$.

Case (2): If $d = 0$, there is nothing to show, since our assumptions imply that $R \cong A_{\mathfrak{p}A}$. We now assume $d \geq 1$. Observe that, if we set $\mathfrak{p} = P \cap A = pA$, then $T \cong A_{\mathfrak{p}}[x_1, \dots, x_d]_P$. If $g_1, \dots, g_s \in T$ is a generating set of I , we can write $g_i = f_i + h_i$, where $f_i \in pT$ and $h_i \in (x_1, \dots, x_d)T$. We may assume that every h_i belongs to P^2T , otherwise T/g_iT is isomorphic to the localization of a polynomial ring over A in one fewer variable, and we are done by induction. Tensoring $R \cong T/I$ with $(A/\mathfrak{p})_{\mathfrak{p}}$ we obtain

$$R \otimes_A (A/\mathfrak{p})_{\mathfrak{p}} \cong \frac{T}{(p, g_1, \dots, g_s)T} \cong \frac{T}{(p, h_1, \dots, h_s)T}.$$

Since $A \rightarrow R \cong T/I$ is formally smooth by assumption, the fiber $R \otimes_A (A/\mathfrak{p})_{\mathfrak{p}}$ must be regular. As we previously reduced to the case $h_i \in P^2T$ for all i , we must have $(h_1, \dots, h_s) \subseteq pT$, otherwise $R \otimes_A (A/\mathfrak{p})_{\mathfrak{p}}$ would not be regular. In particular, we have $g_i \in pT$ for all i , that is, $I = pJ$ for some ideal $J \subseteq T$. Unless $I = 0$, this contradicts the fact that $A \rightarrow T/I$ is formally smooth, hence flat [Gro67, 17.5.1]. It follows that $I = 0$, and we have proved that $R = S_P$ for some polynomial algebra $S = A[x_1, \dots, x_d]$ and some prime ideal $P \in \text{Spec}(A[x_1, \dots, x_d])$. \square

Lemma 2.2. Let either

- (1) $A = k$ be a field, or
- (2) A be \mathbb{Z} or a DVR with uniformizer $p \in \mathbb{Z}$, and $k = A/pA$.

Let (R, \mathfrak{m}, K) be a local ring that is essentially smooth over A , and suppose that $p \in \mathfrak{m}$. Assume that the field extension $k \rightarrow K$ is separable. If y_1, \dots, y_s is a minimal generating set of \mathfrak{m} in Case (1) or p, y_1, \dots, y_s is a minimal generating set of \mathfrak{m} in Case (2), then there is a free basis for $\Omega_{R|A}$ that contains dy_1, \dots, dy_s .

Proof. We can write $R \cong A[x_1, \dots, x_d]_P$ by Lemma 2.1. The module $\Omega_{R|A}$ is free of rank d . It suffices to show that the images of dy_1, \dots, dy_s are linearly independent in $\Omega_{R|A} \otimes_R K$. We apply the second fundamental sequence of Kähler differentials [Mat89, Theorem 25.2] to the maps $A \rightarrow R \rightarrow R/\mathfrak{m}$ to obtain a right-exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\varrho} \Omega_{R|A} \otimes_R K \longrightarrow \Omega_{K|A} \longrightarrow 0,$$

where $\varrho(y) = dy \otimes 1$.

The transcendence degree of K over k is $d - s$. By [Hoc, p. 24] (or see [Mat89, §26]), since $k \rightarrow K$ is a separable field extension, $\Omega_{K|A} = \Omega_{K|k}$ is a K -vector space of dimension at most $d - s$. In Case (1), $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = s$, so ϱ must be injective.

In Case (2), $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = s + 1$, so the kernel of ϱ must be at most one-dimensional. The class of p modulo \mathfrak{m}^2 is in the kernel of ϱ , so the images of the other basis elements y_1, \dots, y_s must be linearly independent. This concludes the proof. \square

2.2. Differential operators. We now review some results regarding differential operators that we use in the rest of the paper. For unexplained facts and notation we refer to [Gro67, Chapter 16].

Definition 2.3. Let $B \rightarrow S$ be a map of rings. The B -linear differential operators on S of order n are defined inductively as follows:

- $D_{S|B}^0 = \text{Hom}_S(S, S) \subseteq \text{Hom}_B(S, S)$.
- $D_{S|B}^n = \{\delta \in \text{Hom}_B(S, S) \mid [\delta, f] \in D_{S|B}^{n-1} \text{ for all } f \in D_{S|B}^0\}$.

Lemma 2.4. Let $B \rightarrow S$ be a formally smooth map of rings. Suppose that $\Omega_{S|B}$ is free, e.g., S is local, and let $\{dz_1, \dots, dz_\Lambda\}$ be a free basis for $\Omega_{S|B}$. Then there exists a family of differential operators $\{D_\alpha\}_{\alpha \in \mathbb{N}^\Lambda}$ such that

- $D_\alpha(z^\beta) = \binom{\beta}{\alpha} z^{\beta-\alpha}$ for all $\beta \in \mathbb{N}^\Lambda$ with $\beta_i \geq \alpha_i$ for all i , and

- $D_\alpha(z^\beta) = 0$ for all $\beta \in \mathbb{N}^\Lambda$ with $\beta_i < \alpha_i$ for some i .

The S -module $D_{S|B}^n$ is a free S -module generated by $\{D_\alpha : |\alpha| \leq n\}$ for each n .

Proof. By [Gro67, Theorem 16.10.2], S is differentially smooth over B . Then, the statement above is the content of [Gro67, Theorem 16.11.2]. \square

Corollary 2.5. In the context of Lemma 2.4, fix α such that $\alpha_{t+1} = \dots = \alpha_\Lambda = 0$, for some $t \leq \Lambda$. Consider the ideal $I = (z_1, \dots, z_t)$, and let $u \notin I$. Then:

- (1) $D_\alpha(uz^\alpha) \notin I$, and
- (2) For all $\beta \neq \alpha$ with $|\beta| = |\alpha|$, $D_\alpha(uz^\beta) \in I$.

Proof. First, note that $z^\beta \in I^{|\beta|}$. By Proposition 3.2 (3), $\partial(z^\beta) \in I$ for all $\partial \in D_{S|B}^{|\beta|-1}$. Therefore, every differential operator $[D_\alpha, u]$ with $|\alpha| = |\beta|$, takes z^β into I . It follows that

- (1) $D_\alpha(uz^\alpha) = [D_\alpha, u](z^\alpha) + uD_\alpha(z^\alpha) = [D_\alpha, u](z^\alpha) + u \notin I$, and
- (2) $D_\alpha(uz^\beta) = [D_\alpha, u](z^\beta) + uD_\alpha(z^\beta) = [D_\alpha, u](z^\beta) \in I$.

\square

2.3. p -Derivations. Fix a prime $p \in \mathbb{Z}$, and let S be a ring on which p is a nonzerodivisor. The following operators were introduced independently in [Joy85] and [Bui95]:

Definition 2.6. We say that a set-theoretic map $\delta : S \rightarrow S$ is a p -derivation if $\phi_p(x) := x^p + p\delta(x)$ is a ring homomorphism. Equivalently, δ is a p -derivation if $\delta(1) = 0$ and δ satisfies the following identities for all $x, y \in S$:

$$(2.3.1) \quad \delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

and

$$(2.3.2) \quad \delta(x + y) = \delta(x) + \delta(y) + C_p(x, y),$$

where $C_p(X, Y) = \frac{X^p + Y^p - (X+Y)^p}{p} \in \mathbb{Z}[X, Y]$. If δ is a p -derivation, we set δ^a to be the a -fold self-composition of δ ; in particular, δ^0 is the identity. We set $\text{Der}_p(S)$ to be the set of p -derivations on S . For all positive integers n , we let

$$\text{Der}_p^n(S) = \{\delta_1 \circ \dots \circ \delta_t \mid \delta_i \in \text{Der}_p(S) \text{ for all } i, \text{ and } t \leq n\}.$$

For a thorough development of the theory of p -derivations, see [Bui05].

Remark 2.7. Repeated application of Equation (2.3.2) shows that a p -derivation sends the prime ring of R (i.e., the canonical image of \mathbb{Z}) to itself. If R has characteristic zero, so that the prime ring is \mathbb{Z} , any p -derivation on R restricts to a p -derivation on \mathbb{Z} ; there is a unique p -derivation on \mathbb{Z} given by the *Fermat difference operator*: $\delta(n) = \frac{n - n^p}{p}$. In particular, when R has characteristic zero, every p -derivation satisfies the rule $\delta(p) = 1 - p^{p-1} \in (\mathbb{Z} \setminus p\mathbb{Z}) \subseteq (R \setminus pR)$.

Moreover, if δ is a p -derivation on R , $\delta(p^t) \in (p^{t-1}\mathbb{Z} \setminus p^t\mathbb{Z}) \subseteq (p^{t-1}R \setminus p^tR)$. The base case $t = 1$ for induction is the previous claim. Then,

$$\delta(p^t) = \delta(p \cdot p^{t-1}) = p^p\delta(p^{t-1}) + p^{pt-p}\delta(p) + p\delta(p)\delta(p^{t-1}).$$

Applying the induction hypothesis, the p -adic orders of the terms are $t + p - 2$, $t(p - 1)$, and $t - 1$, so the p -adic order of the sum is $t - 1$.

Remark 2.8. Let R be a ring, and $I, J \subseteq R$ be ideals. Let δ be a p -derivation. If $a \in I$ and $b \in J$, then $\delta(a + b) \equiv \delta(a) + \delta(b)$ modulo IJ . In fact, we have that

$$C_p(a, b) = \frac{a^p + b^p - (a + b)^p}{p} = \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} a^i b^{p-i} \in IJ,$$

because $\binom{p}{i} \in \mathbb{N}$ for all $1 \leq i \leq p-1$. In particular, we have that $\mathcal{C}_p(a, b) \in (a) \cap (b)$. With similar considerations, one can show that if $a, b \in I$, then $\delta(a + b) \equiv \delta(a) + \delta(b)$ modulo I^p .

Note that having a p -derivation on S is equivalent to having a lift $\phi: S \rightarrow S$ of the Frobenius map $S/pS \rightarrow S/pS$. Indeed, it follows from the definition that if $\phi: S \rightarrow S$ is a map such that the induced map $\bar{\phi}: S/pS \rightarrow S/pS$ is the Frobenius map, then $\delta(x) = \frac{\phi(x) - x^p}{p}$ is a p -derivation. For example, if $R = \mathbb{Z}[x_1, \dots, x_n]$, then the map that sends a polynomial $f(x_1, \dots, x_n)$ to

$$\frac{f(x_1^p, \dots, x_n^p) - f(x_1, \dots, x_n)^p}{p}$$

is a p -derivation.

However, not every ring admits a p -derivation. See [Dup14], or consider the following example:

Example 2.9. Let $S = \mathbb{Z}_p[x_1, \dots, x_n]$, and $R = S/(p - f)$, where $f \in (x_1, \dots, x_n)^2$. Suppose that there is some p -derivation δ on R . Then, considering $p = f$ as elements of R , by Proposition 3.10 (2) below, $\delta(f) \in (x_1, \dots, x_n)R$. However, by Remark 2.7, $\delta(p) \in (\mathbb{Z} \setminus p\mathbb{Z})$, which yields a contradiction.

However, we do have the following:

Proposition 2.10. A ring R admits a p -derivation in each of the following cases:

- (1) $S = \mathbb{Z}$,
- (2) $S = W(C)$ is the Witt vectors over C for some perfect ring of positive characteristic,
- (3) S is a polynomial ring over a ring B that admits a p -derivation, or
- (4) S is p -adically complete and formally smooth over a ring B that admits a p -derivation.

Suppose also that δ is a p -derivation on S and Q is a prime of S containing p . Then there exists a p -derivation ϕ on $S' \supseteq S$ such that $\phi(s) = \delta(s)$ for all $s \in S$ when:

- (a) $S' = S_Q$, or
- (b) $S' = \widehat{S}_Q$.

In Case (a), we write $\phi = \delta_Q$ and in Case (b), we write $\phi = \widehat{\delta}_Q$.

Proof. As we have noted before, showing that a p -derivation exists is equivalent to proving that there exists a lift of Frobenius. Moreover, to verify that a p -derivation ϕ extends δ , it suffices to check that the associated lift of Frobenius extends the other.

- (1): The identity on \mathbb{Z} is a lift of Frobenius.
- (2): This is a well-known property of Witt vectors.
- (3): Extend a lift of Frobenius on A by sending each variable to its p -th power.
- (4): Let $\alpha_1: S \rightarrow S/pS$ be the composition of the quotient map with the Frobenius map on S/pS . Since S is formally smooth, this map lifts to $\alpha_2: S \rightarrow S/p^2S$. Applying formal smoothness inductively, one obtains a family of lifts of α_1 , $\alpha_i: S \rightarrow S/p^iS$. The map $\varprojlim \alpha_i: S \rightarrow \varprojlim S/p^iS \cong S$ is a lift of the Frobenius.

- (a): Given a lift of the Frobenius map on S , its extension to S_Q is a lift of the Frobenius on S_Q .
- (b): Given a lift Φ of the Frobenius on S , to check that it extends to \widehat{S}_Q , it suffices to check that $\Phi(Q) \subseteq Q$, since Φ is Q -adically continuous in this case. For $s \in Q$, $\Phi(s) \equiv s^p \pmod{pS}$, so $\Phi(s) \in Q$. \square

3. RESULTS

3.1. Primes not containing p . In this subsection, we focus on differential and symbolic powers of prime ideals that do not contain any prime integer. To study symbolic powers of such ideals, we use differential operators.

Definition 3.1. Let S be a ring, B be a subring of S , and I be an ideal of S . The n -th (B -linear) differential power of I is

$$I^{(n)}_B := \{f \in S \mid \partial(f) \in I \text{ for all } \partial \in D_{S|B}^{n-1}\}.$$

The following proposition is a generalization of [DDSG⁺17, Proposition 2.4].

Proposition 3.2. Let S be a ring, B be a subring of S , and \mathfrak{a} an ideal of S . The following properties hold:

- (1) $\mathfrak{a}^{(n)}_B$ is an ideal, and $\mathfrak{a}^{(n)}_B \subseteq \mathfrak{a}$.
- (2) $\mathfrak{a}^{(n+1)}_B \subseteq \mathfrak{a}^{(n)}_B$ for all n .
- (3) For any $0 \leq t \leq n$, and any $\partial \in D_{S|B}^t$, we have $\partial(\mathfrak{a}^n) \subseteq \mathfrak{a}^{n-t}$. In particular, $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)}_B$.
- (4) If Q is a prime ideal of S , and \mathfrak{a} is Q -primary, then $\mathfrak{a}^{(n)}_B$ is Q -primary.
- (5) If Q is a prime ideal of S , and \mathfrak{a} is Q -primary, then $\mathfrak{a}^{(n)} \subseteq \mathfrak{a}^{(n)}_B$.

Proof. The proof of (1) and (2) is analogous to that of [DDSG⁺17, Proposition 2.4], where the same claim is made for the case when B is a field.

For Part (3), we first proceed by induction on $t \geq 0$. If $t = 0$, then $\partial \in D_{S|B}^0$ is just multiplication by an element of S , and the statement is clear. If $t \geq 1$, we proceed by induction on $n - t \geq 0$. If $n = t$, then $\partial(\mathfrak{a}^n) \subseteq \mathfrak{a}^0 = S$ is trivial. By induction, assume that $\partial(\mathfrak{a}^{n-1}) \subseteq \mathfrak{a}^{n-1-t}$. To conclude the proof that $\partial(\mathfrak{a}^n) \subseteq \mathfrak{a}^{n-t}$ it suffices to show that $\partial(xy) \in \mathfrak{a}^{n-t}$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{a}^{n-1}$. To see this, observe that $\partial(xy) = [\partial, x](y) + x\partial(y)$, with $[\partial, x] \in D_{S|B}^{t-1}$. By the inductive hypothesis on t , we have that $[\partial, x](y) \in \mathfrak{a}^{n-1-(t-1)} = \mathfrak{a}^{n-t}$. Since $x \in \mathfrak{a}$ and $y \in \mathfrak{a}^{n-1}$, we also have $x\partial(y) \in \mathfrak{a}\mathfrak{a}^{n-1-t} = \mathfrak{a}^{n-t}$, and the claim follows. In particular, this shows that $\partial(\mathfrak{a}^n) \subseteq \mathfrak{a}$ for all ∂ of order up to $n - 1$, so that $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)}_B$.

Part (5) follows from (4), given that $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)}_B$, and that $\mathfrak{a}^{(n)}$ is the smallest Q -primary ideal that contains \mathfrak{a}^n .

To show (4), we first observe that $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)}_B \subseteq \mathfrak{a}$, so that $\sqrt{\mathfrak{a}^{(n)}_B} = Q$. To prove that $\mathfrak{a}^{(n)}_B$ is Q -primary, we proceed by induction on $n \geq 1$. The case $n = 1$ is true by assumption, since $\mathfrak{a}^{(1)}_B = \mathfrak{a}$. Let $xy \in \mathfrak{a}^{(n)}_B$, with $x \notin Q$. Observe that $xy \in \mathfrak{a}^{(n)}_B \subseteq \mathfrak{a}^{(n-1)}_B$, and by the inductive hypothesis the latter is Q -primary. Since $x \notin Q$, it follows that $y \in \mathfrak{a}^{(n-1)}_B$. Let $\partial \in D_{S|B}^{n-1}$, and note that $[\partial, x] \in D_{S|B}^{n-2}$, by definition. It follows that $[\partial, x](y) \in \mathfrak{a}$. On the other hand, we also have $\partial(xy) \in \mathfrak{a}$, and thus $x\partial(y) = \partial(xy) - [\partial, x](y) \in \mathfrak{a}$. Using again that $x \notin Q$, and that \mathfrak{a} is Q -primary, it follows that $\partial(y) \in \mathfrak{a}$. Since $\partial \in D_{S|B}^{n-1}$ was arbitrary, we conclude that $y \in \mathfrak{a}^{(n)}_B$, and thus $\mathfrak{a}^{(n)}_B$ is Q -primary. \square

We now recall some properties of differential powers with respect to localizations.

Lemma 3.3 ([BJNB17]). Let S be a ring, B be a subring of S , W be a multiplicatively closed subset of S , and I be an ideal of S . Suppose that S is essentially of finite type over B . Then

- (1) $(W^{-1}I)^{(n)}_B = (W^{-1}I)^{(n)}_{(W \cap B)^{-1}B}$,
- (2) $I^{(n)}_B = (W^{-1}I)^{(n)}_B \cap S$, and
- (3) $W^{-1}I^{(n)}_B = (W^{-1}I)^{(n)}_B$.

As noted in the introduction, the Zariski-Nagata Theorem can be stated in terms of differential powers of ideals [DDSG⁺17]. Namely, if $S = K[x_1, \dots, x_d]$, K is a perfect field, and Q is a prime ideal of S , then $Q^{(n)}_K = Q^{(n)}$ for all n . We can give a concise proof of the Zariski-Nagata Theorem for algebras over fields by combining the previous two results.

Theorem 3.4 (Zariski-Nagata). Let K be a field, R be essentially smooth over K , and Q be a prime ideal of R . If $K \hookrightarrow R_Q/QR_Q$ is separable, then $Q^{(n)}_K = Q^{(n)}$ for all n . In particular, if K is perfect, then $Q^{(n)}_K = Q^{(n)}$ for every prime Q and all n .

Proof. First, we note that $Q^{(n)} \subseteq Q^{(n)K}$ by Proposition 3.2 (5), so it suffices to prove $Q^{(n)K} \subseteq Q^{(n)}$. Since Q is the only associated prime of $Q^{(n)}$, it is sufficient to check the equality $Q^{(n)}R_Q = Q^{(n)K}R_Q$. By Lemma 3.3, $Q^{(n)K}R_Q = (QR_Q)^{(n)K}$, and since Q is maximal in R_Q , $Q^{(n)}R_Q = Q^nR_Q$. Thus, it suffices to show that $(QR_Q)^{(n)K} \subseteq Q^nR_Q$.

Let y_1, \dots, y_t be a minimal generating set for QR_Q . Suppose that there exists an element $f \in (QR_Q)^{(n)K}$ of order $s < n$, meaning that $f \in Q^sR_Q$, $f \notin Q^{s+1}R_Q$. Then, we can write $f = \sum_i u_i y^{\alpha_i} + g$ for some units $u_i \in R_Q$, some α_i with $|\alpha_i| = s$, and some $g \in Q^{s+1}R_Q$. Fix some multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| = s$ and some unit u such that uy^α appears in the expression of f as above. By Lemma 2.2, dy_1, \dots, dy_t form part of a free basis for $\Omega_{R|K}$. Thus, by Theorem 2.4 and Corollary 2.5, there exists a differential operator $D_\alpha \in D_{R_Q|K}^s$ such that $D_\alpha(uy^\alpha) \notin QR_Q$ and $D_\alpha(u_i y^{\alpha_i}) \in QR_Q$ for each other term $u_i y^{\alpha_i}$ in the expression for f . Additionally, since $g \in Q^{s+1}R_Q \subseteq (QR_Q)^{(s+1)K}$ and $g \in D_{R_Q|K}^s$, we have $D_\alpha(g) \in Q$, so $D_\alpha(f) \notin QR_Q$. This contradicts that $f \in (QR_Q)^{(n)K}$. \square

The following example shows that the conclusion of the Zariski-Nagata Theorem may fail if the field extension $K \rightarrow R_Q/QR_Q$ is not separable.

Example 3.5. Let $K = \mathbb{F}_p(t)$, $R = K[x]$, and $Q = (x^p - t)$. We claim that $Q^{(2)K} = Q$. By Lemma 2.4, $D_{R|K}^1 = R \oplus R \frac{d}{dx}$ (where $\frac{d}{dx} = D_1$ in the notation of Lemma 2.4), so it suffices to show that $\frac{d}{dx}(x^p - t) \in Q$. Indeed, $\frac{d}{dx}(x^p - t) = 0$, so the claim is verified. Since Q is a principal ideal, $Q^{(2)} = Q^2$. In particular, $Q^{(2)} \neq Q^{(2)K}$.

We are now ready to state our first main result: a version of the Zariski-Nagata Theorem for prime ideals that do not contain any prime integer.

Theorem 3.6. Let A be either \mathbb{Z} or a DVR of mixed characteristic. Let R be an essentially smooth A -algebra. If $Q \in \text{Spec}(R)$ is such that $Q \cap A = (0)$, then $Q^{(n)} = Q^{(n)A}$.

Proof. Let $A' = A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R' = R \otimes_A A' = R \otimes_{\mathbb{Z}} \mathbb{Q}$. We note that A' is a field of characteristic zero, and R' is formally smooth and essentially of finite type over A' . Observe that QR' is a prime ideal in R' . We claim that $(QR')^{(n)} \cap R = Q^{(n)}$. Indeed, $A \cap Q = (0)$, so $R_Q \cong R'_{QR'}$, and thus

$$Q^{(n)} = Q^n R_Q \cap R = ((QR')^n R'_{QR'} \cap R') \cap R = (QR')^{(n)} \cap R.$$

We note next that $(QR')^{(n)} = (QR')^{(n)A'}$; indeed, Theorem 3.4 applies. Finally, applying parts (1) and (2) of Lemma 3.3, we conclude that

$$Q^{(n)} = (QR')^{(n)} \cap R = (QR')^{(n)A'} \cap R = Q^{(n)A}. \quad \square$$

As an application of Theorem 3.6, we obtain a generalization of Zariski's Main Lemma on Holomorphic Functions [Zar49, EH79].

Corollary 3.7. Assume that R is smooth over A . Let $A' = A \otimes_{\mathbb{Z}} \mathbb{Q}$, and $R' = R \otimes_A A'$. For a prime $Q \subseteq R$ not containing any prime integer, set $\mathcal{B} = \{\mathfrak{n} \cap R \mid \mathfrak{n} \in \text{Max Spec}(R') \cap \mathcal{V}(QR')\}$. We have $Q^{(n)} = \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}^{(n)}$.

Proof. By Theorem 3.6, it suffices to show that $Q^{(n)A} = \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}^{(n)A}$. Since $Q \subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{B}$, it follows that $Q^{(n)A} \subseteq \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}^{(n)A}$. For the converse, we claim that $Q = \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}$. Let $J = \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}$, and let p be a prime integer. Note that $J \subseteq \bigcap_{\mathfrak{q} \in \mathcal{B}} (\mathfrak{q} + (p)) = Q + (p)$, where the equality follows from the fact that R/pR is a Jacobson ring, and that $\{\mathfrak{q} + (p) \mid \mathfrak{q} \in \mathcal{B}\} = \text{Max Spec}(R) \cap \mathcal{V}(Q + (p))$. Observe that p is a nonzerodivisor on R/J , because $p \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{B}$. Let $a \in J$. It follows from the containments proved above that we can write $a = x + py$, where $x \in Q$ and $y \in R$. Since $Q \subseteq J$, we conclude that $py = a - x \in J$, which implies that $y \in J$. Therefore, we have that

$J \subseteq Q + pJ \subseteq Q + \mathfrak{m}J \subseteq J$ for every maximal ideal \mathfrak{m} of R that contains p . Since the prime integer p was arbitrary, this argument shows that $J = Q + \mathfrak{m}J$ for every maximal ideal \mathfrak{m} of R . Nakayama's Lemma applied to the localization at each such ideal allows us to conclude that $J = Q$, as desired. Now let $f \in \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}^{(n)A}$, and $\delta \in D_{R|A}^{n-1}$ be arbitrary. By assumption, we have that $\delta(f) \in \bigcap_{\mathfrak{q} \in \mathcal{B}} \mathfrak{q}$. Finally, our previous claim shows that $\delta(f) \in Q$, and we conclude that $f \in Q^{(n)A}$. \square

3.2. Primes containing p . Throughout this subsection, p is a prime integer, and we assume that all rings involved are p -torsionfree; we note that this condition is implied by the hypotheses of Setting 3.18 below. We now move our attention to prime ideals that contain a given prime integer p . The following remark shows that, in order to characterize symbolic powers of such ideals, it is not sufficient to rely just on differential operators.

Remark 3.8. Let S be a ring of characteristic zero, $B \subseteq S$ a subring, and Q be a prime ideal of S such that $Q \cap \mathbb{Z} = (p) \neq (0)$. Then, since every B -linear differential operator ∂ on S satisfies $\partial(p) = p\partial(1) \in (p) \subseteq Q$, we have $p \in Q^{(n)B}$ for all n . Because $\bigcap_{n \in \mathbb{N}} Q^{(n)} = 0$, $Q^{(n)B} \neq Q^{(n)}$ for all but finitely many n .

For a simple concrete example, let $S = \mathbb{Z}$ and $Q = (2)$. Then $Q^n = (2^n)$ for all n , whereas $Q^{(n)\mathbb{Z}} = (2)$ for all n .

Remark 3.8 shows that differential powers are too large for our purpose. The issue with this approach is that differential operators cannot decrease the p -adic order of an element. On the other hand, p -derivations have this feature, and this motivates the following definition.

Definition 3.9. Let $p \in \mathbb{Z}$ be a prime, S be a ring with a p -derivation δ , and I be an ideal of S . The n -th p -differential power of I with respect to δ is

$$I^{(n)\delta} := \{f \in R \mid \delta^a(f) \in I \text{ for all } a \leq n-1\}.$$

It immediately follows from the definition that $I^{(n+1)\delta} \subseteq I^{(n)\delta}$ for all n .

Proposition 3.10. Let $p \in \mathbb{Z}$ be a prime, S be a ring with a p -derivation δ , and I be an ideal of S . If I is an ideal, then $I^{(n)\delta}$ is an ideal for all n . Moreover, if $Q \in \text{Spec}(S)$ is a prime ideal, and \mathfrak{a} is a Q -primary ideal that contains p , we have the following properties:

- (1) For any $s, t \in \mathbb{N}$ we have $\mathfrak{a}^{(s)\delta} \mathfrak{a}^{(t)\delta} \subseteq \mathfrak{a}^{(s+t)\delta}$.
- (2) $\delta(\mathfrak{a}^n) \subseteq \mathfrak{a}^{n-1}$ for all n . In particular, $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)\delta}$.
- (3) $\mathfrak{a}^{(n)\delta}$ is a Q -primary ideal.
- (4) $\mathfrak{a}^{(n)} \subseteq \mathfrak{a}^{(n)\delta}$.

Proof. We first show by induction on n that $I^{(n)\delta}$ is an ideal. For $n = 1$ we have $I^{(1)\delta} = I$, and the statement is trivial. Assume that $I^{(n-1)\delta}$ is an ideal, and let $x, y \in I^{(n)\delta}$. By induction, we have that $\delta(x) + \delta(y) \in I^{(n-1)\delta}$, since $\delta(x), \delta(y) \in I^{(n-1)\delta}$. It follows that $\delta(x+y) = \delta(x) + \delta(y) + \mathcal{C}_p(x, y) \in I^{(n-1)\delta}$, because $\mathcal{C}_p(x, y) \in (x) \subseteq I^{(n)\delta} \subseteq I^{(n-1)\delta}$ by Remark 2.8. This shows that $x+y \in I^{(n)\delta}$. Now let $x \in I^{(n)\delta}$ and $b \in S$. By induction, we may assume that $yc \in I^{(n-1)\delta}$ for all $y \in I^{(n-1)\delta}$ and all $c \in S$. Since $\delta(x) \in I^{(n-1)\delta}$, it follows that $\delta(xb) = x^p\delta(b) + b^p\delta(x) + p\delta(b)\delta(x) \in I^{(n-1)\delta}$, because $x \in I^{(n)\delta} \subseteq I^{(n-1)\delta}$. This finishes the proof that $I^{(n)\delta}$ is an ideal.

Now we let \mathfrak{a} be a Q -primary ideal of S that contains p .

(1) We proceed by induction on $s+t \geq 0$. If either $s = 0$, or $t = 0$, then the claim is trivial. This takes care of the base case for the induction, and allows us to assume henceforth that $s \geq 1$ and $t \geq 1$. Let $x \in \mathfrak{a}^{(s)\delta}$ and $y \in \mathfrak{a}^{(t)\delta}$. We observe that

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y) \in \mathfrak{a}^{(s)\delta} \mathfrak{a}^{(t-1)\delta} + \mathfrak{a}^{(t)\delta} \mathfrak{a}^{(s-1)\delta} + \mathfrak{a} \cdot \mathfrak{a}^{(s-1)\delta} \mathfrak{a}^{(t-1)\delta},$$

and by induction we obtain that $\delta(xy) \in \mathfrak{a}^{(s+t-1)\delta}$. This shows that $xy \in \mathfrak{a}^{(s+t)\delta}$, as claimed.

(2) We proceed by induction on $n \geq 1$. For $n = 1$ the statements are clear. Assume that $\delta(\mathfrak{a}^n) \subseteq \mathfrak{a}^{n-1}$ holds true; we want to show that \mathfrak{a}^{n+1} is mapped inside \mathfrak{a}^n by δ . We first show that any element of the form xy , with $x \in \mathfrak{a}$ and $y \in \mathfrak{a}^n$ satisfies $\delta(xy) \in \mathfrak{a}^n$. Using the inductive hypothesis, we get

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y) \in \mathfrak{a}^p \mathfrak{a}^{n-1} + \mathfrak{a}^{pn} + p \cdot \mathfrak{a}^{n-1}.$$

Given that $p \in \mathfrak{a}$, we conclude that $\delta(xy) \in \mathfrak{a}^n$, as desired. Because every element of \mathfrak{a}^{n+1} can be written as a sum of elements xy , with $x \in \mathfrak{a}$ and $y \in \mathfrak{a}^n$, it suffices to show that the sum of any two such elements is mapped by δ inside \mathfrak{a}^n . Let $z, w \in \mathfrak{a}^{n+1}$ be elements of this form, and recall that $\delta(z+w) = \delta(z) + \delta(w) + \mathcal{C}_p(z, w)$. Since $\mathcal{C}_p(z, w) \in (z) \subseteq \mathfrak{a}^{n+1}$, and $\delta(z), \delta(w) \in \mathfrak{a}^n$ by what we have shown above, we conclude that $\delta(z+w) \in \mathfrak{a}^n$. The final claim now follows immediately, since $\delta(\mathfrak{a}^{n+1}) \subseteq \mathfrak{a}^n \subseteq \mathfrak{a}^{(n)\delta}$ by induction, so that $\mathfrak{a}^{n+1} \subseteq \mathfrak{a}^{(n+1)\delta}$.

(3) Since $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)\delta} \subseteq \mathfrak{a}$, it follows immediately that $\sqrt{\mathfrak{a}^{(n)\delta}} = Q$ for all n . To show that $\mathfrak{a}^{(n)\delta}$ is indeed primary, we proceed by induction on n , the base case $\mathfrak{a}^{(1)\delta} = \mathfrak{a}$ being trivial. Let $n \geq 2$, and assume that $xy \in \mathfrak{a}^{(n)\delta}$, but $x \notin Q$. Since $\mathfrak{a}^{(n)\delta} \subseteq \mathfrak{a}^{(n-1)\delta}$, and $\mathfrak{a}^{(n-1)\delta}$ is Q -primary by induction, we necessarily have that $y \in \mathfrak{a}^{(n-1)\delta}$. By assumption, we have that $\delta(xy) \in \mathfrak{a}^{(n-1)\delta}$, therefore

$$(x^p + p\delta(x))\delta(y) = \delta(xy) - y^p \delta(x) \in \mathfrak{a}^{(n-1)\delta}.$$

Because $p \in Q$, we have that $x^p + p\delta(x) \notin Q$, otherwise $x \in Q$. Since $\mathfrak{a}^{(n-1)\delta}$ is Q -primary, we conclude that $\delta(y) \in \mathfrak{a}^{(n-1)\delta}$, that is, $y \in \mathfrak{a}^{(n)\delta}$.

(4) follows from (2) and (3), since $\mathfrak{a}^{(n)}$ is the smallest Q -primary ideal that contains \mathfrak{a}^n . \square

Remark 3.11. Let \mathfrak{a} be a Q -primary ideal, with $p \in Q$. Note that $\mathfrak{a}^n \subseteq \mathfrak{a}^{(n)\delta}$ is not true, in general, if we do not assume that \mathfrak{a} itself contains p .

Example 3.12. Consider the ideal $\mathfrak{a} = (4, x+2, y+2)$ inside the polynomial ring $R = \mathbb{Z}[x, y]$. Then \mathfrak{a} is Q -primary, with $Q = (2, x, y)$. However, we claim that there is a p -derivation δ for which $\mathfrak{a}^2 \not\subseteq \mathfrak{a}^{(2)\delta}$. Set $f = x+2$ and $g = y+2$, and let ϕ be the ring homomorphism

$$\begin{aligned} \phi : \quad \mathbb{Z}[x, y] &\longrightarrow \mathbb{Z}[x, y] \\ h(x, y) &\longmapsto h(x^2, y^2). \end{aligned}$$

Consider the associated p -derivation $\delta \in \text{Der}_p(R)$, defined as $\delta(h) = \frac{\phi(h) - h^p}{p}$. Note that $\delta(2) = -1$, while $\delta(x) = \delta(y) = 0$. Then we have

$$\delta(fg) = f^2 \delta(g) + g^2 \delta(f) + 2\delta(f)\delta(g).$$

Since $f^2, g^2 \in \mathfrak{a}$, we have that $\delta(fg) \in \mathfrak{a}$ if and only if $\delta(f)\delta(g) \in \mathfrak{a} :_R (2) = Q$. We have

$$\delta(f) = \delta(x+2) = \delta(x) + \delta(2) + \frac{x^2 + 2^2 - (x+2)^2}{2} = -1 + \frac{-4x}{2} = -1 - 2x$$

and similarly $\delta(g) = \delta(y+2) = -1 - 2y$. It follows that $\delta(f)\delta(g) \notin Q$. This shows that $\delta(fg) \notin \mathfrak{a}$, and hence $fg \in \mathfrak{a}^2 \setminus \mathfrak{a}^{(2)\delta}$.

Lemma 3.13. Let $p \in \mathbb{Z}$ be a prime, S be a ring with a p -derivation δ , and $Q \in \text{Spec}(S)$ be a prime ideal containing p . Let $\psi = \delta_Q$, in the notation of Proposition 2.10. Then

$$Q^{(n)\delta} S_Q = (QS_Q)^{(n)\psi}.$$

Proof. It suffices to verify that $\frac{r}{1} \in Q^{(n)\delta} R_Q$ if and only if $\frac{r}{1} \in (QR_Q)^{(n)\psi}$, which is immediate from the fact that $\psi^a(\frac{r}{1}) = \frac{\delta^a(r)}{1}$ for all $a \geq 0$. \square

We now make a key definition of this article. We combine the action of differential operators and p -derivations in order to control symbolic powers of prime ideals that contain a given prime integer. The definition we write is very general, but we will later focus on a more restrictive setting.

Definition 3.14. Let δ be a p -derivation on S . Let I be an ideal of S . The n -th mixed differential power of I with respect to δ is

$$\begin{aligned} I^{\langle n \rangle_{\text{mix}}} &:= \bigcap_{a+b \leq n+1} (I^{\langle a \rangle_{\delta}})^{\langle b \rangle_B} \\ &= \{f \in S \mid (\delta^s \circ \partial)(f) \in I \text{ for all } \partial \in D_{S|B}^t \text{ with } s+t \leq n-1\}. \end{aligned}$$

Note that given $a+b \leq n+1$, computing $I^{\langle a \rangle_{\delta}}$ involves applying δ^s with $s \leq a-1$, while taking $J^{\langle b \rangle_B}$ requires taking differential operators $\partial \in D_{S|B}^{b-1}$, so that overall, to compute $I^{\langle n \rangle_{\text{mix}}}$ we need to use combinations of differentials and p -derivations of order up to $(a-1) + (b-1) \leq n-1$.

A word of caution about the order of operations: an element $f \in (I^{\langle a \rangle_{\delta}})^{\langle b \rangle_B}$ is one such that $\partial(f) \in I^{\langle a \rangle_{\delta}}$ for all $\partial \in D_{S|B}^{b-1}$, so that $\delta^s \circ \partial(f) \in I$ for all $s \leq a-1$.

Remark 3.15. Note that the definition of $I^{\langle n \rangle_{\text{mix}}}$ depends, in principle, on both δ and B . However, we will show in Corollary 3.21 that this definition is actually independent of the choice of δ for prime ideals in our main setting.

Proposition 3.16. If I is an ideal, then $I^{\langle n \rangle_{\text{mix}}}$ is an ideal. Moreover, if $Q \in \text{Spec}(S)$ is a prime ideal that contains p , we have the following properties:

- (1) $Q^{\langle n \rangle_{\text{mix}}} \subseteq Q^{\langle m \rangle_{\text{mix}}}$ if $n \geq m$.
- (2) $Q^n \subseteq Q^{\langle n \rangle_{\text{mix}}}$ for all n .
- (3) $Q^{\langle n \rangle_{\text{mix}}}$ is a Q -primary ideal.
- (4) $Q^{\langle n \rangle} \subseteq Q^{\langle n \rangle_{\text{mix}}}$.

Proof. First of all, we show that $I^{\langle n \rangle_{\text{mix}}}$ is an ideal. Let a, b be integers such that $a+b \leq n+1$. Then $I^{\langle a \rangle_{\delta}}$ is an ideal by Proposition 3.10. As a consequence, we have that $(I^{\langle a \rangle_{\delta}})^{\langle b \rangle_B}$ is an ideal by Proposition 3.2 (1). Since $I^{\langle n \rangle_{\text{mix}}}$ is then defined as an intersection of ideals, the claim follows.

(1) Immediate from the definition.

(2) From Proposition 3.2 (3), we deduce that $\partial(Q^n) \subseteq Q^{n-t}$ for all $t \leq n-1$, and $\partial \in D_{S|B}^t$.

Since $Q^{n-t} \subseteq Q^{\langle n-t \rangle_{\delta}}$, it follows that $Q^n \subseteq (Q^{\langle n-t \rangle_{\delta}})^{\langle t+1 \rangle_B}$. Because this holds for all $t \leq n-1$, we conclude that $Q^n \subseteq Q^{\langle n \rangle_{\text{mix}}}$.

Finally, parts (3) and (4) follow from the corresponding properties for differential powers and p -derivation powers. In fact, if a and $b \in \mathbb{N}$ are integers such that $a+b \leq n+1$, then observe that $Q^{\langle a \rangle_{\delta}}$ is a Q -primary ideal by Proposition 3.10 (3). Therefore, the ideal $(Q^{\langle a \rangle_{\delta}})^{\langle b \rangle_B}$ is Q -primary by Proposition 3.2 (4). It then follows that $Q^{\langle n \rangle_{\text{mix}}} = \bigcap_{a+b \leq n+1} (Q^{\langle a \rangle_{\delta}})^{\langle b \rangle_B}$ is Q -primary, since it is a finite intersection of Q -primary ideals. Because the n -th symbolic power $Q^{\langle n \rangle}$ is the smallest Q -primary ideal that contains Q^n , this concludes the proof. \square

Lemma 3.17. Let δ be a p -derivation on R . Let $Q \in \text{Spec}(R)$ be a prime containing p . Then

$$Q^{\langle n \rangle_{\text{mix}}} R_Q = (QR_Q)^{\langle n \rangle_{\text{mix}}},$$

where $(QR_Q)^{\langle n \rangle_{\text{mix}}}$ is the mixed differential power with respect to δ_Q .

Proof. This is a direct consequence of Lemmas 3.3 and 3.13. \square

Setting 3.18. Let p be a prime. Let $A = \mathbb{Z}$ or a DVR with uniformizer p . Let R be an essentially smooth A -algebra that has a p -derivation δ . Let Q be a prime ideal of R that contains p , and assume that the field extension $A/pA \hookrightarrow R_Q/QR_Q$ is separable.

We note that polynomial rings over \mathbb{Z} , over Witt vectors of perfect fields, or over complete unramified DVRs of mixed characteristic and with perfect residue field satisfy Setting 3.18, by Proposition 2.10.

Proposition 3.19. In the context of Setting 3.18, suppose further that (R, \mathfrak{m}) is local. Then $\mathfrak{m}^{\langle n \rangle}_{\text{mix}} = \mathfrak{m}^n$.

Proof. If p, y_1, \dots, y_d is a minimal generating set for the maximal ideal \mathfrak{m} , then, by Lemma 2.2, dy_1, \dots, dy_d is part of a free basis for $\Omega_{R|A}$. It follows from Theorem 2.4 that, for every $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, there exists a differential operator $D_\alpha \in D_{R|A}^{|\alpha|}$ such that $D_\alpha(y^\beta) = \binom{\beta}{\alpha} y^{\beta-\alpha}$ for all $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ such that $\beta_i \geq \alpha_i$ for all i . Furthermore, we have $D_\alpha(y^\beta) = 0$ for all β for which $\beta_i < \alpha_i$ for some i .

The containment $\mathfrak{m}^n \subseteq \mathfrak{m}^{\langle n \rangle}_{\text{mix}}$ always holds. For the converse, assume that there exists $f \in \mathfrak{m}^{\langle n \rangle}_{\text{mix}}$ of order $s < n$, meaning that $f \in \mathfrak{m}^s$, $f \notin \mathfrak{m}^{s+1}$. Then, we can write $f = \sum_i u_i p^{s-|\alpha_i|} y^{\alpha_i} + g$ for some units $u_i \in R$, some α_i with $|\alpha_i| \leq s$, and some $g \in \mathfrak{m}^{s+1}$. Fix some multi-index $\alpha \in \mathbb{N}^d$ and a unit u such that $u p^{s-|\alpha|} y^\alpha$ appears in the expression of f as above, with $|\alpha|$ maximal as such. Observe that multiplying by a unit does not affect whether or not f belongs to the ideals $\mathfrak{m}^{\langle n \rangle}_{\text{mix}}$ and \mathfrak{m}^n . Therefore, after multiplying by u^{-1} , we may assume that $p^{s-|\alpha|} y^\alpha$ appears in the support of f , with $|\alpha|$ maximal. Consider the corresponding differential operator $D_\alpha \in D_{R|A}^s$. Let $t = s - |\alpha|$ and observe that $D_\alpha(p^t y^\alpha) = p^t D_\alpha(y^\alpha) = p^t$. Recall that $p^t \notin \mathfrak{m}^{\langle t+1 \rangle}_\delta$ by Remark 2.7. For the remaining $\alpha_i \neq \alpha$ with $|\alpha_i| = |\alpha|$, we have

$$D_\alpha(u_i p^t y^{\alpha_i}) = p^t D_\alpha(u_i y^{\alpha_i}) \in p^t (y_1, \dots, y_d) \subseteq \mathfrak{m}^{t+1} \subseteq \mathfrak{m}^{\langle t+1 \rangle}_\delta,$$

by Corollary 2.5 and Proposition 3.10 (2). For α_i such that $|\alpha_i| < |\alpha|$, we must have $s - |\alpha_i| \geq t + 1$, so that

$$D_\alpha(u_i p^{s-|\alpha_i|} y^{\alpha_i}) = p^{s-|\alpha_i|} D_\alpha(u_i y^{\alpha_i}) \in (p^{t+1}) \subseteq \mathfrak{m}^{\langle t+1 \rangle}_\delta.$$

Finally, note that $D_\alpha(g) \in \mathfrak{m}^{t+1}$ by Proposition 3.2 (3), and thus $D_\alpha(g) \in \mathfrak{m}^{\langle t+1 \rangle}_\delta$ as well. Combining these facts together, and using that $\mathfrak{m}^{\langle t+1 \rangle}_\delta$ is an ideal, we obtain

$$D_\alpha(f) = p^t + D_\alpha \left(\sum_{\substack{\alpha_i \neq \alpha \\ |\alpha_i| = |\alpha|}} u_i p^t y^{\alpha_i} \right) + D_\alpha \left(\sum_{|\alpha_i| < |\alpha|} u_i p^{s-|\alpha_i|} y^{\alpha_i} \right) + D_\alpha(g) \notin \mathfrak{m}^{\langle t+1 \rangle}_\delta.$$

Thus, $f \notin (\mathfrak{m}^{\langle t+1 \rangle}_\delta)^{\langle |\alpha|+1 \rangle}_A$, so that $f \notin \mathfrak{m}^{\langle s+1 \rangle}_{\text{mix}}$. This contradicts the assumption that $f \in \mathfrak{m}^{\langle n \rangle}_{\text{mix}} \subseteq \mathfrak{m}^{\langle s+1 \rangle}_{\text{mix}}$. \square

Theorem 3.20. In the context of Setting 3.18, one has the equality $Q^{\langle n \rangle} = Q^{\langle n \rangle}_{\text{mix}}$.

Proof. Since both $Q^{\langle n \rangle}$ and $Q^{\langle n \rangle}_{\text{mix}}$ are Q -primary, it is enough to show equality locally at Q . After localizing, we have that $Q^{\langle n \rangle} R_Q \cong Q^n R_Q$. Moreover, Lemma 3.17 implies that

$$Q^{\langle n \rangle}_{\text{mix}} R_Q \cong (Q R_Q)^{\langle n \rangle}_{\text{mix}} = (Q R_Q)^n$$

by Proposition 3.19 applied to the local ring $(R_Q, Q R_Q)$. \square

Recall that the definition of mixed differential powers depends, a priori, on the chosen p -derivation. However, Theorem 3.20 immediately gives that this is not the case in the context of Setting 3.18. We record this fact in the following corollary.

Corollary 3.21. In the context of Setting 3.18, the ideal $Q^{\langle n \rangle}_{\text{mix}}$ is independent of the choice of p -derivation δ . Moreover,

$$Q^{\langle n \rangle} = Q^{\langle n \rangle}_{\text{mix}} = \{f \in R \mid (\delta \circ \partial)(f) \in I \text{ for all } \delta \in \text{Der}_p^a(R), \partial \in D_{R|A}^b \text{ with } a + b \leq n - 1\}.$$

Proof. Write $Q^{\langle n \rangle}_*$ for the ideal on the right-hand side. Set

$$Q^{\langle n \rangle}_p = \{f \in R \mid \text{Der}_p^a(f) \subseteq Q \text{ for all } a \leq n - 1\}.$$

We observe that the conclusions of Proposition 3.10 hold with $Q^{(n)\delta}$ replaced by $Q^{(n)p}$; indeed, essentially the same proofs work for these ideals. Now, $Q^{(n)\star} = \bigcap_{a+b \leq n+1} (Q^{(n)p})^{(a)}_A$. Then, the same proof as Proposition 3.16 shows that $Q^{(n)} \subseteq Q^{(n)\star}$. It is clear from the definition that $Q^{(n)\star} \subseteq Q^{(n)\text{mix}}$. Then, by the previous Theorem, we have that $Q^{(n)\star} = Q^{(n)\text{mix}} = Q^{(n)}$. \square

Corollary 3.22. In the context of Setting 3.18, suppose further that R is smooth over A . Then

$$Q^{(n)} = \bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m}^n,$$

for all $n \geq 1$, where $\mathcal{B} = \text{Max Spec}(R) \cap \mathcal{V}(Q)$.

Proof. By Theorem 3.20, it is sufficient to show that $Q^{(n)\text{mix}} = \bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m}^n$. Since R is smooth over A by assumption, we have that R/pR is an algebra of finite type over the field A/pA . In particular, R/pR is a Hilbert-Jacobson ring, and thus $Q = \bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m}$. Observe that $Q^{(n)\text{mix}} \subseteq \mathfrak{m}^{(n)\text{mix}}$ for all $\mathfrak{m} \in \mathcal{B}$ is clear, since $Q \subseteq \mathfrak{m}$. To prove the converse, let $f \in \bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m}^n$, so that $f \in \mathfrak{m}^{(n)\text{mix}}$ for all $\mathfrak{m} \in \mathcal{B}$, by Proposition 3.19. Therefore $(\delta^a \circ \partial)(f) \in \bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m} = Q$ for any given p -derivation δ , and any differential operator $\partial \in D_{R/A}^{n-1-a}$. It follows that $f \in Q^{(n)\text{mix}}$, as desired. \square

The following example shows that the separability hypothesis in Theorem 3.20 is necessary. Given that a similar hypothesis is required for the Zariski-Nagata Theorem in equicharacteristic $p > 0$, this is to be expected.

Example 3.23. Let $A = \mathbb{Z}[t]_{(p)}$, $R = \mathbb{Z}[t, x]_Q$, where $Q = (p, x^p - t)$. We note that these satisfy all of the conditions of Setting 3.18, except that $A/pA \hookrightarrow R_Q/QR_Q$ is inseparable. In this example, $Q^{(2)} = Q^2$. However, we will show that there exists a p -derivation on R such that $Q^{(2)\text{mix}} \neq Q^{(2)}$.

Set $w = x^p - t$. We can write $\mathbb{Z}[t, x] = \mathbb{Z}[w, x]$. Then, the map

$$\phi(f(w, x)) = \frac{f(w^p, x^p) - f(w, x)^p}{p}$$

is a p -derivation such that $\phi(w) = 0$. Since R is a localization of $\mathbb{Z}[t, x]$, we have by Proposition 2.10 (a) that there is a p -derivation δ on R that extends ϕ . In particular, $\delta(w) = 0 \in Q$. Also, $\frac{d}{dx}(w) = px^{p-1} \in Q$. Thus, $w \in Q^{(2)\setminus} Q^{(2)}$.

ACKNOWLEDGMENTS

We would like to thank Alexandru Buium for a helpful correspondence, and Bhargav Bhatt for useful conversations and suggestions. The third author was partially supported by NSF Grant DMS #1606353.

REFERENCES

- [And16] Yves André. La conjecture du facteur direct. *arXiv:1609.00345*, 2016. 1
- [Bha16] Bhargav Bhatt. On the direct summand conjecture and its derived variant. *arXiv:1608.08882*, 2016. 1
- [BJNB17] Holger Brenner, Jack Jeffries, and Luis Núñez-Betancourt. Differential signature: Quantifying singularities with differential operators. *arXiv*, 2017. 6
- [Bor09] James Borger. Lambda-rings and the field with one element. *arXiv preprint arXiv:0906.3146*, 2009. 1
- [Bui95] Alexandru Buium. Differential characters of abelian varieties over p -adic fields. *Inventiones mathematicae*, 122(2):309–340, 1995. 1, 4
- [Bui05] Alexandru Buium. *Arithmetic differential equations*, volume 118 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. 1, 4
- [Bui15] Alexandru Buium. Differential calculus with integers. In *Arithmetic and geometry*, volume 420 of *London Math. Soc. Lecture Note Ser.*, pages 139–187. Cambridge Univ. Press, Cambridge, 2015. 1
- [DDSG⁺17] Hailong Dao, Alessandro De Stefani, Eloisa Grifo, Craig Huneke, and Luis Núñez-Betancourt. Symbolic powers of ideals. 2017. 1, 6
- [Dup14] Taylor Dupuy. Positivity and lifts of the Frobenius. *Mathematical Research Letters*, 21(2), 2014. 5

- [EH79] David Eisenbud and Melvin Hochster. A Nullstellensatz with nilpotents and Zariski's main lemma on holomorphic functions. *J. Algebra*, 58(1):157–161, 1979. [1](#), [7](#)
- [ELS01] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Invent. Math.*, 144(2):241–252, 2001. [1](#)
- [Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967. [3](#), [4](#)
- [HH02] Melvin Hochster and Craig Huneke. Comparison of symbolic and ordinary powers of ideals. *Invent. Math.*, 147(2):349–369, 2002. [1](#)
- [Hoc] Mel Hochster. Math 615 lecture notes, winter, 2010. [3](#)
- [Joy85] A. Joyal. δ -anneaux et vecteurs de Witt. *C.R. Acad. Sci. Canada*, VII(3):177–182, 1985. [1](#), [4](#)
- [Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid. [3](#)
- [MS17] Linquan Ma and Karl Schwede. Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers. *arXiv:1705.02300*, 2017. [1](#)
- [Nag62] Masayoshi Nagata. *Local rings*. Interscience, 1962. [1](#)
- [Zar49] Oscar Zariski. A fundamental lemma from the theory of holomorphic functions on an algebraic variety. *Ann. Mat. Pura Appl. (4)*, 29:187–198, 1949. [1](#), [7](#)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588-0130, USA
E-mail address: adestefani2@unl.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904-4135, USA
E-mail address: eloisa.grifo@virginia.edu

UNIVERSITY OF MICHIGAN, DEPARTMENT OF MATHEMATICS, USA
E-mail address: jackjeff@umich.edu