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Regularity of Powers and Products of Ideals

Author:
Sepehr Jafari

Adviser:
Prof. Aldo Conca

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Abstract

The aim of this thesis is to study the Castelnuovo-Mumford regularity of powers and products of ideals of polynomial rings. In general, it is known that the regularity of large enough powers of ideals are given by a linear function (see [28] and [45]). Moreover, similar result holds for products of large enough powers of ideals which are generated in the same degrees (see [5] and [15]). These are fascinating results, however, there are still missing mysteries on specifics of the asymptotic linear functions. For example, the stabilizing index (the smallest power for which the regularity is a linear function) and the so called constant (the constant term in the linear function) is unknown in general. We compute these two missing parts for two deferent classes of ideals using two deferent methods.
In the first case, we study the regularity of the products of the family of determinantal ideals of Hankel matrices namely $\mathcal{F}$. The detailed construction of this family can be collected in chapter 3 and [41]. Our main result is the following theorem extending [26] and [30].

Theorem (Theorem 3.2.16)**.** *Let $I_1, \dots, I_r$ be ideals in the family $\mathcal{F}$. The following holds:*

(i) Every product $I_1 \dots I_r$ has linear resolution.

(ii) Computing the initial ideal commutes over products, in particular the natural generators form a Gröbner basis.

(iii) The multi-Rees algebra $\mathscr{R}(I_1, \dots, I_r)$ is defined by a quadratic Gröbner basis with respect to some term order, it is Koszul, normal, Cohen-Macaulay domain. Moreover, the natural algebra generators form a Sagbi basis.

The second case of study of this thesis is the study of regularity of powers of some family of edge ideals of graphs. Let $C_n \cdot P_l \cdot C_m$ be a dumbbell graph with $m, n \geq 3$ where C_n and C_m are n-cycles and m-cycles connected via path P_l. Let a bicyclic graphs be a graph constructed by adding trees to a given dumbbell graph. The chapter 4 of this thesis is aimed at the study of edge ideals of this family. We compute the induced matching number of dumbbell graphs with respect to n, m and l. The induced matching number is a closely related combinatorial invariant of a graph to the regularity of its edge ideal. Moreover, for the family of dumbbell graphs, we compute the regularity of the first power and when $l \leq 2$, we formulate the regularity of given powers in terms of the regularity of first power. The detailed argument of this result can be found in chapter 4. The main results of this study are the following.

Theorem (Theorem 4.1.6)**.** *Let $m, n \geq 3$ and $l \geq 1$. Let $\nu(C_n \cdot P_l \cdot C_m)$ denote the induced matching number of the dumbbell graph $C_n \cdot P_l \cdot C_m$, then*

(i) if $l \equiv 0, 1 \pmod 3$, then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) if $l \equiv 2 \pmod 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Theorem (Theorem 4.2.11)**.** *For the dumbbell graph* $C_n \cdot P_l \cdot C_m$ *with* $l \leq 2$*, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geq 1$*.*

The above results proves a popular conjecture given in [6, Conjecture 7.11] for this family of graphs. Moreover, for the family of bicyclic graphs, we explicitly compute the regularity of the first power in terms of its induced matching number.

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Contents

Chapter 1

Introduction

The problem of the study of the Castelnuovo-Mumford regularity of powers and products of ideals has been attractive for several researchers for past few decades. A well-celebrated result in the behavior of regularity of powers of ideals is given in [28] and independently in [45], that is: Let I be an ideal of polynomial ring S. There exists d, n_0 and b in $\mathbb{N}$ such that $\operatorname{reg}(I^q) = dq + b$ for all $q \geq q_0$. The term d is well-understood. For example, d is equal to the degree of the generators of I when I is generated in a single degree d. However, q_0 (known as the stabilizing index) and b (known as the constant) remain unknown in general. Although it is in general difficult to determine q_0 and b, there exists a vast literature computing or bounding them for various families of ideals (see [33], [22], [53] and [34]).
As the next step, the authors in [5] and [15] showed that the regularity of every product $I_1 \dots I_r$ of ideals of S is asymptotically given by supremum of some linear functions. Note that duplicates are allowed in $I_1 \dots I_r$. In particular, when $I_1, \dots, I_r$ are generated in single degrees $d_1, \dots, d_r$, then there exists a linear function $L(u_1, \dots, u_r) = \sum_{i=1}^{r} d_i u_i + b$ such that $\operatorname{reg}(I_1^{a_1} \dots I_r^{a_r}) = L(a_1, \dots, a_r)$ for all $(a_1, \dots, a_r) \geq (a_1^{(0)}, \dots, a_r^{(0)})$.
This thesis aims at two families of ideals to investigate their regularity of powers and products. The first case which we would like to call the determinantal ideals of "close cuts of Hankel matrices" is motivated by [30], [9], [14]. We encourage the reader to see chapter 3 for detailed construction of "close cuts of Hankel matrices". One standard method to study determinantal ideals is to approach their initial ideals via Gröbner bases. In [55] and [40] the authors described the Gröbner of determinantal ideals of generic matrices. We recall that a generic matrix is a matrix whose entries are pairwise distinct indeterminates over some field $\mathbb{K}$ and the number of its rows are smaller than or equal to the number of its columns. We say a family of ideals, say $\mathcal{F}$, of polynomial ring S has linear powers if for every $I \in \mathcal{F}$ and every $q \geq 1$, I^q has linear resolution. In [16] it was proved that the determinantal ideals of maximal minors of a generic matrix has linear powers. With the perspective of the study of regularity of powers of ideals, while some family $\mathcal{F}$ has linear powers, we have $q_0 = 1$ (the so called stabilizing index) and $b = 0$ (the so called constant) for all $I \in \mathcal{F}$. In the recent work [51], the determinantal ideals of a generic matrix with asymptotic linear powers is classified.

In [9], the authors improved the result in [16] to the study of behavior of the products of determinantal ideals of maximal minors of particular sub-matrices of a given generic matrix. The construction is to fix a generic matrix X_t with t rows. Then consider the family of sub-matrices of X_r with $r \leq t$. The authors proved the family of determinantal ideals of maximal minors of these cuts has linear products. Recall that a family of ideals, say $\mathcal{F}$, has linear products if for the ideals $I_1, \dots, I_r \in \mathcal{F}$, every product has linear resolution. The second and third author of [9] improved their work in [14]. They showed that for a fixed generic matrix X, the family of north-east sub-matrices (defined just as it is called), forms a family of ideals with linear products. On the other hand, in [26], the authors introduced another family of determinantal ideals with linear products. Let X_t be a Hankel matrix with t rows and entries $x_1, \dots, x_n$ the indeterminates of the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. It is known that the determinantal ideals of a Hankel matrix only depends the size of the minors and the entries. The authors proved that the determinantal ideals of the family of matrices X_t where $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$, form a family of determinantal ideals with linear product. Later on, the same result was given in [30] with a different method for the extended Hankel matrices. In all three works [9], [14] and [30] the authors described the Gröbner bases and a nice primary decomposition of the products that is given by determinantal ideals. Moreover, they showed in their cases that the associated multi-Rees algebras are Koszul, Cohen-Macaulay normal domain, and its defining ideal has a Gröbner bases of degree two. In chapter 3, we introduce a new family of determinantal ideals with linear products that is that is constructed from Hankel matrices. Inspired by [9] and [14], we tried to see whether similar "north-east" pattern exists for a given Hankel matrix? It is interesting that we expect to construct a family with linear products for any given sub-Hankel matrix of a given family of Hankel matrices, however, only for the case of "close cuts of Hankel matrices" the standard methodology works (See Remark 3.2.17). For this particular case of ideals, we describe the Gröbner bases of the products and we study the associated multi-Rees algebra. In particular, we show that the associated multi-Rees algebra is Koszul, Cohen-Macaulay normal domain, and its defining ideal has a Gröbner bases of degree two. A detailed investigation of this case is given in chapter 3 and can be collected also in [41]. With point of view of the study of regularity of products of ideals, our result shows that the stabilizing index and constant are zero for the family of close cuts of Hankel matrices.

The other case of study of this thesis is the family of edge ideals of graphs. As we recalled earlier, it is in general difficult to compute the stabilizing index and the constant in the asymptotic function which determines the regularity of large enough powers of an ideal. One natural step to tackle this problem is to try to solve it for the smallest, yet interesting, case of ideals which is the family of edge ideals. Let G be a simple undirected graph. We associate an indeterminate over a given field $\mathbb{K}$ uniquely for every vertex of G. Naturally, we consider every edge of G as a squarefree monomial of degree two. The edge ideal of G denoted by $I(G)$ is the ideal generated by the edges of G. It is interesting that many combinatorial aspects of G can be interpreted in terms of properties of the edge ideal $I(G)$. For example the induced matching number can give a lower bound for the regularity of powers of a given edge ideal (see Theorem 2.5.15). The induced matching number together with decycling number can give an upper bound for the regularity of the first power of a given edge ideal (see Theorem 2.5.26). And, the co-chord number gives an upper bound for the regularity of given powers of any edge ideal (see Theorem 2.5.16). A popular conjecture in this direction is the following: Let G be a simple undirected graph and $I(G)$ be its edge ideal. Then $\operatorname{reg} I(G)^q \leq 2q + \operatorname{reg} I(G) - 2$ (see Conjecture 2.5.18). This conjecture is settled for few families of graphs i.e gap-free graphs (see [7]) and unicyclic graphs (see [1]). In chapter 4, we tackle this conjecture for the family of dumbbell graphs. The problem is motivated by [10], [48], [1] and [36]. For this family of graphs, we explicitly compute the induced matching number, the regularity of first power in terms of the induced matching number and finally we show that the equality above conjecture holds for this family provided the connecting path has two or one vertices. We encourage the reader to find the detailed investigation in chapter 4 or [23].

Chapter 2

Preliminaries

In this chapter, we recall the notions required for this thesis and fix some notations. While presenting some results we try to touch on the historical importance and motivations.

2.1 Basic Setting

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with n indeterminates over arbitrarily field $\mathbb{K}$. We refer to usual grading system induced by $\deg(x_i) = 1$ and $\deg(k) = 0$ for all $k \in \mathbb{K}$ by *standard grading* system. We say polynomial ring S is standard $\mathbb{N}^r$-graded, $r \leq n$, if $\deg(x_i) = e_i$ where $\{e_i\}_{i:1,\ldots,r}$ is the standard $\mathbb{K}$-bases for $\mathbb{N}^r$. Let $R = \mathbb{K}[f_1, \ldots, f_r]$ be a finitely generated $\mathbb{K}$-subalgebra of S. Let $R = \mathcal{A}/\mathcal{I}$ be the finite representation of R where $\mathcal{A} = \mathbb{K}[Y_1, \ldots, Y_r]$. We say R is a *standard* $\mathbb{K}$-algebra if $\mathcal{I}$ is homogeneous and $\mathcal{I} \subseteq (Y_1, \ldots, Y_r)^2$. Throughout this thesis, we adapt the standard graded setting. Meaning we consider every ideal $I \subset S$ to be homogeneous ideal; every S- module M is considered to be graded. Moreover, we consider some term order τ on S which is inherited by R. In case we need to employ different term orders, we clearly define and introduce them.

2.2 Castelnuovo-Mumford Regularity: Generalities and Some Motives

Let M be a graded R-module where R is a standard graded $\mathbb{K}$-algebra. Let $\mathbb{F}$ be the minimal free resolution of M as an R-module:

$$\mathbb{F} : \ldots F_i \to \ldots \to F_1 \to F_0 \to M \to 0$$

where $F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}}$ and $\beta_{i,j} = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)_j$ is the so called (i, j)-th *Betti number* of M. The length of minimal free resolution of M is called the *projective dimension* of M. In general, one can not expect to have a finite projective dimension for a given R-module (see Example 2.3.5). However, due to Hilbert's syzygy theorem, the projective dimension is bounded by n if R is the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. A classification of rings with finite projective dimension is given in the famous Auslander-Buchsbaum-Serre Theorem. The following is the graded version of this theorem.

Theorem 2.2.1. *The following conditions are equivalent:*

(1) $\mathrm{pd}_R(M)$ *is finite for every R-module M,*

(2) $\mathrm{pd}_R(K)$ *is finite,*

(3) R is regular, that is, R is a polynomial ring.

We refer the reader to standard text books like [49] and [17] for more detailed information. One of the important invariants of M which can be read from its minimal free resolution is the Castelnuovo-Mumford regularity:

Definition 2.2.2. *Let M be an R-module. The* Castelnuovo-Mumford regularity *of M is*

$$\mathrm{reg}\,(M) = \sup\{j - i \;:\; \beta_{i,j}(M) \neq 0\}.$$

Roughly speaking, regularity "measures" the complexity of any system of generators of M and its syzygies. The following general results are well-known:

Lemma 2.2.3. *Let N, M and K be R-modules. Let*

$$0 \to K \to N \to M \to 0$$

be a short exact sequence. Then:

$$\begin{aligned} \mathrm{reg}\,(K) &\leq \max\{\mathrm{reg}\,(N), \mathrm{reg}\,(M) + 1\} \\ \mathrm{reg}\,(N) &\leq \max\{\mathrm{reg}\,(K), \mathrm{reg}\,(M)\} \\ \mathrm{reg}\,(M) &\leq \max\{\mathrm{reg}\,(N), \mathrm{reg}\,(K) - 1\} \end{aligned}$$

Let I be an ideal of R. Then $\mathrm{reg}\,(I) = \mathrm{reg}\,(R/I) + 1$.

For the proposes of this thesis, we will always consider ideals and modules over polynomial rings.

Of particular interest, is a special case of minimal free resolutions known as linear resolutions. Let I be an ideal of S generated in a single degree d. It is easy to see that

$$d \leq \mathrm{reg}\,(I).$$

We say I has linear resolution if and only if $\mathrm{reg}\,(I) = d$. In some sense to have a linear resolution is the best case scenario of complexity of generators of I and the syzygies. We

say the family of ideals, say $\mathcal{F}$, has *linear powers* if for every $I \in \mathcal{F}$, all power of I has linear resolution. The family $\mathcal{F}$ has *linear products* if every products of powers of ideals $I_1, \dots, I_r \in \mathcal{F}$ has linear resolution. Note that the ideals need not to be distinct.
An interesting research problem is to find families of ideals with linear products. (i.e [26], [13], [30]).

One standard method to approach problems regarding the study of powers and products of ideals is via Rees algebras and multi-Rees algebras:

Definition 2.2.4. *Let I be an ideal of S. The* Rees algebra *of I is*

$$\mathscr{R}(I) = \bigoplus_{j \in \mathbb{N}} I^j t^j$$

where t is a new indeterminate over $\mathbb{K}$. One can also consider Rees algebras as $\mathscr{R}(I) = S[It] \subset S[t]$.
Let $I_1, \dots, I_r$ be ideals of S. The multi-Rees algebra *of $I_1, \dots, I_r$ is*

$$\mathscr{R}(I_1, \dots, I_r) = \bigoplus_{(a_1, \dots, a_r) \in \mathbb{N}^r} I_1^{a_1} t_1^{a_1} \dots I_r^{a_r} t_r^{a_r}$$

where $t_1, \dots, t_r$ are new indeterminate over $\mathbb{K}$. One can also consider multi-Rees algebras as $\mathscr{R}(I_1, \dots, I_r) = S[I_1 t_1, \dots, I_r t_r] \subset S[t_1, \dots, t_r]$.

Consider $\mathscr{R}(I_1, \dots, I_r) = S[I_1 t_1, \dots, I_r t_r]$. If S is standard graded and each ideal I_i is generated in a single degree, say d_i, the Multi-Rees algebras (or similarly Rees algebras) carries a natural standard $\mathbb{Z} \times \mathbb{Z}^r$-graded setting. In fact one sets $\deg(x_i) = e_0$ for all $i \colon 1, \dots, n$ and $\deg(t_j) = -d_i e_0 + e_j$ for all $j \colon 1, \dots, r$. Note that $\{e_i\}_{i:0,\dots,r}$ is the standard bases for $\mathbb{Z} \times \mathbb{Z}^r$. Clearly, $\mathscr{R}$ is generated as finitely generated algebra by elements of degrees $e_0, e_1, \dots, e_r$.
With this structure in mind, it is not difficult to see that the multi-homogeneous sections of $\mathscr{R} = \mathscr{R}(I_1, \dots, I_r)$ as an S-module encodes products of ideals $I_1, \dots, I_r$. In fact we have

$$\mathscr{R}_{(*,a)} = I^a t^a \cong I^a$$

where $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$, $I = I_1 \dots I_r$ and $t = t_1 \dots t_r$. Hence, multi-Rees algebras are strong tools to simultaneously study all products of powers of ideals $I_1, \dots, I_r$.

Definition 2.2.5. *Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a standard $\mathbb{N}^r$-graded polynomial ring and R be a standard $\mathbb{K}$-subalgebra of S. The partial regularity of R over S is defined as*

$$\operatorname{reg}_j(R) = \sup\{a_j - i : \beta_{i,(a_1,\dots,a_r)}(R) \neq 0\}$$

for all $a : 1, \dots, r$.

Theorem 2.2.6. [53][39] *Let I be an ideal of S generated in single degree d. Then* $\operatorname{reg}(I^j) \leq dj + \operatorname{reg}_0(\mathscr{R}(I))$. *In particular, if* $\operatorname{reg}_0(\mathscr{R}(I)) = 0$*, then all powers of I have linear resolution.*

The above theorem was originally proved in [53] by the study of Rees algebras. Later on in [39], the authors took advantage of the bigraded structure of Rees algebras to significantly shorten the proof given in [53]. Afterwards, the above theorem was improved in [13, Theorem 3.1] using similar method.

Theorem 2.2.7. *Let S be a standard graded polynomial ring over the field $\mathbb{K}$. The family of ideals $I_1, \dots, I_r$ of S, generated in single degrees $d_1, \dots, d_r$, has linear products if and only if* $\operatorname{reg}_0(\mathscr{R}(I_1, \dots, I_r)) = 0$.

Remark 2.2.8. *The above theorem is a strong tool to test conjectures on families of ideals expected to have linear products.*

A historical question which was asked in [21] is whether the following inequality always holds:

$$\operatorname{reg}(I^r) \leq r.\operatorname{reg}(I),$$

when r is arbitrary and I is any ideal of S. A negative answer is given in [57]. Here by we recall this counter example:

Example 2.2.9. *Let $S = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6]$. Consider the ideal*

$$I = (x_1x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_5, x_2x_5x_6, x_3x_4x_6, x_3x_5x_6).$$

One can check with Macaulay2 that $\operatorname{reg}(I) = 3$ *but* $\operatorname{reg}(I^2) = 7$.

On the quest of finding families of ideals with linear powers or products, one might wonder to start from well-known families of ideals with linear first power. It was shown in [35] that a square-free monomial ideal of degree two has linear resolution if and only if it is the edge ideal of a co-chordal graph[1] (The result is also known as Fröberg's characterization.). Let us name this family of edge ideals by co-chordal ideals. In [39], it was proved that every power of co-chordal ideals has linear resolution. A natural question is whether the family of all co-chordal ideals has linear products? Unfortunately, the answer is negative due to an example found by Aldo Conca and given to the author by Alessio D'Alì.

Example 2.2.10. *Let $I = (x_1x_3, x_2x_3, x_2x_4)$ and $J = (x_1x_2, x_2x_3, x_3x_4)$ be ideals of $S = \mathbb{K}[x_1, x_2, x_3, x_4]$. One can check with Macaulay2 that I and J have linear resolutions. Hence by Fröberg's characterization, I and J are co-chordal ideals. But, IJ has regularity equal to 5. Hence, the family of co-chordal ideals does not have linear products. Moreover, Theorem 2.2.7 admits that* $\operatorname{reg}_0 \mathscr{R}(I, J) \neq 0$.

An other nice example is the following:

[1]The reader can refer to section 2.5 for the definition of edge ideals and co-chordal graphs.

Example 2.2.11. [26, Example 2.1] *Let* $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ *and let* $J = (x_1^2x_2, x_1x_2x_3, x_2x_3x_4, x_3x_4^2)$ *and* $I = (x_2, x_3)$*. One can check that* $\operatorname{reg}(J) = 3$ *and* $\operatorname{reg}(I) = 1$*. Hence* I *and* J *both have linear resolution. But* $\operatorname{reg}(IJ) = 5$*.*

Code 2.2.12. *The following Macaulay2 code can be used to calculate the multi-Rees ring of ideals given in Example 2.2.10 and check its partial regularity. The same code can be applied to other examples with minor modifications.*

```
i1 : S=QQ[x_1..x_4,T_1,T_2,Degrees
   =>{4:{1,0,0},{-2,1,0},{-2,0,1}}];
i2 : (S1,f) = selectVariables({0..3},S)
i3 : I=f(ideal(x_2*x_4,x_2*x_3,x_1*x_3));
i4 : J=f(ideal(x_3*x_4,x_2*x_3,x_1*x_2));
i5 : genI = first entries gens I;
i6 : genJ = first entries gens J;
i7 : R=QQ[X_1..X_4,Y_1..Y_(#genI),Z_1..Z_(#genJ),Degrees
   =>{4:{1,0,0},#genI:{0,1,0},#genJ:{0,0,1}}];
i8 : xvars=first entries f(vars S1);
i9 : genAlg= xvars|T_1*genI|T_2*genJ;
i10 : rees = map(S,R,genAlg);
i11 : reesKer = trim ker rees;
i12 : peek betti res reesKer
o12 = BettiTally{(0, {0, 0, 0}, 0) => 1}
                 (1, {0, 2, 2}, 4) => 1
                 (1, {1, 0, 1}, 2) => 2
                 (1, {1, 1, 0}, 2) => 2
                 (1, {2, 1, 1}, 4) => 1
                 (2, {1, 2, 2}, 5) => 4
                 (2, {2, 0, 2}, 4) => 1
                 (2, {2, 1, 1}, 4) => 4
                 (2, {2, 1, 2}, 5) => 1
                 (2, {2, 2, 0}, 4) => 1
                 (2, {2, 2, 1}, 5) => 1
                 (2, {3, 1, 1}, 5) => 2
                 (3, {2, 2, 2}, 6) => 7
                 (3, {3, 1, 2}, 6) => 4
                 (3, {3, 2, 1}, 6) => 4
                 (3, {4, 1, 1}, 6) => 1
                 (4, {3, 2, 2}, 7) => 6
                 (4, {4, 1, 2}, 7) => 1
                 (4, {4, 2, 1}, 7) => 1
                 (5, {4, 2, 2}, 8) => 1
```

Note that in the last tally in the above code, from left to right, the first column is the homological degree the second column is the shifting. Hence it is easy to read the partial regularity with this method.
The above examples already provides that it is difficult to find families of ideals with linear powers or products due to the fact that the counter examples are rather "small". Nevertheless, there are some discoveries on finding such families. Not surprisingly, these families tend to be non-trivial. More examples of this kind are provided in the further sections of this manuscript.

2.3 Koszul Algebras

Koszul algebras are introduced in [50] originally. They are graded $\mathbb{K}$-algebras R whose residue field $\mathbb{K}$ has linear free resolution as an R-module. From certain perspectives, Koszul algebras have similar homological properties as the ones of polynomial rings. For example every R-module has finite regularity as the same holds for polynomial rigs due to Hilbert's syzygy theorem (see Theorem 2.3.4). On the other hand, there are homological properties of polynomial rings which are not shared with Koszul algebras. For example, Poincaŕe series over Koszul algebras can be irrational (see [2]) whereas they are rational over polynomial rings (see [49, Theorem 33.6]). As the authors of [25] say, "This mixture of similarities and differences with the polynomial ring and their frequent appearance in classical constructions are some of the reasons that make Koszul algebras fascinating, studied and beloved by commutative algebraists and algebraic geometers. In few words, a homological life is worth living in a Koszul algebra.".
For the convenience of this thesis, we collect some important results on Koszul algebras from [25].

Definition 2.3.1. *The $\mathbb{K}$-algebra R is Koszul if $\mathbb{K}$ has linear resolution as an R-module, that is, $\operatorname{reg}_R(\mathbb{K}) = 0$ or, equivalently, $\beta_{i,j} = 0$ whenever $i \neq j$.*

Definition 2.3.2. *The $\mathbb{K}$-algebra $R = \mathcal{A}/\mathcal{I}$ is called G-*quadratic *if $\mathcal{I}$ has a Gröbner bases of degree 2 with respect to some coordinate system and some term order on $\mathcal{A}$.*

Remark 2.3.3. *The following are some remarks we need throughout the thesis. Let $R = \mathcal{A}/\mathcal{I}$ be a $\mathbb{K}$-algebra:*

(1) If R is Koszul, then the ideal $\mathcal{I}$ is defined by quadrics (i.e homogeneous polynomials of degree 2). This is due to fact that $\beta_{2,j}(\mathbb{K}) = 0$ for all $j \neq 2$. It is important to note that not every algebra defined by some quadric ideal is Koszul. See, for example, $R = \mathbb{K}[x, y, z, t]/I$ with $I = (x^2, y^2, z^2, t^2, xy + zt)$. One can check that $\beta_{3,4}(\mathbb{K}) = 5$.

(2) If $\mathcal{I}$ is generated by monomials of degree 2 with respect to some coordinate system of $\mathcal{A}_1$, then R is Koszul.

(3) If R is G-quadratic, then R is Koszul. This follows from (2) and from the standard deformation argument showing that $\beta^R_{i,j}(\mathbb{K}) \le \beta^A_{i,j}(\mathbb{K})$ with $A = \mathcal{A}/\operatorname{in}(\mathcal{I})$. One notes that there exists Koszul algebras which are not G- quadratic. See $\mathbb{C}[x,y,z]/(x^2+yz, y^2+xz, z^2+xy)$ and [25, Remark] and [32] for more detail.

In general, it is not an easy task to prove whether a Koszul algebra is G-quadratic. In the nineties, it was asked by Peeva and Sturmfels, if the coordinate ring of pinched Veronese

$$PV = \mathbb{K}[x^3, x^2y, x^2z, xy^2, xz^2, y^3, y^2z, yz^2, z^3]$$

is Koszul. This example used to be the "go to" example to test the new techniques for proving Koszulness. In 2009 [19] and 2013 [20] it was proved that PV is Koszul. It is still open whether PV is G-quadratic.

The following theorem is characterization of the Koszul property in terms of regularity.

Theorem 2.3.4. *(Avramov-Eisenbud-Peeva) The following are equivalent:*

(1) $\operatorname{reg}_R(M)$ *is finite for every R-module M,*

(2) $\operatorname{reg}_R(\mathbb{K})$ *is finite,*

(3) R is Koszul.

Proof. Proof in [3] and [4]. □

Example 2.3.5. *Let $R = \mathbb{K}[x]/(x^r)$ with $r > 1$. Then the minimal free R-resolution of $\mathbb{K}$ is:*

$$\cdots \to R(-2r) \to R(-r-1) \to R(-r) \to R(-1) \to R \to 0$$

where the maps are given by multiplication by x or x^{r-1}. Therefore $F_{2i} = R(-ir)$ and $F_{2i+1} = R(-ir-1)$ so that $P\dim_R(\mathbb{K}) = \infty$ for all $r > 1$. Moreover, $\operatorname{reg}_R(\mathbb{K}) = \infty$ when $r > 2$ and $\operatorname{reg}_R(\mathbb{K}) = 0$ if $r = 2$. Thus, R is Koszul only for $r = 2$ by virtue of Theorem 2.3.4.

As we pointed out in section 2.2, multi-Rees algebras play a crucial role in the study of products of ideals. An other nice result on this matter which connects the Koszulness property with the problem of finding families of ideals with linear products is given in [12]. The result is given for Rees algebras of a single ideal, however, the arguments follows also for multi-Rees algebras of a family of ideals:

Theorem 2.3.6. [12, Corollary 3.6] *Let I be an ideal of S generated in degree d. If $\mathscr{R}(I)$ is Koszul, then every power of I has linear resolution.*

The variant of the above theorem for multiple ideals is the following.

Theorem 2.3.7. *Let $I_1, \ldots, I_r$ be ideals of S each of them generated in a single degree like d_i. If $\mathscr{R}(I_1, \ldots, I_r)$ is Koszul, then every given product $I_1 \ldots I_r$ has linear resolution.*

The above theorem shows a technique to tackle the problem of finding families of ideals with linear powers or linear products.

2.4 Sagbi Bases

The theory of Sagbi bases (i.e one can see it as analogues of Gröbner bases for algebras) was introduced in [52] and independently in [43]. Let $\mathrm{in}(R)$ be the initial algebra of R, that is the $\mathbb{K}$-algebra generated by all $\mathrm{in}(f)$ where $f \in R$. Like Gröbner bases theory, many properties of R are "lifted" from the ones of $\mathrm{in}(R)$. The idea of applying Sagbi bases techniques is to approach R via $\mathrm{in}(R)$ provided the later one has simpler structure. For example it is a Toric algebra which sometimes caries nice combinatorial properties. In some scenarios, this combinatorial properties are nice to analyze. In 1996, Sagbi deformations was introduced in [27] which covers some properties of R that can be derived from $\mathrm{in}(R)$ by means of Sagbi bases. We collect some results from this paper which are crucial in our treatment. First, let us formally define that Sagbi bases.

Definition 2.4.1. *A set of elements in R like $f_1, \ldots, f_n$ is* Sagbi bases *of R with respect to τ if $\mathrm{in}(R) = \mathbb{K}[\mathrm{in}(f_i) \ : \ 1 \leq i \leq r]$.*

Example 2.4.2. [52, Example 1.17] *Let $U = \{u_1, \ldots, u_r\}$ be monomials in polynomial ring S. By definition, it is clear that U is a Sagbi bases for $R = \mathbb{K}[u_1, \ldots, u_r]$ with respect to any term order.*

The algebra generators of some finitely generated $\mathbb{K}$-algebra do not always form a Sagbi bases:

Example 2.4.3. [52, Example 1.18] *Let $\mathbb{K}[x]$ be the polynomial ring in one indeterminate equipped with its usual term order (where $x^j \geq x^i$ if and only if $j \geq i$.). Consider the subset $U = \{x^2 + x, x^2\}$ of $\mathbb{K}[x]$. It is clear that $\mathbb{K}[x]$ is the sub-algebra generated by U since $x = x^2 + x - x^2$. On the other hand, x^2 is the leading term of elements in U. Thus U generates $\mathbb{K}[x]$ but it is not a Sagbi bases for $\mathbb{K}[x]$.*

Unlike Gröbner bases, finitely generated algebras might not necessarily have finite Sagbi bases:

Example 2.4.4. [52, Example 1.20] *Let $R = \mathbb{K}[x + y, xy, xy^2]$ be a $\mathbb{K}$-sublagebra of $S = \mathbb{K}[x, y]$. Then R does not have a finite Sagbi bases with respect to any term order.*

Proof. Hereby we sketch the proof. Since R is generated by homogeneous generators, R is a graded sublagebra of S. The proof takes advantage of two family of elements in S. One is a family of elements which is in R and the other family is some elements which are not in R.

Elements in R: Clearly xy and xy^2 are in R. By induction on n, we see that $xy^n = (x + y)xy^{n-1} - (xy)xy^{n-2}$ is in R for all $1 \leq n$.

Elements not in R: For $j \geq 1$, R has no element with y^j as a homogeneous component, hence it has no elements with non-zero scalar times y^j as a homogeneous component.

For any term order on S, we have either $x > y$ or $y > x$. Suppose $x > y$. We have $U = \{x + y, xy, xy^2, xy^3, \ldots\} \subset R$. U is a Sagbi bases and there exists no finite Sagbi bases for R.
Suppose $y > x$. Note that R is also generated by $x + y, xy, x^2y$. By similar reasoning, one can see $U = \{x + y, xy, x^2y, x^3y, \ldots\} \subset R$ is a Sagbi bases and there exists no finite Sagbi bases for R. □

Nevertheless, every Sagbi bases of R is a system of generators of R. It is necessary, for application purposes, to know whether some finite set of elements of R are Sagbi bases. The answer is given in [52], however, we recall the statements from [27] since it is closer to our notations:

Proposition 2.4.5. [27, Proposition 1.1] *Let $f_1, \ldots, f_r$ be a system of generators of R and let $g_1, \ldots, g_s$ be a set of generators of $\mathcal{J}$ the defining ideal of $\mathbb{K}[\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)]$. Then $f_1, \ldots, f_r$ is a Sagbi bases for R if and only if*

$$g_j(f_1, \ldots, f_r) = \sum_v \lambda_v^{(j)} f^v; \qquad 1 \leq j \leq s$$

with $\mathrm{in}(f^v) \leq_\tau \mathrm{in}(g_j(f_1, \ldots, f_r))$, $\lambda_i \in \mathbb{K}$ is non-zero and $f^v = f_1^{v_1}, \ldots, f_r^{v_r}$.

Corollary 2.4.6. [27, Corollary 2.1] *Adapt the assumptions of Proposition 2.4.5. Consider the polynomial ring $\mathcal{A} = \mathbb{K}[Y_1, \ldots, Y_r]$ the algebraic homomorphism $\phi : \mathcal{A} \to R$ defined by $Y_i \mapsto f_i$. Let $\mathcal{I}$ be the kernel of ϕ. Then $\mathcal{I}$ is generated by the "lifted" polynomials*

$$G_j = g_j(Y_1, \ldots, Y_r) - \sum_v \lambda_v^{(j)} Y^v; \qquad 1 \leq j \leq s.$$

In addition, if $g_1, \ldots, g_s$ form a Gröbner bases for $\mathcal{J}$ with respect to some term order τ of $\mathcal{A}$, then the "lifted" polynomials form a Gröbner bases with respect to τ for $\mathcal{I}$. In particular, $\mathrm{in}(\mathcal{I}) = \mathrm{in}(\mathcal{J})$.

Corollary 2.4.7. [27, Corollary 2,3]

(1) If $\mathrm{in}(R)$ is Cohen-Macaulay of dimension d and type r, then R is Cohen-Macaulay of dimension d and type $\leq r$. In particular, if $\mathrm{in}(R)$ is Gornestein, then so is R.

(2) If $\mathrm{in}(R)$ is normal, then R is normal Cohen-Macaulay domain.

At this point of this thesis, we have the stage ready to present some examples using the material and the theory we have presented so far. In the following we illustrate the technique used in [27]. We apply the same strategy in chapter 3.

Example 2.4.8. [27] *Let* $X = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} \\ x_2 & x_3 & \dots & \dots & x_n \end{pmatrix}$ *be the Hankel matrix with two rows and entries* $x_1, \dots, x_n$ *which are indeterminates over a given field* $\mathbb{K}$*. Consider the polynomial ring* $S = \mathbb{K}[x_1, \dots, x_n]$*. The ideal* I *generated by all minors of size* 2 *of* X *has linear powers. The idea is the following:*

Let $I = (g_1, \dots, g_r)$ *be the natural generators of* I *(the minors of size* 2 *of* X*). By applying Corollary 2.4.6, one proves that* $(g_1, \dots, g_r)$ *is a Sagbi bases for the Rees algebra* $\mathscr{R}(I)$ *with respect to some term order. The authors show that* $\mathscr{R}^{\text{in}}(I) = S[\text{in}(g_i)t : 1 \leq i \leq r]$ *is G-quadratic. Then by "lifting" the kernel of* $\mathscr{R}^{\text{in}}(I)$ *and using the second part of Corollary 2.4.6 it follows that* $\mathscr{R}(I)$ *is G-quadratics. Thus Remark 2.3.3* (3) *yields* $\mathscr{R}(I)$ *is Koszul. Therefore, by Theorem 2.3.6* I *has linear powers.*

Following the same steps, the result of above example was improved in [30].

Example 2.4.9. *Let* $\mathbf{X}_t$ *be the Hankel matrix with* t *rows whose entries are the indeterminates* $x_1, \dots, x_n$ *of* S *and* $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$*. Let* I_t *be the ideal generated by maximal minors of* $\mathbf{X}_t$*. Let* I_t *denote the determinantal ideal generated by minors of size* t *of the matrix* $\mathbf{X}_t$*. Let* $\mathcal{F}$ *be the family of all such ideals for fixed number of indeterminates* $x_1, \dots, x_n$*. It is known that* $\mathcal{F}$ *has linear products. Moreover, for every product* $I_{t_1} \dots I_{t_r}$*, the multi-Rees algebra* $\mathscr{R}(I_{t_1}, \dots, I_{t_r})$ *is Koszul. In particular, the ordinary generators of* $I_{t_1}, \dots, I_{t_r}$ *form a Sagbi bases for the Rees algebra that is equivalent to saying that ordinary generators of* $I_{t_1} \dots I_{t_r}$ *form a Gröbner bases with respect to some term order. See* [24], [30] *and* [26] *for more detail.*

2.5 Graph Theory and Combinatorial Commutative Algebra

Monomial ideals are algebraic objects that are at the intersection of algebra, combinatorics and topology. There exists enormous number of publications and literature on the study of monomial ideals. The interested reader can refer to [38] and [47] as a start point. As we mentioned in section 2.2, the investigation of regularity of powers or products of ideals is a difficult task. Restricting the investigation to the class of edge ideals (see Definition 2.5.2) the goal is to relate regularity with the combinatorial data of graphs. Here, we recall some combinatorial material we need for this thesis from [6].

Let $G = (V, E)$ be a *simple graph*, that is an undirected graph with to torsion edge, by vertex set V and edge set E. Throughout this note, we always consider our graphs to be simple. By abuse of notation, we consider the vertex set of G to be $V = \{x_1, \dots, x_n\}$ and the edge set of G is considered as squarefree monomials of degree 2, however, for an arbitrary edge we still use the letter e. When we need to emphasize on the vertices of some edge, we denote it by $e_{i,j} = x_i x_j$. A pair of arbitrary vertices is denoted by x and x'. Let $A \subseteq V(G)$. The induced subgraph of A is the graph with set of vertices A and edges in G which includes that joins two vertices from A.

In the Figure 2.1 graph H is an induced subgraph of G while H' is not.

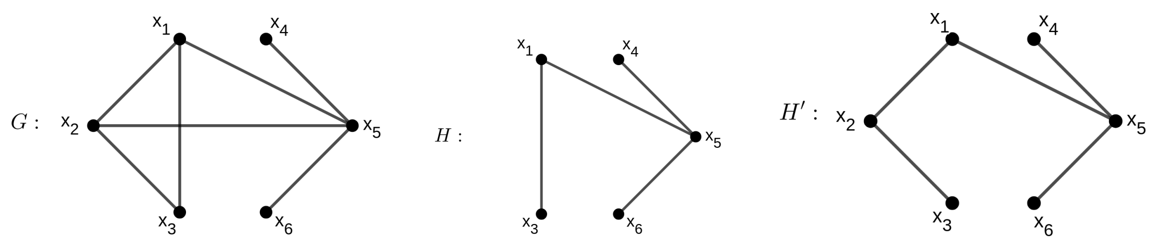


Figure 2.1

For a vertex x in a graph G, let $N_G(x) = \{x' \in V | xx' \in E\}$ define the *open neighborhood* of x, and let $N_G[x] := N_G(x) \cup \{x\}$ define the *closed open neighborhood* of x. An edge e is *incident* to x if e is divisible by x as a monomial. The degree of a vertex $x \in V$, denoted by $\deg_G(x)$, is the number of edges incident to x. When there is no confusion, we will omit G and write $N(x), N[x]$ and $\deg(x)$. For an edge e in a graph $G = (V, E)$, we define $G \setminus e$ to be the subgraph of G obtained by deleting e from E (but the vertices are remained). For a subset $W \subseteq V$ of the vertices in G, we define $G \setminus W$ to be the subgraph of G deleting the vertices of W and their incident edges. When $W = \{x\}$ consists of a single vertex, we write $G \setminus x$ instead of $G \setminus \{x\}$. For an edge $e = xx' \in E$, let $N_G[e] = N_G[x] \cup N_G[x']$ and define G_e to be the induced subgraph of G over the vertex set $V \setminus N_G[e]$. In Figure 2.1, we have $H = G \setminus x_2$ and $H' = G \setminus e_{2,5}$. It is clear from the definition that deleting vertices or from a given graph G, always gives an induced subgraph of G.

Definition 2.5.1. *Let G be a graph:*

(1) A path P_l is a graph with l vertices $V(P_l) = \{x_1, \ldots, x_l\}$ and $\{x_i, x_{i+1}\} \in E(P_l)$ for all $1 \leq i \leq l-1$. P_l is a path in G if it is a subgraph of G.

(2) A cycle C_n is a graph with n vertices with $\{x_i, x_{i+1}\} \in E$ for all $1 \leq i \leq n-1$ and $\{x_n, x_1\} \in E$. C_n is a cycle of G if it is a subgraph of G.

A *chord* in a cycle C_n is an edge $x_i x_j$ with $x_j \neq x_{i-1}x_{i+1}$. A *chordal* graph is a graph for which every cycle of length greater than or equal 4 has a chord. A graph is *co-chordal* if its complement is chordal. The graph G in Figure 2.1 is chordal. By definition, G^c (the complement of G) is co-chordal. See Figure 2.2.

A *forest* is a graph with no cycles and a *tree* is a connected forest. A *bipartite* graph is a graph that splits in to two partition of vertices such that there is no edge contained in one partition. It is clear that bipartite graphs are those graphs with only even cycles. See

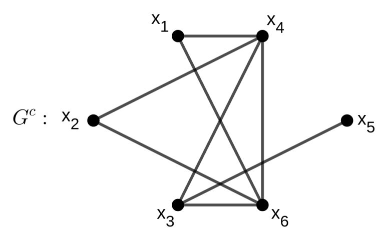


Figure 2.2

Figure 2.3. A *unicyclic* graph is a graph with exactly one cycle. See Figure 2.4. A *leaf* is an edge with one vertex of degree one. In Figure 2.1 graph G the edges $e_{4,5}$ and $e_{5,6}$ are leaves.

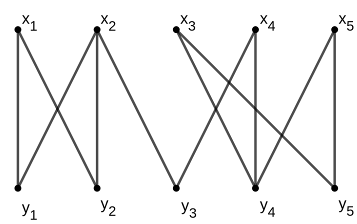


Figure 2.3: A bipartite graph

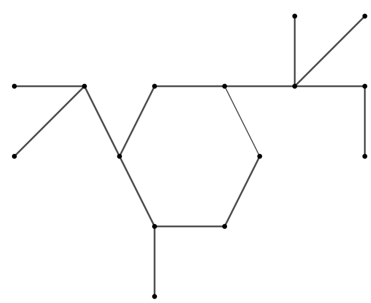


Figure 2.4: A unicyclic graph

Hereby, we formally define the edge ideal of a graph.

Definition 2.5.2. *The edge ideal of the graph* $G = (V, E)$ *is the square free monomial ideal*

$$I(G) = (x_i x_j \mid e_{i,j} \in E).$$

The so called inductive methods and techniques are developed in order to relate regularity of edge ideals with "smaller" edge ideals. The soul of these methods are in performing (often) simple combinatorial operations on a given graph to obtain a new graph with better behavior. Examples of operations of this kind are deleting vertices, edges, leaves etc. With the help of inductive methods and some algebraic techniques researchers have a developed

a nice toolkit to be used in the study of regularity of powers of edge ideals. Here we recall some related results.
For a given ideal $I \subseteq S$ and m a monomial of degree d. The following short exact sequences are broadly used and proved to be helpful in the study of regularity of powers of edge ideals.

$$0 \to \frac{S}{I:m}(-d) \xrightarrow{.m} \frac{S}{I} \to \frac{S}{I+(m)} \to 0$$

Let I and J be ideals in S. The other helpful short exact sequence is

$$0 \to \frac{S}{I \cap J} \to \frac{S}{I} \oplus \frac{S}{J} \to \frac{S}{I+J} \to 0$$

The behavior of regularity over short exact sequences is easy to understand thanks to Lemma 2.2.3.

The following is a consequence of the above exact sequences.

Lemma 2.5.3. [29, Lemma 2.10] *Let $I \subseteq S$ be a monomial ideal, and let m be a monomial of degree d. Then*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I:m)+d, \operatorname{reg}(I,m)\}.$$

Furthermore, if x is a variable appearing in I, then

$$\operatorname{reg}(I) \in \{\operatorname{reg}(I:x)+1, \operatorname{reg}(I,x)\}.$$

It is easy to see that $(I(G):x) = I(G \setminus \mathrm{N}_G[x])$ and $(I(G),x) = I(G \setminus x)$ for a given vertex in G. Hence, Lemma 2.5.3 can be rewritten as the following.

Lemma 2.5.4. *Let x be a vertex in G. then*

$$\operatorname{reg} I(G) \in \{\operatorname{reg} I(G \setminus \mathrm{N}_G[x])+1, \operatorname{reg} I(G \setminus x)\}.$$

As a consequence of the following result, we can break the regularity of edge ideals of graphs into regularity of its subgraphs.

Theorem 2.5.5. [42] *Let $I_1, \ldots, I_s$ be monomial ideals in S, then*

$$\operatorname{reg}\left(\frac{S}{\sum_{i=1}^{s} I_i}\right) \leq \sum_{i=1}^{s} \operatorname{reg}\left(\frac{S}{I_i}\right).$$

An other helpful result is the following. As an application of this result, one can construct nice induced subgraphs of given graph G to obtain a lower bound for regularity of powers of $I(G)$. We strongly take advantage of the following result in the treatments of chapter 4.

Corollary 2.5.6. [10, Corollary 4.3] *Let G be a graph and H be an induced subgraph of G. Then,*

$$\operatorname{reg} I(H)^s \leq \operatorname{reg} I(G)^s$$

for all $s \geq 1$.

In the study of regularity of powers of edge ideals, the notion of even-connection was introduced in [7].

Definition 2.5.7. *Let $G = (V, E)$ be a graph with edge ideal $I = I(G)$. Two vertices u and v in G are called even-connected with respect to an s-fold product $M = e_1 \cdots e_s$, where $e_1, \ldots, e_s$ are edges in G, if there is a path $p_0, \ldots, p_{2l+1}$, for some $l \geq 1$, in G such that the following conditions hold:*

(i) $p_0 = u$ *and* $p_{2l+1} = v$;

(ii) *for all* $0 \leq j \leq l-1, \{p_{2j+1}, p_{2j+2}\} = e_i$ *for some* i;

(iii) *for all* i, $\left|\{j \mid \{p_{2j+1}, p_{2j+2}\} = e_i\}\right| \leq \left|\{t \mid e_t = e_i\}\right|$.

Example 2.5.8. *Consider C_6 shown in Figure 2.5.*

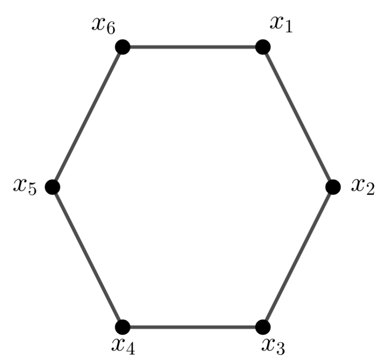


Figure 2.5

The vertices x_1 and x_6 are even-connected with respect to e_1e_2 where $e_1 = x_2x_3$ and $e_2 = x_4x_5$.

Definition 2.5.9. *The edges $e_1 = x_{1,1}x_{1,2}, \ldots, e_q = x_{q,1}x_{q,2}$ are in an even-connected position, if for all $1 \leq i \leq q-1$, the vertex $x_{i,2}$ is connected to the vertex $x_{i+1,1}$ and there exist $x \in N(e_1)$ and $x' \in N(e_q)$ such that x and x' are even-connected with respect to $e_1 \cdots e_q$.*

For the edge ideal $I = I(G)$ of some $G = (V, E)$ and an integer $s \geq 1$, the following holds.

Theorem 2.5.10. [7, Theorems 6.1 and 6.5] *Let $M = e_1 \cdots e_s$ be a minimal generator of I^s. Then $(I^{s+1} \colon M)$ is minimally generated by monomials of degree 2, and xx' (x and x' may be the same) is a minimal generator of $(I^{s+1} \colon M)$ if and only if either $\{x, x'\} \in E$ or x and x' are even-connected with respect to M.*

Remark 2.5.11. *By the above theorem it is easy to see that* $(I^{s+1}\colon M)$ *corresponds to an edge ideal* $I(G')$*. If* x *and* x' *are different and they are even-connected with respect to* M*, then* $E(G') = E(G) \cup \{u, v\}$*. If* $x = x'$*, we have* $x^2 \in (I^{s+1}\colon M)$*. In this case by polarization,* G' *is produced by adding a leaf to* x *in* G*.*

Theorem 2.5.12. [7, Theorem 5.2] *Let* G *be a graph and* $\{m_1, \ldots, m_r\}$ *be the set of minimal monomial generators of* $I(G)^q$ *for all* $q \geq 1$*, then*

$$\operatorname{reg}(I(G)^{q+1}) \leq \max\{\operatorname{reg}((I(G)^q\colon m_l)) + 2q \text{ where } 1 \leq l \leq r, \operatorname{reg}(I(G)^q)\}.$$

To apply the above theorem, one needs explicit understanding of the ideals $I(G)^q\colon m_l$. Meaning that one needs to have control over the graph associated to $I(G)^q\colon m_l$. Therefore, even-connection is mainly helpful while we strict ourselves to particular families of ideals.

As it is mentioned earlier, one of the combinatorial invariants that relates combinatorial structure of G with its regularity is the induced matching number:

Definition 2.5.13. *A collection* $\mathcal{M}$ *of edges of* G *is called an induced matching if* $\mathcal{M}$ *is a matching, and it is induced subgraph of* G*. The maximum size of an induced matching in* G *is called its induced matching number and it is denoted by* $\nu(G)$*.*

The first result bounding regularity with induced matching number was given in [44].

Theorem 2.5.14. [44, Lemma 2.2] *Let* G *be a graph and* $\nu(G)$ *be its induced matching number. Then*

$$\operatorname{reg} I(G) \geq \nu(G) + 1.$$

Later on, a more general result was given in [10] for regularity of powers of edge ideals. The following result is the key ingredient on finding lower bounds for regularity of powers of particular families of graphs. We will get back to it in the future.

Theorem 2.5.15. [10, Theorem 4.5] *Let* G *be a graph and let* $\nu(G)$ *denote its induced matching number. Then, for all* $q \geq 1$*, we have*

$$\operatorname{reg} I(G)^q \geq 2q + \nu(G) - 1$$

An other combinatorial invariant used to bound the regularity is the *co-chordal number* of G which is the minimum number of co-chordal subgraphs of G covering G.

Theorem 2.5.16. [61, Lemma 1] *Let* G *be a graph. Then*

$$\operatorname{reg} I(G) \leq \text{co-chord}(G) + 1.$$

The above result is improved in a recent preprint.

Theorem 2.5.17. [54, Theorem 3.2] *Let G be a graph. Then, for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \leq 2q + \text{co-chord}(G) - 1.$$

Hence the strongest lower bound and upper bound known, so far, for a given graph is

$$2q + \nu(G) - 1 \leq \operatorname{reg} I(G)^q \leq 2q + \text{co-chord}(G) - 1$$

It is interesting that there exists infinitely many connected graphs for which the above inequality is sharp from both sides. A detailed explanation is given in [54, Section 5]. Before the release of [54], one of the popular questions in the aria was whether the regularity of powers of edge ideals can be given in terms of the induced matching number? The reader notes that this was due to the evidences given in most known cases. See [37], [60], [59], [1] and etc. In [54], the authors gave a negative answer to this question.
An other popular conjecture in the aria is given in [6, Conjecture 7.11].

Conjecture 2.5.18. *Let G be a graph. Then for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \leq 2q + \operatorname{reg} I(G) - 2.$$

One notes that by Theorem 2.5.16, the Conjecture 2.5.18 gives a stronger upper bound for the regularity of powers of edge ideals. The above conjecture has been proved for some families of graphs. See [7], [1] and [23]. In the last article, we proved the Conjecture 2.5.18 for the families of dumbbell graphs with connecting path not larger than two and showed that the equality is not the case for given a dumbbell graph. See Figure 2.6.

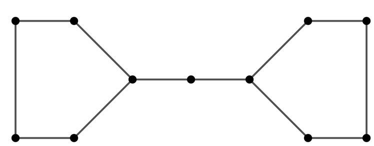


Figure 2.6: A dumbbell graph

To present the arguments, we need a few more known results. Let us recall the notion of *Lozin transformation*. While researching the complexity of the induced matching number of a given graph, the author of [46] introduces an operation (*stretching* as he calls it). In [11], the authors named this operation after Lozin and applied it on the study of regularity problem of edge ideals.

Definition 2.5.19. *Let G be a graph with a vertex x whose open neighborhood splits into two disjoint partitions $\mathrm{N}_G(x) = Y_1 \cup Y_2$. Replace the vertex with a path of length four $\{x_1, x_2, x_3, x_4\}$ together with edges y_1x_1 and y_2x_4 when y_1 and y_2 are any vertices in Y_1 and Y_2. This operation is called Lozin transformation.*

Theorem 2.5.20. *Let G be a graph. Let x be a vertex of G that satisfies the conditions for Lozin transformation. Then,*

(1) $\nu(\mathcal{L}_x(G)) = \nu(G) + 1$. [46, Lemma 1]

(2) $\operatorname{reg} I(\mathcal{L}_x(G)) = \operatorname{reg} I(G) + 1$. [11, Theorem 1.1]

where $\mathcal{L}_x(G)$ is the Lozin transformation of G over vertex x.

See Figure 2.7 and Figure 2.8 as an example of Lozin transformation.

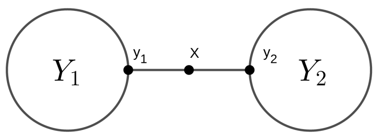


Figure 2.7: A given graph G with vertex x which satisfies the definition.

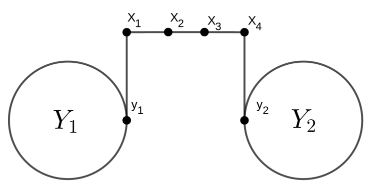


Figure 2.8: The graph $\mathcal{L}_x(G)$

Remark 2.5.21. *([10, Remark 2.12]) Let P_l be a path of l vertices, then we have*

$$\nu(P_l) = \lfloor \frac{l+1}{3} \rfloor$$

Remark 2.5.22. *([10, Remark 2.13]) Let C_n be a cycle of n vertices, then we have*

$$\nu(C_n) = \lfloor \frac{n}{3} \rfloor.$$

A maximal induced matching of C_n is completely determined by just choosing a first edge, and then we go (for instance) in clockwise direction by taking the third consecutive edge after the last one chosen. Thus, we shall use $r = n \bmod 3$ to give a specific characterization of the structure of the maximal induced matching. Depending on r we can assume the following:

(i) when $r = 0$, the edges x_1x_2 and x_1x_n do not belong to a maximal induced matching of C_n;

(ii) when $r = 1$, the edges x_1x_2, x_1x_n and $x_{n-1}x_n$ do not belong to a maximal induced matching of C_n;

(iii) when $r = 2$, the edges x_1x_2, x_2x_3, x_1x_n and $x_{n-1}x_n$ do not belong to a maximal induced matching of C_n.

In [10] the authors provided a formula for the regularity of the edge ideal of a forest or a cycle in terms of its induced matching number.

Theorem 2.5.23. [10, Theorem 4.7] *Let G be a forest with edge ideal $I = I(G)$, then*

$$\operatorname{reg} I(G)^q = 2q + \nu(G) - 1.$$

for all $q \geq 1$, where $\nu(G)$ denote the induced matching number of G.

Theorem 2.5.24. [10, Theorem 5.2]. *Let C_n be a cycle with n vertices, then*

$$\operatorname{reg} I(C_n) = \begin{cases} \nu(C_n) + 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ \nu(C_n) + 2 & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where $\nu(C_n) = \lfloor \frac{n}{3} \rfloor$ denote the induced matching number of C_n. Moreover,

$$\operatorname{reg} I(C_n)^q = 2q + \nu(C_n) - 1.$$

and for all $q \geq 2$.

In addition they prove an upper bound for a graph which contains Hamiltonian path. As a remark, a Hamiltonian path of G is a path that goes through each vertex of G exactly once.

Theorem 2.5.25. [10, Theorem 3.1] *Let G be a graph on n vertices. Assume G contains a Hamiltonian path, then*

$$\operatorname{reg} I(G) \leq \lfloor \frac{n+1}{3} \rfloor + 1$$

For a given graph G, the *decycling* number of G, denoted by $\nabla(G)$, is the minimum number of vertices one needs to delete from G to have a tree. For example, the decycling number of C_n is one.

Theorem 2.5.26. [11, Theorem 4.11] *For any graph G, we have* $\operatorname{reg} I(G) \leq \nu(G)+\nabla(G)+1$.

The authors of [1], gave a through study on the regularity of powers of unicyclic graphs, i.e, any connected graph with exactly one cycle. We need the following notation to state the result.

Notation 2.5.27. *Let G be a unicyclic graph with cycle C_n and $V(C_n) = \{x_1, \ldots, x_n\}$ and $T_1, \ldots, T_m$ be the rooted trees of G with roots $\{x_{i_1}, \ldots, x_{i_m}\} \subseteq V(C_n)$. Consider all the neighbors of $\{x_{i_1}, \ldots, x_{i_m}\}$ in the rooted trees and denote it by $\Gamma(G)$*

$$\Gamma(G) = \bigcup_{j=1}^{m} N_{T_j}(x_{i_j}) := \{y_1, \ldots, y_t\} \subseteq \bigcup_{j=1}^{m} V(T_j).$$

Note that non of the vertices in $\Gamma(G)$ can be a vertex on the cycle C_n. Let H_j be the induced subgraph of T_j obtained by deleting the elements of $\Gamma(G)$ that are varieties in T_j.

$$H_j = T_j \setminus \{z_k | z_k \in V(T_j) \cap \Gamma(G)\}$$

Note that H_j is either a forest or a tree, and H_j's are disjoint. Thus

$$G \setminus \Gamma(G) = C_n \bigcup (\bigcup_{j=1}^{m} H_j)$$

and

$$\nu(G \setminus \Gamma(G)) \nu(C_n) + \sum_{j=1}^{m} \nu(H_j).$$

Theorem 2.5.28. [1, Corollary 3.11, Corollary 3.9] *Let G be unicyclic graph with cycle C_n.*

(1) $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if* $n \equiv 0, 1 (\mod 3)$ *or* $\nu(G \setminus \Gamma(G)) < \nu(G)$.

(2) $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $n \equiv 2 (\mod 3)$ *and* $\nu(G \setminus \Gamma(G)) = \nu(G)$.

In particular they proved the following for the powers of edge ideals of unicyclic graphs.

Theorem 2.5.29. *Let G by a unicyclic graph. Then for all $q \geq 1$,*

$$\operatorname{reg} I(G)^q = 2q + \operatorname{reg} I(G) - 2.$$

Chapter 3

Regularity of Products: Determinantal Ideals of Hankel Matrices

The determinantal ideals have been a central topic in commutative algebra and algebraic geometry for decades. The basic notions of determinantal ideals are described thoroughly in the book [18]. In relatively recent works, [30], [14] and [9] the authors investigated families of determinantal ideal with linear products. The result of the first article is presented in Example 2.4.9. The fascination of the above works is that they give precise information on products of ideals of their case studies like Gröbner basis, a nice primary decompositions and nice description on their associated multi-Rees algebras like Sagbi bases, a Gröbner bases for the defining ideal of multi-Rees algebra, Koszulness, normality and Cohen-Macaulayness. In this chapter, we improve the result of [30] also stated in Example 2.4.9.

3.1 The Study of Standard Forms

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring equipped with standard grading and degree lexicographical term order with respect to $x_1 > \ldots > x_n$. We denote the term order of S by $\leq_\tau$. By $\mathbf{X}_t^{(1,n)}$, where $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$, we denote the *Hankel matrix* with t rows and entries $x_1, \ldots, x_n$:

$$\mathbf{X}_t^{(1,n)} = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_{n-t+1} \\ x_2 & x_3 & \ldots & \ldots & \ldots \\ x_3 & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_t & \ldots & \ldots & \ldots & x_n \end{pmatrix}$$

For $s \leq t$, the standard notation for minors of $\mathbf{X}_t^{(1,n)}$ is $[a_1, \ldots, a_s | b_1, \ldots, b_s]$ where $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ are strictly increasing sequences of row indices and column indices respectively. A minor of the form $[1, \ldots, t | b_1, \ldots, b_t]$ is called a maximal minor of $\mathbf{X}_t^{(1,n)}$. Unless otherwise is stated, $\text{in}(f)$ will always denote the leading term of the polynomial f. Similarly, $\text{in}(I)$ will always denote the leading term ideal of the ideal I. It is clear that the leading term of a minor is the product of the entries laying on the main diagonal. The term orders satisfying this criteria are known as diagonal term orders in the literature. The arguments of this chapter holds identically for any given diagonal term order. Therefore, our results are true for any given diagonal term order. Let i and j be distinct natural numbers. We define the partial order $\leq_1$ by

$$i \leq_1 j \text{ if and only if } i + 1 \leq j.$$

When $i + 1 < j$, we denote the above partial order by $<_1$. A *chain* is a sequence in $\{1, \ldots, n\}$ like $a = a_1, \ldots, a_r$ such that $a_1 <_1 a_2 <_1 \ldots <_1 a_r$. Similarly, we say a monomial $x_a = x_{a_1} \ldots x_{a_r}$ is a chain if its indices form a chain. A given monomial $x_{a_1} \ldots x_{a_r}$ is a chain if and only if it is the leading term of a minor of some $\mathbf{X}_t^{(1,n)}$ with $r \leq t$. This corresponding minor is unique if and only if $r = t$. We will denote the family of all Hankel matrices with entries $x_1, \ldots, x_n$ by $\mathbf{X}^{(1,n)}$. Let $I_{(1,n),t}$ denote the ideal generated by the maximal minors of $\mathbf{X}_t^{(1,n)}$. We denote the family of all determinantal ideals of the matrices of the family $\mathbf{X}^{(1,n)}$ by $\mathcal{F}^{(1,n)}$. It is clear that one can repeat the same constructions for the sequences of indeterminates $x_1, \ldots, x_{n-1}$, $x_2, \ldots, x_n$ and $x_2, \ldots, x_{n-1}$ and construct the families of matrices $\mathbf{X}^{(1,n-1)}$, $\mathbf{X}^{(2,n)}$ and $\mathbf{X}^{(2,n-1)}$ and of course $\mathcal{F}^{(1,n-1)}$, $\mathcal{F}^{(2,n)}$ and $\mathcal{F}^{(2,n-1)}$. We will refer to the family

$$\mathbf{X} = \mathbf{X}^{(1,n)} \cup \mathbf{X}^{(1,n-1)} \cup \mathbf{X}^{(2,n)} \cup \mathbf{X}^{(2,n-1)}$$

by the family of *close cuts Hankel matrices* or in short *close cuts*. In this work, we investigate the following family:

$$\mathcal{F} = \mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$$

We will refer to the tuples $(1, n), (1, n-1), (2, n)$ and $(2, n-1)$ by *labels*. To keep a simple notation, we choose a general notation for them like:

$$\sigma = (\sigma_1, \sigma_2) \in \{(1, n), (1, n-1), (2, n), (2, n-1)\}$$

Given a chain $a = a_1, \ldots, a_r$, there is at least one label, say σ, such that the matrix $\mathbf{X}_r^\sigma$ contains a maximal minor whose initial term is x_a. For many cases, this label is not unique. A *labeled chain* (σ, a) is a chain together with a fixed label. For a given labeled chain, we keep the label fixed unless otherwise is clearly stated.

In our treatment, we need to put an order on the labeled chains. To this end, we order the set of labels lexicographically like the following:

$$(1, n-1) > (1, n) > (2, n-1) > (2, n)$$

Let (σ, a) and (γ, b) be labeled chains. We say $(\sigma, a) \geq_c (\gamma, b)$ if and only if $\sigma > \gamma$ or $\sigma = \gamma$ and $a >_\tau b$. We denote a pair of labeled chains by $(\sigma, a) \geq_c (\gamma, b)$ when we need to emphasis on their order.

Definition 3.1.1. *Let $(\sigma_1, a^{(1)}), (\sigma_2, a^{(2)}), \ldots, (\sigma_k, a^{(k)})$ be a set of labeled chains where $a^{(i)} = a_1^{(i)}, \ldots, a_{r_i}^{(i)}$ for all $i\colon 1, \ldots, l$. A* tabel *$\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ is a set of labeled chains such that $(\sigma_i, a^{(i)}) \geq_c (\sigma_{i+1}, a^{(i+1)})$ for all $i\colon 1, \ldots, l-1$. We will refer to $(\sigma_i, a^{(i)})$ by the i-th row of $\mathcal{A}$.*

Notation 3.1.2. *Throughout this chapter, we reserve r_i to denote the length of the i-th row of some tabel $\mathcal{A}$. The symbol σ_i is reserved for the notation of the label of the i-th row of $\mathcal{A}$. We reserve the letter l to denote the number of the rows of $\mathcal{A}$. In particular, when we mention chains a and b, we denote the length of a by r and the one of b by s.*

Definition 3.1.3. *Let $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ be a tabel. The* shape *of $\mathcal{A}$ is the sequence $(\sigma_1, r_1), \ldots, (\sigma_l, r_l)$.*

Recall that a tabel $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ contains the information on the monomial $\prod_{\mathcal{A}} x_{a_i}$ together with the supporting labels. In the treatment of this section, we need to transform $\mathcal{A}$ to an other tabel, say $\mathcal{B}$, in such a way that both tabels encode the same monomial and have the same shape. The following function Δ controls the shape of $\mathcal{A}$ during this transformation. The functions Ω_d, Ω_c and Ω_ad control the iteration through the transformations of pairs of rows.

Definition 3.1.4. *Let $(\sigma, a) \geq_c (\gamma, b)$ be a pair of labeled chains. We define*

$$\Delta\{(\sigma, a) \geq_c (\gamma, b)\} = \begin{cases} 1 & \text{if} \quad r \leq s, a_r = n \text{ and } \gamma_2 = n-1, \\ & \text{or} \quad r > s, b_s = n \text{ and } \sigma_2 = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Omega_d\{(\sigma, a) \geq_c (\gamma, b)\} = \begin{cases} r - \Delta\{(\sigma, a) \geq_c (\gamma, b)\} & \text{if} \quad r < s, \\ s - 1 & \text{if} \quad r \geq s. \end{cases}$$

$$\Omega_c\{(\sigma, a) \geq_c (\gamma, b)\} = \begin{cases} r - \Delta\{(\sigma, a) \geq_c (\gamma, b)\} & \text{if} \quad r \leq s, \\ s & \text{if} \quad r > s. \end{cases}$$

$$\Omega_{ad}\{(\sigma, a) \geq_c (\gamma, b)\} = \begin{cases} r - 1 & \text{if} \quad r \leq s, \\ s - \Delta\{(\sigma, a) \geq_c (\gamma, b)\} & \text{if} \quad r > s. \end{cases}$$

In case there is no confusion, we use $\Delta, \Omega_d, \Omega_c$ and Ω_{ad} instead of $\Delta\{(\sigma, a), (\gamma, b)\}$, $\Omega_d\{(\sigma, a), (\gamma, b)\}$, $\Omega_c\{(\sigma, a), (\gamma, b)\}$ and $\Omega_{ad}\{(\sigma, a), (\gamma, b)\}$ respectively. We present these functions in the following tabel:

	Ω_d	Ω_c	Ω_{ad}
$r < s$	$r - \Delta$	$r - \Delta$	$r - 1$
$r > s$	$s - 1$	s	$s - \Delta$
$r = s$	$s - 1$	$s - \Delta$	$s - 1$

Let $a = a_1, \ldots, a_r$ be a chain. We define $\mathscr{L}(a) = \bigcup_{2\leq i\leq r}\{a_i - 1, a_i\}$. Let $(\sigma, a) \geq_c (\gamma, b)$ be a pair labeled chains. We have the following relations:

Diagonal relation: The pair $(\sigma, a) \geq_c (\gamma, b)$ has diagonal relations if $a_h > b_k$, for some $1 \leq h \leq \Omega_{\mathrm{d}}$ and $h+1 \leq k \leq s$ and $a_h \notin \mathscr{L}(b)$. If $(\sigma, a) \geq_c (\gamma, b)$ does not have any diagonal relations, we say it is *diagonal sorted*.

Column-wise relations: The pair $(\sigma, a) \geq_c (\gamma, b)$ has column-wise relations if it is diagonal sorted and $a_h > b_h$ for some $1 \leq h \leq \Omega_{\mathrm{c}}$ and $a_h \notin \mathscr{L}(b)$. If $(\sigma, a) \geq_c (\gamma, b)$ does not have any column-wise relations, we say it is *column-wise sorted*.

Anti Diagonal relations: The pair $(\sigma, a) \geq_c (\gamma, b)$ has anti-diagonal relations if it is column-wise sorted and $b_h > a_k$ for some $1 \leq h \leq \Omega_{\mathrm{ad}}$ and $h+1 \leq k \leq r$ and $b_h \notin \mathscr{L}(a)$. If $(\sigma, a) \geq_c (\gamma, b)$ does not have any anti-diagonal relations, we say it is *anti-diagonal sorted*.

With respect to the above relations, we define what we consider as the standard form.

Definition 3.1.5. *The pair $(\sigma, a) \geq_c (\gamma, b)$ is a* standard form *if it satisfies the following conditions:*

(1) $a_i \leq b_{i+1}$ for $i\colon 1, \ldots, \Omega_d$ or $a_h > b_{h+1}$ for some h and $a_h, \ldots, a_{\Omega_d} \in \mathscr{L}(b)$;

(2) $a_i \leq b_i$ for $i\colon 1, \ldots, \Omega_c$ or $a_h > b_h$ for some h and $a_h, \ldots, a_{\Omega_c} \in \mathscr{L}(b)$;

(3) $b_i \leq a_{i+1}$ for $i\colon 1, \ldots, \Omega_{ad}$ or $b_h > a_{h+1}$ for some h and $b_h, \ldots, b_{\Omega_{ad}} \in \mathscr{L}(a)$.

We say a tabel $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ is a standard form *if the following conditions hold for $1 \leq s < t \leq k$:*

(1) $a_i^{(s)} \leq a_{i+1}^{(t)}$ for $i\colon 1, \ldots, \Omega_d$ or $a_h^{(s)} > a_{h+1}^{(t)}$ for some h and $a_h^{(s)}, \ldots, a_{\Omega_d}^{(s)} \in \mathscr{L}(a^{(t)})$;

(2) $a_i^{(s)} \leq a_i^{(t)}$ for $i\colon 1, \ldots, \Omega_c$ or $a_h^{(s)} > a_h^{(t)}$ for some h and $a_h^{(s)}, \ldots, a_{\Omega_c}^{(s)} \in \mathscr{L}(a^{(t)})$;

(3) $a_i^{(t)} \leq a_{i+1}^{(s)}$ for $i\colon 1, \ldots, \Omega_{ad}$ or $a_h^{(t)} > a_{h+1}^{(s)}$ for some h and $a_h^{(t)}, \ldots, a_{\Omega_{ad}}^{(t)} \in \mathscr{L}(a^{(s)})$.

In the second part of the Definition 3.1.5, we refer to a tabel satisfying (1), (2) and (3) by diagonal sorted, column-wise sorted and anti-diagonal sorted respectively.

Proposition 3.1.6. *A pair of labeled chains $(\sigma, a) \geq_c (\gamma, b)$ is a standard form if and only if it is diagonal sorted, column-wise sorted and anti-diagonal sorted.*

Proof. If $(\sigma, a) \geq_c (\gamma, b)$ is a standard form, it is clear that it is diagonal sorted, column-wise sorted and anti-diagonal sorted. It remains to prove the other direction.
Let $d = \Omega_d$. We show Definition 3.1.5 (1) holds. Suppose $a_h > b_{h+1}$ for some $1 \leq h \leq d$. The case $h = d$ is trivial. Since the pair $(\sigma, a) \geq_c (\gamma, b)$ is reduced modulo the relations, we have $a_h \in \mathscr{L}(b)$. Thus, there exists unique $h + 1 < t \leq s$ such that $a_h \in \{b_t - 1, b_t\}$. Since $a_h <_1 a_{h+1}$ we have $a_h \leq b_t < a_{h+1}$. Looking at the bounds of t and the fact that $(\sigma, a) \geq_c (\gamma, b)$ is is diagonal sorted, we have $a_{h+1} \in \mathscr{L}(b)$. By repeating this argument, we obtain $a_{h+1}, \ldots, a_d \in \mathscr{L}(b)$. Therefore the first part of the definition holds.
Let $d = \Omega_c$. We show Definition 3.1.5 (2) holds. Suppose $a_h > b_h$ for some $1 \leq h \leq d$. The case $h = d$ is trivial. Since $(\sigma, a) \geq_c (\gamma, b)$ is reduced modulo the relations, we have $a_h \in \mathscr{L}(b)$. So there exists $h < t \leq s$ such that $a_h \in \{b_t - 1, b_t\}$. From $a_h <_1 a_{h+1}$, we have $a_h \leq b_t < a_{h+1}$. Since $(\sigma, a) \geq_c (\gamma, b)$ is reduced modulo the relations (in particular it is diagonal sorted and column-wise sorted), we have $a_{h+1} \in \mathscr{L}(b)$. Repeating this argument gives $a_{h+1}, \ldots, a_d \in \mathscr{L}(b)$.
Let $d = \Omega_{ad}$. We show Definition 3.1.5 (3) holds. Suppose $b_h > a_{h+1}$ for some $1 \leq h \leq d$. The case $h = d$ is trivial. Since $(\sigma, a) \geq_c (\gamma, b)$ is reduced modulo relations, we have $b_h \in \mathscr{L}(a)$ which gives a unique $h + 1 < t \leq r$ such that $b_h \in \{a_t - 1, a_t\}$. From $b_h <_1 b_{h+1}$, we get $b_h \leq a_t < b_{h+1}$. Since $(\sigma, a) \geq_c (\gamma, b)$ is reduced modulo relations (in particular it is anti-diagonal sorted), we have $b_{h+1} \in \mathscr{L}(a)$. By repeating this argument, we get $b_h, \ldots, b_d \in \mathscr{L}(a)$. □

Remark 3.1.7. *The following are easy to check:*

(1) The tabel $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ is a standard form if and only if the pair $(\sigma_i, a^{(i)}) \geq_c (\sigma_j, a^{(j)})$ is a standard form for all $1 \leq i < j \leq k$.

(2) If the tabel $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ is a standard form and $(\sigma_i, a^{(i)})$ and $(\sigma_j, a^{(j)})$ are some rows of $\mathcal{A}$, then

(i) If $r_i < r_j$, when $a^{(i)}_{r_i} \leq \sigma_{j,2}$ we have $a^{(i)}_{r_i} \leq a^{(j)}_{r_j}$, otherwise we have $a^{(i)}_{r_i} > a^{(j)}_{r_j}$.

(ii) If $r_i = r_j$ and $i < j$, when $\Delta = 0$ we have $a^{(i)}_{r_i} \leq a^{(j)}_{r_j}$, otherwise $a^{(i)}_{r_i} > a^{(j)}_{r_j}$.

Lemma 3.1.8. *Any pair of labeled chains $(\sigma, a) \geq_c (\gamma, b)$ transforms to a diagonal sorted form $(\sigma, c) \geq_c (\gamma, d)$ of the same shape.*

Proof. Suppose $(\sigma, a) \geq_c (\gamma, b)$ is not diagonal sorted. There exists $1 \leq h \leq \Omega_d$ and $h + 1 \leq k \leq s$ such that $a_h > b_k$ and $a_h \notin \mathscr{L}(b)$. Assume (h, k) is minimum with respect $\leq_\tau$. Recall that $\leq_\tau$ is lex order induced by $x_1 > x_2 > \ldots > x_n$.We prove the statement by induction on (h, k). There exists $0 \leq v \leq h - 1$ such that for every $0 \leq i \leq v$, we have $a_{h-i} > b_{k-i}$ and $a_{h-v-1} \leq b_{k-v-1}$ if $v \neq h - 1$. Clearly, for $v \neq h - 1$, we have $a_{h-v-1} \leq b_{k-v-1} <_1 b_{h-v}$ and $b_k < a_h <_1 a_{h+1}$. Therefore, $\tilde{a} = a_1, \ldots, a_{h-v-1}, b_{k-v}, \ldots, b_k, a_{h+1}, \ldots, a_r$ is a chain. By definition of diagonal relations and Ω_d, one can see that $(\sigma, \tilde{a})$ is a well-defined labeled chain. On the other hand, $b_{k-v-1} <_1 b_{k-v} < a_{h-v}$ and $a_h <_1 b_{k+1}$ from $a_h \notin \mathscr{L}(b)$ and the definition of k. Therefore, $\tilde{b} = b_1, \ldots, b_{k-v-1}, a_{h-v}, \ldots, a_h, b_{k+1}, \ldots, b_s$ is a chain. By definition of diagonal relations and Ω_d, one can see that $(\gamma, \tilde{b})$ is well-defined. Moreover, from the above and the definition of $\geq_c$, one can see $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$. Let $(\tilde{h}, \tilde{k})$ be the analogous of (h,k) for the new pair $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$. It remains to show that $(\tilde{h}, \tilde{k}) \geq_\tau (h,k)$, hence the induction yields. By construction of $\tilde{a}$ and $\tilde{b}$, we have $\tilde{a}_{h+1}, \tilde{a}_{h+2}, \ldots, \tilde{a}_r = a_{h+1}, a_{h+2}, \ldots, a_r$ and $\tilde{b}_{k+1}, \tilde{b}_{k+2}, \ldots, \tilde{b}_s = b_{k+1}, b_{k+2}, \ldots, b_s$. Let $\tilde{h} > h$ be the case. Clearly, $h+1 < \tilde{h}+1 \leq \tilde{k}$. If $k < \tilde{k}$, we have $\tilde{b}_{\tilde{k}} = b_{\tilde{k}} < a_{\tilde{h}} = \tilde{a}_{\tilde{h}}$ contradicts the definition of (h,k). If $h+1 < \tilde{h}+1 \leq \tilde{k} \leq k$, from $b_k < a_h < a_{\tilde{h}} = \tilde{a}_{\tilde{h}}$ we have a contradiction with the definition of (h,k). Therefore, $\tilde{h} \leq h$. Suppose $\tilde{h} = h$. If $\tilde{k} > k$ $(\tilde{k} = k)$ we have $b_{\tilde{k}} = \tilde{b}_{\tilde{k}} < \tilde{a}_{\tilde{h}} = \tilde{a}_h = b_k$ $(a_h = \tilde{b}_k = \tilde{b}_{\tilde{k}} < \tilde{a}_{\tilde{h}} = a_{\tilde{h}} = b_k)$ which is a contradiction. Therefore, $\tilde{k} < k$. This proves that the induction on (h,k) yields. □

Lemma 3.1.9. *A diagonal sorted pair like $(\sigma, a) \geq_c (\gamma, b)$ always transforms to a column-wise sorted form like $(\sigma, c) \geq_c (\gamma, d)$ of the same shape.*

Proof. If $(\sigma, a) \geq_c (\gamma, b)$ is not column-wise sorted, there exists $1 \leq h \leq \Omega_c$ such that $a_h > b_h$ and $a_h \notin \mathscr{L}(b)$. Let h be the maximum index with this property. There exists $0 \leq v \leq h-1$ such that $a_{h-i} > b_{h-i}$ for all $0 \leq i \leq v$ and $a_{h-v-1} \leq b_{h-v-1}$ if $v \neq h-1$. By the definition of h, it is clear that $a_{h-v-1} \leq b_{h-v-1} <_1 b_{h-v}$ and $b_h < a_h <_1 a_{h+1}$. Therefore, $\tilde{a} = a_1, \ldots, a_{h-v-1}, b_{h-v}, \ldots, b_h, a_{h+1}, \ldots, a_s$ is a chain. From the definition of column-wise relations and Ω_c, one can see that $(\sigma, \tilde{a})$ is a well-defined labeled chain. On the other hand, $b_{h-v-1} <_1 b_{h-v} < a_{h-v}$ and $a_h <_1 b_{h+1}$ from $a_h \notin \mathscr{L}(b)$ and the definition of h. Therefore, $\tilde{b} = b_1, \ldots, b_{h-v-1}, a_{h-v}, \ldots, a_h, b_{h+1}, b_s$ is a chain. From the definition of column-wise relations and Ω_c, one can deduce that $(\gamma, \tilde{b})$ is a well-defined labeled chain. We need to show that $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ is diagonal sorted and the induction on h converges.
Diagonal relations: Let $\tilde{a}_i > \tilde{b}_j$ for some $1 \leq i \leq \Omega_d$ and $i+1 \leq j \leq s$ and $\tilde{a} \notin \mathscr{L}(\tilde{b})$. The case $h - v \leq i \leq \Omega_d$ leads to trivial contradictions. From the definition of v in the previous part, it implies $a_i < b_{h-v-1}$. Hence $1 \leq i \leq h-v-3$ and $i+1 \leq j \leq h-v-2$. So, $\tilde{a}_i = a_i$ and $\tilde{b}_j = b_j$. Therefore, from the definition of $(\sigma, a) \geq_c (\gamma, b)$, there exists $j+1 \leq t \leq s$ such that $b_t - 1 \leq a_i \leq b_t$. Note that, $a_i < b_{h-v-1}$ again implies $t \leq h-v-2$. Therefore, $b_t = \tilde{b}_t$. This implies $a_i = \tilde{a}_i \in \mathscr{L}(\tilde{b})$ which is a contradiction. Hence, any diagonal relation in $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ admits a contradiction.
Convergence: Let $\tilde{h}$ be the analogues of h for the new pair $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$. From the definition of h, $\tilde{a}$, $\tilde{b}$ and the fact that $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ is diagonal sorted, it is clear that $\tilde{h} < h$. Therefore, the reverse induction on h yields the pair $(\sigma, c) \geq_c (\gamma, d)$. Since every step of the above process gives a smaller h and the pair obtained at every step remains diagonal sorted, one can deduce that $(\sigma, c) \geq_c (\gamma, d)$ is column-wise sorted. □

Lemma 3.1.10. *A column-wise sorted pair like $(\sigma, a) \geq_c (\gamma, b)$ always transforms to an anti-diagonal sorted (standard form) like $(\sigma, c) \geq_c (\gamma, d)$ of the same shape.*

Proof. If $(\sigma, a) \geq_c (\gamma, b)$ is not anti-diagonal sorted, then there exists $1 \leq h \leq \Omega_{ad}$ and $h+1 \leq k \leq r$ such that $b_h > a_k$ and $b_h \notin \mathscr{L}(a)$. Assume (h,k) is minimum with respect $\leq_\tau$.

Recall that $\leq_\tau$ is lex order induced by $x_1 > x_2 > \ldots > x_n$. There exists $0 \leq v \leq h-1$, such that $b_{h-i} > a_{k-i}$ for all $0 \leq i \leq v$ and $b_{h-v-1} \leq a_{k-v-1}$ if $v \neq h-1$. We have $a_{k-v-1} <_1 a_{k-v} < b_{h-v}$. From $b_h \notin \mathscr{L}(a)$ and the definition of Ω_{ad}, we have $b_h <_1 a_{k+1}$ (or $b_h <_1 \sigma_2$). Therefore, $\tilde{a} = a_1, \ldots, a_{k-v-1}, b_{h-v}, \ldots, b_h, a_{k+1}, \ldots, a_r$ is a chain and $(\sigma, \tilde{a})$ is a well-defined labeled chain. On the other hand, we have $b_{h-v-1} \leq a_{k-v-1} <_1 a_{k-v}$. Moreover, $a_k < b_h <_1 b_{h+1}$ (or $a_k < b_h \leq \gamma_2$ when $h = s$). Thus, $\tilde{b} = b_1, \ldots, b_{h-v-1}, a_{k-v}, \ldots, a_k, b_h, \ldots, b_s$ is a chain and $(\gamma, \tilde{b})$ is a well-defined labeled chain. To complete the proof, we need to show that $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ is diagonal sorted and column-wise sorted. Moreover, we need to show that the induction on (h, k) converge.
Diagonal relations: Let $\tilde{a}_i > \tilde{b}_j$ with $1 \leq i \leq \Omega_d$ and $i+1 \leq j \leq s$ and $\tilde{a}_i \notin \mathscr{L}(\tilde{b})$. Assume (i, j) is minimum with respect ot $\leq_\tau$. Recall that v is defined in the first part of the proof. The case $h - v - 1 \leq i \leq \Omega_d$ gives trivial contradictions. Suppose $1 \leq i \leq h - v - 2$. The case $h - v \leq j \leq s$ also gives trivial contradictions. Suppose $1 \leq i \leq h - v - 2$ and $i+1 \leq j \leq h-v-1$. This implies $\tilde{a}_i = a_i$ and $\tilde{b}_j = b_j$. From the definition of $(\sigma, a) \geq_c (\gamma, b)$ and $a_i = \tilde{a}_i > \tilde{b}_j = b_j$ we have $a_i \in \mathscr{L}(b)$. Hence $b_t - 1 \leq a_i \leq b_t$ for unique $i+1 \leq t \leq s$. If $h < t \leq s$, we have a contradiction with $\tilde{a}_i \notin \mathscr{L}(b)$. If $h - v \leq t \leq h$, there exists a unique $0 \leq t' \leq v$ such that $h - t' = t$ and $b_t > a_{k-t'}$. This implies $b_t - 1 \leq a_i <_1 a_{h-t'} < b_t$ which is a contradiction. Hence $1 \leq t \leq h - v$ which is again a contradiction with $\tilde{a}_i \notin \mathscr{L}(\tilde{b})$. Therefore existence of any diagonal relation in $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ leads to a contradiction.
Column-wise relations: We have already seen that $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ is diagonal sorted. Let $\tilde{a}_i > \tilde{b}_i$ for $1 \leq i \leq \Omega_c$ and $\tilde{a}_i \notin \mathscr{L}(\tilde{b})$. Assume i is maximum. The case $h-v \leq i \leq \Omega_c$ implies trivial contradictions. Suppose $1 \leq i \leq h-v-1$. This gives $\tilde{a}_i = a_i$ and $\tilde{b}_i = b_i$. Hence from the definition of $(\sigma, a) \geq_c (\gamma, b)$, we have $a_i \in \mathscr{L}(b)$. Therefore for unique $i+1 \leq t \leq s$, we have $b_t - 1 \leq a_i \leq b_t$. The case $h \leq t \leq s$ implies a contradiction with $\tilde{a}_i \notin \mathscr{L}(\tilde{b})$. Suppose $h - v \leq t \leq h$. Hence there is a unique $0 \leq t' \leq v$ such that $h - t' = t$ and $a_{k-t'} < b_t$. Therefore, $b_t - 1 \leq a_i <_1 a_{h-t'} < b_t$. This bound contradicts the definition of $<_1$. Hence, $1 \leq t \leq h - v$ which, again, contradicts $\tilde{a}_i \notin \mathscr{L}(\tilde{b})$. Therefore any column-wise relation in $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$ leads to a contradiction.
Convergence: It remains to show that the induction on (h, k) converges. Let $(\tilde{h}, \tilde{k})$ be the analogous of (h, k) for the new pair $(\sigma, \tilde{a}) \geq_c (\gamma, \tilde{b})$. It is enough to show $(\tilde{h}, \tilde{k}) \geq_\tau (h, k)$. By the construction of $\tilde{a}$ and $\tilde{b}$, we have $\tilde{a}_{k+1}, \tilde{a}_{k+2}, \ldots, \tilde{a}_r = a_{k+1}, a_{k+2}, \ldots, a_r$ and $\tilde{b}_{h+1}, \tilde{b}_{h+2}, \ldots, \tilde{b}_s = b_{h+1}, b_{h+2}, \ldots, b_s$. Let $\tilde{h} > h$ be the case. Clearly, $h + 1 < \tilde{h} + 1 \leq \tilde{k}$. If $k < \tilde{k}$, we have $\tilde{a}_{\tilde{k}} = a_{\tilde{k}} < b_{\tilde{h}} = \tilde{b}_{\tilde{h}}$ which is a contradiction with the definition of (h, k). If $h+1 < \tilde{h}+1 \leq \tilde{k} \leq k$, from $a_k < b_h < b_{\tilde{h}}$ we have a contradiction with the definition of (h, k). Therefore, $\tilde{h} \leq h$. Suppose $\tilde{h} = h$. If $\tilde{k} > k$ (or $\tilde{k} = k$) we have $a_{\tilde{k}} = \tilde{a}_{\tilde{k}} < \tilde{b}_{\tilde{h}} = \tilde{b}_h = a_k$ (or $b_h = \tilde{a}_k = \tilde{a}_{\tilde{k}} < \tilde{b}_{\tilde{h}} = \tilde{b}_h = a_k$) which is a contradiction with the definition of (h, k). Therefore, $\tilde{k} < k$. This proves that the induction on (h, k) converges. $\square$

Example 3.1.11. *The tabel*

$$\mathcal{A} = \begin{pmatrix} (1,30) & \vline & 1 & 4 & 18 & 24 & 30 & & & \\ (2,29) & \vline & 5 & 7 & 11 & 15 & 17 & 19 & 22 & 28 \end{pmatrix}$$

is not standard. We have $\Delta = 1$ and $(4, 7)$ are the analogues of (h, k) is the proof of

Lemma 3.1.8. After applying Lemma 3.1.8, we obtain the diagonal sorted tabel

$$\mathcal{A}' = \begin{pmatrix} (1,30) & 1 & 4 & 15 & 18 & 30 & & & \\ (2,29) & 5 & 7 & 11 & 17 & 19 & 22 & 24 & 28 \end{pmatrix}$$

The tabel $\mathcal{A}'$ has column-wise relations. The analogous of h in the proof of Lemma 3.1.9 is 3*. By applying Lemma 3.1.9, we obtain the diagonal sorted and column-wise sorted tabel*

$$\mathcal{A}'' = \begin{pmatrix} (1,30) & 1 & 4 & 11 & 18 & 30 & & & \\ (2,29) & 5 & 7 & 15 & 17 & 19 & 22 & 24 & 28 \end{pmatrix}$$

This tabel has anti-diagonal relations. The analogous of (h,k) in the proof of Lemma 3.1.10 is $(1,2)$. Finally, by applying Lemma 3.1.10, we obtain the following standard tabel:

$$\mathcal{B} = \begin{pmatrix} (1,30) & 1 & 5 & 11 & 18 & 30 & & & \\ (2,29) & 4 & 7 & 15 & 17 & 19 & 22 & 24 & 28 \end{pmatrix}$$

Let $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ be a tabel. The coordinates of the "cells" are denoted by (i,j) where i and j are row and column indices respectively.

Algorithm 3.1.12. *Let $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ be a tabel. We assign a label to each coordinate of $\mathcal{A}$ by starting from the cell with coordinate $(1,1)$ (i.e first row and first column) and increasing the labels as we go through every cell in the first column. Then we proceed by increasing the labels for next columns. Let $\mathrm{P}(i,j)$ be the function that assigns a label to the coordinate (i,j) where i is the row index and j is the column index.*

Example 3.1.13. *Let $\mathcal{A}$ be a tabel with shape $(\sigma_1, 8), (\sigma_2, 4), (\sigma_3, 6), (\sigma_4, 5)$. The Algorithm 3.1.12, labels the cells of $\mathcal{A}$ like the following:*

$$\mathcal{A} = \left(\begin{array}{c|cccccccc} \sigma_1 & 1 & 5 & 9 & 13 & 17 & 20 & 22 & 23 \\ \sigma_2 & 2 & 6 & 10 & 14 & & & & \\ \sigma_3 & 3 & 7 & 11 & 15 & 18 & 21 & & \\ \sigma_4 & 4 & 8 & 12 & 16 & 19 & & & \end{array}\right)$$

We say the entry $a_j^{(i)}$ in $\mathcal{A}$ is stable modulo diagonal relations if the coordinate of $a_j^{(i)}$ remains the same after any pairwise diagonal relations transformations of rows of $\mathcal{A}$. Let stable entries modulo column-wise relations and anti-diagonal relations be defined accordingly. It is clear that $\mathcal{A}$ is standard if and only if all the entries of $\mathcal{A}$ are stable modulo diagonal relations, column-wise relations and anti-diagonal relations.

Proposition 3.1.14. *Let $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ be a tabel. Then, $\mathcal{A}$ always reduces to a standard form with the same shape.*

Proof. We treat transformation modulo each type of relations separately. Let $c_1 \leq \ldots \leq c_l$ be the entries of $\mathcal{A}$ in order where the entries are labeled by Algorithm 3.1.12. We argue by revers induction on $1 \leq t \leq l$. Suppose $\mathcal{A}$ has some diagonal relations.

- Let $t = l$. Note that since c_l is the largest entry and the rows of $\mathcal{A}$ are chains, this entry is located at the last cell of some row. Let (i, r_i) be the coordinate of c_l. Suppose $c_l = a^{(i)}_{r_i}$ is not stable modulo diagonal relations. There exists some row index $i < j$ such that $a^{(i)}_{r_i} > a^{(j)}_{r_j}$ and $a^{(i)}_{r_i} \leq \sigma_{j,2}$ and of course $r_i < r_j$. Let j be maximum. By applying Lemma 3.1.8 on the rows (i, j), one can reduce the tabel into a new tabel in which c_l is stable modulo diagonal relations by construction.

- Let c_{t+1} be stable modulo diagonal relations for $t < l$. Let (i, h) be the coordinate of $c_t = a^{(i)}_h$. From the definition of diagonal relations, there exists some row index $i < j$ such that $a^{(i)}_h > a^{(j)}_k$ and $a^{(i)}_h \notin \mathscr{L}(a^{(j)})$ for some k with $h + 1 \leq k \leq \Omega_d(i, j)$. Assume j is maximum. By applying Lemma 3.1.8 on the rows (i, j), one can reduce to a new tabel in which c_t is stable modulo diagonal relations by construction.

Let $\mathcal{A}$ be obtained form the previous step. Suppose $\mathcal{A}$ have some column-wise relations.

- Let $t = l$. Note that since c_l is the largest entry and the rows of $\mathcal{A}$ are chains, this entry is located at the last cell of some row. Let (i, r_i) be the coordinate of $c_l = a^{(i)}_{r_i}$. There exists a row index j such that $r_i = r_j$, $a^{(i)}_{r_i} > a^{(j)}_{r_i}$ and $a^{(i)}_{r_i} \leq \sigma_{j,2}$. Let j be maximum. By applying Lemma 3.1.9 on the rows (i, j) and replacing it in $\mathcal{A}$, one can reduce to a new tabel in which c_l is stable modulo diagonal relations and column-wise relations by construction.

- Let c_{t+1} be stable for $t < l$. Suppose c_t is not stable modulo column-wise relations. Let (i, h) be the coordinate of $c_t = a^{(i)}_h$. There exists some row j such that $a^{(i)}_h > a^{(j)}_h$ and $a^{(i)}_h \notin \mathscr{L}(a^{(j)})$. Assume that j is maximum. By applying Lemma 3.1.9 on the rows (i, j) and replacing it in $\mathcal{A}$, one can reduce to a tabel in which c_t is stable modulo diagonal relations and column-wise relations by construction.

Finally, assume $\mathcal{A}$ is obtained by applying last two steps. Meaning that $\mathcal{A}$ is diagonal sorted and column-wise sorted. Suppose $\mathcal{A}$ has anti-diagonal relations.

- Let $t = l$. Note that since c_l is the largest entry and the rows of $\mathcal{A}$ are chains, this entry is located at the last cell of some row. Let (i, r_i) be the coordinate of $c_l = a^{(i)}_{r_i}$. There exists a row index $j < i$ such that $a^{(i)}_{r_i} > a^{(j)}_{r_j}$ and $a^{(i)}_{r_i} \leq \sigma_{j,2}$ and of course $r_i < r_j$. Let r_j be maximum and j be maximum given r_j. By applying Lemma 3.1.10 on the rows (i, j) and replacing it in $\mathcal{A}$, one can reduce to a new tabel in which c_l is stable modulo diagonal, column-wise and anti-diagonal relations by construction.

- Let c_{t+1} be stable for $t < l$. Let (i, h) be the coordinate of $c_t = a^{(i)}_h$. There exists some row index $j < i$ such that $a^{(i)}_h > a^{(j)}_k$ for some $h + 1 \leq k \leq \Omega_{ad}(i, j)$ and $a^{(i)}_h \notin \mathscr{L}(a^{(j)})$. Let k be maximum and j be maximum given k. By applying Lemma 3.1.10 on the rows (i, j) and replacing it in $\mathcal{A}$, one can reduce to a new tabel in which c_t is stable modulo diagonal, column-wise and anti-diagonal relations by construction.

□

Lemma 3.1.15. *Let $\mathcal{A} = \{(\sigma_i, a^{(i)})\}$ be a tabel. Then, $\mathcal{A}$ always reduces to a unique standard form $\mathcal{B} = \{(\sigma_i, b^{(i)})\}$ of the same shape.*

Proof. Let $\mathcal{A}$ be labeled by Algorithm 3.1.12. We have $\sum_i r_i = l$. Let $\mathcal{B}$ be a standard form reduction of $\mathcal{A}$. Since $\mathcal{A}$ and $\mathcal{B}$ have the same shape, the Algorithm 3.1.12 assigns the same labels to $\mathcal{B}$. Let $c_1 \leq c_2 \leq \ldots \leq c_l$ be the ordered set of the entries of $\mathcal{A}$ and $\mathcal{B}$. Note that the function in Algorithm 3.1.12 is a one to one correspondence. We prove that for all $t\colon 1, \ldots, l$, the entries $b^{(i)}_j$ with $\mathrm{P}(i,j) = t$ is determined uniquely. Therefore, $\mathcal{B}$ is given uniquely. We proceed by revers induction on t.

(1) Let $t = l$. There exists unique coordinate (i,j) such that $\mathrm{P}(i,j) = l$. Recall that by Algorithm 3.1.12, (i,j) is the coordinate of the last cell of the longest row. If $c_l \leq \sigma_{i,2}$, from Algorithm 3.1.12 and part (2) of the Remark 3.1.7, we have $b^{(i)}_j = c_l$. If $c_l > \sigma_{i,2}$, from the fact that $\mathcal{B}$ is a well-defined tabel, there exists c_h such that $c_h < c_{h+1} = \ldots = c_l$. Thus $b^{(i)}_j \leq c_h \leq \sigma_{i,2}$. If $b^{(i)}_j < c_h$, there exists some row index of tabel $\mathcal{B}$ like $i' \neq i$ which contains c_h. This yields that there exists either diagonal relations, column-wise relations or anti diagonal relations in $\mathcal{B}$. Which contradicts the definition of $\mathcal{B}$. Hence, $b^{(i)}_j = c_h$.

(2) Let uniqueness of $b^{(i')}_{j'}$ be given for every coordinate with $t < \mathrm{P}(i', j') \leq l$. Let $c_1 \leq c_2 \leq \ldots \leq c_t$ be the remainder of the entries of $\mathcal{A}$ and $\mathcal{B}$ relabeled by $1, 2, \ldots, t$. There exists a unique coordinate (i,j) with $\mathrm{P}(i,j) = t$. We always have $b^{(i)}_j \leq c_t$.

- If $j = r_i$ and $c_t \leq \sigma_{i,2}$. There exists some coordinate (i_t, j_t) with $\mathrm{P}(i_t, j_t) \leq t$ and $b^{(i_t)}_{j_t} = c_t$. If $b^{(i)}_j < c_t$, we have $i_t \neq i$ since $b^{(i)}$ is a chain. According to the induction hypothesis and the Algorithm 3.1.12, we have $i_t < i$ and $j_t \leq j$ or $i_t > i$ and $j_t > j$. This means that $(\sigma_{i_t}, b^{(i_t)})$ and $(\sigma_i, b^{(i)})$ have either diagonal relations, column-wise relations or anti diagonal relations. This is a contradiction with the definition of $\mathcal{B}$. So $b^{(i)}_j = c_t$.
  If $c_t > \sigma_{i,2}$, since $\mathcal{B}$ is a well-defined tabel, we can find $c_{t'}$ where $c_{t'} < c_{t'+1} = \ldots = c_t$. We have $b^{(i)}_j \leq c_{t'}$. Let $b^{(i)}_j < c_{t'}$. There exist a unique coordinate $(i_{t'}, j_{t'})$ with $\mathrm{P}(i_{t'}, j_{t'}) \leq t$ such that $b^{(i_{t'})}_{j_{t'}} = c_{t'}$. According to the induction hypothesis and the Algorithm 3.1.12, we have $i_{t'} < i$ and $j_{t'} \leq j$ or $i_{t'} > i$ and $j_{t'} > j$. This means that $(\sigma_{i_{t'}}, b^{(i_{t'})})$ and $(\sigma_i, b^{(i)})$ have either diagonal relations, column-wise relations or anti diagonal relations. This is a contradiction with the definition of $\mathcal{B}$. So $b^{(i)}_j = c_{t'}$.

- If $j \neq r_i$ and $c_t + 1 < b^{(i)}_{j+1}$. There exists a unique coordinate (i_t, j_t) with $\mathrm{P}(i_t, j_t) \leq t$ and $b^{(i_t)}_{j_t} = c_t$. If $b^{(i)}_j < c_t$, we have $i_t \neq i$ since $b^{(i)}$ is a chain. According to the induction hypothesis and the Algorithm 3.1.12, we have $i_t < i$ and $j_t \leq j$ or $i_t > i$ and $j_t > j$. From $c_t + 1 < b^{(i)}_{j+1}$ and $b^{(i)}_j < c_t$, we have $c_t \notin \mathscr{L}(b^{(i)})$. This means that $(\sigma_{i_t}, b^{(i_t)})$ and $(\sigma_i, b^{(i)})$ have either diagonal relations, column-wise relations or anti diagonal relations. This is a contradiction with the definition of $\mathcal{B}$. So $b_j^{(i)} = c_t$. If $c_t + 1 \geq b_{j+1}^{(i)}$, from the fact that $\mathcal{B}$ is a well-defined tabel, we can find $c_{t'} <_1 b_{j+1}^{(i)}$. Let t' be the largest label satisfying this condition. Hence, $b_j^{(i)} \leq c_{t'}$. There exists a unique coordinate $(i_{t'}, j_{t'})$ with $\mathrm{P}(i_{t'}, j_{t'}) \leq t$ such that $b_{j_{t'}}^{(i_{t'})} = c_{t'}$. If $b_j^{(i)} < c_{t'}$, we have $i_{t'} \neq i$ since $b^{(i)}$ is a chain. According to the induction hypothesis and the Algorithm 3.1.12, we have $i_{t'} < i$ and $j_{t'} \leq j$ or $i_{t'} > i$ and $j_{t'} > j$. Since $b_j^{(i)} < c_{t'} <_1 b_{j+1}^{(i)}$, we have $c_{t'} \notin \mathscr{L}(b^{(i)})$. This means that $(\sigma_{i_{t'}}, b^{(i_{t'})})$ and $(\sigma_i, b^{(i)})$ have either diagonal relations, column-wise relations or anti diagonal relations. This is a contradiction with the definition of $\mathcal{B}$. So $b_j^{(i)} = c_{t'}$.

□

Example 3.1.16. *The non-standard tabel*

$$\mathcal{A} = \left(\begin{array}{c|cccccccc} (1,29) & 8 & 12 & 18 & 20 & 22 & & & \\ (1,30) & 2 & 7 & 23 & 25 & 27 & 30 & & \\ (1,30) & 1 & 18 & 23 & 27 & 30 & & & \\ (2,29) & 2 & 5 & 7 & 9 & 13 & 16 & 20 & 25 \\ (2,30) & 8 & 10 & 12 & 17 & 25 & 28 & & \end{array} \right)$$

transforms to the following standard tabel:

$$\mathcal{B} = \left(\begin{array}{c|cccccccc} (1,29) & 1 & 7 & 12 & 18 & 23 & & & \\ (1,30) & 2 & 8 & 12 & 20 & 25 & 30 & & \\ (1,30) & 2 & 8 & 13 & 20 & 27 & & & \\ (2,29) & 5 & 9 & 16 & 18 & 20 & 23 & 25 & 28 \\ (2,30) & 7 & 10 & 17 & 22 & 27 & 30 & & \end{array} \right)$$

Remark 3.1.17. *As we saw, the function Δ in fact controls the well-definity of the transformations of labeled chains with respect to our relations. Moreover, $\geq_c$ decides the order of the rows of the tabels. In other words, for a given pair of chains $a = a_1, \ldots, a_r$ and $b = b_1, \ldots, b_s$, we can make a tabel with first row $a = a_1, \ldots, a_r$ and second row $b = b_1, \ldots, b_s$ with out considering any labels. Let us denote this tabel by (a, b). Now, by omitting the role of Δ by setting $\Delta = 0$, we can always perform Lemma 3.1.8,Lemma 3.1.9, Lemma 3.1.10 and Lemma 3.1.15. In particular, the following holds:*

(I) The tabel (a, b) is standard if and only if

(i) $a_i \leq b_{i+1}$ for $i\colon 1, \ldots, \Omega_d$ or $a_h > b_{h+1}$ for some h and $a_h, \ldots, a_{\Omega_d} \in \mathscr{L}(b)$,

(ii) $a_i \leq b_i$ for $i\colon 1, \ldots, \Omega_c$ or $a_h > b_h$ for some h and $a_h, \ldots, a_{\Omega_c} \in \mathscr{L}(b)$,

(iii) $b_i \leq a_{i+1}$ for $i\colon 1, \ldots, \Omega_{ad}$ or $b_h > a_{h+1}$ for some h and $b_h, \ldots, b_{\Omega_{ad}} \in \mathscr{L}(a)$.

(II) In particular, when $r \geq s$, the tabel (a, b) is standard if and only if

(i) $a_i \leq b_i$ *for all* $1 \leq i \leq s$*;*

(ii) $b_i \leq a_{i+1}$ *or* $b_h > a_{h+1}$ *for some* h *and* $b_h, \ldots, b_s \in \mathscr{L}(a)$.

(III) If (a, b) *is standard, we have* $a_r \leq b_s$ *when* $r \leq s$ *and* $b_s \leq a_r$ *when* $r > s$.

3.2 Sagbi Deformations and Multi-Rees Algebra

In this section, we use the machinery introduced in Section 3.1 to study the multi-Rees algebra of ideals of family $\mathcal{F} = \mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$. Let $I_1, \ldots, I_l$ be ideals of the ring $S = \mathbb{K}[\bar{x}]$ where $\bar{x} := x_1, \ldots, x_n$. Consider The multi-Rees algebra

$$\mathscr{R}(I_1, \ldots, I_l) = S[I_1 t_1, \ldots, I_l t_l] \subset S[\,\bar{t}\,]$$

associated to $I_1, \ldots, I_l$.

Notation 3.2.1. *In this chapter, we denote a maximal minor of the Hankel matrix* $\mathbf{X}_r^{\sigma}$ *by* $[a_1, \ldots, a_r]$ *where* $a_1, \ldots, a_r$ *is the chain of the entries on the main diagonal. Moreover, we use* $I_{\sigma,r}$ *to denote the determinantal ideal of maximal minors of* $\mathbf{X}_r^{\sigma}$*.*

Let $\bar{t}$ be the set of all new indeterminates $t_{\sigma,r}$ over S where σ and r go through labels and lengths of all labeled chains (σ, a) with $a = a_1, \ldots, a_r$. Consider the multi-Rees algebra $\mathscr{R} = \mathscr{R}(I_{\sigma,r} t_{\sigma,r} \colon I_{\sigma,r} \in \mathcal{F})$ of all ideals of the family $\mathcal{F}$ and the multi-Rees algebra $\mathscr{R}^{\mathrm{in}} = \mathscr{R}(\mathrm{in}(I_{\sigma,r}) t_{\sigma,r} \colon I_{\sigma,r} \in \mathcal{F})$. We shall consider $\mathscr{R}$ and $\mathscr{R}^{\mathrm{in}}$ as sub rings of $S[\,\bar{t}\,]$. One can also consider the representation of our Rees algebras as quotients of some polynomial ring.
Let $\bar{z}$ be the set of new indeterminates $z_{\sigma,a}$ over S where (σ, a) runs through all labeled chains. Consider the polynomial ring $R = S[\,\bar{z}\,]$. Recall that we reserve the letters r and s for the lengths of chains $a = a_1, \ldots, a_r$ and $b = b_1, \ldots, b_s$ respectively. Consider the following surjective algebraic homomorphisms:

$$\begin{aligned} \varphi \colon R \to & \mathscr{R}(I_{\sigma,r} t_{\sigma,r} \colon I_{\sigma,r} \in \mathcal{F}) \\ x_i \mapsto & x_i \\ z_{\sigma,a} \mapsto & [a] t_{\sigma,r} \end{aligned}$$

and

$$\begin{aligned} \varphi^{\mathrm{in}} \colon R \to & \mathscr{R}(\mathrm{in}(I_{\sigma,r}) t_{\sigma,r} \colon I_{\sigma,r} \in \mathcal{F}) \\ x_i \mapsto & x_i \\ z_{\sigma,a} \mapsto & x_a t_{\sigma,r} \end{aligned}$$

It is known that the maximal minors of $\mathbf{X}_r^\sigma$ form a Gröbner basis for ideal $I_{\sigma,r}$ with respect to any diagonal term order. Therefore φ^{in} is surjective.
Often the structure of multi-Rees algebras are better understood by looking at their representation as a quotient of a polynomial ring. Consider isomorphisms $R/\ker(\varphi) \simeq \mathscr{R}$ and $R/\ker(\varphi^{\text{in}}) \simeq \mathscr{R}^{\text{in}}$ induced by φ and φ^{in}.
Let l be the cardinality of $\bar{z}$. We equip R with $\mathbb{Z}\oplus\mathbb{Z}^l$ graded setting by considering $\deg(x_i) = e$ and $\deg(z_{\sigma,a}) = e_{\sigma,r}$. Note that e and $e_{\sigma,r}$'s are the standard basis for $\mathbb{Z} \oplus \mathbb{Z}^l$. In order to define φ and φ^{in} as multi-homogeneous algebraic homomorphisms, in $S[\,\bar{t}\,]$ we set $\deg(x_i) = e$ and $\deg(t_{\sigma,r}) = -re + e_{\sigma,r}$. This will set $\mathscr{R}$ and $\mathscr{R}^{\text{in}}$ as standard multi-graded algebras. The multi-graded setting is effective in the proof of the following proposition.

Proposition 3.2.2. *Let $mv \in R$ be a monomial where m is a monomial in $\bar{x}$ and v is a monomial in $\bar{z}$. There exists a unique representation $\varphi^{\text{in}}(mv) = u\prod_{\mathcal{A}}(x_a t_{\sigma,r})$ where u is a monomial in $\bar{x}$ and $\mathcal{A}$ is a tabel of shape $\deg(v)$ such that:*

(i) $\mathcal{A}$ is a standard tabel;

(ii) for every indeterminate x_i in u and every row (σ, a) in $\mathcal{A}$, we have:

(a) $i \leq a_1$ or

(b) $i > a_1$ and either $\sigma_2 < i$ or $i \in \mathscr{L}(a)$.

Proof. From the definition of φ^{in} and the multi-graded setting of R there exists a representation $\varphi^{\text{in}}(mv) = c\prod_{\mathcal{B}}(x_b t_{\sigma,r})$ such that $\mathcal{B}$ has shape $\deg(v)$ and c is a monomial in $\bar{x}$. Assume u is the maximum of such c's with respect to $\leq_\tau$. By virtue of Lemma 3.1.15, we take $\mathcal{A}$ to be the unique standard form of $\mathcal{B}$'s. It remains to show that $\varphi^{\text{in}}(mv) = u\prod_{\mathcal{A}}(x_a t_{\sigma,r})$ satisfies (i) and (ii). The condition (i) is clearly satisfied by the construction of $\mathcal{A}$. Let x_i be an indeterminate of u and (σ, a) a row in $\mathcal{A}$ not satisfying (ii). Then, for some $1 \leq t \leq r$ we have $a_t < i \leq a_{t+1}$ (or $a_r < i \leq \sigma_2$). Replace x_i and x_{a_t} (or x_i and x_{a_r}) and denote the new tabel with $\mathcal{A}'$. Consider the presentation $\varphi^{\text{in}}(mv) = x_{a_t}u/x_i\prod_{\mathcal{A}'}(x_{a'}t_{\sigma,r})$. By virtue of Lemma 3.1.15, we can assume $\mathcal{A}'$ is standard. We have $x_{a_t}u/x_i \geq_\tau u$ which is a contradiction with the definition of u. Hence, $\varphi^{\text{in}}(mv) = u\prod_{\mathcal{A}}(x_a t_{\sigma,r})$ satisfies condition (ii). ☐

We will refer to the tabel $\mathcal{A}$ of the above construction by the standard tabel of mv.

Definition 3.2.3. *Let $mv \in R$ be a monomial where m is a monomial in $\bar{x}$ and v is a monomial in $\bar{z}$. We define $u\prod_{\mathcal{A}} z_{\sigma,a}$ to be the standard form of mv, where $\mathcal{A}$ and u are obtained in Proposition 3.2.2. We say mv is a standard monomial if and only if $mv = u\prod_{\mathcal{A}} z_{\sigma,a}$.*

Remark 3.2.4. *The following holds:*

(i) A monomial $u\prod_{\mathcal{A}} z_{\sigma,a}$ in R is standard if and only if every factor $x_i z_{\sigma,a}$ and $z_{\sigma,a}z_{\gamma,b}$ in $u\prod_{\mathcal{A}} z_{\sigma,a}$ is standard.

Proof. Let $u \prod_{\mathcal{A}} z_{\sigma,a}$ in R be standard and let $x_i z_{\sigma,a}$ and $z_{\sigma,a} z_{\gamma,b}$ be factors in $u \prod_{\mathcal{A}} z_{\sigma,a}$. Note that $z_{\sigma,a} z_{\gamma,b}$ defines a pair of rows in $\mathcal{A}$. From Remark 3.1.7, $z_{\sigma,a} z_{\gamma,b}$ is standard provided $u \prod_{\mathcal{A}} z_{\sigma,a}$ satisfies Proposition 3.2.2 part (i). Moreover, Proposition 3.2.2 part (ii) clearly stats that $x_i z_{\sigma,a}$ is standard. Conversely, let every $x_i z_{\sigma,a}$ and $z_{\sigma,a} z_{\gamma,b}$ be standard monomials. From Remark 3.1.7, the tabel $\mathcal{A}$ is standard given every $z_{\sigma,a} z_{\gamma,b}$ is standard. Now, it remains to show that u and $\mathcal{A}$ satisfies Proposition 3.2.2 part (ii). Let that not be the case. There exists x_i dividing u and a row (σ, a) in $\mathcal{A}$ such that $a_t < i \leq_c a_{t+1}$ (or $a_r < i \leq \sigma_2$). This in fact means that $x_i z_{\sigma,a}$ is non-standard which contradicts our hypothesis. □

(ii) Let mv be a monomial in R with standard form $u \prod_{\mathcal{A}} z_{\sigma,a}$. From Proposition 3.2.2, Lemma 3.1.15 and the isomorphism $R/\ker(\varphi^{\mathrm{in}}) \backsimeq \mathscr{R}^{\mathrm{in}}$, it yields that in a class $\overline{mv} \in R/\ker(\varphi^{\mathrm{in}})$, there exists exactly one standard monomial.

Consider a marked polynomial to be a polynomial $f \in R \setminus \{0\}$ together with a specific term $\mathrm{mark}(f)$ in f. Note that $\mathrm{mark}(f)$ can be any term of f. For a given finite set of marked polynomials like $\mathcal{F}$, we define the reduction algorithm modulo $\mathcal{F}$ in the natural sense. We say that $\mathcal{F}$ is marked coherently if there exists a term order $\prec$ on R such that $\mathrm{mark}(f) = \mathrm{in}_{\prec}(f)$ for all $f \in \mathcal{F}$. It is clear that if $\mathcal{F}$ is marked coherently, then the reduction modulo $\mathcal{F}$ is Noetherian. The following is a classic result.

Theorem 3.2.5. *A finite set $\mathcal{F} \subset R$ of marked polynomials is marked coherently if and only if the reduction modulo $\mathcal{F}$ converges.*

Proof. [56, Theorem 3.12] □

Consider the following finite set of marked polynomials where the marked terms are underlined.

$$G = \Big\{ \underline{z_{\sigma,a} z_{\gamma,b}} - z_{\sigma,c} z_{\gamma,d} \,:\, z_{\sigma,a} z_{\gamma,b} \text{ is a non-standard monomial and its standard form is } z_{\sigma,c} z_{\gamma,d}$$
$$\underline{x_i z_{\sigma,a}} - x_j z_{\sigma,c} \,:\, x_i z_{\sigma,a} \text{ is a non-standard monomial and its standard form is } x_j z_{\sigma,c} \Big\}$$

From Lemma 3.1.15, Remark 3.2.4 and Theorem 3.2.5, we can see that there exists a term order on R which picks the underlined monomials of G as the leading terms. We denote this term order by $\leq_\alpha$. The following is a classic result:

Lemma 3.2.6. *Let $K[Y_1, \ldots, Y_n]$ be a polynomial ring equipped with some term order. Let $J \subset K[Y_1, \ldots, Y_n]$ be an ideal and let $f_1, \ldots, f_s$ be polynomials in J. If the set $\Omega = \{Y^a : Y^a \notin (\mathrm{in}(f_1), \ldots, \mathrm{in}(f_s))\}$ are linearly independent in $K[Y_1, \ldots, Y_n]/J$, then $f_1, \ldots, f_s$ is a Gröbner basis of J with respect to the term order.*

Now we have everything we need to prove first main theorem of this section.

Theorem 3.2.7. *The family $\mathcal{F}^{\text{in}} = \{\text{in}(I_{\sigma,a}) : I_{\sigma,a} \in \mathcal{F}\}$ has the following features:*

(1) Every product of ideals of $\mathcal{F}^{\text{in}}$ has linear resolution.

(2) The multi-Rees algebra $\mathscr{R}^{\text{in}} = \mathscr{R}(\text{in}(I_{\sigma,r})\colon I_{\sigma,r} \in \mathcal{F})$ is defined by G with respect to $\leq_\alpha$. In particular, $\mathscr{R}^{\text{in}}$ is Koszul.

Proof. It is enough to prove (2). It is clear that G is in $\ker(\varphi^{\text{in}})$. Let $\Omega = \{mv\colon mv \notin (\text{in}(g)\colon g \in G)\}$. From Lemma 3.2.6, it is enough to prove that the elements of Ω are linearly independent in $R/\ker(\varphi^{\text{in}})$. Let $\sum_i \lambda_i \overline{m_i v_i} = 0$ in $R/\ker(\varphi^{\text{in}})$ where $m_i v_i \in \Omega$. Remark 3.2.4 part (i) shows that $m_i v_i$ is the standard monomial representative of its class. From $R/\ker(\varphi^{\text{in}}) \simeq \mathscr{R}^{\text{in}} \subseteq S[\bar{t}]$, we see that $\overline{m_i v_i}$'s are linearly independent if and only if $\varphi^{\text{in}}(m_i v_i)$'s are pairwise distinct. This is in fact the case from Remark 3.2.4 part (ii). Hence, $\lambda_i = 0$ for every i. Thus $\mathscr{R}^{\text{in}}$ is defined by a quadratic Gröbner basis. Hence it is Koszul. Now the multi-graded version of the theorem of Blum [12] proves (1). □

In the rest of the section, we apply the means of Gröbner basis and sagbi basis to study $\mathscr{R}$. In Section 3.1, we saw that the "data" encoded in the product of some labeled chains can be presented as a tabel. We employ this tools to lift G to a Gröbner basis for $\ker(\varphi)$. Our main tool to perform the lifting is as simple as the observation of Laplace expansion of the minors.

Corollary 3.2.8. *Let $I_1, \ldots, I_l$ be ideals of the family $\mathcal{F}^{(1,n)}$. Then, the natural generators of $I = I_1 \ldots I_l$ form a Gröbner basis with respect to $\leq_\tau$. In particular for $l = 2$, if $\sum_i \lambda_i [a^{(i)}][b^{(i)}]$ is some linear combination, such that $[a^{(i)}]$ and $[b^{(i)}]$ are maximal minors of the matrices $\mathbf{X}_r^\sigma$ and $\mathbf{X}_s^\gamma$ and $\lambda_i \in K$, then there exist chains $e = e_1, \ldots, e_r$ and $f = f_1, \ldots, f_s$ such that*

$$\lambda x_e x_f = \text{in}(\sum_i \lambda_i [a^{(i)}][b^{(i)}])$$

where $\lambda \in K$.

Proof. The first part is proved in [30, Corollary 3.26]. For the second part, it is enough to consider $[a^{(i)}]$ and $[b^{(i)}]$ as maximal minors of the family $\mathbf{X}^{(1,n)}$. Note that this does not affect the polynomials given by this pair of minors. Now from the first part of the statement, the existence of $e = e_1, \ldots, e_r$ and $f = f_1, \ldots, f_s$ follows. □

Notation 3.2.9. *Let u be a monomial in S. We use $\deg_{x_i}(u)$ to denote the degree of x_i in u (i.e the number of copies of x_i in u).*

Observation 3.2.10. *Let $[a]$ and $[b]$ be maximal minors of the matrices $\mathbf{X}_r^\sigma$ and $\mathbf{X}_s^\gamma$. Let $c_1 \leq \ldots \leq c_{r+s}$ be the entries of the chains a and b in order. Let u be any term in $[a][b]$. It is easy to see that $\deg_{x_{c_1}}(u) \leq \deg_{x_{c_1}}(x_a x_b)$ and $\deg_{x_{c_{r+s}}}(u) \leq \deg_{x_{c_{r+s}}}(x_a x_b)$.*
In particular, let $[a]$ be a minor with $a_1 = 1$. From the definition of Hankel matrices, it is

clear that, in the minor $[a]$*, there exists exactly one entry* x_1*. Therefore, for a given pair of minors* $[a]$ *and* $[b]$*, we have* $\deg_{x_1}(u) \leq \deg_{x_1}(x_a x_b) \leq 2$ *for all terms* u *in* $[a][b]$*. Similar argument shows* $\deg_{x_n}(u) \leq \deg_{x_n}(x_a x_b) \leq 2$ *for all terms* u *in* $[a][b]$*.*

Observation 3.2.11. *Let* $[a]$ *be a maximal minor of the matrix* $\mathbf{X}_r^\sigma$*. The Laplace expansion of* $[a]$ *over the first row is*

$$[a] = \sum_{j=1}^{r} (-1)^{j+1} x_{a_j - j+1} [a_1 + 1, \ldots, a_{j-1} + 1, \hat{a}_j, a_{j+1} \ldots, a_r].$$

In particular

$$[a] = x_1[a_2, \ldots, a_r] + \tilde{H}$$

where $\tilde{H}$ *is the remaining factors of the Laplace expansion. Note for all terms* u *of* $\tilde{H}$*, we have* $\deg_{x_1}(u) = 0$*.*
The Laplace expansion of $[a]$ *over the last row is*

$$[a] = \sum_{j=1}^{r} (-1)^{j+1} x_{a_j + r - j} [a_1, \ldots, a_{j-1}, \hat{a}_j, a_{j+1} - 1 \ldots, a_r - 1].$$

In particular

$$[a] = x_n[a_1, \ldots, a_{r-1}] + H$$

where H *is the remaining factors of the Laplace expansion. Note for all terms* u *of* H*, we have* $\deg_{x_n}(u) = 0$*.*

Example 3.2.12. *Let* $n = 10$*. Let* $[4, 7, 10]$ *be a maximal minor in* $\mathbf{X}_3^{(1,10)}$*. The Laplace expansion over the last row is*

$$[4, 7, 10] = x_{10}[4, 7] - x_8[4, 9] + x_6[6, 9].$$

Here, $H = -x_8[4, 9] + x_6[6, 9]$ *is the analogue of the one of Observation 3.2.11.*

Definition 3.2.13. *Let* $(\sigma, a) \geq_c (\gamma, b)$ *be a pair of labeled chains. We say the product* $[a][b]$ *has standard representation if*

$$[a][b] = \sum_i \lambda_i [c^{(i)}][d^{(i)}]$$

such that $\lambda_i \in K$ *and* $(\sigma, c^{(i)}) \geq_c (\gamma, d^{(i)})$ *is a standard form of shape* $(\sigma, r), (\gamma, s)$ *for all* i*. Moreover,*

$$\operatorname{in}([a][b]) >_\tau \operatorname{in}([c^{(i)}][d^{(i)}])$$

for all $i > 1$*.*

Lemma 3.2.14. *Let* $(\sigma, a) \geq_c (\gamma, b)$ *be a pair of labeled chains with* $\sigma \neq \gamma$*. Then* $[a][b]$ *has a standard representation.*

Proof. One needs to repeat the following steps for finitely many times to obtain the standard representation of $[a][b]$. One notes that this process eliminates the leading term in each repetition. Moreover, these terms are products of chains of lengths r and s. Thus they are bounded from below with respect to $\leq_\tau$. Hence, the algorithm converges.

Step(1) Consider

$$\delta = [a][b] - [c^{(1)}][d^{(1)}]$$

where $(\sigma, c^{(1)}) \geq_c (\gamma, d^{(1)})$ is the unique standard form obtained by applying Lemma 3.1.15 on $(\sigma, a) \geq_c (\gamma, b)$. From Lemma 3.1.10 one knows $(\sigma, c^{(1)}) \geq_c (\gamma, d^{(1)})$ has shape $(\sigma, r), (\gamma, s)$. In particular, we have $\mathrm{in}(\delta) <_\tau \mathrm{in}([a][b])$.

Step(2) Consider the pair of standard labeled chains $(\sigma, c^{(2)}) \geq_c (\gamma, d^{(2)})$ by applying Lemma 3.2.15 and Lemma 3.1.15.

Step(3) Update δ with $\delta = \delta - \lambda_2[c^{(2)}][d^{(2)}]$ and return to step(2).

□

Lemma 3.2.15. *Let $(\sigma, a) \geq_c (\gamma, b)$ be a pair of labeled chains with $\sigma \neq \gamma$. Let $\delta = [a][b] - \sum_i \lambda_i [c^{(i)}][d^{(i)}]$ be obtained by finitely many times repeating the steps in Lemma 3.2.14. Then $\mathrm{in}(\delta)$ admits a well-defined pair of labeled chains like $(\sigma, e) \geq_c (\gamma, f)$ of shape $(\sigma, r), (\gamma, s)$.*

Proof. We proceed by double induction on length of $a = a_1, \dots, a_r$ and $b = b_1, \dots, b_s$. When $r = 1$ and $s = 1$, it is trivial. Let $r > 1$ and $s > 1$ be positive integers. The factors of δ have the following properties by construction:

1. $(\sigma, c^{(i)}) \geq_c (\gamma, d^{(i)})$ is standard for all i. Moreover, $(\sigma, c^{(1)}) \geq_c (\gamma, d^{(1)})$ is the standard form of $(\sigma, a) \geq_c (\gamma, b)$.
2. $\mathrm{in}([c^{(j)}][d^{(j)}]) <_\tau \mathrm{in}([c^{(i)}][d^{(i)}])$ for $j > i$. In particular, $\mathrm{in}([c^{(i)}][d^{(i)}]) <_\tau \mathrm{in}([a][b])$ for all $i > 1$.

By virtue of $(1), (2)$ and Lemma 3.1.15, $x_{c^{(i)}} x_{d^{(i)}}$'s are distinct for all i. From Corollary 3.2.8, we have chains $e = e_1, \dots, e_r$ and $f = f_1, \dots, f_s$ such that $\lambda x_e x_f = \mathrm{in}(\delta)$ for some $\lambda \in K$. Note that from Observation 3.2.10, $\deg_{x_1}(x_a x_b) \leq 2$, $\deg_{x_1}(x_{c^{(i)}} x_{d^{(i)}}) \leq 2$, $\deg_{x_n}(x_a x_b) \leq 2$ and $\deg_{x_n}(x_{c^{(i)}} x_{d^{(i)}}) \leq 2$ for all i. It is important to recall that $\lambda x_e x_f$ is nevertheless some term of $[a][b]$ or $[c^{(i)}][d^{(i)}]$ for some i. Thus Observation 3.2.10 implies $\deg_{x_1}(x_e x_f) \leq 2$ and $\deg_{x_n}(x_e x_f) \leq 2$.

When $\deg_{x_1}(x_a x_b) = 0$ or $\deg_{x_1}(x_e x_f) = 0$ or $\sigma_1 = \gamma_1$, we always have $\sigma_1 \leq e_1$ and $\gamma_1 \leq f_1$. Note that, $\deg_{x_1}(x_a x_b) = 2$, requires $\sigma_1 = \gamma_1$ since $(\sigma, a) \geq_c (\gamma, b)$ is a pair of labeled chains. So it falls into the previous case. Therefore, to prove the well-definity on the left, it remains to consider the case $\sigma_1 = 1$, $\gamma_1 = 2$, $\deg_{x_1}(x_a x_b) = 1$ (i.e $a_1 = 1$) and $\deg_{x_1}(x_e x_f) = 1$. Moreover, $\deg_{x_n}(x_a x_b) = 0$ or $\deg_{x_n}(x_e x_f) = 0$ or $\sigma_2 = \gamma_2$ clearly implies $e_r \leq \sigma_2$ and $f_s \leq \gamma_2$.

In particular, $\deg_{x_n}(x_a x_b) = 2$ or $\deg_{x_n}(x_e x_f) = 2$ requires $\sigma_2 = \gamma_2$. Therefore it falls into the previous case. Hence it remains to consider the case $\deg_{x_n}(x_a x_b) = \deg_{x_n}(x_e x_f) = 1$ and $\sigma_2 \neq \gamma_2$.

To show the well-definity on the right, we split the rest of the proof with respect to value of Δ.

(I) Suppose $\Delta\{(\sigma, a) \geq_c (\gamma, b)\} = 0$. By definition of Δ, we have $a_r \leq \gamma_2$ when $r \leq s$ (or $b_s \leq \sigma_2$ when $r > s$).

  (i) When $r \geq s$, consider (e, f) as standard form in the sense of Remark 3.1.17. From Remark 3.1.17 $(II), (i)$, we have $e_1 = 1$. In particular, $\deg_{x_1}(x_e x_f) = 1$, yields $\sigma_1 \leq e_1$ and $\gamma_1 \leq f_1$. Recall that $\deg_{x_n}(x_e x_f) \leq \deg_{x_n}(x_a x_b)$ from Observation 3.2.10. Now, (e, f) being standard, the definition of Δ and Remark 3.1.17 part (III) yields $e_r \leq \sigma_2$ and $f_s \leq \gamma_2$.

  (ii) When $r < s$. By Observation 3.2.11, we have

$$\lambda x_e x_f = \text{in}(x_1[a_2, \ldots, a_r][b] - x_1 \sum_i \lambda_i [c_2^{(i)}, \ldots, c_r^{(i)}][d^{(i)}])$$

where i runs through all $c^{(i)}$'s with $c_1^{(i)} = 1$. Now, Corollary 3.2.8 admits the existence of chains $\tilde{e} = \tilde{e}_1, \ldots, \tilde{e}_{r-1}$ and $\tilde{f} = \tilde{f}_1, \ldots, \tilde{f}_s$ such that

$$\lambda x_{\tilde{e}} x_{\tilde{f}} = \lambda x_e x_f / x_1 = \text{in}([a_2, \ldots, a_r][b] - \sum_i \lambda_i [c_2^{(i)}, \ldots, c_r^{(i)}][d^{(i)}]).$$

Moreover, $\deg_{x_2}(x_{\tilde{e}} x_{\tilde{f}}) = \deg_{x_2}(x_e x_f) \leq \deg_{x_2}(x_a x_b) \leq 1$ by construction and Observation 3.2.10. This implies that if $\deg_{x_2}(x_{\tilde{e}} x_{\tilde{f}}) = 1$, then 2 is the smallest entry in chains $\tilde{e}$ and $\tilde{f}$. Thus, from Remark 3.1.17 part $(II), (i)$, we can assume $2 < \tilde{e}_1$ by considering $(\tilde{f}, \tilde{e})$ as standard form. By reseting notations, we have the chains $e = 1, \tilde{e}_1, \ldots, \tilde{e}_{r-1}$ and $f = \tilde{f}$. It is clear that $\sigma_1 \leq e_1$ and $\gamma_1 \leq f_1$. In particular, Remark 3.1.17 (III) and $\deg_{x_n}(x_{\tilde{e}} x_{\tilde{f}}) \leq \deg_{x_n}(x_a x_b)$ implies $e_r \leq \sigma_2$ and $f_s \leq \gamma_2$. The reader notes that, we can not apply induction hypothesis on $[a_2, \ldots, a_r][b] - \sum_i \lambda_i [c_2^{(i)}, \ldots, c_r^{(i)}][d^{(i)}]$, as $(\sigma, c_2^{(i)}, \ldots, c_r^{(i)}) \geq_c (\gamma, d^{(i)})$'s are not necessarily standard in this context.

(II) Suppose $\Delta\{(\sigma, a) \geq_c (\gamma, b)\} = 1$. By definition of Δ, we have $a_r = n$ and $\gamma_2 = n - 1$ when $r \leq s$ (or $b_s = n$ and $\sigma_2 = n - 1$ when $r > s$).

  (i) When $r \leq s$. Form Observation 3.2.11, we have

$$\lambda x_e x_f = \text{in}(x_n[a_1, \ldots, a_{r-1}][b] - x_n \sum_i [c_1^{(i)} \ldots, c_{r-1}^{(i)}][d^{(i)}])$$

where i runs through all $c^{(i)}$'s with $c_r^{(i)} = n$. By virtue of Corollary 3.2.8, there exists chains $\tilde{e} = \tilde{e}_1, \ldots, \tilde{e}_{r-1}$ and $\tilde{f} = \tilde{f}_1, \ldots, \tilde{f}_s$ such that

$$\lambda x_{\tilde{e}} x_{\tilde{f}} = \lambda x_e x_f / x_n = \text{in}([a_1, \ldots, a_{r-1}][b] - \sum_i [c_1^{(i)} \ldots, c_{r-1}^{(i)}][d^{(i)}]). \tag{3.1}$$

Note that $(\sigma, c_1^{(i)}, \ldots, c_{r-1}^{(i)}) \geq_c (\gamma, d^{(i)})$'s in (3.1) are clearly standard forms here. Thus, the induction hypothesis implies $\sigma_1 \leq \tilde{e}_1$ and $\gamma_1 \leq \tilde{f}_1$, in particular $\tilde{e}_1 = 1$. By applying Remark 3.1.17, one shall consider $(\tilde{e}, \tilde{f})$ as a standard form. One notes that Remark 3.1.17 part (I), implies that $\tilde{e}_1$ is not repositioned after considering $(\tilde{e}, \tilde{f})$ as a standard form. Hence, one still has $\tilde{e}_1 = 1$. On the other hand, $\deg_{x_{n-1}}(x_{\tilde{e}} x_{\tilde{f}}) = \deg_{x_{n-1}}(x_e x_f) \leq \deg_{x_{n-1}}(x_a x_b) \leq 1$ by construction and Observation 3.2.10. From Remark 3.1.17 part (III), one has $\tilde{e}_{r-1} < n-1$ and $\tilde{f}_s \leq n-1$. Hence, by reseting notations, one has chains $e = \tilde{e}_1, \ldots, \tilde{e}_{r-1}, n$ and $f = \tilde{f}_1, \ldots, \tilde{f}_s$. Thus $e_r \leq \sigma_2$ and $f_s \leq \gamma_2$. In particular, $\sigma_1 \leq \tilde{e}_1$ and $\gamma_1 \leq \tilde{f}_1$ is a consequence of $\tilde{e} = 1$.

(ii) When $r > s$. One can argue similar to the last case. By definition of Δ, one has $b_s = n$ and $\sigma_2 = n - 1$. From Observation 3.2.11, one has

$$\lambda x_e x_f = \text{in}(x_n [a][b_1, \ldots, b_{s-1}] - x_n \sum_i [c^{(i)}][d_1^{(i)} \ldots, d_{s-1}^{(i)}])$$

where i runs through all $d^{(i)}$'s with $d_s^{(i)} = n$. By virtue of Corollary 3.2.8, there exists chains $\tilde{e} = \tilde{e}_1, \ldots, \tilde{e}_r$ and $\tilde{f} = \tilde{f}_1, \ldots, \tilde{f}_{s-1}$ such that

$$\lambda x_{\tilde{e}} x_{\tilde{f}} = \lambda x_e x_f / x_n = \text{in}([a][b_1, \ldots, b_{s-1}] - \sum_i [c^{(i)}][d_1^{(i)} \ldots, d_{s-1}^{(i)}]). \tag{3.2}$$

Note that $(\sigma, c^{(i)}) \geq_c (\gamma, d_1^{(i)}, \ldots, d_{s-1}^{(i)})$'s in (3.2) are clearly standard forms here. Thus, the induction hypothesis implies $\sigma_1 \leq \tilde{e}_1$ and $\gamma_1 \leq \tilde{f}_1$, in particular $\tilde{e} = 1$. By applying Remark 3.1.17, one shall consider $(\tilde{e}, \tilde{f})$ as a standard form. Note that Remark 3.1.17 part $(II), (i)$ implies $\tilde{e}_1 = 1$. Hence, one still has $\tilde{e}_1 = 1$ in particular $\sigma_1 \leq \tilde{e}_1$ and $\gamma_1 \leq \tilde{f}_1$. Recall that $\deg_{x_{n-1}}(x_{\tilde{e}} x_{\tilde{f}}) = \deg_{x_{n-1}}(x_e x_f) \leq \deg_{x_{n-1}}(x_a x_b) \leq 1$ by construction and Observation 3.2.10. From Remark 3.1.17 part (III), one has $\tilde{e}_{r-1} \leq n-1$ and $\tilde{f}_s < n-1$. Hence, by reseting notations, one has chains $e = \tilde{e}_1, \ldots, \tilde{e}_r$ and $f = \tilde{f}_1, \ldots, \tilde{f}_{s-1}, n$. Thus $e_r \leq \sigma_2$ and $f_s \leq \gamma_2$.

□

In the following, we present the main theorem of this chapter. The idea of the proof is to take advantage of G which forms a quadratic gröbner bases and compute a quadratic gröbner bases of $\ker(\varphi)$. This will be done by evaluating G via φ and representing it as a combination of algebra generators of $\mathscr{R}$. We will abuse the word "lifting" for this process.

Theorem 3.2.16. *The family $\mathcal{F} = \mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$ has the following features:*

(1) Every product $\prod_{(\sigma,a)} I_{\sigma,a}$ of ideals in $\mathcal{F}$ has linear resolution.

(2) The natural generators of every products $\prod_{(\sigma,a)} I_{\sigma,a}$ form a Gröbner basis. In particular $\mathrm{in}(\prod_{(\sigma,a)} I_{\sigma,a}) = \prod_{(\sigma,a)} \mathrm{in}(I_{\sigma,a})$.

(3) The natural algebra generators of $\mathscr{R}$ form a Sagbi basis. In particular, $\mathrm{in}(\mathscr{R}) = \mathscr{R}^{\mathrm{in}}$.

(4) The multi-Rees algebra $\mathscr{R}(I_{\sigma,a} : I_{\sigma,a} \in \mathcal{F})$ is defined by a quadratic Gröbner basis with respect to $\leq_\alpha$, it is Koszul, normal, Cohen-Macaulay domain.

Proof. (1) is a consequence of Koszulness of $\mathscr{R}$ and the multi-graded version of the theorem of Blum [12]. (2) and (3) are equivalent and they are consequences of [27, Preposition 1.1] and (4). Hence, it is only enough to prove (4). We apply [27, Corollary 2.2]. The binomials of the form $x_i z_{\sigma,a} - x_j z_{\sigma,c}$ and $z_{\sigma,a} z_{\sigma,b} - z_{\sigma,c} z_{\sigma,d}$ in G lifts to $x_i z_{\sigma,a} - \sum_i x_j z_{\sigma,c^{(i)}}$ (or $z_{\sigma,a} z_{\sigma,b} - \sum_i z_{\sigma,c^{(i)}} z_{\sigma,d^{(i)}}$). These polynomials are constructed step by step like the following:

(1) Evaluate $\delta = x_i z_{\sigma,a} - x_j z_{\sigma,c}$ (or $\delta = z_{\sigma,a} z_{\sigma,b} - z_{\sigma,c} z_{\sigma,d}$) by applying φ,

(2) Obtain $x_j[c^{(1)}]$ (or $[c^{(1)}][d^{(1)}]$) by applying Proposition 3.1.14 (or Corollary 3.2.8 and Proposition 3.1.14) on the leading term of the step (1),

(3) Pull back the outcome of step (2) via φ. One notes that the resulting monomial $x_j z_{\sigma,c^{(1)}}$ (or $z_{\sigma,c^{(1)}} z_{\sigma,d^{(1)}}$) is standard by virtue of Proposition 3.1.14. In particular, $x_j z_{\sigma,c^{(1)}}$ (or $z_{\sigma,c^{(1)}} z_{\sigma,d^{(1)}}$) is well-defined by Proposition 3.2.2 (*ii*) (or the fact that σ is fixed).

(4) Update δ with $\delta - x_j z_{\sigma,c^{(1)}}$ (or $\delta - z_{\sigma,c^{(1)}} z_{\sigma,d^{(1)}}$) and return to (1).

For $\sigma \neq \gamma$, the binomials $z_{\sigma,a} z_{\gamma,b} - z_{\sigma,c} z_{\gamma,d}$ in G lifts to $z_{\sigma,a} z_{\gamma,b} - \sum_i z_{\sigma,c^{(i)}} z_{\gamma,d^{(i)}}$ in $\ker(\varphi)$ where the indices are obtained from Lemma 3.2.14. This admits a quadratic Gröbner basis for $\ker(\varphi)$. One notes that in Definition 3.2.13, the shape represents the multi-degree of monomials of $z_{\sigma,a} z_{\gamma,b} - \sum_i z_{\sigma,c^{(i)}} z_{\gamma,d^{(i)}}$. Hence, the lifted polynomials are homogeneous. Therefor, by virtue of [27, Corollary 2.2] $\mathscr{R}$ is Koszul.
To prove that $\mathscr{R}$ is normal, Cohen-Macaulay domain, by [27, Corollary 2.3], is is enough to prove that $\mathscr{R}^{\mathrm{in}}$ is normal. Recall that the term order $\leq_\alpha$ picks non-standard monomials as the leading terms of the elements in G. Moreover, every non-square free monomial of degree two in indeterminates $\bar{z}$ is standard. Thus $\mathrm{in}_{\leq_\alpha}(\ker(\varphi^{\mathrm{in}}))$ is square free. Hence [56, Proposition 13.15] yields the normality of $\mathscr{R}^{\mathrm{in}}$. □

We conclude this chapter by explaining why the family of close cuts of Hankel matrices are interesting.

Remark 3.2.17. *Let* $x_i, \dots, x_j$ *be an interval of indeterminates of* S *where* $i \leq j$. *Let* $\mathbf{X}^{(i,j)}$ *and* $\mathcal{F}^{(i,j)}$ *be defined similar to* $\mathbf{X}^{(1,n)}$ *and* $\mathcal{F}^{(1,n)}$. *Let* $\tilde{\mathcal{F}} = \cup_{i \leq j} \mathcal{F}^{(i,j)}$. *We expect Theorem 3.2.16,* (1) *to extend for* $\tilde{\mathcal{F}}$. *As we saw, one standard approach is via Sagbi deformations. However, it is easy to see that this is not the case for Theorem 3.2.16,* (2). *For* $n \geq 6$, *we have* $\mathrm{in}(I_{(1,n),2} I_{(3,n),2}) \neq \mathrm{in}(I_{(1,n),2}) \, \mathrm{in}(I_{(3,n),2})$ *which is equivalent to* $\mathscr{R}^{\mathrm{in}}(\mathrm{in}(I) : I \in \tilde{\mathcal{F}}) \neq \mathrm{in}(\mathscr{R}(I : I \in \tilde{\mathcal{F}}))$. *Hence, the kernel of* $\mathscr{R}^{\mathrm{in}}(\mathrm{in}(I) : I \in \tilde{\mathcal{F}})$ *does not lift to the kernel of* $\mathscr{R}(I : I \in \tilde{\mathcal{F}})$ *(see* [27, Proposition 1.1]*). Therefore, Sagbi deformation method fails. Nevertheless, We still expect Theorem 3.2.7 to extend for* $\tilde{\mathcal{F}}$.
Conca and Nam, in separated papers, prove that the product $I = I_1 \dots I_l$ *of ideals in* $\mathcal{F}^{(1,n)}$ *have a nice primary decomposition given by intersection of symbolic powers of ideals in* $\mathcal{F}^{(1,n)}$ *containing* I *(see* [24, Theorem 3.12] *and* [30, Theorem 3.25]*). The author refers to standard text books in commutative algebra for the definition of symbolic powers. Similar feature is provided for generic matrices, however, with ordinary powers by Berget, Bruns and Conca (see* [9, Corollary 2.3] *and* [14, Theorem 3.4]*). This might raise the question that whether this behavior is expected for* $\tilde{\mathcal{F}}$. *Unfortunately, this is not the case. Consider* $I = I_{(1,6),2} I_{(3,6),2}$. *We have* $\mathrm{Ass}(I) = \{I_{(1,6),2}, I_{(3,6),2}, I_{(2,6),1}\}$, *where* $\mathrm{Ass}(I)$ *is the associated primes of* I. *Thanks to* [24, Theorem 3.8], *one can check* $I \subset I_{(3,6),2}$, $I \subset I^{(2)}_{(1,6),2}$ *and* $I \subset I^{(3)}_{(2,6),1}$. *The inclusions is strict and increasing the symbolic exponent will defy the inclusion. In particular,* $I \subsetneq I_{(3,6),2} \cap I^{(2)}_{(1,6),2} \cap I^{(3)}_{(2,6),1}$. *Hence, we can not expect a similar behavior of the primary decompositions for* $\tilde{\mathcal{F}}$. *Nevertheless, a nice primary decomposition is expected for* $\mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$.

Chapter 4

Regularity of Powers: Edge Ideals

In this chapter we study the regularity of powers of edge ideals of dumbbell graphs and bicyclic graphs. A dumbbell graph is constructed by attaching two cycles (not necessarily of the same size) via a path of given length (See Figure 4.1). A bicyclic graph is defined to be any graph with exactly two cycles (See Figure 2.4). This work was motivated by [10] and [1].

4.1 Regularity and Induced Matching Number of a Dumbbell Graph

In this section we compute the induced matching number of a dumbbell graph and the regularity of its edge ideal in terms of the induced matching number. By $C_n \cdot P_l \cdot C_m$ we denotes the dumbbell constructed by two cycles C_n and C_m connected via the path P_l, where n, m and l are the number of the vertices. We denote the vertices of C_n, C_m and P_l by $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ and $\{z_1, \ldots, z_l\}$ respectively. Note that we consider $x_1 = z_1$ and $y_1 = z_l$ for notation prepossess. See the following graphs for example. The motivation behind this attempt is apply Theorem 2.5.15 to obtain a lower bound for regularity of powers of edge ideals of dumbbells in term of the regularity of the first power.

Example 4.1.1. *The following are simple examples of dumbbells:*

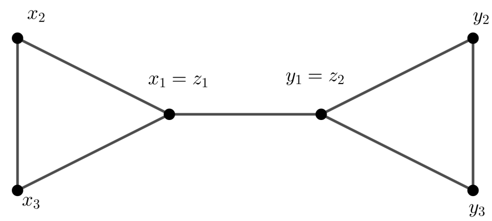


Figure 4.1: Dumbbell $C_2.P_2.C_2$

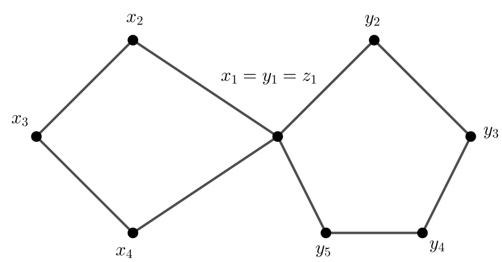
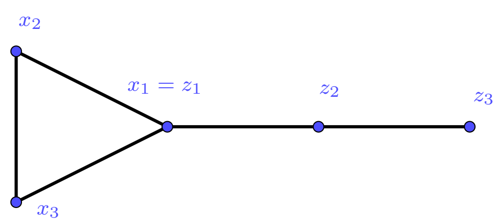


Figure 4.2: Dumbbell $C_4.P_1.C_5$

Notation 4.1.2. *By ξ_3 we shall denote the function*

$$\xi_3(n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

We use $C_n \cdot P_l$ to denote the graph given by attaching C_n and P_l. For example, the graph $C_3 \cdot P_3$ is the following:



Proposition 4.1.3. *Let $n \geq 3$ and $l \geq 1$, then*

$$\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) + 1}{3} \right\rfloor.$$

Proof. The case $l = 1$ is clearly a cycle. Hence we exclude this case in the proof.
Case 1: Suppose $\xi_3(n) = 0$, which is $n \equiv 2 \pmod 3$. From Remark 2.5.22, there exists a maximal induced matching of C_n, say $\mathcal{M}(C_n)$, not containing any edges incident to x_1. This means that $\mathcal{M}(C_n)$ does not put any constrains on $\mathcal{M}(P_l)$ which is a maximal induced matching of P_l. Hence, $\mathcal{M} = \mathcal{M}(C_n) \cup \mathcal{M}(P_l)$ is a maximal induced matching of $C_n \cdot P_l$. Thus $\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l+1}{3} \rfloor$.

Case 2: Suppose $\xi_3(n) = 1$, which is $n \equiv 0, 1 \pmod 3$. Let $\mathcal{M}$ be an induced matching of maximal size in $C_n \cdot P_l$. Let $\mathcal{M}|_{C_n} := \{e \in \mathcal{M} \mid e \in C_n\}$ and $\mathcal{M}|_{P_l} := \{e \in \mathcal{M} \mid e \in P_l\}$. It is clear that $\mathcal{M}$ is disjoint union of $\mathcal{M}|_{C_n}$ and $\mathcal{M}|_{P_l}$ since C_n and P_l have no common edges. Thus $|\mathcal{M}| = |\mathcal{M}|_{C_n}| + |\mathcal{M}|_{P_l}|$.
Suppose $z_1z_2 \notin \mathcal{M}$. Then $\mathcal{M}$ can be considered as the union of a maximal induced matching of C_n as introduced in Remark 2.5.22 and a maximal induced matching of the path $P_l \setminus \{z_1\}$. Therefor,

$$|\mathcal{M}| = \nu(C_n) + \nu(P_{l-1}) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor. \tag{4.1}$$

Suppose $z_1z_2 \in \mathcal{M}$. Then none of the edges incident to vertices in $N_{C_n}[x_1] = \{x_1, x_2, x_n\}$ are in $\mathcal{M}|_{C_n}$. Thus $|\mathcal{M}|_{C_n}| = \nu(P_{n-3}) = \nu(C_n) - 1$ as $n \equiv 0, 1 \pmod 3$. Moreover, $z_1z_2 \in \mathcal{M}$ implies $|\mathcal{M}|_{P_l}| = \nu(P_{l-3}) + 1$. Hence,

$$|\mathcal{M}| = \nu(C_n) + \nu(P_{l-3}) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-3)+1}{3} \rfloor. \tag{4.2}$$

Now from the definition of induced matching number, comparing Equation 4.1 and Equation 4.2 implies $|\mathcal{M}| = \nu(C_n) + \nu(P_{l-1}) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor$. □

Theorem 4.1.4. *Let $n, m \geq 3$ and $l \geq 1$, then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Proof. One can apply the same argument as in Proposition 4.1.3. By Remark 2.5.22 we have that when either $n \equiv 2 \pmod 3$ or $m \equiv 2 \pmod 3$, then there exists a maximal induced matching in C_n or in C_m which does not affect the way we choose edges to obtain a maximal induced matching in P_l.

In the case $n \equiv 0, 1 \pmod 3$ we can choose a maximal induced matching, say $\mathcal{M}$, that does not contain any edge connected to the cycle C_n. Hence the $\mathcal{M}|_{P_l}$ (or $\mathcal{M}|_{C_m}$ when $l = 1$), does not contain the edge z_1z_2 (or the vertex x_1). Therefore $\mathcal{M}|_{C_m \cdot P_l} = \mathcal{M}(C_m \cdot P_{l-1})$ (or $\mathcal{M}|_{C_m} = \mathcal{M}(P_{m-1})$). This will give the required equation. The case $m \equiv 0, 1 \pmod 3$ follows by symmetry. □

Proposition 4.1.5. *Let $n, m \geq 3$ and $l \geq 1$, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) - \nu(C_n \cdot P_l \cdot C_m) = \operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) - \nu(C_n \cdot P_{l+3} \cdot C_m).$$

Proof. Either from Theorem 4.1.4 or Theorem 2.5.20, we have

$$\nu(C_n \cdot P_{l+3} \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1.$$

We can apply the Lozin transformation on any vertex in P_l. Then from Theorem 2.5.20, we have

$$\operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) = \operatorname{reg} I(C_n \cdot P_l \cdot C_m) + 1.$$

Thus, the proof is complete by subtracting the above equalities. □

In the rest of this section, we explicitly compute the regularity of $I(C_n \cdot P_l \cdot C_m)$ in term of the induced matching number. From the previous proposition, it follows that we only need to consider the cases $l = 1$, $l = 2$ and $l = 3$. We treat each case in a separate subsection. In the following theorem we compute the regularity of the edge ideal of the dumbbell $C_n \cdot P_l \cdot C_m$.

Theorem 4.1.6. *Let $m, n \geq 3$ and $l \geq 1$, then*

(i) if $l \equiv 0, 1 \pmod 3$, then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) if $l \equiv 2 \pmod 3$, then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Follows from Proposition 4.1.5, and Theorem 4.1.8, Theorem 4.1.14, and Theorem 4.1.16. □

4.1.1 The case $l = 1$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_1 \cdot C_m$.

Proposition 4.1.7. *Let $n, m \geq 3$, then*

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \in \left\{ \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2 \right\}.$$

Proof. We apply Lemma 2.5.3 on z_1. We have

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \in \left\{ \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus z_1\big), \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus N[z_1]\big) + 1 \right\}.$$

Since $(C_n \cdot P_1 \cdot C_m) \setminus z_1 = P_{n-1} \cup P_{m-1}$ and $(C_n \cdot P_1 \cdot C_m) \setminus N[z_1] = P_{n-3} \cup P_{m-3}$, we get the result by applying Theorem 2.5.5 and Theorem 2.5.23. □

Theorem 4.1.8. *Let $n, m \geq 3$, then*

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_1 \cdot C_m) + 2 & \text{if } n \equiv 2 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_1 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$. Since $\lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k}{3} \rfloor$ when $k \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Thus Proposition 4.1.7 yields

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2. \tag{4.3}$$

Consider the induced sub-graph $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\}$ where x_n is in C_n and it is incident to x_1 (e.g see x_4 in Figure 4.2). In fact, H is the graph given by joining C_m and a path P_{n-1}, that is, $H = C_m \cdot P_{n-1}$. Now from Proposition 4.1.3, we have that $\nu(H) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. By Corollary 2.5.6 , we get $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \geq \operatorname{reg} I(H)$. From Theorem 2.5.28, we have $\operatorname{reg} I(H) = \nu(H)+2$. Therefor, the equality holds in (4.3). The proof of this part is complete since Theorem 4.1.4 admits $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$.
For any case distinct to $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1.$$

Therefore, from Proposition 4.1.7, we have

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1. \tag{4.4}$$

From Theorem 4.1.4, we have $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. Moreover, Theorem 2.5.14 gives $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \geq \nu(C_n \cdot P_1 \cdot C_m) + 1$. Thus, the equality in (4.4) holds. Therefore the proof is complete. □

4.1.2 The case $l = 2$

Throughout this subsection, we consider the dumbbell $C_n \cdot P_2 \cdot C_m$. From Theorem 2.5.14 and the value of $\nu(C_n \cdot P_2 \cdot C_m)$ computed in Theorem 4.1.4, we have

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) \geq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{3 - \xi_3(n) - \xi_3(m)}{3} \rfloor + 1. \tag{4.5}$$

We will perform different constructions to prove that the equality holds above.

Remark 4.1.9. *The regularity of $I(C_n)$ is given in Theorem 2.5.24. For simplicity of notation, we use the equivalent formula* $\operatorname{reg} I(C_n) = \lfloor \frac{n-2}{3} \rfloor + 2$.

Proposition 4.1.10. *Let $n, m \geq 3$, then*

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2. \tag{4.6}$$

Proof. In the dumbbell $C_n \cdot P_2 \cdot C_m$, we delete the edge $e = z_1z_2$ that connects the two cycles C_n and C_m (see Figure 4.1). Also, we denote resulting graph by $C_n \cup C_m$ the disjoint union of C_n and C_m. Consider the short exact sequence

$$0 \longrightarrow \frac{S}{I(C_n \cup C_m) : e}(-2) \xrightarrow{\times e} \frac{S}{I(C_n \cup C_m)} \to \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \to 0,$$

By applying Lemma 2.2.3, we have

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \max\left\{ \operatorname{reg} \frac{S}{I(C_n \cup C_m) : e} + 1, \operatorname{reg} \frac{S}{I(C_n \cup C_m)} \right\}. \tag{4.7}$$

From Theorem 2.5.5, we have

$$\operatorname{reg} \frac{S}{I(C_n \cup C_m)} = \operatorname{reg} \frac{S}{I(C_n)} + \operatorname{reg} \frac{S}{I(C_m)},$$

and using Remark 4.1.9 we get the equality

$$\operatorname{reg} \frac{S}{I(C_n \cup C_m)} = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2. \tag{4.8}$$

On the other hand, the ideal $I(C_n \cup C_m) : e$ corresponds to the edge ideal of the graph $H = \{x_2\} \cup \{x_n\} \cup P_{n-3} \cup \{y_2\} \cup \{y_m\} \cup P_{m-3}$, where x_2 and x_n are neighboring vertices of x_1, and P_{n-3} represents a path of length $n-3$ with the remaining vertices of C_n; also, a similar argument follows for the cycle C_m. Hence from and Theorem 2.5.5 and Theorem 2.5.23 we get

$$\operatorname{reg} \frac{S}{I(C_n \cup C_m) : e} + 1 = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 1, \tag{4.9}$$

The proof is complete by comparing (4.7), (4.8) and (4.9). □

As a result of the previous proposition, we can prove the following corollary.

Corollary 4.1.11. *If $n \equiv 0, 1 \pmod 3$ and $m \equiv 0, 1 \pmod 3$, then*

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor$$

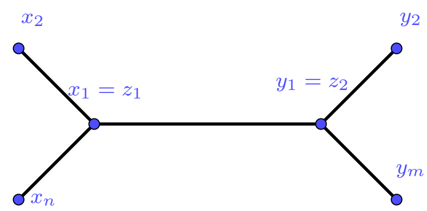
Proof. In (4.6) we have $\nu(C_n \cdot P_2 \cdot C_m) = \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2$ for all these cases. □

It remains to consider three more cases, i.e., the case $n \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$, the case $n \equiv 1 \pmod 3$, $m \equiv 2 \pmod 3$, and the case $n \equiv 2 \pmod 3$, $m \equiv 2 \pmod 3$.

Lemma 4.1.12. *If $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$, then*

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. We shall divide the dumbbell into three subgraphs P_{n-1}, P_{n-2} and H. We make $P_{n-1} = C_n \setminus \{x_1\}$ and $P_{n-2} = C_m \setminus \{y_1\}$. The subgraph H is defined by taking the bridge $e = z_1 z_2$ and the neighboring vertices $\{x_2, x_n, y_2, y_m\}$, i.e. the graph below.



Using this decomposition and Theorem 2.5.5,we have

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \operatorname{reg} \frac{S}{I(P_{n-1})} + \operatorname{reg} \frac{S}{I(P_{n-2})} + \operatorname{reg} \frac{S}{I(H)},$$

Then from Theorem 2.5.23, we have

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Finally, in the present case $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$ we have the equality $\nu(C_n \cdot P_2 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$ from Theorem 4.1.4. Hence

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \nu(C_n \cdot P_2 \cdot C_m).$$

The proof is complete by Theorem 2.5.14. □

Lemma 4.1.13. *If* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*, then*

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} = \nu(C_n \cdot P_2 \cdot C_m) + 1 = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. In this case we will delete the vertex x_1 from the cycle C_n. We have that $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ is an induced subgraph of $C_n \cdot P_2 \cdot C_m$ which is given as the union of a path of length $n-1$ and a cycle m, i.e. $H = P_{n-1} \cup C_m$. From Corollary 2.5.6 and Theorem 2.5.5, we have

$$\operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \geq \operatorname{reg} \frac{S}{I(H)} = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Finally, from Proposition 4.1.10 we have that

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1 \leq \operatorname{reg} \frac{S}{I(C_n \cdot P_2 \cdot C_m)} \leq \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2.$$

Hence the equality holds. □

Theorem 4.1.14. *Let $n, m \geq 3$, then*

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_2 \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_2 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. It follows by Corollary 4.1.11, Lemma 4.1.12 and Lemma 4.1.13. □

4.1.3 The case $l = 3$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_3 \cdot C_m$. We will take advantage of Theorem 2.5.15 and Theorem 4.1.4 in our treatment.

Proposition 4.1.15. *Let $n, m \geq 3$, then*

(i) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \leq \nu(C_n \cdot P_3 \cdot C_m) + 2$, *if $n, m \equiv 2 \pmod 3$;*

(ii) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \nu(C_n \cdot P_3 \cdot C_m) + 1$, *otherwise.*

Proof. Let $E(P_3) = \{e, e'\}$ be the set of the edges of P_3, where $e = z_1z_2$ and $e' = z_2z_3$ are connected to C_n and C_m, respectively. We have the following short exact sequence by deleting e from $C_n \cdot P_3 \cdot C_m$:

$$0 \longrightarrow \frac{S}{I(C_n \cup (e' \cdot C_m)) : e}(-2) \xrightarrow{\times e} \frac{S}{I(C_n \cup (e' \cdot C_m))} \to \frac{S}{I(C_n \cdot P_3 \cdot C_m)} \to 0.$$

We have that $\operatorname{reg} I(C_n \cup (e' \cdot C_m)) : e = \operatorname{reg} I(P_{n-3} \cup P_{m-1})$, and from Proposition 4.1.3 and Theorem 2.5.28 follows that $\operatorname{reg}(I(e' \cdot C_m)) = \lfloor \frac{m}{3} \rfloor + \lfloor \frac{3-\xi_3(m)}{3} \rfloor + 1$. Thus, using Remark 4.1.9, Theorem 2.5.5 and Theorem 2.5.23, we get

$$\begin{aligned} \operatorname{reg} \frac{S}{I(C_n \cdot P_3 \cdot C_m)} &\leq \max\left\{ \operatorname{reg} \frac{S}{I(P_{n-3} \cup P_{m-1})} + 1, \operatorname{reg} \frac{S}{I(C_n \cup (e' \cdot C_m))} \right\} \\ &\leq \max\left\{ \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + 1 + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{3-\xi_3(m)}{3} \right\rfloor \right\}. \end{aligned}$$

On the other hand, from Theorem 4.1.4 we have that $\nu(C_n \cdot P_3 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{4-\xi_3(n)-\xi_3(m)}{3} \rfloor$. Therefore, we can check that $\operatorname{reg} \frac{S}{I(C_n \cdot P_3 \cdot C_m)} \leq \nu(C_n \cdot P_3 \cdot C_m) + 1$ when $n, m \equiv 2 \pmod 3$, and that $\operatorname{reg} \frac{S}{I(C_n \cdot P_3 \cdot C_m)} = \nu(C_n \cdot P_3 \cdot C_m)$ in all the remaining cases. □

Theorem 4.1.16. *Let $n, m \geq 3$, then*

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_3 \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_3 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Using Proposition 4.1.15, then we only need to prove that $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geq \nu(C_n \cdot P_3 \cdot C_m) + 2$ in the case $n, m \equiv 2 \pmod 3$. Hence, we assume $n, m \equiv 2 \pmod 3$. Let z_2 be the middle vertex of $C_n \cdot P_3 \cdot C_m$. By deleting z_2 we see that $H = (C_n \cdot P_3 \cdot C_m) \setminus z_2 = C_n \cup C_m$ is an induced subgraph of $C_n \cdot P_3 \cdot C_m$. From Theorem 2.5.5 and Theorem 2.5.23 and Corollary 2.5.6, we have that

$$\operatorname{reg} I(H) = \operatorname{reg} I(C_n) + \operatorname{reg} I(C_m) - 1 = \nu(C_n) + \nu(C_m) + 3.$$

Since $\nu(C_n \cdot P_3 \cdot C_m) = \nu(C_n) + \nu(C_m) + 1$, then using [10, Corollary 4.3] we get

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geq \operatorname{reg} I(H) = \nu(C_n \cdot P_3 \cdot C_m) + 2. \quad \square$$

4.2 Regularity of Powers of Dumbbell Graphs

In this section, we study the regularity of powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. Our strategy is to show $2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$ is actually an upper bound and a lower bound for $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ for all $q \geq 1$ where $l \leq 2$. To show $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q \leq 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$, we follow the argument of [7, Theorem 5.2]. To prove the revers equality, we proceed by looking at "nice" induced subgraphs of $C_n \cdot P_l \cdot C_m$.

As a side result, we answer an interesting question on the behavior of the constant term of the asymptotic regularity function. Let I be an arbitrary ideal generated in degree d and let $\operatorname{reg} I^q = dq + b_q$ for $q \geq q_0$. An interesting question is the study the sequence $\{b_i\}_{i \geq 1}$. In [31] the authors proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i \geq 1}$ is a weakly decreasing sequence of non-negative integers. In [8, Conjecture 7.11] the authors conjectured that for any edge ideal, $\{b_i\}_{i \geq 1}$ is a weakly decreasing sequence. For the edge ideal of any dumbbell with $l \leq 2$, we prove $b_i = b_1$ for all $i \geq 1$. However, we expect $b_i \leq b_1$ for all $i \geq 1$ for any graph.

Since we focus on dumbbell graphs $C_n \cdot P_l \cdot C_m$ where $l \leq 2$ we recall their regularity with different formulation.

Remark 4.2.1 (Theorem 4.1.4, Theorem 4.1.6)**.** *Let $C_n \cdot P_l \cdot C_m$ be a dumbbell graph where $l \leq 2$ then*

$$\lfloor \frac{n+m+1}{3} \rfloor = \begin{cases} \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) & \text{if } n, m \equiv 1 \pmod 3, \\ \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) - 1 & \text{otherwise;} \end{cases}$$

and

$$\lfloor \frac{n+m}{3} \rfloor = \begin{cases} \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) & \text{if } (n, m) \equiv (1,2), (2,1) \pmod 3, \\ \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) - 1 & \text{otherwise.} \end{cases}$$

In addition, by comparing $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)$ and $\operatorname{reg} I(C_n)$, one can see

$$\operatorname{reg} I(C_n) \leq \operatorname{reg} I(C_n \cdot P_l \cdot C_m).$$

We use the notation of even-connection from [7, Theorem 5.2]. The following lemma is crucial in our treatment of the even-connected vertices.

Lemma 4.2.2. *Let G be a graph. If two vertices u and v are even-connected with respect to $e_1 \cdots e_q$, with $e_i = x_{i,1}x_{i,2}$ for $1 \leq i \leq q$, then*

$$\bigcup_{\substack{1\leq i\leq q\\ j=1,2}} N_{G'}(x_{i,j}) \subset N_{G'}(u) \cup N_{G'}(v),$$

where $I(G') = (I(G)^{q+1} \colon \prod_{1\leq i\leq q} x_{e_i})$.

Proof. Since $(I(G)^{i+1} \colon e_1 \cdots e_i) \subset (I(G)^{q+1} \colon e_1 \cdots e_q)$, then every neighbors of $x_{i,2}$ is connected to u. Since $(I(G)^{q-i+1} \colon e_i \cdots e_q) \subset (I(G)^{q+1} \colon e_1 \cdots e_q)$, then every neighbors of $x_{q-i+1,1}$ is connected to v. □

Remark 4.2.3. *Recall that, as it is been defined in the paragraph before Definition 2.5.1, the symbol G_e stands for the induced subgraph of G over the vertex set $V(G) \setminus N_G[e]$.*

Remark 4.2.4. *Let $G = C_n \cdot P_l \cdot C_m$. If $(I(G)^{q+1} \colon e_1 \cdots e_q)$ is not a square-free monomial ideal and G' be the associated graph, then there exist a vertex x_i which is even-connected to itself. Therefore G' has a leaf. By Lemma 4.2.2 one can see $N_{G'}(x_i)$ contains a cycle. In particular, if we denote the leaf by e then G'_e is an induced subgraph of a unicyclic graph.*

Definition 4.2.5. *Let $G = C_n \cdot P_l \cdot C_m$ be a dumbbell graph. Suppose $(I(G)^{q+1} \colon e_1 \cdots e_q)$ is not a square-free monomial ideal and G' be the associated graph with some leaves. A leaf e is called a **critical leaf** if G'_e is obtained by deleting a cycle in G where e does not belong to.*

Example 4.2.6. *Let $G = C_5 \cdot P_2 \cdot C_5$, $e_1 = y_1y_2$, $e_2 = y_3y_4$, $e_3 = y_5y_1$, then x_1 is even-connected to itself with respect to $e_1e_2e_3$ and one can see the graph G' associated to $(I(G)^4 \colon e_1e_2e_3)$ has a critical leaf attached to x_1.*

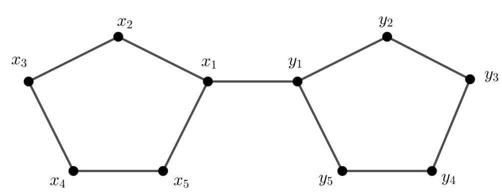


Figure 4.3: The graph G

Remark 4.2.7. *Let e_1 and e_2 be two critical leaf of G on a same cycle (for instance C_n), then by Lemma 4.2.2 edge e_2 is deleted in G'_{e_1}. In addition G'_{e_1} is obtained from G' by deleting at least $m - 3$ vertices.*

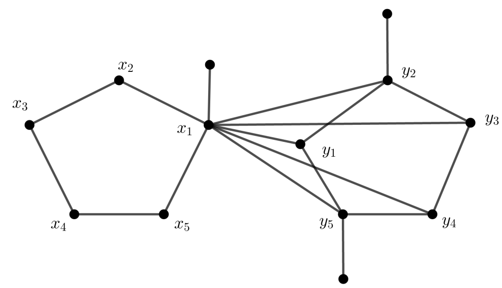


Figure 4.4: The graph G'

Theorem 4.2.8. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leq 2$ and $I = I(G)$ be its edge ideal, then*

$$\operatorname{reg} I^{q+1} \colon e_1 \cdots e_q \leq \operatorname{reg}(I)$$

for any $1 \leq q$ and any edges $e_1, \ldots, e_q \in E(G)$.

Proof. We split the proof into two cases. First, suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is a square-free monomial ideal. In this case $(I^{q+1} \colon e_1 \cdots e_q) = I(G')$ where G' is a graph with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. Let $E(G') = E(G) \cup \{a_1, \ldots, a_r\}$. By Lemma 2.5.4, we have

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G'_{a_1}) + 1\}$$

From Lemma 4.2.2, G'_{a_1} is obtained from G' by removing one of the cycles or deleting at least 6 adjacent vertices. If G'_{a_1} is obtained by removing one of the cycles (for instance remove the whole C_n), then there exists a Hamiltonian path of length $\leq m$ ($\leq m-1$ if $l = 1$). From Theorem 2.5.25 and Remark 4.2.1, we have $\operatorname{reg} I(G'_{a_1}) \leq \operatorname{reg} I(G) - 1$. Suppose G'_{a_1} is obtained by removing 6 adjacent vertices from one of the cycles, say from C_n, and the remaining vertices are $x_1, \cdots, x_{n-6}$. Define H to be the graph obtained by adding a new vertex z and edges $x_{n-6}z$ and zy_1 to G'_{a_1}. Note that G'_{a_1} is an induced subgraph of H. Now, $x_1, \ldots, x_{n-6}, z, y_1, \ldots, y_m$ is a Hamiltonian path in H. By Theorem 2.5.25

$$\operatorname{reg} I(H) \leq \lfloor \frac{n+m-4}{3} \rfloor + 1 = \lfloor \frac{n+m+2}{3} \rfloor - 1.$$

Applying Remark 4.2.1, we get

$$\operatorname{reg} I(G'_{a_1}) \leq \operatorname{reg} I(H) \leq \operatorname{reg} I(G) - 1.$$

Therefore

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G)\}.$$

The same argument gives $\operatorname{reg} I(G' \setminus a_1) \leq \max\{\operatorname{reg} I(G' \setminus \{a_1, a_2\}), \operatorname{reg} I(G)\}$. By continuing this process, we get $\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$.

Suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is not square-free and G' is the graph associated to $(I^{q+1} \colon e_1 \cdots e_q)$. Let G' have r leafs $a_1, \ldots, a_r$ on C_n and s leafs $a'_1, \ldots, a'_s$ on C_m. Note that here we define leafs to be edges. We proceed by induction on r and s. Let $r = 1$ and $s = 0$. From Lemma 2.5.4, we have

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G'_{a_1}) + 1\}.$$

If a_1 is not a critical leaf then by Remark 4.2.4, G'_{a_1} is obtained by removing all vertices in $V(C_n)$ from G'. Remark 4.2.1 implies $\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$. If a_1 be a critical leaf then G'_{a_1} is obtained by removing C_m and at least 3 vertices from C_n so G'_{a_1} is an induced subgraph of a path with $n-3$ vertices therefore $\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$.

Let $r \geq 1$ and $s = 0$. If there exists a non critical leaf, without lose of generality let a_r be a non critical leaf, then $\operatorname{reg}(I)(G') \leq \max\{\operatorname{reg}(I)(G' \setminus a_r), \operatorname{reg}(I)(G'_{a_r}) + 1\}$. Since G'_{a_r} is an induced subgraph of C_m and by induction we get the result. Let all the leaves are critical then by Remark 4.2.7 for all a_i, G'_{a_i} is a induced subgraph of a path with $n-3$ vertices and by using induction we get the result

Let $r \geq 1$ and $s \geq 1$. By Lemma 2.5.4,

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a'_s), \operatorname{reg} I(G'_{a'_s}) + 1\}. \tag{4.10}$$

The induction hypothesis gives $\operatorname{reg} I(G' \setminus a'_s) \leq \operatorname{reg} I(G)$. For simplicity denote $H := G'_{a'_s}$. We consider two cases. Let a'_s be a critical leaf. By Lemma 2.5.4

$$\operatorname{reg} I(H) \leq \max\{\operatorname{reg} I(H \setminus a'_1), \operatorname{reg} I(H_{a'_1}) + 1\}.$$

Since a'_s is a critical leaf in G' by Remark 4.2.7 we deleted all the critical leaves on C_m so a'_1 should be a non critical leaf hence $H_{a'_1}$ is a trivial graph (see Remark 4.2.4). Indeed, we only need to study $\operatorname{reg} I(H \setminus a'_1)$. Applying Lemma 2.5.4 gives

$$\operatorname{reg} I(H \setminus a'_1) \leq \max\{\operatorname{reg} I(H \setminus \{a'_1, a'_2\}), \operatorname{reg} I((H \setminus a'_1)_{a'_2}) + 1\}.$$

With the same argument since a'_s is not a critical leaf, $(H \setminus a'_1)_{a'_2}$ is a trivial graph. Continuing this process we get an induced subgraph of a path with $n-3$ vertices therefore

$$\operatorname{reg} I(H) \leq \lfloor \frac{n-2}{3} \rfloor + 1 \leq \operatorname{reg} I(G) - 1.$$

If a'_s be a non critical leaf then

We use the same methodology by removing leaves on C_n and applying Lemma 2.5.4 inductively. In the i-th step of the process we have to look at $\operatorname{reg}(I(H \setminus \{a_1, \ldots, a_i\}))$ and $\operatorname{reg} I((H \setminus \{a_1, \ldots, a_{i-1}\})_{a_i}) + 1$.

Let $a_1, \ldots, a_k$ are non critical leaves on C_n. Since for all $1 \leq i \leq k$ graph $H \setminus \{a_1, \ldots, a_i\}$ is a trivial graph we have $\operatorname{reg} I(H) \leq \operatorname{reg} I(H \setminus \{a_1, \ldots, a_k\})$. Applying Lemma 2.5.4

$$\operatorname{reg} I(H \setminus \{a_1, \ldots, a_k\}) \leq \{\operatorname{reg} I(H \setminus \{a_1, \ldots, a_{k+1}\}), \operatorname{reg} I((H \setminus \{a_1, \ldots, a_k\})_{a_{k+1}}) + 1\}.$$

Since we already deleted all the non critical leaves by Remark 4.2.7, $(H \setminus \{a_1, \dots, a_k\})_{a_{k+1}})$ is an induced subgraph of a path with $n-5$ vertices. Note that, in H we deleted at least 3 vertices from C_n and we should delete at least 2 more vertices in this step. also, if $n < 5$ then it will become a trivial graph. By Remark 4.2.1 we get $\operatorname{reg} I(P_{n-6}) \leq \operatorname{reg} I(G) - 2$ and $\operatorname{reg} I(P_{n-3}) \leq \operatorname{reg} (I)(G) - 1$. By removing $a_{k+2}, \dots, a_s$ inductively we get

$$\operatorname{reg} I(H) \leq \operatorname{reg} I(G) - 1.$$

Back to (4.10)

$$\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$$

which completes the proof. □

Remark 4.2.9. *The above theorem is a generalization of [36] for the case $l = 0$.*

Theorem 4.2.10. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q \geq 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2,$$

for any $q \geq 1$.

Proof. By virtue of Theorem 2.5.15, we have $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geq 2q + \nu(C_n \cdot P_2 \cdot C_m) - 1$. For the cases where $\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1$ (see Theorem 4.1.6) we get the expected inequality. We divide the proof in two parts, the cases $l = 1$ and $l = 2$.

Case 1. Let $l = 1$. It remains to prove the case $n, m \equiv 2 \pmod 3$. Let H be the induced subgraph of $C_n \cdot P_1 \cdot C_m$ mentioned in the proof of Theorem 4.1.8, i.e. $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\} = C_m \cdot P_{n-1}$. From Theorem 4.1.4, Proposition 4.1.3 and $n, m \equiv 2$ (mod 3), we have

$$\nu(H) = \nu(C_n \cdot P_1 \cdot C_m)$$

and that

$$\nu(H) = \nu(H \setminus \Gamma_H(C_m)).$$

From Theorem 4.1.8 and Theorem 2.5.28 we get

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \nu(C_n \cdot P_1 \cdot C_m) + 2 = \nu(H) + 2 = \operatorname{reg} I(H).$$

Since H is an induced subgraph of $C_n \cdot P_1 \cdot C_m$, then from Theorem 2.5.29 and Corollary 2.5.6 we get the inequality

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)^q \geq \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) - 2.$$

Case 2. Let $l = 2$. It remains to prove the case $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We consider the same induced subgraph H as in Lemma 4.1.13. The induced subgraph $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ of $C_n \cdot P_2 \cdot C_m$ is given as the union of a path of length $n-1$ and the cycle C_m, i.e., $H = P_{n-1} \cup C_m$.

By Theorem 4.1.14, for the cases $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \nu(C_n \cdot P_2 \cdot C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2,$$

and from Theorem 2.5.28 we have

$$\operatorname{reg} I(H) = \nu(H) + 2 = \nu(P_{n-1}) + \nu(C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Hence, we get $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \operatorname{reg} I(H)$. By Theorem 2.5.29 and Corollary 2.5.6, we get the inequality

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geq \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) - 2.$$

□

Theorem 4.2.11. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geq 1$.

Proof. It follows by Theorem 4.2.8, Theorem 2.5.12 and Theorem 4.2.10. □

Remark 4.2.12. *One may ask whether*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

always holds for given n, m, l and q. Unfortunately, this is not the case. In fact, it can be checked that

$$\operatorname{reg} I(C_5 \cdot P_3 \cdot C_5)^2 < 4 + \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5) - 2.$$

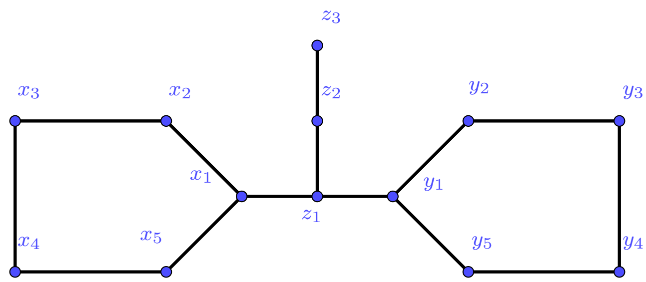
4.3 Characterization of the Regularity of a Given Bicyclic Graph

For a given bicyclic graph G, it is clear that $\nabla(G) \leq 2$, where $\nabla(G)$ is the decycling number of G. From Theorem 2.5.14 and Theorem 2.5.26, we have

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 3.$$

The following is an example of a bicyclic graph meeting the upper bound of the above inequality.

Example 4.3.1. *Consider the graph G:*



One can check that $\operatorname{reg} I(G) = 6$ *and induced matching number* $\nu(G) = 3$.

In this section, we characterize the regularity of bicyclic graphs in terms of their induced matching number. We use the term "dumbbell" of the bicyclic graph G to refer to its unique subgraph of the form $C_n \cdot P_l \cdot C_m$. The following theorem gives the characterization.

Theorem 4.3.2. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following hold:*

(I) Let $n, m \equiv 0, 1 \pmod 3$, *then* $\operatorname{reg} I(G) = \nu(G) + 1$.

(II) Let $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$, *then*

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2,$$

and $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

(III) Let $n, m \equiv 2 \pmod 3$ *and* $l \geq 3$, *then* $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 3$. *Moreover:*

(i) $\operatorname{reg} I(G) = \nu(G) + 3$ *if and only if* $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$.

(ii) $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions hold:*

(a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

(IV) Let $n, m \equiv 2 \pmod 3$ *and* $l \leq 2$, *then* $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2$. *If x is an edge on P_l and $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x, then* $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions are satisfied:*

(a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

(c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. Statement *(I)* follows from Proposition 4.3.4. In Theorem 4.3.12, *(II)* is proved. By Theorem 4.3.18 and Theorem 4.3.21, we get *(III)*. Finally, from Corollary 4.3.23, we obtain *(IV)*. □

The following simple remark will be crucial in our treatment.

Remark 4.3.3. *([1, Observation 2.1]) Let G be a graph with a leaf y and its unique neighbor x, say $e = \{x, y\}$. If $\{e_1, \ldots, e_s\}$ is an induced matching in $G \setminus N[x]$, then $\{e_1, \ldots, e_s, e\}$ is an induced matching in G. So we have $\nu(G \setminus N[x]) + 1 \leq \nu(G)$.*

Proposition 4.3.4. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold.*

(i) When $n, m \equiv 0, 1 \pmod 3$, we have $\operatorname{reg} I(G) = \nu(G) + 1$.

(ii) When $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$, we have $\operatorname{reg} I(G) \leq \nu(G) + 2$.

(iii) When $l \leq 2$, we have $\operatorname{reg} I(G) \leq \nu(G) + 2$.

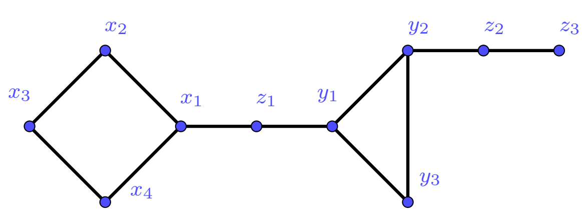
Proof. (i) By virtue of Theorem 2.5.14, it is enough to show $\operatorname{reg} I(G) \leq \nu(G) + 1$. Let E' be the set of edges $E' = E(G) \setminus E(C_n \cdot P_l \cdot C_m)$. We proceed by induction on the cardinality $|E'|$. If $|E'| = 0$ then the statement follows from Theorem 4.1.6. Suppose $|E'| > 0$. There exists a leaf y in G with $N[y] = \{x\}$. Let $G' = G \setminus x$ and $G'' = G \setminus N[x]$, then by Lemma 2.5.4 we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

The graphs G' and G'' are either a bicyclic graph with the same dumbbell $C_n \cdot P_l \cdot C_m$, or a unicyclic graph with cycle C_r ($r = n$ or $r = m$) of the type $r \equiv 0, 1$ (mod 3), or a forest. Using either the induction hypothesis, or Theorem 2.5.28, or Theorem 2.5.23, then we get $\operatorname{reg} I(G') = \nu(G') + 1$ and $\operatorname{reg} I(G'') = \nu(G'') + 1$. Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (by Remark 4.3.3), then we obtain the required inequality.

(ii) and (iii) follow by the same inductive argument, only changing the fact that G' and G'' could be unicyclic graphs with cycle C_r of the type $r \equiv 2$ (mod 3). ☐

Example 4.3.5. *Statement (I) of Theorem 4.3.2. Let G be the graph below.*



Then we have $\operatorname{reg} I(G) = 4$ and $\nu(G) = 3$.

Remark 4.3.6. *The inductive process of the previous proposition cannot conclude $\operatorname{reg} I(G) \leq \nu(G) + 2$ in the case $l \geq 3$. Here one may encounter two disjoint subgraphs G_1 and G_2 with $\operatorname{reg} I(G_i) = \nu(G_i) + 2$, which implies $\operatorname{reg} I(G_1 \cup G_2) = \nu(G_1 \cup G_2) + 3$ (See Example 4.3.1). Nevertheless, with this method an alternative proof of the inequality $\operatorname{reg} I(G) \leq \nu(G) + 3$ follows for an arbitrary bicyclic graph G.*

We take advantage of the following notation in the rest of this section.

Notation 4.3.7. *Let G be a graph and $H \subset G$ be its subgraph. Then by $\Gamma_G(H)$ we denote the set*

$$\Gamma_G(H) = \{v \in G \mid d(v, H) = 1\}.$$

In the case $k > 0$, by $S_{G,k}(H)$ we denote the subgraph induced by the vertex set

$$V(S_{G,k}(H)) = \{v \in G \mid d(v, H) \geq k\}.$$

Moreover, we use $S_{G,0}$ to denote the subgraph induced by the vertex set

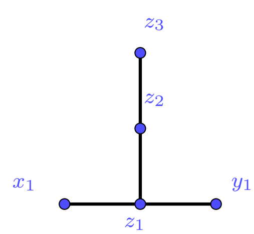
$$V(S_{G,0}(H)) = \{v \in G \mid d(v, H) > 0 \text{ or } \deg(v) \geq 3\}.$$

Here $d(v, H)$ denotes the minimal distance from the vertex v to the subgraph H, and $\deg(v)$ the degree of v (i.e. the number of edges incident to v). We define

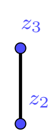
$$d(v, H) = \min\{d(v, w) \mid w \in H\},$$

and $d(v, w)$ denotes the length (i.e., the number of edges) of the shortest path between v and w.

Example 4.3.8. *Let G be the graph of Example 4.3.1 and $H = C_5 \cup C_5$ be the subgraph given by the two cycles of length 5. Then $\Gamma_G(H) = \{z_1\}$. Moreover $S_{G,0}(H)$ is the graph*



and $S_{G,2}(H)$ is the graph:



At this point, the regularity of $I(G)$ in the case $n, m \equiv 0, 1 \pmod 3$ is computed. For the remaining cases we need to consider different cases.

4.3.1 Case I

Throughout this subsection, we consider $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. This case turns out to be almost identical to a unicyclic graph. Our treatment is influenced by the one of [1, Section 3].

Notation 4.3.9. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ and $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We denote by $F_1, \ldots, F_c$ the connected components of $S_{G,0}(C_m)$. In the current case, each F_i is either a tree or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod 3$). Then, the graph $S_{G,2}(C_m)$ is the union of the components $H_1, \ldots, H_c$, where*

$$H_i = F_i \setminus \{v \in G \mid d(v, C_m) \leq 1\}.$$

One notes that each H_i is not necessarily a connected graph, and that it could be even the empty graph.

Lemma 4.3.10. *Adopt Notation 4.3.9. If $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$, then $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$.*

Proof. Follows identically to [1, Lemma 3.5]. □

Proposition 4.3.11. *Adopt Notation 4.3.9. If $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$ then $\operatorname{reg} I(G) = \nu(G) + 1$.*

Proof. By virtue of Theorem 2.5.14, it is enough to show $\operatorname{reg} I(G) \leq \nu(G) + 1$. Using the contrapositive of Lemma 4.3.10, then there exists some i with $\nu(H_i) < \nu(F_i)$. Let x be the vertex in $F_i \cap C_m$. Consider $G' = G \setminus x$ and $G'' = G \setminus N[x]$. By Lemma 2.5.4, we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

Note that both G' and G'' can be either unicyclic graphs with cycle C_n (and $n \equiv 0, 1 \pmod 3$), or forests. Hence, from Theorem 2.5.28 and Theorem 2.5.23, we have $\operatorname{reg} I(G') = \nu(G') + 1$ and $\operatorname{reg} I(G'') = \nu(G'') + 1$.

In the case of G', we have that $\operatorname{reg} I(G') = \nu(G') + 1 \leq \nu(G) + 1$. Let H be the induced subgraph of G obtained by deleting the vertices of $F_i \cup N[x]$. Then we have $G'' = H \cup H_i$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be maximal induced matchings in H and H_i respectively, then $\nu(G'') = |\mathcal{M}_1| + |\mathcal{M}_2|$. By the condition $\nu(F_i) > \nu(H_i)$ then there exists a maximal induced matching $\mathcal{M}_3$ in F_i, such that $|\mathcal{M}_3| > |\mathcal{M}_2|$. From the fact that $H \cup F_i$ is an induced subgraph in G, then we get

$$\nu(G) \geq \nu(H \cup F_i) = |\mathcal{M}_1| + |\mathcal{M}_3| > |\mathcal{M}_1| + |\mathcal{M}_2| = \nu(G'').$$

Hence $\operatorname{reg} I(G'') = \nu(G'') + 1 \leq \nu(G)$. This completes the proof. □

Theorem 4.3.12. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. Then*

(i) $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2$*;*

(ii) $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$*.*

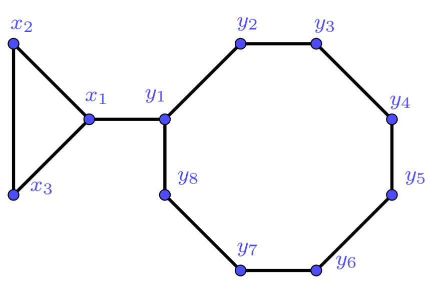
Proof. In Proposition 4.3.4 we proved (i). In order to prove (ii), it is enough to show that $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$ implies $\operatorname{reg} I(G) \geq \nu(G) + 2$, since the inverse follows from Proposition 4.3.11.

As in Notation 4.3.9, let $G \setminus \Gamma_G(C_m) = C_m \cup (\cup_{i=1}^{c} H_i)$ where each H_i is either a forest or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod 3$). Then, from Theorem 2.5.28 and Theorem 2.5.23, we have

$$\begin{aligned}
\operatorname{reg} I(G \setminus \Gamma_G(C_m)) &= \operatorname{reg} I(C_m) + \operatorname{reg} I(\cup_{i=1}^{c} H_i) - 1 \\
&= (\nu(C_m) + 2) + (\nu(\cup_{i=1}^{c} H_i) + 1) - 1 \\
&= \nu(G \setminus \Gamma_G(C_m)) + 2 \\
&= \nu(G) + 2.
\end{aligned}$$

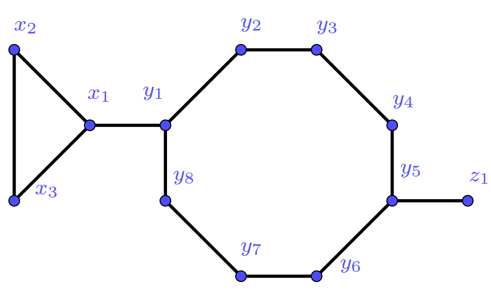
Finally, since $G \setminus \Gamma_G(C_m)$ is an induced subgraph of G then we have $\operatorname{reg} I(G) \geq \nu(G) + 2$ applying Corollary 2.5.6. ☐

Example 4.3.13. *Let G be the graph below.*



Then we have $\operatorname{reg} I(G) = 5$ *and* $\nu(G) = 3$*.*

On the other hand, let G be the graph below.



Then we have $\operatorname{reg} I(G) = 5$ *and* $\nu(G) = 4$.

4.3.2 Case II

In this subsection, we consider the case where $n, m \equiv 2 \pmod 3$, $l \geq 3$, and in particular when $\operatorname{reg} I(G) = \nu(G) + 3$. More specifically, we shall give necessary and sufficient conditions for the equality $\operatorname{reg} I(G) = \nu(G) + 3$.

Notation 4.3.14. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$ *such that* $n, m \equiv 2 \pmod 3$ *and* $l \geq 3$. *As in Notation 4.3.9, let* $F_1, \ldots, F_c$ *be the components of the graph* $S_{G,0}(C_n)$. *We order the* F_i*'s in such a way that* F_1 *is a unicyclic graph with cycle* C_m, *and* F_i *is a tree for all* $i > 1$. *The graph* $S_{G,2}(C_n)$ *can be decomposed in components (not necessarily connected)* $H_1, \ldots, H_c$ *where*

$$H_i = F_i \setminus \{v \in G \mid d(v, C_n) \leq 1\}.$$

Remark 4.3.15. *Due to the assumption* $l \geq 3$, *we have* C_m *is a subgraph of* H_1. *During this subsection and the next one, we fundamentally use this fact. It will allow us to inductively "separate" the two cycles* C_n *and* C_m.

Lemma 4.3.16. *Adopt Notation 4.3.14. If* $\nu(H_i) = \nu(F_i)$ *for all* $1 \leq i \leq c$ *and* $\nu(H_1) = \nu(H_1 \setminus \Gamma_{H_1}(C_m))$, *then*

$$\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G).$$

Proof. Since $G \setminus \Gamma_G(C_n \cup C_m)$ is an induced subgraph of G, we have $\nu(G \setminus \Gamma_G(C_n \cup C_m)) \leq \nu(G)$. To prove the reverse inequality, we observe that

$$G \setminus \Gamma_G(C_n \cup C_m) = C_n \cup \left(\cup_{i=2}^{c} H_i\right) \cup \left(H_1 \setminus \Gamma_{H_1}(C_m)\right). \tag{4.11}$$

Then

$$\begin{aligned}
\nu(G \setminus \Gamma_G(C_n \cup C_m)) &= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\
&= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1) \\
&= \nu(C_n) + \sum_{i=1}^{c} \nu(F_i) \\
&\geq \nu(G).
\end{aligned}$$

Thus $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$. $\square$

Proposition 4.3.17. *Adopt Notation 4.3.14. If $\nu(G \setminus \Gamma_G(C_n \cup C_m)) < \nu(G)$, then*

$$\operatorname{reg} I(G) \leq \nu(G) + 2.$$

Proof. Using the contrapositive of Lemma 4.3.16, then there exists some i with $\nu(H_i) < \nu(F_i)$ or we have $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Therefore, we consider two cases:

Case 1: Suppose $\nu(H_i) < \nu(F_i)$ for some $1 \leq i \leq c$. This case follows similar to Proposition 4.3.11. Let x be the vertex in $F_i \cap C_n$. Consider $G' = G \setminus x$ and $G'' = G \setminus N[x]$. By Lemma 2.5.4, we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

Note that both G' and G'' are unicyclic graphs. Thus by Theorem 2.5.26, we have $\operatorname{reg} I(G') \leq \nu(G') + 2$ and $\operatorname{reg} I(G'') \leq \nu(G'') + 2$. Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (see the proof of Proposition 4.3.11), the inequality in this case follows.

Case 2: Suppose $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Let x be the vertex in $F_1 \cap C_n$. Consider $G' = G \setminus x$ and $G'' = G \setminus N[x]$. By Lemma 2.5.4, we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

The graphs G' and G'' are unicyclic. For the graph G' we have $\operatorname{reg} I(G') \leq \nu(G') + 2 \leq \nu(G) + 2$. The graph G'' can be given as the disjoint union of H_1 and another graph H defined by $H = G \setminus (F_1 \cup N[x])$, that is $G'' = H \cup H_1$. Since H is a forest, then Theorem 2.5.23 gives $\operatorname{reg} I(G'') \leq \nu(G'') + 1$. So $\operatorname{reg} I(G'') + 1 \leq \nu(G'') + 2 \leq \nu(G) + 2$. ☐

At this point, the case where $\operatorname{reg} I(G) = \nu(G) + 3$ can be described.

Theorem 4.3.18. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. Then $\operatorname{reg} I(G) = \nu(G) + 3$ if and only if the following conditions are satisfied:*

(i) $n \equiv 2 \pmod 3$;

(ii) $m \equiv 2 \pmod 3$;

(iii) $l \geq 3$;

(iv) $\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G)$.

Proof. In Proposition 4.3.4 we proved that the conditions (i), (ii) and (iii) are necessary, and from Proposition 4.3.17 we have that the condition (iv) is also necessary. Hence, we only need to prove that $\operatorname{reg} I(G) = \nu(G) + 3$ under these conditions.

From the decomposition in (4.11), and using Theorem 2.5.28 and Theorem 2.5.23, one can compute

$$\begin{aligned}\operatorname{reg}\big(I(G\setminus\Gamma_G(C_n\cup C_m))\big) &= \operatorname{reg}\big(I(C_n)\big)+\operatorname{reg}\big(I\big(\cup_{i=2}^c H_i\big)\big)+\operatorname{reg}\big(I\big(H_1\setminus\Gamma_{H_1}(C_m)\big)\big)-2\\ &= (\nu(C_n)+2)+(\nu(\cup_{i=2}^c H_i)+1)+(\nu(H_1\setminus\Gamma_{H_1}(C_m))+2)-2\\ &= \nu(G\setminus\Gamma_G(C_n\cup C_m))+3\\ &= \nu(G)+3.\end{aligned}$$

Since $G\setminus\Gamma_G(C_n\cup C_m)$ is an induced subgraph of G, by Corollary 2.5.6, one has

$$\operatorname{reg} I(G)\geq \operatorname{reg} I(G\setminus\Gamma_G(C_n\cup C_m))=\nu(G)+3.$$

Thus from Theorem 2.5.26, one has the equality. □

4.3.3 Case III

In this subsection, we assume G is bicyclic graph with dumbbell $C_n\cdot P_l\cdot C_m$ such that $n,m\equiv 2 \pmod 3$ and $l\geq 3$. Now that the case $\operatorname{reg} I(G)=\nu(G)+3$ is characterized, we would like to distinguish between $\operatorname{reg} I(G)=\nu(G)+1$ and $\operatorname{reg} I(G)=\nu(G)+2$.

Lemma 4.3.19. *Adopt Notation 4.3.14. If $\nu(G)-\nu\left(G\setminus\Gamma_G(C_n\cup C_m)\right)=1$ then*

$$\operatorname{reg} I(G)=\nu(G)+2.$$

Proof. From Theorem 4.3.18, we have that $\operatorname{reg} I(G)\leq\nu(G)+2$. Using the same method as in Theorem 4.3.18, we can obtain a lower bound

$$\operatorname{reg} I(G)\geq \operatorname{reg} I(G\setminus\Gamma_G(C_n\cup C_m))=\nu(G\setminus\Gamma_G(C_n\cup C_m))+3=\nu(G)+2,$$

and so the equality follows. □

Lemma 4.3.20. *Adopt Notation 4.3.14. If $\nu(G)=\nu(G\setminus\Gamma_G(C_n))$ then*

$$\operatorname{reg} I(G)\geq\nu(G)+2.$$

Symmetrically, the same argument holds for C_m.

Proof. Making a lower bound similarly to Theorem 4.3.12, we get $\operatorname{reg} I(G)\geq\operatorname{reg} I(G\setminus\Gamma_G(C_n))\geq \nu(G\setminus\Gamma_G(C_n))+2=\nu(G)+2$. □

Taking into account the induced matching numbers $\nu(G)$, $\nu(G\setminus\Gamma_G(C_n\cup C_m))$, $\nu(G\setminus\Gamma_G(C_n))$ and $\nu(G\setminus\Gamma_G(C_m))$, we can give necessary and sufficient conditions for the equality $\operatorname{reg} I(G)=\nu(G)+1$.

Theorem 4.3.21. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geq 3$. Then $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:*

(i) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(ii) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(iii) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

Proof. From Lemma 4.3.19 and Lemma 4.3.20, we have that the conditions (i), (ii) and (iii) are necessary. Hence, it is enough to prove $\operatorname{reg} I(G) \leq \nu(G) + 1$ under these conditions.

For any $x \in G$, consider $G' = G \setminus x$ and $G'' = G \setminus N[x]$. By Lemma 2.5.4, we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

We shall prove that under the conditions (i), (ii) and (iii) there exists a vertex $x \in C_n$ such that $\operatorname{reg} I(G') \leq \nu(G) + 1$ and $\operatorname{reg} I(G'') + 1 \leq \nu(G) + 1$. We divide the proof into three steps.

Step 1. In this step, we prove that for any $x \in C_n$ we have $\operatorname{reg} I(G') \leq \nu(G) + 1$. First we note the following two statements:

- If $\nu(G') < \nu(G)$, then by Theorem 2.5.26, we have $\operatorname{reg} I(G') \leq \nu(G') + 2 \leq \nu(G) + 1$.
- If $\nu(G') > \nu(G' \setminus \Gamma_{G'}(C_m))$, then from Theorem 2.5.28, we get $\operatorname{reg} I(G') \leq \nu(G') + 1 \leq \nu(G) + 1$.

Thus, applying Theorem 2.5.28, we get that

$$\operatorname{reg} I(G') = \nu(G) + 2 \iff \nu(G) = \nu(G') \text{ and } \nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m)).$$

Hence, if we prove that $\nu(G') = \nu(G)$ implies $\nu(G') > \nu(G' \setminus \Gamma_{G'}(C_m))$ then we will get the required inequality $\operatorname{reg} I(G') \leq \nu(G) + 1$. Suppose $\nu(G) = \nu(G')$. From the hypothesis $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$ and the fact that $G' \setminus \Gamma_{G'}(C_m)$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$, then we get

$$\nu(G') = \nu(G) > \nu(G \setminus \Gamma_G(C_m)) \geq \nu(G' \setminus \Gamma_{G'}(C_m)).$$

Step 2. Using $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$ and the same argument of Lemma 4.3.10, then there exists some $1 \leq i \leq c$ such that $\nu(F_i) > \nu(H_i)$. Following Notation 4.3.14, we have that F_1 is a unicyclic graph containing the cycle C_m and that F_i is a tree for all $i > 1$. In this step, let us assume $i > 1$ where F_i is a tree and $\nu(F_i) > \nu(H_i)$.

Let x be the vertex in $F_i \cap C_n$ and H be the induced subgraph $H = G \setminus (F_i \cup N[x])$. We have that $G'' = H \cup H_i$, and we get the inequalities

$$\nu(G'') = \nu(H) + \nu(H_i) < \nu(H) + \nu(F_i) \leq \nu(G)$$

from the condition $\nu(H_i) < \nu(F_i)$ and the fact that $H \cup F_i$ is an induced subgraph of G.

Let K be the induced subgraph defined by $K = (G \setminus \Gamma_G(C_m)) \setminus (F_i \cup N[x])$. Since $i > 1$ then $F_i \cap F_1 = \emptyset$, and so we get the following statements:

- $G'' \setminus \Gamma_{G''}(C_m) = K \cup H_i$.
- $K \cup F_i$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$.
- We have the following inequalities

$$\nu(G'' \setminus \Gamma_{G''}(C_m)) = \nu(K) + \nu(H_i) < \nu(K) + \nu(F_i) \leq \nu(G \setminus \Gamma_G(C_m)).$$

We can apply the same argument as in Step 1 and obtain from Theorem 2.5.28 and Theorem 2.5.26 the following equivalence

$$\operatorname{reg} I(G'') + 1 = \nu(G) + 2 \iff \nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Again, it is enough to prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$. Assuming $\nu(G) = \nu(G'') + 1$ then we can get

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_m)) - 1 \geq \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Step 3. In this last step we assume that $\nu(F_1) > \nu(H_1)$ and that $\nu(F_i) = \nu(H_i)$ for all $i > 1$. Let x be the vertex in $F_1 \cap C_n$, then as in Step 2 we have the statements:

- $\nu(G'') < \nu(G)$.
- $\operatorname{reg} I(G'') + 1 = \nu(G) + 2 \iff \nu(G) = \nu(G'') + 1$ and $\nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))$.

Once more, we shall prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$.

We denote by L the induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$ given by disconnecting all the trees F_i with $i > 1$, that is

$$L = (G'' \setminus \Gamma_{G''}(C_m)) \setminus \Gamma_G(C_n).$$

From the conditions $\nu(F_i) = \nu(H_i)$ for all $i > 1$, then we get $\nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m))$ (see the proofs of Lemma 4.3.10 or Lemma 4.3.16). We also have that L is an induced subgraph of $G \setminus \Gamma_G(C_n \cup C_m)$ because we have the equality

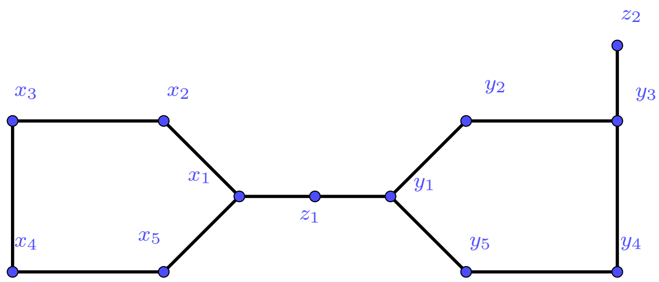
$$L = (G \setminus \Gamma_G(C_n \cup C_m)) \setminus N[x].$$

Finally, from the hypothesis $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$ we can obtain

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_n \cup C_m)) \geq \nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m)). \quad \square$$

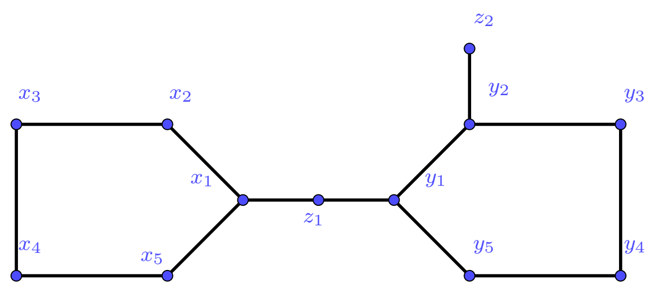
Example 4.3.22. *In Example 4.3.1, we saw a graph G where* $\operatorname{reg} I(G) = 6$ *and* $\nu(G) = 3$.

Let G be the graph below.



Then we have $\operatorname{reg} I(G) = 5$ *and* $\nu(G) = 3$.

But if we move the outer edge to the left, then we get a different result. Let G be the graph below.



Then we have $\operatorname{reg} I(G) = 5$ *and* $\nu(G) = 4$.

4.3.4 Case IV

In this very short subsection we deal with the remaining case. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \leq 2$.

When $l \leq 2$, the two circles are too "close" to each other which makes it difficult to make a direct analysis (with our methods). Fortunately, with the characterization of the case $l \geq 3$, the problem can be solved with the Lozin transformation. Suppose x is a vertex on the bridge P_l. We can perform the Lozin transformation of G on x. This can yield a bicyclic graph $\mathcal{L}_x(G)$ with dumbbell of the type $C_n \cdot P_k \cdot C_m$ where $k \geq 4$. From Theorem 2.5.20, we have

$$\operatorname{reg}\left(I(\mathcal{L}_x(G))\right) - \nu\left(\mathcal{L}_x(G)\right) = \operatorname{reg}\left(I(G)\right) - \nu\left(G\right), \tag{4.12}$$

Hence, we have a characterization in the following corollary.

Corollary 4.3.23. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \leq 2$. Let x be a vertex on P_l and let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x. Then, $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2$. In particular $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:*

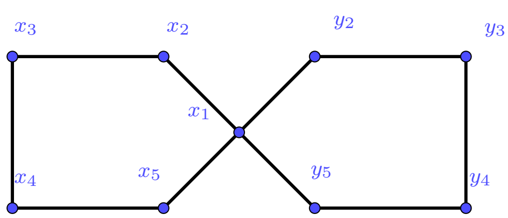
(i) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$*;*

(ii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$*;*

(iii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$*.*

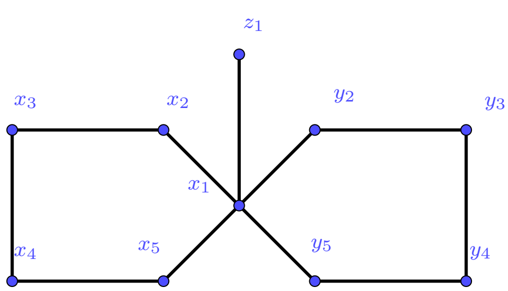
Proof. It follows from Proposition 4.3.4, Equation 4.12, and Theorem 4.3.21. □

Example 4.3.24. *Let G be the graph below.*



Then we have $\operatorname{reg} I(G) = 4$ *and* $\nu(G) = 2$*.*

By adding an edge, let G be the graph below.



Then we have $\operatorname{reg} I(G) = 4$ *and* $\nu(G) = 3$*.*

Chapter 5

Further Research Questions

This chapter is dedicated to collect some questions which arises from chapter 3 and chapter 4. The goal is to introduce some questions to the interested reader and explain our expectations, difficulties and potential strategies to solve the problems. We also include some failed attempts on solving these problems and also wrong expectations on some problems.

5.1 Determinantal Ideals and Linear Products

Question 5.1.1. *Let $\mathcal{F} = \bigcup_{i \leq j} \mathcal{F}^{(i,j)}$ be the family of ideals described in Remark 3.2.17. Does $\mathcal{F}$ have linear products? Can we describe the Gröbner bases of products of ideals $I_1, \dots, I_r \in \mathcal{F}$ with respect to some nice (probably diagonal) term order?*

By several computations with Macaulay2, we expect that $\mathcal{F}$ has linear products. One can use Theorem 2.2.7 to test this conjecture. However, the standard Sagbi deformations approach does not solve this problem(see Remark 3.2.17). Moreover, the sub-family of $\mathcal{F}$ studied in chapter 3 is the "largest" sub-family for which the Sagbi deformation approach is applicable. One can try to prove Theorem 3.2.7 for the case of family $\mathcal{F}$. This could be feasible with carefully expanding the arguments of section 3.1 and following similar strategy to expand Theorem 3.2.7. One notes that since $\mathscr{R}^{\text{in}}(\text{in}(I) : I \in \mathcal{F}) \neq \text{in}(\mathscr{R}(I : I \in \mathcal{F}))$ (see Remark 3.2.17), we can not lift the supposedly proven relations of $\mathscr{R}^{\text{in}}(\text{in}(I) : I \in \mathcal{F})$ to obtain the relations of $\mathscr{R}(I : I \in \mathcal{F})$. Hence, the strategy of Sagbi deformations will fail for this particular family of ideals. On the other hand, it is not an easy task to tackle $\mathscr{R}(I : I \in \mathcal{F})$ directly and prove that it is Koszul. Nevertheless, we expect $\mathscr{R}(I : I \in \mathcal{F})$ to be defined by a quadratic Grb̈ner bases. Due to computations done by Macaulay2, we expect the relations of $\mathscr{R}(I : I \in \mathcal{F})$ to be exactly those that can be lifted from the ones of $\mathscr{R}^{\text{in}}(\text{in}(I) : I \in \mathcal{F})$.

An other way to tackle the problem is to describe the Gröbner bases of the products of ideals of $I_1, \dots, I_r \in \mathcal{F}$ and then try to understand $\text{in}(I_1, \dots, I_r)$. Recall that by the well-known deformation $\beta_{i,j}(I) \leq \beta_{i,j}(\text{in}(I))$, for given ideal I of S, one can address the problem by investigating the reg $(\mathrm{in}(I_1 \dots I_r))$. Unfortunately, this idea is not so straight forward. In the following we try to explain the reason. Let $I_1, \dots, I_r$ be some ideals in the polynomial ring S and let $\mathscr{R}(I_1, \dots, I_r)$ be the associated multi-Rees algebra equipped with an expansion of the term order of S. Recall that $\mathscr{R}(\mathrm{in}(I_1) \dots \mathrm{in}(I_r)) \neq \mathrm{in}(\mathscr{R}(I_1, \dots, I_r))$ is equivalent to $\mathrm{in}(I_1 \dots I_r) \neq \mathrm{in}(I_1) \dots \mathrm{in}(I_r)$. Therefore, for the case of $I_1, \dots, I_r \in \mathcal{F}$, we can not expect the natural generators of the product $I_1 \dots I_r$ to form a Gröbner bases with respect to the term order of S. In the following, we illustrate an example of this case for which the natural generators of product of ideals does not form a Gröbner bases with respect to Lex.

Code 5.1.2. *Let S be the polynomial ring with 6 indeterminates equipped with* Lex *term order. In the following we have $I \in \mathcal{F}^{(1,6)}$ and $J \in \mathcal{F}^{(3,6)}$.*

```
i1 : S = QQ[x_1..x_6, MonomialOrder => Lex ];
i2 : M1 = matrix{{x_1..x_5},{x_2..x_6}}
o2 = | x_1 x_2 x_3 x_4 x_5 |
     | x_2 x_3 x_4 x_5 x_6 |
i3 : M2 = matrix{{x_3..x_5},{x_4..x_6}}
o3 = | x_3 x_4 x_5 |
     | x_4 x_5 x_6 |
i4 : I = minors(2,M1);
i5 : J = minors(2,M2);
i6 : L = first entries gens gb (I*J);
i7 : leadTerm L_24
o7 = x_2^2 x_4 x_6
```

In the above example, it is clear that $x_2^2x_4x_6$ can not be presented as the products of two diagonal terms of the matrices $M1$ and $M2$ since the later one does not include the indeterminate x_2. Recall that in [24], the author clearly identifies the leading term of every minor of a given Hankel matrix by the product of the elements located on the main diagonal. These elements are in fact the so called chains (see chapter 3 for more detail). Hence, there exists only one pair of chains of length two that divides $x_2^2x_4x_6$ which are x_2x_4 and x_2x_6. It is clear that neither of the chains are a diagonal term of a 2-minor of $M2$. Therefore, the natural generators of product of ideals $I_1, \dots, I_r \in \mathcal{F}$ do not always form a Gröbner bases. However, it is still interesting to describe the required Gröbner bases with respect to some term order.

In the following we provide an example in the form a Macaulay2 code illustrating the kernel multi-Rees algebras of I and J from the above example.

Code 5.1.3. *Let I and J be the ideals defined in the above example. Note that the function monvars is simply used to show a better notation for indices of the variables in the presentations of $\mathscr{R}(I, J)$ as a quotient ring. The double indices are the indices of the diagonal terms of their associated minors.*

```
-- the function retruns the indices of a monomial
```

```
--INPUT: a monomial
--OUTPUT: a list
i1 : monvars = (mon) -> (
     RING := ring mon;
     var := first entries vars RING;
     flatten for i to length var -1 list (
     while mon % var_i ==0 list
       i+1
     do
       mon=mon//var_i
       )
  )

o1 = monvars
o1 : FunctionClosure
i2 : S = QQ[x_1..x_6 , T_1 , T_2, MonomialOrder => Lex ,
   Degrees=>{6:{1,0,0},{0,-2,0},{0,0,-2}} ];

-- the function computes the kernel of themulti- Rees
   algebra of I and J
-- INPUT: tow ideals
-- OUTPUT: ideal

i3 : reesKer = (I,J) -> (
         RING := ring I;
         genI := first entries gens I;
         genJ := first entries gens J;
         l := first entries vars RING;
         xvars := l_{0..#l-3};
         Zvars := for i to length genI -1 list (Z_(monvars(
            leadTerm genI_i)));
         Yvars := for i to length genJ -1 list (Y_(monvars(
            leadTerm genJ_i)));
         R := QQ[X_1..X_(#xvars) , Zvars , Yvars , Degrees =>
         {#xvars:{1,0,0} ,#Zvars:{0,1,0} , #Yvars:{0,0,1}}];
         genAlg := xvars|T_1*genI|T_2*genJ;
         rees := map(S,R,genAlg);
         K := trim ker rees;
         return K
         )
o3 = reesKer
o3 : FunctionClosure
```

```
i4 : K = trim reesKer(I,J);
i5 : first entries gens K
```

The above will output the defining ideal of $\mathscr{R}(I,J)$. One needs to focus on the generators containing Z and Y variable together, since the other cases are given by $\mathscr{R}(I)$ and $\mathscr{R}(J)$. They are given in the following:

$$
\begin{aligned}
&Z_{\{4,6\}}Y_{\{3,6\}} - Z_{\{3,6\}}Y_{\{4,6\}},\\
&Z_{\{4,6\}}Y_{\{3,5\}} - Z_{\{3,5\}}Y_{\{4,6\}},\\
&Z_{\{3,6\}}Y_{\{3,5\}} - Z_{\{3,5\}}Y_{\{3,6\}},\\
&Z_{\{2,6\}}Y_{\{3,5\}} - Z_{\{2,5\}}Y_{\{3,6\}} + Z_{\{2,4\}}Y_{\{4,6\}},\\
&Z_{\{1,6\}}Y_{\{3,5\}} - Z_{\{1,5\}}Y_{\{3,6\}} + Z_{\{1,4\}}Y_{\{4,6\}}
\end{aligned}
$$

Question 5.1.4. *Let $\mathcal{F} = \mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$ be the family of determinantal ideals of so called close cuts of Hankel matrices (see chapter 3). Can we describe the primary decomposition of the product of $I_1, \dots, I_r \in \mathcal{F}$?*

For the product of ideals $J_1, \dots, J_s \in \mathcal{F}^{(1,n)}$, an interesting primary decomposition is described in [24] and [30], which is given by symbolic powers of ideals in $\mathcal{F}^{(1,n)}$ containing $J_1 \dots J_s$. We expect similar behavior for the family $\mathcal{F} = \mathcal{F}^{(1,n)} \cup \mathcal{F}^{(1,n-1)} \cup \mathcal{F}^{(2,n)} \cup \mathcal{F}^{(2,n-1)}$.

5.2 Regularity of Powers of Edge Ideals

Turns out that the study of asymptotic behavior of regularity of powers of edge ideals is a quite attractive topic at the time of writing this thesis. For the convenience, we recall Conjecture 2.5.18 here:

Conjecture 5.2.1. *Let G be a simple undirected graph. Then for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \leq 2q + \operatorname{reg} I(G) - 2.$$

The above conjecture has been settled for gap-free graphs, unicyclic graphs and some dumbbell graphs (see [7], [1] and [23]) and it is open in general.

Question 5.2.2. *The natural question on expanding Theorem 4.2.11 could be the following:*

(1) Let $C_n \cdot P_l \cdot C_m$ be a dumbbell graph with $l \geq 3$. Can we prove Conjecture 5.2.1 for this family of graphs?

(2) Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ with $l \leq 2$. Can we prove Conjecture 5.2.1 for this family?

(3) *Let G be a cactus graph i.e a graph constructed by connecting some cycles via some paths in a way that every cycle is connected to at most two other cycles (see Figure 5.1). Can we compute the regularity of $I(G)$ in terms of the induced matching number of G? Can we prove Conjecture 5.2.1 for the case that the connecting paths have at most two vertices?*

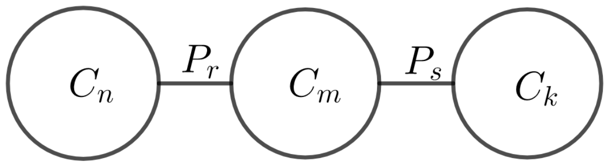


Figure 5.1: A cactus graph with cycles C_n, C_m and C_k and paths P_r and P_s

The above question aims for a very specific class of graphs, however, they are still interesting questions since very few families have settled Conjecture 5.2.1. Regarding the above question, from Remark 4.2.12, we already know that the equality is not always the case for part (1). For part (2), one can try to mimic the proof given in [1], however, the combinatorial detail, as it is expected, will significantly increases. For part (3), one can try to expand the strategy of Theorem 4.1.4 to prove some formula for the induced matching number of a cactus graph in terms of the size of cycles and lengths of paths. By Theorem 2.5.14, and similar strategy as Theorem 4.1.6 one may compute the regularity in term of the induced matching number.
As we mentioned in section 2.2, the study of Rees algebras of ideals have proven to be powerful tools in understanding the behavior of powers of ideals. One natural question to ask is whether one can tackle Conjecture 5.2.1 considering the Rees algebra of edge ideals? It turns out that the relations of $\mathscr{R}(I(G))$ can be read from G. In [58], the author describes the relations of $\mathscr{R}(I(G))$ in terms of combinatorial data of G. In particular note that Theorem 2.2.6 shows that $\operatorname{reg}_0 \mathscr{R}(I)$ bounds the so called constant in the asymptotic formula computing the regularity of large powers of given ideal I in polynomial ring S. Hence, it is interesting to investigate $\operatorname{reg}_0 \mathscr{R}(I(G))$ for graph G.

Question 5.2.3. *Let G be a graph. Can we prove $\operatorname{reg}_0 \mathscr{R}(I(G)) \leq \text{co-chordal}(G) - 1$?*

The above question is trivial when $c = 1$ due to [39]. We expect it to be the case for $c = 2$. A positive answer to the above question will reprove Theorem 2.5.17 and lead to the following question:

Question 5.2.4. *Let G_1 and G_2 be two given graphs. Can we compute or bound $\operatorname{reg}_0 \mathscr{R}(I(G_1)+I(G_1))$ in terms of the $\operatorname{reg}_0 \mathscr{R}(I(G_1))$ and $\operatorname{reg}_0 \mathscr{R}(I(G_2))$?*
In general, when G is a graph, can we compute $\operatorname{reg}_0 \mathscr{R}(I(G))$ in terms of some combinatorial data of G? Most importantly, can we compute or bound $\operatorname{reg}_0 \mathscr{R}(I(G))$ in terms of $\operatorname{reg}(I(G))$?

Tackling the above question can open new approaches in the study of Conjecture 5.2.1. The last part of the above question has a positive answer for unicyclic graphs (see Theorem 2.5.29) and some dumbbell graphs (see Theorem 4.2.11) and paths (it can be deduced from Theorem 2.5.23 and Remark 2.5.21). Due to Theorem 2.2.6, Theorem 2.5.29, Theorem 2.5.23 and Remark 2.5.21 it is clear that $\operatorname{reg}_0 \mathscr{R}(I(G)) \geq \operatorname{reg}(I(G)) - 2$ when G is a path or a cycle. We expect the equality to hold when G is a path or a cycle.

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