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Zeros of L-functions and cancellation of modular coefficients along prime numbers

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Abstract

This thesis is divided into two main parts. In Chapter 1, we consider the average of modular coefficients over prime numbers, using the classical circle method.
In Chapter 2 and 3, which correspond to the second part, we focus on Dirichlet series. In particular, in Chapter 2 we deal with the distribution of the zeros, giving an account of the main examples of Dirichlet series with infinitely many zeros in the region of absolute convergence. We prove the existence of zeros of this type for a generalized version of the Hurwitz zeta function. In Chapter 3, we consider this problem in the framework of the Selberg Class $\mathcal{S}$ of L-functions. We first give a general overview of the theory of $\mathcal{S}$ and its extension $\mathcal{S}^\sharp$. Then, we focus on our main problem. Given a degree 1 function in $\mathcal{S}^\sharp$, we are interested in studying the analytic properties of its linear twists. We prove that the linear twists satisfy a functional equation of Hurwitz-Lerch type and we also give some results on the distribution of the zeros outside the critical strip.

Contents

Introduction

This thesis is divided into two main parts. In the first part (Chapter 1), we consider the average of modular coefficients over prime numbers. The result can be seen as the analogue of the prime number theorem for the convolution of a Hecke-Maass eigenform and the modular form of half-integral weight whose n-th Fourier coefficient is the number of representations of n as the sum of three squares (cf. equation (1.1.11)). Our result (cf. Theorem 1.1, [60, Theorem 1]) improves a recent bound obtained by Hu in [25]. The main technique used is the classical circle method (cf. e.g. [58]) and a key tool in the proof of our estimate is a result of Fouvry and Ganguly [21], based on the absence of Siegel zeros proved by Hoffstein and Ramakrishnan [24]. It is interesting to observe that, since our work is roughly equivalent to proving the cancellation for the correlation of the Möbius function with modular coefficients, our problem can be linked with the so-called *Möbius Randomness Law.* This predicts that the terms 1 and -1 of the Möbius function will cancel on average when multiplied by any "reasonable" function (cf. Section 1.1).

In the second part (Chapters 2 and 3), we deal with Dirichlet series and their analytic properties. In Chapter 3, we consider the Selberg class $\mathcal{S}$ of L-functions. In particular, we focus on its extension $\mathcal{S}^\sharp$, which is the class of the absolutely convergent Dirichlet series admitting a meromorphic continuation to the complex plane and satisfying a functional equation. We give a brief general overview of the theory of the Selberg class, also giving some examples of important functions belonging to it. Our main purpose is to study the analytic properties of the linear twists of a function of degree 1 in $\mathcal{S}^\sharp$. The first step is proving that the linear twists satisfy a functional equation reflecting s into $1-s$, which can be seen as a Hurwitz-Lerch type of functional equation. Going further in the study of the linear twists, we focus on the problem of the distribution of their zeros. In particular, we are interested in analyzing the zeros outside the critical strip. In fact, it is well-known that there exist several examples of Dirichlet series which do not satisfy the Riemann hypothesis. As a first work on this topic, we mention Potter and Titchmarsh's paper on the zeros of the Epstein zeta function [47], where they stated that numerical evidence suggests the existence of a particular Epstein zeta function, without an Euler product, with a zero outside of the critical line. The case of the Epstein zeta function was later studied by Davenport and Heilbronn [16], together with the Hurwitz zeta function. To prove the existence of infinitely many zeros in the region of absolute convergence, the authors introduced a new technique, essentially based on the fact that equivalent Dirichlet series (see Definition 2.4) take the same set of values in vertical strips. Other key ingredients

in their argument are the almost periodicity property of L-functions (cf. Section 2.1.3) and Rouché's theorem (cf. Theorem 2.7). In Chapter 2, we discuss the result of Davenport and Heilbronn, giving a sketch of their proof, and we mention Cassels' work [9], which completes the analysis of the Hurwitz zeta function. We also present other results about Dirichlet series with zeros in the half-plane $\sigma = \Re(s) > 1$. Indeed, the technique of Davenport and Heilbronn has been refined and applied in several cases. For instance, Conrey and Ghosh [14] considered the L-function associated to a cusp form of weight 24 for the full modular group, while Saias and Weingartner [52] studied the case of Dirichlet series with periodic coefficients. Booker and Thorne [7] extended the result to the case of combinations of L-functions coming from automorphic representations, under Ramanujan conjecture. Finally, Righetti [50] generalized the proof to combinations of Dirichlet series with Euler products.
Theorem 2.15 (appearing in our preprint [61, Theorem 1]) also goes in this direction. Indeed, we consider a generalization of the Hurwitz zeta function, defined by

$$F(s, f, \alpha) = \sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s},$$

for $\sigma > 1$, where $0 < \alpha \leq 1$ and $f(n)$ is a non identically zero periodic function. We generalize the results of Davenport-Heilbronn and Cassels to $F(s, f, \alpha)$. Since the proof is obtained without any additional assumption, our result also improves the work of Chatterjee and Gun [10], where some restrictive conditions were required.
In the proof of our result, we distinguish three cases. If α is transcendental, we essentially apply the argument of Davenport and Heilbronn. The case of α rational can be deduced by Saias and Weingartner [52], while for algebraic irrational values of α we will need to prove a suitably modified version of Cassels' lemma [9, Lemma].

Going back to the linear twists, we observe that the two terms on the right-hand side of the functional equation (3.3.2) are Hurwitz zeta functions with periodic coefficients. Then, our result on the distribution of the zeros of generalized Hurwitz zeta functions can be applied to prove the existence of infinitely many zeros for the linear twists, provided that some conditions hold. In particular, thanks to Theorem 2.15, for any value of α we can deduce that the linear twist has infinitely many zeros in the left half-plane $\sigma < 0$. On the other hand, as will be explained in detail in Section 3.6, in the region of absolute convergence the proof of the existence of the zeros is complete only if α is rational.
Our results on the linear twists (cf. [62]) are collected in Sections 3.3-3.6.
We want to remark that in [37], Kaczorowski and Perelli considered a similar problem for degree 2 functions. In particular, they studied the standard twist of L-functions associated to half-integral weight cusp forms, deriving a functional equation and analyzing trivial and non-trivial zeros. In Section 3.2.1, we briefly recall the main results of their paper.

Chapter 1

On the average of modular coefficients over primes

1.1 Introduction

Given $f : \mathbb{N} \to \mathbb{C}$, one of the basic problems in analytic number theory is to estimate sums of the form

$$\sum_{p \le x} f(p)$$

as $x \to +\infty$, where p runs over prime numbers. A very closely related sum, essentially equivalent and more convenient to deal with, is

$$\sum_{n \le x} f(n)\Lambda(n), \tag{1.1.1}$$

where Λ is the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^m \quad \text{with} \quad m \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall the identity

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right), \tag{1.1.2}$$

where μ is the Möbius function,

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ (-1)^k & \text{if} \quad n = p_1 \cdot \dots \cdot p_k, \quad p_i \ne p_j \quad \text{if} \quad i \ne j \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.3}$$

Then, dealing with (1.1.1) is roughly similar to giving an estimate of

$$\sum_{n \leq x} \mu(n) f(n). \tag{1.1.4}$$

Sums of the form (1.1.4) are related to a principle concerning the randomness of μ. It states that summing the Möbius function against any "reasonable" f leads to a significant cancellation. This is the so-called *Möbius Randomness Law* (cf. [28, 13.1]), saying that μ does not correlate with any function of "low complexity". In general it means that

$$\sum_{n \leq x} \mu(n) f(n) = o\bigg(\sum_{n \leq x} |f(n)| \bigg), \tag{1.1.5}$$

and, assuming that f is bounded,

$$\sum_{n \leq x} \mu(n) f(n) = o(x) \quad \text{as} \quad x \to \infty. \tag{1.1.6}$$

An interesting problem is trying to understand for which functions f equations (1.1.5) and (1.1.6) hold. In particular, precise notions of *reasonable* and *low complexity* are required. This is the main purpose of Sarnak's lectures on Möbius function randomness and dynamics [53]. The problem is still open, but there are interesting conjectures in this field. We refer to these lectures for more details.

In this thesis, we focus on the particular case of the sum

$$\pi_{a,\Lambda}(x) = \sum_{m_1^2+m_2^2+m_3^2 \leq x} a(m_1^2+m_2^2+m_3^2)\Lambda(m_1^2+m_2^2+m_3^2), \tag{1.1.7}$$

where $(a(n))_{n \geq 1}$ is the sequence of the normalized Fourier coefficients of a holomorphic or Maass cusp form f. This problem has been studied by Hu in [25]. He proved the bound

$$\pi_{a,\Lambda}(x) = O(x^{3/2} \log^c x), \tag{1.1.8}$$

where c is a suitable positive constant. It can be observed that (1.1.8) could be obtained in a simpler way applying Cauchy-Schwarz's inequality. We also claim that a stronger bound can be obtained, taking into account the absence of exceptional zeros proved by Hoffstein and Ramakrishnan in [24] (see Lemma 1.8 below).

We now give a brief account of the theory of holomorphic and Maass cusp forms. Even if we will focus on forms of level one, we now consider a general level $N \geq 1$. Given the *full modular group*

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$

the Hecke congruence subgroup $\Gamma_0(N)$ is defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod N \right\}.$$

The action of this group on the upper half-plane $\mathbb{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ is

$$\gamma(z) := \frac{az+b}{cz+d}, \quad \text{for all} \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The quotient space $\Gamma_0(N)\backslash\mathbb{H}$ is not compact, but a compactification can be obtained by adding a finite number of points, called *cusps* (cf. e.g. [28, 14.1] for a precise argument).
Let now χ be a Dirichlet character modulo N. For any $\gamma \in \Gamma_0(N)$ as above, we define $\chi(\gamma) := \chi(d)$, thus considering χ as a character on the group $\Gamma_0(N)$. A *modular form* of level N, weight $k \geq 1$ and character χ is a function $f : \mathbb{H} \to \mathbb{C}$ holomorphic on $\mathbb{H}$ and at the cusps, satisfying

$$f_{|\gamma}(z) = \chi(\gamma) f(z) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{1.1.9}$$

where

$$f_{|\gamma}(z) := (cz+d)^{-k} f(\gamma(z)).$$

We spend a few words about holomorphy at the cusps. Since $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, if $f(z)$ satisfies (1.1.9) it is 1-periodic, then it admits a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a(n) n^{\frac{k-1}{2}} e(nz),$$

for $\Re(z) > 0$, where $e(z) := e^{2\pi i z}$ and the $a(n)$, $n \geq 1$ are the normalized Fourier coefficients of f. We say that f is holomorphic at $i\infty$ if $a(n) = 0$ for $n < 0$. If c is another cusp, f is holomorphic at c if

$$f_{|\gamma_c}(z) = \sum_{n=0}^{\infty} a(n, c) e(nz),$$

where $\gamma_c \in SL_2(\mathbb{R})$ is the scaling matrix sending $i\infty$ to c. A *cusp form* is a modular form vanishing at all cusps. Then, in particular, it can be written as a Fourier series of the form

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e(nz).$$

The space of cusp forms of level N, weight k and character χ is denoted by $S_k(N, \chi)$.
Consider now the Laplace operator Δ_k of weight k, defined by

$$\Delta_k = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} \right) - iky \frac{\partial}{\partial x}.$$

A smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying (1.1.9) that is also an eigenfunction of the above Laplace operator, i.e. such that $(\Delta_k + \lambda)f = 0$ for some complex number λ, is a *Maass form* of level N, weight k, character χ and Laplace eigenvalue λ. Moreover, it is a *Maass cusp form* if it vanishes at all cusps (cf. [19, §4]). As shown in [21, 2.2] or in [30, §1.4.6], a Maass form admits a Fourier expansion

$$f(z) = \sum_{n \neq 0} \rho(n) W_{\frac{kn}{2|n|}, ir}(4\pi |n| y) e(nx),$$

where r is the *spectral parameter*, which is related to the Laplace eigenvalue by $r^2 = \lambda - \frac{1}{4}$, and $W_{\alpha,\beta}$ is the Whittaker function [59]. We then define the normalized Fourier coefficients by

$$a(n) := \left(\frac{4\pi |n| y}{\cosh \pi r} \right)^{\frac{1}{2}} \rho(n).$$

Again, see [19, §4] for more details. It can be observed that the space $S_k(N, \chi)$ can be embedded into the space of Maass cusp forms, via the map

$$f \in S_k(N, \chi) \longmapsto y^{k/2} f(z), \quad \text{where} \quad y = \Im(z).$$

We can define the n-th *Hecke operator* acting on the space $S_k(N, \chi)$ as

$$T_{n,\chi} : f(z) \longmapsto (T_{n,\chi} f)(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{b \,(\mathrm{mod}\, N)} f\left(\frac{az + b}{d} \right),$$

and similarly the action of the n-th Hecke operator on the space of Maass cusp forms can be defined by

$$T'_{n,\chi} : f(z) \longmapsto (T'_{n,\chi} f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \,(\mathrm{mod}\, N)} f\left(\frac{az + b}{d} \right).$$

The forms which are common eigenfunctions of the Hecke operators $T_{n,\chi}$ for $(n, N) = 1$ define an orthonormal basis, the so-called *Hecke basis.* The elements of this basis are called *Hecke-Maass cusp forms.* If a Hecke-Maass cusp form is not induced by a modular form of lower level it is called a *primitive form.* Observe that a Hecke-Maass cusp form of level one is trivially a primitive form. In the following, we will assume f to be either a holomorphic or a Maass cusp form of level one, with normalized Fourier coefficients $a(n)$, $n \geq 1$. In particular, we will focus on primitive Maass-cusp form, with the meaning that holomorphic cusp forms are included. Thus, we can proceed with a unified argument.

Remark 1.1. We want to observe that

$$\sum_{m_1^2 + m_2^2 + m_3^2 \leq x} a(m_1^2 + m_2^2 + m_3^2) = \sum_{n \leq x} a(n) r_3(n),$$

where $r_3(n)$ is the number of representations of the integer n as the sum of three squares.

In general, given a positive integer k, if $r_k(n)$ is the number of representations of n as the sum of k squares, we have

$$\sum_{n=0}^{\infty} r_k(n)e(nz) = \theta(z)^k, \tag{1.1.10}$$

where θ denotes the classical *theta function*

$$\theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z).$$

The theta function is a modular form of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$. Recall that, given an odd integer k, a positive integer N divisible by 4 and a character χ modulo N, a modular form of half-integral weight $\frac{k}{2}$, level N and character χ is a holomorphic function defined on the upper half plane $\mathbb{H}$, satisfying

$$f(\gamma z) = \chi(d) j(\gamma, z)^k f(z),$$

for all $\gamma \in \Gamma_0(N)$, and being holomorphic at the cusps. Here the automorphy factor $j(\gamma, z)$ is

$$j(\gamma, z) = \frac{\theta(\gamma(z))}{\theta(z)}.$$

It follows that (1.1.10) is a modular form of weight $\frac{k}{2}$ and, particular, $r_3(n)$ is the n-th Fourier coefficient of a modular form of weight $\frac{3}{2}$. Then, we can write

$$\pi_{a,\Lambda}(x) = \sum_{m_1^2+m_2^2+m_3^2\le x} a(m_1^2+m_2^2+m_3^2)\Lambda(m_1^2+m_2^2+m_3^2) = \sum_{n\le x} a(n) r_3(n)\Lambda(n). \tag{1.1.11}$$

Hence, our result can be seen as the analogue of the prime number theorem for the convolution of a Maass cusp form and a modular form of half-integral weight. For a more detailed theory of modular forms of half-integral weight we refer to Shimura [55], while one can see Iwaniec [27, Chapter 10-11] for an overview of the theta function.

Remark 1.2. We recall that the Ramanujan's conjecture, claiming that

$$|a(p)| \le 2, \quad \text{for} \quad p \text{ prime},$$

is a well-known result for holomorphic cusp forms of integral weight (cf. Deligne [17], [18]). It follows that, for all $\varepsilon > 0$, the normalized coefficients satisfy $a(n) = O(n^{\varepsilon})$. On the other hand, for Maass forms the best bound currently obtained is

$$|a(p)| \le 2p^{\theta} \quad \text{with} \quad \theta = \frac{7}{64},$$

due to Kim and Sarnak [38]. Hence, the coefficients of Maass cusp forms satisfy, for all $\varepsilon > 0$,

$$a(n) = O(n^{\theta+\varepsilon}).$$

In the above described setting, we prove the following theorem (cf. [60, Theorem 1]), which improves the bound (1.1.8) obtained by Hu.

Theorem 1.1. *Let f be either a holomorphic or Maass cusp form for the full modular group and let $a(n)$ be the n-th normalized Fourier coefficient of f. Then, there exists a constant $c > 0$ such that*

$$\pi_{a,\Lambda}(x) = O(x^{3/2}\exp(-c\sqrt{\log x})).$$

In the following sections, we first introduce some notations and preliminary results, then we focus on the proof of our statement.

1.2 Notation and outline of the method

Since we follow Hu's approach to the problem and refer to his paper [25] for some details, we introduce the same notation. For $\alpha \in \mathbb{R}$ and $y > 1$, define

$$S_1(\alpha, y) := \sum_{1\leq m\leq y} e(m^2\alpha), \qquad S_2(\alpha, y) := \sum_{|m|\leq y} e(m^2\alpha) \tag{1.2.1}$$

and the exponential sum

$$T(\alpha, y) := \sum_{1\leq n\leq y} a(n)\Lambda(n)e(n\alpha). \tag{1.2.2}$$

As a first step, we have

$$\pi_{a,\Lambda}(x) = \int_0^1 S_2^3(\alpha, \sqrt{x})T(-\alpha, x)d\alpha.$$

In fact, recalling

$$\int_0^1 e(n\alpha)d\alpha = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

we can easily observe that

$$\begin{aligned}
\int_0^1 S_2^3(\alpha, \sqrt{x})T(-\alpha, x)d\alpha &= \int_0^1 \sum_{\substack{|m_i|\leq\sqrt{x}\\ i=1,2,3}} e((m_1^2+m_2^2+m_3^2)\alpha) \sum_{1\leq n\leq x} a(n)\Lambda(n)e(-n\alpha)d\alpha \\
&= \sum_{\substack{|m_i|\leq\sqrt{x}\\ i=1,2,3\\ n\leq x}} a(n)\Lambda(n)\int_0^1 e((m_1^2+m_2^2+m_3^2-n)\alpha)d\alpha \\
&= \sum_{m_1^2+m_2^2+m_3^2\leq x} a(m_1^2+m_2^2+m_3^2)\Lambda(m_1^2+m_2^2+m_3^2) = \pi_{a,\Lambda}(x).
\end{aligned}$$

Moreover, since $S_2(\alpha, y) = 2S_1(\alpha, y) + 1$, we get, for all $\varepsilon > 0$,

$$\pi_{a,\Lambda}(x) = 8\int_0^1 S_1^3(\alpha, \sqrt{x})T(-\alpha, x)d\alpha + O(x^{1+\theta+\varepsilon}).$$

Let now $P = \exp(C\sqrt{\log x})$, for a positive constant C, and let $Q = xP^{-1} = x\exp(-C\sqrt{\log x})$. Thanks to the periodicity of the integrand,

$$\begin{aligned}\pi_{a,\Lambda}(x) &= 8\int_0^1 S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha + O(x^{1+\theta+\varepsilon})\\ &= 8\int_{1/Q}^{1+1/Q} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha + O(x^{1+\theta+\varepsilon}).\end{aligned} \tag{1.2.3}$$

Dirichlet's theorem on rational approximation (cf. e.g. [58, Lemma 2.1]) assures that for any real α and any $Q \geq 1$ there exists a rational number $\frac{a}{q}$, with $(a,q) = 1$, $1 \leq q \leq Q$ such that

$$\alpha = \frac{a}{q} + \beta \quad \text{with} \quad |\beta| \leq \frac{1}{qQ}.$$

For $1 \leq q \leq P$ and $1 \leq a \leq q$, $(a,q) = 1$ let

$$\mathcal{M}(a,q) = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

The set $\mathcal{M}$ of the *major arcs* is given by the union of the $\mathcal{M}(a,q)$, as a,q run in the above ranges. As usual, the set m of the *minor arcs* is instead

$$\mathsf{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathcal{M}.$$

Hence, by equation (1.2.3)

$$\pi_{a,\Lambda}(x) = 8\int_{\mathcal{M}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha + 8\int_{\mathsf{m}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha + O(x^{1+\theta+\varepsilon}). \tag{1.2.4}$$

Our goal is now to give an estimate of the integrand on major and minor arcs.

Remark 1.3. For an introduction to the classical circle method we refer again to [58]. In particular, in Chapter 1 a historical background is given.

1.3 First estimates

In this section we first recall the estimates of $S_1(\alpha,\sqrt{x})$ on major and minor arcs given by Hu in Lemma 4.1 and 4.2 of [25] respectively. Then, we discuss a key result that allows us to improve the estimate of $T(\alpha,x)$ on major arcs (Lemma 1.8).
We start presenting a list of preliminary lemmas (cf. [25, §3]). For a real number t, we denote by $||t||$ the distance of t from the nearest integer, i.e.

$$||t|| := \min(\{t\}, 1 - \{t\}),$$

where $\{t\} := t - \lfloor t \rfloor$ is the fractional part of t. The following lemma (cf. [25, Lemma 3.1]) is a partial summation result.

Lemma 1.1. *Let* (b_n), $n \geq 1$ *be a sequence of complex numbers such that*

$$B(u) = \sum_{n \leq u} b_n = M(u) + E(u),$$

where M *is continuously differentiable on* $(0, +\infty)$. *Suppose that* f *is a continuously differentiable function on* $[u_1, u_2]$, *where* $u_1 \geq 0$. *Then*

$$\sum_{u_1 < n < u_2} b_n f(n) = \int_{u_1}^{u_2} f(u) M'(u) du + \int_{u_1}^{u_2} f(u) dE(u).$$

Lemma 1.2. *Let* $\alpha \in \mathbb{R}$. *Then,*

$$\sum_{1 \leq n \leq y} e(n\alpha) \ll \min\left(y, \frac{1}{||\alpha||}\right).$$

The above lemma easily follows from the observation that

$$\sum_{1 \leq n \leq y} e(n\alpha) = e(\alpha) \frac{e(\alpha y) - 1}{e(\alpha) - 1} = \frac{\sin(\pi y \alpha)}{\sin(\pi \alpha)} e\left(\frac{\alpha}{2}(y+1)\right).$$

Then, one concludes noticing that $|\sin(\pi\alpha)| \geq 2||\alpha||$ (cf. e.g. [28, eq. (8.7)]).

Lemma 1.3. *Let* $\alpha = \frac{a}{q} + \beta$ *with* $(a, q) = 1$, $q \geq 3$ *and* $|\beta| \leq 1/q^2$. *Then*

$$\sum_{n \leq N} \min\left(y, \frac{1}{||\alpha n||}\right) \ll (y + q \log q)\left(1 + \frac{N}{q}\right).$$

For the proof of Lemma 1.3, Hu refers to [41]. We observe that this lemma is in the spirit of [58, Lemma 2.2]. See also [28, Chapter 8] where similar estimates are given.

Let now t be a real number. We define

$$\psi_1(t) := \{t\} - \frac{1}{2}$$

and for $j \geq 1$

$$\begin{cases} \psi_{j+1}(u) - \psi_{j+1}(0) = \int_0^u \psi_j(t) dt \\ \int_0^1 \psi_{j+1}(u) du = 0. \end{cases}$$

With the above notation, the following results hold.

Lemma 1.4. *For any* $H \geq 2$, *we have*

$$\psi_1(u) = \sum_{1 \leq |h| \leq H} \frac{e(hu)}{2\pi h} + O\left(\min\left(1, \frac{1}{H||u||}\right)\right).$$

Also, we have

$$\min\left(1, \frac{1}{H||u||}\right) = \sum_{-\infty}^{\infty} f(h) e(hu), \tag{1.3.1}$$

where $f : \mathbb{Z} \to \mathbb{R}$ *is a function satisfying*

$$f(0) \ll \frac{\log H}{H} \quad \text{and} \quad f(h) \ll \min\left(\frac{1}{|h|}, \frac{H}{h^2}\right).$$

Lemma 1.5. *Let* $l \geq 1$*. Then*

$$\psi_{2l}(u) = (-1)^{l-1} \sum_{h=1}^{\infty} \frac{2}{(2h\pi)^{2l}} \cos(2\pi u),$$
$$\psi_{2l+1}(u) = (-1)^{l-1} \sum_{h=1}^{\infty} \frac{2}{(2h\pi)^{2l+1}} \sin(2\pi u).$$

For the proofs of Lemma 1.4 and Lemma 1.5 we refer, as in Hu's, to [23] and [41] respectively. We now introduce the quadratic Gauss sum. For $a, b \in \mathbb{Z}$, $q \geq 1$ let

$$G(a, b, q) = \sum_{r=1}^{q} e\left(\frac{ar^2 + br}{q}\right).$$

Assume that $(a, q) = 1$, then the above sum satisfies (see [25, Lemma 3.5] and [28, eq. (8.6)])

$$G(a, b, q) \ll \sqrt{q}. \tag{1.3.2}$$

Below, we give an estimate of $S_1(\alpha, \sqrt{x})$ on major arcs (cf. [25, §4.1]).

Lemma 1.6. *Let* $1 \leq q \leq P$, $1 \leq a \leq q$, $(a, q) = 1$ *and* $\alpha = \frac{a}{q} + \beta \in \mathcal{M}(a, q)$ *with* $|\beta| \leq \frac{1}{qQ}$*. Then,*

$$S_1(\alpha, \sqrt{x}) = \frac{G(a, 0, q)}{q} \sqrt{x} \int_0^1 e(x\beta v^2) dv + O(\sqrt{q} \log(q + 1)).$$

Proof. Recalling (1.2.1), we have

$$S_1(\alpha, \sqrt{x}) = \sum_{1 \leq n \leq \sqrt{x}} e\left(\frac{n^2 a}{q}\right) e(n^2 \beta) = \sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right) \sum_{\substack{1 \leq n \leq \sqrt{x} \\ n \equiv r \ (\mathrm{mod}\ q)}} e(n^2 \beta).$$

Observe that

$$\sum_{\substack{n \leq u \\ n \equiv r \ (\mathrm{mod}\ q)}} 1 = \sum_{0 \leq k \leq \frac{u-r}{q}} 1 = 1 + \left\lfloor \frac{u-r}{q} \right\rfloor = \frac{1}{2} + \frac{u-r}{q} - \psi_1\left(\frac{u-r}{q}\right).$$

We apply Lemma 1.1 with

$$M(u) = \frac{1}{2} + \frac{u-r}{q}, \quad \text{and} \quad E(u) = -\psi_1\left(\frac{u-r}{q}\right),$$

then in the notation of the lemma, $b_n = 1$ if $n \equiv r \pmod q$ and $b_n = 0$ otherwise, while

$f(n) = e(n^2\beta)$. We get

$$\begin{aligned}\sum_{\substack{1\le n\le\sqrt{x}\\ n\equiv r \ (\mathrm{mod}\ q)}} e(n^2\beta) &= \int_0^{\sqrt{x}} e(u^2\beta)M'(u)du + \int_0^{\sqrt{x}} e(u^2\beta)dE(u)\\ &= \frac{1}{q}\int_0^{\sqrt{x}} e(u^2\beta)du - \int_0^{\sqrt{x}} e(u^2\beta)d\psi_1\frac{u-r}{q} = \int_1 - \int_2,\end{aligned}$$

say. Consider $\int_1$. By the change of variables $u = \sqrt{x}v$,

$$\int_1 = \frac{\sqrt{x}}{q}\int_0^1 e(xv^2\beta)dv$$

and hence

$$\sum_{r=1}^{q} e\Big(\frac{r^2a}{q}\Big)\int_1 = \frac{G(a,0,q)}{q}\sqrt{x}\int_0^1 e(xv^2\beta)dv.$$

Let us now focus on the second integral. In order to simplify the notation, put $g(u) = e(u^2\beta)$. Using partial integration,

$$\begin{aligned}\int_2 &= g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}} - \int_0^{\sqrt{x}} g'(u)\psi_1\Big(\frac{u-r}{q}\Big)du\\ &= g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}} - q\int_0^{\sqrt{x}} g'(u)d\psi_2\Big(\frac{u-r}{q}\Big)du,\end{aligned}$$

since, by definition,

$$\psi_2\Big(\frac{u-r}{q}\Big) = \psi_2(0) + \int_0^{(u-r)/q}\psi_1(t)dt$$

and hence

$$d\psi_2\Big(\frac{u-r}{q}\Big) = \frac{d}{du}\psi_2\Big(\frac{u-r}{q}\Big)\frac{1}{q} = \psi_1\Big(\frac{u-r}{q}\Big).$$

Using repeated partial integrations,

$$\begin{aligned}\int_2 &= g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}} - qg'(u)\psi_2\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}} + q\int_0^{\sqrt{x}} g''(u)\psi_2\Big(\frac{u-r}{q}\Big)du\\ &= \sum_{j=0}^{l}(-1)^j q^j g^{(j)}(u)\psi_{j+1}\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}} + (-1)^{l+1}q^l\int_0^{\sqrt{x}} g^{(l+1)}(u)\psi_{l+1}\Big(\frac{u-r}{q}\Big)du,\end{aligned}$$

where $l \ge 1$ is a fixed positive integer (the last equality can be easily verified by induction). Hence,

$$\begin{aligned}\sum_{r=1}^{q} e\Big(\frac{r^2a}{q}\Big)\int_2 &= \sum_{j=0}^{l}(-1)^j q^j g^{(j)}(u)\sum_{r=1}^{q} e\Big(\frac{r^2a}{q}\Big)\psi_{j+1}\Big(\frac{u-r}{q}\Big)\Big|_0^{\sqrt{x}}\\ &\quad + (-1)^{l+1}q^l\int_0^{\sqrt{x}} g^{(l+1)}(u)\sum_{r=1}^{q} e\Big(\frac{r^2a}{q}\Big)\psi_{l+1}\Big(\frac{u-r}{q}\Big)du.\end{aligned} \tag{1.3.3}$$

Assume at first the following facts (which will be proved later).

(i) For any fixed $j \geq 0$ and uniformly for $0 \leq u \leq \sqrt{x}$

$$\sum_{r=1}^{q} e\Big(\frac{r^2 a}{q}\Big)\psi_{j+1}\Big(\frac{u-r}{q}\Big) \ll \begin{cases} \sqrt{q}\log(q+1) & \text{if } \ j=0 \\ \sqrt{q} & \text{if } \ j \geq 1. \end{cases}$$

(ii) For any fixed $k \geq 0$, uniformly for $0 \leq u \leq \sqrt{x}$,

$$g^{(k)}(u) \ll_k \left(\frac{\sqrt{x}}{qQ}\right)^k.$$

Then, equation (1.3.3) gives

$$\begin{aligned}
&\sum_{r=1}^{q} e\Big(\frac{r^2 a}{q}\Big)\int_2 = \sum_{r=1}^{q} e\Big(\frac{r^2 a}{q}\Big)\int_0^{\sqrt{x}} e(u^2\beta)d\psi_1\frac{u-r}{q} \\
&\ll |g(u)|\sqrt{q}\log(q+1)\Big|_0^{\sqrt{x}} + \sum_{j=1}^{l}(-1)^j q^j |g^{(j)}(u)|\sqrt{q}\Big|_0^{\sqrt{x}} + (-1)^{l+1}q^l\int_0^{\sqrt{x}}\sqrt{q}|g^{(l+1)}(u)|du \\
&\ll \sqrt{q}\log(q+1) + \sqrt{q}\sum_{j=1}^{l} q^j\left(\frac{\sqrt{x}}{qQ}\right)^j + q^l\sqrt{q}\Big(\frac{\sqrt{x}}{qQ}\Big)^{l+1}\sqrt{x} \\
&\ll \sqrt{q}\log(q+1) + \sqrt{q}\sum_{j=1}^{l}\left(\frac{\sqrt{x}}{Q}\right)^j + \frac{1}{\sqrt{q}}\sqrt{x}\Big(\frac{\sqrt{x}}{Q}\Big)^{l+1}.
\end{aligned}$$

Let $l = \lfloor 2/\varepsilon \rfloor$, with $0 < \varepsilon < 1$, then

$$\sqrt{x}\Big(\frac{\sqrt{x}}{Q}\Big)^{l+1} \ll 1 \quad \text{and} \quad \sqrt{q}\sum_{j=1}^{l}\left(\frac{\sqrt{x}}{Q}\right)^j \ll \sqrt{q},$$

recalling the choice of the value of Q. It follows that

$$\sum_{r=1}^{q} e\Big(\frac{r^2 a}{q}\Big)\int_2 \ll \sqrt{q}\log(q+1) + \sqrt{q} + \frac{1}{\sqrt{q}} \ll \sqrt{q}\log(q+1).$$

So, we conclude

$$S_1(\alpha,\sqrt{x}) = \sum_{r=1}^{q} e\Big(\frac{r^2 a}{q}\Big)\Big(\int_1 - \int_2\Big) = \frac{G(a,0,q)}{q}\sqrt{x}\int_0^1 e(xv^2\beta)dv + O(\sqrt{q}\log(q+1)).$$

We still have to prove (i) and (ii). A direct calculation shows that (i) holds for $q = 1, 2$.

Assume now $q \geq 3$ and $j = 0$. Then, using Lemma 1.4 with $H = q$ we get

$$\sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right)\psi_1\left(\frac{u-r}{q}\right) = \sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right) \sum_{1\leq|h|\leq q} \frac{e(h(u-r)/q)}{2\pi h}$$
$$+ \sum_{r=1}^{q} O\left(\min\left(1, \frac{1}{q||(u-r)/q||}\right)\right) = \sum_1 + \sum_2,$$

say. Consider the first sum. Thanks to (1.3.2) and using partial summation,

$$\sum_1 = \sum_{1\leq|h|\leq q} \frac{e(hu/q)}{2\pi h} G(a,-h,q) \ll \sqrt{q} \sum_{1\leq|h|\leq q} \frac{1}{h} \ll \sqrt{q}\log q.$$

Moreover, by Lemma 1.4 and the fact that for $q, r \in \mathbb{N}$

$$\sum_{h=1}^{q} e\left(\frac{hr}{q}\right) = \begin{cases} q & \text{if} \quad q \mid r \\ 0 & \text{otherwise,} \end{cases}$$

we obtain by (1.3.1) the estimate

$$\begin{aligned}
\sum_2 &\ll \sum_{r=1}^{q}\sum_{h\in\mathbb{Z}} f(h) e\left(\frac{h(u-r)}{q}\right) = \sum_{h\in\mathbb{Z}} f(h) e\left(\frac{hu}{q}\right) \sum_{r=1}^{q} e\left(\frac{-hr}{q}\right) \\
&= q \sum_{\substack{h\in\mathbb{Z} \\ q|h}} f(h) e\left(\frac{hu}{q}\right) \ll q \sum_{\substack{h\in\mathbb{Z} \\ q|h}} |f(h)| = q \sum_{k\in\mathbb{Z}} |f(kq)| \\
&= q(|f(0)| + \sum_{k=1}^{\infty} |f(kq)| + \sum_{k=-\infty}^{-1} |f(kq)|) \\
&\ll q\frac{\log q}{q} + 2q \sum_{k=1}^{\infty} \min\left(\frac{1}{|kq|}, \frac{q}{q^2k^2}\right) \ll \log q + 2q\frac{1}{q}\sum_{k=1}^{\infty}\frac{1}{k^2} \ll \log q.
\end{aligned}$$

For $j \geq 1$, we apply Lemma 1.5 and again equation (1.3.2). If $j+1$ is even,

$$\sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right)\psi_{j+1}\left(\frac{u-r}{q}\right) \ll \sqrt{q}\sum_{h=0}^{\infty}\frac{1}{h^{2l}},$$

for some $l \geq 1$. On the other hand, if $j+1$ is odd,

$$\sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right)\psi_{j+1}\left(\frac{u-r}{q}\right) \ll \sqrt{q}\sum_{h=0}^{\infty}\frac{1}{h^{2l+1}},$$

again with $l \geq 1$. Then, for some $k \geq 2$ we have

$$\sum_{r=1}^{q} e\left(\frac{r^2 a}{q}\right)\psi_{j+1}\left(\frac{u-r}{q}\right) \ll \sqrt{q}\sum_{h=0}^{\infty}\frac{1}{h^{k}} \ll \sqrt{q},$$

since the series $\sum_{h=0}^{\infty} h^{-k}$ converges. Then (i) follows gathering the above cases.
We now focus on (ii). Let $f(u) = 4\pi i u\beta$. Since, $|g(u)| = |e(u^2\beta)| \le 1$, one immediately sees that (ii) holds for $k = 0$. Moreover, $g'(u) = 4\pi i\beta u e(u^2\beta) = f(u)g(u)$, so

$$|g'(u)| \le |f(u)| \ll \sqrt{x}|\beta| \ll \frac{\sqrt{x}}{qQ}.$$

Now, by Leibniz formula, for $k \ge 1$,

$$g^{(k+1)}(u) = (g(u)f(u))^{(k)} = \sum_{m=0}^{k} \binom{k}{m} g^{(m)}(u) f^{(k-m)}(u) = g^{(k)}(u)f(u) + kg^{(k-1)}(u)f'(u),$$

where the last equality holds since $f^{(m)}(u) = 0$ for $m \ge 2$.
Proceeding by induction on k, suppose that the estimate (ii) holds for $g^{(k)}$. Then,

$$\begin{aligned} g^{(k+1)}(u) &\ll |\beta|\sqrt{x}\left(\frac{\sqrt{x}}{qQ}\right)^{k} + k\left(\frac{\sqrt{x}}{qQ}\right)^{k-1} \\ &\ll \left(\frac{\sqrt{x}}{qQ}\right)^{k+1} + \frac{1}{qQ}\left(\frac{\sqrt{x}}{qQ}\right)^{k-1} \ll \left(\frac{\sqrt{x}}{qQ}\right)^{k+1}, \end{aligned}$$

since the inequality $qQ \le PQ = x$ implies that $\frac{1}{qQ} \le \left(\frac{\sqrt{x}}{qQ}\right)^2$.
Thus (ii) follows and this concludes the proof of the lemma. □

In the following lemma we recall the corresponding bound on minor arcs, (cf. [25, §4.2]).

Lemma 1.7. *Let* $\alpha = \frac{a}{q} + \beta \in \mathsf{m}$ *with* $1 \le a \le q$, $(a, q) = 1$ *and* $|\beta| \le \frac{1}{qQ}$. *Then,*

$$S_1(\alpha, \sqrt{x}) \ll x^{1/2}q^{-1/2} + q^{1/2}(\log q)^{1/2} + x^{1/4}(\log q)^{1/2}. \tag{1.3.4}$$

Proof. By definition of $S_1(\alpha, \sqrt{x})$, we get

$$\begin{aligned} |S_1(\alpha, \sqrt{x})|^2 = S_1(\alpha, \sqrt{x})\overline{S_1(\alpha, \sqrt{x})} &= \sum_{1 \le m,n \le \sqrt{x}} e((m^2 - n^2)\alpha) \\ &= \sum_{\substack{m=n \\ 1 \le m \le \sqrt{x}}} 1 + \sum_{1 \le n < m \le \sqrt{x}} e((m^2 - n^2)\alpha) + \sum_{1 \le m < n \le \sqrt{x}} e((m^2 - n^2)\alpha) \\ &= \lfloor \sqrt{x} \rfloor + T(\sqrt{x}) + \overline{T(\sqrt{x})}, \end{aligned}$$

where

$$\begin{aligned} T(\sqrt{x}) = \sum_{1 \le n < m \le \sqrt{x}} e((m^2 - n^2)\alpha) &= \sum_{1 \le n < m \le \sqrt{x}} e((m - n)(m + n)\alpha) \\ &= \sum_{1 \le v \le \sqrt{x} - 1} \sum_{1 \le n \le \sqrt{x} - v} e((v^2 + 2nv)\alpha), \end{aligned}$$

taking $v = m - n$ and hence $m + n = v + 2n$. Thus, by Lemma 1.2

$$\begin{aligned} T(\sqrt{x}) &= \sum_{1\le v\le\sqrt{x}-1} e(v^2\alpha) \sum_{1\le n\le\sqrt{x}-v} e(2nv\alpha) \ll \sum_{1\le v\le\sqrt{x}-1} \min\left(\sqrt{x}-v, \frac{1}{||2v\alpha||}\right) \\ &\ll \sum_{1\le z\le 2\sqrt{x}} \min\left(\sqrt{x}, \frac{1}{||z\alpha||}\right) \ll (\sqrt{x} + q\log q)\left(1 + \frac{\sqrt{x}}{q}\right) \\ &\ll x^{1/2} + xq^{-1} + q\log q + x^{1/2}\log q, \end{aligned} \tag{1.3.5}$$

applying Lemma 1.3. The wanted estimate now easily follows from (1.3.4) and (1.3.5). □

Remark 1.4. Recalling that on minor arcs $P \le q \le Q$ and that we chose $P = \exp(C\sqrt{\log x})$, by Lemma 1.7, the estimate (1.3.4) becomes

$$S_1(\alpha, \sqrt{x}) \ll x^{1/2}P^{-1/2} = x^{1/2}\exp(-\frac{C}{2}\sqrt{\log x}).$$

Let now χ be a Dirichlet character modulo q and define

$$\psi_f(x,\chi) = \sum_{n\le x} a(n)\chi(n)\Lambda(n).$$

Lemma 1.8. *There exists a constant $A > 0$ such that*

$$\psi_f(x,\chi) \ll \sqrt{q}x\exp(-A\sqrt{\log x}).$$

Proof. The bound follows with standard methods by formula (28) of [21, Theorem 4.1],

$$\sum_{p\le x} a(p)\chi(p)\log p \ll \sqrt{q}x\exp(-A_1\sqrt{\log x}),$$

where A_1 is a positive constant. The proof of the above bound is essentially based on the zero-free region derived by Hoffstein-Ramakrishnan (cf. [24, Theorem C, part (3)] or [21, Theorem C]) and the absence of exceptional zeros for $GL(2)$ L-functions. The result of Hoffstein and Ramakrishnan states that if f is a primitive form of level q, spectral parameter r and weight k, the corresponding L-function $L_f(s)$ (cf. Section 3.1.1, equation (3.1.1)) does not vanish in

$$\sigma \ge 1 - \frac{c}{\log(q(|t| + |r| + 2))},$$

where c is a positive absolute constant. For the complete argument to prove formula (28), we refer to the proof of Theorem 4.1 of [21]. □

We can now improve Hu's bound [25, Lemma 5.1], giving a stronger estimate of $T(\alpha, x)$ on major arcs. As we shall see, the result proved in Lemma 1.8 is a key ingredient in our argument. To prove Lemma 1.9, we also apply some classical identities, briefly recalled below.

Let a, m be coprime integers. Then,

$$e\left(\frac{a}{m}\right) = \frac{1}{\varphi(m)} \sum_{\chi \,(\mathrm{mod}\, m)} \bar{\chi}(a)\tau_\chi, \tag{1.3.6}$$

where τ_χ is the Gauss sum defined as

$$\tau_\chi = \sum_{b \,(\mathrm{mod}\, m)} \chi(b) e\left(\frac{b}{m}\right). \tag{1.3.7}$$

Moreover, note that

$$\chi(a)\tau_{\bar{\chi}} = \sum_{b \,(\mathrm{mod}\, m)} \bar{\chi}(b) e\left(\frac{ab}{m}\right). \tag{1.3.8}$$

For the details and the proof of the above identities, we refer to [28, Section 3.4].

Lemma 1.9. *Let $1 \le q \le P = \exp(C\sqrt{logx})$, $1 \le a \le q$ with $(a, q) = 1$ and $\alpha = \frac{a}{q} + \beta \in \mathcal{M}(a, q)$ with $|\beta| \le \frac{1}{qQ}$. Then, for C small enough, there exists a constant $B > 0$ such that*

$$T(\alpha, x) \ll x \exp(-B\sqrt{\log x}).$$

Proof. By definition (1.2.2), since $\alpha = \frac{a}{q} + \beta$, we have

$$\begin{aligned} T(\alpha, x) &= \sum_{n \le x} a(n)\Lambda(n) e\left(\frac{an}{q}\right) e(n\beta) \\ &= \sum_{\substack{n \le x \\ (n,q)=1}} a(n)\Lambda(n) e\left(\frac{an}{q}\right) e(n\beta) + \sum_{\substack{n \le x \\ (n,q)>1}} a(n)\Lambda(n) e\left(\frac{an}{q}\right) e(n\beta). \end{aligned} \tag{1.3.9}$$

By (1.3.6), (1.3.7) and (1.3.8),

$$\begin{aligned} &\sum_{\substack{n \le x \\ (n,q)=1}} a(n)\Lambda(n) e\left(\frac{an}{q}\right) e(n\beta) = \sum_{\substack{n \le x \\ (n,q)=1}} a(n)\Lambda(n) \left(\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \bar{\chi}(an)\tau_\chi\right) e(n\beta) \\ &= \sum_{\substack{n \le x \\ (n,q)=1}} a(n)\Lambda(n) \left(\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \bar{\chi}(n) \sum_{b \,(\mathrm{mod}\, q)} \chi(b) e\left(\frac{ab}{q}\right)\right) e(n\beta) \\ &= \sum_{b \,(\mathrm{mod}\, q)} e\left(\frac{ab}{q}\right) \left(\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(b) \sum_{n \le x} a(n)\bar{\chi}(n)\Lambda(n) e(\beta n)\right) \\ &= \sum_{b \,(\mathrm{mod}\, q)} e\left(\frac{ab}{q}\right) \left(\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(b) \left(\psi_f(x, \bar{\chi}) e(\beta x) - 2\pi i \beta \int_1^x \psi_f(u, \bar{\chi}) e(\beta u) du\right)\right), \end{aligned}$$

where, for the last equality we apply partial summation. On the other hand, the second sum in (1.3.9) gives rise to an error term of the form $O(x^{\theta+\varepsilon}\log^2 x)$, since

$$\begin{aligned}\sum_{\substack{n\le x\\(n,q)>1}} a(n)\Lambda(n)e(n\alpha) &\ll \sum_{\substack{n\le x\\(n,q)>1}} a(n)\Lambda(n) \ll x^{\theta+\varepsilon}\sum_{\substack{n\le x\\(n,q)>1}}\Lambda(n)\\ &= x^{\theta+\varepsilon}\sum_{p|q}\sum_{p^r\le x}\log p = x^{\theta+\varepsilon}\sum_{p|q}\log p\sum_{p^r\le x}1 \ll x^{\theta+\varepsilon}\log^2 x.\end{aligned}$$

Then, we have

$$\begin{aligned}T(\alpha,x) &\ll \sum_{b \,(\mathrm{mod}\, q)} (1+|\beta|x)\sqrt{q}x\exp(-A\sqrt{\log x})\\ &\ll xP^{3/2}\exp(-A\sqrt{\log x}) \ll x\exp(-B\sqrt{\log x}),\end{aligned}$$

where $B = A - \frac{3}{2}C$ and A is the constant of Lemma 1.8. Hence, taking C in the definition of P to satisfy

$$C < \frac{2}{3}A, \tag{1.3.10}$$

we have $B > 0$, as desired. □

Remark 1.5. In [25], the author claims that, simply normalizing the coefficients, it is possible to generalize [43, Theorem 1] to the case of Maass forms. However, the proof of that result is strongly based on Ramanujan's conjecture (which is not known for Maass forms, as recalled in Remark 1.2), so we are not sure of the estimate stated in [25, Lemma 5.1]. For this reason, our argument for $T(\alpha,x)$ on major arcs follows a different approach.

1.4 Proof of the main result

We are now ready to prove Theorem 1.1. We start by considering the major arcs. By construction,

$$\int_{\mathcal{M}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha = \sum_{1\le q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\frac{a}{q}-\frac{1}{qQ}}^{\frac{a}{q}+\frac{1}{qQ}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha.$$

Assume now that $\alpha\in\mathcal{M}(a,q)$. By Lemma 1.6 and since $(X+Y)^3 \ll X^3+Y^3$ for $X,Y>0$, we have

$$S_1^3(\alpha,\sqrt{x}) \ll \frac{G^3(a,0,q)}{q^3}x^{3/2} + q^{3/2}\log^3(q+1) \ll x^{3/2}q^{-3/2}. \tag{1.4.1}$$

Hence, Lemma 1.9 and equation (1.4.1) give

$$S_1^3(\alpha,\sqrt{x})T(-\alpha,x) \ll x^{3/2}q^{-3/2}x\exp(-B\sqrt{\log x}) = x^{5/2}q^{-3/2}\exp(-B\sqrt{\log x}).$$

Now, recalling that the length of the interval $\mathcal{M}(a,q)$ is $2(qQ)^{-1}$,

$$\int_{\frac{a}{q}-\frac{1}{qQ}}^{\frac{a}{q}+\frac{1}{qQ}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha \ll (qQ)^{-1}x^{5/2}q^{-3/2}\exp(-B\sqrt{\log x})$$
$$= q^{-1}x^{3/2}q^{-3/2}P\exp(-B\sqrt{\log x}),$$

since $Q = xP^{-1}$ and so $Q^{-1} = x^{-1}P$. We sum over the residue classes modulo q, obtaining

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\frac{a}{q}-\frac{1}{qQ}}^{\frac{a}{q}+\frac{1}{qQ}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha \ll x^{3/2}q^{-3/2}P\exp(-B\sqrt{\log x}),$$

and finally over $1 \le q \le P$. Observing that $\sum_{q=1}^{P} q^{-3/2} = O(1)$, we get

$$\int_{\mathcal{M}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha \ll x^{3/2}P\exp(-B\sqrt{\log x}) = x^{3/2}\exp(-(B-C)\sqrt{\log x}), \quad (1.4.2)$$

since $P = \exp(C\sqrt{\log x})$. Note that, for our purpose we have to impose $B - C > 0$, so

$$C < B = A - \frac{3}{2}C \Longleftrightarrow C < \frac{2}{5}A. \quad (1.4.3)$$

We still have to find an estimate for

$$\int_{\mathsf{m}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha.$$

Using Cauchy-Schwarz inequality,

$$\int_{\mathsf{m}} S_1^3(\alpha,\sqrt{x})T(-\alpha,x)d\alpha \ll \max_{\mathsf{m}} |S_1(\alpha,\sqrt{x})| \int_0^1 |S_1(\alpha,\sqrt{x})|^2|T(-\alpha,x)|d\alpha$$
$$\ll \max_{\mathsf{m}} |S_1(\alpha,\sqrt{x})| \Big(\int_0^1 |S_1(\alpha,\sqrt{x})|^4 d\alpha\Big)^{1/2} \Big(\int_0^1 |T(-\alpha,x)|^2 d\alpha\Big)^{1/2}.$$

Now, as observed in Lemma 1.7,

$$\max_{\mathsf{m}} |S_1(\alpha,\sqrt{x})| \ll x^{1/2}P^{-1/2} = x^{1/2}\exp(-\frac{C}{2}\sqrt{\log x}). \quad (1.4.4)$$

Moreover, we have

$$\int_0^1 |T(-\alpha,x)|^2 d\alpha \ll \sum_{n\le x} a^2(n)\Lambda^2(n) \ll x\log^2 x, \quad (1.4.5)$$

since the average bound

$$\sum_{n\le x} |a(n)|^2 \ll x$$

holds for Maass forms (see e.g. [21, Corollary 3.1]).

Finally, we have

$$\begin{aligned}\int_0^1 |S_1(\alpha,\sqrt{x})|^4 d\alpha &= \int_0^1 \sum_{1\le m_1,m_2\le\sqrt{x}} e((m_1^2-m_2^2)\alpha) \sum_{1\le m_3,m_4\le\sqrt{x}} e((m_3^2-m_4^2)\alpha)d\alpha \\ &= \sum_{\substack{1\le m_i\le\sqrt{x}\\ i=1,2,3,4}} \int_0^1 e((m_1^2+m_3^2-m_2^2-m_4^2)\alpha)d\alpha \\ &= \sum_{\substack{1\le m_i\le\sqrt{x}\\ i=1,2,3,4\\ m_1^2-m_2^2=m_4^2-m_3^2}} 1 \ll \sum_{n\le x} d^2(n) \ll x\log^3 x,\end{aligned} \tag{1.4.6}$$

where $d(n)$ is the number of divisors of n (cf. e.g. [26, Theorems 5.3, 5.4] for the last estimate). Then, it follows that, for a positive constant C' such that $C' < \frac{C}{2}$,

$$\begin{aligned}\int_{\mathfrak{m}} S_1^3(\alpha,\sqrt{x})T(-\alpha,\sqrt{x})d\alpha &\ll x^{1/2}P^{-1/2}x^{1/2}\log x x^{1/2}\log^{3/2} x \\ &= x^{3/2}\log^{5/2} x\exp(-\frac{C}{2}\sqrt{\log x}) \ll x^{3/2}\exp(-C'\sqrt{\log x}).\end{aligned} \tag{1.4.7}$$

Now, equations (1.4.2) and (1.4.7) give

$$\pi_{a,\Lambda}(x) \ll x^{3/2}\exp(-(B-C)\sqrt{\log x}) + x^{3/2}\exp(-C'\sqrt{\log x}).$$

Hence, combining (1.3.10) and (1.4.3), we choose $P=\exp(C\sqrt{\log x})$, with $C<\frac{2}{5}A$, getting

$$\pi_{a,\Lambda}(x) \ll x^{3/2}\exp(-c\sqrt{\log x}),$$

where $c=\min(B-C,C')=\min(A-\frac{5}{2}C,C')$.

Before concluding this part, we point out that a similar argument can be applied to Theorem 2 of [43]. Indeed, Theorem 2 therein can be strengthen as follows

Theorem 1.2. *There exists a positive constant c' such that*

$$\sum_{n_1+n_2+n_3=N} \tau(n_1)\Lambda(n_1)\tau(n_2)\Lambda(n_2)\tau(n_3)\Lambda(n_3) \ll N^{37/2}\exp(-c'\sqrt{\log N}),$$

where τ is the Ramanujan's function.

The key point is again Hoffstein-Ramakrishnan's result, since Ramanujan's τ function is a holomorphic cusp form of level 12. This assures the non-existence of exceptional zeros and then an analogous version of Lemma 1.8 holds also in this case. We can apply the circle method choosing $P=\exp(A'\sqrt{\log N})$, where $A'>0$ depends on the constant A of Lemma 1.8. As a consequence, the Corollary in [43] gives

$$S_\tau(\alpha) \ll N^{13/2}\exp(-C\sqrt{\log N}),$$

where again the positive constant C depends on A and

$$S_\tau(\alpha) = \sum_{n \le N} \tau(n)\Lambda(n)e(n\alpha).$$

The result then follows as in [43] (the constant c' of the statement will depend on A).

Chapter 2

Zeros of generalized Hurwitz zeta functions

2.1 Introduction

In the literature, several results show that non-trivial linear combinations of L-functions may have infinitely many zeros in the region of absolute convergence, so they do not satisfy the Riemann hypothesis. For instance, in [47] Potter and Titchmarsh showed that the Epstein zeta function has infinitely many zeros on the critical line $\sigma = \frac{1}{2}$, but they also gave an example of an Epstein zeta function, without an Euler product, which has a zero in the critical strip not lying on the critical line. Later, Davenport and Heilbronn [16] studied the distribution of the zeros of the Epstein and the Hurwitz zeta functions in the region of absolute convergence, proving the existence of infinitely many zeros in some cases. The study of the Hurwitz zeta function was then completed by Cassels in [9]. Other examples in this direction are the works of Conrey and Ghosh [14], Saias and Weingartner [52], Booker and Thorne [7] and Righetti [50], [51]. In the following sections, we give a brief account of these results.

In our work [61], we consider the case of a particular generalization of the Hurwitz zeta function, proving that it admits infinitely many zeros in the region of absolute convergence. The idea is to proceed as Davenport and Heilbronn did for the Epstein zeta function without an Euler product and for the classical Hurwitz zeta function. They were the first to use Bohr's equivalence theorem (cf. Theorem 2.4 below) as a main tool to deal with the existence of zeros. Later, the same technique has been often applied in this kind of problems. We now briefly recall the preliminary definitions and results we need in our work. We also describe, without the complete details, the case of the Hurwitz zeta function and some other examples of Dirichlet series with infinitely many zeros in the right half-plane.

2.1.1 Bohr's equivalence theorem

We give a brief overview of Bohr's theory of generalized Dirichlet series. In particular we will focus on Bohr's equivalence theorem, which is an important ingredient in the proof of the existence of the zeros. Roughly speaking, this theorem allows one to turn the problem of finding the zeros of a function into the problem of finding a function, with a zero in the half-plane $\sigma > 1$, which is equivalent (in Bohr's definition) to the one we are studying. Indeed, as we shall see, equivalent Dirichlet series take the same values on vertical strips, and so it is sufficient to find a zero for an equivalent series to state that our Dirichlet series itself has a zero. The existence of infinitely many zeros can be deduced by almost periodicity (see Section 2.1.3). We refer to Bohr [3] or to Apostol [1, Chapter 8] for a complete treatment of Bohr's theory.

Definition 2.1. Let $\{\lambda(n)\}$ be a strictly increasing sequence of real numbers such that $\lambda(n) \to +\infty$ as $n \to +\infty$. A *generalized Dirichlet series* is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a(n) e^{-\lambda(n)s},$$

where the numbers $a(n) \in \mathbb{C}$ are the *coefficients* and the $\lambda(n)$ are the *exponents* of the series.

Remark 2.1. Observe that if $\lambda(n) = \log(n)$ for $n \geq 1$, then $f(s)$ is the ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

See [1, §8.2] for a discussion on the absolute convergence of generalized Dirichlet series.

Definition 2.2. Let $\Lambda = \{\lambda(n)\}$ be an infinite sequence of distinct real numbers. A *basis* of Λ is a finite or countably infinite sequence $B = \{\beta(n)\}$ of real numbers satisfying the following conditions

(1) the element of B are linearly independent over the rationals,

(2) for all $n \geq 1$, $\lambda(n)$ can be expressed as a linear combination over $\mathbb{Q}$ of elements of B,

(3) for all $n \geq 1$, $\beta(n)$ can be expressed as a linear combination over $\mathbb{Q}$ of elements of Λ.

Remark 2.2. Let $\Lambda = \{\log(n)\}$. In this case, it is easy to verify that a basis for Λ is given by $B = \{\log(p_n)\}$, where p_n is the n-th prime number. It can also be observed that any sequence of exponents has infinitely many bases.

It can be useful to express the above facts in matrix notation. We consider Λ and B as infinite column vectors (notice that B could also be finite). We also consider finite or infinite square matrices R of rational entries, such that all but a finite number of elements in each row are zero. These rational square matrices are called *Bohr matrices*. Addition and multiplication are defined as for finite matrices. In the above notation, we say that B is a basis for Λ if the following conditions hold

(1) if $RB = 0$ for some Bohr matrix R, then $R = 0$,

(2) there exists a Bohr matrix R_B such that $\Lambda = R_B B$,

(3) there exists a Bohr matrix T_B such that $B = T_B\Lambda$.

Remark 2.3. If B, B' are two bases for Λ, then $B' = T_{B'}R_B B$, (cf. [1, Theorems 8.5 and 8.6]).

Given a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s}$ and a basis $B = \{\beta(n)\}$ for the sequence of exponents, we can associate to $f(s)$ a function $F_B(z_1, z_2, \dots)$ of countably many complex variables. Let Z be the column vector with entries z_j for $j \geq 1$ and let R_B be the Bohr matrix such that $\Lambda = R_B B$.

Definition 2.3. The Bohr function $F_B(Z) = F_B(z_1, z_2, \dots)$ associated to $f(s)$ with respect to the basis B is the series

$$F_B(Z) = \sum_{n=1}^{\infty} a(n)e^{-(R_B Z)_n},$$

where $(R_B Z)_n$ is the n-th entry of the column vector $R_B Z$.

Remark 2.4. If B, B' are two basis for Λ, then $F_B(Z) = F_{B'}(T_{B'}R_B Z)$, (cf. [1, Theorem 8.7]).

It can be observed that the formal substitution $Z = sB$ leads to

$$F_B(sB) = \sum_{n=1}^{\infty} a(n)e^{-s(R_B B)_n} = \sum_{n=1}^{\infty} a(n)e^{-s\Lambda_n} = f(s).$$

We deduce that if $f(s)$ converges absolutely for $s_0 = \sigma_0 + it_0$, then the associated Bohr function $F_B(Z)$ converges absolutely for any choice of $z_1, z_2, \dots$ such that $\Re(z_n) = \sigma_0\beta(n)$ for all $n \geq 1$. Note that in this case $\Re(Z) = \sigma_0 B$. Then, we define

$$U_f(\sigma_0; B) = \{\, F(Z) \mid \Re(Z) = \sigma_0 B \,\}.$$

It can be proved that $U_f(\sigma_0, B)$ does not depend on the basis (cf. [1, Theorem 8.8]), then from now on it will be denoted as $U_f(\sigma_0)$. The next step is relating the above set with the set of values taken by $f(s)$ on the vertical line $\sigma = \sigma_0$. So let

$$V_f(\sigma_0) = \{\, f(\sigma_0 + it) \mid t \in \mathbb{R} \,\}.$$

The following theorem holds (see [1, Theorem 8.9]).

Theorem 2.1. *Let σ_0 be such that $f(s)$ is absolutely convergent for $\sigma = \sigma_0$. Then,*

$$V_f(\sigma_0) \subseteq U_f(\sigma_0) \subseteq \overline{V_f(\sigma_0)}, \quad \textit{hence} \quad \overline{U_f(\sigma_0)} = \overline{V_f(\sigma_0)}.$$

Given σ_0 as in Theorem 2.1, let $\delta_0 > 0$ be such that $f(s)$ is absolutely convergent for $|\sigma - \sigma_0| < \delta_0$. For any $0 < \delta < \delta_0$, define

$$W_f(\sigma_0, \delta) = \{\, f(\sigma + it) \mid |\sigma - \sigma_0| < \delta, t \in \mathbb{R} \,\} \quad \text{and} \quad W_f(\sigma_0) = \bigcap_{0<\delta\leq\delta_0} W_f(\sigma_0, \delta).$$

Theorem 8.15 of [1] establishes the relation between the above defined set and the set of values taken by $f(s)$ on the vertical line $\sigma = \sigma_0$, i.e,

$$W_f(\sigma_0) = \overline{V_f(\sigma_0)}.$$

We are now ready to give the definition of equivalent Dirichlet series in Bohr's theory. To this end, let $\Lambda = \{\lambda(n)\}$ be a sequence of exponents, let $B = \{\beta(n)\}$ be a basis for Λ and write $\Lambda = R_B B$. Consider two generalized Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b(n)e^{-\lambda(n)s}.$$

Definition 2.4. The Dirichlet series $f(s)$ and $g(s)$ are equivalent with respect to the basis B if there exists a finite or infinite sequence of real numbers $Y = \{y(n)\}$ such that

$$b(n) = a(n)e^{i(R_B Y)_n} \quad \text{for all} \quad n \geq 1.$$

It can be proved that the above definition does not depend on the basis. Moreover, two equivalent Dirichlet series have the same region of absolute convergence. For both results, the reference is [1, Theorem 8.10]. An important result on equivalent Dirichlet series is used in the proof of Bohr's equivalence theorem, i.e. if $f(s)$ and $g(s)$ are equivalent general Dirichlet series absolutely convergent for $\sigma = \sigma_0$, then

$$U_f(\sigma_0) = U_g(\sigma_0),$$

(see [1, Theorem 8.13]). We can now state

Theorem 2.2 (Bohr's equivalence theorem)**.** *Let $f(s)$ and $g(s)$ be equivalent general Dirichlet series with abscissa of absolute convergence σ_a. Then, in any open half-plane $\sigma > \sigma_1 \geq \sigma_a$ the functions $f(s)$ and $g(s)$ take the same set of values.*

As observed by Righetti in [50], even if the theorem is stated for right half-planes, from the proof one can deduce that the same result holds for vertical strips. In his Ph.D thesis [49, Theorem 2.1.11], Righetti proved a different version of the above theorem, adapting the proof of Theorem 8.15 in [1].

Theorem 2.3. *Let $f(s)$ be a general Dirichlet series absolutely convergent for $\sigma = \sigma_0$. Then,*

$$\overline{V_f(\sigma_0)} = \{\, g(\sigma_0) \mid g(s) \textit{ is a general Dirichlet series equivalent to } f(s) \,\}.$$

For ordinary Dirichlet series, i.e. if $\Lambda = \{\log(n)\}$, the equivalence relation can be formulated as below (cf. [1, Theorem 8.12]).

Theorem 2.4. *The ordinary Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \textit{and} \quad g(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

are equivalent if and only if there exists a completely multiplicative function $\varphi(n)$ *such that*

(i) $b(n) = a(n)\varphi(n)$ *for all* $n \geq 1$,

(ii) $|\varphi(p)| = 1$ *for all the primes* $p \mid n$ *with* $a(n) \neq 0$.

2.1.2 Addition of convex curves

In the proof of our result (cf. Section 2.3), we also need to apply Bohr's results on addition of convex curves. We give the statements of the theorems and refer to the original work of Bohr [2] or to Jessen and Wintner [29] for a complete treatment.
We recall that, given two sets of points A and B in $\mathbb{C}$, the set $A + B$ is the set of points $a + b$ with $a \in A$, $b \in B$. We say that $A + B$ is the vectorial sum of A and B.

Theorem 2.5. *Let* $N \geq 2$ *and let* $C_1, \dots, C_N$ *be circles with centers* c_j *and radii* r_j *for* $j = 1, \dots, N$. *Assume that* $r_1 \geq r_j$ *for all* $j \geq 1$. *Then, if*

$$c = \sum_{j=1}^{N} c_j \quad \textit{and} \quad R = \sum_{j=1}^{N} r_j,$$

the following holds

(1) *if* $2r_1 > R$, *then* $C_1 + \cdots + C_N$ *is the set of the points* z *such that* $2r_1 - R \leq |z - c| \leq R$,

(2) *if* $2r_1 \leq R$, *then* $C_1 + \cdots + C_N$ *is the set of the points* z *such that* $|z - c| \leq R$.

In other words, in the first case the vectorial sum of the circles gives rise to a ring centered in c with internal radius $2r_1 - R$ and external radius R. Otherwise, $C_1 + \cdots + C_N$ is a disk of center c and radius R. Bohr also proved the corresponding result for an infinite sum of curves.

Theorem 2.6. *For* $j \geq 1$ *let* C_j *be a circle with center* c_j *and radius* r_j. *Assume that* $r_1 \geq r_j$ *for all* $j \geq 1$. *Then, denoting*

$$c = \sum_{j \geq 1} c_j \quad \textit{and} \quad R = \sum_{j \geq 1} r_j,$$

if $|c| < +\infty$, *we have*

(1) *if* $R < +\infty$ *and* $2r_1 > R$, *then* $\sum_{j \geq 1} C_j$ *is the set of the points* z *such that* $2r_1 - R \leq |z - c| \leq R$.

(2) *if* $R < +\infty$ *and* $2r_1 \leq R$, *then* $\sum_{j \geq 1} C_j$ *is the set of the points* z *such that* $|z - c| \leq R$.

(3) *if* $R = \infty$, *but* $\sum_{j \geq 1} r_j^2 < \infty$, *then* $\sum_{j \geq 1} C_j$ *is the whole complex plane.*

2.1.3 Bohr almost periodic functions

Definition 2.5. A holomorphic function $f(s) = f(\sigma + it)$ defined in some vertical strip $-\infty \leq \sigma_1 < \sigma < \sigma_2 \leq +\infty$ is *Bohr almost periodic* in (σ_1, σ_2) if for any $\varepsilon > 0$ the set

$$\{ \tau \in \mathbb{R} \mid |f(s+i\tau) - f(s)| < \varepsilon \text{ for all } \sigma_1 < \sigma < \sigma_2, t \in \mathbb{R} \}$$

is *relatively dense*, i.e. there exists $\ell = \ell(f, \varepsilon, \sigma_1, \sigma_2) > 0$ such that any interval of length ℓ contains at least an element of the above set. Moreover, $f(s)$ is Bohr almost periodic in $[\sigma_1, \sigma_2]$ if it is Bohr almost periodic in any interval (σ', σ'') with $\sigma_1 < \sigma' < \sigma'' < \sigma_2$.

It can be proved that a general Dirichlet series is Bohr almost periodic in the region of absolute convergence. For the details one can refer to Bohr [4].

2.1.4 Rouché's theorem

We recall the statement of an important result in complex analysis, which is often a key tool in the proofs of the results we present. See e.g. [1, Lemma 2] for the proof.

Theorem 2.7 (Rouché's theorem)**.** *Let two functions $f(s)$ and $g(s)$ be analytic inside and on a closed simple curve $\mathcal{C}$. Assume that*

$$|f(s)| > |g(s)| \quad on \quad \mathcal{C}.$$

Then, $f(s)$ and $f(s) + g(s)$ have the same number of zeros inside $\mathcal{C}$.

We remark that, from an application of Rouché's theorem to the definition of Bohr almost periodicity, one can deduce the following result.

Theorem 2.8. *Let $f(s)$ be a holomorphic Bohr almost periodic function in $[\sigma_1, \sigma_2]$. For any $\sigma_1 \leq \sigma' < \sigma'' \leq \sigma_2$ and $T_1, T_2 > 0$, define*

$$N(\sigma', \sigma'', T_1, T_2) = |\{ \rho = \beta + i\gamma \in \mathbb{C} \mid f(\rho) = 0, \sigma' < \beta < \sigma'', T_1 < \gamma < T_2 \}|.$$

Then, if $N(\sigma', \sigma'', T_1, T_2) \geq 1$, we get

$$\liminf_{T_2 - T_1 \to \infty} \frac{N(\sigma', \sigma'', T_1, T_2)}{T_2 - T_1} \gg 1.$$

2.1.5 The Hurwitz zeta function

Let $0 < \alpha \leq 1$ be a real number, the *Hurwitz zeta function* is defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \tag{2.1.1}$$

for $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. It is well-known that it admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ (cf. e.g. [22]). If $\alpha = 1$, then $\zeta(s, 1) = \zeta(s)$, while for $\alpha = \frac{1}{2}$, we get $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. In particular, for these values of α, the Hurwitz zeta function does not vanish in the half-plane $\sigma > 1$. In their paper [16], Davenport and Heilbronn proved that if $\alpha \notin \{1, \frac{1}{2}\}$ is either rational or transcendental, then $\zeta(s, \alpha)$ has infinitely many zeros for $\sigma > 1$. They observed that the set of values of $\zeta(s, \alpha)$ in any strip $1 < \sigma_1 < \sigma < \sigma_2$ coincides with the set of values of the series

$$\sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\alpha)^s}, \tag{2.1.2}$$

where $\varphi(n)$ is any function satisfying some conditions. If α is transcendental, the numbers $\log(n + \alpha)$ are linearly independent over the rationals, then the only condition on $\varphi(n)$ is that it is of absolute value 1. Since $\zeta(s, \alpha)$ is absolutely convergent for $\sigma > 1$, for any $\delta > 0$ there exist an integer m such that

$$\sum_{n=0}^{m} \frac{1}{(n+\alpha)^{1+\delta}} > \sum_{n=m+1}^{\infty} \frac{1}{(n+\alpha)^{1+\delta}}.$$

Choosing $\varphi(n) = 1$ if $n \leq m$ and $\varphi(n) = -1$ if $n \geq m + 1$, it can be verified that the series (2.1.2) has a real zero in $(1, 1 + \delta)$. Then, by Rouché's theorem and almost periodicity one concludes that $\zeta(s, \alpha)$ has infinitely many zeros for $\sigma > 1$ (cf. Theorem 2.8).
If $\alpha = \frac{l}{k}$, with $(l, k) = 1$, is rational, then $\zeta(s, \alpha)$ can be written as a linear combination of Dirichlet L-functions $L(s, \chi)$, with χ modulo k, i.e.

$$\zeta(s, \alpha) = \frac{k^s}{\phi(k)} \sum_{\chi \,(\mathrm{mod}\, k)} \bar{\chi}(l) L(s, \chi).$$

The Dirichlet L-functions are replaced by the equivalent series

$$L_{\varphi}(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\varphi(n)}{(n+\alpha)^s},$$

where $\varphi(n)$ is of absolute value 1 and completely multiplicative. The corresponding linear combination of $L_{\varphi}(s, \chi)$ turns out to have a sequence of zeros with real part greater than 1, then one conclude as in the transcendental case. Note that, since in the rational case $\zeta(s, \alpha)$ is a linear combination of Dirichlet L-functions, one could deduce the existence of infinitely many zeros by the result of Saias and Weingartner [52], provided that $\alpha \notin \{1, \frac{1}{2}\}$. See also [6, §1.2] for an overview of the work of Davenport and Heilbronn on the Hurwitz zeta function. The remaining case when α is algebraic irrational required a more complicate argument and was later settled by Cassels in [9]. The key ingredient in this case is a lemma of algebraic number theory. We do not give more details here, since we apply the same argument, with suitable small modifications, in the proof of our Theorem 2.15.

2.1.6 The Epstein zeta function and other results

In [16], the authors also studied the case of the Epstein zeta function. Consider a positive-definite quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2,$$

with integer coefficients and fundamental discriminant $D = b^2 - 4ac$. The associated Epstein zeta function is defined by

$$\zeta(s, Q) = \sum_{(m,n)\neq(0,0)} \frac{1}{Q(m, n)^s}.$$

The equivalence classes of quadratic forms Q of fundamental discriminant D are in one-to-one correspondence with the ideal classes $\mathfrak{Q}$ of the quadratic field $\mathbb{Q}(\sqrt{D})$, (cf. [20, Section VII] and also [6]). Given a quadratic form Q of discriminant D, the correspondence is such that the number of representations of an integer n by Q is ω_D (the number of roots of unity in $\mathbb{Q}(\sqrt{D})$) times the number of integral ideals $I \in \mathfrak{Q}$ such that $\mathrm{Norm}(I) = n$ in the ideal class $\mathfrak{Q}$ corresponding to Q. It follows that

$$\zeta(s, Q) = \omega_D \sum_{I\in\mathfrak{Q}} \frac{1}{\mathrm{Norm}(I)^s},$$

and using the orthogonality of characters

$$\zeta(s, Q) = \frac{\omega_D}{h(D)} \sum_{\chi} \bar{\chi}(\mathfrak{Q}) L(s, \chi),$$

where χ is a character of the ideal class group of $\mathbb{Q}(\sqrt{D})$, $h(D)$ is the class number and

$$L(s, \chi) = \sum_{I} \frac{\chi(I)}{\mathrm{Norm}(I)^s}.$$

If $h(D) = 1$, the Epstein zeta function is expected to satisfy the Riemann hypothesis, since it is a multiple of the Dedekind zeta function of $\mathbb{Q}(\sqrt{D})$ and it has an Euler product. On the other hand, for $h(D) > 1$, Davenport and Heilbronn in [16] proved the following result.

Theorem 2.9 (Davenport–Heilbronn)**.** *Let Q be a positive-definite quadratic forms with integer coefficients and fundamental discriminant D. Then, if $h(D) > 1$, the Epstein zeta function $\zeta(s, Q)$ has infinitely many zeros in the half-plane $\sigma > 1$.*

The argument used for the rational case of the Hurwitz zeta function also applies in this case, provided that $\zeta(s, Q)$ can be written as a linear combination of at least two Hecke L-functions.

In [14], Conrey and Ghosh gave another example of Dirichlet series with infinitely many zeros in the region of absolute convergence. They consider the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\quad\text{where}\quad\sum_{n=1}^{\infty}f(n)e(nz)=\Delta^2(z),$$

and Δ is the Ramanujan cusp form of weight 12 for the full modular group, i.e.

$$\Delta(z)=(2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e(nz),$$

and $\tau(n)$ is the Ramanujan tau function. The authors proceeded using the method of Davenport and Heilbronn, defining a series equivalent to $F(s)$ with a zero for $\sigma>1$ and applying Rouché's theorem. See Theorem 2 of [14] and the related Lemma and Corollary for the details.

Saias and Weingartner considered the general case of Dirichlet series with periodic coefficients (cf. [52]). As a first step, they show that the set of these Dirichlet series is a $\mathcal{P}$-module, where $\mathcal{P}$ is the set of Dirichlet polynomial. A basis for this $\mathcal{P}$-module is the family of the Dirichlet L-functions associated to primitive characters (see [52, Theorem PDCB]). In particular, the fact that a Dirichlet series with periodic coefficients can be written as a finite linear combination of Dirichlet L-functions comes from [12, Lemma 1], while [33, Lemma 8.1] gives the linear independence. Thus, if $(a(n))$ is a periodic sequence of complex numbers, the corresponding Dirichlet series can be written as

$$F(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\sum_{j=1}^{N}P_j(s)L(s,\chi_j),\tag{2.1.3}$$

where for $j=1,\dots,N$, $P_j(s)$ is a Dirichlet polynomial and χ_j is a primitive character. Moreover, let $N_a(\sigma_1,\sigma_2,T)$ and $N_a'(\sigma_1,\sigma_2,T)$ be the number zeros of $F(s)$ in the rectangle $\sigma_1<\sigma<\sigma_2$, $|t|\le T$ counted respectively with and without multiplicity. With the above notation, the following result holds (cf. [52, Theorem]).

Theorem 2.10 (Saias–Weingartner)**.** *Let $(a(n))_{n\ge1}$ be a periodic sequence of complex numbers. If the associated Dirichlet series $F(s)$ is not of the form $P(s)L(s,\chi)$, then there exists $\eta>0$ such that for any $1<\sigma_1<\sigma_2\le1+\eta$, $F(s)$ has infinitely many zeros with $\sigma_1<\sigma<\sigma_2$. In particular, there exist positive numbers c_1,c_2 and T_0 such that, for all $T\ge T_0$,*

$$c_1T\le N_a'(\sigma_1,\sigma_2,T)\le N_a(\sigma_1,\sigma_2,T)\le c_2T.$$

As a consequence of this theorem, the Dirichlet series with periodic coefficients which do not vanish in $\sigma>1$ can be precisely characterized [52, Corollary].

Corollary 2.1. *Let $F(s)$ be a Dirichlet series with periodic coefficients. The following statements are equivalent*

(i) $F(s)$ does not vanish in the half-plane $\sigma>1$.

(ii) $F(s) = P(s)L(s,\chi)$*, where* χ *is a Dirichlet character and* $P(s)$ *is Dirichlet polynomial which does not vanish in* $\sigma > 1$.

Their idea is a reminiscence of the method used by Kaczorowski and Kulas in [31] to show that series of the form (2.1.3) have infinitely many zeros in $\frac{1}{2} < \sigma < 1$ off the critical line. However, the universality property introduced in [31, Theorem 3] does not hold in strips in the half-plane $\sigma > 1$, so the two results are based on rather different techniques. In [52, Lemma 2], Saias and Weingartner proved a sort of *weak joint universality property* of Dirichlet L-functions, using Brouwer fixed-point theorem.

A generalization of the result of Saias and Weingartner was given by Booker and Thorne in [7]. They proved that combinations of L-functions coming from unitary cuspidal automorphic representations, with Dirichlet polynomial as coefficients, have infinitely many zeros with $\sigma > 1$, provided that they do not have Euler product and that they satisfy the generalized Ramanujan conjecture at every finite place.

In [50], Righetti was able to properly modify the proof of [7, Theorem 1.2], in order to obtain the analogous result in a more general setting. He axiomatically defined the class $\mathcal{E}$ of the complex functions $F(s)$ satisfying

(E1) $F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$ is absolutely convergent for $\sigma > 1$,

(E2) $\log F(s) = \sum_p F_p(s) = \sum_p \sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}$, absolutely convergent for $\sigma > 1$,

(E3) there exists a constant k_F such that $|a(p)| \leq k_F$ for all primes p,

(E4) $\sum_p \sum_{k=1}^{\infty} \frac{|b_F(p^k)|}{p^{ks}} < \infty$,

(E5) for any pair of functions $F, G \in \mathcal{E}$ there exists $m_{F,G} \in \mathbb{C}$, with $m_{F,F} > 0$, such that

$$\sum_{p \leq x} \frac{a_F(p)\overline{a_G(p)}}{p} = (m_{F,G} + o(1)) \log\log x \quad \text{as} \quad x \to +\infty.$$

Remark 2.5. Two functions $F, G \in \mathcal{E}$ are said to be *orthogonal* if $m_{F,G} = 0$.

Consider now the set P of the prime numbers and, for $\mathcal{Q} \subseteq P$ let

$$\langle \mathcal{Q} \rangle = \{\, n \in \mathbb{N} \mid \text{every prime factor of } n \text{ is in } Q \,\}.$$

Righetti introduced the ring of *p-finite Dirichlet series* absolutely convergent in $\sigma \geq 1$ (cf. [8]),

$$\mathcal{F} = \left\{ \sum_{n \in \langle \mathcal{Q} \rangle} \frac{a(n)}{n^s} \quad \text{absolutely convergent for } \sigma \geq 1 \;\middle|\; \mathcal{Q} \subseteq P \quad \text{has finitely many elements} \right\}.$$

Observe that the above family contains in particular the Dirichlet polynomials. In this setting, Righetti proved the following result (cf. [50, Theorem 3]).

Theorem 2.11 (Righetti)**.** *Let* $N \geq 1$ *and for* $j = 1, \dots, N$ *let* $F_j \in \mathcal{E}$ *be pairwise orthogonal. Then any polynomial* $P \in \mathcal{F}[X_1, \dots, X_N]$ *is either monomial or* $P(F_1(s), \dots, F_N(s))$ *has* *infinitely many zeros for $\sigma > 1$. In the second case, there exists $\eta > 0$ such that for any $1 < \sigma_1 < \sigma_2 \leq 1 + \eta$*

$$|\{ \rho = \beta + i\gamma \mid P(F_1(\rho), \dots, F_N(\rho)) = 0,\ \sigma_1 < \beta < \sigma_2,\ T_1 < \gamma < T_2 \}| \gg T_2 - T_1,$$

for $T_2 - T_1$ sufficiently large.

The idea of the proof follows the arguments of Saias and Weingartner [52] and Booker and Thorne [7]. So, the first step is reducing the problem of finding the zeros to a problem of value distribution of N-uples of logarithms of Euler products. Then, Righetti proved a weak universality property for orthogonal Euler products, i.e., ([50, Proposition 1])

Proposition 2.1. *Let $N \geq 1$ and for $j = 1, \dots, N$ let $F_j \in \mathcal{E}$ be pairwise orthogonal. Given the real numbers $R, y \geq 1$ there exists $\eta > 0$ such that for every $\sigma \in (1, 1 + \eta]$ we have*

$$\left\{ \left(\prod_{p>y} F_{1,p}(\sigma + it_p), \dots, \prod_{p>y} F_{N,p}(\sigma + it_p) \right) \middle| t_p \in \mathbb{R} \right\} \supset \left\{ (z_1, \dots, z_N) \in \mathbb{C}^N \middle| \frac{1}{R} \leq |z_j| \leq R \right\}.$$

An immediate consequence of Theorem 2.11 is the following corollary [50, Corollary 1].

Corollary 2.2. *Let $N \geq 2$ and let $F_1, \dots, F_N \in \mathcal{E}$ be pairwise orthogonal. Then, given non-zero constants $c_1, \dots, c_N \in \mathbb{C}$, there exists $\tilde{\sigma}$ such that*

$$\left\{ \sigma \in (1, \tilde{\sigma}] \,\middle|\, \exists t \in \mathbb{R} \text{ such that } \sum_{j=1}^{N} c_j F_j(\sigma + it) = 0 \right\} \quad \text{is dense in } (1, \tilde{\sigma}].$$

We remark that Bombieri and Ghosh [6] conjectured that the real parts of the zeros of a linear combination of two or more L-functions are dense in the interval $(1, \sigma^*]$, where σ^* is the least upper bound of the real parts of these zeros, so Corollary 2.2 seems to be a partial result in this direction. However, the conjecture has been disproved by Righetti in [51]. Indeed, even if there are examples for which this fact is known to hold, he showed that it is not true in general. In particular, he proved the following result [51, Theorem 1.1].

Theorem 2.12 (Righetti)**.** *Let $N \geq 2$. For $j = 1, \dots, N$ consider non-identically zero Dirichlet series $F_j(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ absolutely convergent for $\sigma > 1$. Then, if $\sum_{n=1}^{N} |a_j(1)| \neq 0$, there exist infinitely many $\mathbf{c} = (c_1, \dots, c_N) \in \mathbb{C}^N$ such that the Dirichlet series $L_{\mathbf{c}}(s) = \sum_{j=1}^{N} c_j F_j(s)$ has no zeros in some vertical strip $\sigma_1 < \sigma < \sigma_2$, with $1 < \sigma_1 < \sigma_2 < \sigma^*(L_{\mathbf{c}})$.*

2.2 Generalized Hurwitz zeta functions

Let now $f(n)$ be a periodic function of period $q \geq 1$ and let $\alpha \in (0, 1]$. For $\sigma > 1$, we define the generalized Hurwitz zeta function as

$$F(s, f, \alpha) = \sum_{n=0}^{\infty} \frac{f(n)}{(n + \alpha)^s}. \tag{2.2.1}$$

Since the coefficients are periodic, it can be easily observed that

$$F(s,f,\alpha) = \frac{1}{q^s}\sum_{b=0}^{q-1} f(b)\zeta\Big(s, \frac{b+\alpha}{q}\Big).$$

Then, it follows by the well-known properties of the classical Hurwitz zeta function (cf. e.g. [22]) that $F(s,f,\alpha)$ admits a meromorphic continuation to the whole complex plane with a possible simple pole at $s=1$ with residue

$$\mathrm{Res}_{s=1} F(s,f,\alpha) = q^{-1}\sum_{b=0}^{q-1} f(b).$$

In [10], Chatterjee and Gun proved that $F(s,f,\alpha)$ has infinitely many zeros in the half-plane $\sigma>1$ if α is either transcendental or algebraic irrational, under some specific assumptions on the coefficients. Indeed, they proved the following two results [10, Theorems 1.1 and 1.2].

Theorem 2.13 (Chatterjee−Gun)**.** *Let α be a positive transcendental number and let f be a real valued periodic function with period $q \geq 1$. If $F(s,f,\alpha)$ has a pole at $s=1$, then $F(s,f,\alpha)$ has infinitely many zeros for $\sigma > 1$.*

Theorem 2.14 (Chatterjee−Gun)**.** *Let α be a positive algebraic irrational number and let f be a positive valued periodic function with period $q \geq 1$. Moreover, let*

$$c := \frac{\max_n f(n)}{\min_n f(n)} < 1.15. \tag{2.2.2}$$

If $F(s,f,\alpha)$ has a pole at $s=1$, then $F(s,f,\alpha)$ has infinitely many zeros for $\sigma > 1$.

In [61], we show that these assumptions can be removed, proving the result in full generality, including the case of α rational, which can be easily deduced from Theorem 2.10.

Theorem 2.15. *Let $f(n)$ be a non identically zero periodic function with period $q \geq 1$ and let $0 < \alpha \leq 1$ be a real number. If $\alpha \notin \{1, \frac{1}{2}\}$, or if $\alpha \in \{1, \frac{1}{2}\}$ and $F(s,f,\alpha)$ is not of the form $P(s)L(s,\chi)$, where $P(s)$ is a Dirichlet polynomial and $L(s,\chi)$ is the L-function associated to a Dirichlet character χ, then $F(s,f,\alpha)$ has infinitely many zeros with $\sigma > 1$.*

2.3 Proof of Theorem 2.15

2.3.1 Case α rational

If $\alpha = 1$, $F(s,f,1)$ is a Dirichlet series with periodic coefficients. By the result of Saias and Weingartner (Corollary 2.1), we know that it does not vanish in the half-plane $\sigma > 1$ if and only if it is the product of a Dirichlet polynomial and a Dirichlet L-function.

Remark 2.6. Examples of functions $f(n)$ giving rise to series $F(s,f,1)$ which do not vanish in the right half-plane are $f(n) = \chi(n+1)$, where χ is a Dirichlet character modulo q, or $f(n) = (-1)^n$.

If $0 < \alpha < 1$ is rational, $F(s, f, \alpha)$ can be written as a linear combination of Dirichlet L-functions, i.e.

$$F(s, f, \alpha) = \sum_{\chi \in \mathcal{C}} P_\chi(s) L(s, \chi), \tag{2.3.1}$$

where $\mathcal{C}$ is a set of primitive characters and $P_\chi(s)$ is a Dirichlet polynomial. Again by Corollary 2.1, expression (2.3.1) does not vanish in the half-plane $\sigma > 1$ if and only if the sum reduces to a single term. Let now $\alpha = \frac{a}{b} \in \mathbb{Q}$, with $(a, b) = 1$, $1 \leq a < b$. Then,

$$F(s, f, a/b) = b^s \sum_{n=0}^{\infty} \frac{f(n)}{(bn + a)^s} = b^s \sum_{m \equiv a \ (\mathrm{mod}\ b)} \frac{g(m)}{m^s}, \tag{2.3.2}$$

where $g(m)$ is periodic of period bq. We prove the following lemma.

Lemma 2.1. *Let $\alpha = \frac{a}{b}$, with $(a, b) = 1$, $1 \leq a < b$. If $\frac{a}{b} \neq \frac{1}{2}$, then $F(s, f, \frac{a}{b})$ is not of the form $P(s)L(s, \chi)$, where P is a Dirichlet polynomial and $L(s, \chi)$ is the Dirichlet L-function associated to the character χ.*

Proof. Consider a Dirichlet polynomial $P(s) = \sum_{n \in \mathcal{N}} \frac{a(n)}{n^s}$, where $\mathcal{N}$ is a non-empty finite set of positive integers, and let χ be a Dirichlet character modulo k. Then,

$$P(s)L(s, \chi) = \sum_{m} \frac{b(m)}{m^s}, \quad \text{where} \quad b(m) = \sum_{\substack{n \in \mathcal{N} \\ n \mid m}} a(n) \chi\left(\frac{m}{n}\right),$$

and the coefficients $b(m)$ are periodic of period $k \prod_{n \in \mathcal{N}} n$. Assume that there exist two coprime integers h, r with $h < r$, such that $b(m) \neq 0$ only if $m \equiv h \pmod{r}$. Let $n_1 := \min \mathcal{N}$, then $b(n_1) = a(n_1) \neq 0$ and so our assumption implies $n_1 \equiv h \pmod{r}$.
On the other hand, $b(-n_1) = \chi(-1)a(n_1) \neq 0$, then we also have $-n_1 \equiv h \pmod{r}$. It follows that $2h \equiv 0 \pmod{r}$, which implies $r = 2$. Thus, we conclude that expression (2.3.2) can be of the form $P(s)L(s, \chi)$ only if $\alpha = \frac{1}{2}$. □

Remark 2.7. Observe that if $\alpha = \frac{1}{2}$, the sum on the right-hand side of (2.3.1) reduces to a single term for instance if $g(m) = c\chi(m)$, where χ is a Dirichlet character modulo $2q$ and c is a non-zero constant (i.e. $f(n) = c\chi(2n + 1)$). In this case, $F(s, f, \frac{1}{2}) = c2^s L(s, \chi) \neq 0$ in $\sigma > 1$.

2.3.2 Case α transcendental

If α is transcendental, the argument of Davenport and Heilbronn for the Hurwitz zeta function applies also to $F(s, f, \alpha)$, since we have

$$\sum_{n=0}^{\infty} \frac{|f(n)|}{(n + \alpha)^\sigma} \to +\infty \quad \text{as} \quad \sigma \to 1^+. \tag{2.3.3}$$

In fact, let $A = \{ n \in \mathbb{N} \mid f(n) \neq 0 \}$ and $\beta = \min_{n \in A} |f(n)| > 0$. Since $f(n)$ is periodic of period q, there exists $b \in \{ 0, 1, \ldots, q-1 \}$ such that $|f(n)| = \beta$ if $n \equiv b \pmod q$. Moreover, for $n \in A$,

$$|f(n)| \geq \beta \mathbf{1}_B(n), \quad \text{where} \quad B = \{ n \mid n \equiv b \pmod q \} .$$

Then, we have

$$\sum_{n \in A} \frac{|f(n)|}{(n+\alpha)^\sigma} \geq \sum_{n \in A} \frac{\beta \mathbf{1}_B(n)}{(n+\alpha)^\sigma} = \beta \sum_{n \equiv b \ (\mathrm{mod}\ q)} \frac{1}{(n+\alpha)^\sigma} = \beta q^{-\sigma} \zeta_L\Big(\sigma, 0, \frac{\alpha+b}{q}\Big).$$

Thus (2.3.3) follows, since the Hurwitz zeta function has a pole at $s = 1$, and the assumption on the existence of the pole for $F(s, f, \alpha)$ can be avoided. We can now proceed applying Bohr's theory of equivalent Dirichlet series. The set $\{ \log(n+\alpha) \}$ is a basis for the sequence of the exponent of $F(s, f, \alpha)$, since the numbers $\log(n+\alpha)$ are linearly independent over $\mathbb{Q}$, as already observed for the Hurwitz zeta function in Section 2.1.5. Then, for any function $\varphi(n)$ of absolute value 1

$$F^\varphi(s, f, \alpha) = \sum_{n=0}^{\infty} \frac{f(n)\varphi(n)}{(n+\alpha)^s}$$

is equivalent to $F(s, f, \alpha)$ (see Definition 2.4). Then, by Theorem 2.2 the set of values taken by $F(s, f, \alpha)$ on the vertical line $\Re s = \sigma$ is

$$\left\{ \sum_{n \in A} \frac{f(n)\varphi(n)}{(n+\alpha)^\sigma} \,\middle|\, \varphi(n) \text{ arithmetic function of abolute value 1} \right\}. \tag{2.3.4}$$

To prove that $F(s, f, \alpha)$ has infinitely many zeros in $\sigma > 1$, it is then sufficient to show that there exists a function $\varphi(n)$ of absolute value 1 such that $F^\varphi(\sigma, f, \alpha) = 0$ for some $\sigma > 1$. We rearrange the sum over n and rewrite the set (2.3.4) as $\{ \sum_i c_i r_i \}$ where

$$r_i = \frac{|f(n_i)|}{(n_i+\alpha)^\sigma}, \quad |c_i| = 1 \quad \text{and} \quad r_1 = \max_{n \in A} \frac{|f(n)|}{(n+\alpha)^\sigma}.$$

So $r_1 \geq r_j$ for any $j > 1$. Moreover, by (2.3.3) there exists $\delta > 0$ such that for any $1 < \sigma < 1+\delta$,

$$r_1 = \frac{|f(n_1)|}{(n_1+\alpha)^\sigma} \leq \sum_{i>1} \frac{|f(n_i)|}{(n_i+\alpha)^\sigma}.$$

Then, Bohr's results on addition of complex curves (cf. Theorem 2.6 part (2)) imply that

$$\sum_{n \in A} \frac{f(n)\varphi(n)}{(n+\alpha)^\sigma}$$

takes any value z with $|z| \leq \sum_{n \in A} \frac{|f(n)|}{(n+\alpha)^\sigma} = R$. Thus, for any $1 < \sigma < 1+\delta$ and any z in the disk of center 0 and radius R there exists $\varphi(n)$ of absolute value 1 such that $F^\varphi(\sigma, f, \alpha) = z$. The wanted result follows taking $z = 0$.

2.3.3 Case α algebraic irrational

We now focus on the case of α algebraic irrational. The proof of the theorem in this case is based on a modification of Cassels' original lemma (see [9]). A suitable decomposition over the residue classes allows us to remove the assumption (2.2.2).
Let $K = \mathbb{Q}(\alpha)$ and let $\mathcal{O}_K$ be its ring of integers. Denote by $\mathfrak{a}$ the denominator ideal of α, i.e. $\mathfrak{a} = \{ r \in \mathcal{O}_K \mid r \cdot (\alpha) \subseteq \mathcal{O}_K \}$, where (α) is the principal fractional ideal generated by α. Then for any integer $n \geq 0$, $(n+\alpha)\mathfrak{a}$ is an integral ideal. The following result holds.

Lemma 2.2. *Let α be algebraic irrational and let $K = \mathbb{Q}(\alpha)$. Given an integer $q \geq 1$, fix $b \in \{ 0, \dots, q-1 \}$. There exists an integer $N_0 > 10^6 q$, depending on α and q, satisfying the following property:*
for any integer $N > N_0$ put $M = \lfloor 10^{-6} N \rfloor$, then at least $0.54\frac{M}{q}$ of the integers $n \equiv b \pmod{q}$, $N < n \leq N+M$ are such that $(n+\alpha)\mathfrak{a}$ is divisible by a prime ideal $\mathfrak{p}_n$ for which

$$\mathfrak{p}_n \nmid \prod_{\substack{m \leq N+M \\ m \neq n}} (m+\alpha)\mathfrak{a}.$$

In the following sections, we first show how to complete the proof of Theorem 2.15 assuming the above lemma and then we give a proof of the lemma itself.

Proof of the main result

The idea is to rearrange Cassels' argument, applying it to each residue class modulo q. As in [9], or directly by Bohr's theory (see Theorem 2.2), it suffices to show that for any $0 < \delta < 1$ there exist a σ, with $1 < \sigma < 1+\delta$, and a function φ of absolute value 1 multiplicative on the group of ideals of $\mathcal{O}_K$, such that

$$\sum_{n=0}^{\infty} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^{\sigma}} = 0.$$

Notice that it is enough to define $\varphi(\mathfrak{p})$, with $|\varphi(\mathfrak{p})| = 1$, on the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ dividing $(n+\alpha)\mathfrak{a}$, since if $(n+\alpha)\mathfrak{a} = \prod \mathfrak{p}^{v_{\mathfrak{p}}}$, then we have

$$\varphi((n+\alpha)\mathfrak{a}) = \prod \varphi(\mathfrak{p})^{v_{\mathfrak{p}}}.$$

Let $0 < \delta < 1$, $N_1 = \max(N_0, 10^7 q)$ and consider σ such that $1 < \sigma < 1+\delta$ and

$$\sum_{n=0}^{N_1} \frac{|f(n)|}{(n+\alpha)^{\sigma}} < \frac{1}{100} \sum_{n=N_1+1}^{\infty} \frac{|f(n)|}{(n+\alpha)^{\sigma}}. \tag{2.3.5}$$

Observe that such a σ exists by (2.3.3). Now, for $\mathfrak{p} \mid \mathfrak{a}$ or $\mathfrak{p} \mid (n+\alpha)\mathfrak{a}$ with $n \leq N_1$ we choose $\varphi(\mathfrak{p}) = 1$. Proceeding by induction, for $j \geq 1$, we put $M_j = \lfloor 10^{-6} N_j \rfloor$ and $N_{j+1} = N_j + M_j$.

Suppose we have defined $\varphi(\mathfrak{p})$ for any $\mathfrak{p} \mid (n+\alpha)\mathfrak{a}$ with $n \le N_j$ in such a way that

$$\left|\sum_{n=0}^{N_j} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma}\right| < \frac{1}{100} \sum_{n=N_j+1}^{\infty} \frac{|f(n)|}{(n+\alpha)^\sigma}. \tag{2.3.6}$$

We want to define $\varphi(\mathfrak{p})$ for any prime ideal

$$\mathfrak{p} \mid \prod_{n \le N_{j+1}} (n+\alpha)\mathfrak{a} \tag{2.3.7}$$

in such a way that (2.3.6) holds for $j+1$ in place of j. For any $b \in \{0, \ldots, q-1\}$, we divide the integers $N_j < n \le N_{j+1}$, with $n \equiv b \pmod q$ into two sets $\mathfrak{A}(b)$ and $\mathfrak{B}(b)$ according to whether a prime ideal $\mathfrak{p}_n$ as in Lemma 2.2 exists or not for $N = N_j$ and $M = M_j$. It can be easily noticed that $|\mathfrak{A}(b)| \ge 5$, since

$$|\mathfrak{A}(b)| \ge \frac{54}{100}\frac{M_j}{q} = \frac{54}{100}\frac{\lfloor 10^{-6}N_j \rfloor}{q},$$

and $N_j \ge 10^7 q$. Then we have divided the integers $N_j < n \le N_{j+1}$ into the disjoint sets $\mathfrak{A} = \cup_{b=0}^{q-1}\mathfrak{A}(b)$ and $\mathfrak{B} = \cup_{b=0}^{q-1}\mathfrak{B}(b)$. As in Cassels', given a prime ideal as in (2.3.7), we distinguish three cases:

(1) $\mathfrak{p} \mid \prod_{n \le N_j}(n+\alpha)\mathfrak{a}$: in this case $\varphi(\mathfrak{p})$ is fixed by the inductive hypothesis.

(2) $\mathfrak{p} = \mathfrak{p}_n$ for some $n \in \mathfrak{A}$

(3) the remaining $\mathfrak{p}$ with property (2.3.7). In this case, we fix arbitrarily $\varphi(\mathfrak{p}) = 1$.

In particular, $\varphi((n+\alpha)\mathfrak{a})$ is defined for any $n \in \mathfrak{B}$, whereas if $n \in \mathfrak{A}$, we have that $\varphi((n+\alpha)\mathfrak{a}) = c_n\varphi(\mathfrak{p}_n)$, with c_n fixed of modulus 1. Now assume $n \in \mathfrak{A}$ and $n \equiv b \pmod q$ with $b \in \{0, \ldots, q-1\}$, i.e. $n \in \mathfrak{A}(b)$. We apply Bohr's results on addition of convex curves (cf. Theorem 2.5).

For an appropriate choice of $\varphi(\mathfrak{p}_n)$ for all $n \in \mathfrak{A}(b)$, we have that

$$\sum_{n\in\mathfrak{A}(b)} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} = \sum_{n\in\mathfrak{A}(b)} \frac{f(n)c_n\varphi(\mathfrak{p}_n)}{(n+\alpha)^\sigma} = f(b)\sum_{n\in\mathfrak{A}(b)} \frac{c_n\varphi(\mathfrak{p}_n)}{(n+\alpha)^\sigma} \tag{2.3.8}$$

takes any given value z satisfying

$$|z| \le S_{3,b} := |f(b)| \sum_{n\in\mathfrak{A}(b)} \frac{1}{(n+\alpha)^\sigma}.$$

In fact, (2.3.8) is the vectorial sum of circles with center 0 and radius $\frac{|f(b)|}{(n+\alpha)^\sigma}$. To show that this sum is a disk, we observe that for any $n, n' \in \mathfrak{A}(b)$,

$$\left(\frac{n+\alpha}{n'+\alpha}\right)^\sigma < 2$$

Moreover, given $n_1, n_2, n_3 \in \mathfrak{A}(b)$ with $n_1 < n_2 < n_3$, we have

$$\frac{|f(n_1)|}{(n_1+\alpha)^\sigma} - \frac{|f(n_2)|}{(n_2+\alpha)^\sigma} - \frac{|f(n_3)|}{(n_3+\alpha)^\sigma} = |f(b)|\left(\frac{1}{(n_1+\alpha)^\sigma} - \frac{1}{(n_2+\alpha)^\sigma} - \frac{1}{(n_3+\alpha)^\sigma}\right) < 0,$$

then we can apply part (2) of Theorem 2.5, recalling that $|\mathfrak{A}(b)| > 5$. Let now

$$\Lambda(b) := f(b)\Bigg(\sum_{\substack{n \le N_j \\ n \equiv b \ (\mathrm{mod}\ q)}} \frac{\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} + \sum_{n \in \mathfrak{B}(b)} \frac{\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma}\Bigg),$$

and define $\varphi(\mathfrak{p}_n)$ for $n \in \mathfrak{A}(b)$ so that

$$\sum_{n \in \mathfrak{A}(b)} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} = \begin{cases} -\Lambda(b) & \text{if} \quad |\Lambda(b)| \le S_{3,b} \\ -S_{3,b}\frac{\Lambda(b)}{|\Lambda(b)|} & \text{if} \quad |\Lambda(b)| > S_{3,b}. \end{cases}$$

With this choice, it is easy to verify that

$$\Bigg| \sum_{\substack{n \le N_{j+1} \\ n \equiv b \ (\mathrm{mod}\ q)}} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} \Bigg| \le \max(0, |\Lambda(b)| - S_{3,b}). \tag{2.3.9}$$

We introduce the notation

$$S_{1,b} = \Bigg| \sum_{\substack{n=0 \\ n \equiv b \ (\mathrm{mod}\ q)}}^{N_j} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} \Bigg|, \quad S_{4,b} = |f(b)| \sum_{\substack{n > N_{j+1} \\ n \equiv b \ (\mathrm{mod}\ q)}} \frac{1}{(n+\alpha)^\sigma},$$

$$S_{2,b} = |f(b)| \sum_{n \in \mathfrak{B}(b)} \frac{1}{(n+\alpha)^\sigma}.$$

Since $\mathfrak{B}(b)$ contains at most $0.46\frac{M_j}{q}$ elements and $\mathfrak{A}(b)$ at least $0.54\frac{M_j}{q}$, we have

$$\frac{S_{3,b}}{S_{2,b}} \ge \frac{54}{46}\frac{(N_j+\alpha)^\sigma}{(N_{j+1}+\alpha)^\sigma} > \frac{101}{99}.$$

Thus, we deduce

$$S_{3,b} - S_{2,b} > \frac{1}{100}(S_{3,b} + S_{2,b}). \tag{2.3.10}$$

Now, by the equations (2.3.6), (2.3.9) and (2.3.10) we get

$$\Bigg| \sum_{\substack{n=0 \\ n \equiv b \ (\mathrm{mod}\ q)}}^{N_{j+1}} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} \Bigg| < \frac{1}{100} S_{4,b} = \frac{1}{100} \sum_{\substack{n > N_{j+1} \\ n \equiv b \ (\mathrm{mod}\ q)}} \frac{|f(n)|}{(n+\alpha)^\sigma}.$$

Summing over the residue classes modulo q, we finally get that

$$\left|\sum_{n=0}^{N_{j+1}} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma}\right| < \frac{1}{100}\sum_{b=0}^{q-1} S_{4,b} = \frac{1}{100}\sum_{n>N_{j+1}} \frac{|f(n)|}{(n+\alpha)^\sigma}.$$

So, equation (2.3.6) also holds for $j+1$ in place of j, as desired. By induction, it then holds for all $j \geq 1$. Since $F(s,f,\alpha)$ is absolutely convergent for $\sigma > 1$, the right-hand side goes to zeros as $j \to +\infty$. It then follows that $\sum_{n=0}^{\infty} \frac{f(n)\varphi((n+\alpha)\mathfrak{a})}{(n+\alpha)^\sigma} = 0$ and the proof is complete.

Proof of Lemma 2.2

Let $\mathfrak{P}$ be the set of the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ such that if $\mathfrak{p} \mid (n+\alpha)\mathfrak{a}$ for some integer n, then $(n+\alpha)\mathfrak{a}/\mathfrak{p}$ is not divisible by any prime ideal $\mathfrak{p}'$ with $\mathrm{Norm}(\mathfrak{p}') = \mathrm{Norm}(\mathfrak{p})$ and such that $(p,q) = 1$, with $p := \mathrm{Norm}(\mathfrak{p})$. For any integer n we write

$$(n+\alpha)\mathfrak{a} = \mathfrak{b}\prod_{\mathfrak{p}} \mathfrak{p}^{u(\mathfrak{p})}, \tag{2.3.11}$$

where $u(\mathfrak{p})$ is an integer and $\mathfrak{b}$ contains all the prime factors of $(n+\alpha)\mathfrak{a}$ which are not in $\mathfrak{P}$. The norm of the ideal $\mathfrak{b}$ is uniformly bounded and so, if n is large enough,

$$\mathrm{Norm}((n+\alpha)\mathfrak{a}) > cn^2,$$

where c is a positive constant which does not depend on n. So, taking the logarithms,

$$\sum_{\mathfrak{p}} u(\mathfrak{p}) \log p \geq 2\log n - C, \tag{2.3.12}$$

for n sufficiently large and again with C independent of n.
Consider now an integer $N > 10^6 q$ and let $M = \lfloor 10^{-6}N \rfloor$. Given $b \in \{0, \ldots, q-1\}$, define $\mathfrak{S} = \mathfrak{S}(N,q,b)$ as the set of the integers $N < n \leq N+M$, $n \equiv b \pmod q$ such that, for all the primes $\mathfrak{p} \in \mathfrak{P}$ in (2.3.11) one has $p^{u(\mathfrak{p})} < M$. Let $S = S(N,q,b)$ be the cardinality of $\mathfrak{S}$. We want to deduce an upper bound for S. For any prime $\mathfrak{p} \in \mathfrak{P}$ and any integer v, let

$$\phi(\mathfrak{p}^v, n) = \begin{cases} \log p & \text{if} \quad \mathfrak{p}^v \mid (n+\alpha)\mathfrak{a} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(n) = \sum \phi(\mathfrak{p}^v, n),$$

where the sum is over the primes $\mathfrak{p} \in \mathfrak{P}$ and the integers v such that $\mathfrak{p}^v < M$. Then, by (2.3.12), we have

$$\sigma(n) \geq 2\log M - C. \tag{2.3.13}$$

Summing over $n \in \mathfrak{S}$, we get, as $N \to \infty$,

$$\sum_{n\in\mathfrak{S}} \sigma(n) \geq (2+o(1))S\log M. \tag{2.3.14}$$

Moreover, by the definition of $\mathfrak{P}$, if $\mathfrak{p}^v \mid (n_1+\alpha)\mathfrak{a}$ and $\mathfrak{p}^v \mid (n_2+\alpha)\mathfrak{a}$ for some integer v, then

$$n_1 \equiv n_2 \pmod{p^v}. \tag{2.3.15}$$

Since by our assumptions $(p,q)=1$, the Chinese remainder theorem gives $n_1 \equiv n_2 \pmod{p^v q}$. As in [9], we obtain

$$\sum_{n\in\mathfrak{S}} \phi(\mathfrak{p}^v,n) \le \sum_{\substack{N<n\le N+M\\ n\equiv b \ (\mathrm{mod}\ q)}} \phi(\mathfrak{p}^v,n) \le \left(\frac{M}{p^v q}+1\right)\log p, \tag{2.3.16}$$

and, assuming $\mathfrak{p}_1 \ne \mathfrak{p}_2$,

$$\sum_{n\in\mathfrak{S}} \phi(\mathfrak{p}_1^v,n)\phi(\mathfrak{p}_2^v,n) \le \sum_{\substack{N<n\le N+M\\ n\equiv b \ (\mathrm{mod}\ q)}} \phi(\mathfrak{p}_1^v,n)\phi(\mathfrak{p}_2^v,n) \le \log p_1 \log p_2\left(\frac{M}{p_1p_2q}+1\right). \tag{2.3.17}$$

Let now $\sigma(n)=\sigma_1(n)+\sigma_2(n)+\sigma_3(n)$, where $\sigma_1,\sigma_2,\sigma_3$ are the sums over $\mathfrak{p}$ and v in the sets $v>1$ and $p^v<M$, $v=1$ and $M^{\frac{1}{2}} \le p < M$, and $v=1$ and $p<M^{\frac{1}{2}}$ respectively. By the prime ideal theorem, partial summation and equations (2.3.16), (2.3.17), we get

$$\sum_{n\in\mathfrak{S}} \sigma_2(n) \le \left(\frac{1}{2}+o(1)\right)\frac{M}{q}\log M, \tag{2.3.18}$$

$$\sum_{n\in\mathfrak{S}} (\sigma_3(n))^2 \le \left(\frac{3}{8}+o(1)\right)\frac{M}{q}\log^2 M. \tag{2.3.19}$$

Similarly, as in Cassels' proof, we have

$$\sum_{n\in\mathfrak{S}} \sigma_1(n) = O(M) = o(M\log M). \tag{2.3.20}$$

Let now $\rho := \frac{qS}{M} \le 1$. Combining (2.3.14), (2.3.18) and (2.3.20), we have

$$\sum_{n\in\mathfrak{S}} \sigma_3(n) = \sum_{n\in\mathfrak{S}} (\sigma(n)-\sigma_1(n)-\sigma_2(n)) \ge \left(2\rho-\frac{1}{2}+o(1)\right)\frac{M}{q}\log M. \tag{2.3.21}$$

If the last member is negative, then $\rho \le \frac{1}{4}+o(1)$. Otherwise, by (2.3.19) and (2.3.21), the Cauchy-Schwarz inequality gives

$$\left(2\rho-\frac{1}{2}+o(1)\right)^2\frac{M^2}{q^2}\log^2 M \le \left(\sum_{n\in\mathfrak{S}}\sigma_3(n)\right)^2 \le \frac{\rho M}{q}\left(\frac{3}{8}+o(1)\right)\frac{M}{q}\log^2 M,$$

and so $\left(2\rho-\frac{1}{2}\right)^2-\frac{3}{8}\rho \le o(1)$. The left-hand side of the last relation is positive for $\rho > \frac{19+\sqrt{105}}{64}$, hence we get

$$S=\frac{\rho M}{q} \le \left(\frac{19+\sqrt{105}}{64}+o(1)\right)\frac{M}{q}. \tag{2.3.22}$$

Let now $N < n \leq N + M$, $n \equiv b \pmod{q}$ be such that $n \notin \mathfrak{S}$. Then, there exist a prime $\mathfrak{p} \in \mathfrak{P}$ and an integer v such that

$$\mathfrak{p}^v \mid (n+\alpha)\mathfrak{a} \quad \text{and} \quad p^v \geq M. \tag{2.3.23}$$

By (2.3.15), given $\mathfrak{p} \in \mathfrak{P}$ there exists at most one integer $N < n \leq N + M$ with property (2.3.23), while the prime ideal theorem implies that there are only $o(N + M)$ prime ideals $\mathfrak{p}$ with $p \leq N + M$. Hence, by (2.3.22) there are at least

$$\left(\frac{45-\sqrt{105}}{64} + o(1)\right)\frac{M}{q} \tag{2.3.24}$$

integers $n \equiv b \pmod{q}$, $N < n \leq N + M$ such that the prime ideal $\mathfrak{p}_n$ in (2.3.23) satisfies $p_n > N+M$. Thus, for these values of n, by (2.3.15), $\mathfrak{p}_n \nmid (n'+\alpha)\mathfrak{a}$ for any $n' \neq n$, $n' \leq N+M$, as desired. For N sufficiently large, (2.3.24) is greater than $\frac{54}{100}\frac{M}{q}$ and thus the Lemma follows.

Chapter 3

On the linear twist of degree 1 functions in the extended Selberg Class

3.1 Introduction

L-functions are a very important and powerful tool in number theory, since they play a central role in many different problems. The simpler and most famous example of an L-function is the classical Riemann zeta function, defined for $\sigma > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well-known that $\zeta(s)$ admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ and that it satisfies the functional equation

$$\Phi(s) = \Phi(1-s), \quad \text{where} \quad \Phi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Moreover, it is connected to the prime numbers by the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \sigma > 1.$$

It has been observed that most of the known L-functions share these basic analytic properties, namely meromorphic continuation, functional equation and Euler product. For this reason, there is a great interest in trying to describe the common properties of the known L-functions in order to define a reasonable class of such functions. The main questions are

- What is an L-function?
- Are all L-functions already known?

There have been several attempts to define the class of all L-functions and different approaches in establishing the axioms which describe their common properties (see e.g. [8], [11], [42], [46], [48]). The most satisfactory approach seems to be Selberg's definition [54], presented in Section 3.2. As we shall observe, the class described by Selberg contains most of the known L-functions, (for some of them this is known only under the assumption of some conjectures). The main conjecture on the Selberg class states that $\mathcal{S}$ coincides with the class of automorphic L-functions.
In Section 3.1.1 below, we briefly describe the main examples of known L-functions. We do not enter deeply into the details, but we give references for a more complete discussion.

3.1.1 Classical L-functions

Besides the Riemann zeta function, several L-functions have been introduced in number theory. As a first example, we can consider a generalization of $\zeta(s)$. Given a Dirichlet character χ modulo $q \geq 1$, the *Dirichlet L-function* $L(s,\chi)$ is defined for $\sigma > 1$ as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

and by analytic continuation elsewhere. Observe that the Riemann zeta function corresponds to the particular case $q = 1$. A Dirichlet L-function has an Euler product of the form

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

and satisfies a functional equation of Riemann type, reflecting s into $1 - s$. For a detailed overview of the theory of Dirichlet L-functions we refer e.g. to Davenport [15].
Let K be an algebraic number field of degree $n = r_1 + 2r_2$ and discriminant D_K and let $\mathfrak{f}$ be a non-zero integral ideal of the ring of integers $\mathcal{O}_k$. If χ is a Hecke character of modulus $\mathfrak{f}$, the *Hecke L-function* $L(s,\chi)$ is defined for $\sigma > 1$ by

$$L_K(s,\chi) = \sum_I \frac{\chi(I)}{\mathrm{Norm}(I)^s},$$

where I runs over the non-zero ideals of $\mathcal{O}_K$. The Hecke L-function admits analytic continuation and satisfies the functional equation

$$\Phi(s,\chi) = \omega_\chi \Phi(1-s,\bar{\chi}),$$

where the root-number ω_χ is a complex number of absolute value 1 and the function $\Phi(s,\chi)$ is the completed L-function defined as

$$\Phi(s,\chi) = \left(\frac{\mathrm{Norm}(\mathfrak{f})|D_K|}{4^{r_2}\pi^n}\right)^{\frac{s}{2}} \prod_{\nu \text{ real}} \Gamma\left(\frac{s + n_\nu + it_\nu}{2}\right) \prod_{\nu \text{ complex}} \Gamma\left(s + \frac{|n_\nu|}{2} + it_\nu\right) L(s,\chi),$$

where ν runs over the Archimedean primes and $n_\nu \in \mathbb{Z}$, $t_\nu \in \mathbb{R}$ satisfy some suitable conditions (cf. Neukirch [40]). A Hecke L-function also has an Euler product of the form

$$L(s,\chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{Norm}(\mathfrak{p})^s}\right)^{-1},$$

since the characters are multiplicative over the ideals. It can be observed that the Hecke L-functions reduce to Dirichlet L-functions when $K = \mathbb{Q}$. Moreover, if χ is the trivial character $L(s,\chi)$ coincides with the *Dedekind zeta function*,

$$\zeta_K(s) = \sum_I \frac{1}{\mathrm{Norm}(I)^s}.$$

Another example of L-functions with algebraic nature are *Artin L-functions*. Given a Galois extension K/k with Galois group G and a finite dimensional representation ρ of G, the Artin L-function $L(s, K/k, \rho)$ is defined by an Euler product for $\sigma > 1$ and it can be expressed as a product of integer powers of Hecke L-functions, as a consequence of Artin's reciprocity law. Then, some analytic properties of Artin L-functions can be deduced. Artin's conjecture states that $L(s, K/k, \rho)$, with ρ irreducible, is entire if the character corresponding to ρ is non-trivial, otherwise it has a simple pole at $s = 1$. Again, as a reference for the theory of Artin L-functions, we cite Neukirch [40].

In Chapter 1, we introduced holomorphic modular forms for congruence subgroups. There exists a standard way to associate a L-function to such forms. So, given a holomorphic modular form $f(z)$ of weight k with Fourier coefficients (a_n), we consider the corresponding Dirichlet series defined for $\sigma > 1$ as

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{3.1.1}$$

where $a(n) := a_n n^{-\frac{k-1}{2}}$. Under this normalization, the analytic properties of (3.1.1) are reminiscent of those of the Riemann zeta function. Almost the same holds for the L-functions associated to Maass forms, also introduced in Chapter 1. We refer to Iwaniec-Kowalski [28, Section 5.11] for a thorough treatment of these last examples.

Let now f and g be two modular forms. An interesting operation between the corresponding L-functions can be defined, namely the *Rankin-Selberg convolution*. If $a(n)$ and $b(n)$ are normalized Fourier coefficients of f and g respectively, the Rankin-Selberg convolution is defined by the Dirichlet series

$$L_{f\times\bar{g}}(s) = \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

Again, under suitable restrictions, the Ranking-Selberg convolution satisfies analytic properties similar to those of the Riemann zeta function.

The above description shows that some analytic properties shared by L-functions can be deduced, even though they are of very different nature. This fact seems to go in the direction of a deep unifying theory of L-functions.
As a general reference for the classical theory of L-functions we indicate [28, Chapter 5].

3.2 The Selberg Class

The *Selberg Class* $\mathcal{S}$ was introduced by Selberg in [54] as the class of the functions $F(s)$ of a complex variable s satisfying the following axioms.

(i) Dirichlet series. For $\sigma > 1$, F is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

(ii) Analytic continuation. For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order.

(iii) Functional equation. $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1-s), \quad \text{where} \quad \Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

with $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \geq 0$ and $|\omega| = 1$.

(iv) Ramanujan hypothesis. For every $\varepsilon > 0$, $c(n) \ll n^{\varepsilon}$.

(v) Euler product. For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Examples of functions belonging to $\mathcal{S}$ are the Riemann zeta function, the shifted Dirichlet L-functions, the Hecke L-functions associated with algebraic number fields, the Hecke L-functions associated with a modular form (under suitable normalizations) and the Rankin-Selberg convolution of some Hecke L-functions. Moreover, other L-functions belong to $\mathcal{S}$ provided that certain conjectures hold. For instance, the Artin L-functions belong to $\mathcal{S}$ if Artin's conjecture holds, while the automorphic L-functions are in $\mathcal{S}$ if Ramanujan's conjecture is true. We refer to [30] or to [44] for a detailed discussion of these examples.

The *extended Selberg class* $\mathcal{S}^\sharp$ is defined as the larger class of the non identically zero functions satisfying axioms (i), (ii) and (iii), which describe analytic properties of the functions. On the other hand, one can observe that the properties described by axioms (iv) and (v) have an arithmetic nature. In particular, the Euler product implies that the Dirichlet coefficient are multiplicative. From the Euler product we also deduce that $F \in \mathcal{S}$ does not vanish in the half-plane $\sigma > 1$. On the other hand, the functional equation allows to find the zeros in the half-plane $\sigma < 0$, since they are located at the poles of the γ-factor (i.e. the function $\gamma(s)$ defined in (iii)). These zeros are called *trivial zeros*, while the *non-trivial zeros* lie in the critical strip $0 \le \sigma \le 1$. For $F \in \mathcal{S}^\sharp$, we can still define trivial and non-trivial zeros, but in this case the critical strip is $1 - \sigma_F \le \sigma \le \sigma_F$, where $\sigma_F > 0$ is the greatest lower bound of the real numbers such that $F(s) \neq 0$ for $\sigma > \sigma_F$. An analogue of the Riemann Hypothesis is expected to hold in the Selberg class, i.e. (cf. [54])

Conjecture (GRH)**.** Let $F \in \mathcal{S}$, then $F(s) \neq 0$ for $\sigma > \frac{1}{2}$.

Moreover, given $T \ge 0$, we can denote as $N_F(T)$ the number of non-trivial zeros $\rho = \beta + i\gamma$ with $|\gamma| \le T$. Then, an analogue of the Riemann-von Mangoldt formula holds,

$$N_F(T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T), \tag{3.2.1}$$

where c_F is a constant depending on F and d_F is the *degree* of F, defined as

$$d_F = 2 \sum_{j=1}^{r} \lambda_j. \tag{3.2.2}$$

One can observe by (3.2.1) that the degree only depends on the function F. In general, the *data*, i.e. $Q, \omega, \lambda_j, \mu_j$, for $j = 1, \dots, r$, are not uniquely defined in terms of F. For instance, it can be noticed that the shape of the γ-factor changes applying Legendre multiplication formula to the Euler Γ-function. For this reason, the notion of *invariant* plays an important role. An invariant is an expression defined in terms of the data of $F \in \mathcal{S}$, which is uniquely determined by the function itself. Besides the degree, other important invariants associated to $F \in \mathcal{S}$ are the *conductor* and the *ξ-invariant*, defined respectively as

$$q = (2\pi)^d Q^2 \sum_{j=1}^{r} \lambda_j \quad \text{and} \quad \xi = 2 \sum_{j=1}^{r} (\mu_j - \frac{1}{2}) = \eta + i\theta.$$

The following results give a characterization of the invariants.

Theorem 3.1 ([13])**.** *Let $\gamma(s), \gamma'(s)$ be two gamma-factors of $F \in \mathcal{S}^\sharp$. Then, there exists a constant $C = C(\gamma, \gamma') \in \mathbb{C}$ such that $\gamma(s) = C\gamma'(s)$.*

Theorem 3.2 ([34])**.** *Let $\gamma(s), \gamma'(s)$ be two gamma-factors of $F \in \mathcal{S}^\sharp$. Then, $\gamma(s)$ can be transformed into $C\gamma'(s)$ by repeated applications of the Legendre-Gauss multiplication formula and the factorial formula.*

Remark 3.1. We recall that *Legendre-Gauss multiplication formula* is given by

$$\Gamma(s) = m^{s-\frac{1}{2}} (2\pi)^{\frac{1-m}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{s+k}{m}\right) \quad m = 2, 3, \dots$$

and the *factorial formula* is

$$\Gamma(s+1) = s\Gamma(s).$$

As an immediate consequence of the above theorems we get

Corollary 3.1 ([34])**.** *An expression defined in terms of the data of a function $F \in \mathcal{S}$ is an invariant if and only if it is stable by multiplication and factorial formulae.*

Since the degree is an invariant, we can split $\mathcal{S}$ (resp. $\mathcal{S}^\sharp$) into the disjoint union of the subclasses of the functions of fixed degree $d \geq 0$, i.e.

$$\mathcal{S} = \cup_{d\geq 0}\mathcal{S}_d \quad \text{and} \quad \mathcal{S}^\sharp = \cup_{d\geq 0}\mathcal{S}_d^\sharp,$$

where

$$\mathcal{S}_d = \{\, F \in \mathcal{S} \mid d_F = d \,\} \quad \text{and} \quad \mathcal{S}_d^\sharp = \{\, F \in \mathcal{S}^\sharp \mid d_F = d \,\}.$$

The main conjecture in the theory of the Selberg class, which, if true, implies that all L-functions are already known, can be reformulated in terms of two conjectures involving the degree, as shown below.

Conjecture (Main conjecture)**.** The Selberg class $\mathcal{S}$ is the class of automorphic L-functions.

Since automorphic L-functions have integer degree, the above conjecture can be split.

Conjecture (General converse theorem)**.** Let $d \in \mathbb{N}$. Then, $\mathcal{S}_d$ is the class of automorphic functions of degree d.

Conjecture (Degree conjecture)**.** $S_d = \emptyset$ if $d \notin \mathbb{N}$.

The analogous conjecture can be formulated for the extended Selberg class.

Conjecture (Strong degree conjecture)**.** $S_d^\sharp = \emptyset$ if $d \notin \mathbb{N}$.

So far, the degree conjecture is proved in the range $0 < d < 2$. In particular, several authors independently proved that $\mathcal{S}_d^\sharp = \emptyset$ for $0 < d < 1$, (cf. [48], [5], [13], [39]), while the proof for $1 < d < 2$ is due to Kaczorowski and Perelli [36]. For degrees $d = 0$ and $d = 1$, the elements of $\mathcal{S}$ and $\mathcal{S}^\sharp$ have been completely characterized. The results below describe the structure of $\mathcal{S}_0^\sharp$ and $\mathcal{S}_1^\sharp$ (cf. [33, Theorems 1 and 2]).

Theorem 3.3 (Kaczorowski-Perelli)**.** *(i) Let $F \in \mathcal{S}_0^\sharp$. Then $q \in \mathbb{N}$, the pair (q,ω) is an invariant for F and $\mathcal{S}_0^\sharp$ is the disjoint union of the subclasses $S_0^\sharp(q,\omega)$, with $q \in \mathbb{N}$ and $|\omega| = 1$.*

(ii) Let $F \in \mathcal{S}_0^\sharp(q,\omega)$, with q, ω as above. Then $F(s)$ is a Dirichlet polynomial of the form

$$P(s) = \sum_{n|q} \frac{a(n)}{n^s}. \tag{3.2.3}$$

(iii) $V_0^\sharp(q,\omega) = \mathcal{S}_0^\sharp(q,\omega) \cup \{0\}$ is a real vector space of dimension $d(q) = \sum_{d|q} 1$.

Remark 3.2. As shown in [33, §3] the functional equation in the case $d = 0$ implies the following relation on the coefficients

$$a(n) = \frac{\omega}{\sqrt{q}} n\bar{a}\left(\frac{q}{n}\right) \quad \text{for} \quad n \mid q. \tag{3.2.4}$$

Remark 3.3. In [13], starting from (3.2.3) and assuming the Euler product axiom, Conrey and Gosh proved that $\mathcal{S}_0 = \{1\}$.

Let now χ be a Dirichlet character modulo q. Denote by χ^* the primitive character inducing χ and by f_χ its conductor. If τ_{χ^*} is the Gauss sum corresponding to χ^*, let

$$\omega_{\chi^*} = \frac{\tau_{\chi^*}}{i^{\mathfrak{a}}\sqrt{f_\chi}} \quad \text{where} \quad \mathfrak{a} = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1 \\ 1 & \text{if} \quad \chi(-1) = -1. \end{cases}$$

If $d = 1$ the *root-number* is defined as

$$\omega^* = \omega(\beta Q^2)^{i\theta} \prod_{j=1}^{r} \lambda_j^{-2i\Im(\mu_j)}, \quad \text{where} \quad \beta = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$

Moreover, write

$$\mathfrak{X}(q,\xi) = \begin{cases} \{\,\chi \pmod q \mid \chi(-1) = 1\,\} & \text{if} \quad \eta = -1 \\ \{\,\chi \pmod q \mid \chi(-1) = -1\,\} & \text{if} \quad \eta = 0. \end{cases}$$

With the above notation, a complete characterization of $\mathcal{S}_1^\sharp$ is given.

Theorem 3.4 (Kaczorowski-Perelli). *(i) Let $F \in \mathcal{S}_1^\sharp$. Then $q \in \mathbb{N}$, $\eta \in \{-1, 0\}$ and the triple (q, ξ, ω^*) is an invariant. Moreover, $\mathcal{S}_1^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_1^\sharp(q, \xi, \omega^*)$, with $q \in \mathbb{N}$, $\eta \in \{-1, 0\}$, $\theta \in \mathbb{R}$ and $|\omega^*| = 1$.*

(ii) Let $F \in \mathcal{S}_1^\sharp(q, \xi, \omega^*)$, with q, ξ, ω^* as above. Then, $F(s)$ can be uniquely written as

$$F(s) = \sum_{\chi \in \mathfrak{X}(q,\xi)} P_\chi(s + i\theta) L(s + i\theta, \chi^*), \tag{3.2.5}$$

where P_χ is a Dirichlet polynomial in $\mathcal{S}_0^\sharp(q/f_\chi, \omega^* \bar{\omega}_{\chi^*})$ and $L(s, \chi^*)$ the Dirichlet L-function associated to the primitive character χ^*.

(iii) If $c(n)$ is the n-th Dirichlet coefficient of $F \in \mathcal{S}_1^\sharp$, then $\tilde{c}(n) = c(n)n^{i\theta}$ is periodic of period q.

(iv) $V_1^\sharp(q, \xi, \omega^*) = \mathcal{S}_1^\sharp(q, \xi, \omega^*) \cup \{0\}$ is a real vector space of dimension

$$\dim V_1^\sharp(q, \xi, \omega^*) = \begin{cases} \lfloor \frac{q}{2} \rfloor & \textit{if} \quad \xi = -1 \\ \lfloor \frac{q-1-\eta}{2} \rfloor & \textit{otherwise.} \end{cases}$$

Remark 3.4. Observe that, since all the characters in $\mathfrak{X}(q,\xi)$ have the same parity, $\mathfrak{a}$ is completely determined by η, then by ξ, which is an invariant for $F \in \mathcal{S}_1^\sharp$. In particular, we have $\mathfrak{a} = \eta + 1$.

Remark 3.5. In [39], Molteni provided two different approaches to the proof of the degree conjecture for $0 < d < 1$. As a corollary of the second approach, he gave a direct proof of the Ramanujan conjecture for every $F \in \mathcal{S}_d^\sharp$ with $0 \leq d \leq 1$. This result was proved in [33] as a consequence of the characterization of $\mathcal{S}_1^\sharp$, but Molteni's corollary allows one to simplify the proof of Theorem 3.4, since one can proceed as Soundararajan in [56], where the Ramanujan conjecture is assumed to hold.

With the further assumption of an Euler product, Theorem 3 of [33] completely characterizes the functions of degree 1 in the Selberg class $\mathcal{S}$. The result is essentially based on the fact that the coefficients $\tilde{c}(n)$ are both multiplicative and periodic.

Theorem 3.5. *Let $F \in \mathcal{S}_1$. If $q = 1$, then $F(s) = \zeta(s)$. If $q \geq 2$, there exist a primitive character χ modulo q and $\theta \in \mathbb{R}$ such that $F(s) = L(s + i\theta, \chi)$.*

The main tool in the proof of Theorem 3.4 is the so-called *linear twist*, defined for $\sigma > 1$ as

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} e(-n\alpha), \tag{3.2.6}$$

where $F \in \mathcal{S}_1^\sharp$, $\alpha \in \mathbb{R}$ and as usual $e(x) = e^{2\pi i x}$. In [33, Theorem 7.1], Kaczorowski and Perelli also established some analytic properties of the linear twist, such as the meromorphic continuation to the half-plane $\sigma > 0$ and the possible existence of a simple pole at $s = 1 - i\theta$.

In [35], Kaczorowski and Perelli showed that the theorem for the linear twist is a special case of a general result holding for $\mathcal{S}_d^\sharp$ for any degree $d > 0$. Given $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_d^\sharp$, with $d > 0$, for $\sigma > 1$ they introduced the so-called *standard twist*

$$F_d(s,\alpha) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} e(-n^{1/d}\alpha). \tag{3.2.7}$$

Remark 3.6. Observe that for $d = 1$ the standard twist coincides with the linear twist (3.2.6).

Let now $n_\alpha = qd^{-d}\alpha^d$ and $c(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$. The result below (c.f [35, Theorem 1]) summarizes the main analytic properties of $F_d(s,\alpha)$.

Theorem 3.6 (Kaczorowski-Perelli)**.** *Let $d > 0$, $F \in \mathcal{S}_d^\sharp$ and $\alpha > 0$. Then the standard twist $F_d(s,\alpha)$ has a meromorphic continuation to $\mathbb{C}$. Moreover, it is entire if $c(n_\alpha) = 0$, while if $c(n_\alpha) \neq 0$ then $F_d(s,\alpha)$ has at most simple poles at the points*

$$s_k = \frac{d+1}{2d} - \frac{k}{d} - i\frac{\theta}{d}, \quad k = 0, 1, \ldots,$$

with non-vanishing residue at s_0.

Remark 3.7. As already observed, the degree conjecture for $0 < d < 1$ is a well-known result, but it is interesting to notice that a simple proof of this fact can be deduced by Theorem 3.6.

Indeed, suppose that there exists $F \in \mathcal{S}_d^\sharp$, with $0 < d < 1$ and let $c(m) \neq 0$ for some integer m. If we choose α such that $n_\alpha = m$, Theorem 3.6 implies the existence of a pole at $s_0 = \frac{d+1}{2d} - i\frac{\theta}{d}$. Since $\Re(s_0) > 1$, this contradicts the absolute convergence of the standard twist in $\sigma > 1$ and so such a function F cannot exist.

In this work we will focus on the case $d = 1$. We are interested in investigating further analytic properties of the linear twist. In particular, as a first step we derive a functional equation for the linear twist. Then, we go on studying the growth on vertical strips and the distribution of the zeros. Before discussing our problem in detail, we present a similar result for degree 2 functions. In [37], Kaczorowski and Perelli considered the standard twist of a Hecke L-function associated to a cusp form of half-integral weight, deriving a functional equation which can be seen as a degree 2 analogue of the Hurwitz-Lerch functional equation.

3.2.1 The standard twist of L-functions of half-integral weight cusp forms

Let f be a cusp form of half-integral weight $\kappa = \frac{k}{2}$ and level N, where k is a positive odd integer and $4 \mid N$ (cf. Remark 1.1). The corresponding L-function $L_f(s)$ is the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where the a_n are the Fourier coefficients of f. Then, $L_f(s)$ is entire and satisfies the functional equation

$$\Lambda_f(s) = \omega \Lambda_{f^*}(\kappa - s), \quad \text{where} \quad \Lambda_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s), \tag{3.2.8}$$

$\omega = i^{-\kappa}$ and f^* is given by the relation

$$f^*(z) = (\sqrt{N}z)^{-\kappa} f(-1/Nz).$$

Observing equation (3.2.8), it can be easily seen that $L_f(s)$ does not belong to the extended Selberg class. For this reason, Kaczorowski and Perelli considered the normalization $s \mapsto s + \frac{\kappa-1}{2}$. Then, writing

$$F(s) = L_f\left(s + \frac{\kappa - 1}{2}\right) \tag{3.2.9}$$

and

$$F^*(s) = L_{f^*}\left(s + \frac{\kappa - 1}{2}\right), \tag{3.2.10}$$

the functional equation (3.2.8) becomes

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{\kappa - 1}{2}\right) F(s) = \omega \left(\frac{\sqrt{N}}{2\pi}\right)^{1-s} \Gamma\left(1 - s + \frac{\kappa - 1}{2}\right) F^*(1 - s). \tag{3.2.11}$$

Even if it is not exactly the the functional equation of the Selberg class, several results in the theory of $\mathcal{S}^\sharp$ hold in this case. In particular, $F(s)$ and $F^*(s)$ are absolutely convergent for $\sigma > 1$ and $F(s)$ has degree 2 and conductor q. Then, denoting as $a(n)$ and $a^*(n)$ the Dirichlet coefficients of F and F^* respectively, they considered the standard twist with $d = 2$

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n^{1/2}\alpha).$$

Adjusting some details of [35, Theorem 1] and defining

$$n_\alpha = N\alpha^2/4, \quad \mathrm{Spec}(F) = \{\alpha > 0 | a^*(n_\alpha) \neq 0\},$$

with $a^*(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$, it can be proved that $F(s,\alpha)$ is entire if $\alpha \notin \mathrm{Spec}(F)$. Moreover, if $\alpha \in \mathrm{Spec}(F)$, then $F(s,\alpha)$ is meromorphic on $\mathbb{C}$ with at most simple poles at

$$s_l = \frac{3}{4} - \frac{l}{2} \qquad l = 0, 1, \dots,$$

with non-zero residue at $s = s_0$ given by

$$\mathrm{Res}_{s=s_0} F(s,\alpha) = c_0(F) \frac{a^*(n_\alpha)}{n_\alpha^{1/4}}, \qquad c_0(F) \neq 0.$$

The main result of [37], is that $F(s,\alpha)$ satisfies a functional equation reflecting s into $1-s$. We introduce the following notation

$$k = 2h+1 \quad \text{with} \quad h \in \{0,1,2,\dots\}, \quad \mu = \frac{2h-1}{4}, \quad h^* = \max(0, h-1),$$
$$\nu = \pm\sqrt{n} \quad \text{with} \quad n = 1, 2, \dots \quad \text{and} \quad \nu_\alpha = \sqrt{n_\alpha}.$$

Moreover, for $l = 0, \dots, h^*$, let

$$F_l^*(s,\alpha) = e^{-i\pi s} F_l^+(s,\alpha) + e^{i\pi s} F_l^-(s,\alpha), \tag{3.2.12}$$

where $F_l^{\pm}(s,\alpha)$ are generalized Dirichlet series defined respectively as

$$F_l^+(s,\alpha) = \sum_{\nu > -\nu_\alpha} \frac{c^*(\nu^2)}{|\nu|^{1/2+l} |\nu + \nu_\alpha|^{2s-1/2-l}}, \qquad F_l^-(s,\alpha) = \sum_{\nu < -\nu_\alpha} \frac{c^*(\nu^2)}{|\nu|^{1/2+l} |\nu + \nu_\alpha|^{2s-1/2-l}},$$

with

$$c^*(\nu^2) = c_l^*(\nu^2) = \begin{cases} -e^{i\pi\mu} a^*(\nu^2) & \text{if} \quad \nu \geq 1 \\ e^{i\pi(1/2+l-\mu)} a^*(\nu^2) & \text{if} \quad -\nu_\alpha \leq \nu \leq -1 \\ e^{-i\pi\mu} a^*(\nu^2) & \text{if} \quad \nu < -\nu_\alpha. \end{cases}$$

The coefficients $a_l = a_l(h^*)$ are defined using the polynomial identity

$$\prod_{1 \leq j \leq h^*} (X + 2j - 1) = \sum_{l=0}^{h^*} a_l \prod_{0 \leq \nu \leq h^*-1-l} (X + \nu).$$

It can be verified that the series $F_l^{\pm}(s,\alpha)$ are absolutely convergent for $\sigma > 1$, thanks to the convergence properties of $F^*(s)$. With the above notation, Kaczorowski and Perelli proved the following result (cf. [37, Theorem]).

Theorem 3.7 (Kaczorowski-Perelli)**.** *Let $\alpha > 0$ and let $l = 0,\dots,h^*$. Then the functions $F_l^*(s,\alpha)$ are entire and $F(s,\alpha)$ satisfies the functional equation*

$$F(s,\alpha) = \frac{\omega}{i\sqrt{2\pi}}\left(\frac{\sqrt{N}}{4\pi}\right)^{1-2s}\sum_{l=0}^{h^*} a_l\Gamma(2(1-s)-1/2-l)F_l^*(1-s,\alpha). \tag{3.2.13}$$

As the authors observed, this functional equation is not exactly of Riemann type, but it still reflects s into $1-s$ and it can be seen as a degree 2 analogue of the Hurwitz-Lerch functional equation.

Starting from (3.2.13), further properties of the standard twist can be investigated. For instance, a more precise analysis of the order of growth on vertical strips can be done, even if by [35, Theorem 2], it is already known that the standard twist has polynomial growth on vertical strip. Let now,

$$\mu(\sigma) = \mu_F(\sigma,\alpha)$$

be the Lindelöf function associated to $F(s,\alpha)$. In [37], the authors also proved that $F_l^{\pm}(s,\alpha)$ have polynomial growth on vertical strips and defined

$$\mu^{\pm}(\sigma) = \inf\left\{ \xi \;\middle|\; F_0^{\pm}(\sigma+it,\alpha) \ll |t|^{\xi} \text{ as } t \to \pm\infty \right\},$$

observing that this one-side Lindelöf function satisfies the main properties of the classical Lindelöf function. Moreover, they wrote

$$\mu^*(\sigma) = \max(\mu^+(\sigma),\mu^-(\sigma))$$

and proved the following result [37, Corollary 1].

Corollary 3.2. *Let $\alpha > 0$ and let $l = 0,\dots,h^*$. Then the functions $F_l^*(s,\alpha)$ are entire with polynomial growth on vertical strips and*

$$\mu(\sigma) = 1 - 2\sigma + \mu^*(1-\sigma).$$

It follows by the convergence properties and the continuity of the Lindelöf function that

$$\mu(\sigma) = 0 \quad \text{if} \quad \sigma \geq 1 \quad \text{and} \quad \mu(\sigma) = 1-2\sigma \quad \text{if} \quad \sigma \leq 0.$$

Another result proved by Kaczorowski and Perelli involves the so called *trivial zeros*. In this case, they come from the interference of the two terms in (3.2.12). So let

$\nu_+ = \nu_+(\alpha)$ be the value of $\nu > -\nu_\alpha$ such that $c^*(\nu^2) \neq 0$ and $|\nu + \nu_\alpha|$ is minimum,

$\nu_- = \nu_-(\alpha)$ be the value of $\nu < -\nu_\alpha$ such that $c^*(\nu^2) \neq 0$ and $|\nu + \nu_\alpha|$ is minimum and denote

$$m_\pm = m_\pm(\alpha) = |\nu_\pm + \nu_\alpha| \quad \text{and} \quad c_\pm^* = \sqrt{m_\pm}\frac{c_0^*(\nu_\pm^2)}{|\nu_\pm|^{1/2}} = \rho_\pm e^{i\theta_\pm},$$

where $m_\pm, \rho_\pm > 0$, $\theta_\pm \in [0, 2\pi)$. Moreover, with the usual notation $s = \sigma + it$, define the line

$$\ell(\alpha) : t = \frac{\sigma}{\pi}\log\left(\frac{m_+}{m_-}\right) + \frac{1}{2\pi}\log\left(\frac{\rho_+ m_-^2}{\rho_- m_+^2}\right) \tag{3.2.14}$$

and for $\varepsilon > 0$ consider the region

$$\mathfrak{L}_\varepsilon(\alpha) = \{\, s \in \mathbb{C} \text{ with distance } < \varepsilon \text{ from the line } \ell(\alpha) \,\}. \tag{3.2.15}$$

Then, the following result holds [37, Corollary 2].

Corollary 3.3. *Let $\alpha > 0$. There exists $\delta > 0$ and $\sigma^- \geq 0$ such that $F(s,\alpha) \neq 0$ for $\sigma < -\sigma^-$ unless $s \in \mathfrak{L}_\delta(\alpha)$. More precisely, there exists $c_1(\alpha) > 0$ such that for every $0 < \varepsilon < c_1(\alpha)$ there exists $\sigma_\varepsilon \geq 0$ with the following properties*

(i) $F(s,\alpha) \neq 0$ for $\sigma < -\sigma_\varepsilon$ unless $s \in \mathfrak{L}_\varepsilon(\alpha)$,

(ii) there exists infinitely many zeros of $F(s,\alpha)$ in $\mathfrak{L}_\varepsilon(\alpha)$ with real part $\beta < -\sigma_\varepsilon$.

Finally, Kaczorowski and Perelli examined the *non-trivial zeros* of the standard twist. These are the zeros in the vertical strip $-\sigma^- \leq \sigma \leq \sigma^+$, where σ^- is as in Corollary 3.3, and σ^+ is the upper bound of the real parts of the zeros of $F(s,\alpha)$. Denoting as n_0 the smallest integer n such that $a(n) \neq 0$ and defining the counting function of non-trivial zeros as

$$N_F(T,\alpha) = |\{\, \rho = \beta + i\gamma \mid F(\rho,\alpha) = 0, -\sigma^- \leq \beta \leq \sigma^+, |\gamma| \leq T \,\}|,$$

in [37, Corollary 3] an analogue of the von Mangoldt function is obtained.

Corollary 3.4. *Let $\alpha > 0$. Then, as $T \to \infty$ we have*

$$N_F(T,\alpha) = \frac{2}{\pi}T\log T + \frac{T}{\pi}\log\left(\frac{N}{n_0 m_+ m_-(2\pi e)^2}\right) + O(\log T).$$

Remark 3.8. In [37, Remark 2], the authors observed that the functional equation (3.2.13) essentially depends on the shape of the Γ-factors in (3.2.11), since they allow the computation of particular hypergeometric functions. They also pointed out that the same argument can be applied to a certain class of Γ-factors, for instance of the form $\Gamma(ds/2 + \mu)$ with $d \geq 1$ and suitable $\mu \in \mathbb{R}$. So they deduced that, even in some cases of higher degree, the standard twist satisfies a functional equation of Hurwitz-Lerch type. Kaczorowski and Perelli concluded their introduction with a final remark on the case $d = 1$, considered in this work.

3.3 A functional equation for the linear twist

Let $F \in \mathcal{S}_1^\sharp$ and $\alpha \in \mathbb{R}$. Since $F(s,\alpha) = F(s,\{\alpha\})$, we can assume $\alpha \in (0,1]$. For $\beta \in \mathbb{R}$, let

$$F_*(s,\alpha,\beta) := \sum_{n+\beta>0} \frac{\tilde{c}(n)}{(n+\beta)^{s+i\theta}} e(-n\alpha), \tag{3.3.1}$$

where $\tilde{c}(n)$ is as in Theorem 3.4.

Remark 3.9. In the definition above $n \in \mathbb{Z}$ and $\tilde{c}(n)$ is extend to $\mathbb{Z}$ by periodicity.

It can be easily seen that equation (3.3.1) with $\beta = 0$ coincides with the linear twist $F(s,\alpha)$. In the above notation, our goal is now proving the following result.

Theorem 3.8. *Let $F \in \mathcal{S}_1^\sharp$ and let $\alpha \in (0,1]$. Then, the linear twist $F(s,\alpha)$ satisfies the functional equation*

$$F(1-s,\alpha) = \frac{\omega^* \Gamma(s-i\theta) q^{s-i\theta-\frac{1}{2}}}{i^{\mathfrak{a}}(2\pi)^{s-i\theta}} \left(e^{i\frac{\pi}{2}(s-i\theta)} \bar{F}_*(s,0,-\alpha q) + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(s-i\theta)} \bar{F}_*(s,0,\alpha q) \right). \tag{3.3.2}$$

Remark 3.10. As already observed for degree 2 L-functions, the functional equation (3.3.2) is not of Riemann type, even if it still reflects s into $1-s$. Again, it can be seen as a Hurwitz-Lerch type of functional equation, as will be explained in the following.

The key point to derive the functional equation is the structural theorem for $\mathcal{S}_1^\sharp$, i.e. Theorem 3.4, and, in particular, expression (3.2.5). So, assume that for any $\chi \in \mathfrak{X}(q,\xi)$, P_χ is a Dirichlet polynomial with coefficients $a(n) = a_\chi(n)$ for $n \mid \frac{q}{f_\chi}$. We rewrite the linear twist as

$$F(s,\alpha) = \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{\substack{n \mid q/f_\chi \\ m \geq 1}} \frac{a(n)\chi^*(m)}{(mn)^{s+i\theta}} e(-mn\alpha).$$

Then, rearranging the sums over n and m, we get

$$\begin{aligned} F(s,\alpha) &= \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n \mid q/f_\chi} \frac{a(n)}{n^{s+i\theta}} \sum_{m \geq 1} \frac{\chi^*(m)}{m^{s+i\theta}} e(-mn\alpha) \\ &= \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n \mid q/f_\chi} \frac{a(n)}{n^{s+i\theta}} L(s+i\theta, \chi^*, n\alpha), \end{aligned}$$

where $L(s+i\theta,\chi^*,n\alpha)$ is the linear twist of the Dirichlet L-function associated to χ^*.

3.3.1 A functional equation for $L(s,\chi,\alpha)$

As a first step, we derive a functional equation for the linear twist of a Dirichlet L-function associated to a primitive character. So, let χ be a primitive Dirichlet character modulo q and let $\alpha \in \mathbb{R}$. Using the orthogonality properties of characters we get

$$\begin{aligned} L(s,\chi,\alpha) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e(-n\alpha) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e(-n\{\alpha\}) \\ &= \sum_{n=1}^{\infty} \frac{1}{\tau_{\bar\chi}} \sum_{a=0}^{q-1} \bar\chi(a) e\Big(\frac{na}{q}\Big) n^{-s} e(-n\{\alpha\}) \qquad (3.3.3) \\ &= \frac{1}{\tau_{\bar\chi}} \sum_{a=0}^{q-1} \bar\chi(a) \sum_{n=1}^{\infty} e\Big(n\Big(\frac{a}{q} - \{\alpha\}\Big)\Big) n^{-s} = \frac{1}{\tau_{\bar\chi}} \sum_{a=0}^{q-1} \bar\chi(a) \zeta_L(s, a/q - \{\alpha\}, 0), \end{aligned}$$

where $\zeta_L(s,x,y)$ is the *Hurwitz-Lerch zeta function* defined as

$$\zeta_L(s,x,y) = \sum_{n > -\{y\}} \frac{e(n\{x\})}{(n+\{y\})^s}, \qquad (3.3.4)$$

for $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$. Observe that equation (3.3.4) coincides with the Hurwitz zeta function defined in (2.1.1), when $x = 0$ and $y \in (0,1]$. It is well known that the Hurwitz-Lerch zeta function can be analytically continued to a holomorphic function in $\mathbb{C}$ with the possible exception of a simple pole of residue 1 at the point $s = 1$ if and only if $x \in \mathbb{Z}$. Moreover, it satisfies a functional equation of the form

$$\zeta_L(1-s,x,y) = \frac{\Gamma(s)}{(2\pi)^s} \Big(e^{i\frac{\pi}{2}s - 2\pi i\{x\}\{y\}} \zeta_L(s,-y,x) + e^{-i\frac{\pi}{2}s + 2\pi i\{-x\}\{y\}} \zeta_L(s,y,-x) \Big). \qquad (3.3.5)$$

We refer e.g. to Garunkstis-Laurincikas [22] for a detailed discussion on the properties of the Hurwitz-Lerch zeta function. Now, for $\alpha, \beta \in \mathbb{R}$, we introduce the notation

$$L_*(s,\chi,\alpha,\beta) = \sum_{n+\beta>0} \frac{\chi(n)}{(n+\beta)^s} e(-n\alpha),$$

observing that $L_*(s,\chi,\alpha,0) = L(s,\chi,\alpha)$. Then, the following result holds.

Theorem 3.9. *Let $L(s,\chi)$ be the Dirichlet L-function associated to the primitive character χ modulo q. Then, given $\alpha \in \mathbb{R}$, the linear twist $L(s,\chi,\alpha)$ admits a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s = 1$. Moreover, it satisfies the functional equation*

$$L_*(1-s,\chi,\alpha,0) = \frac{\Gamma(s)\tau_\chi\chi(-1)}{(2\pi)^s q^{1-s}} \Big(e^{i\frac{\pi}{2}s} L_*(s,\bar\chi,0,-\alpha q) + \chi(-1) e^{-i\frac{\pi}{2}s} L_*(s,\bar\chi,0,\alpha q) \Big). \qquad (3.3.6)$$

Proof. Writing $L(s,\chi,\alpha)$ as a linear combination of Hurwitz-Lerch zeta function as in (3.3.3), we deduce that the linear twist can be extended to a meromorphic function on $\mathbb{C}$ with a possible simple pole at $s = 1$. Given $a \in \{0,\dots,q-1\}$ with $(a,q) = 1$, the pole at $s = 1$ of $\zeta_L(s, a/q - \{\alpha\}, 0)$ exists if and only if $\frac{a}{q} - \{\alpha\} \in \mathbb{Z}$. Then, $L(s,\chi,\alpha)$ has a pole at $s = 1$ if and only if $\chi(\alpha q) \neq 0$ (we assume that $\chi(x) = 0$ if $x \notin \mathbb{Z}$). In this case, the residue is

$$\operatorname{Res}_{s=1} L(s,\chi,\alpha) = \frac{\bar\chi(\alpha q)}{\tau_{\bar\chi}}.$$

On the other hand, by (3.3.3) and (3.3.5), we get

$$\begin{aligned}
L(1-s,\chi,\alpha) &= \frac{1}{\tau_{\bar\chi}}\sum_{a=0}^{q-1}\bar\chi(a)\zeta_L(1-s,a/q-\{\alpha\},0)\\
&= \frac{1}{\tau_{\bar\chi}}\sum_{a=0}^{q-1}\bar\chi(a)\frac{\Gamma(s)}{(2\pi)^s}\Big(e^{i\frac{\pi}{2}s}\zeta_L(s,0,a/q-\{\alpha\})+e^{-i\frac{\pi}{2}s}\zeta_L(s,0,-a/q+\{\alpha\})\Big)\\
&= \frac{\Gamma(s)}{(2\pi)^s\tau_{\bar\chi}}\Big(e^{i\frac{\pi}{2}s}\sum_{a=0}^{q-1}\bar\chi(a)\sum_{n>-\{\frac{a}{q}-\{\alpha\}\}}(n+\{a/q-\{\alpha\}\})^{-s}\\
&+e^{-i\frac{\pi}{2}s}\sum_{a=0}^{q-1}\bar\chi(a)\sum_{n>-\{-\frac{a}{q}+\{\alpha\}\}}(n+\{-a/q+\{\alpha\}\})^{-s}\Big).
\end{aligned}$$

We now denote the two sums above respectively as

$$\begin{aligned}
\Sigma_1 &= \sum_{a=0}^{q-1}\bar\chi(a)\sum_{n>-\{\frac{a}{q}-\{\alpha\}\}}(n+\{a/q-\{\alpha\}\})^{-s}\\
\Sigma_2 &= \sum_{a=0}^{q-1}\bar\chi(a)\sum_{n>-\{-\frac{a}{q}+\{\alpha\}\}}(n+\{-a/q+\{\alpha\}\})^{-s}.
\end{aligned}$$

Observe that $\frac{a}{q}-\{\alpha\}\in(-1,1)$. Moreover, if $\frac{a}{q}-\{\alpha\}>0$, then $\{\frac{a}{q}-\{\alpha\}\}=\frac{a}{q}-\{\alpha\}$, otherwise $\frac{a}{q}-\{\alpha\}<0$ implies $\{\frac{a}{q}-\{\alpha\}\}=\frac{a}{q}-\{\alpha\}+1$. Then we get

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{a=0,\dots,q-1\\ a>\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(n+a/q-\{\alpha\})^{-s}+\sum_{\substack{a=0,\dots,q-1\\ a\le\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(n+a/q-\{\alpha\}+1)^{-s}\\
&= q^s\sum_{\substack{a=0,\dots,q-1\\ a>\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(nq+a-\{\alpha\}q)^{-s}+q^s\sum_{\substack{a=0,\dots,q-1\\ a\le\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(nq+a-\{\alpha\}q+q)^{-s}\\
&= q^s\Bigg(\sum_{\substack{m\ge 0\\ \{\frac{m}{q}\}>\{\alpha\}}}\frac{\bar\chi(m)}{(m-\{\alpha\}q)^s}+\sum_{\substack{m\ge 0\\ \{\frac{m}{q}\}\le\{\alpha\}}}\frac{\bar\chi(m)}{(m-q(\{\alpha\}-1))^s}\Bigg).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Sigma_2 &= \sum_{\substack{a=0,\dots,q-1\\ a\ge\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(n-a/q+\{\alpha\}+1)^{-s}+\sum_{\substack{a=0,\dots,q-1\\ a<\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(n-a/q+\{\alpha\})^{-s}\\
&= q^s\sum_{\substack{a=0,\dots,q-1\\ a\ge\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(nq-a+\{\alpha\}q+q)^{-s}+q^s\sum_{\substack{a=0,\dots,q-1\\ a<\{\alpha\}q}}\bar\chi(a)\sum_{n\ge 0}(nq-a+\{\alpha\}q)^{-s}\\
&= q^s\chi(-1)\Bigg(\sum_{\substack{m>-q\\ \{-\frac{m}{q}\}\ge\{\alpha\}}}\frac{\bar\chi(m)}{(m+q(\{\alpha\}+1))^s}+\sum_{\substack{m>-q\\ \{-\frac{m}{q}\}<\{\alpha\}}}\frac{\bar\chi(m)}{(m+\{\alpha\}q))^s}\Bigg).
\end{aligned}$$

Then, recalling that $\frac{\tau_\chi \tau_{\bar{\chi}}}{q} = \chi(-1)$, we get

$$\begin{aligned} L(s,\chi,\alpha) =& \frac{\Gamma(s)\tau_\chi \chi(-1)}{(2\pi)^s q^{1-s}} \Bigg(e^{i\frac{\pi}{2}s} \Bigg(\sum_{\substack{m\geq 0 \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\{\alpha\}q)^s} + \sum_{\substack{m\geq 0 \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-q(\{\alpha\}-1))^s} \Bigg) \\ &+ \chi(-1) e^{-i\frac{\pi}{2}s} \Bigg(\sum_{\substack{m> -q \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+(\{\alpha\}+1)q)^s} + \sum_{\substack{m> -q \\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\{\alpha\}q)^s} \Bigg)\Bigg). \end{aligned} \tag{3.3.7}$$

We now proceed analyzing each term of the above functional equation, in order to simplify the expression. Consider the first sum. Since $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$, we have $m - \{\alpha\}q = m + \lfloor\alpha\rfloor q - \alpha q$. Then, changing m into $m - \lfloor\alpha\rfloor q$, we have

$$\begin{aligned} &\sum_{\substack{m\geq 0 \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\{\alpha\}q)^s} = \sum_{\substack{m\geq \lfloor\alpha\rfloor q \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} \\ &= \sum_{\substack{\lfloor\alpha\rfloor q \leq m\leq \alpha q \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} + \sum_{\substack{m>\alpha q \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} = \sum_{\substack{m>\alpha q \\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} \end{aligned}$$

where the last equality holds since $\lfloor\alpha\rfloor q \leq m \leq \alpha q$ is not compatible with the condition $\{\frac{m}{q}\} > \{\alpha\}$. We proceed in a similar way with the second sum, changing m into $m - q(\lfloor\alpha\rfloor + 1)$

$$\begin{aligned} &\sum_{\substack{m\geq 0 \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-q(\{\alpha\}-1))^s} = \sum_{\substack{m\geq q(\lfloor\alpha\rfloor+1) \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} \\ &= \sum_{\substack{m>\alpha q \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} - \sum_{\substack{\alpha q<m<q(\lfloor\alpha\rfloor+1) \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} = \sum_{\substack{m>\alpha q \\ \{\frac{m}{q}\}\leq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s}, \end{aligned}$$

since the conditions $\alpha q < m < q(\lfloor\alpha\rfloor + 1)$ and $\{\frac{m}{q}\} \leq \{\alpha\}$ lead to $\lfloor\alpha\rfloor < \lfloor\frac{m}{q}\rfloor < \lfloor\alpha\rfloor + 1$. On the other hand, with the change $m \longmapsto m - (\lfloor\alpha\rfloor - 1)q$, we have

$$\begin{aligned} &\sum_{\substack{m>-q \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+(\{\alpha\}+1)q)^s} = \sum_{\substack{m>-q\lfloor\alpha\rfloor \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} \\ &= \sum_{\substack{m>-\alpha q \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} - \sum_{\substack{-\alpha q<m\leq -q\lfloor\alpha\rfloor \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} = \sum_{\substack{m>-\alpha q \\ \{-\frac{m}{q}\}\geq\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s}, \end{aligned}$$

since the set $\Big\{ -\alpha q < m \leq -q\lfloor\alpha\rfloor \Bigm| \{-\frac{m}{q}\} \geq \{\alpha\} \Big\}$ is empty. Indeed, combining the conditions we get $\lfloor-\frac{m}{q}\rfloor < \lfloor\alpha\rfloor < \lfloor-\frac{m}{q}\rfloor + 1$. Finally, we apply the same argument to the fourth and last sum in (3.3.7), changing m into $m + \lfloor\alpha\rfloor q$. We get

$$\sum_{\substack{m>-q\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\{\alpha\}q)^s} = \sum_{\substack{m>-q(\lfloor\alpha\rfloor+1)\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s}$$

$$= \sum_{\substack{m>-\alpha q\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} + \sum_{\substack{-q(\lfloor\alpha\rfloor+1)<m\le-\alpha q\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\{\alpha\}q)^s} = \sum_{\substack{m>-\alpha q\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s},$$

where the last equality holds since the conditions $-q(\lfloor\alpha\rfloor+1) < m \le -\alpha q$ and $\{-\frac{m}{q}\} < \{\alpha\}$ are not compatible. Now, observe that

$$\sum_{\substack{m>\alpha q\\ \{\frac{m}{q}\}>\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} + \sum_{\substack{m>\alpha q\\ \{\frac{m}{q}\}\le\{\alpha\}}} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} = \sum_{m>\alpha q} \frac{\bar{\chi}(m)}{(m-\alpha q)^s} = L_*(s,\bar{\chi},0,-\alpha q),$$

and similarly

$$\sum_{\substack{m>-\alpha q\\ \{-\frac{m}{q}\}\ge\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} + \sum_{\substack{m>-\alpha q\\ \{-\frac{m}{q}\}<\{\alpha\}}} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} = \sum_{m>-\alpha q} \frac{\bar{\chi}(m)}{(m+\alpha q)^s} = L_*(s,\bar{\chi},0,\alpha q).$$

Then, equation (3.3.7) can be rewritten as

$$L_*(1-s,\chi,\alpha,0) = \frac{\Gamma(s)}{(2\pi)^s}\frac{\tau_\chi\chi(-1)}{q^{1-s}}\Big(e^{i\frac{\pi}{2}s}L_*(s,\bar{\chi},0,-\alpha q) + \chi(-1)e^{-i\frac{\pi}{2}s}L_*(s,\bar{\chi},0,\alpha q)\Big).$$

□

Remark 3.11. It can be noticed that (3.3.6) has a shape which is similar to (3.3.5). For this reason, we say that (3.3.6) is a Hurwitz-Lerch type of functional equation. The same holds for (3.3.2) and (3.2.13), which is said to be a degree 2 analogue of the Hurwitz-Lerch functional equation.

3.3.2 A functional equation for the linear twist of $P_\chi(s)L(s,\chi^*)$

Our next step is studying the linear twist of the generic term of the sum (3.2.5). To this end, consider a character $\chi \in \mathfrak{X}(q,\xi)$ and let $P_\chi \in \mathcal{S}_0^\sharp(q,/f_{\chi^*},\omega^*\bar{\omega}_{\chi^*})$. We denote

$$F_\chi(s) := P_\chi(s)L(s,\chi^*).$$

Then, given $F \in \mathcal{S}_1^\sharp$, it can be rewritten as

$$F(s) = \sum_{\chi\in\mathfrak{X}(q,\xi)} F_\chi(s+i\theta).$$

Assume that $P_\chi(s) = \sum_{n|q/f_\chi} \frac{a(n)}{n^s}$. The linear twist of $F_\chi(s)$ becomes

$$
\begin{aligned}
F_\chi(s,\alpha) &:= \sum_{\substack{n|q/f_\chi \\ m\geq 1}} \frac{a(n)\chi^*(m)}{(mn)^s} e(-mn\alpha) \\
&= \sum_{n|q/f_\chi} \frac{a(n)}{n^s} \sum_{m\geq 1} \frac{\chi^*(m)}{m^s} e(-mn\alpha) = \sum_{n|q/f_\chi} \frac{a(n)}{n^s} L(s,\chi^*,n\alpha).
\end{aligned}
$$

We recall that the following relation holds

$$
\frac{a(n)}{n} = \frac{\omega^* \bar{\omega}_{\chi^*}}{\sqrt{q/f_\chi}} \bar{a}\left(\frac{q}{nf_\chi}\right), \tag{3.3.8}
$$

as a consequence of the functional equation for $\mathcal{S}_0^\sharp$ (cf. (3.2.4)). Then, combining (3.3.8) with (3.3.6), we get

$$
\begin{aligned}
&F_\chi(1-s,\alpha) = \sum_{n|q/f_\chi} \frac{a(n)}{n^{1-s}} L_*(1-s,\chi^*,n\alpha,0) \\
&= \sum_{n|q/f_\chi} \frac{a(n)}{n^{1-s}} \left(\frac{\Gamma(s)}{(2\pi)^s} \frac{\tau_{\chi^*}\chi(-1)}{f_\chi^{1-s}} \left(e^{i\frac{\pi}{2}s} L_*(s,\bar{\chi}^*,0,-\alpha n f_\chi) + \chi(-1) e^{-i\frac{\pi}{2}s} L_*(s,\bar{\chi}^*,0,\alpha n f_\chi) \right)\right) \\
&= \frac{\Gamma(s)\tau_{\chi^*}\chi(-1)}{(2\pi)^s f_\chi^{1-s}} \left(\sum_{k=\pm 1} \chi(k) e^{ki\frac{\pi}{2}s} \sum_{n|q/f_\chi} \frac{a(n)}{n^{1-s}} \sum_{m-k\alpha n f_\chi>0} \frac{\bar{\chi}^*(m)}{(m-k\alpha n f_\chi)^s} \right) \\
&= \frac{\omega^*\bar{\omega}_{\chi^*}\Gamma(s)\tau_{\chi^*}\chi(-1)}{\sqrt{q/f_\chi}(2\pi)^s f_\chi^{1-s}} \left(\sum_{k=\pm 1} \chi(k) e^{ki\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ m/n-k\alpha f_\chi>0}} \frac{\bar{a}(q/nf_\chi)\bar{\chi}^*(m)}{(m/n-k\alpha f_\chi)^s} \right) \\
&= \frac{\omega^* f_\chi^s}{i^{\mathfrak{a}}\sqrt{q}} \frac{\Gamma(s)}{(2\pi)^s} \left(e^{i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ m/n-\alpha f_\chi>0}} \frac{\bar{a}(q/nf_\chi)\bar{\chi}^*(m)}{(m/n-\alpha f_\chi)^s} + \chi(-1) e^{-i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ m/n+\alpha f_\chi>0}} \frac{\bar{a}(q/nf_\chi)\bar{\chi}^*(m)}{(m/n+\alpha f_\chi)^s} \right),
\end{aligned}
$$

observing that

$$
\bar{\omega}_{\chi^*} = \omega_{\bar{\chi}^*} = \frac{\tau_{\bar{\chi}^*}}{i^{\mathfrak{a}}\sqrt{f_\chi}} \quad \text{and} \quad \frac{\tau_{\chi^*}\tau_{\bar{\chi}^*}}{f_\chi}\chi(-1) = 1.
$$

We now note that for $n \mid \frac{q}{f_\chi}$, $n' = \frac{q}{nf_\chi}$ also divides $\frac{q}{f_\chi}$, so we can rearrange the sum over n

$$
\begin{aligned}
F_\chi(1-s,\alpha) &= \frac{\omega^* f_\chi^s}{i^{\mathfrak{a}}\sqrt{q}} \frac{\Gamma(s)}{(2\pi)^s} \times \\
&\left(e^{i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ mn-\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn-\alpha q)^s f_\chi^s q^{-s}} + \chi(-1) e^{-i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ mn+\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn+\alpha q)^s f_\chi^s q^{-s}} \right) \\
&= \frac{\omega^* q^{s-\frac{1}{2}}\Gamma(s)}{i^{\mathfrak{a}}(2\pi)^s} \left(e^{i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ mn-\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn-\alpha q)^s} + \chi(-1) e^{-i\frac{\pi}{2}s} \sum_{\substack{n|q/f_\chi \\ mn+\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn+\alpha q)^s} \right).
\end{aligned}
$$

For $\alpha \in (0,1]$ and $\beta \in \mathbb{R}$, let

$$F_\chi(s,\alpha,\beta) := \sum_{\substack{n|q/f_\chi \\ mn+\beta>0}} \frac{a(n)\chi^*(m)}{(mn+\beta)^s} e(-mn\alpha).$$

It can be easily observed that

$$\sum_{\substack{n|q/f_\chi \\ mn\pm\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn\pm\alpha q)^s} = \bar{F}_\chi(s,0,\pm\alpha q).$$

Then, for the linear twist of $F_\chi(s)$ we derive the functional equation

$$F_\chi(1-s,\alpha) = \frac{\omega^* q^{s-\frac{1}{2}}\Gamma(s)}{i^{\mathfrak{a}}(2\pi)^s}\left(e^{i\frac{\pi}{2}s}\bar{F}_\chi(s,0,-\alpha q) + \chi(-1)e^{-i\frac{\pi}{2}s}\bar{F}_\chi(s,0,\alpha q)\right). \tag{3.3.9}$$

3.3.3 Proof of Theorem 3.8

Let now $F \in \mathcal{S}_1^\sharp$. With the usual notation, we write

$$F(s,\alpha) = \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{\substack{n|q/f_\chi \\ m\geq 1}} \frac{a(n)\chi^*(m)}{(mn)^{s+i\theta}} e(-mn\alpha) = \sum_{\chi\in\mathfrak{X}(q,\xi)} F_\chi(s+i\theta,\alpha). \tag{3.3.10}$$

The functional equation for $F(s,\alpha)$ comes from the results of the previous sections.

$$\begin{aligned} F(1-s,\alpha) &= \sum_{\chi\in\mathfrak{X}(q,\xi)} F_\chi(1-s+i\theta,\alpha) \\ &= \sum_\chi \left(\frac{\omega^*\Gamma(s-i\theta)q^{s-i\theta-\frac{1}{2}}}{i^{\mathfrak{a}}(2\pi)^{s-i\theta}} \left(e^{i\frac{\pi}{2}s-i\theta}\bar{F}_\chi(s-i\theta,0,-\alpha q) + \chi(-1)e^{-i\frac{\pi}{2}s-i\theta}\bar{F}_\chi(s-i\theta,0,\alpha q)\right)\right). \end{aligned}$$

Recall that all the characters in $\mathfrak{X}(q,\xi)$ have the same parity and $\mathfrak{a}$ only depends on $\xi = \eta + i\theta$, since $\mathfrak{a} = \eta + 1$ (cf. Remark 3.4). So, the linear twist satisfies

$$F(1-s,\alpha) = \frac{\omega^*\Gamma(s-i\theta)q^{s-i\theta-\frac{1}{2}}}{i^{\mathfrak{a}}(2\pi)^{s-i\theta}} \left(e^{i\frac{\pi}{2}(s-i\theta)}\bar{F}_*(s,0,-\alpha q) + (-1)^{\mathfrak{a}}e^{-i\frac{\pi}{2}(s-i\theta)}\bar{F}_*(s,0,\alpha q)\right),$$

since we have

$$\sum_{\chi\in\mathfrak{X}(q,\xi)} \bar{F}_\chi(s-i\theta,0,\pm\alpha q) = \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{\substack{n|q/f_\chi \\ mn+\pm\alpha q>0}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn\pm\alpha q)^{s-i\theta}} = \bar{F}_*(s,0,\pm\alpha q).$$

3.4 Meromorphic continuation and pole

As already observed, it is well-known that the linear twist $F(s,\alpha)$ has a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s = 1 - i\theta$, (cf. [33, Theorem 7.1] or Theorem 3.6).

We now briefly sketch the steps to calculate the residue. Recall that

$$F(s,\alpha) = \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \frac{a(n)}{n^{s+i\theta}} L(s+i\theta,\chi^*,n\alpha).$$

Let $\chi \in \mathfrak{X}(q,\xi)$ and $n \mid \frac{q}{f_\chi}$. Since $L(s+i\theta,\chi^*,n\alpha) = \frac{1}{\tau_{\bar\chi}} \sum_{a=0}^{f_\chi-1} \bar\chi(a)\zeta_L(s,a/f_\chi-\alpha,0)$, the linear twist of the L-function has a pole at $s=1-i\theta$ if and only if $\frac{a}{f_\chi}-\{\alpha n\}\in\mathbb{Z}$ for some $a\in\{1,\dots,f_\chi\}$, with $(a,f_\chi)=1$, then if and only if $\chi^*(\{\alpha n\}f_\chi)=\chi(\alpha n f_\chi)\neq 0$ (as already pointed out in Theorem 3.9). In particular, the residue is

$$\mathrm{Res}_{s=1-i\theta} L(s+i\theta,\chi^*,\alpha n) = \frac{\bar\chi^*(\alpha n f_\chi)}{\tau_{\bar\chi^*}}.$$

Then, using again (3.3.8) and writing $m=\frac{q}{nf_\chi}$, we get

$$\begin{aligned}\mathrm{Res}_{s=1-i\theta} F(s,\alpha) &= \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \frac{a(n)}{n}\frac{\bar\chi^*(\alpha n f_\chi)}{\tau_{\bar\chi^*}}\\ &= \frac{\omega^*\omega^*_{\bar\chi}}{\sqrt{q/f_\chi}} \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \bar a\left(\frac{q}{nf_\chi}\right)\frac{\bar\chi^*(\alpha n f_\chi)}{\tau_{\bar\chi^*}}\\ &= \frac{\omega^*}{i^{\mathfrak{a}}\sqrt{q}} \sum_{\chi\in\mathfrak{X}(q,\xi)} \sum_{m|q/f_\chi} \bar a(m)\bar\chi^*\left(\frac{\alpha q}{m}\right) = \frac{\omega^*}{i^{\mathfrak{a}}\sqrt{q}}\overline{\tilde c(\alpha q)}.\end{aligned}$$

Remark. Note that, as stated in Theorem 3.6, the pole exists if and only if $c(\alpha q)=c(n_\alpha)\neq 0$.

3.5 The order of growth

Once we have the functional equation, we can go on studying the analytic properties of the linear twist. We start investigating the order of growth on vertical strips. It is already known by [35, Theorem 2] that the linear twist has polynomial growth on vertical strips. However, we consider the Lindelöf function associated to $F(s,\alpha)$,

$$\mu(\sigma,\alpha) = \inf\{\,\xi\in\mathbb{R} \mid F(\sigma+it,\alpha) \ll |t|^\xi \text{ as } |t|\to+\infty\,\}. \tag{3.5.1}$$

We recall that the Lindelöf function is continuous, convex, non-negative and strictly decreasing until it becomes identically zero (cf. e.g. [57, Section 9.41]). We now define

$$\tilde F(s,\alpha) := e^{i\frac{\pi}{2}(s-i\theta)}\bar F_*(s,0,-\alpha q) + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(s-i\theta)}\bar F_*(s,0,\alpha q),$$

and

$$\mu^\pm(\sigma,\alpha) = \inf\{\,\xi\in\mathbb{R} \mid \bar F_*(\sigma+it,0,\mp\alpha q) \ll |t|^\xi \text{ as } t\to\pm\infty\,\}.$$

Moreover, let

$$\mu^*(\sigma,\alpha) := \max(\mu^+(\sigma,\alpha),\mu^-(\sigma,\alpha)).$$

With the above notation, we prove the following result.

Theorem 3.10. *Let $F \in \mathcal{S}_1^\sharp$ and $\alpha \in (0,1]$. Then, the linear twist $F(s,\alpha)$ has polynomial growth on vertical strips and the corresponding Lindelöf function satisfies*

$$\mu(\sigma,\alpha) = \frac{1}{2} - \sigma + \mu^*(1-\sigma,\alpha). \tag{3.5.2}$$

Proof. As a first step, we combine the functional equation (3.3.2) with Stirling's formula for the Γ-factor, getting

$$\begin{aligned}|F(\sigma+it,\alpha)| \ll & q^{\frac{1}{2}-\sigma}(2\pi)^{\sigma-\frac{1}{2}}|t|^{\frac{1}{2}-\sigma}e^{-\frac{\pi|t+\theta|}{2}} \times \\ & (e^{\frac{\pi(t+\theta)}{2}}|\bar{F}_*(1-\sigma-it,0,-\alpha q)| + e^{-\frac{\pi(t+\theta)}{2}}|\bar{F}_*(1-\sigma-it,0,\alpha q)|).\end{aligned}$$

Then we can state

$$\begin{aligned}&\inf\left\{\, \xi\in\mathbb{R} \;\middle|\; F(\sigma+it,\alpha) \ll |t|^\xi \,\right\} \\ &\le \begin{cases} \frac{1}{2}-\sigma+\inf\left\{\,\xi\in\mathbb{R}\mid \bar{F}_*(1-\sigma-it,0,-\alpha q)\ll|t|^\xi\,\right\} & t\to+\infty \\ \frac{1}{2}-\sigma+\inf\left\{\,\xi\in\mathbb{R}\mid \bar{F}_*(1-\sigma-it,0,\alpha q)\ll|t|^\xi\,\right\} & t\to-\infty,\end{cases}\end{aligned}$$

i.e., the above relation can be summarized as

$$\mu(\sigma,\alpha) \le \frac{1}{2} - \sigma + \mu^*(1-\sigma,\alpha). \tag{3.5.3}$$

On the other hand, again by the functional equation and Stirling's formula we have

$$e^{-\frac{\pi}{2}|t+\theta|}\tilde{F}(1-\sigma-it,\alpha) \ll (2\pi)^{\frac{1}{2}-\sigma}q^{\sigma-\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}|F(\sigma+it,\alpha)|.$$

Then, as $t\to+\infty$,

$$\inf\left\{\,\xi \;\middle|\; \bar{F}_*(1-\sigma-it,0,-\alpha q)\ll|t|^\xi\,\right\} \le \sigma-\frac{1}{2}+\mu(\sigma,\alpha),$$

while if $t\to-\infty$ we get

$$\inf\left\{\,\xi \;\middle|\; \bar{F}_*(1-\sigma-it,0,\alpha q)\ll|t|^\xi\,\right\} \le \sigma-\frac{1}{2}+\mu(\sigma,\alpha).$$

Gathering the above inequalities we have

$$\mu^*(1-\sigma,\alpha) = \max(\mu^+(1-\sigma,\alpha),\mu^-(1-\sigma,\alpha)) \le \sigma-\frac{1}{2}+\mu(\sigma,\alpha). \tag{3.5.4}$$

Thus, by (3.5.3) and (3.5.4), we conclude

$$\mu(\sigma,\alpha) = \frac{1}{2} - \sigma + \mu^*(1-\sigma,\alpha).$$

□

Remark 3.12. Since the linear twist $F(s,\alpha)$ is absolutely convergent for $\sigma > 1$, then $\mu(\sigma,\alpha)$ vanishes in the half-plane $\sigma > 1$. Moreover, the Lindelöf function is continuous, thus we get

$$\mu(\sigma,\alpha) = 0 \quad \text{if} \quad \sigma \geq 1.$$

On the other hand, for $\sigma < 0$ we have $\mu^*(1-\sigma,\alpha) = 0$ and by continuity,

$$\mu(\sigma,\alpha) = \frac{1}{2} - \sigma \quad \text{if} \quad \sigma \leq 0.$$

Moreover, by the convexity of the Lindelöf function we deduce the upper bound

$$\mu(\sigma,\alpha) \leq \frac{1-\sigma}{2} \quad \text{if} \quad 0 < \sigma < 1.$$

3.6 Distribution of the zeros

We are now interested in studying the distribution of the zeros of the linear twist outside the critical strip $0 < \sigma < 1$. Our first result concerns the zeros in the left half-plane $\sigma < 0$ coming from the interaction between the two terms on the right-hand side of the functional equation. Since the Γ-factor does not vanish, the zeros of the linear twist are the zeros of the function

$$H(s) := \tilde{F}(1-s,\alpha) = e^{i\frac{\pi}{2}(1-s-i\theta)}\bar{F}_*(1-s,0,-\alpha q) + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(1-s-i\theta)}\bar{F}_*(1-s,0,\alpha q).$$

The theorem below shows that, for σ sufficiently small, the linear twist has infinitely many zeros which are all located inside circles whose centers are in arithmetic progression (and in particular they lie on the same line).

Theorem 3.11. *There exist infinitely many circles C_h, $h \geq 0$, of center $s_h = \alpha h + \beta$, with $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha), \Re(\beta) < 0$, and radius $\eta^{-\Re(s_h)}$ for some $0 < \eta < 1$, such that $F(s,\alpha)$ has exactly one zero inside each circle.*

Proof. Let $m_1 = \min\{\, m > \alpha q \mid \tilde{c}(m) \neq 0 \,\}$ and $m_2 = \min\{\, m > -\alpha q \mid \tilde{c}(m) \neq 0 \,\}$, then

$$\bar{F}_*(1-s,0,-\alpha q) = \sum_{m-\alpha q>0} \frac{\overline{\tilde{c}(m)}}{(m-\alpha q)^{1-s-i\theta}} = \frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-s-i\theta}} + \sum_{m>m_1} \frac{\overline{\tilde{c}(m)}}{(m-\alpha q)^{1-s-i\theta}}$$

and similarly

$$\bar{F}_*(1-s,0,\alpha q) = \sum_{m+\alpha q>0} \frac{\overline{\tilde{c}(m)}}{(m+\alpha q)^{1-s-i\theta}} = \frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-s-i\theta}} + \sum_{m>m_2} \frac{\overline{\tilde{c}(m)}}{(m+\alpha q)^{1-s-i\theta}}.$$

Moreover, we write $H(s) = W(s) + V(s)$, where

$$W(s) = e^{i\frac{\pi}{2}(1-s-i\theta)} \frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-s-i\theta}} + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(1-s-i\theta)} \frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-s-i\theta}} \tag{3.6.1}$$

and

$$V(s)=e^{i\frac{\pi}{2}(1-s-i\theta)}\sum_{m>m_1}\frac{\overline{\tilde{c}(m)}}{(m-\alpha q)^{1-s-i\theta}}+(-1)^{\mathfrak{a}}e^{-i\frac{\pi}{2}(1-s-i\theta)}\sum_{m>m_2}\frac{\overline{\tilde{c}(m)}}{(m+\alpha q)^{1-s-i\theta}}. \quad (3.6.2)$$

The idea is now to study the zeros of $W(s)$ and then to apply Rouché's theorem to localize those of $H(s)$. Let $\overline{\tilde{c}(m_1)}=\rho_1e^{i\theta_1}$ and $\overline{\tilde{c}(m_2)}=\rho_2e^{i\theta_2}$, with $\rho_1,\rho_2>0$ and $\theta_1,\theta_2\in[0,2\pi)$. Then $W(s)=0$ if and only if

$$\begin{aligned}&e^{i\frac{\pi}{2}(1-\sigma)+\frac{\pi}{2}(t+\theta)}e^{\log\rho_1+i\theta_1}e^{(\sigma-1+i(t+\theta))\log(m_1-\alpha q)}\\&=e^{\pi(\mathfrak{a}+1)}e^{-i\frac{\pi}{2}(1-\sigma)-\frac{\pi}{2}(t+\theta)}e^{\log\rho_2+i\theta_2}e^{(\sigma-1+i(t+\theta))\log(m_2+\alpha q)}.\end{aligned}$$

The equality of the moduli of the two sides gives

$$\ell=\ell(\alpha): t+\frac{1}{\pi}\sigma\log\left(\frac{m_1-\alpha q}{m_2+\alpha q}\right)+\frac{1}{\pi}\log\left(\frac{\rho_1(m_2+\alpha q)}{\rho_2(m_1-\alpha q)}\right)+\theta=0, \quad (3.6.3)$$

while from the arguments we get, for $k\in\mathbb{Z}$,

$$\ell_k=\ell_k(\alpha): t\log\left(\frac{m_1-\alpha q}{m_2+\alpha q}\right)-\pi\sigma+\theta_1-\theta_2+\theta\log\left(\frac{m_1-\alpha q}{m_2+\alpha q}\right)+(2k+\mathfrak{a})\pi=0. \quad (3.6.4)$$

Observe that the above lines are orthogonal. Then, as k runs over the integers, $W(s)$ has infinitely many zeros in the half-plane $\sigma<0$ lying on the non-vertical line ℓ. We denote these zeros as $s_k=\sigma_k+it_k\in\ell$, $k\in\mathbb{Z}$, observing that they are in arithmetic progression.
Equation (3.6.3) can be rewritten as

$$\frac{\pi}{2}(t+\theta)=\log\left(\frac{\rho_2}{\rho_1}\left(\frac{m_2+\alpha q}{m_1-\alpha q}\right)^{\sigma-1}\right)^{\frac{1}{2}},$$

hence, on the line ℓ the moduli of the two terms of $W(s)$ have value

$$(\rho_1\rho_2)^{\frac{1}{2}}((m_1-\alpha q)(m_2+\alpha q))^{\frac{\sigma-1}{2}}.$$

Let now $s=\sigma+it\in\ell$ and $\delta>0$. Define $t^*=t+\delta$ and $s^*=\sigma+it^*$. For δ sufficiently small,

$$\begin{aligned}|W(s^*)|&=\left|e^{\frac{\pi}{2}(1-s^*-i\theta)}\frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-s^*-i\theta}}+(-1)^{\mathfrak{a}}e^{-\frac{\pi}{2}(1-s^*-i\theta)}\frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-s^*-i\theta}}\right|\\&\ge e^{\frac{\pi}{2}(t^*+\theta)}\frac{\rho_1}{(m_1-\alpha q)^{1-\sigma}}-e^{-\frac{\pi}{2}(t^*+\theta)}\frac{\rho_2}{(m_2+\alpha q)^{1-\sigma}}\\&=e^{\frac{\pi}{2}\delta}e^{\frac{\pi}{2}(t+\theta)}\frac{\rho_1}{(m_1-\alpha q)^{1-\sigma}}-e^{-\frac{\pi}{2}\delta}e^{-\frac{\pi}{2}(t+\theta)}\frac{\rho_2}{(m_2+\alpha q)^{1-\sigma}}\\&=((m_1-\alpha q)(m_2+\alpha q))^{\frac{\sigma-1}{2}}(\rho_1\rho_2)^{\frac{1}{2}}(e^{\frac{\pi}{2}\delta}-e^{-\frac{\pi}{2}\delta})\\&\gg((m_1-\alpha q)(m_2+\alpha q))^{\frac{\sigma-1}{2}}(\rho_1\rho_2)^{\frac{1}{2}}\delta.\end{aligned} \quad (3.6.5)$$

We now want an upper bound for $|V(s^*)|$. The idea is to write the explicit expression of the two series in (3.6.2) and to use the integral criterion to estimate them. We denote

$$\tilde{m}_1 = \min\{\, m > m_1 \mid \tilde{c}(m) \neq 0 \,\} \quad \text{and} \quad \tilde{m}_2 = \min\{\, m > m_2 \mid \tilde{c}(m) \neq 0 \,\}.$$

Consider the first series in the definition of $V(s)$ and recall that

$$\sum_{m \geq \tilde{m}_1} \frac{\overline{\tilde{c}(m)}}{(m-\alpha q)^{1-s-i\theta}} = \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \sum_{m \geq \frac{\tilde{m}_1}{n}} \frac{\bar{a}(n)\bar{\chi}^*(m)}{(mn-\alpha q)^{1-s-i\theta}}.$$

Moreover, the sums over χ and n are finite and the set $\left\{ \frac{|\bar{a}(n)|}{n} \;\middle|\; \chi \in \mathfrak{X}(q,\xi), n|\frac{q}{f_\chi} \right\}$ is bounded. Then, we get the following estimate

$$\begin{aligned}
\left| \sum_{m \geq \tilde{m}_1} \frac{\overline{\tilde{c}(m)}}{(m-\alpha q)^{1-s-i\theta}} \right| &\ll \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \frac{|\bar{a}(n)|}{n^{1-\sigma}} \sum_{m \geq \frac{\tilde{m}_1}{n}} \frac{1}{(m-\frac{\alpha q}{n})^{1-\sigma}} \\
&\ll \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \frac{|\bar{a}(n)|}{n^{1-\sigma}} \left(\int_{\frac{\tilde{m}_1}{n}}^{+\infty} \frac{1}{(x-\frac{\alpha q}{n})^{1-\sigma}} dx + \frac{1}{(\frac{\tilde{m}_1}{n}-\frac{\alpha q}{n})^{1-\sigma}} \right) \\
&\ll \sum_{\chi \in \mathfrak{X}(q,\xi)} \sum_{n|q/f_\chi} \frac{|\bar{a}(n)|}{n^{1-\sigma}} \frac{1}{n^{\sigma}(\tilde{m}_1-\alpha q)^{-\sigma}} \ll \frac{1}{(\tilde{m}_1-\alpha q)^{-\sigma}},
\end{aligned}$$

Similarly, the same argument applied to the second term gives

$$\sum_{m \geq \tilde{m}_2} \frac{|\overline{\tilde{c}(m)}|}{(m+\alpha q)^{1-\sigma}} \ll \frac{1}{(\tilde{m}_2+\alpha q)^{-\sigma}}.$$

It follows that

$$\begin{aligned}
|V(s^*)| &\leq e^{\frac{\pi}{2}(t^*+\theta)} \frac{1}{(\tilde{m}_1-\alpha q)^{-\sigma}} + e^{-\frac{\pi}{2}(t^*+\theta)} \frac{1}{(\tilde{m}_2+\alpha q)^{-\sigma}} \\
&= e^{\frac{\pi}{2}\delta} e^{\frac{\pi}{2}(t+\theta)} \frac{1}{(\tilde{m}_1-\alpha q)^{-\sigma}} + e^{-\frac{\pi}{2}\delta} e^{-\frac{\pi}{2}(t+\theta)} \frac{1}{(\tilde{m}_2+\alpha q)^{-\sigma}} \ll (1+\delta+o(\delta^2)) \times \\
&\left(\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}} \left(\frac{m_2+\alpha q}{m_1-\alpha q}\right)^{\frac{\sigma-1}{2}} \frac{1}{(\tilde{m}_1-\alpha q)^{-\sigma}} + \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{2}} \left(\frac{r_1-\alpha q}{r_2+\alpha q}\right)^{\frac{\sigma-1}{2}} \frac{1}{(\tilde{m}_2+\alpha q)^{-\sigma}} \right) \\
&\ll ((m_1-\alpha q)(m_2+\alpha q))^{\frac{\sigma-1}{2}} \left(A\left(\frac{m_1-\alpha q}{\tilde{m}_1-\alpha q}\right)^{1-\sigma} + B\left(\frac{m_2+\alpha q}{\tilde{m}_2+\alpha q}\right)^{1-\sigma} \right),
\end{aligned} \tag{3.6.6}$$

where $A = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}}(\tilde{m}_1-\alpha q)$ and $B = \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{2}}(\tilde{m}_2+\alpha q)$. We now observe that $\frac{m_1-\alpha q}{\tilde{m}_1-\alpha q} < 1$, so $\left(\frac{m_1-\alpha q}{\tilde{m}_1-\alpha q}\right)^{1-\sigma} \to 0$ as $\sigma \to -\infty$ and similarly the other term. Then, if $\delta = \eta_0^{-\sigma}$, with

$$\max\left(\frac{m_1-\alpha q}{\tilde{m}_1-\alpha q}, \frac{m_2+\alpha q}{\tilde{m}_2+\alpha q}\right) < \eta_0 < 1, \tag{3.6.7}$$

combining equations (3.6.5) and (3.6.6), for σ sufficiently small we have $|W(s^*)| - |V(s^*)| > 0$. The same result follows by the same argument with $s^* = \sigma + i(t-\delta)$, $\delta > 0$.

Since $s = \sigma + it$ varies on the line ℓ, we have proved that for $\sigma > -\sigma'$, with a suitable $\sigma' > 0$, $|W(s)| - |V(s)| > 0$ on the boundary of the region

$$\mathfrak{L}_\delta = \mathfrak{L}_\delta(\alpha) = \{\, s \in \mathbb{C} \text{ with distance } < \delta \text{ from the line } \ell \,\}.$$

Let now $k \in \mathbb{Z}$ and consider the line l_k, where we know that

$$\arg\left(e^{\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-\sigma}}\right) = \arg\left((-1)^{\mathfrak{a}+1}e^{-\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-\sigma}}\right) + 2k\pi.$$

Assume that for $\delta > 0$ the following relation holds

$$\arg\left(e^{\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-\sigma}}\right) = \arg\left((-1)^{\mathfrak{a}+1}e^{-\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-\sigma}}\right) + 2k\pi + \delta$$

(then we are in a neighborhood of the line ℓ_k). Let us denote

$$e^{\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-s-i\theta}} = \alpha_1 e^{i\beta_1}, \quad \text{with} \quad \alpha_1 > 0, \quad \beta_1 \in [0, 2\pi),$$

$$(-1)^{\mathfrak{a}+1}e^{-\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-s-i\theta}} = \alpha_2 e^{i\beta_2}, \quad \text{with} \quad \alpha_2 > 0, \quad \beta_2 \in [0, 2\pi).$$

Since we are on the line $\ell_k - \delta$, we know that $\beta_2 = \beta_1 - \delta$. Then we have

$$|W(s)| = |\alpha_1 e^{i\beta_1} - \alpha_2 e^{i(\beta_1-\delta)}| = |e^{i\beta_1}(\alpha_1 - \alpha_2 e^{-i\delta})| > \Im|\alpha_1 - \alpha_2 e^{-i\delta}| = \alpha_1 \Im|1 - \frac{\alpha_2}{\alpha_1}e^{-i\delta}| \gg \delta\alpha_2,$$

for $\delta > 0$ sufficiently small. Proceeding as above with $\beta_1 = \beta_2 + \delta$, we can state that $W(s) \gg \delta\alpha_1$. The same argument applies on $\ell_k + \delta$, with $\delta > 0$. So we have proved that on $\ell_k \pm \delta$, with $\delta > 0$ sufficiently small,

$$\begin{aligned} W(s) &\gg \delta \max\left(\left|e^{\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_1)}}{(m_1-\alpha q)^{1-s-i\theta}}\right|, \left|e^{-\frac{\pi}{2}(1-s-i\theta)}\frac{\overline{\tilde{c}(m_2)}}{(m_2+\alpha q)^{1-s-i\theta}}\right|\right) \\ &\gg \delta \max\left(e^{\frac{\pi}{2}(t+\theta)}\rho_1(m_1-\alpha q)^{\sigma-1}, e^{-\frac{\pi}{2}(t+\theta)}\rho_2(m_2+\alpha q)^{\sigma-1}\right). \end{aligned} \tag{3.6.8}$$

Now, as already observed

$$V(s) \ll e^{\frac{\pi}{2}(t+\theta)}\frac{1}{(\tilde{m}_1-\alpha q)^{1-\sigma}} + e^{-\frac{\pi}{2}(t+\theta)}\frac{1}{(\tilde{m}_2+\alpha q)^{1-\sigma}}.$$

Then,

$$V(s) \ll \max\left(e^{\frac{\pi}{2}(t+\theta)}(m_1-\alpha q)^{\sigma-1}\left(\frac{m_1-\alpha q}{\tilde{m}_1-\alpha q}\right)^{1-\sigma}, e^{-\frac{\pi}{2}(t+\theta)}(m_2+\alpha q)^{\sigma-1}\left(\frac{m_2+\alpha q}{\tilde{m}_2+\alpha q}\right)^{1-\sigma}\right).$$

So by (3.6.8) and the above upper bound, there exist $\sigma'' \geq 0$ such that for $\sigma < -\sigma''$

$$|W(s)| > |V(s)|,$$

when $s \in \ell_k \pm \delta$ for any sufficiently small $\delta > 0$.
Let now $\bar{\sigma} = \max(\sigma', \sigma'')$. By Rouché's theorem, there exists $\eta \in (\eta_0, 1)$ with η_0 as in (3.6.7) such that for each zero $s_k = \sigma_k + it_k$ of $W(s)$ with $\sigma_k < -\bar{\sigma}$, $F(s, \alpha)$ has exactly one zero in a circle of center s_k and radius $\eta^{-\sigma_k}$. Re-parameterizing the zeros the statement follows. □

Remark 3.13. The above result corresponds to Corollary 3.3, which describes the distribution of the trivial zeros of the standard twist in degree 2. However, it can be noticed that our theorem is slightly more precise. Indeed, Kaczorowski and Perelli showed that the zeros are located around the line (3.2.14), but, as they observed, the problem of the finer location of the zeros is open. On the other hand, we prove that the zeros in our case are located inside circles centered on the line ℓ with radius which tends to zero as $\sigma \to -\infty$.

We now present another theorem on the distribution of the zeros. In this case, the proof is complete only if α is rational, while for irrational values of α only partial results are known.

Theorem 3.12. *Let $0 < \alpha \leq 1$ be rational. If $F(s, \alpha)$ and $\bar{F}_*(s, 0, \pm\alpha q)$ are not of the form $P(s)L(s, \chi)$, where $P(s)$ is a Dirichlet polynomial and $L(s, \chi)$ is a Dirichlet L-function, then*

(i) there exist $\sigma_1, \sigma_1' > 0$ such that the set

$$\{\sigma \in (1, 1 + \sigma_1] \mid F(\sigma + it, \alpha) = 0, t \in \mathbb{R}\}$$

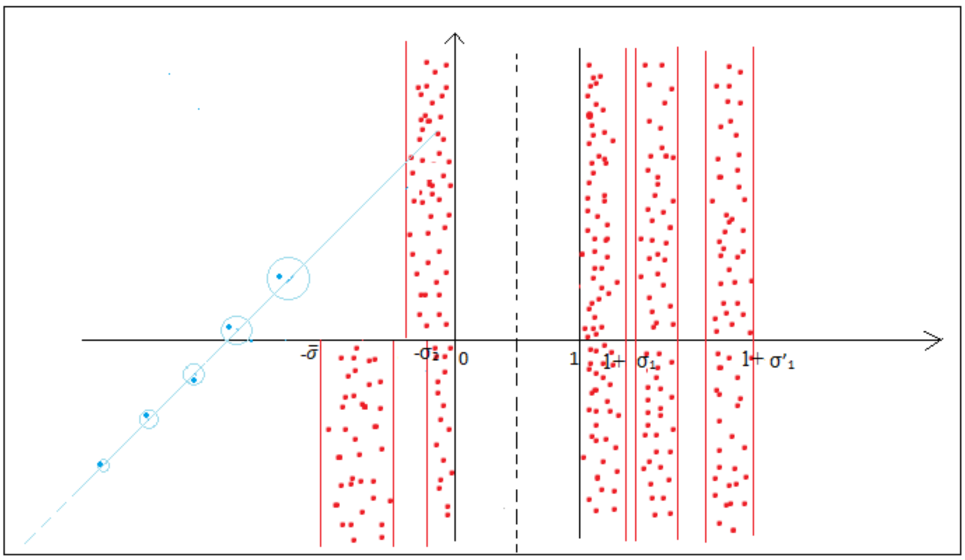
is dense in $(1, 1 + \sigma_1]$ and $F(\sigma + it, \alpha) \neq 0$ if $\sigma > 1 + \sigma_1'$.

(ii) there exist $\sigma_2 > 0$ such that the set

$$\{\sigma \in [-\sigma_2, 0) \mid F(\sigma + it, \alpha) = 0, t \in \mathbb{R}\}$$

is dense in the interval $[-\sigma_2, 0)$.

The following picture represents a typical distribution of the zeros outside the critical strip of the linear twist $F(s, \alpha)$ when α is rational (see Theorems 3.11 and 3.12).



Proof. If α is rational, $F(s,\alpha)$, and $\bar{F}(s,0,\pm\alpha q)$ can be written as linear combinations of Dirichlet L-functions, since the coefficients $\tilde{c}(n)$ are periodic (cf. [52, Theorem PDCB]). Then, by the result of Saias and Weingartner (Theorem 2.10), if these linear combinations do not reduce to a single term of the form $P(s)L(s,\chi)$, they have infinitely many zeros in the half-plane $\sigma > 1$. The density of the real parts and the possible existence of gaps in the region where the zeros exist are consequences of Theorem 2.12. As already observed, with this result Righetti proved that there exist infinitely many Dirichlet series which do not vanish in strips contained in the region where the zeros exist, disproving a conjecture of Bombieri and Ghosh. Therefore, part (i) is proved.

Let now $\sigma < 0$. The main tool in the proof of assertion (ii) is the functional equation. Consider again the function

$$\begin{aligned} H(s) &= e^{i\frac{\pi}{2}(1-s-i\theta)}\bar{F}_*(1-s,0,-\alpha q) + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(1-s-i\theta)}\bar{F}_*(1-s,0,\alpha q) \\ &= e^{i\frac{\pi}{2}(1-s-i\theta)}F_1(s) + (-1)^{\mathfrak{a}} e^{-i\frac{\pi}{2}(1-s-i\theta)}F_2(s). \end{aligned} \tag{3.6.9}$$

Observe that, if they are not of the form $P(s)L(s,\chi)$, $F_1(s)$ and $F_2(s)$ have infinitely many zeros, since $1-\sigma > 1$. Moreover, the exponential factors imply that if one of the two terms of $H(s)$ tends to infinity, the other tends to zero.

Assume $t = \Im(s) > 0$. Let ρ be a zero of $F_1(s)$ and consider $\delta > 0$ sufficiently small such that F_1 does not vanish on a circle of center ρ and radius δ. Define

$$\gamma = \min_{|s|=\delta} |F_1(s+\rho)| > 0. \tag{3.6.10}$$

Since we are working with generalized Dirichlet series, by almost periodicity (cf. Definition 2.5) we can state that, for any $\varepsilon > 0$, the set of $\tau \in \mathbb{R}$ such that

$$\max_{|s|=\delta} |F_1(s+\rho+i\tau) - F_1(s+\rho)| < \varepsilon \tag{3.6.11}$$

is relatively dense. Moreover, we have

$$|e^{-\frac{\pi}{2}(1-s-i\theta)}H(s) - F_1(s)| = |e^{-\pi(1-s-i\theta)}F_2(s)| = e^{-\pi(t+\theta)}|F_2(s)|.$$

Then, the polynomial growth on vertical strips (Theorem 3.10) implies, for some positive A,

$$\max_{|s|=\delta} |H(s+\rho+i\tau)e^{-i\frac{\pi}{2}(1-s-\rho-i\tau)} - F_1(s+\rho+i\tau)| \ll e^{-\pi\tau}\tau^A. \tag{3.6.12}$$

We gather equations (3.6.10), (3.6.11) and (3.6.12), choosing ε and τ such that $\varepsilon + e^{-\pi\tau}\tau^A < \gamma$. Then, by triangular inequality we get

$$\max_{|s|=\delta} |H(s+\rho+i\tau)e^{-i\frac{\pi}{2}(1-s-\rho-i\tau)} - F_1(s+\rho)| < \gamma = \min_{|s|=\delta} |F_1(s+\rho)|.$$

Applying Rouché's theorem, we deduce that $F_1(s)$ and $H(s)$ have the same number of zeros

inside the circle of center ρ and radius δ. Since $F_1(\rho) = 0$, we conclude that $H(s)$, then $F(s,\alpha)$, has a zero inside the considered circle.
The same argument applies for $t < 0$, replacing $F_1(s)$ with $F_2(s)$, since in this case the second term is dominating in $H(s)$. This concludes the proof of part (ii). □

If α is irrational, $\bar{F}_*(s, 0, \pm\alpha q)$ can be seen as generalized Hurwitz zeta functions with periodic coefficients. Thus, by Theorem 2.15 we deduce that they have infinitely many zeros for $\sigma > 1$. Therefore, if $\sigma < 0$, $\bar{F}_*(1-s, 0, \pm\alpha q)$ have infinitely many zeros (cf. equation (3.6.9)) and the same argument used in the proof of (ii) applies. Thus, part (ii) of Theorem 3.12 even holds for α irrational. On the other hand, part (i), i.e. the existence of infinitely many zeros of the linear twist $F(s,\alpha)$ in the right half-plane, is still an open problem if α is not rational, since the analogue for the classical Hurwitz-Lerch zeta function is still not known.

Remark 3.14. Observe that the linear twist $F(s,\alpha)$ is of the form $P(s)L(s,\chi)$ only if $\alpha \in \{1, \frac{1}{2}\}$. In particular, if $\alpha = 1$, the linear twist does not vanish in the region of absolute convergence if and only if the sum (3.2.5) reduces to a single term. On the other hand, $\bar{F}_*(s, 0, \pm\alpha q)$ reduce to the product of a Dirichlet polynomial and a Dirichlet L-function if and only if $\alpha q \in \mathbb{Z}$ or $\alpha q = a + \frac{1}{2}$ with $a \in \mathbb{Z}$. For a more detailed discussion, we refer to Theorem 2.15, where we consider the case of the Hurwitz zeta function with periodic coefficients. In particular, one can see Section 2.3.1.

Bibliography

[1] T. M. Apostol, *Modular functions and Dirichlet series in Number Theory*, 2nd ed., Springer-Verlag, 1990.

[2] H. Bohr, *Om Addition af uendelig mange konvekse Kurver*, Dansk Videnskab. Selsk. Forh. 4 (1913), 325–366.

[3] H. Bohr, *Zur Theorie der allgemeinen Dirichletschen Reihen*, Math. Ann. 79 (1918), 136–156.

[4] H. Bohr, *Zur Theorie der fastperiodischen Funktionen III*, Acta Math. 47 (1926), no. 3, 237–281, Dirichletentwicklung analytischer Funktionen.

[5] S. Bochner, *On Riemann's functional equation with multiple gamma factors*, Ann. of Math. 67 (1958), 29-41.

[6] E. Bombieri D. Ghosh, *Around the Davenport-Heilbronn function*, Russ. Math. Surv. 66 (2011), no. 2, 221-270.

[7] A. R. Booker, F. Thorne, *Zeros of L-functions outside the critical strip*, Algebra Number Theory 8 (2014), no. 9, 2027–2042.

[8] E. Carletti, G. Monti Bragadin, A. Perelli,*On general L-functions*, Acta Arith. 66 (1994), 147-179.

[9] J.W.S. Cassels, *Footnote to a note of Davenport and Heilbronn*, J. London Math. Soc. 36 (1961), 177-184.

[10] T. Chatterjee, S. Gun, *On the zeros of generalized Hurwitz zeta functions*, J. Number Theory, 145 (2014), 352-361.

[11] K. Chandrasekharan, R. Narasimhan, *Functional equations with multiple gamma factors and the average order of arithmetical functions*, Ann. of Math. 76 (1962), 93-136.

[12] P. Codecà, R. Dvornicich and U. Zannier, *Two problems related to the non-vanishing of* $L(1,\chi)$, J. Théor. Nombres Bordeaux 10 (1998), 49-64.

[13] J. B. Conrey, A. Ghosh, *On the Selberg class of Dirichlet series: small degrees*, Duke Math. J. 72 (1993), 673-693.

[14] J. B. Conrey, A. Ghosh, *Turán inequalities and zeros of Dirichlet series associated with certain cusp forms*, Trans. Amer. Math. Soc. 342 (1994), no. 1, 407–419.

[15] H. Davenport, *Multiplicative Number Theory*, Second edition, revised by H.L. Montgomery, Springer Verlag (1980).

[16] H. Davenport, H. Heilbronn, *On the zeros of certain Dirichlet series I, II*, Journal London Math. Soc. 11 (1936), 181-185, 307-312.

[17] P. Deligne, *Formes modulaires et representations l-adiques* , Seminaire Bourbaki vol. 1968/69 Exposes 347-363, Springer Berlin Heidelberg (1971), 139-172.

[18] P. Deligne, *La conjecture de Weil. I*, Publications Mathematiques de l'Institut des Hautes Etudes Scientifiques 43 (1974), no. 1, 273-307.

[19] W. Duke, J.B. Friedlander, H. Iwaniec, *The subconvexity problem for Artin L-functions*, Invent. Math., 149 (2002), 489–577.

[20] A. Fröhlich, M. J. Taylor, *Algebraic Number Theory*, Cambridge Studies in Advanced Mathematics, vol. 27, Cambridge University Press, Cambridge (1993).

[21] E. Fouvry and S. Ganguly. *Orthogonality between the Möbius function, additive characters, and Fourier coefficients of csp forms*, Compos. Math. (5) 150 (2014), 763–797.

[22] R. Garunkstis, A. Laurincikas, *The Lerch Zeta Function*, Kluwer Academic Publishers: Dordrecht (2002).

[23] D. R. Heath-Brown, *The Pjateckii-Shapiro prime number theorem*, J. Number Theory 16 (1983), 242-266.

[24] J. Hoffstein, D. Ramakrishnan, *Siegel zeros and cusp forms*, Internat. Math. Res. Notices (1995), 279–308.

[25] L. Hu, *Quadratic forms connected with Fourier coefficients of Maass cusp forms*, Front. Math. China (2015), 1101-1112.

[26] L. K. Hua, *Introduction to Number theory*, Berlin : Springer-Verlag (1982).

[27] H. Iwaniec, *Topics in classical automorphic forms*, AMS Graduated Studies v. 17 (1997).

[28] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS, Providence (2004).

[29] B. Jessen, Aurel Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. 38 (1935), no. 1, 48–88.

[30] J. Kaczorowski, *Axiomatic theory of L-functions: the Selberg class* Analytic number theory, Lecture Notes in Math., vol. 1891, Springer, Berlin (2006), 133 -209.

[31] J. Kaczorowski, M. Kulas, *On the non-trivial zeros off the critical line for L-functions from the extended Selberg class*, Monatsh. Math. 150 (2007), no. 3, 217–232.

[32] J. Kaczorowski, G. Molteni, A. Perelli, *Linear independence in the Selberg class*, C. R. Math. Rep. Acad. Sci. Canada 21 (1999), 28–32.

[33] J. Kaczorowski A. Perelli, *On the structure of the Selberg class, I:* $0 \leq d \leq 1$, Acta Math. 182 (1999), 207-241.

[34] J. Kaczorowski A. Perelli, *On the structure of the Selberg class, II: invariants and conjectures*, J. reine angew. Math. 524 (2000), 73-96.

[35] J. Kaczorowski A. Perelli, *On the structure of the Selberg class, IV: non-linear twist*, Acta Arith. 116 (2005), 315-341.

[36] J. Kaczorowski A. Perelli, *On the structure of the Selberg class, VII:* $1 < d < 2$, Ann. of Math. 173 (2011), 1397-1441.

[37] J. Kaczorowski A. Perelli, *On the standard twist of the L-functions of half-integral weight cusp forms*, to appear in Nagoya Math. J.

[38] H. Kim, P. Sarnak, *Refined estimates towards the Ramanujan and Selberg conjectures*, Journal of the American Mathematical Society 16 (2003), no. 1, 175-181.

[39] G. Molteni, *A note on a result of Bochner and Conrey-Ghosh about the Selberg class*, Arch. Math. (Basel) 72(3) (1999), 219-222.

[40] J. Neukirch, *Algebraic Number Theory*, Springer-Verlag, 322 Berlin (1999).

[41] Pan Chengdong, Pan Chengbiao, *Goldbach Conjecture*, Beijing: Science Press (1981).

[42] A. Perelli, *General L-functions*, Ann. Mat. Pura Appl. (4) 130 (1982), 287-306.

[43] A. Perelli, *On some exponential sums connected with Ramanujan's* τ *function*, Mathematika 31 (1984), 150-158.

[44] A. Perelli, *A survey of the Selberg class of L-functions, part I*, Milan J. Math. 73 (2005), no. 1, 19-52.

[45] A. Perelli, *A survey of the Selberg class of L-functions, part II*, Riv. Mat. Univ. Parma 3* (2004), 83-118.

[46] I. Piateski-Shapiro, *Arithmetic Dirichlet series: Conjectures*, in *Structure Theory of Set Addition*, CIRM Marseille (1993), 271-281.

[47] H. S. A. Potter, E. C. Titchmarsh, *The zeros of Epstein's zeta functions*, Proc. London Math. Soc. S2-39 (1935), no. 1, 372-384.

[48] H.-E. Richert, *Uber Dirichletreihen mit Funktionalgleichung*, Acad. Serbe Sci. Publ. Inst. Math. 11 (1957), 73-124.

[49] M. Righetti, *Zeros of combinations of Euler products for* $\sigma > 1$, Ph.D. Thesis, Università degli Studi di Genova (2016).

[50] M. Righetti, *Zeros of combination of Euler product for* $\sigma > 1$, Monath. Math. (2016) 180: 337. https://doi.org/10.1007/s00605-015-0773-0.

[51] M. Righetti, *On the density of zeros of linear combinations of Euler products for* $\sigma > 1$, Algebra Number Theory, vol. 11 (2017), no. 9, 2131-2163.

[52] E. Saias A. Weingartner, *Zeros of Dirichlet series with periodic coefficients*, Acta Arithmetica 140 (2009), 335-344.

[53] P. Sarnak, *Three Lectures on the Möbius Function Randomness and Dynamics*, publications.ias.edu/sarnak/paper/512 (2010).

[54] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi conference on analytic number theory, held at Maiori, Amalfi, Italy, from 25 to 29 September 1989, Salerno: Università di Salerno (1992), 367-385.

[55] G. Shimura, *On Modular Forms of Half Integral Weight*, Annals of Mathematics Second Series, vol. 97 (1973), no. 3, 440-481.

[56] K. Soundararajan, *Degree 1 elements of the Selberg class*, Expo. Math. 23 (2005), 65-70.

[57] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, Oxford (1939).

[58] R.C. Vaughan, *The Hardy-Littlewood method*, Cambridge tracts in Mathematics 80, Cambridge University Press (1981).

[59] E. T. Whittaker, *An expression of certain known functions as generalized hypergeometric functions*, Bull. Amer. Math. Soc. 10(3) (1903), 125–134.

[60] G. Zaghloul, *Quadratic forms connected with Fourier coefficients of holomorphic and Maass cusp forms*, J. Number Theory 167 (2016), 118-127.

[61] G. Zaghloul, *A note on the zeros of generalized Hurwitz zeta functions*, J. Number Theory 196 (2019), 197-204. https://doi.org/10.1016/j.jnt.2018.09.016.

[62] G. Zaghloul, *On the linear twist of degree 1 functions in the extended Selberg class*, preprint.

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