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Transfer Reactions, Neutrinoless Double Beta Decay and Double Charge Exchange

Candidato

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Preface

The construction of the interactions in the nuclear structure coming from the first principles is something that from the origin of the quantum mechanics is pursuit, but it is not an easy task to be implemented. Developing phenomenological models is important because will help to understand physical processes which can be analyzed systematically in the experimental data available. The knowledge of the internal degrees of freedom is crucial to understand nuclear structure features like collectivity states, single particle states, pairing properties, etc. There are many experiments and a great deal of research toward the study of the nuclei that helps the development of the compression of the nucleus. Research on transfer reactions, charge exchange reactions plays an essential role to improve the nuclear models. This thesis aims to cover the essential aspects of the description of microscopic interacting boson model applied to different types of problems. Such as the calculation of spectroscopic amplitudes in transfer reactions, nuclear matrix elements of double beta decay without neutrinos and double charge exchange reactions. The knowledge of the nuclear structure allows us to explore and study contemporary problems in modern physics.

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Introduction

In this thesis, for the first time, the formalism for calculating the the spectroscopic amplitudes of the two-nucleon transfer process were calculated within the microscopic IBM-2. The formalism was submitted to Physical Review C [1]. The application of this formalism, which allows to calculate the spectroscopic amplitudes for the nikel isotopes was published in the journal Physical Review C [2]. The experimental cross sections of the two-nucleon transfer reactions were recently measured at the Instituto Nazionale di Fisica Nucleare -Laboratori Nazionali del Sud (Italy). Our theoretical description of two-proton transfer reaction in collaboration with J. Lubian for the reaction ^{116}Cd to ^{116}Sn, agrees with preliminary experimental data, and this calculation will be published in another article [3]. It was found that the two-proton transfer cross section is three orders of magnitude smaller than the double charge exchange cross section. Those are important results since the two proton transfer is a competitive process in respect to the double charge exchange. Thanks to the theoretical work on two-nucleon transfer process we became part of the NUMEN collaboration [4]. Finally, since the goal of the NUMEN collaboration is to arrive to extract the double charge exchange nuclear matrix elements, a simple model of double charge exchange cross section has been developed in this thesis. In particular, it has been demonstrated for the first time the possibility to factorize the nuclear matrix elements from the reaction part [5]. The content of the thesis is given as follows:

In chapter 1, the formalism for calculating the spectroscopic in IBM-2 has been developed and it has been applied to the two proton transfer ^{116}Cd to ^{118}Sn, ^{114}Cd to ^{116}Sn and two neutron transfer ^{118}Sn to ^{116}Sn , ^{116}Cd to ^{114}Cd and ^{64}Ni to ^{66}Ni. The spectroscopic amplitudes of the two-nucleon transfer reactions are important in the forthcoming experiments of NUMEN Collaboration, where the two transfer reaction are competitive process of the Double charge exchange reactions. In the second part of this chapter is presented the calculation of the spectroscopic amplitudes of two-neutron transfer reaction ^{64}Ni(^{18}O,^{16}O)^{66}Ni considering one- and two-step processes, in order to check which type of mechanism is more important.

In chapter 2, we discuss the nuclear matrix elements for neutrinoless double beta decay in the scheme of the microscopic interacting boson model. It has been red-

erived the $0\nu\beta\beta$- decay operator starting from the basic theory of the two body transition densities. The nuclear matrix elements $0\nu\beta\beta$- decay have been calculated between the ^{76}Ge and ^{76}Se ground states, using IBM-2 and Generalized Seniority.

In chapter 3, we present a simple model of double charge exchange by using the Eikonal approximation. It has been demonstrated for the first time the possibility to factorize the nuclear matrix elements, in the low momentum transfer limit, from the reaction part [5]. The charge exchange nuclear matrix elements are Double Gamow Teller (DGT) type and they are in linear correlation with those of $0\nu\beta\beta$ [5].

The general formula to calculate the DGT matrix elements has been derived using IBM2.

Chapter 1
Transfer reactions in Microscopic IBM-2

A better understanding of the nuclear structure and reactions will help to improve the model calculation of nuclear reactions and decays such as single beta decay, double beta decay with and without emission of neutrinos. The nuclear structure exhibits features like collective and single particle states, pairing correlations, clustering and more [6, 7]. Knowledge of relevant internal degrees of freedom is important to understand the main features of the nuclei. Such information can be extracted from direct nuclear reactions like elastic and inelastic scattering, nucleon transfer reactions etc. In particular , transfer reactions in which the projectile transfer one or more nucleons to or from the target nucleus provide access to nuclear spectroscopic amplitudes. F. Capuzzello [8] has been noticed that direct two neutron are important in light nuclei like ^{12}C (^{18}O,^{16}O)^{14}C. Thus we got interested to study the collectivity properties for heavier nuclei. The two nucleon transfer can be used as a test of pairing correlations in nuclei. Reactions with heavy ions have been extensively used to study pairing correlations in proton- rich and neutron rich nuclei to study the nuclear matrix elements for neutrinoless double beta decay by measurements of heavy ion reactions. In 2017, experimentalists of the NUMEN project in the INFN-Laborati Nazionali del Sud in Catania, Italy, have studied the two neutron transfer between ^{118}Sn to ^{116}Sn and ^{64}Ni to ^{66}Ni and two proton transfer ^{116}Cd to ^{118}Sn using a 15 MeV for ^{20}Ne, and 84 MeV ^{18}O beams detecting ^{20}O and ^{18}Ne. The Spectroscopic amplitudes for two nucleon transfer reaction is computed using the formalism of microscopic interacting boson fermion model [2]. The nuclear structure information is contained in the spectroscopic amplitudes or intensities, which is a measure of the overlap of the wave functions of the final and initial nucleus in the reaction [9].

The microscopic IBM-2 is a way to calculate realistic matrix elements for medium and heavy nuclei, that has been applied recently in neutrinoless double beta decay [10]. We want to provide a method to compute the spectroscopic amplitudes for two nucleon transfer reactions between even-even using the microscopic interacting boson model two (IBM-2). We present the theoretical part of the calculation of the spectroscopic amplitude of two nucleon transfer reaction.

1.1 Microscopic Interacting Boson Model

We start our discussion about the origin of the description of the Microscopic Interacting Boson Model. Let us consider states of identical $n = 2N$ nucleons (N number of pairs of nucleons) outside closed shells and a ground state of a nucleus with a $J = 0$ and seniority $\nu = 0$, in a pure j^nconfiguration. These states may be described with Generalized Seniority introduced by I. Talmi [11]. Historically, these states were shown by Mottelson to be composed with 2N particles of the ground state wave functions of the Bardeen, Cooper, and Schrieffer in the theory of superconductivity [12]. They were later used as variational wave functions. The structure constants α_j were taken to be variational parameter computed by Gambhir, Y. K., Rimini, A. and Weber [13] and later were computed differently by I. Talmi [11].

In the calculation of the nuclear operator which provides the transition between two pairs of nucleons, we can assume that they are in different orbitals j and the S^+ operator can be used if the orbits are degenerated. All the information about the structure of the shells should be contained in the structure coefficient; the α_j constants is a delicate point if the orbits are non-degenerate.The Seniority scheme should consider not only operators with degenerated orbits but also non-degenerated orbits thus we require put something to the operator which should indicate that it is created a pair of particles over different orbits j. So the natural way to do this extension is given by generalizing the concept of Seniority including a parameter alpha which carries this information concerned to the energetic structure of the nucleons due to several orbits. However, this is not all the problem. Assuming an operator that creates two fermions over all possible j orbits like $S_+ = \sum_j \alpha_j S_j^+$ there is something else that we should consider, the internal manner how the of the pair of nucleons are coupled.

One possible way is to consider pairs of nucleons are coupled to zero, or pairs coupled to a higher momentum. Under the situation that you have this pairs of nucleons over the same orbital, which corresponds to the formalism of quasi-spin where is a limit case of the generalized seniority, the value of alphas is the unity. However, this assumption is not physically because they the final state may be populated from different orbitals; therefore the alphas should be different.

A more realistic situation, we required a Hamiltonian which their eigenvalues for the ground state should be connected to the binding energies given by the experiments. Igal Talmi [11] considered a Hamiltonian that contains a single particle energy contribution and two body particle contribution. Applying the double commutator between this Hamiltonian and the operator S we obtain some restrictions allowing us determine the values of our constants. This approach is too general, but this condition includes the parameters alphas and other parameters which can be assumed.

Taking into account pairs of nucleons coupled to $J = 0$and $J = 2$(relevant term due to quadrupole interaction) which is essential link this new coupling to the quadrupole interaction, which is the work initiated in shell model and seniority scheme by Talmi, Shalit [14], Bohr , Sorensen, and later applied to General Se-

niority [11], Otsuka Arima [16], and then applied to IBM by Pittel[24], Duval and Barret [25], and started by Scholten [17].

The relationship between the matrix elements of the quadrupole operator in the generalized seniority and bosonic space is the structure parameter beta which is the "new alpha" but for the coupling to $J = 2$. Igal Talmi [18] provides this fact and uses this condition, in this case, the relevant commutator relation is given by $[Q^F, S^F] = D^+$, where D^+ is nothing but

$$D_+(j_1 j_2) = \frac{1}{\sqrt{1+\delta j_1, j_2}} a^+_{j_1} a^+_{j_2} \tag{1.1}$$

which is the operator of creation of the pair of nucleons coupled to $J = 2$ and

$$D^+ = \sum_{j_1 \leq j_2} \beta_{j_1 j_2} D_+(j_1 j_2) \tag{1.2}$$

So we can obtain the parameters alpha and beta, and by putting single particle energies and quadrupole interactions, however, we are missing other contributions which are explicitly written in the Talmi Hamiltonian.

The phenomenological Hamiltonian of the Nucleon-Nucleon interaction the major contributions are:

Table 1.1: Different extensions of the structure constants in the Generalized Seniority.

Interaction	Generalized Seniority Model		
	Talmi(1971)	Pittel (1982)	Otsuka,et al (1977)
Pairing (alpha)	yes	yes	yes
Quadrupole (beta)	no	yes	yes
Symmetry Energy (gamma)	no	no	no
Scholten Hamiltonian (others)	no	no	no

The first correction to the operator of general seniority should consider the structure coefficient called gamma which is connected with the symmetry energy.(see Iachello Pag 140 Eq. 4.43)

It is convenient develop the IBM compatible with seniority scheme in a way to use the same formulas of matrix elements of two body matrix elements used in seniority scheme. There are some recursion relation formulas in seniority scheme and also some matrix elements in seniority scheme like a single particle irreducible

tensors and two particle scalar operators and even some paring and average interaction energies which were explicitly developed by de Shalit and Talmi, Feshbach and others.[1] in the late of 50s and mid of 60s.

1.1.1 Nucleon Pairing and Generalized Seniority

The seniority scheme was introduced by Racah [19] for the classification of states in atomic spectra. The original idea was to find additional quantum numbers to distinguish between states of an electron which have the same values of S, L, and J. We shall consider a generalization of the formalism of j^n configurations of identical nucleons of quasi-spin by [20]. The case of several non degenerate single particle orbits the seniority scheme can be introduced via the operators

$$\begin{aligned} S_+(j) &= \tfrac{1}{2}\sqrt{2j+1}(c_j^\dagger \times c_j^\dagger)^0 \\ S_-(j) &= \tfrac{1}{2}\sqrt{2j+1}(\tilde{c}_j \times \tilde{c}_j)^0 \end{aligned} \tag{1.3}$$

where $c_j^\dagger$ $(\tilde{c}_j)$is the creation(annhilation) operator of a single fermion on the orbit j and $S_0 = \frac{1}{2}[S_+, S_-] = \frac{1}{2}\hat{n} - \frac{2j+1}{2}$ where the $\hat{n}$ is the fermion number operator. Those operators above close under commutation and generates an algebra quasi-spin SU(2) allowing to use the powerful techniques of group theory to obtain reduction formulas of matrix elements of operators between states of nonmaximal seniority to those of maximal seniority. The scheme introduced by Racah is based on the idea of pairing of particles into J=0 pairs, entering the seniority quantum number $.\nu$ (vetek in Hebrew) which is equal to the number of unpaired particles in the j^n configuration, counts the number of particles not pairwise. This configuration is however only valid for single j- configurations. In most problems in the description of heavy nuclei, such transfer reactions, many single particle orbits enter. It is known that an arbitrary system of fermions can be mapped onto a system of bosons [21]. There are many ways to describe relevant physical quantities given initially regarding fermionic degrees of freedom by bosons. One usually starts in the space of fermions with a well-defined interaction, typically an effective two-body interaction. The space under consideration can be the entire Hilbert space of many-fermion states or so-called- collective subspace. The states and operators are realized in the space of bosons. This transition from fermion space to boson space is usually called a mapping. The basic idea is, that the mapping procedure does not change the physical quantities, what we want to describe. If this property did, one describes in both spaces the same physics. The mapping process is no way unique and going from fermion to boson space one has to choose among many different options. As long as

[1] Just a note of history, during the times of the first papers of F. Iachello with Feshbach in 1973 in the Politecnico di Torino was important consider the latest results of Seniority scheme making it a key point for the construction of the Interacting Boson Model.

the mapping is exact, it usually does not matter, which option is selected, since the physics is not changed by the mapping procedure. In this way, one first maps basis states. i.e. one makes a correspondence between a set of orthonormalized states in fermion space and a set of orthonormalized states in boson space. In a second step, one maps operators in a manner that the value of the matrix elements between corresponding states in fermion and boson-space are identical. This mapping of states introduced initially by Marumori, Yamamura and Tokunaga [22] and also has been applied for a particle pairs in the seniority coupling scheme of a single j-shell by Otsuka, Arima and Iachello [23] and is therefore often referred as OAI- mapping and for non-degenerate spherical system by Pittel, Duval and Barret [24] and Gambhir Ring and Schuck [26].

To do a mapping of the fermionic space on the bosonic space, we need to represent the fermionic states into the SD fermionic subspace. The single-particle orbitals have are several possible pair states with $\lambda = 0$ and $\lambda = 2$ thus it is introduced a pair creation operators S_+ and D_+ which by definition create the energetically lowest 0^+ and 2^+ pairs, respectively [11]. The pair-creation operators can be expressed concerning operators that create pairs of nucleons(or nucleons holes) in specific active orbits by

$$\begin{aligned} S_+ &= \sum_j \alpha_j S_+(j) \\ D_+ &= \sum_{j \leq j'} \beta_{j,j'} D_+(j,j') \end{aligned} \tag{1.4}$$

with

$$\begin{aligned} S_+(j) &= \sqrt{\tfrac{\Omega_j}{2}}[c_j^\dagger \times c_j^\dagger]^0 \\ D_+(j,j') &= \sqrt{\tfrac{1}{1+\delta_{jj'}}}[c_j^\dagger \times c_{j'}^\dagger]^2 \end{aligned} \tag{1.5}$$

where α and β are structure constants that depends on the shell selected , S_+ and D_+ create the energetically the lowest 0+ and 2+ paired fermion states, the creation operators $a_j^\dagger$ refers to valence particle outside closed shells and also to valence holes in closed shell.

1.2 Fermionic Transfer Operator

Let us consider identical nucleons in the fermionic space, either neutrons or protons, which are restricted to a set of non-degenerate single particle levels j.

The operator that creates an antisymmetric state of two particles coupled to angular momentum λ is defined as

$$T_+^{(j_1,j_2,\lambda,M)}|0\rangle = |j_1,j_2,\lambda,M\rangle \tag{1.6}$$

and hence in second quantization may be written like[2]

$$\begin{aligned} T_{+}^{(j_1,j_2,\lambda,M)} &= N_{12}\sum_{m_1m_2}\langle j_1,m_1,j_2,m_2|\lambda M\rangle c^{\dagger}_{j_2m_2}c^{\dagger}_{j_1m_1} \\ &= -N_{12}\sum_{m_1m_2}\langle j_1m_1j_2m_2|\lambda M\rangle c^{\dagger}_{j_1m_1}c^{\dagger}_{j_2m_2} \\ &= -N_{12}[c^{\dagger}_{j_1}\times c^{\dagger}_{j_2}]^{\lambda}_{M} \end{aligned} \tag{1.7}$$

where $N_{12}=\sqrt{\frac{1}{1+\delta_{j1,j_2}}}$ the two-particle destruction operator is

$$T_{-}^{(j_1,j_2,\lambda,M)} = N_{12}\sum_{m_1m_2}\langle j_1,m_1,j_2,m_2|\lambda M\rangle c_{j_1m_1}c_{j_2m_2} \tag{1.8}$$

and the tensor form of the two-particle destruction operator is

$$\begin{aligned} T_{-}^{(j_1,j_2,\lambda,M)} &= (-1)^{\lambda-M}\{T_{+}^{(j_1,j_2,\lambda,M)}\}^{+} \\ &\quad (-1)^{\lambda-M}N_{12}\sum_{m_1m_2}\langle j_1,m_1,j_2,m_2|\lambda -M\rangle \\ &\quad \times c_{j_1m_1}c_{j_2m_2} \\ &\quad = N_{12}[\tilde{c}_{j_1}\times\tilde{c}_{j_2}]^{\lambda}_{M} \end{aligned} \tag{1.9}$$

1.3 Bosonic transfer operator

The two nucleon transfer calculations requires matrix elements of number non conserving operators. The operators will be denoted by $T_{+}^{(L)}$ and $T_{-}^{(L)}$. If we retain only the one body operators

$$\begin{aligned} T_{+}^{(L)} &= \sum_k p_k^{(L)} b_k^{\dagger} \\ T_{-}^{(L)} &= \sum_k p_k^{(L)} \tilde{b}_k, \end{aligned} \tag{1.10}$$

and introducing the s and d bosons

$$\begin{aligned} T_{+,0}^{(0)} &= p_0 s^{\dagger} \quad T_{-,0}^{(0)} = p_0\tilde{s} \\ T_{+,\mu}^{(2)} &= p_2 d_{\mu}^{\dagger} \; T_{-,\mu}^{(2)} = \overline{p}_2\tilde{d}_{\mu}. \end{aligned} \tag{1.11}$$

The two nucleon transfer operator in the microscopic interacting boson model requires the computation of matrix elements between shell model states that belong to the SD subspace. In the generalized seniority Scheme introduced by Frank and Van Isacker [27] , we can obtain the matrix elements for given pair operators $(\tilde{c}_j\times\tilde{c}_j)^{(0)}$ and $(\tilde{c}_j\times\tilde{c}_{j'})^{(2)}$, introducing the creation and annihilation operators for single particle states $c^{\dagger}_{nlj}$ as,

[2] A short notation to the two nucleon two nucleon transfer $T_{\pm}^{(j_1,j_2,\lambda,M)}=T_{\pm}^{(\lambda)}$

$$|(n_1,l_1,j_1)(n_2,l_2,j_2);J,M\rangle = \frac{(c^{\dagger}_{n_1l_1j_1}\times c^{\dagger}_{n_2l_2j_2})^{(J)}_M}{\sqrt{1+(-1)^J\delta_{n_1,n_2}}}, \tag{1.12}$$

the second quantizer two nucleontransfer operator $T^{(\lambda)} \sim (c^{\dagger}_{j_1}\times c^{\dagger}_{j_2})^{(J)}$. In the microscopic IBM [28], the shell-model *SD* pair states are mapped onto *sd* boson states with $J^P = 0^+$ and $J^P = 2^+$.

$$\begin{aligned} S^{\dagger} &\to s^{\dagger} \\ D^{\dagger} &\to d^{\dagger} \end{aligned} \tag{1.13}$$

The spectroscopic amplitudes of the two nucleon transfer reactions are calculated by using the microscopic IBM2, which consider a mapping of the two body matrix elements for non conservative operator in the generalized seniority scheme[14]. They can be computed by the commutator method introduced by Frank and van Isacker [27] and by Lipas et al. [15] and Iachello and Barea[10]. Fermionic operators are similarly mapped into bosonic operators by Otsuka, Arima, and Iachello (OAI) method [23]. In this method, one is assured that the matrix elements between fermionic states in the collective subspace are identical to the matrix elements in the bosonic space. The OAI method, when it is carried out to all orders, produces results that are identical to the fermionic results. To investigate to what order our calculations are reliable, we have to consider the OAI expansion to next to leading order (NLO) .

$$\begin{aligned} (c^{\dagger}_j\times c^{\dagger}_j)^{(0)} &\to A_\rho(j)s^{\dagger}_\rho \\ (c^{\dagger}_j\times c^{\dagger}_{j'})^{(2)}_M &\to B_\rho(j,j')(d^{\dagger}_\rho)_M + C_\rho(j,j')s^{\dagger}_\rho(s^{\dagger}\tilde{d}_\rho)^{(2)}_M \\ &\quad + D_\rho(j,j')s^{\dagger}_\rho(d^{\dagger}_\rho\tilde{d}_\rho)^{(2)}_M \end{aligned} \tag{1.14}$$

we obtain the boson image of the fermion operator for the two nucleon transfer for $\lambda = 0$ and $\lambda = 2$

$$\begin{aligned} T^{(0)}_{+\rho} &\to \sqrt{\tfrac{1}{2}}A_\rho(j)s^{\dagger}_\rho \\ T^{(2)}_{+\rho} &\to \sqrt{\tfrac{1}{1+\delta_{j,j'}}}[B_\rho(j,j')d^{\dagger}_\rho \\ &\quad + C_\rho(j,j')s^{\dagger}_\rho(s^{\dagger}\tilde{d}_\rho)^{(2)} \\ &\quad + D_\rho(j,j')s^{\dagger}_\rho(d^{\dagger}_\rho\tilde{d}_\rho)^{(2)}] \end{aligned} \tag{1.15}$$

where the coefficients $A_\rho, B_\rho, D_\rho, C_\rho$ with $(\rho = \nu,\pi)$ are discussed in the Appendix C Matrix elements involving $T^{\lambda}_{-\rho}$ can be converted into $T^{\lambda}_{+\rho}$ as

$$\begin{aligned} &\langle (n-2)\nu J\|T^{(\lambda)}_{-\rho}\|n\nu' J'\rangle \\ &= (-)^{\lambda+J-J'}\langle n\nu' J'\|T^{(\lambda)}_{+\rho}\|(n-2)\nu J\rangle \end{aligned} \tag{1.16}$$

The limit case of a degenerate orbit theory in the quasi-spin formalism for a single j-shell may be obtained putting $\alpha_j = \beta_{j,j'} = 1$ in Eq. 1.15 and 1.7

$$\begin{aligned} -\sqrt{\tfrac{1}{2}}A_\rho(j)s^\dagger_\rho &= -\sqrt{\tfrac{1}{2}}\frac{\langle S^{\frac{n}{2}+1}\|(c^\dagger_j\times c^\dagger_j)^{(0)}\|S^{\frac{n}{2}}\rangle}{\langle s^{\frac{n}{2}+1}\|s^+\|s^{\frac{n}{2}}\rangle}s^\dagger_\rho \\ &= \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{2\Omega-n}{\Omega}}s^\dagger_\rho \\ &= \sqrt{\tfrac{\Omega-N}{\Omega}}s^\dagger_\rho \end{aligned} \tag{1.17}$$

The two nucleon transfer operators as the limit case of a single-j shell with $\lambda = 0$,which is in agreement with the zero order approximation to the two particle adding operator in the Ref [23], and for the case of $\lambda = 2$, the first order contribution we obtain

$$\begin{aligned} -\sqrt{\tfrac{1}{2}}B_\rho(j,j)d^\dagger_\rho &= -\sqrt{\tfrac{1}{2}}\frac{\langle DS^{\frac{n}{2}}\|(c^\dagger_j\times c^\dagger_j)^{(2)}\|S^{\frac{n}{2}}\rangle}{\langle ds^{\frac{n}{2}}\|d^+\|s^{\frac{n}{2}}\rangle}s^\dagger_\rho \\ &= \sqrt{\tfrac{(\Omega-N-1)(\Omega-N)}{\Omega(\Omega-1)}}d^\dagger_\rho \end{aligned} \tag{1.18}$$

which corresponds to the zero order approximation of the boson imagen of the to particle creation operator of Otsuka, Arima and Iachello in Ref [23]. Let $|[N+1],\alpha,J\rangle$ to be any J^+ collective state in the nucleus with 2N+2 particles (or holes), where N is the number of bosons and α represent the remaining quantum numbers to classify the state. The transfer intensity can be defined as [29].

$$I(N\to N+1) = |[\langle N+1],\alpha,J,\|T^{(\lambda)}_{+\rho}\|[N],\alpha',J'\rangle|^2 \tag{1.19}$$

the spectroscopic strength is defined as

$$S(N\to N+1) = \frac{1}{2J+1}I(N\to N+1) \tag{1.20}$$

where the $(2J+1)$ factor is by convention associated with the heavier mass . The spectroscopic factors are related to the expansion of the wave functions for a specific state Φ^n_i with n nucleons in terms of a summation over the complete set of states $\Psi^{n-1}_{f'}$ in the final nucleus with n-1 nucleons $\Phi^A_i = \sum_{f'lj}\theta_{if'lj}(\mathbf{r})\Psi^{n-1}_f$ and for a final nucleus with n-2 nucleons Ψ^{A-2}_f therefore:

$$\Phi^n_i = \sum_{f'lj}\sum_{fl'j'}\theta_{if'lj}(\mathbf{r}_1)\theta_{ffl'j'}(\mathbf{r}_2)\Psi^{n-2}_f \tag{1.21}$$

In the reaction fo the removal of particles from state Ψ_i to a specific state Ψ_f one requieres the overlap functions θ

$$\langle\Psi^{n-2}_f|\Psi^n_i\rangle = \sum_{l,j}\sum_{l',j'}\theta_{i,f',l,j}(\mathbf{r}_1)\theta_{f,f',l',j'}(\mathbf{r}_2) \tag{1.22}$$

where an explicit summation over all l,l' and j,j' values of the two single particle overlap function θ is made. Often the dependence on j is small in which case one measures the sum over possible j values for a given l

$$S_{ll'} = \sum_{j,j'}S_{l,j,l',j'} \tag{1.23}$$

The spectroscopic amplitude $\mathscr{A}$ is defined by the normalization of the overlap functions

$$\mathscr{A}(n \to n+2) = \int\int \theta_{l,j}(\mathbf{r}_1)\theta_{l',j'}(\mathbf{r}_2)d\tau_1 d\tau_2 \tag{1.24}$$

and the related spectroscopic factor is

$$S(N \to N+1) = |\mathscr{A}(n \to n+2)|^2 \tag{1.25}$$

the spectroscopic amplitudes can be written in terms of intensity as

$$\begin{aligned} |\mathscr{A}|^2 &= \tfrac{1}{2J+1} I(N \to N+1) \\ &= \tfrac{1}{2J+1} |[\langle N+1], \alpha, J, \| T^{(\lambda)}_{+\rho} \| [N], \alpha', J' \rangle|^2 \end{aligned} \tag{1.26}$$

and the spectroscopic amplitudes for the two nucleon transfer with $\lambda = 0$ between two $0+$ states and using Eq.(1.26) and Eq. (1.15) we obtain[3]

$$\begin{aligned} \mathscr{A}_0 &= \langle N+1], \alpha, 0, \| T^{(0)}_{+\rho} \| [N], \alpha', 0 \rangle \\ &= \sqrt{\tfrac{1}{2}} A_\rho(j) \langle N+1], \alpha, 0, \| s^\dagger_\rho \| [N], \alpha', 0 \rangle \end{aligned} \tag{1.27}$$

and the first contribution of two nucleon transfer with $\lambda = 2$ from Eq. (1.15) between $0+$ and $2+$ states

$$\begin{aligned} \mathscr{A}_2 &= \tfrac{1}{\sqrt{5}} \langle N+1], \alpha, 2, \| T^{(2)}_{+\rho} \| [N], \alpha', 0 \rangle \\ &= \sqrt{\tfrac{1}{5(1+\delta_{j,j'})}} B_\rho(j,j') \langle N+1], \alpha, 0, \| d^\dagger_\rho \| [N], \alpha', 0 \rangle \end{aligned} \tag{1.28}$$

1.4 Structure Constants

The mapping coefficients $A_\rho(j)$ and $B_\rho(j,j')$ depends on the structure coefficients $\alpha_{\rho j}$, $\beta_{\rho j,j'}$ and can be estimated by diagonalizing a surface delta paring interaction [30], which is given by

$$H_{jj'} = \varepsilon_{0j} + \varepsilon_{0j'} - V_{SDI}, \tag{1.29}$$

in the appropriate shell model space. In the above $\varepsilon_{0j}, \varepsilon_{0j'}$ are the single-particle energies of the levels j, j' and the V_{SDI} is the surface delta function pairing interaction. For S pair, we diagonalize the eq. 1.29 in the basis $|(j)^2 J = 0\rangle$ and use the lowest eigenvalue of the S pair state. We do the same for the D pair using the basis. $|(jj')^2 J = 2\rangle$ where j and j' can be any of the single particle levels in the major shell we are considering. Let us write the complete form of the matrix which we want to diagonalize with all the quantum numbers,

$$\langle (n_a l_a j_a)(n_b l_b j_b)|H|(n_c l_c j_c)(n_d l_d j_d)\rangle, \tag{1.30}$$

[3] The formulation of the spectroscopic amplitudes are being to publish [1].

and the V_{SDI} function may be written like

$$\begin{array}{l} V_{SDI} = \frac{1}{2}A_t(-1)^{n_a+n_b+n_c+n_d}\sqrt{\frac{(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)}{(2J+1)^2(\delta(j_a,j_b)+1)(\delta(j_c,j_d)+1)}} \\ \left(1-(-1)^{J+l_a+l_b+T}\right)(-1)^{j_b+j_d+l_b+l_d} \\ \left\langle j_b,-\frac{1}{2},j_a,\frac{1}{2},J,0\right\rangle\left\langle j_d,-\frac{1}{2},j_c,\frac{1}{2},J,0\right\rangle-\left((-1)^T+1\right) \\ \left\langle j_b,\frac{1}{2},j_a,\frac{1}{2},J,1\right\rangle\left\langle j_d,\frac{1}{2},j_c,\frac{1}{2},J,1\right\rangle, \end{array} \tag{1.31}$$

where T is the isospin, j_a, j_b final angular momentum for each single orbital , j_c, j_d initial angular momentum for each single orbital, J coupled angular momentum, n_a, n_b initial radial quantum numbers, n_c, n_d final radial quantum numbers, A_t strenght of the Surface delta interaction, l_a, l_b initial angular momentum, l_c, l_dfinal angular momentum.

1.5 Two nucleon transfer for heavy nuclei

We are going to describe the calculation of the two-nucleon transfer in the scheme of microscopic IBM-2.

The two-nucleon transfer of ^{116}Cd and ^{118}Sn, for typical midshell values of the single particle energy for protons as holes and the single particle energy for neutrons as holes in the 50-82 shell showed in the tables 1.2 and 1.3 the structure constants α_j and $\beta_{jj'}$ can be estimated and showed in the tables 1.4 , 1.5 1.6,1.7. We can notice that depending on the order of the couplings single nucleons we can introduce a relative sign in the values of the β parameter, considering the order of the states $|(J_1,J_2)^2J=2\rangle$ with $J_1 \geq, J_2$. This can be one reason why there are different signs of the structure parameters with the Barea and Iachello [49], and Pittel and Duval [24]. In this analysis, we present the cases where $J_1 \geq J_2$.

The low-lying spectra of the ^{114,116}Cd and ^{116,118}Sn nuclei is shown in Figures 1.1 and 1.2.

The order of the basis which we considered to compute the structure constants, starts from the lowest angular momentum j, for example, the first $|0^+\rangle$ in the basis to the $S-pairs$ taken here $|(j)^2J=|0^+\rangle$ may be written like,

$$\begin{array}{rl} |0^+\rangle &= b_1|2p1/2\rangle \\ &= b_2|2p3/2\rangle \\ &= b_3|1f5/2\rangle \\ &= b_4|1g9/2\rangle. \end{array} \tag{1.32}$$

The relation between amplitudes probabilities in the shell model and generalized seniority was made by Duval and Barrett in Ref. [25], considering

$$S_+ = \sum \alpha_i S_i, \tag{1.33}$$

where

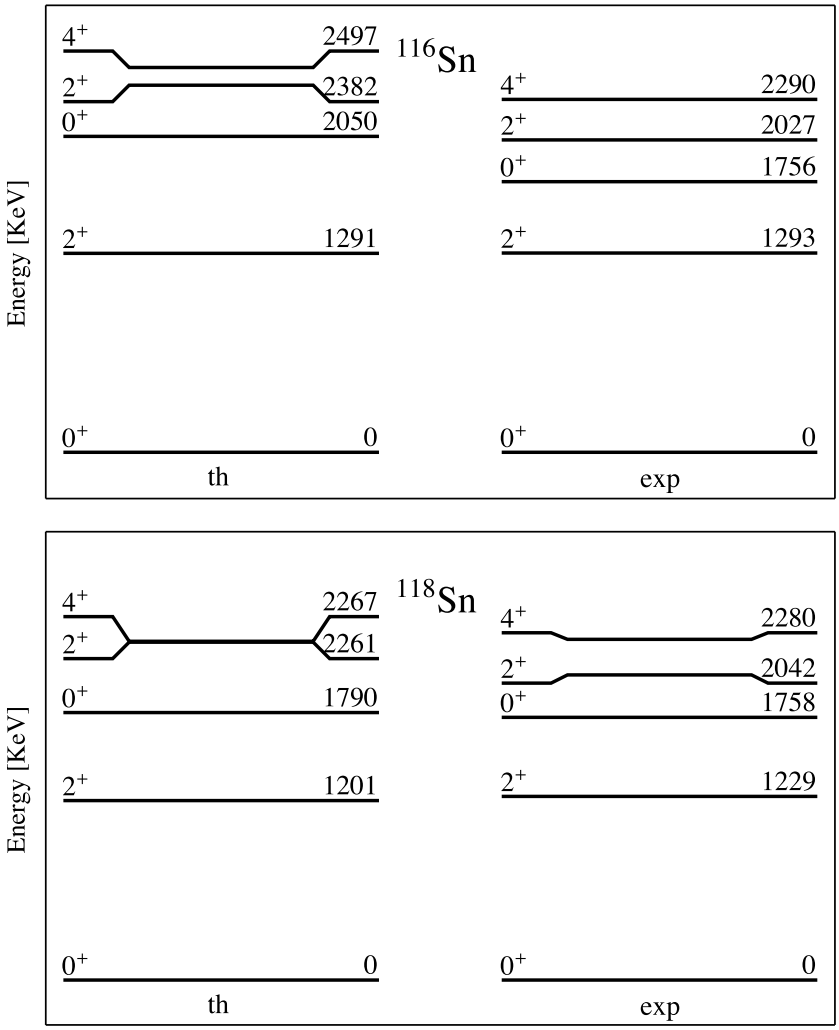


Fig. 1.1: Comparison between the experimental low-lying spectra for the pair of nuclei ^{116}Sn and ^{118}Sn [3] and the theoretical ones calculated by us.

$$S_i^+ = (-1)^{i-m} a_{im}^\dagger a_{i-m}^\dagger = \sqrt{\frac{\Omega}{2}} (a_i^\dagger a_i^\dagger)^{(0)}, \tag{1.34}$$

where the subindex i refers to the corresponding orbit j and

$$\alpha_i = \frac{a_i}{\sqrt{\frac{\Omega_i}{\Omega}}}, \tag{1.35}$$

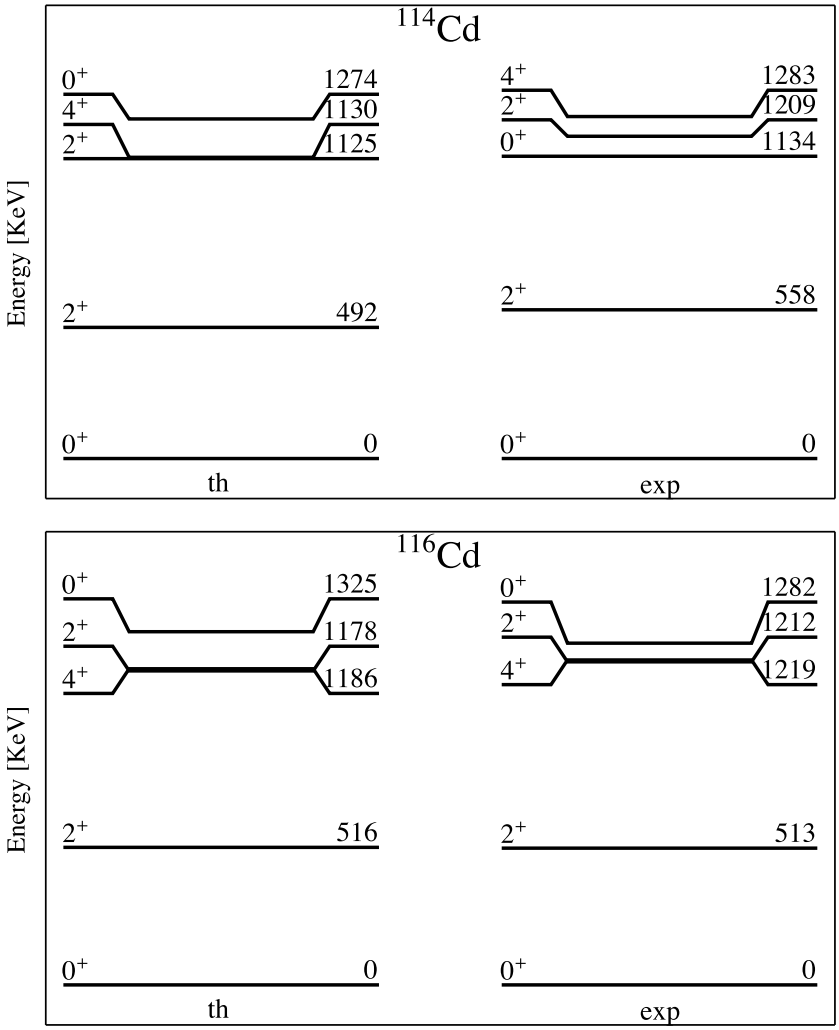


Fig. 1.2: Comparison between the experimental low-lying spectra for the pair of nuclei ^{114}Cd and ^{116}Cd.[3] and the theoretical ones calculated by us.

if we consider two-non degerated j-levels,

$$b_i^2 = \sum a_i^2, \tag{1.36}$$

where a_i are the amplitudes of probabilities obtained by the diagonalization of the Hamiltonian in the fermionic space and b_i is the subset of probabilities associated with one of two non - degenerated levels, the generalization of this and for the exact approach, the relation becomes [31]

$$b_i^2 = a_i^2, \tag{1.37}$$

therefore for each non degenerated j level it follows that

$$\alpha(j,a) = \sqrt{\frac{\Omega}{\Omega(j)}}a, \tag{1.38}$$

and from the pair degeneracy (the half of the level degeneracy)

$$\Omega(j) = j + 1/2, \tag{1.39}$$

where the $\Omega = \sum_i \Omega_i$ for example, in the shell 28-50 $\Omega = 11$, thus the alpha values are :

$$\alpha(j,a) = \sqrt{\frac{11}{\Omega(j)}}a \tag{1.40}$$

Table 1.2: Proton holes single particle energy for 28-50 shell calculated in this thesis.

n	l	j	ε_{0j}
2	1	$\frac{1}{2}$	0.931
2	1	$\frac{3}{2}$	2.198
1	3	$\frac{5}{2}$	2.684
1	4	$\frac{9}{2}$	0.000

Table 1.3: Neutron hole single particle energy for 50-82 shell calculated in this thesis.

n	l	j	ε_{0j}
3	0	$\frac{1}{2}$	0.332
2	2	$\frac{3}{2}$	0.000
2	2	$\frac{5}{2}$	1.655
1	4	$\frac{7}{2}$	2.434
1	5	$\frac{11}{2}$	0.070

The α_i are weighting factors for the single-particle levels. Thus, the total pair creation operator S_+ is designed to create a coherent linear combination of paired fermions frequently called “the correlated pair”. The structure of bosons is, in general not given by the pair creation operator S_+. The reason is due to the Pauli exclusion principle. To be sure, one always makes the correspondence:

$$\frac{S_+^N|0\rangle_F}{\eta_F} = \frac{s_+^n|0\rangle_B}{\eta_B}, \tag{1.41}$$

Table 1.4: Alpha parameter for the 28-50 shell calculated in this thesis (protons holes), with $A_1 = 0.264$.

$A_1 = 0.264$	
J	α_J
1/2	0.68934
3/2	0.40753
5/2	0.3522
9/2	-1.40145

Table 1.5: Beta parameter for the 28-50 shell calculated in this thesis (protons holes), with $A_1 = 0.264$.

$A_1 = 0.264$		
J_1	J_2	β_{J_1,J_2}
3/2	1/2	0.09185
3/2	3/2	-0.04780
5/2	1/2	0.23397
5/2	3/2	0.09889
5/2	5/2	-0.05205
9/2	9/2	0.987507

Table 1.6: Alpha parameter for the 50-82 shell calculated in this thesis (neutrons holes), with $A_1 = 0.163$.

$A_1 = 0.163$	
J	α_J
$\frac{1}{2}$	-0.999
$\frac{3}{2}$	-1.395
$\frac{5}{2}$	-0.469
$\frac{7}{2}$	-0.357
$\frac{11}{2}$	1.287

where N is the number of paired fermions and n the number of fermions, we are going to discuss more this correspondence in the following section. Now for the $J = 2$, we introduce the correlated pair creation operatorD_+ which by definition create the energetically lowest 2^+ pairs. These correlated pair creation operator can be expressed in terms of creation of fermions in specific orbits as

$$D_+ = \sum_{j \leq j'} \beta_{jj'} \sqrt{\frac{1}{1+\delta_{jj'}}} (a_j^+ a_{j'}^+)^2. \tag{1.42}$$

Table 1.7: Beta parameter for the 28-50 shell calculated in this thesis (neutrons holes), with $A_1 = 0.280$.

$A_1 = 0.280$		
J_1	J_2	β_{J_1,J_2}
3/2	1/2	-0.402
3/2	3/2	0.492
5/2	1/2	-0.159
5/2	3/2	0.098
5/2	5/2	0.078
7/2	3/2	0.176
7/2	5/2	-0.037
7/2	7/2	0.065
7/2	7/2	-0.721

The weighting factors β can be calculated into different ways. Let us consider the method used by Pittel and Duval in Ref [31], where the weighting factors β can be determined by diagonalizing an appropriate effective interaction between identical nucleons and looking at the $J = 2$ lowest state as

$$|\Psi_{\text{lowest}}\rangle = |2_1^+\rangle = \sum_{j\leq j'} \beta_{jj'} \Theta^2 (a_j^+ a_{j'}^+)^2, \tag{1.43}$$

where $\beta_{jj'} = b_{jj'}$, and with $\sum_{j\leq j'} \beta_{jj'}^2 = 1$.

1.6 Spectroscopic Amplitudes

The calculation of the spectroscopic amplitudes for the two nucleon transfer reactions in the scheme of IBM-2 using the OAI mapping requires two body matrix elements given by a non conservative operator number, in the scheme the Generalized Seniority Scheme. When the two particle transferred are pairs of nucleons with the same orbital momentum j, in this case one pair of nucleons is added to the initial state with angular momentum zero , the reduced matrix element may be written as,

$$\begin{aligned} &\langle S^{\frac{n}{2}+1} \| (c_j^\dagger \times c_j^\dagger)^0 \| S^{\frac{n}{2}} \rangle = -\langle n,0,0 \| (\tilde{c}_j \times \tilde{c}_j)^{(0)} \| n+2,0,0 \rangle \\ &= \frac{(\frac{n}{2}+1)(\frac{n}{2}!)^2}{\eta_{n,0,0}\eta_{n+2,0,0}} \sqrt{2J+1}\alpha_j \sum_{s=0}^{\frac{n}{2}} (-1)^s \left(\frac{\alpha_j^s \eta_{n-2s,0,0}}{(\frac{n}{2}-s)!} \right). \end{aligned} \tag{1.44}$$

The sign minus appears naturally when we consider the annihilation operators in tensorial form. The matrix elements in the generalized seniority scheme is equivalent to is the commutator method introduced by Frank and van Isacker [27] and by

Lipas et al. [15] and Iachello and Barea[10]. We have used the method by Iachello and Barea[10] method to evaluate the matrix elements of the pair operator $(\tilde{c}\times\tilde{c})^{(0)}$.

When a pair of nucleons of the initial state is removed then

$$\langle S^{\frac{n-2}{2}}\|(\tilde{c}_j\times\tilde{c}_j)^0\|S^{\frac{n}{2}}\rangle = \langle n-2,0,0\|(\tilde{c}_j\times\tilde{c}_j)^{(0)}\|n,0,0\rangle. \tag{1.45}$$

The character hole or particle character of the transferred particles depends on the type of nuclei that we want to describe. During the transfer process the pair of nucleons which are transferred can have different values of angular momentum. Moreover they can belong to different orbitals, so they can be coupled to angular momentum higher than zero. In this case the addition of two particles over an initial state with $J=0$ the two body matrix element is given by

$$\begin{aligned}
&\langle DS^{\frac{n}{2}}\|(c^\dagger_j\times c^\dagger_{j'})^{(2)}\|S^{\frac{n}{2}}\rangle\\
&=(-1)^{j+j'}\langle n,0,0\|(\tilde{c}_{j'}\times c_j)^{(2)}\|n+2,2,2\rangle\\
&=-\sqrt{5(1+\delta_{jj'}}\frac{\eta^2_{n+2,2,2}(jj')}{\eta_{n,0,0}\eta_{n+2,2,2}}\beta_{jj'},
\end{aligned}$$

and, when two particles are removed:

$$\langle S^{\frac{n-2}{2}}\|(\tilde{c}_j\times\tilde{c}_{j'})^{(2)}\|DS^{\frac{n-2}{2}}\rangle = \langle n-2,0,0\|(\tilde{c}_j\times c_{j'})^{(2)}\|n,2,2\rangle,$$

the nuclear wave functions depends on the number of nucleons of the shell, holes or particles[4].

The normalization coefficients are given by

$$\eta^2_{n,0,0}=\left(\frac{n}{2}!\right)^2\sum_{\substack{m_1,\ldots,m_k;\\ \sum_i m_i=\frac{n}{2}}}\left\{\prod_{i=1}^{k}\alpha_{j_i}^{2m_i}\binom{\Omega_{j_i}}{m_i}\right\}, \tag{1.46}$$

$$\begin{aligned}
&\eta^2_{n,2,2}=\sum_{j\le j'}\beta^2_{jj'}\eta^2_{n,2,2}(j,j'),\\
&n^2_{n,2,2}(j,j')=\sum_{p=0}^{\frac{n}{2}-1}\left[\frac{(\frac{n}{2}-1)!}{p!}\right](-1)^{\frac{n}{2}-1-p}\eta^2_{2p,0,0}\\
&\times\sum_{q=0}^{\frac{n}{2}-1-p}\alpha_j^{n-2-2p-2q}\alpha_{j'}^{2q}.
\end{aligned} \tag{1.47}$$

The selection of the shells is very important for the calculation of the boson image of the operator.

[4] We consider as a hole particles when the number of nucleons is smaller than the upper shell and bigger than the middle shell.

1.6.1 Bosonic image operator

Let us consider first the simplest case where the two transferred nucleons belongs to the same orbital and for simplicity we start with the seniority scheme approach and further we extend it to generalized seniority. In the SD subspace the two particle operator is given by

$$T_+^{(\lambda)} = \sqrt{\tfrac{1}{2}}\left(c_j^\dagger \times c_j^\dagger\right)^{(\lambda)}; S_+ = \sqrt{\Omega}T_+^{(0)}, \tag{1.48}$$

where $\Omega = j + 1/2$. Using the reduction formulas given by De Shalit[14] the two body reduced matrix element can be written as,

$$\begin{aligned}&\langle j^{n+2}, v, \alpha, L = 0 \| T_+^{(0)} \| j^n, v, \alpha', L\rangle \\ &= \delta v, v' \delta n_\Delta n_{\Delta'} \sqrt{\tfrac{1}{2}n - \tfrac{1}{2}v + 1}\sqrt{\tfrac{2\Omega - n - v}{2\Omega}} \\ &\langle j^2(S), L = 0 \| T_+^{(0)} \| 0\rangle.\end{aligned} \tag{1.49}$$

In the SD fermionic subspace, the states can be written as $|j^n(S^{\frac{1}{2}(n-v}D^{\frac{1}{2}v)})v_d n_\Delta L, M\rangle$ while in the SD bosonic subspace the bosonic states can be written as

$$|s^{n_s} d^{n_d} v_d n_\Delta L, M\rangle = \frac{1}{\sqrt{n_s!}} S^{+n_s} |d^{n_d} v_d n_\Delta L M\rangle, \tag{1.50}$$

where $n_d = \frac{1}{2}v, N = \frac{1}{2}n$ and $n_s = \frac{1}{2}(n - v)$. We introduce the operator in the SD bosonic space

$$T_+^{(0)} = T_s^+ + s^\dagger T_d^+, \tag{1.51}$$

with

$$\begin{aligned}T_s^+ &= p_0 s^\dagger + p_0' s^\dagger s^\dagger s + \ldots \\ T_d^+ &= q_0 [d^\dagger \tilde{d}]^{(0)} + \sum_L q_0^L [[d^\dagger d^\dagger]^{(0)} [\tilde{d}\tilde{d}]^{(0)}]^{(0)} + \ldots,\end{aligned} \tag{1.52}$$

from eq. (1.49) using eqs. (1.51)and (1.52) the bosonic operator becomes[5]

$$T_+^{(0)} = p_0 s^\dagger = \sqrt{\tfrac{\Omega - N - n_d}{\Omega}} \langle j^2(S) L = 0 \| T_+^{(0)} \| 0\rangle s^\dagger \tag{1.53}$$

The mapping coefficients $A(j)$ for the case of a single-j shell in which $\alpha_j = 1$ is given by

$$\langle s^{\frac{n}{2}} \| s^\dagger \| s^{\frac{n}{2}}\rangle A(j) = \langle S^{\frac{n}{2}+1} \| (c_j^\dagger \times c_j^\dagger)^0 \| S^{\frac{n}{2}}\rangle, \tag{1.54}$$

[5] The derivation is shown in appendix E.2

$$\begin{aligned} A(j) &= \frac{\langle S^{\frac{n}{2}+1}\|(c_j^\dagger\times c_j^\dagger)^0\|S^{\frac{n}{2}}\rangle}{\langle s^{\frac{n}{2}}\|s^\dagger\|s^{\frac{n}{2}}\rangle} \\ &= -\frac{\langle n,0,0\|(\tilde{c}_j\times\tilde{c}_j)^0\|n+2,0,0\rangle}{\sqrt{\frac{n}{2}+1}} \\ &= \frac{\sqrt{\frac{(n+2)(2\Omega_j-n)}{2\Omega_j}}}{\sqrt{\frac{n+2}{2}}} \\ &= \sqrt{\frac{2\Omega_j-n}{\Omega_j}}. \end{aligned} \tag{1.55}$$

in which $\beta_{ij}=1$ by,

$$\langle ds^{\frac{n}{2}}\|d^\dagger\|s^{\frac{n}{2}}\rangle B(j,j') = \langle DS^{\frac{n}{2}}\|(c_j^\dagger\times c_j^\dagger)^{(2)}\|S^{\frac{n}{2}}\rangle \tag{1.56}$$

$$\begin{aligned} B(j,j') &= \frac{\langle DS^{\frac{n}{2}}\|(c_j^\dagger\times c_j^\dagger)^2\|S^{\frac{n}{2}}\rangle}{\langle ds^{\frac{n}{2}}\|d^\dagger\|s^{\frac{n}{2}}\rangle} \\ &= (-1)^{j+j'}\frac{\langle n,2,2\|(\tilde{c}_j\times\tilde{c}_j)^2\|n+2,2,2\rangle}{\sqrt{5}} \\ &= (-1)^{j+j'+1}\sqrt{\frac{(2\Omega-n-2)(2\Omega-n)}{2\Omega(\Omega-1)}}. \end{aligned} \tag{1.57}$$

Now we can proceed in the same way with non degenerate orbits following the previous procedure, but using the reduced matrix elements in the generalized seniority. Therefore we get the mapping coefficients $A(j)$ and $B(j,j')$,

$$A(j) = \begin{cases} \frac{\langle S^{\frac{n}{2}+1}\|(c_j^\dagger\times c_j^\dagger)^{(0)}\|S^{\frac{n}{2}}\rangle}{\langle s^{\frac{n}{2}}\|s^\dagger\|s^{\frac{n}{2}}\rangle} = -\frac{\langle n,0,0\|(\tilde{c}_j\times\tilde{c}_j)^{(0)}\|n+2,0,0\rangle}{\sqrt{\frac{n}{2}+1}} & \text{if } \frac{n}{2}\geq 0, \\ \frac{\langle S^{\frac{n}{2}-1}\|(\tilde{c}_j\times\tilde{c}_j)^{(0)}\|S^{\frac{n}{2}}\rangle}{\langle s^{\frac{n}{2}-1}\|s^\dagger\|s^{\frac{n}{2}}\rangle} = \frac{\langle n-2,0,0\|(\tilde{c}_j\times\tilde{c}_j)^{(0)}\|n,0,0\rangle}{\sqrt{\frac{n}{2}+1}} & \text{if } \frac{n}{2}\geq 1, \end{cases}$$

$$B(j,j') = \begin{cases} \frac{\langle DS^{\frac{n}{2}}\|(c_j^\dagger\times c_{j'}^\dagger)^{(2)}\|S^{\frac{n}{2}}\rangle}{\langle ds^{\frac{n}{2}}\|d^\dagger\|s^{\frac{n}{2}}\rangle} = (-1)^{j+j'}\frac{\langle n,2,2\|(\tilde{c}_{j'}\times\tilde{c}_j)^{(2)}\|n+2,2,2\rangle}{\sqrt{5}} & \text{if } \frac{n}{2}\geq 0, \\ \frac{\langle S^{\frac{n}{2}-1}\|(\tilde{c}_j\times\tilde{c}_{j'})^{(2)}\|DS^{\frac{n}{2}-1}\rangle}{\langle s^{\frac{n}{2}-1}\|\tilde{d}\|s^{\frac{n}{2}}\rangle} = \frac{\langle n-2,2,2\|(\tilde{c}_{j'}\times\tilde{c}_j)^{(2)}\|n,2,2\rangle}{\sqrt{5}} & \text{if } \frac{n}{2}\geq 1, \end{cases}$$

where n is the number of active nucleons.

1.7 Two neutron and two proton transfer

We are going to discuss the calculation of the two proton and two neutron transfer in the scheme of microscopic IBM-2. As an example of calculation, we choose the heavy nuclei ^{114,118}Cd ^{116,118}Sn. The situation is depicted in Fig. 1.3.

The two proton transfer reaction between ^{116}Cd and ^{118}Sn may be computed considering the initial nucleus, ^{116}Cd, has 48 protons, and the final nucleus, ^{118}Sn, has 50 protons so there are two active protons which can be treated as holes. In the scheme of generalized seniority, two possibilities have to be taken into account, the transition between 0+ to 0+ and transition 0+ to 2+ states. For first one the reduced

Fig. 1.3: Two proton transfer between ^{116}Cd to ^{118}Sn and ^{114}Cd to ^{116}Sn. and two neutron transfer ^{118}Sn to ^{116}Sn and ^{116}Cd to ^{114}Cd.

$$\begin{array}{ccc} & b^{\dagger}_{\overline{\nu}} & \\ {}^{116}_{50}\mathrm{Sn}_{66} & \leftarrow & {}^{118}_{50}\mathrm{Sn}_{68} \\ b_{\overline{\pi}} \quad \uparrow & & \uparrow \quad \widetilde{b}_{\overline{\pi}} \\ {}^{114}_{48}\mathrm{Cd}_{66} & \leftarrow & {}^{116}_{48}\mathrm{Cd}_{68} \\ & b^{\dagger}_{\overline{\nu}} & \end{array}$$

matrix element is given by

$$\begin{aligned} &\langle {}^{118}\mathrm{Sn}; J=0 \| c^{\dagger}_{\pi} \times c^{\dagger}_{\pi} \| {}^{116}\mathrm{Cd}; J=0 \rangle \\ &= \langle 0,0,0 \| \tilde{c}_{\overline{\pi}} \times \tilde{c}_{\overline{\pi}} \| 2,0,0 \rangle, \end{aligned} \tag{1.58}$$

which, in the Interacting Boson Model scheme, becomes

$$A_{\pi}(j) \langle {}^{118}\mathrm{Sn}_{gs}; (0+) \| s^{\dagger}_{\pi} \| {}^{116}\mathrm{Cd}_{gs}; (0+) \rangle, \tag{1.59}$$

where the operator of Eq (1.15) was used. For this particular reaction, we have chosen the shell 28-50 with the active orbits $j = f7/2$, $1f5/2$, $2p3/2$,$2p1/2$ and $1g9/2$. For the transition 0+ to 2+ the reduced matrix element is given by

$$\begin{aligned} &\langle {}^{118}\mathrm{Sn}; J=2 \| \tilde{c}_{\pi} \times \tilde{c}_{\pi} \| {}^{116}\mathrm{Cd}; J=0 \rangle \\ &= \langle 0,0,0 \| \tilde{c}_{\overline{\pi}} \times \tilde{c}_{\overline{\pi}} \| 2,2,2 \rangle. \end{aligned}$$

The pair of proton transfer can carry different momentum for each proton, the mapping coefficient $B(j,j')$ depends on two momenta j and j', where $j \leq j'$. Thus, the spectroscopic amplitudes can be computed by

$$B_{\pi}(j,j') \langle {}^{118}\mathrm{Sn}; (2+) \| d^{\dagger}_{\pi} \| {}^{116}\mathrm{Cd}_{gs}; (0+) \rangle. \tag{1.60}$$

The two neutron transfer between ^{118}Sn and ^{116}Sn is analogous to the previous case. In this case, the nearest closed shell is $2d3/2$, and the addition of two neutron holes gives the two neutron transfer. The reduced matrix element for the transition between 0+ to 0+ is given by

$$\begin{aligned} &\langle {}^{116}\mathrm{Sn}; J=0 \| \tilde{c}_{\nu} \times \tilde{c}_{\nu} \| {}^{118}\mathrm{Sn}; J=0 \rangle \\ &= \langle 16,0,0 \| c^{\dagger}_{\overline{\nu}} \times c^{\dagger}_{\overline{\nu}} \| 14,0,0 \rangle \\ &= \langle 14,0,0 \| \tilde{c}_{\overline{\nu}} \times \tilde{c}_{\overline{\nu}} \| 16,0,0 \rangle, \end{aligned} \tag{1.61}$$

which, in the Interacting Boson Model scheme, becomes

$$A_{\nu}(j) \langle {}^{116}\mathrm{Sn}_{gs}; (0+) \| s^{\dagger}_{\nu} \| {}^{118}\mathrm{Sn}_{gs}; (0+) \rangle, \tag{1.62}$$

The transition 0+ to 2+ the reduced matrix element is given by

$$\begin{aligned}&\langle^{116}\mathrm{Sn};J=2\|\tilde{c}_{\nu}\times\tilde{c}_{\nu}\|^{118}\mathrm{Sn};J=0\rangle\\&=\langle 14,0,0\|\tilde{c}_{\overline{\nu}}\times\tilde{c}_{\overline{\nu}}\|16,2,2\rangle,\end{aligned}\tag{1.63}$$

which, in the Interacting Boson Model scheme, becomes

$$B_{\nu}(j,j')\langle^{116}\mathrm{Sn};(2+)\|d^{\dagger}_{\nu}\|^{118}\mathrm{Sn};(0+)\rangle,\tag{1.64}$$

and for the second path corresponding to the two transfer neutron between ^{116}Cd to ^{114}Cd and two transfer proton between ^{114}Cd to ^{116}Sn are calculations are similar like the previous one path.

The differential cross sections of the two-nucleon transfer reactions are an essential ingredient the study of the heavy ion- DCX because can be compared with the experimental DCX cross sections measured by INFN-LNS and examine if the two-nucleon transfer reactions are or not a competitive process which is subject of current research[1].

The spectroscopic amplitudes were computed using the operator in Eq (1.15). The spectroscopic amplitudes computed corresponds to the Eq. (1.27) and Eq. (1.28). The results of the numerical calculations of the spectroscopic amplitudes are reported in the Tables 1.9- 1.14.

Preliminary results, inserting the spectroscopic amplitudes into a FRESCO code indicate that the transfer cross section is two or three orders of magnitude smaller than the single transfer. This significant result, which will be the subject of a future article by the NUMEN collaboration, will help to identify the more competitive processes in the charge exchange reactions. It was seen[2] that the two-nucleon transfer is a dominant process over the sequential transfers between ground to ground between ^{64}Ni and ^{66}Ni. In the following section, we will discuss further this result.

Table 1.8: Spectroscopic amplitudes as calculated by us in the in the microscopic interacting boson model 2 (IBM-2) for two proton transfer, for ^{116}Cd to ^{118}Sn, j_1, j_2 are the spins of the proton orbitals , J is the angular momentum of the two-proton system.

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
$116Cd_{gs}$ (0+)	1f5/2, 1f5/2	0	$118Sn_{gs}$(0+)	0.1562
	2p3/2 ,2p3/2			0.1476
	2p1/2, 2p1/2			0.1765
	1g9/2, 1g9/2			-0.8024
$116Cd_{gs}$ (0+)	1f5/2, 1f5/2	2	$118Sn_{1.23}$(2+)	-0.0117
	1f5/2 ,2p3/2			-0.0090
	2p3/2, 2p1/2			0.0206
	1f5/2, 2p1/2			0.0222
	2p3/2, 2p3/2			-0.0107
	1g9/2, 1g9/2			0.2212
$116Cd_{gs}$(0+)	1f5/2, 1f5/2	0	$118Sn_{1.758}$(0+)	-0.0164
	2p3/2, 2p3/2			-0.0155
	2p1/2, 2p1/2			-0.0186
	1g9/2, 1g9/2			0.0844
$116Cd_{gs}$(0+)	1f5/2, 1f5/2	2	$118Sn_{2.043}$(2+)	0.0027
	1f5/2, 2p3/2			0.0021
	1f5/2, 2p1/2			-0.0051
	2p3/2, 2p3/2			0.0025
	2p3/2, 2p1/2			0.0844
	1g9/2, 1g9/2			-0.0511
$116Cd_{gs}$(0+)	1f5/2, 1f5/2	0	$118Sn_{2.057}$(0+)	-0.0119
	2p3/2, 2p3/2			-0.0113
	2p1/2, 2p1/2			-0.0135
	1g9/2, 1g9/2			0.0614

Table 1.9: Two proton amplitudes as calculated by us for the ^{114}Cd to ^{116}Sn in the microscopic interacting boson model 2 (IBM-2).

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
$114Cd_{gs}(0+)$	1f5/2, 1f5/2	0	$116Sn_{gs}(0+)$	-0.1504
	2p3/2, 2p3/2			-0.1421
	2p1/2, 2p1/2			-0.1700
	1g9/2, 1g9/2			0.7727
$114Cd_{gs}$ (0+)	1f5/2, 1f5/2	2	$116Sn_{1.29}(2+)$	0.0122
	1f5/2, 2p3/2			0.0094
	1f5/2, 2p1/2			-0.0231
	2p3/2, 2p3/2			0.0112
	2p3/2, 2p1/2			-0.0214
	1g9/2, 1g9/2			-0.2306
$114Cd_{0.558}$ (2+)	1f5/2, 1f5/2	2	$116Sn_{gs}(0+)$	-0.0188
	1f5/2, 2p3/2			-0.0145
	1f5/2, 2p1/2			0.0357
	2p3/2, 2p3/2			-0.0173
	2p3/2, 2p1/2			0.0332
	1g9/2, 1g9/2			0.3567
$114Cd_{1.135}(0+)$	1f5/2, 1f5/2	0	$116Sn_{gs}(0+)$	0.0606
	2p3/2,2p3/2			0.0572
	2p1/2, 2p1/2			0.0684
	1g9/2, 1g9/2			-0.3111
$114Cd_{1.210}(2+)$	1f5/2, 1f5/2	2	$116Sn_{gs}(0+)$	0.0045
	1f5/2, 2p3/2			0.0035
	1f5/2, 2p1/2			-0.0086
	2p3/2, 2p3/2			0.0041
	2p3/2, 2p1/2			-0.0080
	1g9/2,1g9/2			-0.0857
$114Cd_{1.306}(0+)$	1f5/2, 1f5/2	0	$116Sn_{gs}(0+)$	0.0227
	2p3/2, 2p3/2			0.0215
	2p1/2, 2p1/2			0.0257
	1g9/2, 1g9/2			-0.1167

Table 1.10: Spectroscopic amplitudes as calculated by us in the in the microscopic interacting boson model 2 (IBM-2) for two neutron transfer, for ^{118}Sn to ^{116}Sn j_1, j_2 are the spins of the neutron orbitals J is the angular momentum of the two-neutron system.

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
$118Sn_{gs}(0+)$	1h11/2 1h11/2	0	$116Sn_{gs}(0+)$	-1.4483
	1g7/2, 1g7/2			-0.7317
	2d5/2, 2d5/2			-0.7648
	2d3/2, 2d3/2			-0.6662
	3s1/2, 3s1/2			-0.5320
$118Sn_{1.23}(2+)$	1h11/2, 1h11/2	2	$116Sn_{gs}(0+)$	0.0181
	1g7/2, 1h11/2			0.000
	1g7/2, 1g7/2			-0.0141
	1g7/2, 2d5/2			0.0074
	1g7/2, 2d3/2			0.0123
	2d5/2, 2d5/2			-0.0142
	2d5/2, 2d3/2			-0.0062
	2d5/2, 3s1/2			0.0160
	2d3/2, 2d3/2			-0.0095
	2d3/2, 3s1/2			0.0128
$118Sn_{1.758}(0+)$	1h11/2, 1h11/2	0	$116Sn_{gs}(0+)$	-0.0199
	1g7/2, 1g7/2			-0.0857
	2d5/2, 2d5/2			-0.0896
	2d3/2, 2d3/2			-0.0780
	3s1/2, 3s1/2			-0.0623
$118Sn_{2.043}(2+)$	1h11/2, 1h11/2	2	$116Sn_{gs}(0+)$	-0.0044
	1g7/2, 1h11/2			0.00
	1g7/2, 1g7/2			0.0035
	1g7/2, 2d5/2			-0.0018
	1g7/2, 2d3/2			-0.0030
	2d5/2, 2d5/2			0.0035
	2d5/2, 2d3/2			0.0015
	2d5/2, 3s1/2			-0.0039
	2d3/2, 2d3/2			0.0023
	2d3/2, 3s1/2			-0.0032
$118Sn_{2.057}(0+)$	1h11/2, 1h11/2	0	$116Sn_{gs}(0+)$	-0.0019
	1g7/2, 1g7/2			-0.0262
	2d5/2, 2d5/2			-0.0274
	2d3/2, 2d3/2			-0.0239
	3s1/2, 3s1/2			-0.0191
$118Sn_{gs}(0+)$	1h11/2, 1h11/2	2	$116Sn_{1.29}(2+)$	0.4385
	1g7/2, 1h11/2			0.000
	1g7/2, 1g7/2			-0.3431
	1g7/2, 2d5/2			0.1805
	1g7/2, 2d3/2			0.2980
	2d5/2, 2d5/2			-0.3454
	2d5/2, 2d3/2			-0.1500
	2d5/2, 3s1/2			0.3877
	2d3/2, 2d3/2			-0.2314
	2d3/2, 3s1/2			0.3119

Table 1.11: Two neutron amplitudes as calculated by us for the ^{116}Cd to ^{114}Cd in the microscopic interacting boson model 2 (IBM-2).

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
$116Cd_{gs}(0+)$	1h11/2,, 1h11/2	0	$114Cd_{gs}(0+)$	-1.1672
	1g7/2, 1g7/2			0.7117
	2d5/2, 2d5/2			0.7438
	2d3/2, 2d3/2			0.6479
	3s1/2,3s1/2			0.5174
$116Cd_{gs}(0+)$	1h11/2, 1h11/2	2	$114Cd_{0.558}(2+)$	0.1895
	1g7/2, 1h11/2			0.000
	1g7/2, 1g7/2			-0.1482
	1g7/2, 2d5/2			0.0780
	1g7/2, 2d3/2			0.1288
	2d5/2, 2d5/2			-0.1493
	2d5/2, 2d3/2			-0.0648
	2d5/2, 3s1/2			0.1675
	2d3/2, 2d3/2			-0.1000
	2d3/2, 3s1/2			0.1348
$116Cd_{gs}(0+)$	1h11/2, 1h11/2	0	$114Cd_{1.134}(0+)$	0.1117
	1g7/2, 1g7/2			-0.0681
	2d5/2, 2d5/2			-0.0712
	2d3/2, 2d3/2			-0.0620
	3s1/2, 3s1/2			-0.0495
$116Cd_{gs}(0+)$	1h11/2, 1h11/2	2	$114Cd_{1.210}(2+)$	-0.0275
	1g7/2, 1h11/2			0.00
	1g7/2, 1g7/2			0.0215
	1g7/2, 2d5/2			-0.0113
	1g7/2, 2d3/2			-0.0187
	2d5/2, 2d5/2			0.0216
	2d5/2, 2d3/2			0.0094
	2d5/2, 3s1/2			-0.0243
	2d3/2, 2d3/2			0.0145
	2d3/2, 3s1/2			-0.0196
$116Cd_{gs}(0+)$	1h11/2, 1h11/2	0	$114Cd_{1.305}(0+)$	0.0305
	1g7/2, 1g7/2			-0.0186
	2d5/2, 2d5/2			-0.0194
	2d3/2, 2d3/2			-0.0169
	3s1/2, 3s1/2			-0.0135

1.8 Two nucleon transfer ^{64}Ni to ^{66}Ni

We have performed exact finite range cross section calculations using the coupled channel Born approximation (CCBA) and coupled reaction channel (CRC) method for the sequential and direct two-neutron transfers, respectively. The microscopic interacting boson model (IBM-2) and interacting boson-fermion model (IBFM) has been applied to two-neutron transfer reactions. From our results we conclude that for two-neutron transfer to the ground state of ^{66}Ni the direct transfer is the dominant reaction mechanism, whereas for the transfer of the first excited state of ^{66}Ni the sequential process dominates. A competition between long-range and short-range correlations is discussed.

High-quality angular distributions for the ^{64}Ni(^{18}O,^{16}O)^{66}Ni transfer reactions at 84 MeV incident energy were measured at the INFN- Laboratori Nazionali del Sud (Italy). Two-nucleon transfer can be used as a test of pairing correlations in nuclei. We are interested in identifying which is the dominant process in the two neutron transfer, it occurs in one step (direct), under the strong influence of pairing correlations or in two steps (sequentially). If the ground state of the residual nucleus is feed from the ground state of the target. Transfer reactions are the best ways to explore the collectivity properties of nuclei. For the first time, we compute the spectroscopic amplitudes of two neutron transfer reactions between ^{64}Ni and ^{66}Ni by using the microscopic interacting boson model two (IBM-2) [2]. The microscopic IBM-2 is a way to calculate realistic matrix elements for medium and heavy nuclei. It has also been applied in neutrinoless double beta decay [10].To study the ^{64}Ni to ^{66} reaction, we need to compute the spectroscopic factors and cross sections for two neutron transfer. The nuclear wave functions of the even-even nuclei ^{66,64} Ni will be constructed by using the IBM-2 with the parameters given in table 1.12.

Table 1.12: Coefficients used in the microscopic IBM-2 Calculations [2]

nucleus	N_ν	N_π	ε_d (MeV)	$c_\nu^{(0)}$ (MeV)	$c_\nu^{(2)}$ (MeV)	$c_\nu^{(4)}$ (MeV)	v_{ln} (MeV)
^{64}Ni	4	0	1.2	0.62	-0.27	-0.274	0.0242
^{66}Ni	5	0	1.2	0.45	0.29	0.036	0.0374

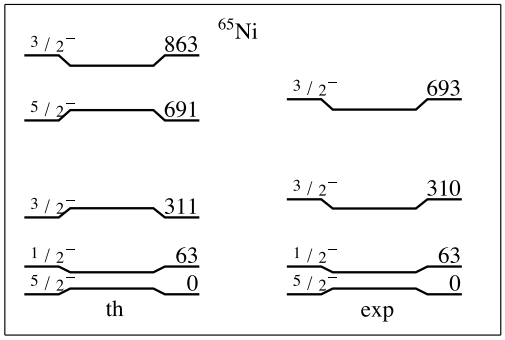
The spectra are shown in Fig 1.4, and we can see that the low-lying states of those nuclei are in good agreement with the experimental data. These nuclei are interesting since they are at the boundary region in which collectivity should start to play a role, but still they can be described with the single particle degrees of freedom (Shell Model).

For the ^{64}Ni and ^{66}Ni it is required to considered the basic features of the effective nucleon-nucleon interaction, that emerges from pairing, quadrupole, and symmetry energy (IBM2) as

$$H_B = E_0 + \varepsilon_\pi \hat{n}_{d_\pi} + \varepsilon_\nu \hat{n}_{d_\nu} + \kappa \hat{Q}^\chi_\pi \cdot \hat{Q}^\chi_\nu + \lambda' \hat{M}_{\pi\nu} + V_{\pi\pi} + V_{\nu\nu}.$$

A detailed description of the IBM is given in the Appendix C. We have calculated the theoretical spectrum of the ^{64,66}Ni nuclei coming from IBM2 which is in agreement with the experimental data.

We will use the eigenstates of the Hamiltonian in Eq. 1.8 to calculate the nuclear matrix elements of the two-nucleon transfer operator.



We have calculated the theoretical spectrum of the the ^{65}Ni nucleus using the interacting boson fermion model (IBFM) [33]

$$H_{IBFM} = H_B + H_{BF} + H_F \tag{1.65}$$

where H_B is given in Eq. 1.8 and H_F and H_{BF} are respectively :

$$\begin{aligned} H_F &= E_0 + \sum_{j_\pi} \varepsilon_{j_\pi} \hat{n}_{j_\pi} + \sum_{j_\nu} \varepsilon_{j_\nu} \hat{n}_{j_\nu} \\ V_{BF} &= \sum_{j_\pi} A_{j\pi} (\hat{n}_{d_\pi} \hat{n}_{j_\pi}) + \sum_{j_\nu} A_{j\nu} (\hat{n}_{d_\nu} \hat{n}_{j_\nu}) \\ &\quad + \Gamma_{\pi\nu} \hat{Q}^\chi_\nu \cdot \hat{q}_\pi + + \Gamma_{\nu\pi} \hat{Q}^\chi_\pi \cdot \hat{q}_\nu + \Gamma_{\nu\nu} \hat{Q}^\chi_\nu \cdot \hat{q}_\nu + \Gamma_{\pi\pi} \hat{Q}^\chi_\pi \cdot \hat{q}_\pi \\ &\quad + \Lambda_{\nu\pi} F_{\pi\nu} + \Lambda_{\pi\nu} F_{\nu\pi} \end{aligned}$$

The parameters used in the Eq. 1.8 can be found in the Ref. [2]

We propose the following coupling scheme: for the two-neutron direct transfer reaction we use microscopic IBM-2, while for sequential transfer we use IBFM.

In order to compute the spectroscopic amplitudes we need the structure constants of the 28-50 neutron shell (see Sec C.2).

The pair structure coeficients, α_j and $\beta_{j,j'}$ are reported in Table 1.13. The single particle energies are obtained by solving the Woods-Saxon (see the Appendix C.9) .

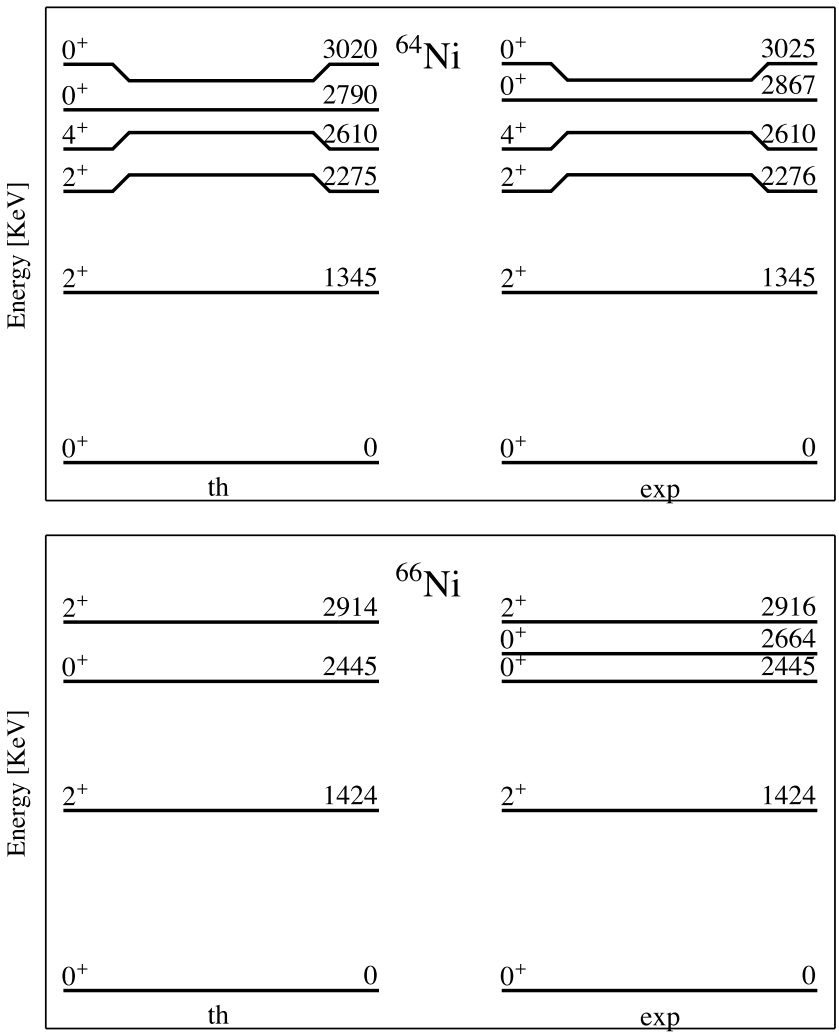


Fig. 1.4: Comparison between calculated and experimental low-lying spectra for the pair of nuclei ^{64}Ni and ^{66}Ni.

Fig. 1.5: Coupling scheme for direct transfer.

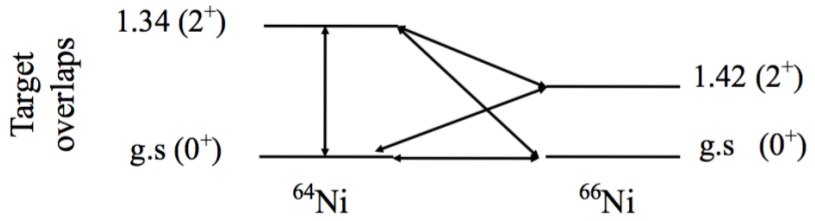
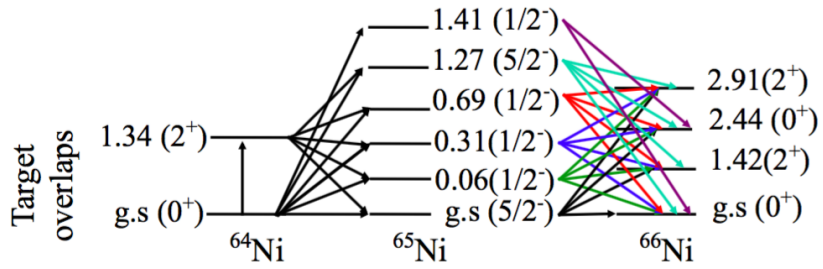


Fig. 1.6: Coupling scheme for sequential transfer.



Once calculated the of the spectroscopic amplitudes of the two neutron transfer with the operator Eq. 1.15, we insert into the FRESCO code (program for direct reactions) in oder to obtain the cross sections (see Fig. 1.7).

1.9 Discussion of results

For the transfer reaction to the ground state ^{66}Ni, both two reaction mechanisms are important. For the transfer to the ground state of ^{66}Ni, the pairing correlation seems to be relevant, especially at the bell shape maximum region. For the two-neutron transfer to the first excited state of the ^{66}Ni, the two-step processes are dominant. For the two-neutron transfer to the first excited state of ^{66}Ni, the results of the sequential mode for the angular distribution are in good agreement.

We have compared the differential cross sections with by using Shell model [2] as we see in the Fig 1.8 and gives a lower value of the angular distribution with respect to the IBM2 calculations and experimental results.

Table 1.13: Pair structure coefficients α_j and $\beta_{jj'}$ used in the current calculation.

	Neutrons
$\alpha_{1/2}$	0.461
$\alpha_{3/2}$	2.218
$\alpha_{5/2}$	0.408
$\alpha_{9/2}$	-0.299
$\beta_{1/2,3/2}$	-0.117
$\beta_{3/2,3/2}$	0.986
$\beta_{1/2,5/2}$	0.069
$\beta_{5/2,5/2}$	-0.049
$\beta_{9/2,9/2}$	0.045

Table 1.14: Spectroscopic amplitudes as calculated by us of the two neutron transfer reaction from ^{64}Ni to ^{66}Ni in the microscopic IBM2. The j_1 and j_2 are the spins of the neutron orbitals and J is the angular momentum of the two-neutron system.

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
^{64}Ni$_{gs}$(0+)	1f5/2, 1f5/2	0	^{66}Ni$_{gs}$(0+)	0.9417
	2p3/2, 2p3/2			0.3253
	2p1/2, 2p1/2			0.5605
	1g9/2, 1g9/2			-1.0463
^{64}Ni$_{gs}$ (0+)	1f5/2, 1f5/2	2	^{66}Ni$_{1.424}$(2+)	-0.2728
	1f5/2, 2p3/2			0.0256
	1f5/2, 2p1/2			0.3476
	2p3/2, 2p3/2			-0.0234
	2p3/2, 2p1/2			-0.0396
	1g9/2, 1g9/2			0.3653
^{64}Ni$_{gs}$(0+)	1f5/2, 1f5/2	0	^{66}Ni$_{2.445}$(0+)	-0.0005
	2p3/2, 2p3/2			-0.0002
	2p1/2, 2p1/2			-0.0003
	1g9/2, 1g9/2			0.0006
^{64}Ni$_{gs}$ (0+)	1f5/2, 1f5/2	2	^{66}Ni$_{2.916}$(2+)	0.0000
	1f5/2, 2p3/2			0.0000
	1f5/2, 2p1/2			0.0000
	2p3/2, 2p3/2			0.0000
	2p3/2, 2p1/2			0.0000
	1g9/2 ,1g9/2			0.0000

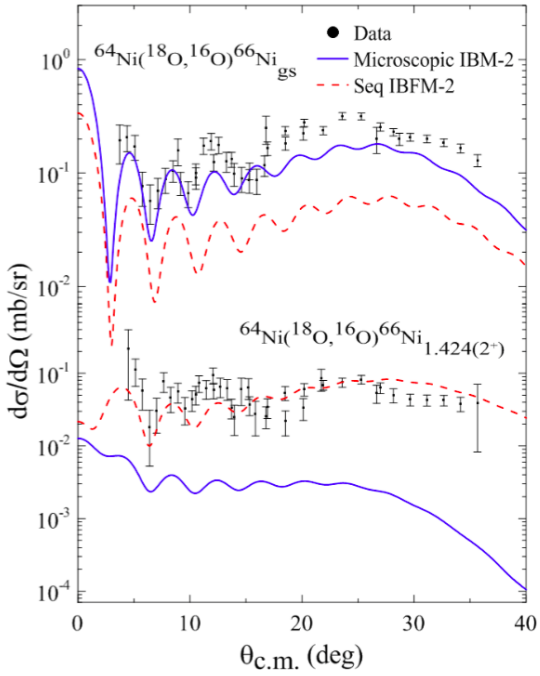
Table 1.15: Continuation

Initial state	j_1, j_2	J	Final state	Spectr. Amp.
$^{64}Ni_{1.34}$ (2+)	1f5/2, 1f5/2	2	$^{66}Ni_{gs}$(0+)	0.1359
	1f5/2, 2p3/2			-0.0961
	1f5/2, 2p1/2			0.2048
	2p3/2, 2p3/2			0.6084
	2p3/2, 2p1/2			0.1746
	1g9/2, 1g9/2			-0.0828
$^{64}Ni_{1.34}$ (2+)	1f5/2, 1f5/2	2	$^{66}Ni_{2.445}$(0+)	-0.0001
	1f5/2, 2p3/2			0.0001
	1f5/2, 2p1/2			-0.0001
	2p3/2, 2p3/2			-0.0004
	2p3/2, 2p1/2			-0.0001
	1g9/2 ,1g9/2			0.0001
$^{64}Ni_{1.34}$ (2+)	1f5/2, 1f5/2	2	$^{66}Ni_{1.42}$(2+)	-0.1112
	1f5/2, 2p3/2			0.0122
	1f5/2, 2p1/2			-0.2788
	2p3/2,2p3/2			-0.0123
	2p3/2, 2p1/2			-0.2065
	1g9/2, 1g9/2			-0.0655
$^{64}Ni_{1.34}$ (2+)	1f5/2, 1f5/2	2	$^{66}Ni_{2.916}$(2+)	0.0000
	1f5/2, 2p3/2			0.0000
	1f5/2, 2p1/2			0.0000
	2p3/2, 2p3/2			0.0000
	2p3/2, 2p1/2			0.0000
	1g9/2, 1g9/2			0.0000

Concerning the first excited state of 66 Ni, the results of the sequential transfer calculations with shell-model amplitudes are closer to the experimental data with respect to the independent coordinate angular distribution, but not satisfactory. Instead, the sequential process obtained within the IBFM-2 describes well the experimental data, while the direct process angular distribution is more than one order of magnitude lower than the data. For the transition to the 0+ ground state the direct mechanism dominates, while for the 2+ state the sequential contribution is more important. This allows us to conclude that the pairing correlations among the two transferred neutrons are relevant mainly for the ground state. This state has a weak collectivity because the 66 Ni is an even-even spherical nucleus. Conversely, the 2+ state is a collective state, so that the long-range correlations between nucleons are dominant over the short-range pairing correlations of the two neutrons.

For the two-neutron transfer reactions to the ground state of the residual nucleus 66Ni, we have evidence of the short-range pairing correlations, using both nuclear structure models. This conclusion is more evident when we use the microscopic

Fig. 1.7: 64Ni(18O,16O)66Ni reaction: the experimental angular distribution is compared with the theoretical calculation of this thesis, obtained by using microscopic IBM-2 and IBFM, respectively, for the direct and sequential process.



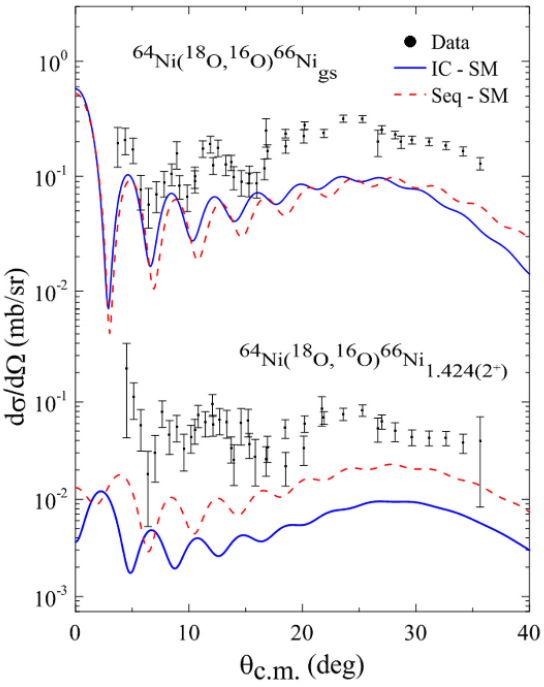
IBM-2 and the IBFM-2. In the two-neutron transfer to the first excited state of 66 Ni, where the collectivity is known to be important, it was verified the predominance of the two-step reaction mechanism. This confirmed the long-range predominance in the wave function of this state. This conclusion was again independent on the nuclear structure model used for the calcuations of the spectroscopic amplitudes of the target overlaps.

These results allows us to conclude that the pairing correlation effects is present in the two transferred neutrons to the ground state.

However for the 2_1^+ state of ^{66}Ni, the long range correlation between nucleons are dominant over short range paring correlations of two neutrons.

In previous experiments the forward angle oscillations were not observed because they were not able to measure the forward angles [34]. It is interesting to observe that for the same nucleus different states prefer different transfer mechanisms.

Fig. 1.8: Comparison of the experimental angular distribution with Shell model for the 64Ni(18O,16O)66Ni reaction.



Chapter 2
Nuclear matrix elements of neutrinoless double beta decay

A neutrino that is its own antiparticle is referred to as a Majorana neutrino, and one that is not is referred as a Dirac neutrino. The neutrino's nature (i.e. Dirac or Majorana) can be explored via the very rare nuclear process in which two neutrons in a nucleus simultaneosly undergo beta decay (double beta decay). For standard double beta decay, if the neutrino is Dirac-type, we would expect the release of two electrons and two anti-neutrinos. The observation of neutrinoless double-beta decay would therefore unambiguosly prove the neutrino to be a Majorana-type particle. Neutrinoless double-beta decay is an extremely rare nuclear transition, possible only for a few tens of isotopes [35]. The simplest mechanism enabling it to occur relates to the rate of the decay process to the square of the so-called 'effective Majorana mass', which is essentially a linear combination of the three neutrino masses. In nuclear physics, double beta decay is a type of radioactive decay in which two protons are simultaneously transformed into two neutrons, or vice versa, inside an atomic nucleus. As in single beta decay, this process allows the atom to move closer to the optimal ratio of protons and neutrons. As a result of this transformation, the nucleus emits two detectable beta particles, which are electrons or positrons. There are two types of double beta decay: ordinary double beta decay and neutrinoless double beta decay. In ordinary double beta decay, which has been observed in several isotopes, two electrons and two electron antineutrinos are emitted from the decaying nucleus. In neutrinoless double beta decay, a hypothesized process that has never been observed (except for one controversial claim [36]), only electrons would be emitted. Experimental half-life lower limits have been obtained for several isotopes: 76Ge, 82Se, 100Mo, 130Te, and 136Xe. Contemporary efforts are focused on so-called second generation experiments (CUORE [38], SuperNEMO [39], nEXO [40], NEXT [41], LUCIFER [42], GERDA II [43], SNO+ [44]) with the goal of approaching the IH region at$|m_{\nu\nu}| \leq 50$meV.

Double beta decay with the emission of two neutrinos is a second order nuclear weak process and it corresponds to the transition from a nucleus (A, Z) to its isobar (A, Z + 2) with the emission of two electrons. The transition may occur via a Standard Model allowed process in which two electron antineutrinos are emitted along with the electrons:

$$(A,Z) \rightarrow (A,Z+2) + 2e^- + 2\nu_e.$$

In principle, a nucleus (A, Z) can decay by double beta decay as long as the nucleus (A, Z + 2) is lighter. However, if the nucleus can also decay by single beta decay, (A, Z + 1), the branching ratio for the double beta decay will be too difficult to be observed due to the overwhelming background rate from the single beta decay. Therefore, candidate isotopes for detecting double beta decay are even-even nuclei that, due to the nuclear pairing force, are lighter than the odd-odd (A, Z + 1) nucleus, making single beta decay kinematically forbidden.

If neutrinos are Majorana fermions, the process of decay is the following:

$$(A,Z) \rightarrow (A,Z+2) + 2e^-,$$

with no (anti)neutrinos in the final state. This decay mode is known as neutrinoless double beta decay $0\nu\beta\beta$. The first calculations of the rate for $0\nu\beta\beta$ was performed by Furry [37]. The observation of neutrinoless double beta-decay would unambiguosly prove the neutrino to be Majorana-type particle. This would have a tremendous impact on our vision of nature, involving discovery of new type of matter (Majorana), and the so-called lepton number would no longer be a symmetry of nature.

2.1 Nuclear matrix elements of Neutrinoless double beta decay

The Nuclear matrix elements of Neutrinoless double beta decay can be determined from the ratio of the zero- and two-neutrino decay widths. However, to extract evidence on the rates of the two possible decay processes from nuclear measurements, the nuclear many-body matrix element involved has to be known accurately. Extensive calculations of the nuclear matrix elements have been done in different models ([45], [46],[49]) as is shown in Fig. 2.1, among them the Interacting Boson Model, the Quasiparticle Random Phase Approximation (QRPA), and the Shell Model (SM).

For medium-heavy nuclei, there is a large difference between the nuclear matrix elements calculated with different models. Part of this discrepancy could arise from the fact that in the shell-model calculations, for technical reasons, it is assumed that the nuclei are very close to a spherical shape, but this is not a good approximation. The effect of this deformation has been studied using the Nilsson model. This calculation indicates that for well-deformed nuclei the double beta decay rate vanishes. This suggests that the influence of deformation has a strong impact on the double beta decay rate. To investigate the effects of nuclear deformation, we have therefore repeated the calculation in the framework of the interacting boson approximation (IBM) [47][48] . In this framework, the problem is divided into two parts: one part is the calculation of the coefficients that appear in the boson image of the operator. This is solely related to the microscopic structure of the two-fermion states, which are the equivalent of the bosons. The second part is the calculation of the nuclear

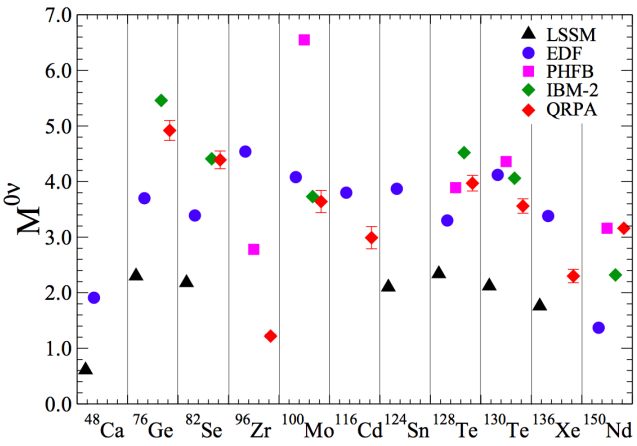


Fig. 2.1: Matrix elements of the neutrinoless double beta decay for the different approaches [46].

matrix elements (NMEs). In this thesis, the calculation of the nuclear matrix elements is done in the closure approximation.

2.2 Neutrinoless double beta decay operator

The half life time of $0\nu\beta\beta$ decay is given by the following formula, which holds for light neutrinos [50, 35] :

$$[T^{0\nu}_{1/2}]^{-1} = G^{0\nu}(Q_{\beta\beta},Z)|M^{0\nu}|^2|f(m_i,U_{ei})|^2$$

The first term $G_{0\nu}(Q_{\beta\beta},Z)$, is a kinematical phase space factor, which is related to the atomic physics. The second term $|M^{0\nu}|^2$ is the nuclear matrix element. Third term contains physics beyond the standard model through the neutrino masses m_i and mixing matrix elements U_{ei} of neutrino species.

The standard operator for the $0\nu\beta\beta$ is given by [51] :

$$T_{s_1s_2} = \frac{1}{2}\sum_{n,n'} \tau^+_n \tau^+_{n'} [\Sigma^{s_1}_n \times \Sigma^{s_2}_{n'}]^\lambda \cdot V(r_{nm'}) C^\lambda(\Omega_{nn'}). \tag{2.1}$$

where s_1 and s_2 can be only 0 or 1, $\Sigma^0 = 1$ and $\Sigma^1 = \sigma$. The operator $T_{s_1s_2}$ has thee contributions: Fermi $(s_1 = s_2 = \lambda = 0)$, Gamow-Teller $(s_1 = s_2 = 1, \lambda = 0)$ and Tensor $(s_1, s_2 = 1, \lambda = 2)$. [1] $V(r)$ is a generic radial form that depends on the

[1] The Gamow Teller contribution is multiplied by a factor of $-\sqrt{3}$ and the tensor part $\sqrt{\frac{2}{3}}$ (see [49]).

mechanism of $0\nu\beta\beta$ and $C^{\lambda} = \sqrt{4\pi/(2\lambda+1)}Y^{\lambda}$.
We observe that Eq. 2.1 is valid under the closure approximation, which is good for $0\nu\beta\beta$ decay since the average neutrino momentum is of the order of 100 MeV/c.

The two body transition operator between an initial $J_I^+ = 0^+$ and final J_F^+ nuclear states can be written in terms of two body matrix elements. Since it is well known that the matrix elements of two body interactions for many particle configurations can be reduced into spin and ordinary space parts the operator $T_{s_1 s_2}$ can be written in terms of the product of a spatial part and a spin part as

$$\begin{aligned}\hat{T}_i^{(\lambda)} = -\frac{1}{4}\sum_{j_1,j_2,J_0}\sum_{j_{1'},j_{2'},J_{0'}}&\sqrt{1+(-1)^{J_0}\delta_{j_1,j_2}}\\ &\sqrt{1+(-1)^{J_{0'}}\delta_{j_{1'},j_{2'}}}\langle j_1,j_2;J_0\|T_i^{(K)}\|j_{1'},j_{2'};J_{0'}\rangle\\ &[A^{\dagger}(j_1j_2;J_0)\otimes\tilde{A}(j_{1'}j_{2'};J_{0'})]^{(K)},\end{aligned} \tag{2.2}$$

which reduces to Eq. 2.1 for a transition between ground states. In Eq. 2.2 $\tilde{A}(j_{1'}j_{2'};J_{0'})$ is the annihilation operator of two neutrons and $A^{\dagger}(j_1j_2;J_0)$ is the creation operator of two protons. The two particle states can be written as

$$|(n_1,l_1,j_1)(n_2,l_2,j_2);J,M\rangle = \frac{(c^{\dagger}_{n_1l_1j_1}\otimes c^{\dagger}_{n_2l_2j_2})^J_M|0\rangle}{\sqrt{1+(-1)^J\delta_{n_1,n_2}\delta_{l_1,l_2}\delta_{j_1,j_2}}}. \tag{2.3}$$

The two body operator, T_i^K, contains the information of the $0\nu\beta\beta$ decay mechanism. For the case of 0+ to 0+ the transition operator becomes [49]:

$$\begin{aligned}T^{\lambda}_{\mathrm{s1s2}} = &-\tfrac{1}{2}\textstyle\sum_{\mathrm{j1}}\sum_{\mathrm{j1'}}G^{\lambda}_{\mathrm{s1s2}}(\mathrm{j1},\mathrm{j1},\mathrm{j1'},\mathrm{j1'};0)A_{\pi}(j1)A_{\nu}(j1')s^{+}{}_{\pi}\cdot\tilde{s}_{\upsilon}\\ &-\tfrac{1}{4}\textstyle\sum_{\mathrm{j1j2}}\sum_{\mathrm{j1'j2'}}\sqrt{\delta_{\mathrm{j1j2}}+1}\sqrt{\delta_{\mathrm{j1'j2'}}+1}\\ &G^{\lambda}_{\mathrm{s1s2}}(\mathrm{j1},\mathrm{j2},\mathrm{j1'},\mathrm{j2'};2)B_{\pi}(j1)B_{\nu}(j1')d^{+}{}_{\pi}\cdot\tilde{d}_{\upsilon},\end{aligned} \tag{2.4}$$

where

$$G^{(\lambda)}_{s1,s2}(j_1,j_2,j_{1'},j_{2'};J) \equiv \langle j_1,j_2;J||T_J^{(K)}\|j_{1'},j_{2'};J\rangle \tag{2.5}$$

are the two-body matrix elements, which can be computed by using Eq. A.5 of the Appendix A.

2.3 ^{76}Ge $\rightarrow$ ^{76}Se matrix elements

For the case of the transition ^{76}Ge $\rightarrow$ ^{76}Se, we obtained the nuclear matrix elements of $0\nu\beta\beta$ decay. The different contributions to the nuclear matrix elements, Axial Vector (AA), Vector Vector (VV), Axial Pseudo scalar (AP), Pseudo scalar Pseudoscalar, (PP) and Weak Magnetism, MM are listed in Table 2.1 (for more details, see App. C.6)). Once taken into account the finite nucleon size (FNS) and the short

Fermi Matrix elements $M_F^{(0\nu)}$[fm^{-1}]				
AA+VV	AP	PP	MM	Sum
-0.2845	0.	0.	0.	-0.2845

Gamow-Teller Matrix elements $M_{GT}^{(0\nu)}$[fm^{-1}]				
AA+VV	AP	PP	MM	Sum
0.5418	-0.1164	0.0346	0.0362	0.4962

Tensor Matrix elements $M_T^{(0\nu)}$ [fm^{-1}]				
AA+VV	AP	PP	MM	Sum
0.	-0.0367	0.0120	-0.0061	-0.0308

$M^{0\nu} = -(\frac{g_V}{g_A})^2 M_F^{(0\nu)} + M_{GT}^{(0\nu)} + M_T^{(0\nu)}$				
AA+VV	AP	PP	MM	Sum
0.7239	-0.1531	0.0466	0.0301	0.6475

Table 2.1: ^{76} Ge $\rightarrow$ ^{76}Se $0\nu\beta\beta$ matrix elements (in fm $^{-1}$) in IBM-2.

range correlations (SRC) the nuclear matrix elements is $M^{0\nu}$=0.5376 fm^{-1}. In order to compare this result with the other models, it is useful to to convert this value, expressed in fm^{-1}, in dimensionless unity multiply by $2R$ (see Tab. 2.2.

IBM-2				QRPA [52]	SM [53] dim.less
$M_F^{(0\nu)}$	$M_{GT}^{(0\nu)}$	$M_T^{(0\nu)}$	$M_{calc}^{0\nu}$	$M^{0\nu}$	
-0.2845	4.096	-0.250	**5.465**	4.680	2.220

Table 2.2: Comparison of the theoretical nuclear matrix elements with different models.

We have computed the nuclear matrix elements again using elements of between ^{76}Ge and ^{76}Se using IBM-2 with different corrections as already done in [32, 54, 55, 56].

The original value using finite nucleon size and short-range correlations give us the value in dimensionless units of 5.46, however removing both corrections we got 6.58.

For this case, the corrections for this particular pair of nucleons makes that the nuclear matrix element is reduced by 17%. Also, we noticed that changing the value

of the axial coupling constant g_A=1.25 to g_A=1.269 the value of the nuclear matrix element is 5.41 and 6.52 without corrections. There is a small reduction by less than 1% of the nuclear matrix element considering an increment of the coupling constant of 1.5%.

Moreover considering only the Generalized Seniority scheme, in the SD subspace, the value of the nuclear matrix elements is 5.29, if we consider only the S contribution the nuclear matrix element has the value of 8.07, therefore, the fact to consider a bigger subspace gives a considerable reduction.

Once the new generation experiments will have enough sensitivity to access the experimental region of interest for the detection of the $0\nu\beta\beta$, they will not be able to extract any physical information because of the huge theoretical uncertainties which affect the $0\nu\beta\beta$ nuclear matrix elements (they are squared so the error increases even more). For this reason, the Numen experiment proposes to study the heavy ion charge exchange in order to extract the double charge exchange nuclear matrix elements. Although the double charge exchange is a process mediated by the strong interaction (it is due to meson exchange), and the neutrinoless double beta decay is a process mediated by the weak interaction, nevertheless the nuclear matrix elements involved in both processes have strong similarities. The initial and final nuclear states of the nuclei that can be used for a double charge exchange reaction can be chosen to be double beta decay nuclei, and, more important, the spin and isospin operators in both processes are the same. Obviously, in the double charge exchange the neutrino potential is not present, however there is a correlation between the double charge exchange nuclear matrix elements and the neutrinoless double beta nuclear matrix elements [5].

Chapter 3
Double charge exchange in the low momentum limit

Double-charge exchange (DCE) reactions are the object of a worldwide renewed interest, also for the information that one could extract on the nuclear matrix elements entering the expression of the life time of the double beta decay. In fact, it is known that the nuclear matrix elements involved in the beta-decays are connected to the charge-exchange reaction ones and consequently to the Fermi or Gamow-Teller (GT) transition strengths. In the same way, the matrix elements involved in the double beta decay should be connected to the DCE reactions ones. In this context, heavy-ion DCE could play a precious role. An intense experimental activity on double charge exchange reactions is planned at the LNS-Catania, according to the NUMEM project [57]. The reaction proceed both via a direct mechanism or a sequential one. The direct mechanism is related to the double isospin-flip direct process, while the successive two-proton plus two-neutron transfer o vice-versa is referred as the sequential mechanism. On the basis of the information present in literature, a reasonably good candidate for the investigation of the double charge exchange nuclei is the reaction ^{116}Sn(^{18}O, ^{18}Ne) ^{116} Cd. The heavy ion DCE transition rate between the ground states of ^{18}O, ^{18}Ne is expected to be large due to the overlap of their wave functions in rspace, being among the same super-multiplet members. The advantage of a T=0 target is that the GT transition is concentrated only on T=2 states of the residual nucleus. Moreover the sequential transfer processes are very mismatched and will scarcely contribute to the cross section.

In this chapter we discuss the treatment of the heavy-ion double charge exchange reactions (DCX) within the eikonal approach at small angles and low momentum transfer.

3.1 Microscopic DCX process

We are going to describe the DCX process as a low-momentum limit of the effective microscopic theory proposed in Ref. [5]. In the approach propose in the [5], the

leading order of the DCX process is given by two different microscopic process: the two-pion exchange process and a short range interaction.

3.1.1 Two-pion exchange contribution

We are going to extend the description of single charge-exchange to double charge exchange reactions following the approach by Bertulani [58], and we will use the microscopic π exchange potentials. The DCX process that we are studying is schematically shown in Fig. 3.1.

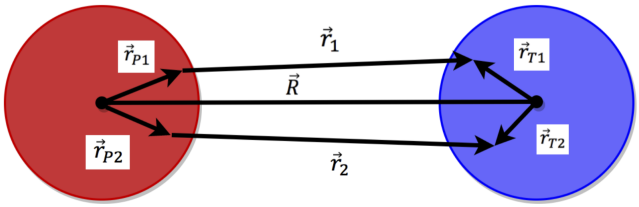


Fig. 3.1: Coordinate system used in the calculations. R is the distance between the center of masses of the two nuclei, target (T) and projectile (P). r_{P1} and r_{P2} (r_{T1} and r_{T2}) are the distances between the nucleons involved in the DCE process and the center of the projectile (target) nucleus. The coordinates $\mathbf{r}_{1,2} = \mathbf{R} - \mathbf{r}_{\mathrm{T1,2}} - \mathbf{r}_{\mathrm{P1,2}}$ are the relative positions of the interacting nucleons.

To introduce the idea of the calculation, we shall start from the leading-order π exchange contribution to the DCX mechanism. In Ref. [5] is shown that, in the non-relativistic limit, the effective potential given by the pion-exchange process

$$V_{2\pi}(\mathbf{q}_1,\mathbf{q}_2) = \frac{f_\pi^4}{m_\pi^4}\left(\frac{(\sigma_{P1}\cdot\mathbf{q}_1)(\sigma_{T1}\cdot\mathbf{q}_1)}{m_\pi^2+q_1^2}(\tau_{P1}\cdot\tau_{T1})\right) \times\left(\frac{(\sigma_{P2}\cdot\mathbf{q}_2)(\sigma_{T2}\cdot\mathbf{q}_2)}{m_\pi^2+q_2^2}(\tau_{P2}\cdot\tau_{T2})\right) \tag{3.1}$$

in the momentum representation. Where q_1 is the momentum transferred between the first pairs of nucleons and q_2 refers to the second pairs of the nucleons , and the indices P,T refer to the Projectile and Target, respectively. The pion coupling constant and the pion mass used in the calculations are $f_\pi^2/4\pi$=0.08 and $m_\pi c^2$=145 MeV, respectively. The Feynman diagram which corresponds to the two pion exchange process is reported in Fig. 3.2.

The whole potential can be written as follows:

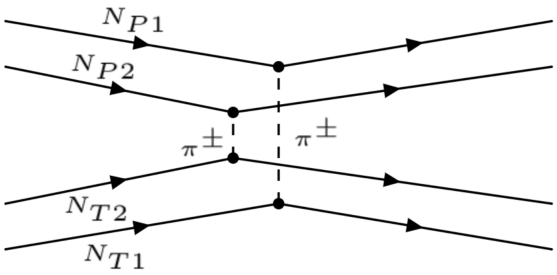


Fig. 3.2: Feynman diagram for a double-charge-exchange process induced from two pion exchange.

$$V(\mathbf{q}_1,\mathbf{q}_2) = V_{2\pi}(\mathbf{q}_1,\mathbf{q}_2) + V_{\rm sr} = [v(\mathbf{q}_1)v(\mathbf{q}_2)(\sigma_{P'}\cdot\hat{q}_2)(\sigma_{T'}\cdot\hat{q}_2)(\sigma_P\cdot\hat{q}_1)(\sigma_T\cdot\hat{q}_1) + w(\mathbf{q}_1)(\sigma_P\cdot\sigma_T)w(\mathbf{q}_2)(\sigma_{P'}\cdot\sigma_{T'})](\tau_{P'}\cdot\tau_{T'})(\tau_P\cdot\tau_T) \tag{3.2}$$

where the functions $v(q)$ are defined as

$$v(q) = -J_\pi \frac{q^2}{m_\pi^2 + q^2}, \tag{3.3}$$

and $w(q)$ is simple the constant

$$w(q) = J_\pi g_\pi \tag{3.4}$$

The parameters are used the previous functions are $J_\pi = f_\pi^2/m_\pi^2 \simeq 400\text{MeV}\cdot\text{fm}^3$. In the the simplest case of one pion exchange, the matrix elements between initial and final states of the projectile and target are written in terms of radial and angular parts and spin-isospin matrix elements. The other matrix elements can be computed analogously. In the isospin and spin spaces we can use the next relations

$$\begin{aligned} \tau_P\cdot\tau_T &= \tau_P^0\tau_T^0 + \tau_P^+\tau_T^- + \tau_P^-\tau_T^+ \\ \sigma_P\cdot\sigma_T &= \sigma_P^0\sigma_T^0 + \sigma_P^+\sigma_T^- + \sigma_P^-\sigma_T^+ \end{aligned} \tag{3.5}$$

We can extract a simple and more compact form for the transition amplitude in the low-momentum scattering limit, $q_1 = q_2 = q \sim 0$. In this specific limit, we get:

$$\mathscr{M}^{DCX}(q\sim 0) \sim \sum_{\mu\lambda\xi\theta} \langle \phi_f^T\phi_f^P\phi_f^{T'}\phi_f^{P'}|\sigma_\mu^P\tau_\lambda^P\sigma_\xi^{P'}\tau_\theta^{P'}\sigma_{-\mu}^T\tau_{-\lambda}^T\sigma_{-\xi}^{T'}\tau_{-\theta}^{T'} w(q_1)w(q_2)|\phi_i^T\phi_i^P\phi_i^{T'}\phi_i^{P'}\rangle \tag{3.6}$$

where $\mu,\xi,\lambda,\theta = 0,+,-$.

3.2 Low momentum limit case

At low-momentum transfer, the pion exchange contribution vanishes (Eq. 3.1), so only the short range interaction have to be considered. In the limit case, where $q_1 = q_2 = 0$ and the potential of the pion exchange vanish, $w(q1)w(q2)$ becomes a constant: $A = (J_\pi g_\pi)^2$. Given this, we can factorize the transition amplitude matrix elements and separate those referring to the target and projectile:

$$\mathscr{M}^{DCX}(q\sim 0) \sim \sum_{\mu\lambda,\xi,\theta} A^2 \langle \phi_f^P \phi_f^{P'} | \sigma_\mu^P \sigma_\xi^{P'} \tau_\lambda^P \tau_\theta^{P'} | \phi_i^P \phi_i^{P'} \rangle \times \langle \phi_f^T \phi_f^{T'} | \sigma_{-\mu}^T \sigma_{-\xi}^{T'} \tau_{-\lambda}^T \tau_{-\theta}^{T'} | \phi_i^T \phi_i^{T'} \rangle$$

We can write the amplitude in terms of Double Gamow-Teller transition densities for the nucleus A:

$$\mathscr{B}_{DGT}^{\mu\xi\lambda\theta}(A \to A') = \langle \Phi_f^{(A')} \| \sigma_\mu \sigma_\xi \tau_\lambda \tau_\theta \| \Phi_i^{(A)} \rangle \tag{3.7}$$

In the low momentum limit we get:

$$\mathscr{M}^{DCX}(b) \sim A^2 \sum_{\mu\lambda\xi\theta} \mathscr{B}_{DGT}^{\mu\xi\lambda\theta}(T \to T') \mathscr{B}_{DGT}^{-\mu-\xi-\lambda-\theta}(P \to P')$$

$$\frac{d\sigma}{d\Omega}(q\sim 0) = \frac{k}{k'}\left(\frac{\mu}{4\pi^2\hbar^2}\right)^2 [A]^4 F(\theta) \left| \sum_{\mu\lambda\xi\theta} \mathscr{B}_{DGT}^{\mu\xi\lambda\theta}(T \to T') \mathscr{B}_{DGT}^{-\mu-\xi-\lambda-\theta}(P \to P') \right|^2 \tag{3.8}$$

Here, $F(\theta)$ is a function which describes the scattering angular distribution.

3.3 Gamow teller matrix elements

In this section, we shall discuss the Gamow Teller matrix elements for the double charge exchange using the Eikonal approximation (see App. D). The Double Gamow Teller matrix elements is given by the operators $\sigma_1 \cdot \sigma_2 \tau_1 \tau_2$ over the initial nuclear states $\Phi^{(A)_i}$ to final nuclear states $\Phi_f^{A'}$, and the general transition of reduced Double Gamow Teller matrix elements $\mathscr{M}(\mathrm{GT} : A \to A')$ over a particular states, may be written as,

$$\langle \Phi_{J'}^{(A')} \| \tfrac{1}{2} \textstyle\sum_{n,n'} [\sigma_n \times \sigma_{n'}]^{(0)} \tau_n \tau_{n'} V(r_{n,n'}) \| \Phi_J^{(A)} \rangle \tag{3.9}$$

In the double charge exchange, there are two Double Gamow Teller transitions, one given by the target and another by the projectile, for each Double Gamow Teller

transitions, there are two nucleons such as protons(neutrons) that becomes two neutrons(protons), for the target(projectile). Therefore the nuclear states are represented by the direct product of the proton and neutron part.

$$\begin{aligned}|\Phi_J^{(A)}\rangle = [&|(n_1 l_1 j_1)(n_2 l_2 j_2); J_{12} M_{12}\rangle_\pi \\ &\times |(n_{1'} l_{1'} j_{1'})(n_{2'} l_{2'} j_{2'}); J_{12'} M_{12'}\rangle_\nu]^{(J)}\end{aligned} \tag{3.10}$$

let us first consider the initial nuclear states as ground states, that for the case even-even nuclei four our interest, can be represented by the first 0_1^+

$$\Phi_{J=0}^{(A)} = |0_1^+\rangle_A. \tag{3.11}$$

Therefore, the general transition of the double Gamow Teller can be written in the standard second quantized form ([14],[49])

$$\begin{aligned}\mathscr{M}(\mathrm{GT}: A \to A') = \langle \Phi_{J'}^{(A')}| - \frac{1}{4} \sum_{j_1 j_2} \sum_{j_{1'} j_{2'}} \sum_J (-1)^J \sqrt{1+(-1)^J \delta_{j_1 j_2}} \sqrt{1+(-1)^J \delta_{j'_1 j'_2}} \\ \times M(j_1, j_2 j'_1, j'_2; J)(c^\dagger_{n_1 l_1 j_1} \times c^\dagger_{n_2 l_2 j_2})^{(J)} \cdot (\tilde{c}_{n'_1 l'_1 j'_1} \times \tilde{c}_{n'_2 l'_2 j'_2})^{(J)} |\Phi_{J=0}^{(A)}\rangle\end{aligned} \tag{3.12}$$

that for the case of DCX the raising operator creates a proton (neutron), and the annihilation operator destroys a neutron (proton) for the projectile (target). From Eq (3.12), the nuclear matrix elements of DCX in the scheme of Microscopic Interacting Boson Model can be written as:

$$\begin{aligned}\mathscr{M}(\mathrm{GT}: A \to A') = {}_{\mathrm{IBM}}\langle \Psi_{J'}^{(A')}| - \frac{1}{2} \sum_{j_1} \sum_{j_{1'}} M(j_1, j_1 j'_1, j'_1; 0) A_\pi(j) A_\nu(j) s^\dagger_\pi \cdot \tilde{s}_\nu \\ - \frac{1}{4} \sum_{j_1 j_2} \sum_{j_{1'} j_{2'}} \sqrt{1+\delta_{j_1 j_2}} \sqrt{1+\delta_{j'_1 j'_2}} M(j_1, j_2 j'_1, j'_2; 2) \\ \times B_\pi(j, j') B_\nu(j, j') d^\dagger_\pi \cdot \tilde{d}_\nu |\Psi_{J=0}^{(A)}\rangle_{\mathrm{IBM}}\end{aligned} \tag{3.13}$$

where $|\Psi_J^{(A)}\rangle_{\mathrm{IBM}}$ are the wave functions of the even-even nuclei that are generated by diagonalizing the IBM-2 Hamiltonian, and and we have consider the mapping of the operators

$$\begin{aligned}(c^\dagger_{j_\pi} \times c^\dagger_{j_\pi})^{(0)} &\to A_\pi(j) s^\dagger_\pi \\ (c^\dagger_{j_\nu} \times c^\dagger_{j_\nu})^{(0)} &\to A_\nu(j) s^\dagger_\nu \\ (c^\dagger_{j_\pi} \times c^\dagger_{j'_\pi})^{(2)} &\to B_\pi(j, j') d^\dagger_\pi \\ (c^\dagger_{j_\nu} \times c^\dagger_{j'_\nu})^{(2)} &\to B_\nu(j, j') d^\dagger_\nu.\end{aligned} \tag{3.14}$$

The matrix element between two fermion states is given by

$$
\begin{aligned}
M(j_1,j_2,j_1',j_2';J) = & \sum_{k_1=|l_1-l_{1'}|}^{l_1+l_1'} \sum_{k_2=|l_2-l_2'|}^{l_2+l_2'} \sum_{k=\max(|j_1-j_1'|,|j_2-j_2'|)}^{\min(j_1+j_1',j_2+j_2')} i^{k_1-k_2}(2k_1+1)(2k_2+1) \\
& \times \langle k_1 0 k_2 0|00\rangle (-1)^{1+k_1} \begin{Bmatrix} k_1 & 1 & k \\ 1 & k_2 & 0 \end{Bmatrix} (-1)^{j_2+j_1'+J} \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & k \end{Bmatrix} \\
& \times (2k+1)\sqrt{2j_1+1}\sqrt{2j_1'+1} \begin{Bmatrix} \frac{1}{2} & l_1 & j_1 \\ \frac{1}{2} & l_1' & j_1' \\ 1 & k_1 & k \end{Bmatrix} \\
& \times \sqrt{2j_2+1}\sqrt{2j_2'+1} \begin{Bmatrix} \frac{1}{2} & l_2 & j_2 \\ \frac{1}{2} & l_2' & j_2' \\ 1 & k_2 & k \end{Bmatrix} \\
& \times \langle \tfrac{1}{2}\|\sigma_1\|\tfrac{1}{2}\rangle (-1)^{-k_1}\sqrt{2l_1+1}\langle l_1 0 k_1|l_1' 0\rangle \\
& \times \langle \tfrac{1}{2}\|\sigma_2\|\tfrac{1}{2}\rangle (-1)^{-k_2}\sqrt{2l_2+1}\langle l_2 0 k_2|l_2' 0\rangle \\
& \times R(n_1,l_1,n_2,l_2,n_1',l_1',n_2',l_2'),
\end{aligned}
\tag{3.15}
$$

where $\sigma_{1,(2)}$ refers to the the first nucleon (second)that interacts in the single gamow teller and the value of this reduced matrix for the sigma is $\sqrt{6}$ and the $R(n_1,l_1,n_2,l_2,n_1',l_1',n_2',l_2')$ refers to the radial integrals for central interactions in which depend on a microscopic ponential discussed before. We are interested to compare the different Gamow Teller contributions from DCX and $0\nu\beta\beta$-decay matrix elements. The radial integral are given as follows

$$
\begin{aligned}
& R(n_1,l_1,n_2,l_2,n_1',l_1',n_2',l_2') \\
& = \int_0^\infty v(q) q^2 dq \int_0^\infty R_{n_1 l_1}(r_1) R_{n_1' l_1'}(r_1) j_{k_1}(pr_1) r_1^2 dr_1 \\
& \times \int_0^\infty R_{n_2 l_2}(r_2) R_{n_2' l_2'}(r_2) j_{k_2}(pr_2) r_1^2 dr_2
\end{aligned}
\tag{3.16}
$$

where j_k are the spherical Bessel functions, and the potential is given by the Fourier Bessel transformation

$$
v(q) = \frac{2}{\pi}\int_0^\infty V(r) j_\lambda(qr) r^2 dr. \tag{3.17}
$$

The usual method to compute the two-body matrix elements, in this case, consists of the Moshinsky transformation and the associated transformation brackets. Besides, the radial matrix elements can be evaluated in closed form by using the Horie method[**?**]:

$$
\begin{aligned}
R = & \int_0^\infty w(q)^2 q^2 dq \\
& \times \int_0^\infty R_{n_1 l_1}(r_1) R_{n_1' l_1'}(r_1) j_{k_1}(pr_1) r_1^2 dr_1 \\
& \times \int_0^\infty R_{n_2 l_2}(r_2) R_{n_2' l_2'}(r_2) j_{k_2}(pr_2) r_1^2 dr_2
\end{aligned}
\tag{3.18}
$$

where the subscript in $1,2$ in the first two integrals refers to the first and second exchanged nucleons meanwhile the last two integrals represents the overlap between the initial and final nucleon states respectively.

We have showed a simple approximation to compute the DCX using the Eikonal approximation. We have presented the essential ingredients that should be taken in to account if we want to compute the double charge exchange cross section using the Glauber approach. We have noticed that it is possible to factorize the reaction part and the structure part under certain conditions. Moreover we stressed the relationship between the nuclear matrix elements in neutrinoless double beta decay and double charge exchange Gamow-Teller matrix elements and also we showed that the microscopic IBM-2 is a great model that allows us to study complex processes.

Conclusions

In this thesis has been derivated the general expressions of the spectroscopic amplitudes two-nucleon transfer reactions within the microscopic IBM-2. For the first time has been calculated the spectroscopic amplitudes of the nucleon transfer process within the microscopic IBM-2. In particular, the spectroscopic amplitudes of the two-nucleon transfer reactions are important in the forthcoming experiments of NUMEN Collaboration, where the two-nucleon transfer process is a competitive process of the Double charge exchange reactions. The new formalism introduced in this thesis has been applied to the cadmium isotopes [3]. The formalism introduced in Chap. 1 was applied to compute the spectroscopic amplitudes of two-neutron transfer reaction ^{64}Ni(^{18}O,^{16}O)^{66}Ni considering one- and two-step processes and we compared with the experimental data). The reaction part was performed using the DWBA. For the two-neutron transfer reactions to the ground state of the residual nuclei ^{66}Ni, we have evidence of the short-range pairing correlations. In the two-neutron transfer to the first excited state of the residual ^{66}Ni nucleus, were the collectivity is known to be important, it was verified the predominance of the two-step reaction mechanism. This confirmed the long-range predominance in the wave function of this state. The calculations for the two-neutron transfer reaction was done considering one- and two-step processes to check which type of mechanism is more important and consequently the effect of pairing correlation on this reaction mechanism.

From our results we conclude that for two-neutron transfer to the ground state of ^{66}Ni, the direct transfer is the dominant reaction mechanism, whereas, for the transfer to the first excited state of ^{66}Ni, the sequential process dominates. It would be interesting to measure and perform similar calculations on another system with well not collectivity of their excitation spectrum to verify our conclusions.

The possibility to study the different one- and two-step processes that helps the understanding of the effect of pairing correlations, as well as a competition between long-range and short-range correlations. The spectroscopic amplitudes needed in microscopic calculations were calculated using two structure models: the shell model and the IBM's. The importance of studying the nickel isotopes is that they are in the upper limit of confident applicability of the shell model and the IBM is about

its lower limit. In the two-neutron transfer to the first excited state of ^{66}Ni, where the collectivity is known to be important, it was verified the predominance of the two-step reaction mechanism. This confirmed the long-range predominance in the wave function of this state, which was well described within IBM framework. The results of this research were published in Phys. Rev. C 96, 044612 (2017). The theoretical work of two-nucleon transfer process is an important study since the two-nucleon transfer is a competitive process in respect to the double charge exchange process.

Finally, since the goal of the NUMEN collaboration is to arrive to extract the double charge exchange nuclear matrix elements, a simple model of double charge exchange cross section has been developed in the present work. In particular, it was demonstrated, for the first time, the possibility to factorize the nuclear matrix elements in the low-momentum transfer limit [5]. The factorization is one of the main results of this thesis.

We have been derived the neutrinoless double beta decay operator starting from the basic theory of the two body transition densities allowing us reproduced the results of the literature [32], following the theory of the neutrinoless double beta decay of several previous autors such as Doi, Tomoda and Simkovic and collaborators [54],[55] [56]. It was computed the nuclear matrix elements for $0\nu\beta\beta$ -decay ^{76}Ge and ^{76}Se and we noticed that changing the value of the axial coupling constant g_A=1.25 to g_A=1.269 the nuclear matrix element decreases by less than 1%. We compared the IBM-2 and Generalized seniority cases and in both cases decreases the nuclear matrix elements. In addition considering the Generalized Seniority case the fact to consider consider only the S contribution the nuclear matrix elements are bigger than consider SD.

We presented the formalism to calculate Double Charge Exchange (DCX) cross-sections in the eikonal approximation. In heavy-ion scattering processes, DCE reactions, where two protons are replaced by two neutrons or vice versa, can take place. The calculation of the double for heavy ion double charge exchange reactions was done within the IBM-2 framework. This result is important since a linear correlation between the double Gamow-Teller part of the $0\nu\beta\beta$ nuclear matrix elements and the Gamow-Teller matrix elements of double charge exchange process has been demonstrated [5]. The calculation of the total cross sections of the DCX reactions is underway since some parameters have to be fixed to the experimental data that will be provided in the forthcoming experiments [3].

In conclusion, because of the introduction of the microscopic description of the operators in the IBM-2, easily should be possible deal with different problems in nuclear structure allowing derive new operators in the scheme of microscopic IBM-2, opening new ways to do further research in different directions in nuclear physics.

Appendix A
Two body matrix elements

The matrix elements of two-body interactions are necessary in the research of nuclear properties by means of the independent particle model. The calculations of these matrix elements are usually carried out by expanding the interactions into series of Legendre polynomials [59].

For the central interaction, it is well known that this procedure is quite easy if one applies the methods of tensor operator proposed by Racah [60]. For the non-central interactions, the situations are somewhat complicated and the interactions have been discussed by Talmi [61].

The method that we will use, was introduced by Horie and Sasaki [62], where the two-body interactions can be easily expanded into series of the products of spherical harmonics by considering Fourier transforms of the interactions, and the matrix elements can be obtained by straightforward application of the method of tensor operators. This procedure is proposed so far for the treatment of the non-central interactions. Specially for the case of tensor interactions. The radial integrals which appear in this expansion involve the variables r_1 and r_2, the distances of the two particles from the origin, in separated forms in the integrands, because the Fourier transform is considered.

The two body transition density contains two body matrix elements of the two body interactions of many particle configurations.

Let us first decompose the tensor interaction into spin and angular parts respectively,

$$[[\Sigma^{(s_1)} \times \Sigma^{(s_2)}]^{(\lambda_1)} \times [C_1^{(k_1)} \times C_2^{(k_2)}]^{(\lambda_2)}]_{\mu}^{(\lambda)} \tag{A.1}$$

where $C_q^{\lambda_2}(\Omega) = [4\pi/(2\lambda_2+1)]^{1/2} Y_q^{\lambda_2}(\Omega)$ are the unnormalized spherical harmonics, with λ_2 be positive integer and it it will be denoted as Ω the direction of the vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $r = |\mathbf{r}|$.

The spin operators is written in terms of a tensor product operators rank s_1 and s_2 coupled to λ_1 and the angular is a product of a tensors of rank k_1 and k_2 coupled to λ_2. It is convenient recouple the s_1 with k_1 and s_2 with k_2. Thus one can relate the operators as the states $|S_1 S_2(\lambda_1) k_1 k_2(\lambda_2); \lambda\mu\rangle$ to $|s_1 k_1(k) s_2 k_2(k'); \lambda\mu\rangle$ by the

transformation.

$$|S_1S_2(\lambda_1)k_1k_2(\lambda_2);\lambda\mu\rangle = \sum_{kk'} \hat{k}\hat{k}'\hat{\lambda}_1\hat{\lambda}_2 \begin{Bmatrix} s_1 & k_1 & k \\ s_2 & k_2 & k' \\ \lambda_1 & \lambda_2 & \lambda \end{Bmatrix} |s_1k_1(k)s_2k_2(k');\lambda\mu\rangle \quad \text{(A.2)}$$

Therefore making the basis transformation, Eq. (A.1) becomes,

$$\begin{aligned} &[[\Sigma^{(s_1)} \times \Sigma^{(s_2)}]^{(\lambda_1)} \times [C_1^{(k_1)} \times C_2^{(k_2)}]^{(\lambda_2)}]_\mu^{(\lambda)} \\ &= \sum_{kk'} \hat{k}\hat{k}'\hat{\lambda}_1\hat{\lambda}_2 \begin{Bmatrix} s_1 & k_1 & k \\ s_2 & k_2 & k' \\ \lambda_1 & \lambda_2 & \lambda \end{Bmatrix} [\Sigma^{s_1} \times C_1^{k_1}]^{(k)} \times [\Sigma^{s_2} \times C_2^{k_2}]^{(k')}]_\mu^{(\lambda)} \end{aligned} \quad \text{(A.3)}$$

The interaction of the spatial part of which has the form $V(r)C^{(\lambda_2)}(\Omega)$. Considering an interaction with a spin and spatial part , it may be decomposed as

$$\begin{aligned} &[[\Sigma^{(s_1)} \times \Sigma^{(s_2)}]^{(\lambda_1)} \times V(r)C^{(\lambda_2)}]_\mu^{(\lambda)} \\ &= \sum_{k_1k_2} i^{k_1-k_2+\lambda_2} \frac{(2k_1+1)(2k_2+1)}{2\lambda_2+1} \\ &\langle k_1 0 k_2 0 \lambda_2 0\rangle v^{k_1,k_2;\lambda_2}(r_1,r_2) \\ &\times [[\Sigma^{(s_1)} \times \Sigma^{(s_2)}]^{(\lambda_1)} \times [C_1^{(k_1)} \times C_2^{(k_2)}]^{(\lambda_2)}]_\mu^{(\lambda)} \\ &= \sum_{kk'} \sum_{k_1k_2} i^{k_1-k_2+\lambda_2} \frac{(2k_1+1)(2k_2+1)}{2\lambda_2+1} \\ &\langle k_1 0 k_2 0 \lambda_2 0\rangle v^{k_1,k_2;\lambda_2}(r_1,r_2) \\ &\hat{k}\hat{k}'\hat{\lambda}_1\hat{\lambda}_2 \begin{Bmatrix} s_1 & k_1 & k \\ s_2 & k_2 & k' \\ \lambda_1 & \lambda_2 & \lambda \end{Bmatrix} [\Sigma^{s_1} \times C_1^{k_1}]^{(k)} \times [\Sigma^{s_2} \times C_2^{k_2}]^{(k')}]_\mu^{(\lambda)} \end{aligned} \quad \text{(A.4)}$$

where a simplified notation $\hat{J}$ represent $\sqrt{2J+1}$. This general interaction of rank λ is a tensor product of tensor operators rank λ_1 and λ_2 which operate spin and ordinary spaces, respectively. This is a generalization of the tensor interaction used by H. Horie(see Eq. 1 in Ref. [62])

By using the Eq (A.4) the reduced two body matrix elements of this operator may be computed as:

$$
\begin{aligned}
&\langle j_1, j_2, J \| [[\Sigma^{(s_1)} \times \Sigma^{(s_2)}]^{(\lambda_1)} \times V(r) C^{(\lambda_2)}]^{(\lambda)} \| j_1', j_2' J' \rangle \\
&= \sum_{kk'} \sum_{k_1 k_2} i^{k_1 - k_2 + \lambda_2} \frac{(2k_1+1)(2k_2+1)}{2\lambda_2 + 1} \\
&\langle k_1 0 k_2 0 \lambda_2 0 \rangle v^{k_1, k_2; \lambda_2}(r_1, r_2) \\
&\hat{k}\hat{k}'\hat{\lambda}_1\hat{\lambda}_2 \begin{Bmatrix} s_1 & k_1 & k \\ s_2 & k_2 & k' \\ \lambda_1 & \lambda_2 & \lambda \end{Bmatrix} \hat{f}\hat{\lambda}\hat{J}' \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ k & k' & \lambda \end{Bmatrix} \\
&\hat{j}_1\hat{k}\hat{j}_1' \begin{Bmatrix} \frac{1}{2} & l_1 & j_1 \\ \frac{1}{2} & l_1' & j_1' \\ s_1 & k_1 & k \end{Bmatrix} \langle \tfrac{1}{2} \| \Sigma^{(s_1)} \| \tfrac{1}{2} \rangle (-1)^{-k_1} \hat{l}_1 \langle l_1 0 k_1 | l_1' 0 \rangle \\
&\hat{j}_2\hat{k}'\hat{j}_2' \begin{Bmatrix} \frac{1}{2} & l_2 & j_2 \\ \frac{1}{2} & l_2' & j_2' \\ s_2 & k_2 & k \end{Bmatrix} \langle \tfrac{1}{2} \| \Sigma^{(s_2)} \| \tfrac{1}{2} \rangle (-1)^{-k_2} \hat{l}_2 \langle l_2 0 k_2 | l_2' 0 \rangle \\
&R^{(k_1 k_2 \lambda_2)}(n_1, l_1, n_2, l_2, n_1', l_1', n_2', l_2')
\end{aligned}
\tag{A.5}
$$

where the radial part is given by

$$
R^{(k_1 k_2 \lambda_2)} = \int_0^\infty V_{\lambda_2}(p) p^2 dp \int_0^\infty R_{n_1 l_1}(r_1) R_{n_1' l_1'}(r_1) j_{k_1}(pr_1) r_1^2 dr_1 \int_0^\infty R_{n_2 l_2}(r_2) R_{n_2' l_2'}(r_2) j_{k_2}(pr_2) r_2^2 dr_2 \tag{A.6}
$$

The allowed quantum numbers should satisfy the following conditions

$$
\begin{aligned}
&|l_1 - l_1'| \le k_1 \le l_1 + l_1' \\
&|l_2 - l_2'| \le k_2 \le l_2 + l_2' \\
&|j_1 - j_1'| \le k \le j_1 + j_1' \\
&|j_2 - j_2'| \le k \le j_2 + j_2' \\
&|k_1 - k_2| \le \lambda_2 \le k_1 + k_2 \\
&|k - k'| \le \lambda \le k + k' \\
&|J - J'| \le \lambda \le J + J'
\end{aligned}
\tag{A.7}
$$

The Eq. (A.5) is a generalized version of the two body matrix elements of the $0\nu\beta\beta$, which can be used to compute $0\nu\beta\beta$ two body matrix elements also for excited states. This formula in Eq. (A.5) can be used to compute $0\nu\beta\beta$, matrix elements between $0+$ to $0+$ in shell model calculations and in the Interacting Boson Model.

Appendix B
Two body transition densities

The nuclear Matrix element of the generalized operator in the Appendix A is computed in the tensor coupled form. The advantage of the tensorial form allows simplifying the calculations. The two body transition operator in tensorial form.

$$T^{\lambda}_{\mu} = \frac{1}{4} \sum_{k_\alpha k_\beta k_\gamma k_\delta} \sum_{m_\alpha m_\beta m_\gamma m_\delta} \langle k_\alpha m_\alpha k_\beta m_\beta | T^{\lambda}_{\mu} | k_\gamma m_\gamma k_\delta m_\delta \rangle a^{\dagger}_{k_\alpha m_\alpha} a^{\dagger}_{k_\beta m_\beta} a_{k_\delta m_\delta} a_{k_\gamma m_\gamma} \tag{B.1}$$

$$T^{\lambda}_{\mu} = \frac{1}{4} \sum_{k_\alpha k_\beta k_\gamma k_\delta} \sum_{m_\alpha m_\beta m_\gamma m_\delta} \langle J_0 M_0 | j_\alpha m_\alpha j_\beta m_\beta \rangle \langle J'_0 M'_0 | j_\gamma m_\gamma j_\delta m_\delta \rangle a^{\dagger}_{k_\alpha m_\alpha} a^{\dagger}_{k_\beta m_\beta} a_{k_\delta m_\delta} a_{k_\gamma m_\gamma} \tag{B.2}$$

$$\begin{aligned} T^{\lambda}_{\mu} = & \frac{1}{4} \sum_{k_\alpha k_\beta k_\gamma k_\delta} \sum_{J_0 M_0 J'_0 M'_0} \sqrt{1+\delta_{k_\alpha k_\beta}\delta_{k_\gamma k_\delta}} \langle k_\alpha k_\beta J_0 M_0 | T^{\lambda}_{\mu} | k_\gamma k_\delta J'_0 M'_0 \rangle \\ & \times \sum_{m_\alpha m_\beta m_\gamma m_\delta} \langle J_0 M_0 | j_\alpha m_\alpha j_\beta m_\beta \rangle \langle J'_0 M'_0 | j_\gamma m_\gamma j_\delta m_\delta \rangle a^{\dagger}_{k_\alpha m_\alpha} a^{\dagger}_{k_\beta m_\beta} a_{k_\delta m_\delta} a_{k_\gamma m_\gamma} \end{aligned} \tag{B.3}$$

$$T^{\lambda}_{\mu} = \frac{1}{4} \sum_{k_\alpha k_\beta k_\gamma k_\delta} \sum_{J_0 M_0 J'_0 M'_0} (1+\delta_{k_\alpha k_\beta})(1+\delta_{k_\gamma k_\delta}) A^{\dagger}(k_\alpha, k_\alpha J_0 M_0) A(k_\gamma, k_\delta J_{0'} M_{0'}) \tag{B.4}$$

$$T^{\lambda}_{\mu} = \frac{1}{4} \sum_{k_\alpha k_\beta k_\gamma k_\delta} \sum_{J_0 J'_0} \langle k_\alpha k_\beta \| T^{\lambda} \| k_\gamma k_\delta \rangle (1+\delta_{k_\alpha k_\beta})(1+\delta_{k_\gamma k_\delta}) \frac{[A^{\dagger}(k_\alpha, k_\alpha J_0 M_0) \otimes A(k_\gamma, k_\delta J_{0'} M_{0'})]^{\lambda}_{\mu}}{\sqrt{2\lambda+1}} \tag{B.5}$$

Appendix C
Interacting Boson Model

It is reviewed the essential ingredients of the interacting boson-fermion model. We are interested to cover the fundamental concepts of this model to introduce , single and double beta decay, single and double charge exchange.

The Interacting Boson Model model was proposed in 1974 by Iachello and Arima [63], inspired by previous works by Feshbach and Iachello [64] [65] and Jasen et al. [66]. Which attempts to describe the collective nuclear excitations with an algebraic formulation. This model have provided a bridge between individual and collective nuclear behavior based on the approximation of the bosonic nature of identical nucleons that dominate the valence dynamics of nucleons and that arises from nuclear forces. This is similar to the BCS theory of semiconductors that have electron coupling with Cooper pairs of zero spin which leads to behaviors collectives and superconductivity.

The collectivity degrees of freedom can be associated with the coupling of pairs of particles. The collective character is a consequence of the degrees of freedom possessed by the nuclei [1]. This allows us to classify these nuclei by the number of protons and neutrons. This classification allows us to develop different nuclear models with fascinating properties. Collective excitations of nuclei are described by bosons. The active pairs of protons and neutrons can be associated with bosons. The number of n-pairs of nucleons corresponds to the number of bosons. Low-lying collective states of nuclei can be described in terms of monopole bosons with angular momentum and parity 0^+ represented by s, and quadrupole bosons 2^+ represented by d. The conceptual basis of IBM has led to the unified description of the collective properties of a medium and of developed pair-pair heavy cores. In this formulation, they belong to the regions of transitions of several dynamic symmetries [67].

The microscopic Interacting Boson Model its the origin from the generalization of the seniority scheme [11],[68], where is inspired from the works of Mottelson and Bardeen, Cooper and Schierieffer [69] [12] at the end of the 50ths.

[1] Strictly speaking the degree of collective freedom are the degrees of freedom of movement of the quadrupoles that are formed in the nucleus.

The relationship between Generalized Seniority and the Interacting Boson Model was developed by Otsuka, Arima, Talmi, and Iachello in 1978. [16]. The manner to connect different nuclear matrix elements between different spaces, like fermionic and bosonic spaces are the way to connect the microscopic description to a phenomenological bosonic description of the nucleus [23], [13].

A manner to relate the bosons of the IBM to the underlying fermion space for nondegenerate j shells is given by the method of of Otsuka and Arima and Iachello[23] who used the so called "number-operator approximation (NOA)"

In the microscopic IBM-2 calculations used in this thesis, we did not used this method, since it "averages" over the subshell effects in which we are interested and has not been tested in the midshell region. We will followed the empirical IBM parameters by allowing the valence nucleons to occupy many nondegenerate j shells. Duval, Barret and Pittel have developed an exact formalism for calculating low-generalized-seniority matrix elements [25], [24]. In this formalism, no approximations are made in the treatment of sub-shell effects.

This generalization allows us to construct the microscopic Interacting Boson Model. The method to calculate two-body matrix elements in the SD subspace may be computed using the commutator technique by A. Frank P. Van Isacker and Lipas et al [27], [15], this allows us to construct a generalized operator in the scheme of the Interacting boson Model which helps to study more complex problems with great accuracy.

On the other hand the nuclei with odd masses may be studied, incorporating the degrees of freedom of a fermion [70]. In 1980, Iachello suggested the simultaneous description of even-numbered and odd-numbered nuclei through the introduction of superalgebras with energy levels in both nuclei that belong to the same supermultiplet [71]. The concept of nuclear supersymmetry was extended in 1985 to include the degree of freedom of the neutron-proton [72]. This new formulation allows the supermultiplet to have even-odd, proton-odd, neutron-odd and odd-odd nuclei.

Table C.1: Different models of IBM where the total number of bosons is given by $N = \sum_i b_i^\dagger b_i$ and total number of fermions $M = \sum_\mu a_\mu^\dagger a_\mu$

Model	Year	Generator	Invariant	Symmetry
IBM	1975	$b_i^\dagger b_j$	N	$U(6)$
IBFM	1979	$b_i^\dagger b_j$, $a_k^\dagger a_l$	N, M	$U(6) \otimes U(m)$
SUSY	1980	$b_i^\dagger b_j$, $a_k^\dagger a_l$, $b_i^\dagger a_k$, $a_k^\dagger b_i$	$\mathscr{N} = N + M$	$U(6/m)$

Spectroscopic studies of heavy cores odd-odd numbers are challenging because of the high density of states. However, after 15 years of the prediction of odd-odd cores in the nuclear supersymmetry, in 1999 has been experimentally preserved the spectrum of the $196Au$, which is according to the theory [73]. However now with the microscopic interacting boson model, it is possible to go beyond and improve microscopic description of the odd-odd and odd-even wave functions that is a work in

progress with J. Barea and J. Kotila. In this way, one body and two body transitions densities becomes a reliable test for the nuclear wave functions and go further to the study the intercorrelation fo weak interactions and strong interactions in a nuclear matter.

C.1 The model

The interacting boson-fermion model is a straightforward extension of the interacting boson model where collective excitations of nuclei are described by bosons, in which an additional single-particle degree of freedom is introduced and coupled to the system of s- and d-bosons. The operators in this model are written in second quantization formalism, the boson creation(and annihilation) operators of multipolarity l and z-component m are represented by $b^{\dagger}_{l,m}(b_{l,m})$ and the creation (annihilation) operator for a nucleon by $a^{\dagger}_{jm}$ (a_{jm}) in one of the valence single-particle spherically symmetric orbitals.

The boson operators satisfy Bose commutation relations,

$$[b_{l,m}, b^{\dagger}_{l',m'}] = \delta_{ll'}\delta_{mm'},$$

$$[b_{l,m}, b_{l',m'}] = [b^{\dagger}_{l,m}, b^{\dagger}_{l',m'}] = 0 \tag{C.1}$$

The fermion operators satisfy Fermi anticommutation relations,

$$[a_{j,m}, a^{\dagger}_{j',m'}] = \delta_{jj'}\delta_{mm'},$$

$$[a_{j,m}, a_{j',m'}] = [a^{\dagger}_{j,m}, a^{\dagger}_{j',m'}] = 0 \tag{C.2}$$

Spherical tensors can be constructed from the creation and annihilation operators in the usual way. The creation operators already transform in the appropriate way. The annihilation operators do not but one can introduce the operators for fermions

$$\tilde{a}_{j,m} = (-)^{j-m} a_{j,-m}, \tag{C.3}$$

and for bosons

$$\tilde{b}_{l,m} = (-)^{l-m} b_{l,-m}. \tag{C.4}$$

The determination of the properties of a quantal system of N interacting particles among them bosons and fermions, considering the particle number conservation, requires the solution of the eigenvalue equation associated with the Hamiltonian

$$H = H_B + H_F + V_{BF}, \tag{C.5}$$

where

$$H_B = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b^\dagger_\alpha b_\beta + \sum_{\alpha\alpha'} \frac{1}{2} u_{\alpha,\alpha',\beta,\beta'} b^\dagger_\alpha b^\dagger_{\alpha'} b_\alpha b_\beta + \dots \tag{C.6}$$

$$H_F = \varepsilon_0 + \sum_{ik} \eta_{i,k} a^\dagger_i a_k + \sum_{ii'kk'} \frac{1}{2} v_{i,i',k,k'} a^\dagger_i a^\dagger_{i'} a_i a_k + \dots \tag{C.7}$$

$$V_{BF} = \sum_{\alpha i\beta k} w_{\alpha,i,\beta,k} b^\dagger_\alpha a^\dagger_i b_\alpha a_k + \dots \tag{C.8}$$

and containing one body terms $\varepsilon_{\alpha,\beta}$, $\eta_{i,k}$, two-body interactions $u_{\alpha,\alpha',\beta,\beta'}$,$v_{i,i',k,k'}$, $w_{\alpha,i,\beta,k}$ and so on; higher order interaction can be included in the expansion if is needed, H_B is the Hamiltonian for even-even core, H_F the single-particle Hamiltonian for odd-nucleon, and V_{BF} the coupling between these degrees of freedom.

This Hamiltonian can be rewritten in such a way that its invariance under rotation becomes evident.

$$H_B = E_0 + \sum_l \varepsilon_l \sqrt{2l+1} [b^\dagger_l \times \tilde{b}_l]^{(0)}_0 + \sum_{L_B,l,l',l'',l'''} u^{(L_B)}_{ll'l''l'''} [[b^\dagger_l \times b^\dagger_{l'}]^{(L_B)}]^{(0)}_0 + \cdots, \tag{C.9}$$

$$H_F = \varepsilon_0 + \sum_j \eta_j \sqrt{2j+1} [a^\dagger_j \times \tilde{a}_j]^{(0)}_0 + \sum_{L_F,j,j',j'',j'''} u^{(L_F)}_{jj'j''j'''} [[a^\dagger_{j''} \times a^\dagger_{j'''}]^{(L_F)}]^{(0)}_0 + \cdots, \tag{C.10}$$

$$V_{BF} = \sum_{J,l,jl'j'} w_{ljl'j'} \sqrt{2J+1} [[b^\dagger_l \times \tilde{a}_j]^{(J)} \times [\tilde{b}_l \times \tilde{a}_{j'}]^{(J)}]^{(0)}_0 + \cdots, \tag{C.11}$$

the coefficients $w_{ljl'j'}$ are the boson-fermion interaction matrix elements $w_{ljl'j'} = \langle b_l a_j; J | V_{BF} | b_{l'} a_{j'}; J \rangle$.

There are other parametrizations of the boson-fermion interactions, referred as multipole expansions,

$$V_{BF} = \sum_{L,ll'jj'} w'_{ll'jj'} (-)^L \sqrt{2L+1} [[b^\dagger_l \times \tilde{b}_{l'}]^{(L)} \times [a^\dagger_j \times \tilde{a}_{j'}]^{(L)}]^{(0)}_0 + \cdots, \tag{C.12}$$

$$w'_{ll'jj'} = -\sum_J (-)^{j+l'+J} (2J'+1) \begin{Bmatrix} l & j & J \\ j' & l' & L \end{Bmatrix} w^{(J)}_{ljl'j'} \tag{C.13}$$

or exchange expansion.

$$V_{BF} = \sum_{J,l,jl'j'} w''_{ljl'j'} (-)^J \sqrt{2J+1} : [[b^\dagger_l \times \tilde{a}_j]^{(J)} \times [\tilde{b}_{l'} \times a^\dagger_{j'}]^{(J)}]^{(0)}_0 : + \cdots, \tag{C.14}$$

$$w''_{ljl'j'} = \sum_{J'} (2J'+1) \begin{Bmatrix} l & j' & J' \\ l' & j & J \end{Bmatrix} w^{(J')}_{lj'l'j} \tag{C.15}$$

where the colons $(: \cdot :)$ means the normal ordering. Normal ordering implies that $a^\dagger_{j'}$ should stand on the left of $\tilde{a}_j$ with a minus sign.

The form of the Hamiltonian in eq (C.8) is general, now written up to two body terms of bosons b_l with angular momentum $l = 0, 2$ and an extra nucleon in one of the valence single-particle spherically symmetric orbitals with energy ε_j we obtain the fermion interaction as [74]

$$H_F = \sum_{j,m} \varepsilon_j a^\dagger_{j,m} a_{jm}, \tag{C.16}$$

with the boson interaction

$$H_B = E_0 + \varepsilon_s (s^\dagger \cdot \tilde{s}) + \varepsilon_d (d^\dagger \cdot \tilde{d}) + \sum_{L=,0,2,4} \frac{1}{2} c_L [[d^\dagger \times d^\dagger]^{(L)} \times [\tilde{d} \times \tilde{d}]^{(L)}]^{(0)}_0 \tag{C.17}$$

$$+ \frac{1}{\sqrt{2}} v_2 [[d^\dagger \times d^\dagger]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(2)} + [d^\dagger \times s^\dagger]^{(2)} \times [\tilde{d} \times \tilde{d}]^{(2)}]^{(0)}_0 \tag{C.18}$$

$$+ \frac{1}{2} v_0 [[d^\dagger \times d^\dagger]^{(0)} \times [\tilde{s} \times \tilde{s}]^{(0)} + [s^\dagger \times s^\dagger]^{(2)} \times [\tilde{d} \times \tilde{d}]^{(0)}]^{(0)}_0 \tag{C.19}$$

$$+ u_2 [[d^\dagger \times s^\dagger]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(0)}]^{(0)}_0 + \frac{1}{2} u_0 [[s^\dagger \times s^\dagger]^{(0)} \times [\tilde{s} \times \tilde{s}]^{(0)}]^{(0)}_0 \tag{C.20}$$

and the boson-fermion interaction

$$\begin{aligned} V_{BF} = &\textstyle\sum_j u_j [(s^\dagger s)^{(0)} (a^\dagger_j \tilde{a}_j)^{(0)}]^{(0)}_0 + \sum_{jj'} v_{jj'} \left\{ [(d^\dagger s)^{(2)} (a^\dagger_j \tilde{a}_{j'})^{(2)}]^{(0)}_0 + \text{h.c.} \right\} \\ &+ \textstyle\sum_{jj'k} w^k_{jj'} [(d^\dagger \tilde{d})^{(k)} (a^\dagger_j \tilde{a}_{j'})^{(k)}]^{(0)}_0, \end{aligned} \tag{C.21}$$

where $\tilde{a}_{jm} = (-1)^{j-m} a_{j-m}$.

C.1.1 Algebras in IBFM

The General linear group $GL(n, \mathbb{C})$ of complex matrices of degree n is the most significant linear matrix group, The other groups listed below are subgroups of this groups. The order of the $GL(n, \mathbb{C})$ is given by the twice the number, n^2 of matrix elements. On the other hand, the $GL(n, \mathbb{R})$ is the order n^2.

In addition to the collective degrees of freedom, in IBMF one wants to describe the single-particle degrees of freedom. In nuclei, the single particles are protons and neutrons. These are fermions. The angular momentum and parity of these particles depend on the allowed orbits.

The generators of $U(n)$ is a coupled tensor notation

$$g : A^{(\lambda)}_\mu (j, j') = [a^\dagger_j \times \tilde{a}_{j'}]^{(\lambda)}_\mu \tag{C.22}$$

and they satisfy commutation relations

$$[A^{(\lambda)}_{\mu}(j,j'),A^{(\lambda')}_{\mu'}(j'',j''')] = -\sum_{\lambda''\mu''}\sqrt{(2\lambda+1)(2\lambda'+1}\langle\lambda\mu\lambda'\mu'|\lambda''\mu''\rangle \times[(-)^{\lambda''+j+j''}\begin{Bmatrix}\lambda & \lambda' & \lambda''\\ j''' & j & j''\end{Bmatrix}\delta_{j'j'''}A^{(\lambda'')}_{\mu''}(j,j''') - (-)^{\lambda'+\lambda'+j'+j''}\begin{Bmatrix}\lambda & \lambda' & \lambda''\\ j'' & j' & j\end{Bmatrix}\delta_{jj'''}A^{(\lambda'')}_{\mu''}(j'',j'')] \tag{C.23}$$

Considering a single orbit j, the value of n is

$$n = 2j+1 \tag{C.24}$$

The operators with odd λ form a closed algebra. They generates a compact algebra $Sp(n,\mathbb{C})$ This algebra has $\frac{n(n+1)}{2}$ generators. Considering the algebra of rotation around the z-axis, $O(2)$, generated by the operator C.22, with $j' = j$ and $\Lambda = 1$ and $\mu = 0$, the chain of algebras is

$$U(2j+1) \supset SU(2j+1) \supset Sp(2j+1) \supset SU(2) \supset O(2) \tag{C.25}$$

therefore for the case of $j = \frac{1}{2}$ there are $2^2 = 4$ generators, which yield to

$$U(2) \supset Sp(2) \supset O(2) \tag{C.26}$$

for example,

$$\begin{aligned} A^{(1)}_{\mu}(\tfrac{1}{2},\tfrac{1}{2}) &= [a^{\dagger}_{\frac{1}{2}} \times \tilde{a}_{\frac{1}{2}}]^{(1)}_{\mu} \\ A^{(1)}_{\mu}(\tfrac{1}{2},\tfrac{1}{2}) &= [a^{\dagger}_{\frac{1}{2}} \times \tilde{a}_{\frac{1}{2}}]^{(0)}_{\mu} \end{aligned} \tag{C.27}$$

where the first equation makes 3 generators and the second 1 generator obtaining in total 4 generators. In similar way for the case of $j = 3/2$ there are $4^2 = 16$ operators generating the Lie algebra $U(4)$

$$U(4) \supset SU(4) \supset Sp(4) \supset Sp(2) \supset O(2) \tag{C.28}$$

and for $j = 5/2$ $6^2 = 36$ generators with

$$U(6) \supset SU(6) \supset Sp(6) \supset SU(2) \supset O(2) \tag{C.29}$$

C.1.2 Diagonalization

The hamiltonian (C.5) can be diagonalized in the full

$$U^{(\nu)}(5) \otimes U^{(\pi)}(5) \otimes U^{(F)}(m), \tag{C.30}$$

where $m = \sum_j (2j+1)$ spherical basis.

It is clear that both (C.16) and (C.21) are conveniently expressed in a coupled notation since they should be angular momentum scalars; however, H includes, in general, a large number of parameters, especially when many orbits are essential. There are many ways to treat this Hamiltonian, and one is given by the reduction of the unitary group which comes from product$U(6) \times U(m)$ for proper orbits. The basic group structure associated to IBFM Hamiltonian of a particular nucleus whose valence shell contains the sp orbits $j = 1/2$, $j = 3/2$ and $j = 5/2$ are $U^{(B)}$ and $U^{(F)}(\sum_j (2j+1) = U^{(F)}(12)$, where $U^{(B)}(6)$ is the usual boson group describing the collective excitations and $U^{(F)}(12)$ is the fermionic group associated to the sp degrees of freedom. The eigenvalue problem for the Hamiltonian of eq (C.5) can be in general be solved numerically on a basis carrying the irreducible representations of the group

$$U^{(B)}(6) \times U^{(F)}(12) \tag{C.31}$$

The mixing of the boson and fermion degrees can be solved analytically, for example, considering the sp orbits $j = 1/2, 3/2$ and $5/2$ as arising from the combination of pseudo-orbital parts $l = 0, 2$ and pseudo-spin part with $s = 1/2$. That corresponds to the reduction

$$U^{(F)}(12) \supset U^{(B)}(6) \times SU^{(F)}(2) \tag{C.32}$$

This scheme can be extended, by embedding the direct product group into a larger group, the Hamiltonian with this larger symmetry will be mapped onto a graded space of bosons and fermions and consequently can be treated with supergroups. In our case $U(6/12)$ with different subgroups chains that were well studied in the literature [75].

An alternative way to solve the equation by considering restricted Hamiltonians with only a few terms, dictated by microscopic considerations [17] that we will consider in the next section.

C.2 Effective Interacting Boson-Fermion model

There are several other equivalent ways of writing the Hamiltonian. One form often used is write the Hamiltonian in terms of elements of multipole expansions. It is convenient to write a Hamiltonian within a space of the fixed number of bosons N_B, the number of independent terms is reduced. For example let us consider consider first component the bosonic situation , since the total number of bosons N_B is the the the sum of number of s-bosons and d-bosons denoted by n_s and n_d therefore the term $\varepsilon_s(s^\dagger \cdot \tilde{s})$ is equal to

$$\varepsilon_s \hat{n}_s = \varepsilon_s(\hat{N} - \hat{n}_d) = \varepsilon_s \hat{N} - \varepsilon_s(d^\dagger \cdot \tilde{d}) \tag{C.33}$$

so following this idea we can use a parametrizations given by

$$\begin{aligned}
\hat{n}_d &= (d^\dagger \cdot \tilde{d}), \\
\hat{L} &= \sqrt{10}[d^\dagger \times \tilde{d}]^{(1)} \\
\hat{Q}^\chi &= [d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d}]^{(2)} + \chi[d^\dagger \times \tilde{d}]^{(2)} \\
\hat{U} &= [d^{\dagger \times \tilde{d}}]^{(3)} \\
\hat{V} &= [d \times \tilde{d}]^{(4)}
\end{aligned} \tag{C.34}$$

therefore the bosonic Hamiltonian can be written as

$$H_B = E_0' + \varepsilon \hat{n}_d + c_1(\hat{L} \cdot \hat{L}) + c_2(\hat{Q}^\chi \cdot \hat{Q}^\chi) + c_3(\hat{U} \cdot \hat{U}) + c_4(\hat{V} \cdot \hat{V}) \tag{C.35}$$

and the transition operators initially can be introduced for the bosonic part is given by

$$\begin{aligned}
T^{(E0)} &= \gamma_0' + \beta_0'' \hat{n}_d \\
T^{(M1)} &= g' \hat{L} \\
T^{(M2)} &= \alpha_2 \hat{Q}^\chi \\
T^{(M3)} &= \beta_3 \hat{U} \\
T^{(E4)} &= \beta_4 \hat{V}
\end{aligned} \tag{C.36}$$

and the transfer operators for the only bosonic case if one retains only the one-body operators can be written

$$\begin{aligned}
P^L_{B+} &= \sum_\alpha p_\alpha b^\dagger_\alpha \\
P^L_{B-} &= \sum_\alpha p_\alpha b_\alpha
\end{aligned} \tag{C.37}$$

For practical applications like computation of transitions of beta-decay , EC or even more complex situations, the operators (C.36) and (C.37) are the main ingredients that will be extended in real situations where is needed the fermion degrees of freedom and more complex interactions coming from microscopic theories.

We can introduce the quasi-particles formalism to deal the extra fermions coupled to the bosons which can be associated with states of single particles in the shell model. The fact that the bosons in the interacting boson model can be regarded as a collective fermion pair states, a natural link between IBM and the shell model can be established through the Generalized Seniority scheme (GS). A generalization of the seniority scheme to the case of non-degenerate orbits it is introduced a pair of creation operators

$$S^\dagger = \sum \alpha_j S^\dagger_j \tag{C.38}$$

where

$$S^\dagger_j = \frac{1}{2}\sqrt{2j+1}(c^\dagger_j c^\dagger_j)^{(0)}. \tag{C.39}$$

where this operator creates a collective pair with $J = 0$, where its Hermitian conjugate S^- their commutator S_0 are the generators of a $SU(2)$ lie algebra, making the

seniority scheme easy to apply since reduction formulas for matrix elements can be performed.

A state with generalized seniority $\nu = 0$ and $n = 2N$ particles can be expressed as

$$|n, J = 0, \nu = 0\rangle = (S^{\dagger})^{N}|0\rangle \tag{C.40}$$

and to describe an excited 2^{+} state one introduces the operator

$$D^{\dagger} = \sum_{j,j'} \frac{1}{2} \beta_{j,j'} \sqrt{1 + \delta_{jj'}} (c_j^{\dagger} c_{j'}^{\dagger})^{(2)} \tag{C.41}$$

which creates a collective state with $J = 2$ and $\nu = 2$, where the state with $n = 2N$ particles and $J = 2$ can be constructed by operating with (C.41)

$$|n, J = 1, \nu = 1\rangle = D^{\dagger}(S^{\dagger})^{N-1}|0\rangle \tag{C.42}$$

where the structure coefficients α_j and β_{ij} of collective pairs can be obtained by diagonalizing the shell model interaction in the space of all seniority $\nu = 2$ states.

The description of the odd- mass nuclei can be realized by introducing explicitly single particle degrees of freedom, in order to construct an orthogonal basis , however should be introduced the odd nucleon operator $a_j^{\dagger}$ as generalized seniority raising operators,

$$a_j^{\dagger}|s^N\rangle = |js^N\rangle \leftrightarrow |n = 2N + 1, J = j, \nu = 1\rangle, \tag{C.43}$$

and,

$$(a_j^{\dagger} d^{\dagger})^{(J)}|s^{N-1}\rangle = |(jd)^{(J)} s^{N-1}\rangle \leftrightarrow |n = 2N + 1, J, \nu = 3\rangle \tag{C.44}$$

Since the odd-particle operator $a_j^{\dagger}$ is defined as a seniority raising operator, its matrix elements will, in general, be different from those of a shell-model single nucleon creation operator $c_j^{\dagger}$.

Let us introduce the pseudoparticle creation operators [76] and the number operator approximation (NOA). The pseudo particle operator $\check{c}_j^{\dagger}$ is defined as the equivalent of the single-particle operator $c_j^{\dagger}$ in the shell model space. The pseudo particle operator $\check{c}_j^{\dagger}$ should be distinguished from the odd nucleon operator $a_j^{\dagger}$ in sense that the latter, should be regarded as a seniority step operator. In order to obtain an expression for the pseudo particle operator , matrix elements have to be calculated in the shell-model space for stares with generalized seniority $\nu \leqslant 3$. In the NOA, the coefficients α_j which enter in the definition of S pair operator of the scheme of generalized seniority are normalized such

$$\hat{n} = \sum_j \alpha^2 \sum_m c_{jm}^{\dagger} c_{jm} = \sum_j \alpha_j^2 \hat{n}_j \tag{C.45}$$

, therefore in this approximation the matrix element of the number operator is simple twice of the number of pairs ,

$$\langle S^N|\hat{n}|S^N\rangle = 2N \tag{C.46}$$

In this approximation the algebra for GS operators becomes equal to that of conventional seniority therefore a normalization constants can be derived [77] for S states,

$$\begin{aligned}\eta^2_{N,0} &= \langle 0|S^N S^{\dagger N}|0\rangle \approx N!\frac{\Gamma(\Omega_e+1)}{\Gamma(\Omega_e-N+1)}\\ \eta^2_{N,1} &= \langle j|S^N S^{\dagger N}|j\rangle \approx \eta^2_{N,0}(1-\frac{N\alpha_j^2}{\Omega_e})\end{aligned} \tag{C.47}$$

where the effective degeneracy for the major shell is defined

$$\Omega_e = \sum_j \alpha_j^2 \Omega_j \tag{C.48}$$

It is convenient to introduce the occupation probabilities

$$v^2 \equiv \frac{n_j}{2j+1} \tag{C.49}$$

where

$$n_j = \langle S^N|\hat{n}_j|S^N\rangle \approx 2N\alpha_j^2\Omega_j/\Omega_e \tag{C.50}$$

the spherical shell model occupancies can be related to the structure coefficients of the S- pair state α_j as

$$v_j^2 = \alpha_j^2 N/\Omega_e \tag{C.51}$$

Since the spherical shell-model occupation probabilities v_j^2have s simpler physical interpretation in the subsequent formulas all factors α_j are replaced by v_j and $u_j = (1-v_j^2)^{1/2}$.

Therefore we can make a relationship between matrix elements of the shell model creation operators $c_j^\dagger$ and IBFM operators $a_j^\dagger$ between states with seniority $\nu \leqslant 1$ as,

$$\begin{aligned}\langle S^N j'\|c_j^\dagger\|S^N\rangle &= -\hat{j}u_j\delta_{jj'} = u_j\langle s^N j'\|a_j^\dagger\|s^N\rangle\\ \langle S^N\|c_j^\dagger\|S^N j'\rangle &= \hat{j}v_j\delta_{jj'} = v_j\langle s^N\|(a_j^\dagger\tilde{a}_j)^{(j)}\|s^{N-1}j'\rangle/\sqrt{N}\end{aligned} \tag{C.52}$$

Similar expression can be also obtained if $\nu \leqslant 3$ states [78] like,

$$\begin{aligned}\langle D^\dagger S^{N-1} j'\|\check{c}_j^\dagger\|D^\dagger S^{N-1}\rangle &= u_j\langle d^\dagger s^{N-1}j'\|a_j^\dagger\|d^\dagger s^{N-1}\rangle\\ &- \textstyle\sum_{j'}\frac{v_j}{\sqrt{N}}\sqrt{\frac{10}{2j+1}}\beta_{j'j}(\frac{1}{\eta_\beta})\langle d^\dagger s^{N-1}j'\|s^\dagger(\tilde{d}a_j^\dagger)^{(J)}\|d^\dagger s^{N-1}\rangle\\ &+ \textstyle\frac{v_j}{\sqrt{N}}\langle d^\dagger s^{N-1}j'\|(s^\dagger\tilde{a}_j)^{(j)}\|d^\dagger s^{N-1}\rangle\\ &+ \textstyle\sum_{j'}u_j\sqrt{\frac{10}{2j+1}}\beta_{jj'}(\frac{1}{\eta_\beta})\langle d^\dagger s^{N-1}j'\|(d^\dagger\tilde{a}_{j'})^{(j)}\|d^\dagger s^{N-1}\rangle\end{aligned} \tag{C.53}$$

with

$$\eta_\beta^2 = \sum_{jj'}\beta_{jj'}^2 \tag{C.54}$$

where the coefficients $\beta_{jj'}$ define the microscopic two particle structure of d-boson coming from eqn. (C.41).

The coefficients α_j and $\beta_{jj'}$ can, in principle, be obtained from microscopic calculations.

Assuming that D-pair state exhausts the full E2 sum rule strength, $D^\dagger \sim [Q, S^\dagger]$ in which case it cab be shown that

$$\beta_{jj'} = (u_j v_{j'} + v_j u_{j'}) Q_{jj'}, \tag{C.55}$$

where the $Q_{jj'}$ are the single particle matrix elements of the quadrupole operator. In most of the applications in nuclear structure, is convenient takes the radial matrix elements equal to the quadruple operator,

$$Q_{jj'} = \langle l \frac{1}{2} j \| Y^{()2} \| l \frac{1}{2} j' \rangle \tag{C.56}$$

Returning to the description of the boson-fermion Hamiltonian it is convenient to write it regarding monopole, quadrupole, and exchange interactions, the monopole interaction

$$V_{BF} = V_{BF}^{mon} + V_{BF}^{quad} + V_{BF}^{exc}, \tag{C.57}$$

the monopole interaction has the effect of changing the d-boson energy given by

$$V_{BF}^{mon} = \sum_j A_j \hat{n}_d \hat{n}_j, \tag{C.58}$$

the product of the boson quadrupole operator with the pure fermion part gives rise the boson-fermion quadrupole interaction given by,

$$V_{BF}^{quad} = \sum_{jj'} \Gamma_{jj'} [a_j^\dagger \tilde{a}_{j'}]^{(2)} \tag{C.59}$$

where

$$\Gamma_{jj'} = \Gamma_0 (u_j u_{j'} - v_j v_{j'}) Q_{ij} \tag{C.60}$$

Retaining only term up to second order in the d-boson operators

$$Q_{BF}^{(2)} = Q_B^{(2)} + Q_F^2 \tag{C.61}$$

where

$$Q_B^{(2)} = (s^\dagger \tilde{d} + d^\dagger s)^{(2)} + \chi (d^\dagger \tilde{d})^{(2)} \tag{C.62}$$

and

$$\begin{aligned} Q_F^{(2)} = & \sum_{jj'} Q_{jj'} (u_j u_{j'} - v_j v_{j'}) (a_j^\dagger \tilde{a}_{j'})^{(2)} \\ & - \sqrt{\tfrac{10}{N}} \sum_{jj'j''} Q_{jj'} (u_j v_{j'} + v_j u_{j'}) \beta_{j''j} [(d^\dagger \tilde{a}_{j''})^{(j)} (s a_{j'}^\dagger)^{(j')}]^{(2)} \frac{1}{\hat{j} \eta_\beta} \end{aligned} \tag{C.63}$$

where χ gives the nuclear shape given by

$$\chi = -10\frac{1}{\eta_\beta^3}\sqrt{2N}\sum_{jj'}\beta_{j''j}\beta_{j'j''}Q_{jj'}\begin{Bmatrix} 2 & 2 & 2 \\ j & j' & j'' \end{Bmatrix}(v_j v_{j'} - u_j u_{j'}) \tag{C.64}$$

the mixed term in the quadrupole arises from the fact that the bosons are built up from fermions occupying the same orbits as the odd fermion to raise the exchange term. It appears from the action of the Pauli principle between the bosons and the odd fermion and therefore is referred to as the exchange interaction up to

$$V_{BF}^{exc} = \sum_{jj'j''}\Lambda_{jj'}^{j''} : [[d^\dagger \tilde{a}_j]^{(j'')} \times (\tilde{d}a_{j'}^\dagger)^{(j'')}]_0^{(0)} : /\hat{j}'' \tag{C.65}$$

where

$$\Lambda_{jj'}^{j''} = -2\sqrt{5}\Lambda_0\beta_{jj''}\beta_{j'j''} \tag{C.66}$$

C.3 Interacting boson-fermion model with two types of fermion

In the interacting boson model with two types of fermion, collective excitations of nuclei are described by bosons and two fermions. Second quantization provides an appropriate formalism to describe this kind of system. One thus introduces boson creation(and annihilation) operators for protons and neutrons $b_{\rho,l,m}^\dagger$ and $b_{\rho,l,m}$, with $\rho = \pi, \nu$ and $l = 0, 2$, with $-l \leqslant m \leqslant l$. fermion creation(and annihilation) operators for the fermions $a_{\rho,j,m}^\dagger$ and $a_{\rho,j,m}$, with $\rho = \pi, \nu$ and $j = j_1, j_2, \ldots, j_n$, with $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm j$.

As in the odd-even nucleus, the boson operators satisfy the bose commutation relations and the fermion operators satisfy Fermi anticommutation relations. The single particle levels here are denoted by nlj with n being the principal quantum number, l, the orbital angular momentum and j the total angular momentum $j = l \pm \frac{1}{2}$. The approximation what we will consider is a truncation assuming that the closed shells are inert and second by considering parity $J^{P+} = 0^+$ and $J^{P+} = 2^+$. In odd-even nuclei at least one particle remains unpaired, this particle could be proton on neutron.

The basis states are written in terms of the fermion and boson operators, the angular momentum couplings are chosen in such a way that bosons and fermions are first coupled among themselves, followed by the final coupling,

$$\begin{array}{c}[[[a_{\pi,j}^\dagger \times a_{\pi,j'}^\dagger \times \cdots]^{(J_\pi)} \times [a_{\nu,j}^\dagger \times a_{\nu,j'}^\dagger \times \cdots]^{(J_\nu)}]^{(L_F)} \\ \times [b_{\pi,l}^\dagger \times b_{\pi,l'}^\dagger \times \cdots]^{(L_\pi)} \times [b_{\nu,l}^\dagger \times b_{\nu,l'}^\dagger \times \cdots]^{(L_\nu)}]^{(L_B)}|0\rangle\end{array} \tag{C.67}$$

The Hamiltonian (C.5) can be extended considering the two types of fermions,

$$\begin{aligned} H_B &= H_{\pi B} + H_{\nu B} + V_{\pi\nu B} \\ H_F &= H_{\pi F} + H_{\nu F} + V_{\pi\nu F} \\ V_{BF} &= V_{\pi\pi BF} + V_{\pi\nu BF} + V_{\nu\pi BF} + V_{\nu\nu BF} \end{aligned} \tag{C.68}$$

The bosons and fermion Hamiltonians that we are going to consider contains the basic features of the effective nucleon-nucleon interaction, that emerge from pairing, quadrupole and symmetry energy.

$$\begin{aligned}
H_B &= E_0+\varepsilon_\pi\hat{n}_{d_\pi}+\varepsilon_\nu\hat{n}_{d_\nu}+\kappa\hat{Q}^\chi_\pi\cdot\hat{Q}^\chi_\nu+\lambda'\hat{M}_{\pi\nu}+V_{\pi\pi}+V_{\nu\nu}\\
H_F &= E_0+\sum_{j_\pi}\varepsilon_{j_\pi}\hat{n}_{j_\pi}+\sum_{j_\nu}\varepsilon_{j_\nu}\hat{n}_{j_\nu}\\
V_{BF} &= \sum_{j_\pi}A_{j\pi}(\hat{n}_{d_\pi}\hat{n}_{j_\pi})+\sum_{j_\nu}A_{j\nu}(\hat{n}_{d_\nu}\hat{n}_{j_\nu})\\
&\quad+\Gamma_{\pi\nu}\hat{Q}^\chi_\nu\cdot\hat{q}_\pi++\Gamma_{\nu\pi}\hat{Q}^\chi_\pi\cdot\hat{q}_\nu+\Gamma_{\nu\nu}\hat{Q}^\chi_\nu\cdot\hat{q}_\nu+\Gamma_{\pi\pi}\hat{Q}^\chi_\pi\cdot\hat{q}_\pi\\
&\quad+\Lambda_{\nu\pi}F_{\pi\nu}+\Lambda_{\pi\nu}F_{\nu\pi}
\end{aligned}\tag{C.69}$$

where

$$\begin{aligned}
\hat{n}_{d_\rho} &= \textstyle\sum_\mu d^\dagger_{\rho,\mu}d_{\rho,\mu}\\
\hat{Q}^\chi_{\rho,\mu} &= [s^\dagger_\rho\times\tilde{d}_\rho+d^\dagger_\rho\times\tilde{s}_\rho]^{(2)}_\mu+\chi_\rho[d^\dagger_\rho\times\tilde{d}_\rho]^{(2)}_\mu\\
\hat{M}_{\pi\nu} &= [s^\dagger_\nu\times\ d^\dagger_\pi-s^\dagger_\pi\times d^\dagger_\nu]^{(2)}\cdot[\tilde{s}_\nu\times\tilde{d}_\pi-\tilde{s}_\pi\times\tilde{d}_\nu]^{(2)}\\
&\quad-2\sum_{k=1,3}\xi_k[d^\dagger_\nu\times d^\dagger_\pi]^{(k)}\cdot[\tilde{d}_\nu\times\tilde{d}_\pi]^{(k)}\\
V_{\rho\rho} &= \sum_{L=0,2,4}\frac{1}{2}c^{(\rho)}_L[d^\dagger_\rho\times d^\dagger_\rho]^{(L)}\cdot[\tilde{d}_\rho\times\tilde{d}_\rho]^{(L)}\\
&\quad+\tfrac{1}{2}v^\rho_0\left[\left((d^\dagger_\rho d^\dagger_\rho)^{(0)}(s_\rho s_\rho)^{(0)}\right)^{(0)}+h.c\right]\\
&\quad+\sqrt{\tfrac{5}{2}}v^\rho_2\left[\left((d^\dagger_\rho d^\dagger_\rho)^{(2)}(\tilde{d}_\rho s_\rho)^{(2)}\right)^{(0)}+h.c\right]\\
\hat{n}_{j_\rho} &= \sum_{m_\rho}a^\dagger_{\rho,j_\rho,m_\rho}a_{\rho,j_\rho,m_\rho}\\
\hat{q}_\rho &= \sum_{k,k'}(u_{j_k}v_{j_{k'}}+v_{j_k}u_{j_{k'}})Q_{j_kj_{k'}}[a^\dagger_{j_k}\times\tilde{a}_{j_{k'}}]^{(2)}_\mu\\
F_{\rho,\rho'} &= -\hat{Q}^\chi_\rho\cdot\{\textstyle\sum_{kk'k''}Q_{j_kj_{k'}}(u_{j_k}v_{j_{k'}}+v_{j_k}u_{j_{k'}})\beta_{j_{k''}j_k}\\
&\quad\sqrt{\tfrac{10}{N_{\rho'}(2j_k+1}}:[(d_{\rho'}\tilde{a}_{j_{k''}})^{(j_k)}(a^\dagger_{j_{k'}}s_{\rho'})^{(j_{k'})}]^{(2)}:+h.c.\}
\end{aligned}\tag{C.70}$$

where the coefficients v_{j_k} are the occupation probabilities of the single- particle orbit j_k and $u_{j_k}=\sqrt{1-v^2_{j_k}}$. The coefficients $Q_{j_kj_{k'}}$ are related to the single particle element of the quadrupole operator $\langle n_kl_k\frac{1}{2}j_k\|r^2Y^{(2)}\|n_{k'}l_{k'}\frac{1}{2}j_{k'}\rangle$, which we choose

$$Q_{j_kj_{k'}}=\langle j_k\|Y^{(2)}\|j_{k'}\rangle.\tag{C.71}$$

As we discussed in the previous section the coefficients $\beta_{j_kj_{k'}}$ can be related to the microscopic structure of the d-boson. Under the assumption that the $|D\rangle$ state absorbs the full $E2$-strenght , based upon a completely degenerate single particle level scheme, $\beta_{j_kj_{k'}}=(u_{j_k}v_{j_{k'}}+v_{j_k}u_{j_{k'}})Q_{j_kj_{k'}}$, however the effect of the non-degenerate single particle orbits can be taken into account by an energy

$$\beta_{j_k j_{k'}} = \frac{1}{E_{j_k} + E_{j_{k'}} - w}(u_{j_k} v_{j_{k'}} + v_{j_k} u_{j_{k'}}) Q_{j_k j_{k'}} \tag{C.72}$$

where the w denotes the energy of the $|D\rangle$ state relative to the $|S\rangle$ state and can be obtained from the excitation energy of the 2_1^+ state in a semi magic nucleus. The quasi energies occupancies v_{j_k} and u_{j_k} can be computed using the BCS approximation,

$$\begin{aligned} E_{j_k} &= [(\varepsilon_{j_k - \lambda)^2}) + \triangle^2]^{\frac{1}{2}} \\ V_{j_k} &= [\tfrac{1}{2}(1 - \tfrac{\varepsilon_{j_k} - \lambda}{E_{j_k}})]^{\frac{1}{2}} \\ u_{j_k} &= [1 - v_{j_k}^2]^{\frac{1}{2}} \end{aligned} \tag{C.73}$$

The full Hamiltonian (C.5) is diagonalized in the truncated basis

$$|(\alpha_c L_c, j) JM\rangle. \tag{C.74}$$

By construction H_B and H_F are diagonal in the basis (C.109). The matrix elements of the boson-fermion interaction

$$V_{BF} = \sum_{\lambda} \sum_{ik} v_{ik}^{(\lambda)} B_i^{(\lambda)} \cdot F_k^{(\lambda)} \tag{C.75}$$

can be separated into a boson and a fermion part

$$\begin{aligned} &\langle (\alpha_c, L_c, j) JM | V_{BF} | (\alpha_c', L_c', j') JM \rangle \\ &= \sum_{\lambda} \sum_{i,k} v_{ij}^{(\lambda)} (-1)^{j+J+L_c'} \begin{Bmatrix} L_c & j, & J \\ j' & L_c' & \lambda \end{Bmatrix} \\ &\times \langle (\alpha_c, L_c) \| B_{(\lambda)_i} \| (\alpha_c', L_c') \rangle \langle (j \| F_k^{(\lambda)} \| j' \rangle \end{aligned} \tag{C.76}$$

The boson matrix elements $\langle (\alpha_c, L_c) \| B_{(\lambda)_i} \| (\alpha_c', L_c') \rangle$ only depend on the eigenvectors of the core Hamiltonian and therefore can be computed separately.

C.4 Effective Operators

In order study the nuclear properties and transitions like electromagnetic transitions, beta decay, double beta decay with or without neutrino, electron capture, transfer reactions, charge exchange reactions, double charge exchange, up to weakly interacting massive particles scattering in our nuclear algebraic model we require first at all, have a realistic wave function that is obtained by the diagonalization of the Hamiltonian in the Eq. (C.5), and later we need take as starting point the operators (C.36) and (C.37) and rewriting them to our basis that we are considering. There are some standard procedures that provide some hints about how we should compute those observables that we are interested in predicting and comparing with the experimental data. For example, if we are dealing with even-even nuclei, sometimes

is enough only work with the bosonic part of the Hamiltonian and also for the operators, however, there are situations that are needed deal with the fermionic degrees of freedom and we require the fermionic and bosonic-fermionic interaction. In addition, if we want to be more precisely for certain types of transitions is not enough only include microscopic information about the nuclear interactions also is needed includes more elemental information about the interactions such as weak interactions, or even more complex interactions where are needed include effective potentials, that can be obtained for example by effective field models to include more explicitly this microscopic dependence in the construction of the operators of the interactions. To begin our discussion about the effective operators in the interacting boson-fermion model let us consider first the transition operators.

C.4.1 Electromagnetic Transitions

The transition operator can be described in terms of proton and neutron bosons and fermions,

$$T^{(L)}_{\mu} = T^{(L)}_{\pi B,\mu} + T^{(L)}_{\nu B,\mu} + T^{(L)}_{\pi F,\mu} + T^{(L)}_{\nu F,\mu} \tag{C.77}$$

where boson terms are given by the equation (C.36) and the fermion terms, to the lowest order can be written as:

$$\begin{aligned} T^{(L)}_{\pi F\mu} &= f^{(0)}_{\pi,0}\delta_{L,0} + \sum_{j_\pi j'_\pi} f^{(L)}_{j_\pi j'_\pi}[a^{\dagger}_{j_\pi} \times \tilde{a}_{j'_\pi}]^{(L)}_{\mu} \\ T^{(L)}_{\nu F\mu} &= f^{(0)}_{\nu,0}\delta_{L,0} + \sum_{j_\nu j'_\nu} f^{(L)}_{j_\nu j'_\nu}[a^{\dagger}_{j_\nu} \times \tilde{a}_{j'_\nu}]^{(L)}_{\mu} \end{aligned} \tag{C.78}$$

We can separate the dependence on the angular momenta j_π and j_ν from the coefficients that determine the strengths of the transitions. For that reason it is introduced an effective charge and moments, for the $E2$ transition,

$$f^{(2)}_{j_\rho j'_\rho} = -e^F_\rho \langle n_\rho, l_\rho | r^2 | n'_\rho, l'_\rho \rangle \langle l_\rho, \frac{1}{2}, j_\rho \| Y^{(2)} \| l'_\rho, \frac{1}{2}, j'_\rho \rangle, \tag{C.79}$$

where e^F_ρ are the fermion effective charges. The units given in electron charge, of e^B_ρ are different from e^F_ρ since the radial integral is already included in (C.79). The boson effective charge e^B_ρ have units of $e\,fm^2$,

$$e'^F_\rho = e^F_\rho \langle n_\rho, l_\rho | r^2 | n'_\rho, l'_\rho \rangle \tag{C.80}$$

and for M1 transitions, the fermion part of the operators is written in the form,

$$f^{(1)}_{j_\rho j'_\rho} = -\sqrt{\frac{3}{4\pi}} \langle l_\rho, \frac{1}{2}, j_\rho \| g^F_{l,\rho}\mathbf{l} + g^F_{s,\rho}\mathbf{s} \| l'_\rho, \frac{1}{2}, j'_\rho \rangle \delta_{l_\rho l_{\rho'}} / \sqrt{3} \tag{C.81}$$

the $g^F_{l,\pi}$, $g^F_{l,\nu}$, $g^F_{s,\pi}$, $g^F_{s,\nu}$, are single-particle g-factors.The free values are $g^F_{l,\pi}=1$, $g^F_{l,\nu}=0$ $g^F_{s,\nu}=5.58$, $g^F_{s,\nu}=-3.82$ units of nuclear magnetons μ_N. Therefore summarizing the general one-body $E2$ electromagnetic operator written more compactly is given by

$$\begin{aligned} T(E2) = & e_\nu[\alpha_\nu(s^\dagger_\nu\tilde{d}_\nu + d^\dagger s_\nu)^{(2)} + \chi^{(2)}_\nu(d^\dagger_\nu\tilde{d}_\nu)^{(2)}] + \\ & + e_\pi[\alpha_\pi(s^\dagger_\nu\tilde{d}_\pi + d^\dagger s_\pi)^{(2)} + \chi^{(2)}_\pi(d^\dagger_\pi\tilde{d}_\pi)^{(2)}] \\ & + e^{(2)}_{sp}\sum_{k\leqslant k'} e^{(2)}(N_{kk'})[(a^\dagger_{j_k}\tilde{a}_{j'_k}) + h.c] \end{aligned} \tag{C.82}$$

with $N_{kk'} = \frac{1}{2}k'(k'-1)+k, k\leqslant k'$ and for multipolarities E0, M1, M3 and M4 the operator is given by

$$\begin{aligned} T(\lambda) = & \chi^{(\lambda)}_\nu(d^\dagger\tilde{d}_\nu)^{(\lambda)} + \chi^{(\lambda)}_\pi(d^\dagger\tilde{d}_\pi)^{(\lambda)} \\ & + e^{(\lambda)}_{sp}\sum_{k\leqslant k'} e^{(\lambda)}(N_{kk'})[(a^\dagger_{j_k}\tilde{a}_{j'_k}) + h.c] \end{aligned} \tag{C.83}$$

with $\lambda = 0,1,3,4$ where the bososn effective charges e_ν and e_π can be taken from a study of absolute $B(EM\lambda)$ in the core nucleus. The coefficients $e^{(\lambda)}(N_{kk'})$ are the single particle matrix elements of the transition operators, and for the $E2$ case given by

$$e^{(2)}(N_{kk'}) = -\frac{1}{\sqrt{5}}(u_{j_k}v_{j_{k'}} - v_{j_k}u_{j_{k'}})\langle l_k, \frac{1}{2}, j_k \| Y^{(2)} \| l'_k, \frac{1}{2}, j'_k\rangle \tag{C.84}$$

and for $M1$ transitions by

$$e^{(1)}(N_{kk'}) = -\frac{1}{\sqrt{3}}(u_{j_k}v_{j_{k'}} + v_{j_k}u_{j_{k'}})\langle l_k, \frac{1}{2}, j_k \| g^F_{l,k}\mathbf{l} + g^F_{s,k}\mathbf{s} \| l'_k, \frac{1}{2}, j'_k\rangle \tag{C.85}$$

C.4.2 Transfer operators

There are two types of one-nucleon transfer operators, those that change the boson number by unit and those that do not. The transfer operators of the first kind can be expanded in terms of creation an annhihilation operators,

$$P^+_{j_\rho} = \theta_{j_\rho}[s^\dagger_\rho \times \tilde{a}_{j\rho}]^{(j_\rho)} + \sum_{j'_\rho} q^{j_\rho j'_\rho}_{\rho'}[d_{\rho^\dagger} \times \tilde{a}_{j\rho'}] + ... \tag{C.86}$$

this kind of operators the number of bosons ins conserved $(N_\rho, N_\rho, j \leftrightarrow N_\rho + 1, N_\sigma, \rho \neq \sigma)$ and the reactions where the number of bosons is conserved $(N_\nu, N_\pi \leftrightarrow N_\nu, N_\pi, j)$

$$\begin{aligned} P^{+}_{j_\rho} &= \xi_{j_\rho} a^{\dagger}_{j_\rho} + \sum_{j'_\rho} \xi_{j_\rho j'_\rho} [[s^{\dagger} \times \tilde{d}_{j_\rho}]^{(2)} \times a_{j'_\rho}]^{(j_\rho)} \\ &+ \sum_{j'_\rho} \xi'_{j_\rho j'_\rho} [[s^{\dagger} \times \tilde{d}_{j_\rho}]^{(2)} \times a_{j'_\rho}]^{(j_\rho)} \\ &+ \sum_{j'_\rho k} \xi''_{j_\rho j'_\rho k} [[d^{\dagger}_{j_\rho} \times \tilde{d}_{j_\rho}]^{(k)} \times a_{j'_\rho}]^{(j_\rho)} + \dots \end{aligned} \tag{C.87}$$

and for practical computations between even-even and odd even nuclei as a first order we take only the first and second summation, in this case, we can related the coefficients θ and ξ in eqs. (C.86) and (C.87) to the occupation probabilities of the single particle orbits like

$$\begin{aligned} \xi_j &= u_j b_1 \\ \xi_{jj'} &= v_j \beta_{jj'} \sqrt{\tfrac{10}{N_\rho (2j+1}} b_2 \\ \theta'_j &= \tfrac{v_j}{\sqrt{N_\rho + 1}} b_1 \\ \theta_{jj'} &= u_j \beta_{j'j} \sqrt{\tfrac{10}{2j+1}} b_2 \end{aligned} \tag{C.88}$$

in the eq. (C.88) we have assumed that the odd-even nucleus is described as particle coupled to an even-even core nucleus. For a hole coupled to the core the coefficients are given by

$$\begin{aligned} \xi_j &= v_j b_1 \\ \xi_{jj'} &= -u_j \beta_{j'j} \sqrt{\tfrac{10}{N_\rho (2j+1}} b_2 \\ \theta'_j &= \tfrac{u_j}{\sqrt{N_\rho + 1}} b_1 \\ \theta_{jj'} &= v_j \beta_{j'j} \sqrt{\tfrac{10}{2j+1}} b_2 \end{aligned} \tag{C.89}$$

the coefficientes b_1 and b_2 can in principle be determined from a the semi-microscopic derivation, the ratio of b_2 and b_1 given by $\frac{b_2}{b_1} = \frac{1}{K_\beta}$ with $K_\beta = (\sum_{jj'} \beta^2_{jj'})^{(1/2)}$ and in addition the remaining coefficient b_1 can be used to satisfy the sum rules for and for particle-coupling

$$\begin{aligned} &\textstyle\sum_{\text{odd}} \langle \text{odd}, N = N_\nu + N_\pi | P^{+}_j | \text{even}, N = N_\nu + N_{pi} \rangle^2 = (2j+1) u^2_j \\ &\textstyle\sum_{\text{odd}} \langle \text{even}, N = N_\nu + N_\pi | P^{+}_j | \text{odd}, N = N_\nu + N_{pi} \rangle^2 = (2j+1) v^2_j \end{aligned} \tag{C.90}$$

and for hole coupling

$$\begin{aligned} &\textstyle\sum_{\text{odd}} \langle \text{odd}, N = N_\nu + N_\pi | P^{+}_j | \text{even}, N = N_\nu + N_{pi} \rangle^2 = (2j+1) v^2_j \\ &\textstyle\sum_{\text{odd}} \langle \text{even}, N = N_\nu + N_\pi | P^{+}_j | \text{odd}, N = N_\nu + N_{pi} \rangle^2 = (2j+1) u^2_j \end{aligned} \tag{C.91}$$

C.4.3 Beta decay operators

In the beta decay a proton is transformed into a neutron (or viceversa) with emsion of an electron and an antineutrino. In the ground state of an even-even nucleus where all particles are paired, one must break a pair to have a beta-decay process.

Operators connecting even-even and odd-odd nuclei are a bit more complex. The most straightforward case when the decay of odd-even nuclei through the conversion of the unpaired particle from proton to a neutron or from neutron to proton. The two main contributions of this process are given by the Fermi decay with no change of angular momentum and Gamow-Teller decay with a change of one unit of angular momentum. For the case of β^- decay, the operators as a first approach has the form

$$\begin{aligned} \hat{O}_F &= \sum_{j_\pi j_\nu} \eta^F_{j_\pi j_\nu} [P^+_{j_\pi} \times \tilde{P}_{j_\nu}]^{(0)} \\ \hat{O}_{GT} &= \sum_{j_\pi j_\nu} \eta^{GT}_{j_\pi j_\nu} [P^+_{j_\pi} \times \tilde{P}_{j_\nu}]^{(1)}_\mu \end{aligned} \tag{C.92}$$

where the coefficients $\eta^F_{j_\pi j_\nu}$ and $\eta^{GT}_{j_\pi j_\nu}$depend on the form of the beta-decay operator. If the beta decay takes place between proton and neutron orbits with the same orbital quantum number, the lowest order term of the decay operator are given by

$$\begin{aligned} \eta^F_{j_\pi j_\nu} &= -\sqrt{2j_\pi + 1}\delta_{j_\pi j_\nu} \\ \eta^{GT}_{j_\pi j_\nu} &= (-1)^{l_\pi + j_\pi - \frac{3}{2}} \sqrt{\tfrac{(2j_\pi+1)(2j_\nu+1)}{2}} \begin{Bmatrix} 1/2 & 1/2 & 1 \\ j_\pi & j_\nu & l_{pi} \end{Bmatrix} \end{aligned} \tag{C.93}$$

in the lowest order transitions between orbits with different orbital quantum numbers ($l_\pi \neq l_\nu$) are forbidden, in the case non-zero beta decay from higher order terms involves radial integrals which contains microscopic information about the interaction.

C.5 Interacting boson-fermion-fermion with two types of fermions

The Hamiltonian of the extension of the IBFM to consider odd-odd nuclei may be written like

$$H = H^B + H^F_\pi + V^{BF}_\pi + H^F_\nu + V^{BF}_\nu + V_{RES}, \tag{C.94}$$

the boson and fermion Hamiltonian parameter are those given in the previous sections. The last term is the residual interaction between the odd proton and odd neutron and is given by [79]

$$H_{RES} = H_\delta + H_{\sigma\sigma\delta} + H_{\sigma\sigma} + H_T + H_{MM} \tag{C.95}$$

where the including terms contains strengths of the delta interaction(V_δ), the spin-spin interaction ($V_{\sigma\sigma}$), the spin-spin-delta interaction ($V_{\sigma\sigma\delta}$), and tensor interaction (V_T) are determined from a fit to the experimental levels,

$$\begin{aligned}
&H_\delta = 4\pi V_\delta \delta(\mathbf{r}_\pi - \mathbf{r}_\nu)\delta(r_\pi - R_0)\delta(r_\nu - R_0),\\
&H_{\sigma\sigma\delta} = 4\pi V_{\sigma\sigma\delta}\delta(\mathbf{r}_\pi - \mathbf{r}_\nu)\sigma_\pi\cdot\sigma_\nu\delta(r_\pi - R_0)\delta(r_\nu - R_0),\\
&H_{\sigma\sigma} = -\sqrt{3}V_{\sigma\sigma}\sigma_\pi\cdot\sigma_\nu,\\
&H_T = V_T[3(\sigma_\pi\cdot r_{\pi\nu})(\sigma_\nu\cdot\mathbf{r}_{\pi\nu})\tfrac{1}{r_{\pi\nu}^2} - \sigma_\pi\cdot\sigma_\nu],\\
&H_{MM} = 4\pi\tfrac{\delta(r_\pi - r_\nu)}{r_\pi r_\nu}\textstyle\sum_{km} V_k Y^*_{km}(\pi)Y_{km}(\nu)
\end{aligned} \tag{C.96}$$

where $\mathbf{r}_{\pi\nu} = \mathbf{r}_\pi - \mathbf{r}_\nu$ and $R_0 = 1.2A^{1/3}$fm,

This residual interaction can be diagonalized in the quasi-particle scheme basis, where

the state $(\tilde{j}_\pi, \tilde{j}_\nu)j_{\pi\nu}$, with quasiproton j_π and quasineutron j_ν coupled to the angular momentum $j_{\pi\nu}$; $|n_d\nu I\rangle$ dtenotes the standard IBM basis states with n_d d-bosons coupled to angular momentum I, here ν are the additional quantum numbers needed to specify the sate, and $n_s = N - n_d$ s-bosons; the angular momenta $j_{\pi\nu}$ and I are coupled to the total angular momentum J, the odd-odd wavefunctions can be expressed in the form

$$|J_k\rangle = \sum \xi_{(j_\pi j_\nu)j_{\pi\nu}n_d\nu I}(\tilde{j}_\pi, \tilde{j}_\nu)j_{\pi\nu}, n_d\nu I; J\rangle \tag{C.97}$$

where

$$|(\tilde{j}_\pi, \tilde{j}_\nu)j_{\pi\nu}, n_d\nu I; J\rangle \tag{C.98}$$

is the quasi-proton-quasi-neutron -boson basis and k represents here the k-th state of the angular momentum J. The relationship between two particles with two quasi-particles matrix elements are given by

$$\begin{aligned}
\langle j'_\nu j'_\pi; J|V_{RES}|j'_\nu j'_\pi; J\rangle_{qs} = {} & (u_{j'_\nu}u_{j'_\pi}u_{j_\nu}u_{j_\pi} + v_{j'_\nu}v_{j'_\pi}v_{j_\nu}v_{j_\pi})\langle j'_\nu j'_\pi; J|V_{RES}|j_\nu j_\pi; J\rangle\\
& -(u_{j'_\nu}v_{j'_\pi}u_{j_\nu}v_{j_\pi} + v_{j'_\nu}u_{j'_\pi}v_{j_\nu}u_{j_\pi})\sum_{J'}(2J'+1)\begin{Bmatrix} j'_\nu & j_\pi & J'\\ j_\nu & j'_\pi & J\end{Bmatrix}\\
& \langle j'_\nu j'_\pi; J|V_{RES}|j_\nu j_\pi; J\rangle
\end{aligned} \tag{C.99}$$

The electromagnetic transition $E2$ operator for odd-odd nuclei can be written like

$$T^{(E2)} = e^B_\pi Q^B_\pi + e^B_\nu Q^B_\nu - \sum_{\rho=\pi,\nu}\frac{1}{\sqrt{5}}\sum_{jj'}(u_{j'}u_j - v_{j'}v_j)\langle j'\|e_{\mathrm{eff},\rho}r^2Y^{(2)}\|j\rangle[a^\dagger_{j'}\times\tilde{a}_j]^{(2)} \tag{C.100}$$

and the $M1$ transitions

$$T^{(M1)} = \sqrt{\tfrac{3}{4\pi}} \left(g_\pi^B L_\pi^B + g_\nu^B L_\nu^B - \sum_{\rho=\pi,\nu} \frac{1}{\sqrt{3}} \sum_{j,j'} (u_{j'} u_j + v_{j'} v_j) \langle j' \| g_{l,\rho} \mathbf{l} + g_{s,\rho} \mathbf{s} \| j \rangle [a_{j'}^\dagger \times \tilde{a}_j]^{(1)} \right), \tag{C.101}$$

where L_ρ^B is the boson angular momentum, $\mathbf{l}$ is the fermion orbital angular momentum, and $\mathbf{s}$ is the fermion spin.

The effective coefficients are taken from the Ref [80]:

Coefficient	Value
e_π^B	$0.13eb$
e_ν^B	$0.13eb$
$e_{\text{eff},\pi}$	$0.405e$
$e_{\text{eff},\nu}$	$0.135e$
g_π^B	$1.3\mu_N$
g_ν^B	$-023\mu_N$
$g_{l,\pi}$	$1.0\mu_N$
$g_{s,\pi}$	$3.910\mu_N$
$g_{l,\nu}$	$0\mu_N$
$g_{s,\nu}$	$-2.678\mu_N$

where the g- have been quanched bya factor of 0.7.

C.6 High Order Contributions and corrections

The partial nuclear matrix elements in neutrinoless double beta decay, VV, MM, AA, PP, PP have their origin from the vector, the weak-magnetism, the axial, the pseudoscalar coupling and the interference of the axial-vector and pseudoscalar coupling, respectively. They can be expressed in relative coordinates by using the second quantization [81]

$$\begin{aligned} h^F(p) &= h^F_{VV}(p) \\ h^{GT}(p) &= h^{GT}_{AA}(p) + h^{GT}_{AP}(p) + h^{GT}_{PP}(p) + h^{GT}_{MM}(p) \\ h^T(p) &= h^T_{AP}(p) + h^T_{PP}(p) + h^T_{MM}(p) \end{aligned} \tag{C.102}$$

The finite nucleon size (FNS) correction

$$\begin{aligned} g_v(p^2) &= g_v \frac{1}{(1+\frac{p^2}{M_v^2})^2} \\ g_A(p^2) &= g_A \frac{1}{(1+\frac{p^2}{M_v^2})^2} \end{aligned} \tag{C.103}$$

fixed by the electromagnetic form factor of the proton

$$\begin{aligned} M_v^2 &= 0.71(GeV/c^2)^2 \\ M_A &= 1.09 GeV/c^2 \end{aligned} \tag{C.104}$$

The short-range correlations are usually taken into account by multiplying the potential H(r) by Jastrow function squared,

$$\begin{aligned} f(r) &= 1 - e^{-ar^2}(1-br^2) \\ a &= 1.1.fm^{-2} \text{ and } b = 0.68 fm^{-2} \end{aligned} \tag{C.105}$$

C.7 Transition Densities

The transition operator of this work applied to odd-even nuclei include the bosonic and fermionic operator part. The development of the transition operator comes from the generalized seniority scheme which we perform the mapping of the single particle operator in the space of boson-fermion.

The one body transition operator in the m scheme may be written

$$\hat{T}^{(\lambda)}_{\mu} = \sum_{\alpha\beta} \langle \alpha | T^{(\lambda)}_{\mu} | \beta \rangle c^{\dagger}_{\alpha} c_{\beta} \tag{C.106}$$

the tensor coupled form for this operator

$$\hat{T}^{(\lambda)}_{\mu} = \sum_{m_\alpha m_\beta} (-1)^{j_\alpha - m_\alpha} \begin{pmatrix} j_\alpha & \lambda & j_\beta \\ -m_\alpha & \mu & m_\beta \end{pmatrix} c^\dagger_\alpha c_\beta \\ \sum_{k_\alpha k_\beta} \langle k_\alpha \| T^{(\lambda)} \| k_\alpha \rangle \frac{[c^\dagger_{k_\alpha} \times \tilde{c}_{k_\beta}]^{(\lambda)}_\mu}{2\lambda+1} \tag{C.107}$$

where α stands for n_α, l_α ,j_α, m_α, the one nucleon creation operator (proton or neutron) in the shell k_α with single particle level quantum numbers n_α l_α $1/2$ and j_α. The annihilation operator with good tensor character is given by $\tilde{c}_{j\alpha m_\alpha} = (-1)^{j_\alpha - m_\alpha} c_{j_{\alpha - m_\alpha}}$ The reduced matrix element for the n-particle wave function can be written as a product over one-body transition densities (OBTD) times reduced single-particle matrix elements

$$\langle \Psi_f \| \hat{T}^{(\lambda)} \| \Psi_i \rangle = \sum_{k_\alpha k_\beta} \frac{\langle nwJ \| [c^\dagger_{k_\alpha} \times \tilde{c}_{k_\beta}]^\lambda \| nw' j' \rangle}{\sqrt{2\lambda+1}} \tag{C.108}$$

C.8 Diagonalization IBFM

The full Hamiltonian is diagonalized in the truncated basis

$$|(\alpha_c L_c, j) JM\rangle. \tag{C.109}$$

By construction H_B and H_F are diagonal in the basis (C.109). The matrix elements of the boson-fermion interaction

$$V_{BF} = \sum_\lambda \sum_{ik} v^{(\lambda)}_{ik} B^{(\lambda)}_i \cdot F^{(\lambda)}_k \tag{C.110}$$

can be separated into a boson and a fermion part

$$\langle (\alpha_c, L_c, j) JM | V_{BF} | (\alpha'_c, L'_c, j') JM \rangle \\ = \sum_\lambda \sum_{i,k} v^{(\lambda)}_{ij} (-1)^{j+J+L'_c} \begin{Bmatrix} L_c & j, & J \\ j' & L'_c & \lambda \end{Bmatrix} \\ \times \langle (\alpha_c, L_c) \| B_{(\lambda)_i} \| (\alpha'_c, L'_c) \rangle \langle (j \| F^{(\lambda)}_k \| j' \rangle \tag{C.111}$$

C.9 Woods-Saxon potential

The WS is based upon the sum of a spin-independent central potential, a spin-orbit potential, and the Coulomb potential,

$$V(r) = V_0(r) + V_{so}(r) \mathbf{l} \cdot \mathbf{s} + V_c(r), \tag{C.112}$$

where V_o(r) is the spin-independent central potential:

$$V_o(r) = V_o f_o(r) \tag{C.113}$$

with a fermi shape

$$f_o(r) = \frac{1}{1+[exp(r-R_0)/a_0]}, \tag{C.114}$$

$V_{so}(r)$ is the spin-orbit potential:

$$V_{so}(r) = V_{so}\frac{1}{r}\frac{df_{so}(r)}{dr}, \tag{C.115}$$

with

$$f_{so}(r) = \frac{1}{1+[exp(r-R_{so})/a_0]}, \tag{C.116}$$

$V_c(r)$ is the Coulomb potential for protons based upon the Coulomb potential for a sphere of radius R_c:

$$V_c(r) = \frac{Ze^2}{r} \text{ for } r \geqslant R_c \tag{C.117}$$

and

$$V_c(r) = \frac{Ze^2}{R_c}\left[\frac{3}{2} - \frac{r^2}{2R_c^2}\right] \text{ for } r \leqslant R_c. \tag{C.118}$$

The radii R_0, R_{so} and R_c are usually expressed as:

$$R_i = r_i A^{1/3} \tag{C.119}$$

Appendix D
Eikonal Scattering

In the eikonal approximation, we assume the projectile follows a straight-line trajectory. For a plane wave along the beam direction, chosen as $\hat{z}$ and decomposing the vector q into the component of the direction perpendicular q_t and the parallel to the collision axis $\hat{z}$ we have $q = q_t + Q_z\hat{z}$. Making a decomposing the position vector R into a component in the direction of the incident beam $\hat{n}$ and another perpendicular to it as $R = z\hat{n} + b$, and typically for forwarding angle scattering,$Q \ll k$. Follows that $Q \cdot \hat{n} \approx 0$, for small energy transfers, the momentum transfer $\mathbf{Q}$ is predominantly transverse making an easy way to perform the numerical calculations.

We are assuming that at high enough energies the trajectory of the scattered particle is a straight line and there is a very small momentum transfer along the collision. This means that the momentum distribution of the scattering wave function is sharply peaked at the incident momentum. For an incident wave along the z-axis, with a wave vector $\mathbf{k} = k\hat{z}$ if $\triangle k_z \ll k_z$ and $\triangle k_t \ll k_z$, where $\triangle k_z$ and $\triangle k_t$ denote respectively the widths of the momentum distributions along longitudinal and transversal directions. These conditions for enough energies makes $E \gg |V| \rightarrow k_z \gg \triangle k_z$ and small scattering angles means $tan\theta = k_t/k_z \ll 1$. To describe the eikonal approximation, we use the cylindrical coordinates $r \equiv (z, b, \phi)$. We can write $\mathbf{r} = z\hat{z} + b$, where z extends from $-\infty$ to ∞, and the impact parameter vector $\mathbf{b}$ subtends the azimuthal angle ϕ with the range $(0, 2\pi)$. Accordingly the volume element is

$$d^3\mathbf{r} = bdbdzd\phi \tag{D.1}$$

Considering the scattering factorized wave function

$$\Psi^{(+)}(\mathbf{k}, \mathbf{r}) = \frac{e^{ikz}}{(2\pi)^{3/2}} \hat{\Psi}^{(+)}(z, \mathbf{b}) \tag{D.2}$$

Inserting the factorized wave function into the Schrodinger equation

$$\left[2ik\frac{\partial}{\partial z} - \frac{2\mu}{\hbar^2}U(z, \mathbf{b})\right] \hat{\Psi}^{(+)}(z, b) = \bigtriangledown^2 \hat{\Psi}^{(+)}(z, \mathbf{b}) \tag{D.3}$$

and since th eikonal wave function is assumed to be a sharp wave packet in momentum space , with $\hat{\Psi}^{(+)}$ giving the dispersion around $k\hat{z}$ we can assume that the relevant Fourier components of $\hat{\Psi}^{(+)}$ have negligible momentum, as compared with its incident value. This leads to the relation $\bigtriangledown^2\hat{\Psi}^{(+)}(z,\mathbf{b}) \ll 2q\partial\hat{\Psi}^{(+)}/\partial z$ making $\bigtriangledown^2\hat{\Psi}^{(+)}(z,\mathbf{b})$very small, therefore neglecting this contribution

$$\left[2ik\frac{\partial}{\partial z} - \frac{2\mu}{\hbar^2}U(z,\mathbf{b})\right]\hat{\Psi}^{(+)}(z,b) = 0 \tag{D.4}$$

gives

$$\hat{\Psi}^{(+)}(z,\mathbf{b}) = \exp[-\frac{i}{\hbar v}\int_{-\infty}^{z} dz' U(z',b)] \tag{D.5}$$

where v is the velocity along the z-axis , which within thee eikonal appoximation has a constant value. We can introduce the eikonal phase is given by

$$\chi(b) = -\frac{k}{2E_k}\int_{-\infty}^{\infty} U(z',b)dz' \tag{D.6}$$

or simply

$$\chi(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} U(z',b)dz' \tag{D.7}$$

where $U(z',b)$ is the optical potential which includes the coulomb phase. However if the projectile is composed system of n particles the eikonal phase should be generalized nevertheless in principle if we consider in the DCX procedure one interacting nucleon per exchange,

Appendix E
Elastic Scattering

Let us now obtain explicitly the scattering amplitude, in case the projectile is not composed system, following the standard procedure [82]. The scattering amplitude within the eikonal approximation

$$f(Q) = -\frac{\mu}{2\pi\hbar^2}\int d^2b e^{-iQ\cdot b}\int_{-\infty}^{\infty} dz U(z,b)\hat{\Psi}^{(+)}(z,b) \tag{E.1}$$

whe have used the approximation

$$\mathbf{Q}\cdot\mathbf{r} = zQ_z + Q_t\cdot\mathbf{r} \simeq \mathbf{Q}_t\cdot\mathbf{r} = Q\cdot\mathbf{b}$$

which corresponds to neglecting the longitudinal momentum transfer, using Eq. (E.1) and Eq. (D.5), we obtain

$$f(Q) = -\frac{\mu}{2\pi\hbar^2}\int d^2\mathbf{b} e^{-iQ\cdot\mathbf{b}}\int_{-\infty}^{\infty} dz U(z,b)\exp[-i\frac{k}{2E_k}\int_{-\infty}^{z} dz' U(z',\mathbf{b})] \tag{E.2}$$

and changing the variable

$$z \to w = -i\frac{k}{E_k}\int_{-\infty}^{z} dz' U(z',\mathbf{b}) \tag{E.3}$$

with this transformation,

$$dz = i\frac{2E_k}{kU(z,\mathbf{b})}dw \tag{E.4}$$

$$z = -\infty \to w = 0 \tag{E.5}$$

$$z = \infty \to w = i\chi(b) \tag{E.6}$$

therefore Eq. (E.2) becomes

$$f(Q) = -\frac{ik}{2\pi}\int d^2b e^{-iQ\cdot b}\int_0^{i\chi(b)} dw e^w \tag{E.7}$$

and performing the integration over w and using the limits of Eq. (E.6) we arrives to the standard expression of the elastic scattering amplitude

$$f(Q) = -\frac{ik}{2\pi}\int d^2b e^{-iQ\cdot b}[e^{i\chi(b)} - 1] \tag{E.8}$$

for practical calculations it is convenient to express the above integrals over **b** in terms of its modulus b and azimuthal angle ϕ_b

$$d^2\mathbf{b} = bdbd\phi_b \tag{E.9}$$

$$-\mathbf{Q}\cdot\mathbf{b} = bQcos(\phi - \phi_b) \tag{E.10}$$

where ϕ is the azimuthal angle associated with $-\mathbf{Q}$. ϕ is also the angle between the scattering plane and the plane x-z. So changing the variable

$$\phi \to \varphi = \phi - \phi_Q \tag{E.11}$$

and therefore

$$f(Q) = -\frac{ik}{2\pi}\int bdb\int_0^{2\pi} d\varphi e^{iQbcos(\varphi)}[e^{i\chi(b)} - 1] \tag{E.12}$$

when the eikonal phase has axial symmetry we can makes the the square brackets even independent of φ and we obtain

$$f(Q) = -ik\int bdbe^{iQbcos(\varphi)}[e^{i\chi(b)} - 1]\left(\frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{iQbcos(\varphi)}\right) \tag{E.13}$$

it is convenient express the integral into a one dimensional integral over cylindrical Bessel function, therefore the round brackets can be written in terms of Bessel function order zero where

$$J_0(Qb) = \frac{1}{2\pi}\int_0^{2\pi} e^{Qbcos\varphi}d\varphi \tag{E.14}$$

therefore

$$f(Q) = -ik\int bdbJ_0(Qb)[e^{i\chi(b)} - 1] \tag{E.15}$$

it is convenient express the scattering amplitude in terms of θ, so taking taking $Q = 2ksin(\theta/2)$ we arrives to well known formula [83]

$$f(\theta) = -ik\int bdbJ_0(2kbsin(\theta/2))[e^{i\chi(b)} - 1] \tag{E.16}$$

E.1 Elastic cross section

The elastic scattering cross section is

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta)|^2 \tag{E.17}$$

For numerical purposes, it is convenient to make use of the analytical formula the Coulomb scattering amplitude in the with the Eikonal approximation [84]

$$f_C(\theta) = \frac{Z_P Z_T e^2}{2\mu v^2 \sin(\frac{\theta}{2})} \exp\{-i\eta ln[sin^2(\theta/2)] + i\pi + 2i\phi_0\} \tag{E.18}$$

where

$$\phi_0 = arg\Gamma(1 + i\frac{\eta}{2}) \tag{E.19}$$

but can be used

$$\phi_0 = -\eta C + \sum_{j=0}^{\infty} \left(\frac{\eta}{j+1} - \arctan\frac{\eta}{j+1} \right) \tag{E.20}$$

where $C = 0.05772156...$ is the Euler constant, and $\eta = Z_p Z_T e^2/\hbar v$ and Z_P, Z_T are the charges of projectile and target and v is their relative velocity. It is convenient for the numerical calculations since the elastic scattering amplitude can be with a separate contribution of the Coulomb scattering amplitude. The problem of the integration of Eq. (E.16) is that the Coulomb part of the optical potential diverges. Including the both Coulomb and nuclear interactions

$$V(r) \equiv V_N(r) + V_C(r) \tag{E.21}$$

One solves this by using

$$\chi = \chi_N + \chi_C \tag{E.22}$$

where the χ_N is given by Eq. (D.7) without the Coulomb potential, and writing the Coulomb eikonal phase, χ_C as

$$\chi_C(b) = 2\eta ln(kb) \tag{E.23}$$

where k is the wavenumber in the center of mass of the system. . This Coulomb phase of the above equation diverges at $b = 0$, however there the scattering at small impact parameters there is strong absorption suppress. In order to evaluate the integral (E.16), we can adds and subtracts the Coulomb amplitude $f_C(\theta)$ in Eq. (E.16), therefore we obtain ,

$$f(\theta) = f_C(\theta) + ik \int_0^{\infty} dbJ_0(2k\sin(\theta/2)b)\exp[i\chi_C(b)]\{1 - \exp[i\chi_N(b)]\} \tag{E.24}$$

The advantage in using this formula is that the term $1 - \exp[i\chi_N(b)]$ becomes zero for impact parameters larger than the sum of the nuclear radii, thus the integral needs to be performed only within a small range.

E.2 Derivation of the first order term of the bosonic operator

From Eq. (1.49) using Eqs. (1.51)and (1.52)

$$\begin{aligned}
&\langle j^{n+2}(S^{\frac{1}{2}n-\frac{1}{2}\nu+1}D^{\frac{1}{2}\nu}),\nu_d,n_\Delta,L=0\|T_+^{(0)}\| \\
&j^n(S^{\frac{1}{2}n-\frac{1}{2}\nu}D^{\frac{1}{2}\nu}),\nu_d,n_\Delta,L=0\rangle \\
&=\langle s^{\frac{1}{2}n-\frac{1}{2}\nu+1}d^{\frac{1}{2}\nu}\|p_0s^\dagger+p_0's^\dagger s^\dagger s+\ldots q_0s^\dagger[d^\dagger\tilde{d}]^{(0)} \\
&+\sum_L q_0^L s^\dagger[[d^\dagger d^\dagger]^{(0)}[\tilde{d}\tilde{d}]^{(0)}]^{(0)}+\ldots\|s^{\frac{1}{2}n-\frac{1}{2}\nu}d^{\frac{1}{2}\nu}\rangle \\
&=q_0\langle s^{\frac{1}{2}n-\frac{1}{2}\nu+1}d^{\frac{1}{2}\nu}\|s^\dagger\|s^{\frac{1}{2}n-\frac{1}{2}\nu}d^{\frac{1}{2}\nu}\rangle \\
&+\langle s^{\frac{1}{2}n-\frac{1}{2}\nu+1}d^{\frac{1}{2}\nu}\|s^\dagger[d^\dagger\tilde{d}]^{(0)}\|s^{\frac{1}{2}n-\frac{1}{2}\nu}d^{\frac{1}{2}\nu}\rangle+\cdots \\
&=\sqrt{\tfrac{1}{2}n-\tfrac{1}{2}\nu+1}\sqrt{\tfrac{2\Omega-n-\nu}{2\Omega}}\langle j^2(S),L=0\|T_+^{(0)}\|0\rangle,
\end{aligned} \tag{E.25}$$

due to the reduced matrix element

$$\langle s^{n_s}\|s\|s^{n_s+1}\rangle=\sqrt{n_s+1}=\sqrt{\tfrac{1}{2}n-\tfrac{1}{2}\nu+1}$$

and using the relations

$$\sqrt{\tfrac{2\Omega-n-\nu}{2\Omega}}=\sqrt{\tfrac{\Omega-N-n_d}{\Omega}}$$

the as a first approximation

$$p_0=\sqrt{\frac{\Omega-N-n_d}{\Omega}}\langle j^2(S)L=0\|T_+^{(0)}\|0\rangle \tag{E.26}$$

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