X International Conference on Structural Dynamics, EURODYN 2017

Cut-off frequencies and correction factors of equivalent single layer theories

Roberta Massabò*

DICCA; University of Genoa, Genova, 16145, Italy

Abstract

Shear and normal correction factors are used within equivalent single-layer first-order theories for homogeneous and layered structures, e.g. first-order shear deformation plate/shell theories, in order to improve the description of the transverse strains. Dynamic correction factors may be derived, following the original treatment for homogeneous plates (Mindlin, R.D., J. Appl. Phys. 1951), by imposing that certain wave propagation frequencies, e.g. the cut-off frequencies of the lowest cut-off frequency modes, match those of three-dimensional elasticity, which are typically obtained through computational procedures. In this paper we consider multilayered plates with principal material directions parallel to the geometrical axes and propose the use of a multiscale structural theory which allows the accurate closed-form derivation of the cut-off frequencies of the first thickness modes associated to plane-strain Rayleigh-Lamb waves. The theory captures the effects of the inhomogeneities on local fields and global behavior through homogenized equilibrium equations which depend on a limited number of variables independent of the number of layers. Explicit expressions are obtained for the lowest cut-off frequencies of multilayered wide plates with an arbitrary number of layers, with arbitrary layup and elastic constants, which are then used to define the dynamic correction factors of classical equivalent single-layer theories. The comparison with predictions obtained by matching exact elasticity solutions for highly-inhomogeneous bilayer media highlights the accuracy and potentials of the proposed approach.
© 2017 The Authors. Published by Elsevier Ltd.
Peer-review under responsibility of the organizing committee of EURODYN 2017.

Keywords: homogenization; structural theories; wave propagation; correction factors; layered media; composites

1. Introduction

Equivalent Single Layer (ESL) theories are extensively being used by the engineering community for anisotropic

---

* Corresponding author. Tel.: +39-010-353-2956; fax: +39-010-353-2534.
  E-mail address: roberta.massabo@unige.it

layered structures, such as laminated and sandwich beams, plates and shells. First-order shear deformation theory uses kinematic assumptions which imply constant transverse displacements and transverse shear strains in the through-thickness direction. First-order shear, first-order normal deformation theory additionally accounts for the transverse compressibility of the layers and assumes thickness-wise constant transverse-normal strains [1,2]. Both approaches introduce correction factors, to partly overcome the limitations of the kinematic assumptions for certain loadings and frequency ranges. Many different methods have been proposed to derive the correction factors. In dynamic problems they are often derived as suggested in the original work of Mindlin for homogeneous plates [3] by imposing that the cut-off frequencies of the first cut-off frequency modes predicted by the approximate structural theories match those of 3D elasticity. Explicit expressions for the cut-off frequencies using 3D elasticity can be obtained only for homogeneous plates and yield the well-known correction factors $\pi^2/12$, for both normal and shear strains. A numerical solution of the frequency equation is required for layered media, even with only two or three layers, and in media with a larger number of layers the cut-off frequencies are typically obtained using a matrix formalism and systematic computational methods, e.g. Transverse Matrix Method. Other approaches define the shear correction factors by fitting the dispersion relations of the first flexural modes obtained using the ESL theory with those of the first modes of the Rayleigh-Lamb waves in the long wavelength limit [4,5]. The procedure to define the exact cut-off frequencies or the exact dispersion curves is problem specific and quite complex; because of this, the engineering community often extends the correction factors derived for homogeneous plates to layered plates. However, the correction factors required to match the exact cut-off frequencies or dispersion curves of layered systems may substantially differ from those of homogeneous plates, especially if the mismatch of the elastic constants of the layers is high, and this procedure may affect strongly predictions of all modes of propagation, including the lowest flexural modes.

In this paper we examine the potentials of a homogenized first-order structural theory [6], which is based on the original zig-zag theory formulated in [7] for fully bonded plates and extended in [8] to systems with interfacial imperfections (e.g., thin elastic interlayers or damaged interfaces), for the accurate closed-form derivation of the cut-off frequencies of the two lowest modes of propagation in layered media. The theory couples an equivalent single layer theory (global scale) and a detailed discrete-layer cohesive-crack model (local scale) using homogenization and imposing a priori continuity conditions at the layer interfaces. It captures the effects of the inhomogeneous material structure and interfacial imperfections on local fields and global behavior through homogenized equilibrium equations which depend on the kinematic variables of the global theory only. The equations are then tractable and can be solved in closed-form for many relevant problems, e.g. thermo-mechanical loading [9].

Explicit expressions are presented for the cut-off frequencies of the two lowest cut-off frequency modes of propagation of plain-strain Rayleigh-Lamb waves in multilayered plates with principal material directions parallel to the geometrical axes and arbitrary number of layers with arbitrary layup and elastic constants. Correction factors are defined by matching to these frequencies those calculated using the equivalent first-order single layer theories. The accuracy of the predictions is highlighted through comparison with exact elasticity solutions for bi-layers with highly inhomogeneous properties [10].

2. Model and wave propagation analysis

The propagation of plane-strain Rayleigh-Lamb waves in the multilayered plate of infinite in plane extent and thickness h shown in Fig. 1 has been studied in [6] using a multiscale homogenized model. In this paper the model in [6] is extended to account for the transverse compressibility of the layers and used to define correction factors of ESL theories. The plate in Fig. 1 consists of n layers joined by $n-1$ zero-thickness interfaces. The thickness of the layer k, with $k = 1, .., n-1$ numbered from bottom to top, is $^{(k)}h$ and the coordinates x_3^{k-1} and x_3^k defines its lower and upper interfaces. The interfaces are assumed as mathematical surfaces where the interfacial tractions are continuous while the materials properties may change discontinuously (the more general case of imperfect interfaces has been treated in [6]). The results have been obtained under the following assumptions: the layers have different mechanical properties, are linearly elastic and orthotropic with principal material directions parallel to the geometrical axes and $^{(k)}C_{ij}$ the stiffness coefficients with respect to the axes of material symmetry,

$i, j = 2,3,5$; the in plane displacements are piece-wise linear functions of the through-thickness coordinate (first-order theory); plane-strain parallel to the plane $x_2 - x_3$.
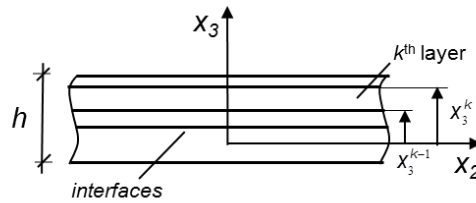

Fig. 1. Layered plate with *n* unequal orthotropic layers, principal material directions parallel to the geometrical axes, and *n*-1 interfaces.

A small-scale displacement field in the layer *k* at the coordinate $\mathbf{x} = \{x_1, x_2, x_3\}^T$ is assumed:

$$^{(k)}v_2(x_2, x_3, t) = v_{02}(x_2, t) + x_3\varphi_2(x_2, t) + \sum_{i=1}^{k-1}\Omega_2^i(x_2, t)(x_3 - x_3^i)$$

$$^{(k)}v_3(x_2, x_3, t) = w_0(x_2, t) + x_3\varphi_3(x_2, t) + \sum_{i=1}^{k-1}\Omega_3^i(x_2, t)(x_3 - x_3^i)$$

(1)

where $v_{02}(x_2, t)$, $w_0(x_2, t)$, $\varphi_2(x_2, t)$ and $\varphi_3(x_2, t)$ define the displacement field of the global model, which is continuous with continuous derivatives in the thickness direction and coincides with that of first-order shear and normal deformation theory. This field is enriched by local contributions, given by the terms in the summations on the left hand side. The functions $\Omega_j^i(x_2)(x_3 - x_3^i)$ for $i = 1, ..., k-1$ and $j = 2,3$ define a piece-wise linear displacement field in x_3, which is introduced to reproduce the zig-zag pattern due to the multilayered material structure. The small-scale kinematic description of the problem is defined by a total of 4+2×(n-1) unknown functions in the *n* layers. The 2×(*n*-1) small-scale unknowns may be defined as functions of the global unknowns, v_{02}, w_0, φ_2 and φ_3 by imposing continuity of shear and normal tractions at the *n*-1 layer interfaces. This yields the macro-scale displacement field, which is then used, along with the compatibility and constitutive equations of the layers to derive the weak form of the dynamic equilibrium equations using Hamilton principle [8]. The system of equations can be decoupled and to show that the transverse behavior is controlled by a sixth order equation in w_0, which reduces to the classical fourth order equation of first-order shear deformation theory when the zig-zag effect is negligible [9]. A similar behavior is observed in [11] where highly contrasted laminated plates are studied through a multi-scale asymptotic approach.

To analyze the propagation of continuous, straight-crested waves in the x_2 direction, solutions of the form $y(x_2, t) = Y\exp i(kx_2 - \omega t)$ are assumed, with $y(x_2, t)$ the displacement variable, Y an unknown constant; k, the wave number, and ω, the radian frequency, are related to the wavelength, Λ, frequency, f, and the wave speed, c, by $\Lambda = 2\pi/k$, $f = \omega/2\pi$ and $c = \omega/k$. Substitution into the dynamic equilibrium equations yields a system of homogeneous equations in the unknown constants whose non-trivial solution gives the frequency equation of the problem. The method and relevant results have been presented in [6] for plates with transverse incompressibility of the layers and $^{(k)}v_3(x_2, x_3, t) = w_0(x_2, t)$ in Eq. (1).

According to the proposed theory, four wave-types are operative in a multilayered plate. These types correspond to the two lowest branches of all frequency curves and to the branches of the first two cut-off frequencies. The asymptotic limits of the first and second cutoff frequency modes for infinitely long wavelengths, $k \to 0$, are defined by the first and second thickness modes, which are propagation modes characterized by the absence of transverse displacements $w_0(x_2, t) = 0$ and by in-plane displacements independent of the in-plane

coordinates (thickness-shear mode) and by non-zero transverse displacements and the absence of longitudinal displacements and rotations, $v_{02}(x_2,t) = 0$ and $\varphi_2(x_2,t) = 0$ (thickness-stretch mode). These modes can be used to easily calculate in closed form the cut-off frequencies of the lowest cut-off frequency modes in plates with complex layups. In the limit of the long wavelengths, for $k \rightarrow 0$, the frequency equations of the first thickness modes as predicted by the homogenized model are:

$$(\omega^{SH})^4 \left(R_2 - \frac{R_1^2}{R_0} \right) - (\omega^{SH})^2 A_{55} = 0 \ , \qquad (\omega^{ST})^4 \left(R_2^N - \frac{\left(R_1^N \right)^2}{R_0} \right) - (\omega^{ST})^2 A_{33} = 0 \tag{2}$$

with ω^{SH} and ω^{ST} the radian frequencies of the thickness-shear and -stretch modes and the coefficients R_0, R_1, R_2, R_1^N, R_2^N A_{55}, A_{33}, which depend on the elastic constants, inertia and stacking sequence, given in Appendix A [6].

3. Cut-off frequencies of thickness-shear and thickness-stretch modes

The cut-off frequencies of the lowest thickness-shear and thickness-stretch modes associated to plane-strain Rayleigh-Lamb waves propagating along one of the principal material directions, are obtained through the solution of the frequency equations given in Eq. (2):

$$\omega_{c.o.}^{SH} = \left(\frac{A_{55}}{R_2 - R_1^2 / R_0} \right)^{1/2} \text{ (thickness-shear mode)} \tag{3}$$

$$\omega_{c.o.}^{ST} = \left(\frac{A_{33}}{R_2^N - (R_1^N)^2 / R_0} \right)^{1/2} \text{ (thickness-stretch mode)} \tag{4}$$

The cut-off frequencies derived using first-order shear deformation theory and first-order shear and normal deformation theory, $\omega_{c.o.}^{SH,ESL}$ and $\omega_{c.o.}^{ST,ESL}$, can be easily recovered from Eqs. 3,4 by neglecting the zig-zag contributions, $\Lambda_{22}^{(1;k)} = 0$ for $k = 1,..,n-1$, in Eq. (7) which leads to the modified coefficients given in Eq. 8. Explicit expressions for the cut-off frequencies similar to those presented above have been derived in [12] by means of a higher-order layer-wise zig-zag theory. A comparison between the two results will be presented elsewhere.

4. Shear and normal correction factors for beams and wide plates

Shear and normal correction factors for the solution of plane strain problems using equivalent single layer theories in multilayered wide plates can be calculated by imposing that the cut-off frequencies of the first thickness modes must equate the cut-off frequencies in Eqs. 3,4, $\omega_{c.o.}^{SH,ESL} = \omega_{c.o.}^{SH}$ and $\omega_{c.o.}^{ST,ESL} = \omega_{c.o.}^{ST}$, which yields:

$$\frac{k_g^{ESL}}{\pi^2 / 12} = \frac{C_{55}^P}{\int_h C_{55} dx_3} \frac{R_2^{ESL} R_0 - (R_1^{ESL})^2}{R_2 R_0 - (R_1)^2} \text{ (shear correction factor)} \tag{5}$$

$$\frac{k_3^{ESL}}{\pi^2 / 12} = \frac{C_{33}^P}{\int_h C_{33} dx_3} \frac{R_2^{N,ESL} R_0 - (R_1^{N,ESL})^2}{R_2^N R_0 - (R_1^N)^2} \text{ (normal correction factor)} \tag{6}$$

The correction factors have been normalized using the correction factors of homogeneous plates, $k_g = k_3 = \pi^2 / 12$

[3] to highlight the influence of an inhomogeneous material structure on the behavior. The accuracy of the correction factors in Eqs. 5,6 is verified through comparison with the correction factors obtained by matching the exact cut-off frequencies of 2D elasticity for a bi-layer medium with isotropic layers [10]. Figure 2 shows the shear correction factors which should be used in a first-order shear deformation theory to match the lowest cut-off frequencies of a plate with two fully bonded layers of thicknesses, f and c, and shear moduli in the plane $x_2 - x_3$, G_l and G_u. The correction factors are shown on varying the normalized thickness of the lower layer, f/h. The results compare predictions obtained using the exact cut-off frequency and the cut-off frequencies given by Eq. 3 (homogenized model). The elastic constants used in Fig. 2a, $G_u = G_l / 2$ and $\rho_u = \rho_l$ could describe a classical [0/90] cross-ply laminate with unidirectionally reinforced carbon-epoxy layers; those used in Fig. 2.b, $G_u = 0.02 G_l$ and $\rho_u = 0.15 \rho_l$ could describe the carbon-epoxy face sheet and foam core of a composite sandwich plate. The correction factors obtained through the homogenized structural model (Eq. 5, solid lines) are very accurate over most of the domain but for $c/h=(0 \div 0.15)$, which describes a plate with a very thin softer layer. In this limiting configuration, the effects of the inhomogeneity are negligible and the required correction factor, as predicted by 2D elasticity, coincides with that of a homogeneous plate. This behavior, which is probably a consequence of the assumed first-order kinematic description, needs to be further investigated. A comparative study on the correction factors derived here and obtained by other static/dynamic approaches is in progress.

5. Conclusions

Explicit expressions for the correction factors of Equivalent Single Layer theories have been derived for multilayered plates with principal material axes parallel to the geometrical axes by matching the cut-off frequencies of the first cut-off frequency modes of propagation of plane-strain Rayleigh-Lamb waves with those obtained using a multiscale homogenized structural model. Applications to highly inhomogeneous bilayers show that the correction factors virtually coincide with those obtained numerically using the cut-off frequencies of 2D elasticity but for some limit configurations where the effects of the material inhomogeneity become negligible and the correction factors coincide with those of homogeneous plates. The explicit expressions can be used for systems with many layers, $n \geq 3$, for which 2D elasticity solutions would instead require complex computational procedures.

Acknowledgements

Appendix A. Coefficients

The coefficients in the expressions of the cut-off frequencies derived using the homogenized structural approach and assuming perfect bonding of the layers are given below.

$$(R_0, R_1, R_2, R_1^N, R_2^N) = \int_h \rho(1, x_3, x_3^2, x_3, x_3^2) dx_3 + (0, R^{0S}, 2R^{1S} + R^{S2}, R^{0N}, 2R^{1N} + R^{N2})$$

$$A_{55} = k_g C_{55}^P, \qquad A_{33} = k_3 C_{33}^P$$

$$C_{55}^P = \sum_{k=1}^{n} {}^{(k)}C_{55} \int_{x_3^{k-1}}^{x_3^k} \left(1 + \sum_{i=1}^{k-1} \Lambda_{22}^{(1;i)}\right)^2 dx_3, \quad C_{33}^P = \sum_{k=1}^{n} {}^{(k)}C_{33} \int_{x_3^{k-1}}^{x_3^k} \left(1 + \sum_{i=1}^{k-1} \Lambda_{33}^{(j)}\right)^2 dx_3,$$

$$R^{rS} = \sum_{k=1}^{n} {}^{(k)}\rho \int_{x_3^{k-1}}^{x_3^{k}} (x_3)^r \sum_{i=1}^{k-1} \left[\Lambda_{22}^{(1;i)} \left(x_3 - x_3^{i} \right) \right] dx_3 , \quad R^{S2} = \sum_{k=1}^{n} {}^{(k)}\rho \int_{x_3^{k-1}}^{x_3^{k}} \left(\sum_{i=1}^{k-1} \left[\Lambda_{22}^{(1;i)} \left(x_3 - x_3^{i} \right) \right] \right)^2 dx_3 \quad (7)$$

$$R^{rN} = \sum_{k=1}^{n} {}^{(k)}\rho \int_{x_3^{k-1}}^{x_3^{k}} (x_3)^r \sum_{i=1}^{k-1} \Lambda_{33}^{(i)} \left(x_3 - x_3^{i} \right) dx_3 \quad R^{N2} = \sum_{k=1}^{n} \int_{x_3^{k-1}}^{x_3^{k}} {}^{(k)}\rho \left\{ \sum_{i=1}^{k-1} \Lambda_{33}^{(i)} \left(x_3 - x_3^{i} \right) \right\}^2 dx_3$$

$$\Lambda_{22}^{(1;i)} = {}^{(1)}C_{55} \left(\frac{1}{{}^{(i+1)}C_{55}} - \frac{1}{{}^{(i)}C_{55}} \right), \qquad \Lambda_{33}^{(i)} = -{}^{(1)}C_{33} \left(\frac{{}^{(i+1)}C_{33} - {}^{(i)}C_{33}}{{}^{(i+1)}C_{33}\,{}^{(i)}C_{33}} \right)$$

Under the assumptions of the first-order Equivalent Single Layer theories they simplify as:

$$(R_0 , R_1^{ESL}, R_2^{ESL}, R_1^{N,ESL}, R_2^{N,ESL}) = \int_h \rho(1, x_3, x_3^2, x_3, x_3^2) dx_3 , \quad A_{55}^{ESL} = k_g^{ESL} \int_h C_{55} dx_3 , \quad A_{33}^{ESL} = k_3^{ESL} \int_h C_{33} dx_3 \quad (8)$$
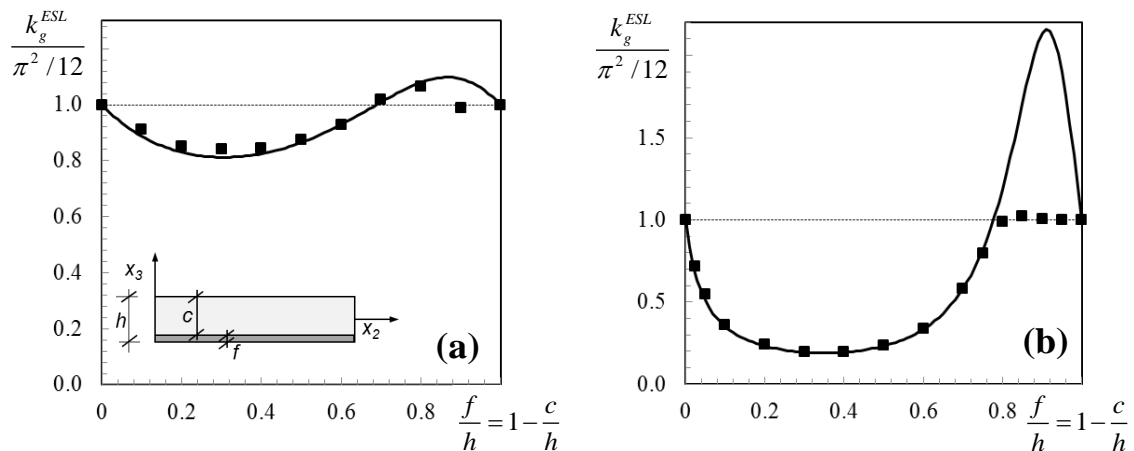

Fig. 2. Correction factors of first-order shear deformation theory required to match the lowest cut-off frequencies of plane-strain Rayleigh-Lamb waves in bi-layers; solid line: matching frequencies calculated using the homogenized model, Eq. 3; boxes: matching frequencies from 2D elasticity [10]; dotted line: homogeneous plate. (a) bi-layer with limited mismatch of the elastic properties. (b) bi-layer with high mismatch.

References

[1] P. C. Yang, C. H. Norris and Y. Stavsky, Elastic wave propagation in heterogeneous plates, Int. Journal Solids and Struct. 1966, 2, 665-684

[2] Whitney J. M. and Sun C.T., A higher order theory for extensional motion of laminated composites, J. Sound and Vibration, 30,1971, p. 85

[3] R.D. Mindlin, Thickness-shear and flexural vibrations of crystal plates, J. Appl. Phys. 22, 316, 1951.

[4] Liu, L., Bhattacharya, K., Wave propagation in a sandwich structure, Int. Journal of Solids and Structures, 46, 2009, 3290-3300

[5] Stephen, N.G., Mindlin plate theory: best shear coefficient and higher spectra validity. J. Sound Vibr., 1997, 202, 539–553.

[6] Massabò, R., Propagation of Rayleigh-Lamb waves in multilayered plates through a multiscale structural model, 2016, submitted.

[7] Di Sciuva, M., Bending, vibration and buckling of simply supported thick multilayered orthotropic plates: an evaluation of a new displacement model, J. Sound and Vibrations, 1986, 105 (3), 425-442

[8] Massabò R., Campi F., An efficient approach for multilayered beams and wide plates with imperfect interfaces and delaminations, Compos Struct, 2014, 116, 311-324

[9] Pelassa M., and Massabò, R., (2015), Explicit solutions for multi-layered wide plates beams with perfect and imperfect bonding and delaminations under thermo-mechanical loading, Meccanica, 50, 2497-2524.

[10] Jones, J.P., Wave propagation in two-layered medium, J. Appl. Mech., 1964, 31, 213.

[11] Viverge, K., Boutin, C., Sallet, F., Model of highly contrasted plates versus experiments on laminated glass, Int. J. Solids Struct., 2016, 102-103, 238-258.

[12] Nanda, N., Kapuria, S., Gopalakrishnan, S., Spectral finite element based on an efficient layerwise theory for wave propagation analysis of composite and sandwich beams, Journal of Sound and Vibration, 2014, 3120-3137.