Syzygies of *GS* monomial curves and Weierstrass property.

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Abstract

We find a resolution for the coordinate ring R of an algebraic monomial curve associated to a *GS* numerical semigroup (i.e. generated by a generalized arithmetic sequence), by extending a previous paper (Gimenez, Sengupta, Srinivasan) on arithmetic sequences, A consequence is the "determinantal" description of the first syzygy module of R. By this fact, via suitable deformations of the defining matrices, we can prove the smoothability of the curves associated to a large class of semigroups generated by arithmetic sequences, that is the Weierstrass property for such semigroups.

Keywords : Numerical semigroup, Generalized arithmetic sequence, Monomial curve, Free resolution, Deformation, Weierstrass semigroup.

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0 Introduction

Let $k[x_0, ..., x_n]/I \simeq k[t^s,\ s \in S]$ be the affine coordinate ring of a monomial algebraic curve $X \subseteq \mathbb{A}_k^{n+1}$ defined by a numerical semigroup S and let $R = k[x_0, ..., x_n]/I$ (k field). Several properties of these rings have been studied in the recent years; on the other side, some questions are still open and, among them, the problem of classifying the Weierstrass semigroups, which are appropriate for the construction of algebraic geometric codes (see [3], [9] and e.g. [13]). By a well-known Pinkham's Theorem in [15], the *smoothability* of a monomial curve, i.e., the existence of flat deformations with smooth fibres, assures the "Weierstrass property" of the associated numerical semigroup.

We consider semigroups generated by *generalized arithmetic sequences* (*GS* for short), i.e., $S =< m_0, \dots, m_n >$ with $m_i = \eta m_0 + i\, d$, for $\eta, i \geq 1$; it can be noted that these sequences are particular cases of *almost arithmetic sequences* that also appear in the literature (if $\eta \geq 2$, the first generator of a *GS* semigroup is the one that is not in arithmetic progression). The defining ideal I of a *GS* curve X is generated by the 2×2 minors of two matrices (first noted in [4] in case $\eta = 1$, when S is generated by an arithmetic sequence, *AS* for short). By adapting to *GS* semigroups a procedure shown in [5] for *AS* semigroups and based on iterated mapping cone technique, we obtain a minimal free resolution and the Betti numbers for *GS* semigroup rings. Further we give a "determinantal" description of the first syzygy module based on the explicit definition of the boundary maps (Theorem 1.6) .

In view of these facts and some results of Shaps and Pinkham, one can easily prove the Weierstrass property of *GS* semigroups of maximal embedding dimension (in particular the *Arf* ones). More generally, for a *GS* semigroup, it is quite natural to deal with the smoothability of the associated curve X by finding suitable compatible deformations of the matrices defining I and by considering the variety Y defined by the 2×2 minors of the deformed matrices: the "determinantal" description of the first syzygy module immediately gives the flatness of the induced morphism $\pi : Y \longrightarrow \Sigma$ (base space). By these tools, we prove the Weierstrass property for some classes of *AS* semigroups. Let $m_0 \equiv b\ (mod.\, n)$; when $n \leq 3b$, we construct a deformation of the curve X and, via Bertini's Theorem with some more technical trick, we show this deformation has smooth fibres, as concerns *AS* semigroups (Theorem 3.2 and Theorem 3.4). In particular, this result and the well known "determinantal" case $b = 1$ proved in [17], ensure the Weierstrass property of all the *AS* semigroups with embedding dimension ≤ 7 and of every *AS* semigroup with $b \neq 2$ and embedding dimension ≤ 10.

1 Free resolution and syzygies.

Notation and preliminary results 1.1 We fix the following Notation.

(a) Let S be a semigroup minimally generated by a *generalized arithmetic sequence* (*GS* for short):

$$S = \sum_{0 \leq i \leq n} \mathbb{N} m_i \text{ , where } m_i = \eta m_0 + id, \quad (1 \leq i \leq n) \text{ and } GCD(m_0, d) = 1,$$

let $a, b, \mu \in \mathbb{N}$ be such that

$$m_0 = an + b\,, \quad a \geq 1, \quad 1 \leq b \leq n, \quad \mu := a\eta + d.$$

Let $P := k[x_0, ..., x_n]$ (k field), with $weight(x_i) := m_i$, and let $k[S] = k[t^s,\, s \in S]$.

(b) We shall denote by S_0 the *AS* semigroup generated by the sequence $m_0, m_0 + d, ..., m_0 + nd$, and by X_0 the associated *AS* monomial curve $Spec\, k[S_0]$.

(c) As in the case of *AS* curves the defining ideal $I \subseteq P$ of a *GS* curve is generated by the 2×2 minors of the following two matrices:

$$A := \begin{pmatrix} x_0^{\eta} & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix} \quad A' := \begin{pmatrix} x_n^{a} & x_0^{\eta} & \dots & x_{n-b} \\ x_0^{\mu} & x_b & \dots & x_n \end{pmatrix}.$$

and a minimal set of generators for I can be obtained by the $\binom{n}{2}$ maximal minors $\{f_1, ..., f_{\binom{n}{2}}\}$ (we choose lexicographic order) of the matrix A and the $n-b+1$ maximal minors M_{1j} containing the first column of the matrix A'. For $h=1$ this fact is well-known, see [4, Theorem 1.1].

If $h \geq 1$, let $\mathfrak{m} = (x_0, x_1, ..., x_n) \subseteq R = P/I$, and let $Gr(k[S]) := Gr_{\mathfrak{m}}(R) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring (with $deg(x_i) = 1, \ \forall i$).

For $\phi \in R$ define its initial form ϕ^* as the homogeneous component of the least degree; the graded ideal I^* generated by $\{f^*, f \in I\}$ is called the initial ideal of I and $Gr(k[S]) \simeq P/I^*$. If $h > 1$, it is straightforward that the $\binom{n}{2} + (n-b+1)$ elements described in (c) are in I, further their initial forms generate the ideal I^* corresponding to the graded ring , as proved in [18, Corollary 3.5]. Then by applying [8, Theorem 1.2] one can deduce that the above elements constitute a *standard basis* of I.

(d) We call $\mathfrak{C}$ the codimension two ideal generated by the 2×2 minors of the matrix A.

(e) For $h = 0, ..., n-b$, we shall denote by g_h the minor formed by the columns $|c_1, \, c_{n-b+2-h}|$ of A':
$g_0 := x_n^{a+1} - x_0^{\mu}x_{n-b}, \quad g_h = x_n^a x_{n-h} - x_0^{\mu}x_{n-b-h}, \quad g_{n-b} := x_n^a x_b - x_0^{(\mu+\eta)},.$
and by $\delta_h = am_n + m_{n-h}$ its *weight*.

1.1 Resolution and syzygies of I.

Via mapping cone, quite analogously to what is done in [5] for *AS* monomial curves, we can construct a free minimal resolution for the ideal I generating a *GS* curve, see Theorem 1.4 below, which extends [5, Theorem 3.8]. We briefly recall the main steps and the changes to adapt the proof of [5, Theorem 3.8] to the case of *GS* curves.
An interesting corollary is that the Betti numbers of the ideal I are the same as the Betti numbers of I_0, the ideal of the related *AS* curve X_0 (as in Notation1.1b); further these values are maximal, since coincide with the Betti numbers of the associated graded ring to $k[S]$. Another consequence is the determinantal shape of the syzygies among the generators of I. To show this property we shall describe explicitly the maps

$$R_2 \stackrel{d_2^*}{\longrightarrow} R_1 \stackrel{d_1^*}{\longrightarrow} P$$

considered in Theorem 1.4 below.

The starting point of the construction is the existence of the well-known exact sequence

$$0 \longrightarrow R/(I:z) \stackrel{\pi}{\longrightarrow} R/I \longrightarrow R/I+(z) \longrightarrow 0$$

where I is an R-ideal, $z \in R$, π is the multiplication by z (See [18] and [5]). Hence we start with a generalization of [5, Lemma 3.1].

Lemma 1.2 *With Notation* 1.1*: assume* $1 \leq b \leq n-1$: *then* $\big(\mathfrak{C} + (g_0, ..., g_h)\big) : g_{h+1} = (x_1, ..., x_n)$, *for each* $0 \leq h \leq n-b-1$.

Proof. To prove the inclusion " $\supseteq$ ", we divide the proof in several subcases:

(a) $0 \leq h \leq n-b-2, \quad 1 \leq i \leq n-1$:
First observe that $x_{i+1}g_{h+1} - x_i g_h = x_{i+1}(x_n^a x_{n-h-1} - x_0^{\mu}x_{n-b-h-1}) - x_i(x_n^a x_{n-h} - x_0^{\mu}x_{n-b-h}) =$
$= x_n^a(x_{i+1}x_{n-h-1} - x_i x_{n-h}) - x_0^{\mu}(x_{i+1}x_{n-b-h-1} - x_i x_{n-b-h}) \in \mathfrak{C}$, hence the inclusion $\supseteq$ is clear.

(b) $0 \leq h \leq n-b-2, \quad i = 0$:
$x_1 g_{h+1} - x_0^{\eta} g_h = x_1(x_n^a x_{n-h-1} - x_0^{\mu}x_{n-b-h-1}) - x_0^{\eta}(x_n^a x_{n-h} - x_0^{\mu}x_{n-b-h}) =$
$= x_n^a(x_1 x_{n-h-1} - x_0^{\eta}x_{n-h}) - x_0^{\mu}(x_1 x_{n-b-h-1} - x_0^{\eta}x_{n-b-h}) \in \mathfrak{C}$.

(c) $h = n-b-1, \quad 1 \leq i \leq n-1$:
$x_{i+1}g_{n-b} - x_i g_{n-b-1} = x_{i+1}(x_n^a x_b - x_0^{(a+1)\eta+d}) - x_i(x_n^a x_{b+1} - x_0^{\mu}x_1) =$
$= x_n^a(x_{i+1}x_b - x_i x_{b+1}) - x_0^{\mu}(x_{i+1}x_0^{\eta} - x_i x_1) \in \mathfrak{C}$.

(d) $h = n-b-1, \quad i = 0$:
$x_1 g_{n-b} - x_0^{\eta} g_{n-b-1} = x_1(x_n^a x_b - x_0^{(a+1)\eta+d}) - x_0^{\eta}(x_n^a x_{b+1} - x_0^{\mu}x_1) = x_n^a(x_1 x_b - x_0^{\eta}x_{b+1})x_1) \in \mathfrak{C}$.

To prove " $\subseteq$ ", first note that by the above items we deduce

$$(e): \left[\begin{array}{ll} (a), (c) \Longrightarrow & x_i g_h = x_{i+1}g_{h+1} + \phi_h, \quad \phi_h \in \mathfrak{C}, \quad \forall\, 0 \leq h \leq n-b-1 \\ (b), (d) \Longrightarrow & x_0^{\eta} g_h = x_1 g_{h+1} + \psi_h, \quad \psi_h \in \mathfrak{C}, \quad \forall\, 0 \leq h \leq n-b-1, \quad 1 \leq i \leq n-1. \end{array} \right.$$

Now assume that $x_0 g_{h+1} \in \mathfrak{C} + (g_0, ..., g_h)$. Then $x_0 g_{h+1} = \beta + \alpha_0 g_0 + ...\alpha_h g_h,\ \alpha_i \in P,\ \beta \in \mathfrak{C}$. Hence, by (e):

$$x_0^{\eta+1} g_{h+1} = x_0^\eta \beta + \alpha_0 x_0^\eta g_0 + ... + \alpha_h x_0^\eta g_h = \beta_1 + \alpha_0 x_1 g_1 + ... + \alpha_h x_1 g_{h+1} =$$
$$\beta_2 + \alpha_0 x_2 g_2 + ... + (\alpha_{h-1} x_2 + \alpha_h x_1) g_{h+1} = ... = \beta_h + \alpha g_{h+1},\ \text{with } \beta_h \in \mathfrak{C},\ \ \alpha \in (x_1, ..., x_n).$$

This would imply that $(x_0^{\eta+1} - \alpha) g_{h+1} \in \mathfrak{C}$, impossible since $\mathfrak{C}$ is prime, $g_{h+1} \notin \mathfrak{C}$, $x_0^{\eta+1} - \alpha = x_0^{\eta+1} - (\alpha_0 x_{h+1} + \alpha_1 x_h + ... + \alpha_h x_1) \notin \mathfrak{C}$ (because $t^{(\eta+1)m_0} \notin (t^{m_1}, ..., t^{m_n})$). $\diamond$

The mapping cone construction starts with two well-known complexes, which are still exact for *GS* curves:

1.3 *Assume R weighted graded with* $deg(x_i) = m_i, \quad 0 \le i \le n$.

(1) The weighted-graded Eagon-Northcott, minimal free resolution of the determinantal R-ideal $\mathfrak{C} = \left(f_1, ..., f_{\binom{n}{2}}\right)$, is

$$\mathbb{E}: \quad 0 \longrightarrow E_{n-1} \overset{d_{n-1}}{\longrightarrow} E_{n-2} \overset{d_{n-2}}{\longrightarrow} \longrightarrow E_2 \overset{d_2}{\longrightarrow} E_1 \overset{d_1}{\longrightarrow} E_0$$

where $E_0 = P$, $\quad E_s = \wedge^{s+1} P^n \bigotimes \left(Sym_{s-1}(P^2)\right)^* \simeq P^{\beta_s}(-s-1)$, for $1 \le s \le n-1$, $\quad \beta_s = s\binom{n}{s+1}$, $Sym_{s-1}(P^2)$ free P-module of $rank\ s$.

(2) The Koszul complex $\mathbb{K}$, minimal free resolution for $P/(x_1, ..., x_n)$, is

$$\mathbb{K}: \quad 0 \longrightarrow K_n \overset{\Delta_1}{\longrightarrow} \ ... \ \longrightarrow K_1 \overset{\Delta_1}{\longrightarrow} K_0 \longrightarrow P/(x_1, ..., x_n) \longrightarrow 0$$

where $K_s = \wedge^s P^n, K_0 = P (= E_0)$.

(3) The maps in $\mathbb{E}$, $\mathbb{K}$ are defined as follows. Let

$\{\lambda_0^{v_0} \lambda_1^{v_1} | \, v_0 + v_1 = s-1\}$, $(1 \le s \le n-1)$ be a basis of $Sym_{s-1}(P^2)$;
$< e_{i_1} \wedge ... \wedge e_{i_{s+1}} \otimes \lambda_0^{v_0} \lambda_1^{v_1},\ (1 \le i_1 < i_2 < ... < i_{s+1} \le n, \quad v_0 + v_1 = s-1) >$ be a basis of E_s;

$$d_1 : E_1 \longrightarrow E_0, \ \begin{cases} e_1 \wedge e_{i_2} \mapsto (x_0^\eta x_{i_2} - x_1 x_{i_2 - 1}) \\ e_{i_1} \wedge e_{i_2} \mapsto (x_{i_1 - 1} x_{i_2} - x_{i_1} x_{i_2 - 1}), \quad if \quad 2 \le i_1 < i_2 \le n \end{cases};$$

$$d_s\left((e_{i_1} \wedge ... \wedge e_{i_{s+1}}) \otimes \lambda_0^{v_0} \lambda_1^{v_1}\right) = \Delta_0(e_{i_1} \wedge ... \wedge e_{i_{s+1}}) \otimes \lambda_0^{v_0 - 1} \lambda_1^{v_1} + \Delta_1(e_{i_1} ... \wedge e_{i_{s+1}}) \otimes \lambda_0^{v_0} \lambda_1^{v_1 - 1}, \quad s \ge 2,$$

where only the summands with non-negative powers of λ_0, λ_1 are considered, and, for $q = 0, 1, s \ge 1$, the maps $\Delta_q : \wedge^s P^n \longrightarrow \wedge^{s-1} P^n$ are defined as:

$$\Delta_q(e_i) := x_{i+q-1}, \quad (1 \le i \le n, \quad s = 1), \ \ \Delta_0(e_1) = x_0^\eta$$
$$\Delta_q(e_{i_1} \wedge ... \wedge e_{i_s}) := \sum_{j=1}^s (-1)^{j+1} x_{i_j + q - 1} \, e_{i_1} \wedge ... \widehat{e_{i_j}} ... \wedge e_{i_s}, \quad (1 \le i_1 < ... < i_s \le n, \quad 2 \le s \le n).$$

and when $j_i + q - 1 = 0$, the monomial x_0 must be replaced with x_0^η.

(4) Further for $(0 \le h \le n-b)$ denote respectively by

$$\left[\begin{array}{lll} \varepsilon_h & \text{the basis of} & K_0(-\delta_h), \quad 0 \le h \le n-b \\ e_{i_1}^{(h)} \wedge ... \wedge e_{i_s}^{(h)} & \text{the basis of} & K_s(-\delta_h), \quad (1 \le h \le n-b, \quad 1 \le i_1 < ... < i_s \le n) \\ e_{i_1}^{(0)} \wedge ... \wedge e_{i_{s+1}}^{(0)} \otimes \lambda_0^{v_0} \lambda_1^{v_1} & \text{the basis of} & E_s(-\delta_0), \quad (1 \le i_1 < i_2 < ... < i_{s+1} \le n, \quad v_0 + v_1 = s-1. \end{array} \right.$$

$$\text{with} \left[\begin{array}{l} weight(x_i) = m_i = \eta m_0 + id \ (0 \le i \le n) \\ weight(\lambda_0) = 0, \qquad weight(\lambda_1) = d, \\ weight(\varepsilon_h) = -\delta_h, \\ weight(e_{i_1}^{()} \wedge ... \wedge e_{i_s}^{()}) = m_{i_1} + ... + m_{i_s} - (s-1)d. \end{array} \right.$$

Then the Eagon-Northcott and Koszul complexes are naturally weighted graded, with the following grading of modules K_s, E_s:

$$K_s(-\delta_h) = \bigoplus_{1 \le i_1 < ... < i_s \le n} P\left(-\delta_h - m_{i_1} - ... - m_{i_s} + (s-1)d\right), \ \text{for} \quad 1 \le s \le n+1, \quad h = 0, ..., n-b,$$

$$E_s = \bigoplus_{0 \le v_1 \le s-1} \left[\bigoplus_{1 \le i_1 < ... < i_{s+1} \le n} P\left(-m_{i_1} - ... - m_{i_{s+1}} + (s - v_1)d\right) \right], \ \text{for} \quad 1 \le s \le n-1, \quad E_0 = P.$$

According to 1.3, quite similarly as done in [5], one can construct a resolution for the generating ideal I of a *GS* curve:

Theorem 1.4 [5, Theorems 3.8 and 4.1].

(1) *With Notation* (1.1), (1.3)*, the following complex is a free minimal resolution of the ideal I defining a GS curve:*

$$\mathcal{R}: \quad 0 \longrightarrow R_n \xrightarrow{d_n^*} R_{n-1} \; ... \; R_2 \xrightarrow{d_2^*} R_1 \xrightarrow{d_1^*} P$$

$R_1 = K_0(-\delta_{n-b}) \oplus ... \oplus K_0(-\delta_1) \oplus K_0(-\delta_0) \oplus E_1$ *and for* $s \geq 2$,

$$R_s = \left[\begin{array}{ll} E_{s-1}(-\delta_0) \oplus E_s & if \quad b = n \\ \Big(K_{s-1}(-\delta_{n-b}) \oplus ... \oplus K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0) \oplus E_s\Big)\Big/D_s & if \quad 1 \leq b \leq n-1 \\ D_s \subseteq K_{s-1}(-\delta_{n-b}) \oplus ... \oplus K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0), & \\ dim_k(D_s) = \nu_s\binom{n}{s} + \nu_{s-1}\binom{n}{s-1}, where\ \nu_p := min\{p-1, n-b\} & \end{array} \right.$$

(2) *The Betti numbers of the ideal I are*

$$\beta_s = dim(R_s) = \left[\begin{array}{ll} (n-b+2-s)\binom{n}{s-1} + s\binom{n}{s+1}, & if \quad 1 \leq s \leq n-b+1 \\ & \\ (s-1-n+b)\binom{n}{s} + s\binom{n}{s+1}, & if \quad n-b+2 \leq s \leq n \end{array} \right.$$

(3) *In particular, if* $b < n$: $\left[\begin{array}{l} R_2 = K_1(-\delta_{n-b}) \oplus ... \oplus K_1(-\delta_1) \oplus E_2 \\ dim\, R_2 = (n-b)n + 2\binom{n}{3} \end{array} \right.$, *if* $b = n$: $\left[\begin{array}{l} R_2 = E_1(-\delta_0) \oplus E_2 \\ dim\, R_2 = \binom{n}{2} + 2\binom{n}{3} \end{array} \right.$.

We note that if $n = 3$, the above Betti numbers could be deduced also from [16].
It is well-known that the Betti numbers satisfy $\beta_i(R) \leq \beta_i(Gr_{\mathfrak{m}}(R))$, in our *GS* case, the equalities hold for each i:

Corollary 1.5 *Let $R = k[S]$ where S is a GS semigroup. Then $\beta_i(R) = \beta_i(Gr_{\mathfrak{m}}(R))$ for each $i = 1, ..., n$.*

Proof. It is immediate by (1.4.2) and by [18, Theorem 4.1].

Corollary 1.6 *With Notation* 1.1*, the first syzygies of the generating ideal I of a GS curve can be described as follows:*

(1) *The* $2\binom{n}{3}$ *syzygies concerning the ideal $\mathfrak{C}$ are given as determinants of the 3×3 minors obtained by doubling a row in the matrix A.*

(2) *If* $1 \leq b \leq n-1$, *the remaining* $(n-b)n$ *syzygies can be written by expanding the determinants of the following matrices along the first column and the third row:*

$$\left\{ \begin{array}{l} 1 \leq h < n-b \\ 2 \leq i \leq n \end{array} \right. : \begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu & x_{n-h} & x_{n-h+1} \\ 0 & x_{i-1} & x_i \end{pmatrix}; \quad \left\{ \begin{array}{l} h = n-b \\ 2 \leq i \leq n \end{array} \right. : \begin{pmatrix} x_n^a & x_0^\eta & x_{n-b-h+1} \\ x_0^\mu & x_b & x_{b+1} \\ 0 & x_{i-1} & x_i \end{pmatrix};$$

$$\left\{ \begin{array}{l} 1 \leq h < n-b \\ (i = 1) \end{array} \right. : \begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu & x_{n-h} & x_{n-h+1} \\ 0 & x_0^\eta & x_1 \end{pmatrix}; \quad \left\{ \begin{array}{l} h = n-b \\ i = 1 \end{array} \right. : \begin{pmatrix} x_n^a & x_0^\eta & x_1 \\ x_0^\mu & x_b & x_{b+1} \\ 0 & x_0^\eta & x_1 \end{pmatrix}$$

(3) *If* $b = n$ *the remaining* $\binom{n}{2}$ *syzygies are trivial:* $f_i g_0 - f_i g_0 = 0$.

Proof. From the technical construction via mapping cone in the proof of Theorem 1.4, we deduce in particular that:

(1). The construction of the $2\binom{n}{3}$ determinantal first syzygies of $\mathfrak{C}$ by doubling one row of the matrix A, is well-known [10] (and comes out as $d_2^*(E_2)$).

(2). Let $1 \leq b \leq n-1$: the remaining $n(n-b)$ syzygies are obtained as

$$d_2^*(e_i^{(h)}), \quad \text{for } 1 \leq h \leq n-b, \quad 1 \leq i \leq n\,, \quad \text{where}$$

$$R_2 = K_1(-\delta_{n-b}) \oplus ... \oplus K_1(-\delta_1) \oplus E_2 \xrightarrow{d_2^*} R_1 = K_0(-\delta_{n-b}) \oplus ... \oplus K_0(-\delta_1) \oplus K_0(-\delta_0) \oplus E_1, \quad \text{and}$$

$d_2^* = \oplus_{1 \leq h \leq n-b}\ d_2^*(e_i^{(h)}) \ \oplus\ d_2(E_2)$, *with* $d_2^*(e_i^{(h)}) = (\Delta_1 - \Delta_0)(e_i^{(h)}) + e_i \wedge \big(x_n^a e_{n-h+1} - x_0^\mu e_{n-b-h+1}\big)$

$$\left[\begin{array}{l} d_2^*(e_1^{(h)}) = x_1\varepsilon_h - x_0^\eta\varepsilon_{h-1} + e_1 \wedge \big(x_n^a e_{n-h+1} - x_0^\mu e_{n-b-h+1}\big), \quad 1 \leq h \leq n-b \\ d_2^*(e_i^{(h)}) = x_i\varepsilon_h - x_{i-1}\varepsilon_{h-1} + e_i \wedge \big(x_n^a e_{n-h+1} - x_0^\mu e_{n-b-h+1}\big), \quad if \quad 2 \leq i \leq n, \quad 1 \leq h \leq n-b \\ d_2(e_h \wedge e_k \wedge e_l \otimes \lambda_q) = \Delta_q((e_h \wedge e_k \wedge e_l), \quad (q = 0, 1). \end{array} \right.$$

By noting that the map $d_1^* : R_1 \longrightarrow P$ is such that

$$\left[\begin{array}{l} d_1^*(\varepsilon_h) = g_h \\ d_1^*(e_1 \wedge e_j) = x_0^\eta x_j - x_1 x_{j-1} \\ d_1^*(e_i \wedge e_j) = x_{i-1}x_j - x_i x_{j-1} \quad 2 \le i < j \le n \end{array}\right.$$

we deduce for $i \geq 2$, that:

$$\left[\begin{array}{l} x_i d_1^*(\varepsilon_h) = x_i g_h = x_i(x_n^a x_{n-h} - x_0^\mu x_{n-b-h}), \\ d_1^*\big(e_1 \wedge (x_n^a e_{n-h+1} - x_0^\mu e_{n-b-h+1})\big) = x_n^a(x_0^\eta x_{n-h+1} - x_1 x_{n-h}) - x_0^\mu(x_0^\eta x_{n-b-h+1} - x_1 x_{n-b-h}), \\ d_1^*\big(e_i \wedge (x_n^a e_{n-h+1} - x_0^\mu e_{n-b-h+1})\big) = x_n^a(x_{i-1}x_{n-h+1} - x_i x_{n-h}) - x_0^\mu(x_{i-1}x_{n-b-h+1} - x_i x_{n-b-h}). \end{array}\right.$$

Therefore we have: if $1 \le h < n-b$,

$$\left[\begin{array}{l} d_1^* d_2^*(e_1^{(h)}) = x_1 g_h - x_0^\eta g_{h-1} + x_n^a(x_0^\eta x_{n-h+1} - x_1 x_{n-h}) - x_0^\mu(x_0^\eta x_{n-b-h+1} - x_1 x_{n-b-h}) = 0, \\ d_1^* d_2^*(e_i^{(h)}) = x_i g_h - x_{i-1} g_{h-1} + x_n^a(x_{i-1}x_{n-h+1} - x_i x_{n-h}) - x_0^\mu(x_{i-1}x_{n-b-h+1} - x_i x_{n-b-h}) = 0\,; \end{array}\right.$$

if $h = n-b$,

$$\left[\begin{array}{l} d_1^* d_2^*(e_1^{(n-b)}) = x_1 g_h - x_0^\eta g_{h-1} + x_n^a(x_0^\eta x_{b+1} - x_1 x_b) = 0, \\ d_1^* d_2^*(e_i^{(h)}) = x_i g_h - x_{i-1} g_{h-1} + x_n^a(x_{i-1}x_{b+1} - x_i x_b) + x_0^\mu(x_0^\eta x_i - x_1 x_{i-1}) = 0\ . \end{array}\right.$$

In each case, $d_1^* d_2^*(e_i^{(h)})$ is the expansion of the determinant of one of the above matrices , along the first column and the third row.

(3). If $b = n$, the remaining $\binom{n}{2}$ syzygies are obtained as $d_2^*\big(e_i^{(0)}\big)$, $1 \le i \le n$, with

$$R_2 = E_1(-\delta_0) \oplus E_2 \longrightarrow R_1 = K_0(-\delta_0) \overset{d_2^*}{\oplus} E_1 = E_0(-\delta_0) \oplus E_1 \quad , \quad d_2^*(e_i \wedge e_j) = d_1(e_i \wedge e_j) - g_0 \cdot (e_i \wedge e_j).$$

Hence $d_1^* d_2^*(e_i \wedge e_j) = d_1(e_i \wedge e_j) g_0 - g_0 d_1(e_i \wedge e_j)$. $\diamond$

2 Arf-GS monomial curves and their smoothability.

The definition of Arf semigroups comes from the classical one given by Lipman [11] for a semi-local ring R. When R is analitically irreducible, residually rational there exists a finite sequence $R = R_0 \subseteq R_1 \subseteq ... \subseteq R_{m-1} \subseteq R_m = \overline{R}$ of one dimensional local noetherian rings such that for each $1 \le i \le m$, the ring R_i is obtained from R_{i-1} by blowing up the maximal ideal $\mathfrak{m}_{i-1}$ of R_{i-1}. Then R is called *Arf* if for each $i = 1, ..., m$ the embedding dimension of R_i is equal to the multiplicity of R_i, i.e. $embdim(R_i) = e(R_i)$. In particular, if a local ring R is Arf, then R has maximal embedding dimension. According to Lipman's definition, for $X = Spec\, k[[S]]$, S numerical semigroup minimally generated by $m_0 < m_1 < ... < m_n$, the *blowing-up* $L(S)$ of S along the maximal ideal $M = S \setminus \{0\}$ is $L(S) = \cup_{h \ge 1}\big(hM - hM\big)$, where $hM := M + \cdots + M$ (h summands, $h \ge 1$), $hM - hM := \{z \in \mathbb{Z} \mid z + hM \subseteq hM\}$. It is well-known by [1, (1.2.4), and (1.3.1)] that:

(a) $L(S) =< m_0, m_1 - m_0, ..., m_n - m_0 >$.

(b) There exists a finite sequence of blowing-ups : $S \subseteq S_1 = L(S) \subseteq ... \subseteq S_m = L(S_{m-1}) = \mathbb{N}$.

Definition 2.1 *A numerical semigroup S is called an Arf semigroup if the sequences of its blowing-up $S \subseteq S_1 = L(S) \subseteq ... \subseteq S_m = L(S_{m-1}) = \mathbb{N}$ satisfy $embdim(S_i) = e(S_i)$ $\forall i = 1, ..., m$.*

According to a well-known result of Shaps [17] and Pinkham's Theorem [15], we get

Proposition 2.2 *Every GS semigroup as in Notation 1.1 of maximal embedding dimension is Weierstrass. In particular, each Arf semigroup generated by a generalized arithmetic sequence has the Weierstrass property .*

Proof. If S is a *GS* semigroup of maximal embedding dimension , then $n+1 = embdim(S) = e(S) = m_0$. Therefore $a = b = 1$, and so the defining ideal I is determinantal generated by the 2×2 minors of the matrix A'. Then the associated curve is smoothable according to [17] and so S is Weierstrass. In particular, it holds for the *GS* semigroups which are also *Arf* since (as above recalled) they have maximal embedding dimension.

As regard *Arf-GS* semigroups we recall the following characterization

Proposition 2.3 *We have:*

(1) *A numerical GS semigroup is Arf if and only if either* $e(S) = 2$, *or* $d = 1, 2$ [12, Prop.2.4 and its proof] .

(2) *Given a semigroup of maximal embedding dimension, minimally generated* $m_0 < m_1 < ... < m_n$, *if* $m_1 \equiv 1$, *then* S *is Arf if and only if it is GS (with* $d = 1$*).*

Proof. (1) follows by noting that for S minimally generated by $< m_0, \eta m_0 + d, ..., \eta m_0 + nd >$, with $\eta \geq 2$, then its blowing-up is $L(S) =< m_0, (\eta - 1)m_0 + d, ..., (\eta - 1)m_0 + nd >$; further $L(S_\eta) =< m_0, d >$.
(2). The proof is in [20, Prop.3.1]. $\diamond$

3 Smoothability of AS curves via modifications of the defining matrices.

In general, if X and Σ are schemes over a field k, a *deformation* of X over Σ is a k-scheme Y, flat over Σ, together with a closed immersion $X \hookrightarrow Y$ such that the induced map $X \to Y \times_\Sigma k$ is an isomorphism: namely there is a cartesian diagram $\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \\ \{0\} & \hookrightarrow & \Sigma \end{array}$ with π flat morphism.

A variety X is said to be *smoothable* if there exists an integral scheme Σ of finite type and a deformation Y of X over Σ admitting non-singular fibres.

This section deals with the *smoothability* of *AS* monomial curves (*GS* curves with $\eta = 1$); this topic is strictly closed to the classification of Weierstrass semigroups, see [3] for a survey. In fact, by Pinkham's Theorem [15], if the field k is algebraically closed of characteristic 0, the semigroup associated to any smoothable monomial curve is Weierstrass. We refer to [7] and [13], for the basic tools on deformations and Weierstrass semigroups.
For an *AS* semigroup S as in Notation 1.1, let $m_0 \equiv b \ (mod.\, n)$ and let X be the associated monomial curve. If $b = 1$ or $b = n$, then X is smoothable: for $b = 1$, the defining ideal I of X is determinantal and the result follows by [17], for $b = n$, see [13]. Further for $n \leq 4$, the smoothability of X is proved in [14].
Now we extend the above result to the curves verifying $n \leq 3b$. The bideterminantal shape of the ideal I and the determinantal description of the first syzygy module explained in (1.6), allow to construct suitable deformations of the matrices defining I and by this way to get immediately the flatness of the induced morphism, since the first syzygies naturally lift to the set of deformed generators. The proof of the smoothability is based on the following version of the classical Bertini's Theorem:

Theorem 3.1 *Let X be a nonsingular variety over an algebraically closed field k of characteristic 0. Let D be a finite dimensional linear system.Then almost every element of D, considered as a closed subscheme of X, is nonsingular (but maybe reducible) outside the base points of D.*[6, III Corollary 1.9, Remark 1.9.2]

We suppose the field k algebraically closed of characteristic 0, even if some of the following results hold under more general assumptions.

3.1 Case $n \leq 2b$.

We set up explicitly a 2-dimensional family of curves containing X as special fibre, whose generic member is regular.

Theorem 3.2 *Assume that* $n \leq 2b$. *Deform the matrices* A *and* A' *in Notation 1.1, respectively as*

$$A_{def} = \begin{pmatrix} x_0 & \dots & x_{n-b-1} & \dots & x_{n-1} \\ x_1 & \dots & x_{n-b} - V & \dots & x_n \end{pmatrix} \qquad A'_{def} = \begin{pmatrix} x_n^a & x_0 & \dots & x_{n-b} - V \\ x_0^\mu - U & x_b & \dots & x_n \end{pmatrix}.$$

Let $Y \subseteq \mathbb{A}^{n+3}$ *be the variety defined by the union of the* 2×2 *minors of* A_{def} *and* A'_{def}. *Then*
(1) *The ideal* $I_Y \subseteq k[x_0, ..., x_n, U, V]$ *is minimally generated by the* 2×2 *minors* $\{F_1, ..., F_{\binom{n}{2}}\}$ *of* A_{def} *(lexicographically ordered) and by the minors* $\{G_0, ..., G_{n-b}\}$ *of* A'_{def} *containing the first column.*
(2) *The induced morphism* $\pi : Y \longrightarrow Spec\, k[U, V]$ *is a deformation, with smooth fibres, of the monomial curve* X.

Proof. (1). The 2×2 minors of the matrix A'_{def} not containing the first column, have two possible shapes:

$$M_{i,j} = det \begin{pmatrix} x_i & x_j \\ x_{b+i} & x_{b+j} \end{pmatrix} \quad \text{with} \quad 0 \leq i < j < n - b,$$

$$M_{i,n-b} = det \begin{pmatrix} x_i & x_{n-b} - V \\ x_{b+i} & x_n \end{pmatrix} \quad \text{with} \quad 0 \leq i < n - b.$$

These minors belong to the ideal generated by the 2×2 minors of A_{def}. In fact:

$M_{i,j} = (x_i x_{b+j} - x_{i+1}x_{b+j-1}) + (x_{i+1}x_{b+j-1} - x_{i+2}x_{b+j-2}) + ... + (x_{j-1}x_{b+i+1} - x_j x_{b+i})$,
since $b \geq n-b$ and $b+j > b+j-1 > ... > b+i+1 > n-b$;
$M_{i,n-b} = (x_i x_n - x_{i+1}x_{n-1}) + (x_{i+1}x_{n-1} - x_{i+2}x_{n-2}) + ... + (x_{n-b-2}x_{b+i+2} - x_{n-b-1}x_{b+i+1}) + (x_{n-b-1}x_{b+i+1} - x_{b+i}(x_{n-b} - V))$, since $b+i+k > n-b$, for $1 \leq k \leq n-b-i$.

(2). According to this "compatibility" among the minors of the two matrices A_{def}, A'_{def} and by Theorem1.6, we can see that the expansions of the determinants of the following matrices along the first column and the third row, are relations among the chosen generators of I_Y which lift those of I found in (1.6):

$$\begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu - U & x_{n-h} & x_{n-h+1} \\ 0 & x_{i-1} & x_i \end{pmatrix} \quad \text{with} \quad 1 < h \leq n-b,\ 1 \leq i \leq n,\ i \neq n-b,$$

$$\begin{pmatrix} x_n^a & x_{n-b-1} & x_{n-b} - V \\ x_0^\mu - U & x_{n-1} & x_n \\ 0 & x_{i-1} & x_i \end{pmatrix} \quad \text{with} \quad h = 1, \quad 1 \leq i \leq n, \quad i \neq n-b,$$

$$\begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu - U & x_{n-h} & x_{n-h+1} \\ 0 & x_{n-b-1} & x_{n-b} - V \end{pmatrix} \quad \text{with} \quad 1 < h \leq n-b, \quad i = n-b,$$

$$\begin{pmatrix} x_n^a & x_{n-b-1} & x_{n-b} - V \\ x_0^\mu - U & x_{n-1} & x_n \\ 0 & x_{n-b-1} & x_{n-b} - V \end{pmatrix} \quad \text{with} \quad h = 1, \quad i = n-b.$$

Therefore there exists a flat morphism $\pi : Y \longrightarrow Spec\, k[U,V]$ with special fibre the curve X.
It remains to verify that this deformation has smooth fibres, equivalently, that the rank of the jacobian matrix of the generic fibre is n at every point. For this, we fix $V = V_0 \neq 0$ and we first prove that the *two*-dimensional variety Z defined by the minors of A_{def}, (with $V = V_0$), is non singular (Z is a deformation of the cone on the rational normal curve). This fact allows us to apply Bertini's (3.1) to Z and to the divisor D on Z defined by the element $G_0 = x_n^{a+1} - (x_0^\mu - U)(x_{n-b} - V_0)$: a fortiori the generic fibre X' of π is smooth outside the fixed points of D. Finally, by choosing other suitable generators of X', we shall deduce the regularity of X' at the above fixed points.
To show that the variety Z is regular, we prove that the Jacobian matrix $J_Z(P)$ has rank $n-1$ for each $V \neq 0$ and for each $P \in Z$. Consider the submatrix J'_Z of J_Z formed by the rows corresponding to the elements $F_1, \dots F_{n-1}$:

$$\begin{pmatrix} x_2 & -2x_1 & x_0 & 0 & \dots & 0 & 0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 & \dots & 0 & 0 & 0 \\ & \dots & \dots & & & & & \\ x_{n-b} - V_0 & -x_{n-b-1} & 0 & ... & x_0 & \dots & 0 & 0 \\ & \dots & & & \dots & & & \\ x_{n-1} & -x_{n-2} & & 0 & \dots & -x_1 & x_0 & 0 \\ x_n & -x_{n-1} & 0 & 0 & \dots & 0 & -x_1 & x_0 \end{pmatrix}$$

If $x_0 \neq 0$ the rank is $n-1$. The points belonging to Z with $x_0 = 0$ are

$$P(0,0,...,0,x_n), \qquad Q(0,0,...,0,V_0,x_{n-b+1},...,x_n).$$

At the points P a non vanishing minor with size $(n-1)$ of J_Z comes from the subset of generators of A_{def} of the shape

$$det \begin{pmatrix} x_i & x_{n-b-1} \\ x_{i+1} & x_{n-b} - V_0 \end{pmatrix} = x_i(x_{n-b} - V_0) - x_{i+1}x_{n-b-1}, \quad i = 0,...,n-1, \ i \neq n-b-1.$$

Now, to achieve the proof, consider the $(n \times (n+1))$ submatrix J''_Y of J_Y related to the minors

$$det \begin{pmatrix} x_i & x_{n-b} \\ x_{i+1} & x_{n-b+1} \end{pmatrix}, 0 \leq i \leq n-1, i \neq n-b \ \text{(of } A_{def}\text{)}, \quad \text{and} \quad det \begin{pmatrix} x_{n-b-1} & x_{n-b} \\ x_{n-b} - V_0 & x_{n-b+1} \end{pmatrix} = G_{n-b} \ :$$

$$
\left(\begin{array}{cccccccccccc}
x_{n-b+1} & -x_{n-b} & \dots & 0 & 0 & -x_1 & x_0 & \dots & 0 & & & \\
0 & x_{n-b+1} & \dots & 0 & 0 & -x_2 & x_1 & \dots & 0 & & & \\
0 & \dots & \dots & \dots & & & \dots & & & & & \\
0 & 0 & \dots & -x_{n-b} & 0 & -x_{n-b-2} & \dots & \dots & 0 & & & \\
0 & 0 & \dots & x_{n-b+1} & -x_{n-b} & -x_{n-b-1} & \dots & \dots & 0 & & & \\
0 & 0 & \dots & 0 & x_{n-b+1} & -2x_{n-b}+V_0 & x_{n-b-1} & \dots & 0 & & & \\
0 & 0 & \dots & 0 & \dots & x_{n-b+2} & -2x_{n-b+1} & & \dots & & & \\
0 & 0 & \dots & \dots & \dots & & & & \dots & & & \\
0 & 0 & \dots & \dots & & x_n & -x_{n-1} & \dots & & \dots & -x_{n-b+1} & x_{n-b} \\
-(\mu+1)x_0^{\mu}+U & 0 & \dots & & & 0 & \dots & \dots & x_n^a & \dots & & a x_n^{a-1} x_b
\end{array}\right)
$$

To see the regularity of Z at the points of type Q, one can consider the minor of J''_Y obtained by deleting the first, the $(n-b+2)$-th column and the last row. Therefore we conclude that Z is a regular variety.

Now by Bertini's Theorem 3.1, applied to Z and to the linear system D defined by G_0, it remains to prove that the generic fibre X' is smooth at the fixed points of D, which are

$$R(x_0, 0, ..., , 0, V_0, 0..., 0) \text{ (where } V_0 \text{ is the } (n-b+1)\text{-}th \text{ component of } R).$$

This fact holds because the Jacobian matrix $J_{X'}$ evaluated at the points R has rank n. In fact a $(n \times n)$ non-null minor in R is: $-V^{n-1}(\mu+1)x_0^{\mu}+U$ (the minor of J''_Y obtained by deleting the $(n-b+2)$-th column), for each R as above and for each $U \neq 0$ $\big($since $R \in X' \Longrightarrow G_{n-b}(R) = 0 \Longrightarrow x_0^{\mu} = U\big)$. $\diamond$

3.2 Case $2b < n \leq 3b$.

The preceding algorithm of deforming the matrices A, A' cannot always be used, because, in general, the conditions of compatibility which hold for $n \leq 2b$ are more than the number of parameters. In this section, we first note that if $hb < n \leq (h+1)b$ for each $h \geq 2$, there exists a matrix A'' such that the 2×2 minors of A, A'' are still a system of generators for the ideal I. In the case $h = 2$ by deforming the matrices A, A'', we still get a deformation of the curve X with smooth fibres.

Lemma 3.3 *With Notation* 1.1*, let* $A = \begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix}$ *and for* $h \geq 2$ *consider*

$$A'' = \left(\begin{array}{cccccc|ccc} x_n^{a-1} & x_0 & x_1 & \dots & x_{n-hb} & & x_{n-hb+1} & \dots & x_{n-b} \\ x_0^{\mu} & x_{n-hb+b}x_{hb} & x_{n-hb+b}x_{hb+1} & \dots & x_{n-hb+b}x_n & & x_{n-hb+1+b}x_n & \dots & x_n^2 \end{array}\right).$$

Then the 2×2 *minors of* A, A'' *are still a system of generators for the ideal* I*.*

Proof.

- for $0 \leq i \leq n-hb$: $det \begin{pmatrix} x_n^a & x_i \\ x_0^{\mu} & x_{i+b} \end{pmatrix} =$
  $x_n^{a-1}[(x_{i+b}x_n - x_{i+b+1}x_{n-1}) + ... + (x_{n-hb+b-1}x_{i+hb+1} - x_{n-hb+b}x_{i+hb})] +$
  $x_n^{a-1}x_{n-hb+b}x_{i+hb} - x_0^{\mu}x_i = x_n^{a-1}\alpha + det \begin{pmatrix} x_n^{a-1} & x_i \\ x_0^{\mu} & x_{n-hb+b}x_{i+hb} \end{pmatrix}$, where $\alpha \in \mathfrak{C}$ (see Notation 1.1);

- for $0 \leq i < j \leq n-hb$: $det \begin{pmatrix} x_i & x_j \\ x_{n-hb+b}x_{i+hb} & x_{n-hb+b}x_{j+hb} \end{pmatrix} =$
  $x_{n-h+b}[(x_ix_{hb+j} - x_{i+1}x_{hb+j-1}) + ... + (x_{j-1}x_{hb+i+1} - x_jx_{hb+i})] \in \mathfrak{C}$.

- for $0 \leq i \leq n-hb, n-hb+1 \leq j \leq n-b$: $det \begin{pmatrix} x_i & x_j \\ x_{n-hb+b}x_{i+hb} & x_nx_{j+b} \end{pmatrix} =$
  $x_{j+b}[(x_ix_n - x_{i+1}x_{n-1}) + ... + (x_{n-hb-1}x_{hb+i+1} - x_{n-hb}x_{hb+i})] +$
  $x_{hb+i}[(x_{n-hb}x_{b+j} - x_{n-hb+1}x_{b+j-1}) + ... + (x_{j-1}x_{n-hb+b+1} - x_jx_{n-hb+b})] \in \mathfrak{C}$.

- if $n-hb+1 \leq i < j \leq n-b$: $det \begin{pmatrix} x_i & x_j \\ x_{i+b}x_n & x_{j+b}x_n \end{pmatrix} \in \mathfrak{C}$. $\diamond$

Note also that the syzygies obtained with the method described in (1.6.b) by considering A, A'', instead of A, A' still generate the first syzygies module of I.
If $2b < n \leq 3b$ we can again find suitable "compatible" deformations of A, A'': the ideal I_Y generated by the union of the 2×2 minors of A_{def}, A''_{def} defines a deformation of X with smooth fibres, as in Theorem (3.2).

Theorem 3.4 *Assume $2b < n \leq 3b$. Deform the matrices A, A'' as*

$$A_{def} = \begin{pmatrix} x_0 & \dots & x_{n-b-2} & x_{n-b-1} & x_{n-b} & x_{n-b+1} & \dots & x_{n-1} \\ x_1 & \dots & x_{n-b-1} & x_{n-b} - V & x_{n-b+1} & x_{n-b+2} & \dots & x_n \end{pmatrix}$$

$$A''_{def} = \left(\begin{array}{ccccc|ccc} x_n^{a-1} & x_0 & x_1 & \dots & x_{n-2b} & x_{n-2b+1} & \dots & x_{n-b} - V \\ x_0^{\mu} - U & x_{n-b}x_{2b} & x_{n-b}x_{2b+1} & \dots & x_{n-b}x_n & x_{n-b+1}x_n & \dots & x_n^2 \end{array} \right).$$

Let $Y \subseteq \mathbb{A}^{n+3}$ be the variety defined by the union of the 2×2 minors of A_{def} and A''_{def}. Then
(1) *The ideal $I_Y \subseteq k[x_0, ..., x_n, U, V]$ is minimally generated by the 2×2 minors $\{F_1, ..., F_{\binom{n}{2}}\}$ of A_{def}, (lexicographically ordered), and by the minors $\{G_0, ..., G_{n-b}\}$ of A''_{def} containing the first column.*
(2) *The induced morphism $\pi : Y \longrightarrow Spec\, k[U, V]$ is a flat deformation of the monomial curve X with smooth fibres.*

Proof. The result follows with the same arguments as in Theorem 3.2. It suffices to prove the "compatibility" among the minors of A_{def}, A''_{def}.

- for $0 \leq i < j \leq n - 2b$, $det \begin{pmatrix} x_i & x_j \\ x_{n-b}x_{i+2b} & x_{n-b}x_{j+2b} \end{pmatrix} =$
$x_{n-b}[(x_i x_{2b+j} - x_{i+1}x_{2b+j-1}) + ... + (x_{j-1}x_{2b+i+1} - x_j x_{2b+i})] \in \mathfrak{C}_{def}$ since $2b + j > 2b + i \geq n - b$, by assumption.

- for $0 \leq i \leq n - 2b, n - 2b + 1 \leq j \leq n - b - 1$: $det \begin{pmatrix} x_i & x_j \\ x_{n-b}x_{i+2b} & x_n x_{j+b} \end{pmatrix} =$
$x_{j+b}[(x_i x_n - x_{i+1}x_{n-1}) + ... + (x_{n-2b-1}x_{2b+i+1} - x_{n-2b}x_{2b+i})] +$
$x_{2b+i}[(x_{n-2b}x_{b+j} - x_{n-2b+1}x_{b+j-1}) + ... + (x_{j-1}x_{n-2b+b+1} - x_j x_{n-2b+b})] \in \mathfrak{C}_{def}$, since
$n > ... > 2b + i + 1 > n - b$ and $b + j > ... > n - b + 1$.

- for $0 \leq i \leq n - 2b, j = n - b$: $det \begin{pmatrix} x_i & x_{n-b} - V \\ x_{n-b}x_{i+2b} & x_n^2 \end{pmatrix} =$
$x_n[(x_i x_n - x_{i+1}x_{n-1}) + ... + (x_{n-2b-1}x_{2b+i+1} - x_{n-2b}x_{2b+i})] +$
$x_{2b+i}[(x_{n-2b}x_n - x_{n-2b+1}x_{n-1}) + ... + (x_{n-b-2}x_{n-b+2} - x_{n-b-1}x_{n-b+1}) + (x_{n-b-1}x_{n-b+1} - x_{n-b}(x_{n-b} - V))] \in \mathfrak{C}_{def}$.

- if $n - 2b + 1 \leq i < j \leq n - b - 1$: $det \begin{pmatrix} x_i & x_j \\ x_{i+b}x_n & x_{j+b}x_n \end{pmatrix} \in \mathfrak{C}_{def}$.

- if $n - 2b + 1 \leq i, j = n - b$: $det \begin{pmatrix} x_i & x_{n-b} - V \\ x_{i+b}x_n & x_n^2 \end{pmatrix} = x_n[(x_i x_n - x_{i+1}x_{n-1}) + ... + (x_{n-b-2}x_{b+i+2} - x_{n-b-1}x_{b+i+1}) + (x_{n-b-1}x_{b+i+1} - x_{b+i}(x_{n-b} - V))] \in \mathfrak{C}_{def}$. $\diamond$

Recalling that if $b = 1$ the ideal I is determinantal and so the curve X is smoothable by [17], we deduce the following

Corollary 3.5 *With Notation 1.1 : the AS semigroups with $n \leq 3b$ are Weierstrass. In particular every AS semigroup with embedding dimension less or equal to seven and every semigroup with $b \neq 2$ and embedding dimension ≤ 10 have this property.*

Remark 3.6 When $n > 3b$ (e.g. for the *AS* semigroup with $n = 7, a = 1, b = d = 2$) we cannot find a compatible deformation of the two matrices A, A'. Hence it is more complicated to check the flatness of the possible maps. In these cases one can consider the module T^1_R and try to lift the infinitesimal deformations as described e.g. in [19] (see also the procedure we use in [13]). To this end we are implementing an algorithm with Cocoa [2].

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