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2015 J. Phys.: Conf. Ser. 588 012039

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# Probabilistic inversion: a preliminary discussion

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## Abstract.

We continue the discussion on the possibility of interpreting probability as a logic, that we have started in the previous IMEKO TC1-TC7-TC13 Symposium. We show here how a probabilistic logic can be extended up to including direct and inverse functions. We also discuss the relationship between this framework and the Bayes-Laplace rule, showing how the latter can be formally interpreted as a probabilistic inversion device. We suggest that these findings open a new perspective in the evaluation of measurement uncertainty.

## 1. Introduction

In the last Joint IMEKO TC1-TC7-TC13 Symposium (2013) we proposed a logical approach to probability [3-5;7-10] that proved to be quite effective in treating measurement representations accounting for uncertainty as an inherent feature [1-2;12-13]<sup>1</sup>.

We now proceed in this research line by providing tools for effectively describing the measurement process. It has been noted that the measurement process can be modelled in fairly general terms as the concatenation of an observation and a restitution phase. The former considers the "physical" behaviour of the measuring system, the latter includes the data processing required to produce a measurement result including a statement on measurement uncertainty [6]. If a probabilistic model is assumed for describing the observation phase, restitution may be interpreted as a "probabilistic inversion" of observation [11].

Yet, so far this interpretation has been provided on an intuitive basis mainly, assuming that the Bayes-Laplace rule can be understood as providing some kind of probabilistic inversion. Here we reconsider this question in a more formal way, following the logical approach to probability that we have recently developed, at least in an initial way.

## 2. Probabilistic functions

Let us briefly recall the general reference framework [13]. Consider a finite universe  $A$ , a finite collection  $\mathcal{E} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M\}$  of structures over  $A$  and a probabilistic distribution over  $\mathcal{E}$ ,  $P(\mathcal{A}_i)$ , such that

$$P(\mathcal{E}) = P\{\mathcal{A} \in \mathcal{E}\} = \sum_{i=1}^M P(\mathcal{A}_i) = 1. \quad (1)$$

Let  $\phi$  be any statement whose truth can be assessed on any element (structure) of  $\mathcal{E}$  [5]. Then

<sup>1</sup> References are listed in chronological order to give a feeling of the historical development of the subject



**Table 1.** An illustrative example of a probabilistic function from  $A$  to  $B$ .

Structure	Function	$f_i(a)$	$f_i(b)$	$P(\mathcal{A}_i)$
$\mathcal{A}_1$	$f_1$	$c$	$c$	0.2
$\mathcal{A}_2$	$f_2$	$d$	$d$	0.2
$\mathcal{A}_3$	$f_3$	$c$	$d$	0.5
$\mathcal{A}_4$	$f_4$	$d$	$c$	0.1

$$P(\phi) = P\{\mathcal{A} \in \mathcal{E}|\phi\} = \sum_{\mathcal{A}_i|\phi} P(\mathcal{A}_i), \tag{2}$$

where  $\mathcal{A} \in \mathcal{E}|\phi$  denotes a structure where the statement  $\phi$  is true.

In Refs. [12, 13] we have amply discussed the application of this framework to order and to generic m-ary relations. Let us now focus to *functions*. To this purpose, note that the statement

$$v = f(u), \tag{3}$$

referring to a function  $f : A \rightarrow B$ , may be seen as denoting a binary relation on  $A \times B$ , such that

$$\forall u \in A \exists v \in B (v = f(u)), \tag{4}$$

$$\forall u, v, z \in A (v = f(u) \wedge z = f(u) \Rightarrow v = z). \tag{5}$$

Therefore we can consider a finite collection of structures  $\mathcal{E} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M\}$ , where  $\mathcal{A}_i = (A, B, f_i)$ , and an associated probability distribution  $P(\mathcal{A}_i)$  over  $\mathcal{E}$ , that satisfies condition (1). Then the probability that the elements  $a \in A$  and  $b \in B$  are linked by the function  $f$  is given by

$$P(f(a) = b) = P\{\mathcal{A} \in \mathcal{E}|f(a) = b\} = \sum_{\mathcal{A}_i|f(a)=b} P(\mathcal{A}_i). \tag{6}$$

Consider, as an example,  $A = \{a, b\}$  and  $B = \{c, d\}$ . There are thus four possible functions from  $A$  to  $B$ , as shown in Table 1.

Suppose that a probability is assigned to each of them, as shown in the same table. Then it is possible to calculate  $P(f(u) = v)$ , for each  $u \in A$  and each  $v \in B$ . We obtain, in this case:

- $P(f(a) = c) = 0.7$ ,
- $P(f(a) = d) = 0.3$ ,
- $P(f(b) = c) = 0.3$ ,
- $P(f(b) = d) = 0.7$ .

### 3. Probabilistic inversion

Consider now the probabilistic inverse function to the above,  $g : B \rightarrow A$ . Note that in the deterministic case, only two of the four functions above are invertible, namely  $f_3$  and  $f_4$ . Instead, in the probabilistic case the notion of inversion becomes more general as we will see in the following development.

To approach the notion of probabilistic inversion, consider first that we can calculate directly the probabilities associated to each value of  $g$ , on the basis of the (complete) knowledge

**Table 2.** Probabilistic inverse function, from  $B$  to  $A$ .

Structure	Function	$g_i(c)$	$g_i(d)$	$P(\mathcal{B}_i)$
$\mathcal{B}_1$	$g_1$	$a$	$a$	0.21
$\mathcal{B}_2$	$g_2$	$b$	$b$	0.21
$\mathcal{B}_3$	$g_3$	$a$	$b$	0.49
$\mathcal{B}_4$	$g_4$	$b$	$a$	0.09

associated to the (direct) probabilistic function  $f$ , through the very definition of inverse function. In fact we can establish the following rule:

$$P(g(b) = a) \propto P\{\mathcal{A} \in \mathcal{E} | f(a) = b\} = \sum_{\mathcal{A}_i | f(a)=b} P(\mathcal{A}_i), \quad (7)$$

where anyway a proportionality, rather than an equality, holds between the first and the second member of the equation. In fact this rule does not provide directly a probability distribution; yet the proportionality factor can be calculated by imposing the following closure condition:

$$\sum_{v \in A} P(g(v) = a) = 1. \quad (8)$$

If we apply this rule to our numerical example, we obtain<sup>2</sup>:

- $P(g(c) = a) = 0.7$ ,
- $P(g(c) = b) = 0.3$ ,
- $P(g(d) = a) = 0.3$ ,
- $P(g(d) = b) = 0.7$ .

Under a different perspective, since  $g$  is a function from  $B$  to  $A$ , we recognise that we have four such functions, as shown in Table 2, where the corresponding structures,  $\mathcal{B}_k = (B, A, g_k)$ , are also defined. Their probabilities can thus be calculated by the rule

$$P(g_k) = P\left(\bigwedge_{b \in B} g(b) = g_k(b)\right). \quad (9)$$

As an example, we can apply this rule to our example and we obtain:

- $P(g_1) = P(g(c) = a \wedge g(d) = a) = P(g(c) = a)P(g(d) = a) = 0.7 \cdot 0.3 = 0.21$ ,
- $P(g_2) = P(g(c) = b \wedge g(d) = b) = P(g(c) = b)P(g(d) = b) = 0.3 \cdot 0.7 = 0.21$ ,
- $P(g_3) = P(g(c) = a \wedge g(d) = b) = P(g(c) = a)P(g(d) = b) = 0.7 \cdot 0.7 = 0.49$ ,
- $P(g_4) = P(g(c) = b \wedge g(d) = a) = P(g(c) = b)P(g(d) = a) = 0.3 \cdot 0.3 = 0.09$ ,

These results are summarized in the table. So rules (7-9) allow us to define the probabilistic inverse function  $g$ .

<sup>2</sup> In this example the proportionality becomes an equality, but this is due to the simplicity of the example and does not happen in the general case.

**Table 3.** Probabilistic function  $f$ , from  $X$  to  $Y$ .

Structure	Function	$f_i(1)$	$f_i(2)$	$f_i(3)$	$P(\mathcal{A}_i)$
$\mathcal{A}_1$	$f_1$	4	4	4	0.1
$\mathcal{A}_2$	$f_2$	4	4	5	0.4
$\mathcal{A}_3$	$f_3$	4	5	4	0.1
$\mathcal{A}_4$	$f_4$	4	5	5	0.2
$\mathcal{A}_5$	$f_5$	5	4	4	0.0
$\mathcal{A}_6$	$f_6$	5	4	5	0.1
$\mathcal{A}_7$	$f_7$	5	5	4	0.0
$\mathcal{A}_8$	$f_8$	5	5	5	0.1

**4. Relationship with the Bayes-Laplace rule**

Consider now the special but important case, where  $A$  and  $B$  are two finite sets of integers, that is  $A = X \in \mathbb{N}$  and  $B = Y \in \mathbb{N}$ . Then the function  $f : X \rightarrow Y$  can now be regarded as linking two probabilistic variables,  $x \in X$  and  $y \in Y$ . The corresponding probabilistic inverse function  $g : Y \rightarrow X$  can still be obtained by formulae (7-9).

From another standpoint, note that the *conditional distribution*  $P(y|x)$  can be defined by

$$P(y = j|x = i) = P\{\mathcal{A} \in \mathcal{E}|f(i) = j\} = \sum_{\mathcal{A}_k|f(i)=j} P(\mathcal{A}_k). \tag{10}$$

So there is a close link between the probabilistic function  $f$  and the conditional distribution  $P(y|x)$ .

Now, the (inverse) conditional distribution  $P(x|y)$  can be obtained by the Bayes-Laplace rule, assuming a uniform distribution for  $x$ . We obtain

$$P(x = i|y = j) \propto P(y = j|x = i), \tag{11}$$

where the proportionality factor can be fixed through the closure condition

$$\sum_y P(x|y) = 1. \tag{12}$$

On the other hand, such distribution can be obtained from the inverse function  $g$  through rule (8), now rewritten as

$$P(x = i|y = j) = P\{\mathcal{B} \in \mathcal{E}|g(j) = i\} = \sum_{\mathcal{B}_k|g(j)=i} P(\mathcal{B}_k), \tag{13}$$

where  $\mathcal{B}_k = (B, A, g_k)$  and the two paths yield consistent results.

An example is provided in Table 3, where  $X = \{1, 2, 3\}$  and  $Y = \{4, 5\}$ .

The corresponding inverse function, calculated through rules (7-9), is reported in Table 4. Alternatively, we can proceed by following the Bayes-Laplace rule. We thus calculate the (direct) conditional probability distribution  $P(y|x)$  and we obtain:

$$\begin{aligned} P(y = 4|x = 1) &= P(f(1) = 4) = 0.8 \\ P(y = 5|x = 1) &= P(f(1) = 5) = 0.2 \\ P(y = 4|x = 2) &= P(f(2) = 4) = 0.6 \\ P(y = 5|x = 2) &= P(f(2) = 5) = 0.4 \\ P(y = 4|x = 3) &= P(f(3) = 4) = 0.2 \end{aligned}$$

**Table 4.** Probabilistic inverse function  $g$ , from  $Y$  to  $X$ .

Structure	Function	$g_i(4)$	$g_i(5)$	$P(\mathcal{B}_i)$
$\mathcal{B}_1$	$g_1$	1	1	0.07
$\mathcal{B}_2$	$g_2$	1	2	0.14
$\mathcal{B}_3$	$g_3$	1	3	0.29
$\mathcal{B}_4$	$g_4$	2	1	0.05
$\mathcal{B}_5$	$g_5$	2	2	0.11
$\mathcal{B}_6$	$g_6$	2	3	0.21
$\mathcal{B}_7$	$g_7$	3	1	0.02
$\mathcal{B}_8$	$g_8$	3	2	0.04
$\mathcal{B}_9$	$g_9$	3	3	0.07

$$P(y = 5|x = 3) = P(f(3) = 5) = 0.8$$

Then we calculate the inverse probability distribution  $P(x|y)$ , through rules (9-10), and we obtain

$$P(x = 1|y = 4) = P(g(4) = 1) = 0.500$$

$$P(x = 2|y = 4) = P(g(4) = 2) = 0.375$$

$$P(x = 3|y = 4) = P(g(4) = 3) = 0.125$$

$$P(x = 1|y = 5) = P(g(5) = 1) \cong 0.143$$

$$P(x = 2|y = 5) = P(g(5) = 2) \cong 0.286$$

$$P(x = 3|y = 5) = P(g(5) = 3) \cong 0.571$$

Where some rounding of numerical values has been done to obtain a greater readability of the results. On the other hand, we can also calculate this distribution through (13), and the two procedures yield consistent results, apart from rounding effects.

To sum up,

- a probabilistic function,  $f$ , can be defined by formulae (3-6);
- the corresponding probabilistic inverse function,  $g$ , by formulae (7-9);
- the conditional distribution,  $P(y|x)$ , corresponding to function  $f$ , can be obtained according to formula (10)<sup>3</sup>;
- the inverse conditional distribution,  $P(x|y)$ , corresponding to the inverse function  $g$ , can be obtained thanks to the Bayes-Laplace rule, reported in formulae (11-12);
- the same inverse conditional distribution,  $P(x|y)$ , can also be obtained directly from the corresponding inverse function  $g$  through the rule expressed by (13), which thus established a linked between the Bayes-Laplace rule and the notion of probabilistic inverse function, here obtained after considering probability as a logic.

## 5. Conclusion

We have shown how a probabilistic logic can be developed up to including direct and inverse functions. In particular, we have shown that in a framework where probability is interpreted as a (first order) logic it is possible to formally define the notion of *probabilistic function* and of

<sup>3</sup> Here the (special) case where  $y = f(x)$  and  $x$  and  $y$  are probabilistic variables, i.e., numerical entities, is considered, yet a more general formulation is also possible in a very similar way.

*probabilistic inverse function*, and that the Bayes Laplace rule provides an alternative consistent way for performing *probabilistic inversion*.

Therefore, the Bayes-Laplace rule can be *formally*, and not only intuitively, interpreted as a probabilistic inversion device. This opens a new perspective in the development of a probabilistic approach to measurement, that frees it from unnecessary links with subjective interpretations of probability [13]. The full exploitation of these ideas will be the object of future studies.

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