K3 SURFACES WITH A NON-SYMPLECTIC AUTOMORPHISM AND PRODUCT-QUOTIENT SURFACES WITH CYCLIC GROUPS

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ABSTRACT. We classify all the K3 surfaces which are minimal models of the quotient of the product of two curves $C_1 \times C_2$ by the diagonal action of either the group $\mathbb{Z}/p\mathbb{Z}$ or the group $\mathbb{Z}/2p\mathbb{Z}$. These K3 surfaces admit a non-symplectic automorphism of order p induced by an automorphism of one of the curves C_1 or C_2. We prove that most of the K3 surfaces admitting a non-symplectic automorphism of order p (and in fact a maximal irreducible component of the moduli space of K3 surfaces with a non-symplectic automorphism of order p) are obtained in this way.

In addition, we show that one can obtain the same set of K3 surfaces under more restrictive assumptions namely one of the two curves, say C_2, is isomorphic to a rigid hyperelliptic curve with an automorphism δ_p of order p and the automorphism of the K3 surface is induced by δ_p.

Finally, we describe the variation of the Hodge structures of the surfaces constructed and we give an equation for some of them.

1. INTRODUCTION

One of the main themes of interest in the study of K3 surfaces S regards their automorphisms. We call an automorphism $\mathbf{g}$ of S *non-symplectic* if it acts non-trivially on the nowhere vanishing holomorphic 2-form ω. In the case $|\mathbf{g}| = p$, a prime number, $\mathbf{g}(\omega) = \zeta_p \omega$, where ζ_p is a primitive p-th root of unity. The pairs $(S, \mathbf{g})$ are quite rare, in the sense that there are strict restrictions on both p, which must be smaller then or equal to 19, and on the K3 surface S, which cannot be generic in the moduli space. More precisely, the families of K3 surfaces with a non-symplectic automorphism of odd prime order have a finite number of connected components and the biggest of them has dimension 9 while the moduli space of the K3 surfaces is 20-dimensional.

Several authors worked on the classification of K3 surfaces admitting a non-symplectic automorphism of odd prime order (see, e.g., [AS08], [AST11],[K92], [OZ98], [OZ99], [OZ00]) and eventually their complete classification is given in [AST11]. The classification is based on the following procedure: First, a non-symplectic automorphism $\mathbf{g}$ of order p acting on a K3 surface S, determines an action on the lattice $T := (H^2(S, \mathbb{Z})^{\mathbf{g}})^{\perp}) \subset H^2(S, \mathbb{Z})$. The lattice T satisfies several conditions. Second, one lists all the lattices with such conditions. Third, for each lattice T in the list one has to actually construct an example of a K3 surface admitting a non-symplectic automorphism $\mathbf{g}$ and such that $T \simeq (H^2(S, \mathbb{Z})^{\mathbf{g}})^{\perp})$. Each example is given by an *ad hoc* construction.

The aim of this paper is to give a systematical way to construct most of these K3 surfaces by showing that they are the minimal models of product-quotient surfaces (i.e., of the minimal resolution of the quotient $(C_1 \times C_2)/G$ where the C_i are curves of genus $g(C) \geq 1$ and G is a finite group acting diagonally on the product, see Definition 3.2). In addition, as we have already observed, the pairs $(S, \mathbf{g})$ are quite "special", we give here a geometrical interpretation of this "speciality" in many cases: S is the minimal model of a product-quotient surface. Moreover, we prove that there is a curve which seems to play a central rôle in this construction. We define the curve D_p as the hyperelliptic curve with equation $v^2 = u^p - 1$. It clearly admits an automorphism δ_p of order p, acting on u as the multiplication by ζ_p, and an automorphim τ_p of order $2p$ which is the composition of δ_p with the hyperelliptic involution. We prove the following theorem.

Theorem 1.1. *If S is a K3 surface admitting a non-symplectic automorphism of order $p = 3$ (resp. $p = 5, 7, 11, 13$, $p = 17, 19$) whose fixed locus contains at least 2 (resp. 1, 0) curves, then it is the minimal model of a resolution of the quotient $(C_1 \times D_p)/(\mathbf{g_1} \times \tau_p)$, where $\mathbf{g_1}$ is an automorphism of C_1 of order $2p$. The non-symplectic automorphism of order p on S is induced by the automorphism* id $\times \delta_{\mathrm{p}}$.

Any K3 surface S constructed in the Theorem 1.1 admits two isotrivial families of curves, whose general member is isomorphic to C_1 and D_p respectively. The non-symplectic automorphism of order p on S is in fact given by the action of an automorphism on each member of one of these families.

We observe that the K3 surfaces given by the Theorem 1.1 admit a non-symplectic automorphism of order $2p$ too, the one induced by $\mathrm{id} \times \tau_p$. This gives a geometrical explanation of the following significative result on non-symplectic automorphisms: If a K3 surface S admits a non-symplectic automorphism of order p, under certain conditions on the fixed locus it admits in fact a purely non-symplectic automorphism of order $2p$ (see [GS13, Theorems 1.6, 1.7] and [Di12] for the precise statement).

The proof of the Theorem 1.1 is based on the construction of the K3 surfaces that we now explain. We first bound the genus of ther curves C with a cyclic group of automorphisms G of order p (resp. $2p$), having the properties that $C/G \simeq \mathbb{P}^1$, and such that there exists an eigenspace $H^{1,0}(C)_{\zeta_p^i}$ (resp. $H^{1,0}(C)_{-\zeta_p^i}$) of dimension 1 of the induced action. This is achieved by exploiting the Riemann Existence Theorem, the Holomorphic Lefschetz Fixed-Point formula, and the Chevalley–Weil formula. Second, we classify all these curves and we couple them by choosing the action of G in such a way that the singular surfaces $(C_1 \times C_2)/G$ have $p_g = 1$ and $q = 0$. Third, we resolve the singularities and we get product-quotient surfaces X which are not minimal models, but in two cases. We observe that K_X^2 could be very negative. After having found all the (-1)-curve on X (this is a quite delicate task see e.g., [BP12]) we carefully contract them to produce a minimal model S of $(C_1 \times C_2)/G$. Finally, we prove that S is a K3 surface.

As a byproduct of the proof we classify the K3 surfaces which are minimal models of the product-quotient surfaces with the groups $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/2p\mathbb{Z}$. Hence, we obtain also a "negative" result: if a K3 surface does not satisfy the hypothesis of Theorem 1.1 (i.e., either it does not admit a non-symplectic automorphism of order p, or it admits a non-symplectic automorphism of order p, but its fixed locus does not satisfy the condition of the Theorem 1.1), then this K3 surface is not the minimal model of a product-quotient with group $\mathbb{Z}/2p\mathbb{Z}$. However, we cannot exclude that such a K3 surface is the minimal model of a product-quotient with a different group. Indeed, we are aware that this is the case for at least certain families of K3 surfaces admitting a non-symplectic automorphism of order 3. We shall analyze this problem in a forthcoming article.

This paper is organized as follows.

In Section 2 we briefly recall three classical results: the Riemann Existence Theorem, the Holomorphic Lefschetz Fixed-Point formula, and the Chevalley–Weil formula. We establish the upper and lower bound for the genus of a curve C with a cyclic group of automorphisms of odd prime order, and the property that there exists an eigenspace $H^{1,0}(C)_{\zeta_p^i}$ of dimension 1. We state similar results for cyclic groups of automorphisms of order $2p$. Moreover, we give some explanatory examples introducing the curve D_p.

Section 3 is divided into three parts. In the first part we give the definition of product-quotient surfaces and we recall the properties of these surfaces that are needed for our purposes. In the second part we calculate the Hodge numbers of product-quotient surfaces. In the last subsection we describe the automorphisms of the minimal model of a product-quotient surface.

In Section 4 we describe, first, the procedure we used to construct product-quotient surfaces with group either $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/2p\mathbb{Z}$ and $p_g = 1$ and $q = 0$. Second, we give a method to prove that these surfaces are K3 surfaces.

Sections 5 and 6 present the main results of the paper on K3 surfaces which are minimal models of product-quotient surfaces with group either $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/2p\mathbb{Z}$ respectively. Moreover, in these two sections one can find the tables with the surfaces we constructed (see `Table 1` and `Table 2`).

In Section 7 we give the equations for the singular models of the K3 surfaces constructed for $p = 3, 5, 7, 11$. In particular if $p = 5, 7, 11$ we describe these surfaces as hypersurfaces in weighted projective spaces.

In the last section we describe the variation of the Hodge structures of the K3 surfaces constructed in terms of the Hodge structure of $H^1(C_1)$ relating them by a half twist.

In the Appendix there is the `MAGMA` script of the program that we used. There are essentially two programs `Surfacesp` and `t1t2PtsSurfaces`. The former one gives a list of all product-quotient surfaces X with group $\mathbb{Z}/p\mathbb{Z}$ (p any odd prime), $p_g(X) = 1$ and $q(X) = 0$, as well as the singularities of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$, this program becomes very slow as p increases. The latter program gives a similar list, it is much faster, it works also for the group $\mathbb{Z}/2p\mathbb{Z}$, but it requires two additional data which are the number of ramification points of the two coverings $C_i \to C_i/G$.

Notation: We work over the field of complex numbers $\mathbb{C}$. We will denote by $\zeta_n := e^{\frac{2\pi i}{n}}$ a primitive n-th root of unity.

By "curve" or "Riemann surface" we mean a projective, non-singular curve C, we denote by $H^{1,0}(C) = H^0(C, \Omega_C^1)$ and by $g(C) := h^{1,0}(C) := dim H^{1,0}(C)$ the *genus* of the curve.

By "surface" we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^{2,0}(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^{1,0}(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic*. The *Noether formula* is $12\chi(S) = K_S^2 + e(S)$, where $e(S)$ is the Euler number of S.

By abuse of notation by "(-1)-curve" we mean a curve C with $C \simeq \mathbb{P}^1$ and $C^2 = -1$.

2. Curves

This section is devoted to recall some classical results: a reformulation of the Riemann Existence Theorem, the Holomorphic Lefschetz Fixed-Point formula, and the Chevalley–Weil formula, as well as fixing the notation. Moreover, here we establish all the properties for the curves that we will use to construct K3 surfaces in the next sections.

2.1. The Riemann Existence Theorem.

Definition 2.1. *Let $m_1, \ldots, m_r$ be positive integers with $m_i \geq 2$ for all i. A* polygonal group *of type $(m_1, \ldots, m_r)$ is a group presented as follows:*

$$\Gamma(m_1, \ldots, m_r) := \langle \gamma_1, \ldots, \gamma_r | \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = \gamma_1 \cdot \ldots \cdot \gamma_r = 1 \rangle.$$

Definition 2.2. *Let Γ be a polygonal group and G be a finite group. An epimorphism $\theta : \Gamma(m_1, ..., m_r) \to G$ is called* admissible *if $\theta(\gamma_i) = x_i$ has order exactly m_i for all i. If an admissible epimorphism exists, then the image $(x_1, \ldots, x_r)$ of a set of generators of Γ is called a* spherical system of generators *for G.*

The following is a reformulation of *Riemann's Existence Theorem* (see e.g., [M95] chapter III, sections 3 and 4):

Theorem 2.3. *A finite group G acts as a group of automorphisms on some compact Riemann surface C such that $C/G \simeq \mathbb{P}^1$ if and only if there are natural numbers $m_1, \ldots, m_r$, and an admissible epimorphism*

$$\theta : \Gamma(m_1, \ldots, m_r) \to G.$$

The genus $g(C)$ is determined by the Riemann–Hurwitz relation:

$$(1) \qquad 2g(C) - 2 = |G| \left(-2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

The $G-$cover $C \to \mathbb{P}^1$ is branched in r points $p_1, \ldots, p_r$ with branching number $m_1, \ldots, m_r$, respectively. Moreover, the cyclic subgroups $\langle x_i \rangle$ and their conjugates are the non-trivial stabilizers of the action of G on C.

In this paper we always assume G to be cyclic, so any spherical system of generators is of the form $(x^{\xi_1}, \ldots, x^{\xi_r})$, where $\xi_i \in \mathbb{Z}/n\mathbb{Z}$. Moreover, we are interested only in the cases:

$$G = \mathbb{Z}/n\mathbb{Z} \text{ with } n = p \text{ or } n = 2p \text{ and } p \text{ prime.}$$

From now on we shall assume that we are in one of these cases. Notice that some of the results are true also for more general cases.

As an automorphism group of C, $G = \langle \mathbf{g} \rangle$ gives a conformal self-mapping $\mathbf{g} : C \to C$ of order n. Please notice that the abstract group G has also other realizations: as image of an admissible epimorphism $\langle x \rangle$ or as local action $\langle \zeta_n \rangle$ near a point $P \in C$. Suppose that $\mathbf{g}^k$ fixes a point $P \in C$, then in a suitable local coordinate z near P we must have $\mathbf{g}^k(z) = \zeta_n^i z$. Thus $\mathbf{g}^k$ is locally a rotation at P and the rotation angle is determined in the following proposition.

Proposition 2.4. [Ha71, Theorem 7] *Let C be a curve associated to the spherical system of generators $(x^{\xi_1}, \ldots, x^{\xi_r})$. Let P_j be a point with a non trivial stabilizer in $\langle \mathbf{g} \rangle \subset \mathrm{Aut}(C)$ of order m generated by $\mathbf{g}^{\xi_j}$. Then $\mathbf{g}^{\xi_j}(z) = \zeta_n^{\eta_j} z$ where z is the local coordinate near the point P_j and $\xi_j \eta_j \equiv n/m \mod n$ and $0 < \eta_j < m$.*

Assume that $n = p$ with p prime, denote by a_i the number of ramification points of C where the local action is given by ζ_p^i. Notice that in this case we have total ramification over each branch point. The condition of Theorem 2.3 is equivalent to the condition

$$(2) \qquad \sum_{i=i}^{p-1} i^{-1} a_i \equiv 0 \bmod p,$$

where the inverse of i is taken $\bmod p$. We shall call this condition the *numerical monodromy condition*.

2.2. **The Holomorphic Lefschetz Fixed-Point Formula.** Let $\mathbb{Z}/n\mathbb{Z} \simeq G \subset Aut(C)$, then there is an induced linear action of G on the Dolbeault cohomology $H^{*,*}(C)$. Let us recall the holomorphic analogous of the Lefschetz Fixed-Point formula.

Theorem 2.5. (Holomorphic Lefschetz Fixed-Point Formula [GH, p. 426]) *Let G be a group of automorphism of a smooth complex curve C. Let $\mathbf{g} \in G$ then it holds*

$$1 - Tr(\mathbf{g} \mid_{H^0(X, \Omega_X^1)}) = \sum_{\substack{P \in C \\ \mathbf{g}(P) = P}} \frac{1}{1 - \zeta_P},$$

where ζ_P is the local action of $\mathbf{g}$ in P.

In particular in the case $G \simeq \mathbb{Z}/p\mathbb{Z}$ we have a decomposition in eigenspaces

$$H^{1,0}(C) = \bigoplus_{j=1}^{p-1} H^{1,0}(C)_{\zeta_p^j}.$$

The dimension of each eigenspace will be denoted by

$$\alpha_j := dim H^{1,0}(C)_{\zeta_p^j}.$$

With this piece of notation we can rewrite the Holomorphic Lefschetz Fixed-Point Formula in the following way.

Corollary 2.6. *Let C be a curve and $G = \langle \zeta_p \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ then it holds*

$$(3) \qquad -\sum_{j=1}^{p-1} \zeta^j \alpha_j = \sum_{l=1}^{p-1} a_l \frac{1}{1 - \zeta^l} - 1.$$

Proposition 2.7. (The Chevalley–Weil Formula) *Let C be a cyclic $p : 1$ cover $\mathbb{P}^1$. Let a_i be the number of ramification points of C where the local action is given by $z \mapsto \zeta_p^i z$, and $\alpha_j := dim H^{1,0}(C)_{\zeta_p^j}$. Then α_j are determined by the a_i's:*

$$(4) \qquad \alpha_{p-r} = -1 + \frac{1}{p} \sum_{l=1}^{p-1} a_l l^{-1} (p - r),$$

where the product $l^{-1}(p-1)$ is taken $\bmod p$, and $r = 1, \ldots, n - 1$.

Proof. First let us consider the LHS of (3)

$$-\sum_{j=1}^{p-1} \zeta^j \alpha_j = -\sum_{j=1}^{p-2} \alpha_j \zeta^j + \alpha_{p-1} + \sum_{j=1}^{p-2} \alpha_{p-1} \zeta^j = \alpha_{p-1} + \sum_{j=1}^{p-2} (\alpha_{p-1} - \alpha_j) \zeta^j.$$

We can rewrite the above equation as

$$(5) \qquad \begin{bmatrix} -1 & 0 & \ldots & \ldots & 1 \\ 0 & -1 & 0 & \ldots & 1 \\ \vdots & & \ddots & & 1 \\ 0 & & & -1 & 1 \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_{p-1} \end{bmatrix} = \begin{bmatrix} \text{coeff. of } \zeta \\ \text{coeff. of } \zeta^2 \\ \vdots \\ \vdots \\ \text{coeff. of } 1 \end{bmatrix}.$$

Second looking at the RHS of (3) we have,

$$(6) \qquad \sum_{l=1}^{p-1} a_l \frac{1}{1-\zeta^l} = \frac{1}{p} \sum_{l=1}^{p-1} a_l \Big(\sum_{k=0}^{p-1} (p-1-k)\zeta^{lk} \Big) = \frac{1}{p} \sum_{l=1}^{p-1} a_l \Big(\sum_{h=0}^{p-1} (p-1-hl^{-1})\zeta^h \Big) =$$

$$= \frac{1}{p} \sum_{l=1}^{p-1} a_l \Big[\sum_{h=0}^{p-2} (p-1-hl^{-1})\zeta^h - \big(p-1-(p-1)l^{-1}\big) \sum_{h=0}^{p-2} \zeta^h \Big] = \frac{1}{p} \sum_{l=1}^{p-1} a_l \Big(\sum_{h=0}^{p-2} l^{-1}(p-h-1)\zeta^h \Big),$$

where the product $l^{-1}(p-h-1)$ is taken mod p. Comparing the coefficient of ζ^0 from the LHS and the RHS of (3) we obtain

$$(7) \qquad \alpha_{p-1} = -1 + \frac{1}{p} \sum_{l=1}^{p-1} a_l l^{-1}(p-1).$$

More generally we have by (5)

$$\alpha_{p-r} = \alpha_{p-1} - \text{coeff of } \zeta^r.$$

By substituting the expression of the coefficient of ζ^r given in (6) we obtain (4). $\qquad \square$

Notice that the above proposition is a special case of the well known Chevalley–Weil Formula, see [CW34].

The above proposition has many important applications.

Corollary 2.8. *Let r be the number of ramification points for $\pi \colon C \to C/(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{P}^1$. If there exists $i \in \{1, \ldots, p-1\}$ such that $\alpha_i = 1$, then $(p-1)/2 \leq g(C) \leq (p-1)^2$ and $3 \leq r \leq 2p$.*

Proof. By (4) we have

$$g(C) = \sum_{k=1}^{p-1} \alpha_k = 1 - p + \sum_{k=1}^{p-1} \Big[\frac{1}{p} \sum_{l=1}^{p-1} a_l l^{-1}(p-k) \Big].$$

Notice that $\sum_{k=1}^{p-1} l^{-1}(p-k)$ is the sum of the first $p-1$ integers, hence it is equal to $p(p-1)/2$, and this gives $g(C)$. The equation $\alpha_i = 1$ gives conditions on the a_i's. Substituting these values in $g(C)$ one obtains $(p-1)^2 - \sum_i \lambda_i a_i$ with $\lambda_i \in \mathbb{N}$, which is less then or equal to $(p-1)^2$. The minimal genus is realized by the curve in Example 2.9.
The case $p = 3$ was already studied in [vG92]. $\qquad \square$

Notice that by (7) the condition: α_{p-1} being a natural number, is equivalent to the numerical monodromy condition (2). Hence, by the Riemann Existence Theorem we have a curve with group of automorphism $\mathbb{Z}/p\mathbb{Z}$ and quotient $\mathbb{P}^1$ once we provide a integral soltution of (7).

Example 2.9. Let $G \simeq \mathbb{Z}/p\mathbb{Z}$. The triple (x^{p-1}, x^{p-1}, x^2) is a spherical system of generators for G. Hence, by Theorem 2.3 there exists a curve, D_p, such that D_p is a $p : 1$ cover of $\mathbb{P}^1$ branched in 3 points and the cover automorphism δ_p acts locally as ζ_p^{p-1} near two fixed points and as $\zeta_p^{(p+1)/2}$ near the other one. This means that the only non zero a_i's are $a_{p-1} = 2$, $a_{(p+1)/2} = 1$. For every choice of three points in $\mathbb{P}^1$ there exists an involution of $\mathbb{P}^1$ switching the first two points and fixing the other, so there exists an involution of $\mathbb{P}^1$ acting in this way on the branch points of $D_p \to \mathbb{P}^1$. This induces an involution on D_p, which will be denoted by ι_p. We observe that $\iota_p \delta_p = \delta_p \iota_p$ and we denote by $\tau_p = \iota_p \delta_p$. An equation of D_p and the corresponding equation for its automorphisms are:

$$D_p : u^p = v^2 - 1, \quad \delta_p : (u,v) \mapsto (\zeta_p u, v), \quad \iota_p : (u,v) \mapsto (u,-v), \quad \tau_p : (u,v) \mapsto (\zeta_p u, -v).$$

The genus of D_p, computed by the Riemann–Hurwitz formula, is $g(D_p) = (p-1)/2$. The curve D_p is hyperelliptic over $\mathbb{P}^1_{[u]}$ and so a basis for $H^{0,1}(D_p)$ is given by $\{u^j du/v\}$, $j = 0, \ldots, (p-3)/2$. Therefore the eigenspaces decomposition of $H^{0,1}(D_p)$ for the induced action of δ_p is $\alpha_i = 1$ if $i = 1, \ldots, (p-1)/2$ and $\alpha_i = 0$ if $i = (p+1)/2, \ldots, p-1$. This agrees with the results stated in Proposition 2.7.
Finally we observe that D_3 is the elliptic curve with complex multiplication of order 3, which is associated to the automorphism δ_3.

There are analogous results for $G \simeq \mathbb{Z}/2p\mathbb{Z}$. The calculations are similar to the ones done above, which is why we omit it here. To get the formulae one can use the `MAGMA` script `MaxGenus` in the Appendix. Despite the similarities, there is a difference between the case $\mathbb{Z}/p\mathbb{Z}$ and the case $\mathbb{Z}/2p\mathbb{Z}$. In the latter case we succeeded in finding a maximal genus only if we ask that the eigenspace $H^{0,1}(C)_{\zeta_{2p}^i} = 1$ is relative to an element of maximal order. Indeed, we obtain the following.

Corollary 2.10. *Let r be the number of ramification points for $\pi\colon C \to C/(\mathbb{Z}/2p\mathbb{Z}) \simeq \mathbb{P}^1$. If there exists $i \in \{1,\ldots,p-1\}$ such that $dim(H^{1,0}(C))_{-\zeta_p^i} = 1$, then $(p-1)/2 \leq g(C) \leq (2p-1)^2$ and $3 \leq r \leq 4p$.*

Example 2.11. Let us consider the curve D_p and the automorphism τ_p of order $2p$ (cf. Example 2.9) whose associated spherical system of generators is (x^{p+2}, x^{2p-2}, x^p). The action of τ_p on the form $u^i du/v$ is the multiplication by $-\zeta_p^{i+1}$, $0 \leq i \leq (p-3)/2$. Hence, the eigenspaces decomposition of $H^{1,0}(D_p)$ for τ_p is the following: if $0 \leq i \leq (p-3)/2$ and $i \equiv 0 \mod 2$, then $\alpha_{i+1} = 1$; if $0 \leq i \leq (p-3)/2$ and $i \equiv 1 \mod 2$, then $\alpha_{p+i+1} = 1$; $\alpha_j = 0$ otherwise. We observe that τ_p switches two of the fixed points of δ_p and fixes the third fixed point of δ_p. Clearly the two points switched by τ_p are the ones where the local action of δ_p is ζ_p^{p-1} and the point fixed both by τ_p and by δ_p is the one where the local action of δ_p is $\zeta_p^{(p+1)/2}$. The automorphism τ_p^2 coincides with δ_p^2 and this identifies the local action of τ_p: τ_p fixes one point with local action $-\zeta_p^{(p+1)/2}$; τ_p^2 fixes other two points (switched by τ_p) with local action ζ_p^{p-2}; τ_p^p fixes other p points (permuted by τ_p) with local action $-\zeta_p^p = -1$.

3. Surfaces

In this section we recall the properties of the product-quotient surfaces and their minimal models and we calculate their numerical invariants.

3.1. **Product-Quotient Surfaces.** Let us consider two curves C_1 and C_2 of genera greater then or equal to 1, and their product $C_1 \times C_2$. Then $\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2) \subset \mathrm{Aut}(C_1 \times C_2)$. One can say even more in the case $g(C_i) \geq 2$.

Lemma 3.1. [C00, Corollary 3.9] *Let us assume $g(C_1) \geq 2$ and $g(C_2) \geq 2$. If $C_1 \not\simeq C_2$, then $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$, otherwise $\mathrm{Aut}(C_1 \times C_2) = (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)) \rtimes \mathbb{Z}/2\mathbb{Z}$*

Let $G \subset \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$ be a finite group and consider $(C_1 \times C_2)/G$, where G acts diagonally on the product $C_1 \times C_2$.

Definition 3.2. *The minimal resolution X of the singularities of $(C_1 \times C_2)/G$, where G is a finite group with a diagonal action on the direct product of two smooth curves C_1 and C_2 of respective genera at least 1, is called a* product-quotient surface *with group G.*
We call $(C_1 \times C_2)/G$ the quotient model *of the product-quotient surface.*

Product-quotient surfaces were intensively studied, we refer to [BP12], [MP10] and [BCGP09] for a detailed account of them. We recall only the facts that are important for our purposes.

Remark 3.3. The following facts hold:
  (1) There are only finitely many points on $C_1 \times C_2$ with non trivial stabilizer, which is cyclic. Therefore the quotient model has only a finite number of cyclic quotient singularities.
    A *cyclic quotient singularity* is locally analytic isomorphic to the quotient of $\mathbb{C}^2$ by the diagonal linear automorphism with eigenvalues $e^{\frac{2\pi i}{n}}$, $e^{\frac{2\pi i q}{n}}$ with $(q,n) = 1$. This singularity is called of *type* $\frac{1}{n}(1, q)$, or $\frac{q}{n}$ for short.
  (2) The exceptional divisor E on the minimal resolution of a cyclic quotient singularity is given by a Hirzebruch-Jung string (see e.g., [BHPV, Chapter III, Section 5]). A *Hirzebruch-Jung string* (HJ-string, for short) is a union $\widetilde{E} := \cup_i^k E_i$ of smooth rational curves E_i such that:
    • $E_i^2 = -b_i \leq -2$ for all i,
    • $E_i E_j = 1$ if $\mid i - j \mid = 1$,
    • $E_i E_j = 0$ if $\mid i - j \mid \geq 2$,
    where the b_i's are given by the *continued fraction* associated to $\frac{1}{n}(1, q)$. Indeed, by the formula:
$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\dots - \frac{1}{b_k}}}.$$
  (3) A product-quotient surface comes together with two isotrivial fibrations. Let us consider one of them: $\pi_2 : X \to C_2/G$. Take any point $b \in C_2/G$, and let F denote the fibre of π_2 over b. Then (see [S96, Theorem 2.1]):
    • The reduced structure of F is the union of an irreducible smooth curve Y, called the *central component* of F, and either none or at least two mutually disjoint HJ-strings, each one meeting Y at one point. These strings are in one-to-one correspondence with the branch points of $C_1 \to (C_1/H)$, where $H \subset G$ is the stabilizer of b.

- The intersection of a string with Y is transversal and it takes place at only one of the end component of the string.

(4) There are formulae for calculating the self intersection of the canonical divisor of a product-quotient surface.

$$K_X^2 \geq \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in Sing(X)} h_x,$$

where h_x depends on the type of singularity at x and the equality holds if $g(C_i) \geq 2$, for at least one value of $i \in \{1,2\}$. If x is a cyclic quotient singularity of type $\frac{1}{n}(1,q)$ then:

$$h_x := 2 - \frac{2+q+q'}{n} - \sum_{i=1}^{k}(b_i - 2),$$

where $q' \in \{1,\ldots,n-1\}$ is such that $qq' \equiv 1 \ mod \ n$, and $[b_1,\ldots,b_k]$ is the continued fraction associated to $\frac{1}{n}(1,q)$.

(5) Finally, notice that X is in general not a minimal model. Indeed, we will treat mostly examples with X not minimal.

3.2. **Hodge Structure of X.** Sometimes it will be useful to keep track of the action of an automorphism on each curve separately; that is why in the following we assume $G \simeq \mathbb{Z}/n\mathbb{Z} \simeq \langle x \rangle \hookrightarrow \mathrm{Aut}(C_1)$ $x \mapsto \mathbf{g_1}$, and $G \hookrightarrow \mathrm{Aut}(C_2)$, $x \mapsto \mathbf{g_2}$. We have $G = \langle \mathbf{g_1} \times \mathbf{g_2} \rangle \subset \mathrm{Aut}(C_1 \times C_2)$, and we write $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ for $(C_1 \times C_2)/(\langle \mathbf{g_1} \times \mathbf{g_2} \rangle)$.

We shall describe the Hodge structure of the product-quotient surface X whose quotient model is $(C_1 \times C_2)/G$. The Hodge numbers of X are determined by the action of G on the cohomology of C_1 and C_2.

As in the previous section we denote by α_i, $i = 1,\ldots n$ the dimension of the eigenspace $H^{1,0}(C_1)_{\zeta_n^i}$ with eigenvalue ζ^i w.r.t. the action of $\mathbf{g_1}$ and by β_i, $i = 1,\ldots n$, the dimension of the eigenspace $H^{1,0}(C_2)_{\zeta_n^i}$ with eigenvalue ζ_n^i w.r.t. the action of $\mathbf{g_2}$.

By [F71, Satz 1] we have

$$H^0(X, \Omega_X^i) \simeq H^0(C_1 \times C_2, \Omega_{C_1 \times C_2}^i)^G.$$

with $i = 0,1,2$

Thus by the Künneth formula, we have

- $H^{0,0}(X) = H^{0,0}(C_1 \times C_2)^G = H^{0,0}(C_1) \otimes H^{0,0}(C_2)$;
- $H^{1,0}(X) = H^{1,0}(C_1 \times C_2)^G = H^{1,0}(C_1)_{id} \otimes H^{0,0}(C_2) \oplus H^{0,0}(C_1) \otimes H^{1,0}(C_2)_{id}$,
  in particular $h^{1,0}(X) = h^{0,1}(X) = \alpha_n + \beta_n$;
- $H^{2,0}(X) = H^{2,0}(C_1 \times C_2)^G = \sum_{i=1}^{n} H^{1,0}(C_1)_{\zeta_n^i} \otimes H^{1,0}(C_2)_{\zeta_n^{n-i}}$,
  in particular $h^{2,0}(X) = h^{0,2}(X) = \sum_{i=1}^{n} \alpha_i \beta_{n-i}$.

In order to find $h^{1,1}(X)$, one has to know the number and the type of singularities of $(C_1 \times C_2)/G$. Indeed, the desingularization of $(C_1 \times C_2)/G$ introduces some exceptional divisors which increase the Picard number of the surface and thus $h^{1,1}(X)$.

Here we describe the structure of $H^{1,1}(X)$ starting from the description of the action of $\mathbf{g_i}$ on C_i, $i = 1,2$.

Let us consider the set of points of C_1 (resp. C_2) with a non trivial stabilizer w.r.t. the action of $\mathbf{g_1}$ (resp. $\mathbf{g_2}$). Let us denote by $a_{i,h}$ (resp. $b_{i,h}$) the number of points on C_1 (resp. C_2) whose stabilizer has order h and such that the local action of $\mathbf{g_1}^{|\mathbf{g_1}|/h}$ (resp. $\mathbf{g_2}^{|\mathbf{g_2}|/h}$) is ζ_h^i. Let $P \in C_1$ (resp. $Q \in C_2$) be a point with stabilizer of order h (resp. k) where local action of $\mathbf{g_1}$ (resp. $\mathbf{g_2}$) is ζ_h^i, $i \in \{1\ldots h\}$ (resp. ζ_k^j, $j \in \{1\ldots k\}$). We assume h (resp. k) are divisors of $|\mathbf{g_1}|$ different from $|\mathbf{g_1}|$. We will say that $P \times Q$ is a point of type $((i,h),(j,k))$ and clearly the number of these points is $a_{i,h}b_{j,k}$. The stabilizer of these points has order $d_{(h,k)} := \gcd(h,k)$ and the orbit of $P \times Q$ w.r.t. $\mathbf{g_1} \times \mathbf{g_2}$ contains $|\mathbf{g_1}|/d_{(h,k)}$ distinct points. In particular we observe that if $d_{(h,k)} = 1$ the stabilizer of $P \times Q$ is empty and its orbit contains exactly $|\mathbf{g_1}|$ points and if $i = j = |\mathbf{g_1}|$, then the stabilizer of $P \times Q$ is $\mathbb{Z}/|\mathbf{g_1}|\mathbb{Z}$ and the orbit of $P \times Q$ consists only of $P \times Q$. Let us now consider the quotient model $(C_1 \times C_2)/G$ and the image $\overline{P \times Q}$ of the point $P \times Q$ for the quotient map: the point $\overline{P \times Q}$ is smooth if $d_{(h,k)} = 1$; otherwise it is a singular points of type $\frac{1}{d_{(h,k)}}(1,q)$ where q can be computed as in [BHPV, Prop. 5.3]. The images of the $a_{i,h}b_{j,k}$ points of type $((i,h),(j,k))$ under the quotient map consists of $a_{i,h}b_{j,k}d_{h,k}/|\mathbf{g_1}|$ points.

Now we assume that $n = p$ is a prime number: then h (resp. k) is necessarily 1, hence $a_{i,h} = a_i$ (resp. $b_{j,k} = b_j$) with the notation of Section 2. The singular point $\overline{P \times Q} \in (C_1 \times C_2)/G$ is of type $\frac{1}{p}(1, ij^{-1})$, where ij^{-1} is computed in mod p. Moreover, the orbit of every point with non trivial stabilizer consists only of one point and thus we have exactly $a_i b_j$ singular points on $(C_1 \times C_2)/G$ which are of type $\frac{1}{p}(1, ij^{-1})$.

Once one knows the number and the type of singularities of $(C_1 \times C_2)/G$ one can easily compute $h^{1,1}(X)$, recalling that every singularity we introduce a HJ-string, by Remark 3.3 (2), and that

$$H^{1,1}(C_1 \times C_2)^G = \left(H^{1,1}(C_1) \otimes H^{0,0}(C_2)\right) \oplus \left(\oplus_i \left(H^{1,0}(C_1)_{\zeta_n^i} \otimes H^{0,1}(C_2)_{\zeta_n^{n-i}}\right)\right) \oplus$$

$$\oplus \left(\oplus_i \left(H^{0,1}(C_1)_{\zeta_n^i} \otimes H^{1,0}(C_2)_{\zeta_n^{n-i}}\right)\right) \oplus \left(H^{0,0}(C_1) \otimes H^{1,1}(C_2)\right),$$

In particular, since $H^{0,1}(C_j)_{\zeta_n^i} \simeq \overline{H^{1,0}(C_j)_{\zeta_n^{n-i}}}$, $j = 1, 2$, $h^{1,1}(C_1 \times C_2)^G = 2(1 + \sum_{i=1}^{n} \alpha_i \beta_i)$.

The following proposition recap the results proved in this section.

Proposition 3.4. *Let C_i, $i = 1, 2$ be a curve with an automorphism $\mathbf{g_i}$ of order n. Let $\alpha_l := dim(H^{1,0}(C_1)_{\zeta_n^l})$ and $\beta_m := dim(H^{1,0}(C_2)_{\zeta_n^m})$. Let $a_{(i,h)}$, $b_{(j,k)}$ and $d_{(h,k)}$ as above. Note that i and j are invertible in $\mathbb{Z}/d_{(h,k)}\mathbb{Z}$ and we denote by r_i and s_j their inverse. Let X be the minimal resolution of the quotient surface $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. Then the Hodge numbers of X are:*

$$h^{0,0}(X) = 1, \quad h^{1,0}(X) = \alpha_n + \beta_n, \quad h^{2,0}(X) = \sum_{i=1}^{n} \alpha_i \beta_{n-1}$$

$$h^{1,1}(X) = 2(1 + \sum_{i=1}^{n} \alpha_i \beta_i) + \sum_{h,k} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (a_{i,h} b_{j,k} d_{(h,k)}/n) k(i,j)$$

where:
- *we pose $k(i,j) = 0$ if $d_{(h,k)} = 1$;*
- *$k(i,j)$ is the number of curves introduced by a singularity of type $\frac{1}{d_{(h,k)}}(1,q)$;*
- *$q := is_j \in \mathbb{Z}/d_{(h,k)}\mathbb{Z}$ (or equivalently $q := jr_i \in \mathbb{Z}/d_{(h,k)}\mathbb{Z}$).*

Remark 3.5. We recall that α_i and β_i are uniquely determined by a_i and b_i and viceversa, thus the Hodge numbers of X depend only on the set of values $\{\alpha_i \ \beta_i\}$, $i = 1, \dots, n$ (or equivalently on the set of values $\{a_i \ b_i\}$, $i = 1, \dots, n$).

3.3. The minimal model S. As observed in the last point of Remark 3.3 the surface X is in general non minimal. Let us denote by S the minimal model of X. Since $h^{i,0}$ are birational invariant, the Hodge numbers $h^{0,0}$, $h^{1,0}$, $h^{2,0}$ of the product-quotient X coincide with the ones of its minimal minimal model S. More complicate is the computation of $h^{1,1}(S)$.

In order to determine the minimal model S of a product-quotient X, we have to find all the (-1)-curves on X. In the cases we will treat the (-1)-curves are central components of reducible fibres of one or both isotrivial fibrations of X. Then, after contractions of these, the (-1)-curves could be images of some divisors in the HJ-strings. Note that this is in general not true, see e.g., [BP12]. A quick method to calculate the self intersection of the central components is given in [P10]. We shall recall it.

Definition 3.6. *We say that a reducible fibre F_1 of $\pi_2 \colon X \longrightarrow C_2/G$ is of type $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$ if it contains exactly r HJ-strings $\widetilde{E}_1, \dots, \widetilde{E}_r$, where each $\widetilde{E}_i$ is of type $\frac{1}{n_i}(1, q_i)$. The same definition holds for a reducible fibre F_2 of $\pi_1 \colon S \longrightarrow C_1/G$.*

Proposition 3.7. *[P10, Proposition 2.8] Let F_1 be of type $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$ and let Y_1 be its central component. Then*

$$(Y_1)^2 = -\sum_{i=1}^{r} \frac{q_i}{n_i}.$$

If F_2 is of type $\left(\frac{q_1'}{n_1}, \dots, \frac{q_r'}{n_r}\right)$ then $(Y_2)^2 = -\sum_{i=1}^{r} \frac{q_i'}{n_i}$.

In the following example we construct two surfaces, S_1 and S_2, both of them are the minimal model of quotients of $D_3 \times D_3$ by a diagonal action of $\mathbb{Z}/3\mathbb{Z}$ but the action of this group is different in the two cases. As a consequence the minimal resolution of one quotient has an infinite number of (-1)-curves, the minimal resolution of the other has no (-1)-curves and the minimal models of these two resolutions are totally different: one of them is a rational surface, one is a K3 surface.

Example 3.8. Let us consider the product of two elliptic curves $D_3 \times D_3$ and its automorphisms $\delta_3 \times \delta_3^1$ and $\delta_3 \times \delta_3^2$. We will denote by X_i the minimal resolution of $(D_3 \times D_3)/(\delta_3 \times \delta_3^i)$ and by S_i its minimal model. We recall that δ_3^i acts on $H^{1,0}(D_3)$ as the multiplication by ζ_3^i. We obtain $h^{0,0}(X_i) = h^{0,0}(S_i) = 1$, $h^{1,0}(X_i) = h^{1,0}(S_i) = 0$ for $i = 1, 2$; $h^{2,0}(X_1) = h^{2,0}(S_1) = 0$ and $h^{2,0}(X_2) = h^{2,0}(S_2) = 1$.

Now we compute $h^{1,1}(X_i)$: on $(D_3 \times D_3)/(\delta_3 \times \delta_3)$ there are 9 singularities of type $\frac{1}{3}(1,1)$ and on $(D_3 \times D_3)/(\delta_3 \times \delta_3^2)$ there are 9 singularities of type $\frac{1}{3}(1,2)$. The resolution of a point of type $\frac{1}{n}(1,1)$ introduces 1 curve of self intersection $-n$, and thus $h^{1,1}(X_1) = 9 + dim\left(H^{1,1}(D_3 \times D_3)^{\delta_3 \times \delta_3}\right) = 13$. The desingularization of singularities of type $\frac{1}{n}(1, n-1)$ introduces $n - 1$ rational curves with self intersection -2, whose dual diagram is A_{n-1}, and thus $h^{1,1}(X_2) = 2 \cdot 9 + dim\left(h^{1,1}(D_3 \times D_3)^{\delta_3 \times \delta_3^2}\right) = 20$.

With the same method of the Section 7 we obtain that an equation for $(D_3 \times D_3)/(\delta_3 \times \delta_3^i)$ is given by $y^2 = x^3 + (v^2 - 1)^{2i}$, which is the equation of an elliptic fibration over $\mathbb{P}^1_{[v]}$. If $i = 1$, this elliptic fibration is a rational elliptic fibration (this depend on the degree of $(v^2 - 1)^{2i}$). By the standard theory of elliptic fibration (cf. [M89]), it has 3 reducible fibers (over 1, -1 and ∞) of type IV (coming from the contraction of the central components of the reducible fibers on X_1). The rank of the Mordell–Weil group of this elliptic fibration is $10 - 2 - 6 = 2$. This implies that there are infinite sections of this elliptic fibration, and thus an infinite number of (-1)-curves. Since the minimal elliptic fibration is obtained by contractions of X_1, we obtain that X_1 contains an infinite number of (-1)-curve. A minimal model, S_1, of X_1 is birational to $\mathbb{P}^2$, so $h^{1,1}(S_1) = 1$.

The elliptic fibration $y^2 = x^3 + (v_2^2 - 1)^4$ has a K3 surface as minimal model. The reducible fibers of this elliptic fibration are 3 fibers of type IV^* each of them consists of the central component (which is a (-2)-curve, by Proposition 3.7) and of three copy of A_2. So X_2 coincides with S_2 and $h^{1,1}(S_2) = 20$. This K3 surface was constructed in [SI77].

3.4. Automorphisms of S and quotient surfaces.

By construction the surface $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ always admits an automorphism of order n induced by $\mathrm{id} \times \mathbf{g_2} \in \mathrm{Aut}(C_1 \times C_2)$ (or equivalently by $\mathbf{g_1} \times id \in \mathrm{Aut}(C_1 \times C_2)$). This automorphism lifts to an automorphism of X and of S. Thus one can consider the quotient $((C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2}))/(\mathrm{id} \times \mathbf{g_2})$. Since $\langle \mathbf{g_1} \times \mathbf{g_2}, \mathrm{id} \times \mathbf{g_2}\rangle = \langle \mathbf{g_1} \times \mathrm{id}, \mathrm{id} \times \mathbf{g_2}\rangle$, we have the following commutative diagram:

$$
\begin{array}{ccc}
 & C_1 \times C_2 & \\
\swarrow n:1 & & \searrow n:1 \\
(C_1 \times C_2)/\mathbf{g_1} \times \mathbf{g_2} & & (C_1 \times C_2)/\mathrm{id} \times \mathbf{g_2} \\
\searrow n:1 & & \swarrow n:1 \\
 & C_1/\mathbf{g_1} \times C_2/\mathbf{g_2}. &
\end{array}
$$

This diagram lifts to the minimal resolution of all the surfaces we are considering, and so one obtains that the surface X has a generically $n^2 : 1$ map to $C_1/\mathbf{g_1} \times C_2/\mathbf{g_2}$, and in particular to $\mathbb{P}^1 \times \mathbb{P}^1$, if we assume $C_i/\mathbf{g_i} \simeq \mathbb{P}^1$. Moreover, the map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ induces a rational $n^2 : 1$ map $S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

We can explicitly describe the action of $\mathrm{id} \times \mathbf{g_2}$ on the cohomology of X (we keep the assumption $C_i/\mathbf{g_i} \simeq \mathbb{P}^1$): $\mathrm{id} \times \mathbf{g_2}$ acts as the identity on the spaces $H^{0,0}(X)$ and $H^{1,0}(X)$. The invariant subspace of $H^{2,0}(X)$ under $\mathrm{id} \times \mathbf{g_2}$ is the image of the space $H^{1,0}(C_1)^{\mathbf{g_1}} \otimes H^{1,0}(C_2)^{\mathbf{g_2}}$ and thus has dimension $\alpha_n \beta_n = 0$. As we saw in Section 3.2, the space $H^{1,1}(X)$ splits into two parts: the image of $H^{1,1}(C_1 \times C_2)^{\mathbf{g_1} \times \mathbf{g_2}}$ and a direct summand, say R, which comes from the resolution of the singularities of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. Hence, $H^{1,1}(X)^{\mathrm{id} \times \mathbf{g_2}}$ splits into the direct sum of the image of $(H^{0,0}(C_1) \otimes H^{1,1}(C_2)) \oplus (H^{1,0}(C_1)^{\mathbf{g_1}} \otimes H^{0,1}(C_2)^{\mathbf{g_2}}) \oplus (H^{0,1}(C_1)^{\mathbf{g_1}} \otimes H^{1,0}(C_2)^{\mathbf{g_2}}) \oplus (H^{1,1}(C_1) \otimes H^{0,0}(C_2))$ and $R^{\mathrm{id} \times \mathbf{g_2}}$. The dimension of the first term is $2 + 2\alpha_n \beta_n = 2$. We note that, if every point in the branch locus of C_i, $i = 1, 2$ is of total ramification, then the action of $\mathrm{id} \times \mathbf{g_2}$ is the identity on R, and one finds $H^{1,1}(X)^{\mathrm{id} \times \mathbf{g_2}} = (H^{0,0}(C_1) \otimes H^{1,1}(C_2)) \oplus (H^{1,1}(C_1) \otimes H^{0,0}(C_2)) \oplus (R \otimes \mathbb{C})$.

4. K3 SURFACES

This section is devoted to the construction of K3 surfaces S, which are minimal model of product-quotient surfaces.

We recall that by definition a *K3 surface* S has $h^{1,0}(S) = 0$ and trivial canonical bundle. The Hodge numbers of S are uniquely determined by these properties and are $h^{0,0}(S) = h^{2,0}(S) = 1$, $h^{1,0}(S) = 0$, $h^{1,1}(S) = 20$.

4.1. **Product-quotient surfaces with $p_g = 1$ and $q = 0$.** Let S be the minimal model of a product-quotient X with quotient model $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. If S is a K3 surface, then $q(S) = h^{1,0}(S) = 0$ and $p_g(S) = h^{2,0}(S) = 1$. Since $h^{i,0}$ are birational invariants, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 1$. Therefore, by Proposition 3.4, $\alpha_n + \beta_n = 0$ and $\sum_{i=1}^{n} \alpha_i \beta_{n-i} = 1$. In particular

(8) $\qquad \alpha_n = \beta_n = 0$ and there exists $i \in \{1 \ldots n\}$ such that $\alpha_i = \beta_{n-i} = 1$, $\quad \alpha_j \beta_{n-j} = 0$ if $j \neq i$.

Condition (8) is divided in two: a condition on each factor of the product, namely the action of $\mathbf{g_i}$ on $H^{1,0}(C_i)$; and a condition on the whole product, namely the action of $\mathbf{g_1} \times \mathbf{g_2}$ on $H^{2,0}(C_1 \times C_2)$.

Now we shall give a procedure to construct product-quotient surfaces with $p_g = 1$ and $q = 0$. Since, in general, this algorithm requires excessively long calculations, many of them are performed using the `MAGMA` script in the Appendix. While describing the procedure we also indicate which part of the program does what.

First we search for pairs $(C_1, \mathbf{g_1})$ which satisfy the following two conditions: $\alpha_n = 0$, or equivalently $C_i/\mathbf{g_i} \simeq \mathbb{P}^1$; and there exists an index $0 < j < n$ such that $\alpha_j = 1$. The first condition is obtained by giving spherical systems of generators for G (see Theorem 2.3). The second one is obtained exploiting the Chevalley–Weil formula and calculating the rotation angles as in Proposition 2.4. Analogous conditions must hold for a second pair $(C_2, \mathbf{g_2})$.

Second we couple the curves by requiring that there exists an $i \in \{1 \ldots n\}$ such that $\alpha_i = \beta_{n-i} = 1$ and $\alpha_j \beta_{n-j} = 0$ if $j \neq i$.

Let us assume that $n = p$ a prime. The curves C_i, admitting an automorphism $\mathbf{g_i}$ of order p such that $C_i/\mathbf{g_i} \simeq \mathbb{P}^1$ and $\alpha_j = 1$ for some $j \in \{1, \ldots, p-1\}$, have genus at most $(p-1)^2$ by Corollary 2.8. The number of the ramification points of the cover $C_i \to C_i/\mathbf{g_i}$ is at most $2p$. This implies that the number of curves with these properties is finite. These curves are classified by the `MAGMA` program given in the Appendix. The calculation can be found in the function `Surfacesp`. The function calculates all the partitions of all the numbers from 3 up to $2p$, giving all the admissible ramification data $(a_1, \ldots, a_{n-1})$ of the coverings $C_i \to C_i/g_i \simeq \mathbb{P}^1$. Afterwards it evaluates the α_j, as in Proposition 2.7, and lists only the ones with $\alpha_j = 1$ for at least one $j \in \{1, \ldots, p-1\}$.

Remark 4.1. We observe that for every prime p there exists at least one curve with the required properties, the one with $a_{p-1} = 2$ and $a_{(p+1)/2} = 1$ (cf. Example 2.9).

The condition $\alpha_j \beta_{n-j} = 0$ if $j \neq i$ implies that the list $(\alpha_1, \ldots, \alpha_{p-1}, \beta_1, \ldots \beta_{p-1})$ contains at least $p - 2$ zeros. This condition is verified in the function `MaybeSur1` in the Appendix. Then, the function `TheSur` tests if a surface given by `MaybeSur1` has also $p_g = 1$.

Proposition 4.2. *There exists a finite number of surfaces S which are the minimal model of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ with $|(\mathbf{g_1} \times \mathbf{g_2})| = p$ and $p_g(S) = 1$, $q(S) = 0$.*

Proof. For a given p, the number of surfaces S is finite, since the numbers of pairs $(C_i, \mathbf{g_i})$ is so (cf. Corollary 2.8).

The automorphism $\mathrm{id} \times \mathbf{g_2}$ induces an automorphism on S which acts non trivially on $H^{2,0}(S)$, see Section 3.4. In order to give a bound for p we prove that there exists no a surfaces Z with $p_g(Z) = 1$, $q(Z) = 0$ and an automorphism of order $p > 19$ acting non trivially on $H^{2,0}(Z)$.

Let Z be a minimal surface with $h^{2,0}(Z) = 1$ which admits an automorphism σ of order p and let $H^2(Z)_{\zeta_p^i}$ be the eigenspace of the eigenvalue ζ_p^i for the action of σ, $i = 0, \ldots, p-1$. The dimension of $H^2(Z)_{\zeta_p^i}$ does not depend on i if $i \neq 0$. Thus if there exists $i \neq 0$, $i \in \{1 \ldots p-1\}$ such that $dim(H^2(Z)_{\zeta_p^i}) \geq 1$, $b_2(Z) \geq p - 1$.

Since $h^{2,0}(Z) = 1$, $K_Z^2 \geq 0$. By $h^{1,0}(Z) = 0$ follows that $\chi(Z) = 2$ and that $e(Z) \leq 24$ by Noether equality. So $b_2(Z) \leq 22$.

Since there exists no a surface Z with $p_g(Z) = 1$ $q(Z) = 0$ and $b_2(Z) > 22$, there exists no a surfaces Z with $p_g(Z) = 1$, $q(Z) = 0$ and an automorphism of order $p > 19$ acting non trivially on $H^{2,0}(Z)$. $\qquad \square$

Remark 4.3. Since a minimal surface with $p_g = 1$, $q = 0$ can not admit an automorphism of order $p > 19$, there exists no pairs $(C_1 \times C_2, \mathbf{g_1} \times \mathbf{g_2})$ such that $|\mathbf{g_1} \times \mathbf{g_2}| = p > 19$, $dim(H^1(C_1 \times C_2)^{\mathbf{g_1} \times \mathbf{g_2}}) = 0$ and $dim(H^{2,0}(C_1 \times C_2)^{\mathbf{g_1} \times \mathbf{g_2}}) = 1$.

Lemma 4.4. *Let X be the minimal resolution of $(C_1 \times C_2)/G$. Let F_i be the fiber of $\pi_j : X \to C_j/G$, $\{i, j\} = \{1, 2\}$. We recall $F_i \simeq C_i$. If $q(X) = 0$, the linear systems $|F_i|$ on X, $i = 1, 2$ are complete and of dimension 1.*

Proof. Since $F_i^2 = 0$, we have the short exact sequence

$$0 \to \mathcal{O}(K_X - F_i) \to \mathcal{O}(K_X) \to \mathcal{O}_{F_i}(K_{F_i}) \to 0.$$

This induces the long exact sequence in cohomology

$$H^1(X, \mathcal{O}(K_X)) \to H^1(F_i, \mathcal{O}(K_{F_i})) \to H^2(X, \mathcal{O}(K_X - F_i)) \to H^2(X, \mathcal{O}(K_X)) \to 0.$$

By Serre duality $\dim(H^1(X, \mathcal{O}(K_X))) = q(X) = 0$ and $H^2(X, \mathcal{O}(K_X - F_i)) = H^0(X, \mathcal{O}(F_i))$. Therefore $\dim(H^0(X, \mathcal{O}(F_i))) = \dim(H^1(F_i, \mathcal{O}(K_{F_i}))) + \dim(H^2(X, \mathcal{O}(K_X))) = \dim(H^0(F_i, \mathcal{O})) + 1 = 2$. $\square$

Proposition 4.5. *Let X be the minimal resolution of $(C_1 \times C_2)/G$. If $q(X) = 0$, the dimension of the family of product-quotient surfaces of $(C_1 \times C_2)/G$ is $r_1 + r_2 - 6$, where $C_i \to C_i/G$ is ramified in r_i points.*

Proof. By Teichmüller theory the number η of parameters of the family of product-quotient surfaces is less then or equal to $r_1 + r_2 - 6$. For simplicity we assume $r_2 = 3$. If $\eta < r_1 - 3$, then there exists a positive dimensional family of curves isomorphic to C_1 which induces a family of isotrivial fibrations on X, whose fibers are isomorphic to C_1. Since $q(X) = 0$, $Pic(X) = NS(X)/\text{Tors} \simeq \mathbb{Z}^N/\text{Tors}$ for a certain positive integer N. By Lemma 4.4, the linear system $|F_1|$ is complete and of dimension 1. Since $Pic(X)$ is discrete there is no positive dimensional family of such linear systems. Therefore $\eta = r_1 - 3$. $\square$

There is a finite number of surfaces as described in Proposition 4.2. These surfaces are given by the program `Surfacesp`. However the number of permutations of the ramification points increases too rapidly with the growth of p for a computation in a short time. Since our aim was the construction of K3 surfaces, and we know the dimension of the families we are searching for, we wrote another program with a fixed number of ramification points, and hence with fixed dimension of the family. The program `t1t2PtsSurfaces` in the Appendix – given a cyclic group G of order p or $2p$ and the numbers ti, $i = 1, 2$, of ramification points of $C_i \to C_1/G \simeq \mathbb{P}^1$ – returns a list of product-quotient surfaces X with $p_g(X) = 1$ and $q(X) = 0$, as well as the singularities of X.

Remark 4.6. If $|\mathbf{g_1}| = 2p$, then by Remark 2.10 there exists a finite list of curves with at least one $\alpha_j = 1$ and ζ_{2p}^j is a primitive $2p$-root of unity. If the action of $\mathbf{g_1}$ on C_1 is of this type, the same must be true for the action of $\mathbf{g_2}$ on C_2, thus we have a finite list for $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. Hence, we obtain a complete classification of such surfaces as in Proposition 4.2. Otherwise, if we assume that the action of $\mathbf{g_1}$ on C_1 is such that $\alpha_j = 1$ and ζ_{2p}^j is a either a primitive p-root of unity or (-1), then the same must be true for the action of $\mathbf{g_2}$ on C_2. In this case we can not construct a complete list of the curves C_1 and C_2, since we have no an upper bound for their genera, and so for the number of ramification points of the map $f_i : C_i \to C_i/\mathbf{g_i} \simeq \mathbb{P}^1$. Anyway, if we fix the maximal number n of ramification points for f_i, then we obtain a finite list of curves C_i and thus a finite number of surfaces $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ as in Proposition 4.2. Our aim is to construct K3 surfaces, so the bound on n depends on the dimension of the moduli space of K3 surfaces. More precisely, the moduli space of projective K3 surfaces has dimension 19, so the sum of the ramification points of f_1 and f_2 can not be grater then 25.

4.2. **K3 surfaces.** Let S be the minimal model of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. Let us assume $\langle \mathbf{g_1} \times \mathbf{g_2} \rangle \simeq \mathbb{Z}/n\mathbb{Z}$. If S is a K3 surface, then by definition $p_g(S) = 1$ and $q(S) = 0$. Therefore the K3 surfaces obtained as minimal model of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ are among the ones listed in Section 4.1. In order to prove that S is a K3 surface one has to verify that the canonical bundle is trivial.

Lemma 4.7. *Let Z be a surface obtained contracting $-K_X^2$ (-1)-curves on X. We recall that $p_g(Z) = 1$ and $q(Z) = 0$. Let F_1 be the class of the fiber of the fibration $\pi_2 : X \to C_2/\langle \mathbf{g_2} \rangle$ and F_2 be the class of the fiber of the fibration $\pi_1 : X \to C_1/\langle \mathbf{g_1} \rangle$. Let E be the sum of all the exceptional divisors of the blow up $X \to Z$. If $(K_X - E)F_1 = 0$ and $(K_X - E)F_2 = 0$, then K_Z is trivial. In this case Z is minimal.*

Proof. We shall denote by P_k the singular points of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$, and by $A_{j,k}$ the exceptional curves of the blow up $\pi : X \to (C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ over P_k.
Let D be an effective divisor on X, then $D = \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 B + \sum_{j,k} \lambda_{j,k} A_{j,k}$ with $\lambda_i, \lambda_{j,k} \geq 0$ and $BF_1 > 0$, $BF_2 > 0$. If $DF_1 = 0$ and $DF_2 = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. For every k, $A_{j,k}$ is a HJ-string, hence $DA_{j,k} = 0$ for every j and k. Therefore $DF_1 = DF_2 = 0$ give a homogeneuos linear system in $\lambda_{j,k}$. The corresponding matrix is a diagonal block matrix and each block is invertible, being associated to the resolution of the quotient singularity P_k. Thus $D = 0$.
The divisor $K_X - E$ is an effective divisor being the pullback of the canonical divisor of Z, which has $p_g(Z) = 1$. Applying the previous result to $D = K_X - E$ we obtain $K_X - E = 0$ hence K_Z is trivial. $\square$

Necessary conditions to obtain a K3 surface S as minimal model of the minimal resolution X of the quotient $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ are the following:

(1) $h^{i,0}(X) = 1$, for $i = 0, 2$;
(2) $h^{1,0}(X) = 0$;
(3) there are exactly $-K_X^2$ (-1)-curves on X.

Thus, in order to classify the K3 surfaces S, we list the surfaces X satisfying the conditions (1), (2). If $|\mathbf{g_1} \times \mathbf{g_2}| = p$ is prime number, as already said, this is done by the program `Surfacesp`.

Next we consider the (-1)-curves which are either central components Y of reducible fibers (we calculate Y^2 using Proposition 3.7) or appear as image of curves in HJ-string after some contractions. In this way we find $-K_X^2$ (-1)-curves. After the contraction of all these curves we always obtain a surface which satisfy the condition of Lemma 4.7 and so a surface Z with trivial canonical bundle. This implies that there are no other (-1)-curves on Z, which is thus the minimal model S of X and in particular S is a K3 surface.

4.3. **Non-symplectic automorphisms.** We saw in Section 3.4 that every surface S which is the minimal model of a product-quotient with $\mathbf{g_1} \times \mathbf{g_2}$ admits an automorphism induced by $\mathrm{id} \times \mathbf{g_2}$ which acts non trivially on $H^{2,0}(S)$. This means that if S is a K3 surface, the automorphism induced on S by $\mathrm{id} \times \mathbf{g_2}$ is a purely non-symplectic automorphism on S. Thus the surface S admits a non-symplectic automorphisms of prime order.

Definition 4.8. *Let W be a K3 surface. Let ω_W be a generator of $H^{2,0}(W)$. An automorphism $\mathbf{g} \in Aut(W)$ of order n is called* symplectic *if $\mathbf{g}(\omega_W) = \omega_W$,* purely non-symplectic *if $\mathbf{g}(\omega_W) = \zeta_n^i \omega_W$ and ζ_n^i is a primitive n-root of unity.*

We observe that an automorphism of prime order p which is non-symplectic is purely non-symplectic. This type of automorphism are classified [AST11]. In this section we summarize the main results on non-symplectic automorphisms of prime order, which will be considered in the following.

For every prime number $2 \le p \le 19$ there exists a K3 surface W admitting a non-symplectic automorphism $\mathbf{g}$ of order p. Let us assume $3 \le p \le 19$. The fixed locus $Fix_{\mathbf{g}}(W) = \{w \in W \text{ such that } \mathbf{g}(w) = w\}$ consists of the disjoint union of n isolated points and $k + 1$ curves. At most one of the fixed curves has a positive genus, and we denote by $g(C)$ the genus of the curve with highest genus. Hence, the fixed locus consists of n isolated points, k rational curves and another curve C with a possibly positive genus. For each prime number $3 \le p \le 19$ there exists a finite number of possibilities for the fixed locus $Fix_{\mathbf{g}}(W)$, and the fixed locus is uniquely determined by the three invariants $(n, g(C), k+1)$. The admissible choices for $(n, g(C), k + 1)$ are listed in [AST11], where it is also proved that there exists a K3 surface with a non-symplectic automorphism of order p with fixed locus determined by $(n, g(C), k+1)$ for every admissible choice of the invariants.

More precisely, the invariants $(p, n, g(C), k + 1)$ determine uniquely the two lattices $S^p_{(n,g(C),k+1)} := H^2(W, \mathbb{Z})^g$ and $T^p_{(n,g(C),k+1)} := (H^2(W, \mathbb{Z})^g)^\perp$ and a K3 surface admits a non-symplectic automorphism of order p with fixed locus determined by $(n, g(C), k + 1)$ only if $S^p_{(n,g(C),k+1)}$ is primitively embedded in its Néron-Severi group. This allows one to describe the family of K3 surfaces with a non-symplectic automorphism of order p and a prescribed fixed locus in terms of the period map of K3 surfaces. We will denote by $\mathcal{M}^p_{(n,g(C),k+1)}$ the family of K3 surfaces admitting a non-symplectic automorphism of order p with fixed locus determined by $(n, g(C), k+1)$. It has one connected component of dimension $\left(\mathrm{rk}(T^p_{(n,g(C),k+1)})/(p-1)\right) - 1$. To give a more precise description of the moduli space of the K3 surfaces that admit a non-symplectic automorphism of order p and a prescribed fixed locus, we consider the action of $\mathbf{g}$ on $T^p_{(n,g(C),k+1)} \otimes \mathbb{C}$, which does not depend on the K3 surface considered. By definition $\mathbf{g}$ has no eigenvalue 1 on $T^p_{(n,g(C),k+1)} \otimes \mathbb{C}$ and the decomposition in eigenspaces consists of $p - 1$ equidimensional eigenspaces (of the eigenvalues ζ_p^i, $i = 1, \ldots p - 1$). Let $(T^p_{(n,g(C),k+1)} \otimes \mathbb{C})_{\zeta_p}$ be the unique eigenspace such that $(T^p_{(n,g(C),k+1)} \otimes \mathbb{C})^{2,0}_{\zeta_p} \ne 0$. Set $\mathcal{B} := \{z \in \mathbb{P}((T^p_{(n,g(C),k+1)} \otimes \mathbb{C})_{\zeta_p}) \text{ such that } (z,z) = 0, \quad (z,\overline{z}) > 0\}$. The space $\mathcal{B}$ is a ball of dimension $(\mathrm{rk}(T^p_{(n,g(C),k+1)})/(p-1)) - 1$. Let $\Gamma := \{\gamma \in \mathrm{O}(T^p_{(n,g(C),k+1)} \otimes \mathbb{C}) \text{ such that } \gamma\mathbf{g} = \mathbf{g}\gamma\}$. The generic point of $\mathcal{B}/\Gamma$ corresponds to a K3 surface admitting a non-symplectic automorphism as required and there is a birational map between the space of such K3 surfaces and $\mathcal{B}/\Gamma$ (see [DK07, Section 11]).

For a fixed prime number $3 \le p \le 19$, there are some inclusions among the families $\mathcal{M}^p_{(n,g(C),k+1)}$, for example $\mathcal{M}^3_{(9,0,6)} \subset \mathcal{M}^3_{(8,0,5)}$. All these inclusions are described in [AST11] and the maximal components are determined in [AST11, Theorem 9.5]:

if $p = 3$, there are three maximal components: $\mathcal{M}^3_{(3,-,0)}$, $\mathcal{M}^3_{(0,4,1)}$, $\mathcal{M}^3_{(0,5,2)}$ and for every admissible data $(n, g(C), k+1)$, we have $\mathcal{M}^3_{(n,g(C),1)} \subset \mathcal{M}^3_{(0,4,1)}$ and $\mathcal{M}^3_{(n,g(C),k+1)} \subset \mathcal{M}^3_{(0,5,2)}$ if $k + 1 \ge 2$. The dimension of each family is $m = 10 - n$;

if $p = 5, 7, 11$, there are two maximal components: $\mathcal{M}^p_{(n,-,0)}$, and $\mathcal{M}^3_{(n,g(C),1)}$ and for every admissible data $(n, g(C), k+1)$, we have $\mathcal{M}^3_{(n,g(C),k+1)} \subset \mathcal{M}^3_{(n,g(C),1)}$ if $k + 1 \neq 0$. The dimension of each family is $m = (13 - n)/(p - 2)$ if $p = 5, 7$ and $m = (11 - n)/(p - 2)$ if $p = 11$;

if $p = 13, 17, 19$, there is only one (rigid) K3 surface admitting a non-symplectic automorphism of order p. Thus there is one maximal component (in fact one component of dimension 0) which is $\mathcal{M}^{13}_{(9,0,1)}$, $\mathcal{M}^{17}_{(7,-,0)}$, $\mathcal{M}^{19}_{(5,-,0)}$ respectively.

The K3 surfaces we are constructing are members of the families of K3 surfaces admitting a non-symplectic automorphism of prime order. In certain cases it turns out they are the general member of some of these families. In the following sections we will construct K3 surfaces S and we will determine the fixed locus of the non-symplectic automorphism induced by $\text{id} \times \mathbf{g_2}$ (or by some of its powers), and thus we identify on which component of the family of K3 surfaces with a non-symplectic automorphisms they lie.

The following remark is used to determine the fixed locus.

Remark 4.9. Let us assume $|\mathbf{g_2}| = p$ is a prime number. The automorphism of S induced by $\text{id} \times \mathbf{g_2}$ fixes all the central components of all the reducible fibers of both the fibrations π_1 and π_2 and all the singular points of the HJ-strings introduced resolving the singularities of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$. It is possible that the automorphism fixes some disjoint components of certain HJ-strings which do not meet any other fixed curves. It induces an automorphism $\mathbf{g}_S$ on S which is non-symplectic of order p.

5. K3 SURFACES WHICH ARE MINIMAL MODELS OF $(C_1 \times C_2)/(\mathbb{Z}/p\mathbb{Z})$

The aim of this section is to list and to describe the K3 surfaces obtained as minimal model of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ with $|\mathbf{g_1} \times \mathbf{g_2}| = p$.

Theorem 5.1. *The K3 surfaces S admitting a non-symplectic automorphism of odd prime order p with fixed locus listed in the column $(n, g, k+1)$ of the* Table 1 *are all minimal models of product-quotient surfaces with group $\mathbb{Z}/p\mathbb{Z}$. Moreover, for each such surface S we can choose $(C_2, \mathbf{g_2}) \simeq (D_p, \delta_p)$ and the non-symplectic automorphism on S is always induced by $\text{id} \times \delta_{\mathrm{p}}$.*

Viceversa all the K3 surfaces which are minimal models of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$, $|\mathbf{g_1} \times \mathbf{g_2}| = p$ admit a non-symplectic automorphism of order p whose fixed locus is one of those listed in Table 1.

Proof. Let us fix p. By Section 4.3 we know the dimension M_p of the maximal components of the family of K3 surfaces with a non-symplectic automorphim of order p. Since every K3 surface minimal model of a product-quotient with group $\mathbb{Z}/p\mathbb{Z}$ admits a non-symplectic automorphism of order p (see Section 3.4), we can bound the number of moduli of the pairs $(C_1, \mathbf{g_1})$, $(C_2, \mathbf{g_2})$ by M_p, see also Proposition 4.5.

The first step consists in listing product-quotient surfaces with $p_g = 1$, $q = 0$. This is done using the program t1t2PtsSurfaces giving the group $G \simeq \mathbb{Z}/p\mathbb{Z}$ and the numbers $t1$, $t2$ such that $t1 \ge 3$, $t2 \ge 3$ and $t1 + t2 = m - 6$, where $m \le M_p$. Indeed, recall that $t1$ and $t2$ are the numbers of branching points of the projections $C_i \to C_i/\mathbf{g_i}$ respectively. Then $t1 - 3$ and $t2 - 3$ are the moduli of the pairs $(C_i, \mathbf{g_i})$, see e.g. [C00], and m is the dimension of family of the product-quotient surfaces, by Proposition 4.5.

Second step: For every product-quotient surface in the list one has to check if the minimal model is a K3 surface and has to calculate the fixed locus, determining $(n, g, k+1)$, of the induced automorphism. This is done for every entry in the list exactly as in Example 5.3.

Every member of the family, $\mathcal{F}_{PQ}(C_1, C_2)$, of K3 surfaces minimal models of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$, is also a member of a family $\mathcal{M}^p_{(n,g,k+1)}$. This implies that $\mathcal{F}_{PQ}(C_1, C_2) \subset \mathcal{M}^p_{(n,g,k+1)}$. For every $(n, g, k+1)$, there exists always a choice of $(C_1, \mathbf{g_1})$ and $(C_2, \mathbf{g_2})$ such that $\dim(\mathcal{F}_{PQ}(C_1, C_2)) = \dim(\mathcal{M}^p_{(n,g,k+1)})$ which implies $\mathcal{F}_{PQ}(C_1, C_2) = \mathcal{M}^p_{(n,g,k+1)}$.

Moreover, we observe that different (up to isomorphism) admissible choices for $(C_1, \mathbf{g_1})$, $(C_2, \mathbf{g_2})$ correspond to the same component $\mathcal{M}^p_{(n,g,k+1)}$. In Table 1 we give one example for each component. It is remarkable that one can always construct this example choosing $C_2 \simeq D_p$, $\mathbf{g_2} \simeq \delta_p$. In Table 1 one can find: the properties which characterize the pair $(C_1, \mathbf{g_1})$; the singularities of $(C_1 \times D_p)/(\mathbf{g_1} \times \delta_p)$;

the value of K_X^2; the fixed locus $(n, g, k + 1)$ of the automorphism induced on S; and this identifies $\mathcal{M}_{(n,g,k+1)}^p$ whose dimension is m. $\qquad\square$

The quotients $(C_1 \times D_3)/(\mathbf{g_1} \times \delta_3)$ which admit a minimal model which is a K3 surface are classified in [GvG10, Remark 3.1] and are listed in `Table 1`.

p	$g(C_1)$	$(\alpha_1, \ldots \alpha_{p-1})$	$(a_1, \ldots, a_{p-1})$	$Sing(C_1 \times D_p/\mathbf{g_1} \times \delta_p)$	K_X^2	$(n, g, k+1)$	m
3	4	$(3,1)$	$(0,6)$	$\left(\frac{1}{3}\right)^{18}$	-6	$(6,0,3)$	3
3	3	$(2,1)$	$(1,4)$	$\left(\frac{1}{3}\right)^{12}, \left(\frac{2}{3}\right)^3$	-4	$(7,0,4)$	2
3	2	$(1,1)$	$(2,2)$	$\left(\frac{1}{3}\right)^6, \left(\frac{2}{3}\right)^6$	-2	$(8,0,5)$	1
3	1	$(0,1)$	$(3,0)$	$\left(\frac{2}{3}\right)^9$	0	$(9,0,6)$	0
5	6	$(3,2,1,0)$	$(0,0,0,5)$	$\left(\frac{1}{5}\right)^{10}, \left(\frac{3}{5}\right)^5$	-12	$(7,0,1)$	2
5	4	$(2,1,1,0)$	$(0,1,0,3)$	$\left(\frac{1}{5}\right)^6, \left(\frac{2}{5}\right)^5, \left(\frac{4}{5}\right)^5$	-8	$(10,0,2)$	1
5	2	$(1,0,1,0)$	$(0,2,0,1)$	$\left(\frac{1}{5}\right)^2, \left(\frac{2}{5}\right)^5, \left(\frac{4}{5}\right)^2$	-4	$(13,0,3)$	0
7	6	$(2,2,1,1,0,0)$	$(0,0,0,0,1,3)$	$\left(\frac{1}{7}\right)^6, \left(\frac{3}{7}\right)^3, \left(\frac{4}{7}\right)^2, \left(\frac{5}{7}\right)$	-14	$(8,0,1)$	1
7	3	$(0,1,1,0,0,1)$	$(0,0,0,2,1,0)$	$\left(\frac{1}{7}\right)^2, \left(\frac{2}{7}\right)^2, \left(\frac{3}{7}\right)^5$	-7	$(13,0,2)$	0
11	5	$(1,1,0,1,1,$ $0,0,1,0,0)$	$(0,0,1,0,0,$ $0,1,0,0,1)$	$\left(\frac{1}{11}\right)^2, \left(\frac{2}{11}\right), \left(\frac{3}{11}\right)^2,$ $\left(\frac{4}{11}\right), \left(\frac{5}{11}\right), \left(\frac{7}{11}\right)^2$	-13	$(11,0,1)$	0
13	6	$(1,1,1,0,1,1,$ $0,0,1,0,0,0)$	$(0,0,0,1,0,0,$ $0,0,0,1,0,1)$	$\left(\frac{1}{13}\right)^2, \left(\frac{2}{13}\right), \left(\frac{3}{13}\right)^2,$ $\left(\frac{5}{13}\right), \left(\frac{6}{13}\right), \left(\frac{9}{13}\right)^2$	-17	$(9,0,1)$	0
17	8	$(1,1,1,1,1,1,0,1,$ $0,0,1,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,$ $0,0,0,0,1,0,1,1)$	$\left(\frac{1}{17}\right)^2, \left(\frac{4}{17}\right), \left(\frac{5}{17}\right),$ $\left(\frac{7}{17}\right)^2, \left(\frac{8}{17}\right), \left(\frac{9}{17}\right)^2$	-23	$(7,-,0)$	0
19	9	$(1,1,1,1,1,1,$ $0,1,1,0,0,1,)$ $0,0,0,0,0,0)$	$(0,0,0,0,0,0,$ $0,0,0,0,0,0,$ $0,0,1,1,0,1)$	$\left(\frac{1}{19}\right)^2, \left(\frac{3}{19}\right), \left(\frac{5}{19}\right)^2,$ $\left(\frac{7}{19}\right), \left(\frac{9}{19}\right), \left(\frac{13}{19}\right)^2$	-25	$(5,-,0)$	0

`Table 1`

Remark 5.2. The example listed in `Table 1` are all the examples obtained as described if $p \leq 7$. For $p \geq 11$ there are other admissible pairs $(C_1, \mathbf{g_1})$, $(C_2, \mathbf{g_2})$, such that $C_i \not\simeq D_p$, $i = 1, 2$, which correspond to the components $\mathcal{M}_{(11,0,1)}^{11}, \mathcal{M}_{(9,0,1)}^{13}, \mathcal{M}_{(7,-,0)}^{17}, \mathcal{M}_{(5,-,0)}^{19}$.

Example 5.3. As example (line 5 `Table 1`) we consider the pairs $(C_1, \mathbf{g_1})$ and $(C_2, \mathbf{g_2})$ such that $|\mathbf{g_i}| = 5$ and:
• $g(C_1) = 6$, $C_1 \to C_1/\langle \mathbf{g_1} \rangle$ is branched in 5 points and the local action of $\mathbf{g_1}$ near these points is ζ_5^4. The dimension of the eigenspaces for the induced action in cohomology is $(\alpha_1, \ldots \alpha_4) = (3, 2, 1, 0)$;
• $C_2 \simeq D_5$, $\mathbf{g_2} = \delta_5$ (cf. Example 2.9).
We will denote by P_i, $i = 1, 2, 3, 4, 5$ the branch points of $C_1 \to \mathbb{P}^1$, by Q_j, $j = 1, 2, 3$ the branch points of $C_2 \to \mathbb{P}^1$ and we assume the local action of $\mathbf{g_2}$ is the same near the points Q_1 and Q_2.
The singularities of the quotient $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$ are 10 singularities of type $\frac{1}{5}(1, 1)$ (over $P_i \times Q_j$, $i = 1 \ldots, 5$, $j = 1, 2$) and 5 of types $\frac{1}{5}(1, 3)$ (over $P_i \times Q_3$, $i = 1, \ldots, 5$). The resolution of the 10 singularities of type $\frac{1}{5}(1, 1)$ consists of the 10 (-5)-curves $\widetilde{B_{i,j}}$, $i = 1, \ldots, 5$, $j = 1, 2$. The resolution of each singularity of type $\frac{1}{5}(1, 3)$ consists of 2 curves, $\widetilde{B_{i,3}^h}$, $h = 1, 2$, meeting in one point and with self intersection -2 and -3, respectively. The resolution of the singularities is as in Figure 1.
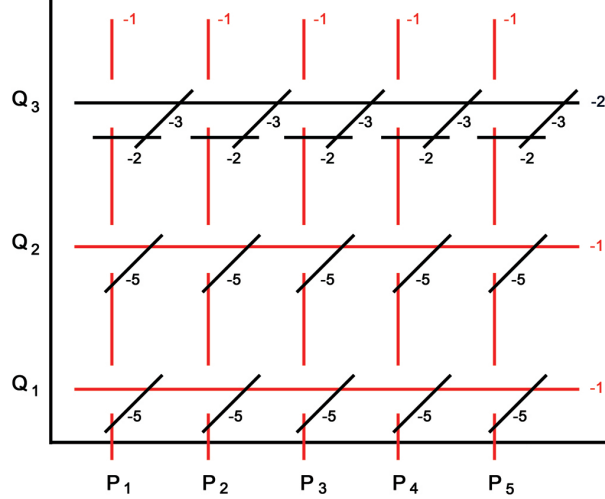
Figure 1.

The central components F_{P_i} of the 5 reducible fibers of the fibration $\pi_1 \colon X \to C_1/\langle \mathbf{g_1} \rangle$ are (-1)-curves. The central components F_{Q_j} of the reducible fibers of $\pi_2 \colon X \to C_2/\langle \mathbf{g_2} \rangle$ over Q_j with $j = 1, 2$ are (-1)-curves and the central component F_{Q_3} of π_2 over Q_3 is a (-2)-curve (cf. Proposition 3.7).

In order to construct the minimal model S we first contract the 5 (-1)-curves F_{P_i} (we call this contraction map σ_1). The images $B^1_{i,3}$ of the curves $\widetilde{B^1_{i,3}}$ are 5 (-1)-curves and thus we contract them (we call this contraction map σ_2). After the contraction of the 2 (-1)-curves F_{Q_1} and F_{Q_2} we are left with only (-2)-curves from the configuration we started with. We call this surface S (see Figure 2).
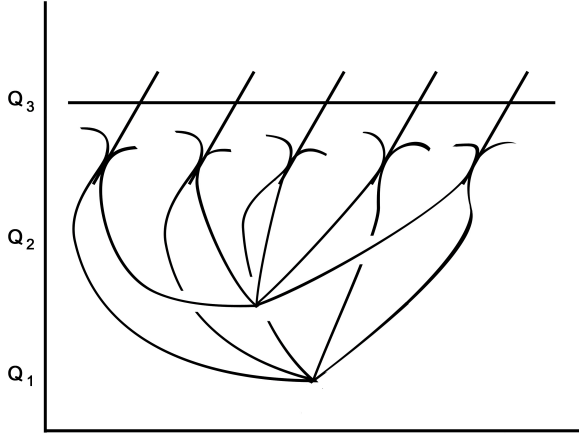
Figure 2.

We contracted 12 curves and since the canonical bundle of X has self-intersection $K_X^2 = -12$, we obtain $K_S^2 = 0$. Now we verify the surface S satisfy the hypothesis of Lemma 4.7 and this proves S is a K3 surface:

By adjunction, $2g(C_1) - 2 = (K_X - F_2)F_2 = K_X F_2 = 10$ and $2g(C_2) - 2 = (K_X - F_1)F_1 = K_X F_1 = 2$. Since $\sigma_1^*(B^1_{i,3}) = \widetilde{B^1_{i,3}} + F_{P_i}$, the exceptional divisor E of the blow up $X \to S$ is

$$E = \sum_{i=1}^{5} (2F_{P_i} + \widetilde{B^1_{i,3}}) + F_{Q_1} + F_{Q_2}.$$

The curves F_{P_i} are sections of the fibration π_2 and F_{Q_j} are sections of the fibration π_1, hence $F_{P_i} F_2 = F_{Q_j} F_1 = 1$. The curves $\widetilde{B^h_{i,j}}$ are components of the reducible fibers of both the fibration π_1 and π_2, hence

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$\widetilde{B_{i,j}^h} F_1 = \widetilde{B_{i,j}^h} F_2 = 0$. So $(K_X - E)F_i = 0$ for $i = 1, 2$.

By Remark 4.9 the fixed locus of g_S on S consists of one rational curve (the image of the central components F_{Q_3}) and 7 points (see Figure 2).

Since C_2 is a rigid curve and C_1 varies in an irreducible 2-dimensional family, we in fact construct an irreducible 2-dimensional family $\mathcal{F}$ of K3 surfaces S with a non-symplectic automorphism of order 5 and fixed locus $(n, g, k+1) = (7, 0, 1)$. Thus $\mathcal{F} \subset \mathcal{M}_{(7,0,1)}^5$. By [AST11], $\mathcal{M}_{(7,0,1)}^5$ is an irreducible 2-dimensional family of K3 surfaces with the required automorphism and hence $\mathcal{F}$ coincides $\mathcal{M}_{(7,0,1)}^5$.

6. K3 SURFACES WHICH ARE MINIMAL MODELS OF $(C_1 \times C_2)/(\mathbb{Z}/2p\mathbb{Z})$

In Section 5 we described K3 surfaces which are minimal models of a product-quotient surfaces with group $\mathbb{Z}/p\mathbb{Z}$. If $p \leq 11$ we never get the maximal irreducible components of the moduli space of K3 surfaces with a non-symplectic automorphism of order p. In order to find at least one maximal irreducible component of such moduli space, we consider product-quotient with the $\mathbb{Z}/2p\mathbb{Z}$.

Theorem 6.1. *Let $p = 3$ (resp. $3 < p \leq 13$, $13 < p \leq 19$). The K3 surface S admitting a non-symplectic automorphism of order p which fixes at least 2 (resp. 1, 0) curves are all minimal models of product-quotient surfaces with the group $\mathbb{Z}/2p\mathbb{Z}$. In particular we obtain the irreducible maximal component $\mathcal{M}_{(n,g,2)}^3$ (resp. $\mathcal{M}_{(n,g,1)}^p$, $\mathcal{M}_{(n,-,0)}^p$).*
Moreover, we can always choose $(C_2, \mathbf{g_2}) \simeq (D_p, \tau_p)$ and hence the non-symplectic automorphism on S is always induced by $\mathrm{id} \times \delta_{\mathrm{p}}$.
In `Table 2` *we list an example for each family assuming $(C_2, \mathbf{g_2}) \simeq (D_p, \tau_p)$. Viceversa all the K3 surfaces which are minimal models of $(C_1 \times C_2)/(\mathbf{g_1} \times \mathbf{g_2})$, $|\mathbf{g_1} \times \mathbf{g_2}| = 2p$ admit a non-symplectic automorphism of order p whose fixed locus is one of those listed in* `Table 2`.

Proof. The proof is analogous to the one of Theorem 5.1. We remark that in `Table 2` one can find: the properties which characterize the pair $(C_1, \mathbf{g_1})$; the singularities of $(C_1 \times D_p)/(\mathbf{g_1} \times \tau_p)$; the value of K_X^2; the fixed locus $(n, g, k+1)$ of the automorphism induced on S; and this identifies $\mathcal{M}_{(n,g,k+1)}^p$ whose dimension is m. $\square$

Corollary 6.2. *All the K3 surfaces admitting a non-symplectic automorphism of order $p = 3$ (resp. $3 < p \leq 13$, $13 < p \leq 19$) which fixes at least 2 (resp. 1,0) curves admit a non-symplectic automorphism of order $2p$ induced by $\mathrm{id} \times \tau_{\mathrm{p}}$.*

We observe that the results in Corollary 6.2 were already proved in [Di12] for $p = 3$ and [GS13] for $p > 3$.

p	$g(C_1)$	$(\alpha_1,\ldots\alpha_{2p-1})$	$(a_1,\ldots,a_{2p-1})$	$Sing(C_1\times D_p/\mathbf{g_1}\times\tau_p)$	K_X^2	$(n,g,k+1)$	m
3	25	$(9,7,5,3,1)$	$(0,12,0,0,0)$	$(\frac16)^{12},(\frac13)^{12},(\frac12)^{12}$	-36	$(0,5,2)$	9
3	22	$(8,6,4,3,1)$	$(0,10,2,0,0)$	$(\frac16)^{10},(\frac13)^{13},(\frac12)^{10}$	-31	$(1,4,2)$	8
3	19	$(7,5,3,3,1)$	$(0,8,4,0,0)$	$(\frac16)^{8},(\frac13)^{14},(\frac12)^{8}$	-26	$(2,3,2)$	7
3	16	$(6,4,2,3,1)$	$(0,6,6,0,0)$	$(\frac16)^{6},(\frac13)^{15},(\frac12)^{6}$	-21	$(3,2,2)$	6
3	17	$(6,5,3,2,1)$	$(0,8,0,2,0)$	$(\frac16)^{8},(\frac13)^{8},(\frac23)^{3},(\frac12)^{8}$	-24	$(3,3,3)$	6
3	13	$(5,3,1,3,1)$	$(0,4,8,0,0)$	$(\frac16)^{4},(\frac13)^{16},(\frac12)^{4}$	-16	$(4,1,2)$	5
3	14	$(5,4,2,2,1)$	$(0,6,2,2,0)$	$(\frac16)^{6},(\frac13)^{9},(\frac23)^{3},(\frac12)^{6}$	-19	$(4,2,3)$	5
3	15	$(5,4,3,2,1)$	$(1,7,0,0,0)$	$(\frac16)^{7},(\frac56),(\frac13)^{7},(\frac23),(\frac12)^{8}$	-21	$(4,3,4)$	5
3	10	$(4,2,0,3,1)$	$(0,2,10,0,0)$	$(\frac16)^{2},(\frac13)^{17},(\frac12)^{2}$	-11	$(5,0,2)$	4
3	11	$(4,3,1,2,1)$	$(0,4,4,2,0)$	$(\frac16)^{4},(\frac13)^{10},(\frac23)^{3},(\frac12)^{4}$	-14	$(5,1,3)$	4
3	12	$(4,3,2,2,1)$	$(1,5,2,0,0)$	$(\frac16)^{5},(\frac56),(\frac13)^{8},(\frac23),(\frac12)^{6}$	-16	$(5,2,4)$	4
3	7	$(3,1,0,2,1)$	$(0,1,8,0,3)$	$(\frac16),(\frac13)^{13},(\frac12)^{5}$	-7	$(6,0,3)$	3
3	9	$(3,2,1,2,1)$	$(1,3,4,0,0)$	$(\frac16)^{3},(\frac56),(\frac13)^{9},(\frac23),(\frac12)^{4}$	-11	$(6,1,4)$	3
3	4	$(2,0,0,1,1)$	$(0,0,6,0,6)$	$(\frac13)^{9},(\frac12)^{8}$	-3	$(7,0,4)$	2
3	7	$(2,2,1,1,1)$	$(1,3,0,2,0)$	$(\frac16)^{3},(\frac56),(\frac13)^{3},(\frac23)^{4},(\frac12)^{4}$	-9	$(7,1,5)$	2
3	3	$(1,0,0,1,1)$	$(1,0,4,0,3)$	$(\frac56),(\frac13)^{6},(\frac23)^{2},(\frac12)^{5}$	-2	$(8,0,5)$	1
3	5	$(1,1,1,1,1)$	$(2,2,0,0,0)$	$(\frac16)^{2},(\frac56)^{2},(\frac13)^{2},(\frac23)^{2},(\frac12)^{4}$	-6	$(8,1,6)$	1
3	1	$(0,0,0,0,1)$	$(0,1,2,0,3)$	$(\frac56),(\frac23)^{4},(\frac12)^{5}$	0	$(9,0,6)$	0
5	22	$(4,1,0,2,\ 2,4,5,3,1)$	$(0,0,6,0,\ 0,0,2,0,0)$	$(\frac{1}{10})^{6},(\frac15)^{2},(\frac25)^{6},(\frac35),(\frac12)^{6}$	-28	$(1,2,1)$	4
5	17	$(3,2,0,2,\ 1,1,4,3,1)$	$(0,0,4,0,\ 2,0,2,0,0)$	$(\frac{1}{10})^{4},(\frac15)^{3},(\frac25),(\frac35)^{6},(\frac12)^{4}$	-21	$(4,1,1)$	3
5	12	$(3,1,1,2,\ 0,2,2,1,0)$	$(0,0,1,1,\ 0,0,6,0,0)$	$(\frac{1}{10}),(\frac{7}{10}),(\frac15)^{7},(\frac25)^{3},(\frac35),(\frac45),(\frac12)^{2}$	-13	$(7,1,2)$	2
5	8	$(2,1,1,1,\ 0,1,1,1,0)$	$(0,0,1,1,\ 0,0,2,2,0)$	$(\frac{1}{10}),(\frac{7}{10}),(\frac15)^{3},(\frac25)^{3},(\frac35),(\frac45),(\frac12)^{2}$	-9	$(10,0,2)$	1
5	4	$(1,0,1,1,\ 0,1,0,0,0)$	$(0,1,0,1,\ 0,0,2,0,0)$	$(\frac{3}{10}),(\frac{9}{10}),(\frac15)^{3},(\frac35)^{2},(\frac12)^{2}$	-5	$(13,0,3)$	0
7	19	$(2,2,3,1,0,2,\ 1,0,3,1,0,3,1)$	$(0,0,0,4,0,0,\ 0,0,2,0,0,0,0)$	$(\frac{1}{14})^{4},(\frac47)^{2},(\frac57)^{5},(\frac12)^{4}$	-25	$(3,1,1)$	2
7	12	$(1,1,2,1,0,1,\ 0,1,2,1,0,1,1)$	$(0,0,0,1,1,0,\ 2,0,0,0,2,0,0)$	$(\frac{1}{14}),(\frac{5}{14}),(\frac17)^{3},(\frac37),(\frac47),(\frac57)^{3},(\frac12)^{2}$	-14	$(8,1,2)$	1
7	6	$(0,1,1,0,0,0,\ 0,0,1,1,0,1,1)$	$(1,0,0,1,0,0,\ 2,0,0,0,0,0,0)$	$(\frac{1}{14}),(\frac{9}{14}),(\frac17),(\frac37)^{3},(\frac67),(\frac12)^{2}$	-10	$(13,0,2)$	0
11	21	$(1,2,2,0,2,1,0,1,0,2,\ 1,0,2,0,2,1,0,1,0,\ 2,1)$	$(0,0,0,0,0,3,0,0,1,\ 0,0,0,0,0,0,0,0,0,0,\ 0,0)$	$(\frac{1}{22})^{3},(\frac{19}{22}),(\frac{6}{11}),(\frac{9}{11})^{3},(\frac12)^{4}$	-31	$(2,1,1)$	1
11	10	$(1,0,1,1,0,1,0,0,0,1,\ 0,1,1,0,1,1,1,0,0,\ 0,0)$	$(0,0,0,0,0,0,1,0,1,\ 0,0,0,0,0,0,0,0,0,2,\ 0,0)$	$(\frac{3}{22}),(\frac{7}{22}),(\frac{1}{11})^{2},(\frac{2}{11}),(\frac{4}{11}),(\frac{9}{11}),(\frac12)^{2}$	-13	$(11,0,1)$	0
13	12	$(0,1,1,0,1,0,0,1,0,1,\ 0,0,0,0,1,1,1,0,1,\ 0,0,1,0,1,1,1)$	$(0,0,0,0,0,0,1,0,1,\ 0,0,0,0,0,0,0,0,2,\ 0,0,0,0,0)$	$(\frac{1}{26}),(\frac{5}{26}),(\frac{3}{13})^{2},(\frac{4}{13}),(\frac{6}{13}),(\frac{4}{13}),(\frac12)^{2}$	-18	$(9,0,1)$	0
17	16	$(1,1,1,0,1,0,1,0,0,1,\ 0,1,0,0,1,1,0,0,0,\ 0,1,1,1,0,1,0,0,0,\ 1,0,1,0,1,0)$	$(0,0,0,0,0,0,0,0,1,\ 1,0,0,0,0,0,0,0,0,0,\ 0,0,0,2,0,0,0,0,\ 0,0,0,0,0,0,0)$	$(\frac{1}{34}),(\frac{23}{34}),(\frac{5}{17})^{2},(\frac{7}{17})^{2},(\frac{15}{17}),(\frac12)^{2}$	-22	$(7,-,0)$	0
19	18	$(1,0,1,1,1,1,0,0,1,1,\ 0,1,0,0,0,0,0,1,0,\ 1,1,0,1,1,1,1,1,1,0,\ 0,0,1,1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0,\ 1,0,0,0,1,0,0,0,0,0,\ 0,0,0,0,0,0,0,0,0,\ 0,0,0,0,2,0,0)$	$(\frac{3}{38}),(\frac{11}{38}),(\frac{1}{19})^{2},(\frac{4}{19}),(\frac{8}{19}),(\frac{17}{19}),(\frac12)^{2}$	-22	$(5,-,0)$	0

Table 2

Remark 6.3. The K3 surfaces constructed in the `Table 2` admit an automorphism of order $2p$, induced by $\mathrm{id}\times\tau_\mathrm{p}$ (Corollary 6.2) and one of order 2, induced by $\mathrm{id}\times\iota_\mathrm{p}$. The fixed locus of these automorphisms can be computed case by case. In Section 7 we compute it in certain cases, by using a projective model of the surfaces.

6.1. Intermediate quotients. The $2p:1$ map $C_1\times D_p\to (C_1\times D_p)/(\mathbf{g_1}\times\tau_\mathbf{p})$ clearly factorizes through

$$C_1\times D_p \xrightarrow{p:1} (C_1\times D_p)/(\mathbf{g_1}\times\tau_p)^2 \xrightarrow{2:1} (C_1\times D_p)/(\mathbf{g_1}\times\tau_p).$$

This induces a 2:1 rational map between the minimal model, Q, of $(C_1\times D_p)/(\mathbf{g_1}\times\tau_p)^2$ and the K3 surface S. In particular Q is a 2-cover of a K3 surface. We observe that $p_g(Q)\geq p_g(S)$. This immediately implies that the Kodaira dimension of Q, $k(Q)$, is non negative. The following examples show that all the other three possibilities, $k(Q)=0,1,2$, appear in our classification. First we notice that the genus of the quotient $C_1/\mathbf{g_1}^2$ is α_p and so $q(Q)=\alpha_p$.

Let us consider the line 18 of `Table 2`, it corresponds to the quotient $(D_3 \times D_3)/(\tau_3^5 \times \tau_3)$. The quotient $(D_3 \times D_3)/(\tau_3^5 \times \tau_3)^2$ is isomorphic to $(D_3 \times D_3)/(\delta_3 \times \delta_3^2)$. The minimal model of such a surface is described in [SI77] (see Example 3.8 the fourth line of `Table 1`) and is a K3 surface. In particular in this case $k(Q) = 0$.

Let us consider the line 9 of `Table 2`. The map $C_1 \to C_1/\mathbf{g_1}^2 \simeq \mathbb{P}^1$ is branched in 12 points and an equation of C_1 is $w^3 = p_{12}(t)$ where $p_{12}(t)$ is a polynomial with 12 simple roots. With the same method we will apply in Section 7, case $p = 3$, we obtain the equation $y^2 = x^3 + p_{12}^2(t)$ of $(C_1 \times D_p)/(\mathbf{g_1} \times \tau_p)^2$. So the surface Q admits an elliptic fibration, its birational invariant are $q(Q) = 0$, $p_g(Q) = 3 = \alpha_2 + \alpha_5$, and we obtain $k(Q) = 1$.

Let us consider the line 19 of `Table 2`. It corresponds to the quotient $(C_1 \times D_5)/(\mathbf{g_1} \times \tau_5)$ where $g(C_1) = 22$. Let Y be the minimal resolution of $(C_1 \times D_5)/(\mathbf{g_1} \times \tau_5)^2$. As in Example 5.3 , one proves that the singularities of $(C_1 \times D_5)/(\mathbf{g_1} \times \tau_5)^2$ are 10 singularities of type $\frac{1}{5}(1,1)$, 12 singularities of type $\frac{1}{5}(1,2)$ and 2 singularities of type $\frac{1}{5}(1,3)$. The computation of K_Y^2 can be done as explained in Remark 3.3 (4) and it gives $K_Y^2 = 10 > 0$. Since Q is the minimal model of Y, $K_Q^2 \geq K_Y^2$ and we conclude that $K_Q^2 > 0$, Q is a surface of general type, and so $k(Q) = 2$.

7. EQUATIONS

7.1. Automorphisms of order $p = 3$.

In [AS08, Proposition 4.2] it is proved that every K3 surfaces admitting a non-symplectic automorphism of order 3 fixing at least two curves is in fact an isotrivial elliptic fibration with generic fiber isomorphic to the elliptic curve E_{ζ_3} with complex multiplication of order 3. Indeed, every such a K3 surface is described as en elliptic K3 surface with an equation of type $y^2 = x^3 + f_{12}(t)$. In view of our construction this is very natural: we proved that every such a K3 surface is the minimal model of the quotient $(C_1 \times D_3)/(\mathbf{g_1} \times \tau_3)$ where $D_3 \simeq E_{\zeta_3}$ and τ_3 are described in Example 2.9 and $(C_1, \mathbf{g_1})$ varies. The maximal component is obtained by $(C_1, \mathbf{g_1})$ as in the first line of the `Table 2`. In this case C_1 is a $6 : 1$ cover of $\mathbb{P}^1$ whose ramification consists of 12 points of order 6. An equation of C_1 is $w^6 = f_{12}(t)$, where $\deg(f_{12}(t)) = 12$ and $f_{12}(t)$ does not have multiple roots. The local action near the fixed points is $-\zeta_3^2$ (see `Table 2`) and thus we can assume that the automorphism $\mathbf{g_1}$ is $\mathbf{g_1} : (w, t) \mapsto (-\zeta_3^2 w, t)$. The new functions $x := uw^2$, $y := vw^3$ and t are invariant for $\mathbf{g_1} \times \tau_3$ and satisfy the equation

$$y^2 = x^3 + f_{12}(t).$$

Moreover, if W is the surface defined by this equation, then the generic fiber of the map $C_1 \times D_3 \to W$ consist of 6 points, thus we have the following commutative diagram:

$$
\begin{array}{ccc}
 & C_1 \times D_3 & \\
6:1 \swarrow & & \searrow 6:1 \\
W & \dashrightarrow & (C_1 \times D_3)/(\mathbf{g_1} \times \tau_3)
\end{array}
$$

which shows that W and S are birational and so W is a singular model of the K3 surface S. This construction was also considered in [vG01, Example 3.11].

More in general the curve C_1 has an equation of type $w^6 = f_{12}(t)$ where f_{12} does not admit roots with multiplicity greater then 5 and there exists no a polynomial $h(t)$ such that either $f_{12}(t) = h^2(t)$ or $f_{12}(t) = h^3(t)$. If some of the roots of $f_{12}(t)$ have multiplicity higher then 1, then the fixed locus of $\text{id} \times \delta_3$ changes and we obtain a member of a more special family (cf. lines from 2 to 18 `Table 2`).

We saw in Section 5 that certain K3 surfaces admitting a non-symplectic automorphism of order 3, can be obtained as quotient of $(C_1 \times D_3)$ by an automorphism of order 3, $\mathbf{g_1} \times \delta_3$. So we obtain a different equation for these K3 surfaces. Indeed, in this case one can assume C_1 to have the following equation $w^3 = f_6(t)$, such that $f_6(t)$ does not admit roots with multiplicity greater then 2. In a very similar way as before this gives the following equation for the quotient surface:

$$y^2 = x^3 + f_6(t)^2.$$

These equations were already considered in [GvG10] and [GS13].

7.2. Automorphisms of order $p = 5$.

In Theorem 6.1 and `Table 2` we proved that the K3 surfaces admitting a non-symplectic automorphism of order 5 with at least one curve in the fixed locus are the minimal models of quotients $(C_1 \times D_5)/(\mathbf{g_1} \times \tau_5)$ for a certain choices of the pair $(C_1, \mathbf{g_1})$. In particular the maximal component (with fixed locus $(n, g(C), k + 1) = (1, 2, 1)$) is obtained choosing C_1 to be a $10 : 1$ cover of $\mathbb{P}^1$ branched along 6 points of order 10 and 1 point of order 5. An equation of C_1 is $w^{10} = f_6(t)$ where $\deg(f_6(t)) = 6$ and $f_6(t)$ does not have multiple roots (we are assuming the branch point of order 5 is at infinity). The local action near the fixed points is $-\zeta_5^3$ (see `Table 2`) and so we

can assume that the automorphism $\mathbf{g_1}$ is $\mathbf{g_1} : (w, t) \mapsto (-\zeta_5^2 w, t)$. The functions $x := uw^2$, $y := vw^5$ and t are invariant under $\mathbf{g_1} \times \tau_5$ and satisfy the equation

$$y^2 = x^5 + f_6(t).$$

As in the case $p = 3$, one shows that this gives in fact a (singular) model of the K3 surface S. The equation exhibit S as double cover of $\mathbb{P}^2_{[x,t]}$ branched along the (non homogenous) sextic $x^5 + f_6(t) = 0$. A similar model for this K3 surface is described in [AST11, Example 5.1], where the relation with the curves C_1 and D_5 was not observed.

More in general, we observe that every curve C_1 in the `Table 2` admits an equation of the type $w^{10} = f_6(t)$ with $f_6(t)$ which is not a square, such that $\mathbf{g_1} : (w, t) \to (-\zeta_5^2 w, t)$. If $f_6(t)$ is generic we find the previous equation and so the maximal component of the moduli space of K3 surfaces admitting a non-symplectic automorphism of order 5 fixing at least one curve. Specializations of the polynomial $f_6(t)$ induce specializations of the K3 surface S. For example the line 20 of the `Table 2` corresponds to the curve C_1 given by $w^{10} = t^2 g_4(t)$, $\deg(g_4(t)) = 4$ and g_4 does not have multiple roots. The corresponding K3 surface is to the double cover of $\mathbb{P}^2_{[x,t]}$ branched along the sextic $x^5 + t^2 g_4(t) = 0$, which has a singular point of type A_4 in the point $(x, t) = (0, 0)$.

The automorphism τ_5 (resp. δ_5) on D_5 induces the non-symplectic automorphism $\mathrm{id} \times \tau_5$ (resp. $\mathrm{id} \times \delta_5$) of order 10 (resp. 5) on the K3 surface S which acts on the coordinates (x, y, t) as $(x, y, t) \to (\zeta_5 x, -y, t)$ (resp. $(x, y, t) \to (\zeta_5 x, y, t)$). The fixed locus of $\mathrm{id} \times \delta_5$ consists of one curve of genus 2 if $f_6(t)$ is generic and specializes to different fixed locus when $f_6(t)$ specializes (see also [AST11, Example 5.1]).

We observe that the non-symplectic automorphism $(id \times \tau_5)^5$ of order 2 is exactly the cover involution of the double cover of $\mathbb{P}^2$ and this allows one to compute easily its fixed locus.

We saw in Section 5 that certain K3 surfaces admitting a non-symplectic automorphism of order 5, can be obtained from the quotient $(C_1 \times D_5)/(\mathbf{g_1} \times \delta_5)$. So we obtain a different equation for these K3 surfaces. The surfaces obtained in this way are listed in `Table 1`. In the case of the 1-dimensional and 0-dimensional families the equation of the curve C_1 is $w^5 = f_3(t)$, $\deg(f_3(t)) = 3$ and f_3 is not a cube, and the automorphism is $\mathbf{g_1}: (w, t) \to (\zeta_5^2 w, t)$. The functions $x := uw^2$, $y := vw^5$ and t are invariant and give a (singular) model of the K3 surface S, with equation $y^2 = x^5 s + f_3^2(t)$.

Every K3 surface that is the double cover of $\mathbb{P}^2$ branched along a sextic can be viewed as a hypersurface in the weighted projective space $W\mathbb{P}(3, 1, 1, 1)$. In particular the homogeneous equation of S can be written as $y^2 = x^5 s + f_6(t : s)$ where $(y : x : s : t)$ are the homogeneous coordinates of $W\mathbb{P}(3, 1, 1, 1)$ (y is the coordinate of weight 3).This remark will be useful in view of the equations we found in cases $p = 7, 11$.

7.3. Automorphisms of order $p = 7$.

In Theorem 6.1 and `Table 2` we proved that the K3 surfaces admitting a non-symplectic automorphism of order 7 with at least one curve in the fixed locus are the minimal model of the quotient $(C_1 \times D_7)/(\mathbf{g_1} \times \tau_7)$ for a certain choice of the pair $(C_1, \mathbf{g_1})$. In particular the maximal component (with fixed locus $(n, g(C), k + 1) = (3, 1, 1)$) is obtained choosing C_1 to be a $14 : 1$ cover of $\mathbb{P}^1$ branched along 4 points of order 14 and 1 point of order 7. An equation of C_1 is $w^{14} = t(t - 1)(t - \lambda_1)(t - \lambda_2)$. The local action near the fixed points is $-\zeta_7^4$ (see `Table 2`) and thus the automorphism is $\mathbf{g_1} : (w, t) \mapsto (-\zeta_7^3 w, t)$. The functions $x := uw^2$, $y := vw^7$, t are invariant for $\mathbf{g_1} \times \tau_7$ and satisfy the equation $y^2 = x^7 + t(t - 1)(t - \lambda_1)(t - \lambda_2)$. As in case $p = 3$, one shows that this gives in fact a (singular) model of the K3 surface S: The equation can be homogeneized to

$$(9) \qquad y^2 = x^7 s + t(t - s^2)(t - \lambda_1 s^2)(t - \lambda_2 s^2) \subset W\mathbb{P}(4, 2, 1, 1)_{(y:t:x:s)}.$$

In order to show that the equation (9) corresponds in fact to a (singular model of a) K3 surface we observe that the surface defined by (9) is well formed (cf. [Fl00, Definition 6.9]) and quasismooth (cf. [Fl00, Definition 6.3]). If a hypersurface Z of degree d in a weighted projective space $W\mathbb{P}(a_0, a_1, a_2, a_3)$ is well formed and quasismooth, then the adjunction formula generalizes and the canonical sheaf is $\omega_Z \simeq \mathcal{O}_Z(d - \sum_{i=0}^3 a_i)$ (cf. [Fl00, Paragraph 6.14]). In particular if $d = \sum_{i=0}^3 a_i$, then Z is a K3 surface and so the surface defined by (9) is a singular model of a K3 surface.

We recall that the generic hypersurface of degree 8 in $W\mathbb{P}(4, 2, 1, 1)$ is a singular model of a K3 surface ([R79, Section 4.5]) with two singularities of type $\frac{1}{2}(1, 1)$ at the points $(1 : 1 : 0 : 0)$, $(-1 : 1 : 0 : 0)$. The surface defined by (9) (which is not general) has no other singular points.

The automorphism induced on S by $\mathrm{id} \times \tau_7$ acts on the coordinates of $W\mathbb{P}(4, 2, 1, 1)$ in the following way: $(y : t : x : s) \mapsto (-y : t : \zeta_7^2 x : s)$. It has order 14 and its fixed locus consists of 5 points: $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$, $(0 : 1 : 0 : 1)$, $(0 : \lambda_1 : 0 : 1)$, $(0 : \lambda_2 : 0 : 1)$. The singular points of $W\mathbb{P}(4, 2, 1, 1)$ are

switched by the automorphism. The fixed locus of the non-symplectic automorphism of order 7 induced by $\mathrm{id} \times \delta_7$ consists of the point $(0:0:1:0)$ and of the curve $y^2 = t(t-s^2)(t-\lambda_1 s^2)(t-\lambda_2 s^2) \subset W\mathbb{P}(4,2,1)$. The well formed expression (cf. [Fl00, Definition 5.11]) of this curve is $y^2 = t(t-s)(t-\lambda_1 s)(t-\lambda_2 s) \subset W\mathbb{P}(2,1,1)$ (cf. [Fl00, Lemma 5.7]) and the genus of such a curve is 1 (e.g., [D82, Corollary 3.4.4]).

We observe that the singular points of $W\mathbb{P}(4,2,1,1)$ are contained in the fixed locus of the automorphism. By equation (9) one sees that S is a $2:1$ ramified cover of $W\mathbb{P}(2,1,1)$, with branch locus given by $x^7 s + t(t-s^2)(t-\lambda_1 s^2)(t-\lambda_2 s^2) = 0$. The weighted projective plane $W\mathbb{P}(2,1,1)$ has a natural embedding in $\mathbb{P}^3$ with coordinates $(x_0 : x_1 : x_2 : x_3) = (t : x^2 : s^2 : xs)$, whose image is a cone Q of equation $x_3^2 = x_1 x_2$. The branch locus of the covering is now given by the intersection of Q and the curve $x_1^3 x_3 + x_0(x_0 - x_2)(x_0 - \lambda_1 x_2)(x_0 - \lambda_2 x_2) = 0$, which does not pass through the vertex of Q. The automorphism descends to $\mathbb{P}^3$ with the action $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \zeta_7^2 x_1 : x_2 : \zeta_7 x_3)$. The fixed locus is the isolated point $(0:1:0:0)$ and the curve $x_1 = x_3 = 0$, which passes through the vertex of Q. We blow up the vertex of Q introducing a copy of $\mathbb{P}^1$. The induced automorphism leaves invariant the exceptional divisor E and fixes the strict transform of the fixed curve B. Since it restricts to an automorphism of E, it fixes two points on it, one of them is $E \cap B$. Above the other fixed point on E we find two fixed point on S.

The fixed locus of the non-symplectic involution $\mathrm{id} \times \iota$ is the curve $x^7 s + t(t-s^2)(t-\lambda_1 s^2)(t-\lambda_2 s^2) \subset W\mathbb{P}(2,1,1)$. This is a curve of genus 9 in $W\mathbb{P}(2,1,1)$ by [D82, Corollary 3.4.4].

In `Table 1` we showed that certain K3 surfaces admitting a non-symplectic automorphism of order 7, can be obtained from the quotient $(C_1 \times D_7)/(\mathbf{g_1} \times \delta_7)$. In the case of the 0-dimensional family the equation of the curve C_1 is $w^7 = t(t-1)$ and the automorphism is $\mathbf{g_1} : (w,t) \to (\zeta_7^3 w, t)$. The functions $x := uw^2$, $y := vw^7$ and t are invariant and gives a (singular) model of the K3 surface S, with equation $y^2 = x^7 s + t^2(t-s^2)^2 \subset W\mathbb{P}(4,2,1,1)_{(y:t:x:s)}$.

7.4. **Automorphisms of order** $p = 11$. If $p = 11$, one can obtain an equation for a (singular model) of S, minimal model of $(C_1 \times D_{11})/(\mathbf{g_1} \times \tau_{11})$, as in cases $p = 3, 5, 7$: an equation for the curve C_1 is $w^{11} = t(t-1)(t-\lambda)$ (where if $\lambda \neq 0, 1$ the curve C_1 is the one described in line 27 of `Table 2`, if either $\lambda = 1$ or $\lambda = 0$, the curve C_1 is the one described in line 28 of `Table 2`) and the automorphism $\mathbf{g_1} : (w,t) \to (-\zeta_{11}^5 w, t)$. An equation of S is $y^2 = x^{11} s - t(t-s^4)(t-\lambda s^4)$ where $y := vw^{11}$, t, $x := uw^2$ and s are coordinates of the weighted projective space $W\mathbb{P}(6,4,1,1)$. As in case $p = 7$, one shows that this equation define in fact a singular model W of a K3 surface. The surface W is singular in the point $(1:1:0:0)$.

The automorphism $\mathrm{id} \times \tau_{11}$ induces the non-symplectic automorphism $(y : t : x : s) \mapsto (-y : t : \zeta_{11} x : s)$ on the surface W whose fixed locus consists of the points $(0:0:0:1)$, $(0:1:0:1)$, $(0:\lambda:0:1)$, $(0:0:1:0)$ (which are all distinct if $\lambda \neq 0$ and $\lambda \neq 1$).The point $(0:0:1:0)$ is a singular point of type of the surface.

The automorphism $\mathrm{id} \times \delta_{11}$ induces the non-symplectic automorphism $(y : t : x : s) \mapsto (y : t : \zeta_{11} x : s)$ on the surface W whose fixed locus consists of the point $(0:0:1:0)$ and of the curve $y^2 = t(t-s^4)(t-\lambda s^4) \subset W\mathbb{P}(6,4,1)$. The well formed expression of this curve is $y^2 = t(t-s^2)(t-\lambda s^2) \subset W\mathbb{P}(3,2,1)$ which is quasismooth if $\lambda \neq 0, \lambda \neq 1$. In this case the genus of the curve is 1 [Fl00, Theorem 7.2].

The automorphism $\mathrm{id} \times \iota$ induces the non-symplectic involution $(y : t : x : s) \mapsto (-y : t : x : s)$ on the surface W whose fixed locus consists of the curve $x^{11} s - t(t-s^4)(t-\lambda s^4) \subset W\mathbb{P}(4,1,1)$ whose genus is 10, if $\lambda \neq 0, \lambda \neq 1$ [Fl00, Theorem 7.2].

8. Moduli of K3 surfaces

By Theorems 5.1 and 6.1 certain components $\mathcal{M}^p_{(n,g(C),k+1)}$ of the moduli space of the K3 surfaces with an automorphism of order p coincide with certain moduli spaces $\mathcal{F}_{PQ}(C_1, D_p)$ of the K3 surfaces which are minimal models either of the quotients $(C_1 \times D_p)/(\mathbf{g_1} \times \delta_p)$ or of the quotients $(C_1 \times D_p)/(\mathbf{g_1} \times \tau_p)$. Since both $D_p \to D_p/\delta_p \simeq \mathbb{P}^1$ and $D_p \to D_p/\tau_p \simeq \mathbb{P}^1$ are branched in 3 points, the parameters of the family depend only on the parameters of the curve C_1. In particular the dimension of the family of K3 surfaces is $r - 3$, where $r = \sum a_i$ is the number of ramification points of the cover $C_1 \to C_1/\mathbf{g_1} \simeq \mathbb{P}^1$, cf. Proposition 4.5. Here we describe the relation between the moduli of the curve C_1 and the moduli of the surface S. In particular we relate the variation of the Hodge structure of weight 2 of S with the one of $H^1(C_1, \mathbb{Q})$.

A particular case is the one with $p = 3$ and the quotient $(C_1 \times D_3)/(\mathbf{g_1} \times \delta_3)$. In this case the variation of the Hodge structure of S (and of a Calabi–Yau 3-fold constructed from S and D_3) is described in

[GvG10]. Moreover, if the family is 1-dimensional, the Picard–Fuchs equation of the surface S is found from the one of the curve C_1 (cf. [GvG10, Section 2.5]).

We now assume S to be the minimal model of $(C_1 \times D_p)/(\mathbf{g_1} \times \delta_p)$. By construction S admits a non-symplectic automorphism $\mathbf{g}_S$ induced by $\mathrm{id} \times \delta_\mathrm{p}$. The moduli space of the K3 surfaces S obtained in such a way is determined by the variation of the period of S in a certain eigenspace $H^2(S,\mathbb{C})_{\zeta_p^j}$, cf. Section 4.3. Indeed, the choice of the period of S determines the Hodge structure of $H^2(S,\mathbb{C})$ completely.

Proposition 8.1. *Let S be a generic K3 surface in the family $\mathcal{F}_{PQ}(C_1 \times D_p)$ of the surfaces minimal models of $(C_1 \times D_p)/(\mathbf{g_1} \times \delta_p)$ listed in* `Table 1`. *The weight 2 Hodge structure of the transcendental lattice of S, $T_S \otimes \mathbb{Q}$, is induced by the weight 1 Hodge structure of $H^1(C_1, \mathbb{Q})$. In particular, the half twist $(T_S \otimes \mathbb{Q})_{1/2}$ is $H^1(C_1, \mathbb{Q})$. As a consequence the dimension of the family $\mathcal{F}_{PQ}(C_1 \times D_p)$ is $2g(C_1)/(p-1) - 1$.*

Proof. Since S is generic, the transcendental lattice of S carries a weight 2 Hodge structure of type $(1, (p-1)(m+1) - 2, 1)$. Since the K3 surface S admits a non-symplectic automorphism $\mathbf{g}_S$, the Hodge structure of $T_S \otimes \mathbb{Q}$ is of CM-type with the field $K \simeq \mathbb{Q}(\zeta_p)$ (cf. [vG92]). In order to perform a half twist on the Hodge structure one has to fix a CM-type, i.e. a set Σ of $(p-1)/2$ distinct embeddings of K in $\mathbb{C}$ with the property that no two of them are conjugate. By abuse of notations we put $\Sigma = \{\zeta_p, \ldots, \zeta_p^{(p-1)/2}\}$. The eigenspaces decomposition (for the action of δ_p) of $H^1(D_p)$ consists of $p-1$ 1-dimensional vector spaces. Therefore $K_{-1/2} \simeq H^1(D_p)$ as Hodge structure of weight 1, where $K_{-1/2}$ is the negative half twist of K (see [vG92, Section 1.4]).

Let us denote by $\nu : C_1 \times D_p \to S$ the map induced by the quotient map. The pull-back ν^* maps $T_S \otimes \mathbb{Q}$ in the $(\mathbf{g_1} \times \delta_p)$-invariants in $H^1(C, \mathbb{Q}) \otimes H^1(D_p, \mathbb{Q})$. For dimensional reason

$$T_S \otimes \mathbb{Q} \simeq \nu^*(T_S \otimes \mathbb{Q}) = \left(H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(D_p, \mathbb{Q})\right)^{\mathbf{g_1} \times \delta_p}.$$

Let us consider the half twist of both the members of the above equation:

$$(10) \qquad (T_S \otimes \mathbb{Q})_{1/2} \simeq \left(\left(H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(D_p, \mathbb{Q})\right)^{\mathbf{g_1} \times \delta_p}\right)_{1/2}.$$

In order to compute the second member of (10), we first consider the $(\mathbf{g_1} \times \delta_p)$-invariant subspace of $H^1(C_1, \mathbb{C}) \otimes H^1(D_p, \mathbb{C}) \subset H^2(C_1 \times D_p)$. We recall that $H^{1,0}(D_p)_{\zeta_p^i}$ is an eigenspace of dimension 1 if $i \leq (p-i)/2$ and is trivial if $i > (p-1)/2$. By the fact that $p_g(S) = 1$, the pair $(C_1, \mathbf{g_1})$ is such that there exists only one value $\bar{h}$ such that $\bar{h} > (p-1)/2$ and $H^{1,0}(C_1)_{\zeta_p^{\bar{h}}}$ is non zero. So the Hodge decomposition of an eigenspace $H^1(C_1)_{\zeta_p^k}$ is trivial, in the sense that $H^1(C_1)_{\zeta_p^k} = H^{a,b}(C_1)_{\zeta_p^k}$ with $(a,b) \in \{(1,0),(0,1)\}$, except for $k = \bar{h}$. Hence we obtain:

$$\left(\left(H^1(C, \mathbb{C}) \otimes H^1(D_p, \mathbb{C})\right)\right)^{\mathbf{g_1} \times \delta_p} = \sum_{i=1}^{p-1} \left(\left(H^1(C_1, \mathbb{C})_{\zeta_p^i} \otimes H^1(D_p, \mathbb{C})_{\zeta_p^{p-i}}\right)\right) =$$

$$= \quad H^{1,0}(C_1)_{\zeta_p^{\bar{h}}} \otimes H^{1,0}(D_p)_{\zeta_p^{p-\bar{h}}} \oplus \sum_{i=1}^{(p-1)/2} \left(H^{1,0}(C_1)_{\zeta_p^i} \otimes H^{0,1}(D_p)_{\zeta_p^{p-i}}\right) \oplus$$

$$\oplus \quad H^{0,1}(C_1)_{\zeta_p^{p-\bar{h}}} \otimes H^{0,1}(D_p)_{\zeta_p^{\bar{h}}} \oplus \sum_{i=(p+1)/2}^{(p-1)} \left(H^{0,1}(C_1)_{\zeta_p^i} \otimes H^{1,0}(D_p)_{\zeta_p^{p-i}}\right).$$

Now we consider the splitting given by the choice of $\Sigma = \{\zeta_p, \ldots \zeta_p^{(p-1)/2}\}$ and we recall that the action of $\mathbf{g}_S$ on T_S is induced by the action of δ_p on D_p. So $\left(\sum_{i=(p-1)/2}^{p-1} \left(H^{0,1}(C_1)_{\zeta_p^i} \otimes H^{1,0}(D_p)_{\zeta_p^{p-i}}\right)\right)_{\bar{\Sigma}} = 0$. Hence

$$\left(\left(H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(D_p, \mathbb{Q})\right)^{\mathbf{g_1} \times \delta_p}\right)_{1/2}^{1,0} = \left(H^{1,0}(C_1)_{\zeta_p^h} \otimes H^{1,0}(D_p)_{\zeta_p^{p-h}}\right) \oplus \left(\sum_{i=1}^{(p-1)/2} H^{1,0}(C_1)_{\zeta_p^i} \otimes H^{0,1}(D_p)_{\zeta_p^{p-i}}\right).$$

Since $H^{1,0}(D_p)_{\zeta_p^i} \simeq \mathbb{C}$ if $i \leq (p-1)/2$, we obtain

$$\left(\left(H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(D_p, \mathbb{Q})\right)^{\mathbf{g_1} \times \delta_p}\right)_{1/2}^{1,0} \simeq H^{1,0}(C_1)_{\zeta_p^h} \oplus \sum_{i=1}^{(p-1)/2} H^{1,0}(C_1)_{\zeta_p^i} = H^{1,0}(C_1).$$

By conjugacy, $\left(\left(H^1(C_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(D_p, \mathbb{Q})\right)^{\mathbf{g_1} \times \delta_p}\right)_{1/2}^{0,1} = H^{0,1}(C_1)$. Substituting in (10), we obtain $(T_S \otimes \mathbb{Q})_{1/2} \simeq H^1(C_1, \mathbb{C})$ as Hodge structure.

In particular $\mathrm{rk}(T_S \otimes \mathbb{Q}) = 2g(C_1)$ and by the computation of the moduli of the K3 surface S with a non-symplectic automorphism of order p, it follows $m = 2g(C_1)/(p-1) - 1$.

$\square$

Remark 8.2. The eigenspaces decomposition (for the action of $\mathbf{g_1}$) of $H^1(C_1, \mathbb{Q})$ splits this space in $(p-1)$ equidimensional subspaces and thus $(p-1)(\dim(H^1(C_1)_{\zeta_p^j})) = 2g(C_1)$ for every $j \in \{1, \ldots, p-1\}$. By the previous proposition we have $\dim(H^1(C_1)_{\zeta_p^j}) = 2g(C_1)/(p-1) = m+1$ for every $j \in \{1, \ldots, p-1\}$. Moreover, in the proof of the previous proposition we saw that there exists a unique value $\bar{h}$ such that $\bar{h} > p - 1$ and $H^{1,0}(C_1)_{\zeta_p^{\bar{h}}}$ is non zero. In particular $\dim(H^{1,0}(C_1)_{\zeta_p^{\bar{h}}}) = 1$, because $p_g(S) = 1$. So $m = \dim(H^1(C_1)_{\zeta_p^{\bar{h}}}) - 1 = \dim(H^{1,0}(C_1)_{\zeta_p^{\bar{h}}} \oplus H^{0,1}(C_1)_{\zeta_p^{\bar{h}}}) - 1 = \dim(H^{1,0}(C_1)_{\zeta_p^{p-\bar{h}}}) = \alpha_{p-\bar{h}}$.

Remark 8.3. In case S is the minimal model of $(C_1 \times D_p)/(\mathbf{g_1} \times \tau_p)$ one can obtain a result similar to the one of Proposition 8.1: the half twist of $T_S \otimes \mathbb{Q}$ is a sub-Hodge structure of $H^1(C_1, \mathbb{Q})$ and in fact the one of $\sum_{i=1}^{p-1} H^1(C_1, \mathbb{C})_{-\zeta_p^i}$. An explicit example is given in [vG92, Section 3.11].

The variation of the period of S is described by the Picard–Fuchs equation of ω_S, and so by the Picard–Fuchs equation of certain holomorphic 1-form on C_1. In particular if C_1 varies in a 1-dimensional family, then it admits an equation of type $y^N = t^a(t-1)^b(t-\lambda)^b$. The forms of these curves and their Picard–Fuchs equations are described in [GvG10, Section 2.5] and this immediately gives the Picard–Fuchs equations of S.

For example the Picard-Fuchs equation of the 1-dimensional family $\mathcal{M}^5_{(10,0,2)}$ is the Picard–Fuchs equation of the 1-holomorphic form ω_C of C_1 such that $\mathbf{g_1}(\omega_C) = \zeta_p^3 \omega_C$. Since an equation for C_1 is $y^5 = t(t-1)(t-\lambda)$, the holomorphic form we are interested in is $\omega_1 := dt/y^2$ and its Picard Fuchs equation is

$$\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + \left(\frac{4}{5} - \frac{8}{5}\lambda\right)\frac{\partial}{\partial\lambda} - \frac{2}{25} = \frac{2}{5}\frac{t(t-1)}{(t-\lambda)y^2}.$$

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```
Q:=Rationals();                              Li:=[];
Z:=Integers();                               for i in [1..#G-1] do
                                              for j in [1..#G-1] do
                                               Append(~Li,Au[i][j]);
/////////////////////////////////////////    end for; end for;
                                             M:=Matrix(R,#G-1,Li);
MaxGenus:=function(G)                         return Li;
                                            end function;

// Given a cyclic group of order n =  p or
// 2*p it returns the max genus            TerminiNoti:=function(G)
//  of a curve C with group of              R:=PolynomialRing(Q,#G-1);
// automorphisms Z/n, C/Z/n=P1, and         if IsPrime(#G) then
// such that the induced action on H^{1,0}(C) p:=#G;
// has an eigenspace of dimension 1.        else
                                             x:=(#G)/2;
// Check if the data are correct.            p:=Z ! x;
                                            end if;
 if not IsCyclic(G) then                    sf:=(p-1)/2;
  return 0;                                 st:=Z ! sf;
 end if;                                    F<s>:=CyclotomicField(p);
 if IsCoercible(Z, #G/2) then
  x:=#G/2;                                  // First case |G|=p
  y:=Z ! x;
  if not IsPrime(y) then                     if IsPrime(#G) then
  return 0;                                   Li:=[];
 end if; end if;                             for i in [1..p-1] do
 if not IsCoercible(Z, #G/2) then             Li[i]:=1/(1-s^i);
  if not IsPrime(#G) then                     end for;
  return 0;                                   P:=[];
 end if; end if;                             List:=[];
 R:=PolynomialRing(Q,#G-1);                  for i in [1..p-1] do
                                              List[i]:=Eltseq(Li[i]);
 Preparation:=function(G)                    end for;
                                             K<s>:=PolynomialRing(Q,1);
// Given a cyclic group G of order          for i in [1..p-1] do
// 2*p or p it return the matrix of hol.     P[i]:=&+[List[i][j]*s^(j-1): j in [1..p-1]];
// Lefschetz.                                end for;
// WARNING: the eigenspaces have a peculiar  ww:=[i*0: i in [1..p-1]];
// ordering. The integers are always first. w:=Matrix(R,1,p-1,ww);
                                             for i in [1..p-1] do
  T:=CharacterTable(G);                       w:=w+R.i*Matrix(R,1,p-1,Coefficients(P[i]));
  if (#G mod 2) eq 0 then                     end for;
   x:=#G/2;                                   v:=[];
   y:=Z ! x;                                  v[1]:=-1+w[1,p-1];
   F:=CyclotomicField(y);                     for i in [2..p-1] do
  else                                         v[i]:=w[1,p-i];
   y:=#G;                                      end for;
   F:=CyclotomicField(#G);                     return v;
  end if;                                    else
  L:=[];
  Append(~L,T[#G]);                         // Second case |G|=2*p
 if #G ge 5 then
  Append(~L,T[#G-3]);                        Li:=[];
  Append(~L,T[2]);                           for i in [1..p-1] do
 else                                         Li[i]:=1/(1+s^i);
  Append(~L,T[2]);                            end for;
  Append(~L,T[1]);                           P:=[];
 end if;                                     List:=[];
  R:=PolynomialRing(F,#G-1);                 for i in [1..p-1] do
  List:=[];                                   List[i]:=Eltseq(Li[i]);
  for i in [1..#G-1] do                       end for;
   List[i]:=R.i;                             TN:=[];
  end for;                                   for hh in [1..p-1] do
  v:=Vector(R,List);                          TN[hh]:=[];
  g:=[];                                      end for;
  for j in [1..3] do                         for i in [1..p-1] do
   List2:=Eltseq(L[j]);                        for k in [1..p-1] do
   w:=Vector(R,List2);                          Append(~TN[k],List[i][k]);
   g[j]:=&+[w[i+1]*v[i]: i in [1..#G-1]];      end for;end for;
  end for;                                    w:=[];
  Au:=[];                                     for i in [1..p-1] do
  for k in [1..3] do                           w[i]:=&+[TN[i][j]*R.j: j in [1..p-1]];
   C:=Coefficients(g[k]);                      end for;
   TN:=[];                                     Li:=[];
   for hh in [1..y-1] do                       for i in [1..p-1] do
    TN[hh]:=[];                                 Li[i]:=1/(1-s^i);
   end for;                                     end for;
   for i in [1..#G-1] do                        P:=[];
    pip:=Eltseq(C[i]);                          List:=[];
    for h in [1..y-1] do                        for i in [1..p-1] do
     Append(~TN[h],-pip[h]);                     List[i]:=Eltseq(Li[i]);
    end for;                                     end for;
   end for;                                      TN:=[];
   for kk in [1..y-1] do
    Append(~Au,TN[kk]);
   end for; end for;
```

```
    for hh in [1..p-1] do
     TN[hh]:=[];
    end for;
    for i in [1..p-1] do
     for k in [1..p-1] do
      Append(~TN[k],List[i][k]);
    end for;end for;
    ww:=[];
    for i in [1..p-1] do
     ww[i]:=(&+[TN[i][j]*R.((j+j*st) mod p): j in [1..p-1]])
+(&+[TN[i][j]*R.(((j+j*st) mod p)+p-1): j in [1..p-1]]);
    end for;
    t:=2*p-1;
    www:=(&+[R.i: i in [1..p-1]]+R.t)/2;
    v:=[];
    for i in [1..p-1] do
     if i eq 1 then
      v[i]:=-1+w[i];
     else
      v[i]:=w[i];
     end if;
    end for;
    for i in [1..p-1] do
     if i eq 1 then
      v[p-1+i]:=-1+ww[i];
     else
      v[p-1+i]:=ww[i];
     end if; end for;
    v[2*p-1]:=www-1;
   end if;
  return v;
 end function;


// Find the correct eigenspace, which will
// have dimension 1

 if IsPrime(#G) then
  if #G ge 7 then
   p1:=1;
  else p1:=#G-1;
  end if;
 end if;
 if not IsPrime(#G) then
  if #G ge 7 then
   x:=(#G)/2+1;
   p1:=Z ! x;
  else
   p1:=#G-1;
  end if;
 end if;

// MAIN RUTINE MAXGENUS

 z:=#G-1;
 Li:=Preparation(G);
 v:=TerminiNoti(G);
 M:=Matrix(R,#G-1,Li);
 v1:=Vector(R,v);
 M1:=Transpose(M);
 sol:=Solution(M1,v1);
 g1:=&+[sol[i]: i in [1..z]];
 printf "======================== \n";
 printf "The general genus is: \n";
 printf "%o, \n", g1;
 printf "======================== \n";
 printf "\n";
 printf "======================== \n";
 printf "The dim of eigsp. are:  \n";
 for i in [1..z] do
 printf" %o, \n", sol[i];
 end for;
 printf "======================== \n";
 Coeff:=Coefficients(sol[p1]-1);
 ass:=-1/Coeff[1]*(sol[p1]-1-Coeff[1]*R.1);

// Get a bound on g
 g:=Evaluate(g1,1,ass);
 printf "\n";
 printf "======================== \n";
 printf "The special genus is: \n";
 printf "%o, \n", g;
 printf "======================== \n";
 printf "\n";
 Coefs:=Coefficients(g);
 MaxGen:=Coefs[#G-1];
```

```
 printf "======================== \n";
 printf "The max genus is: %o \n", MaxGen;
 printf "======================== \n";
 printf "\n";
 return MaxGen,sol;
end function;


/////////////////////////////////////////
//
//              END OF MAXGENUS
//
/////////////////////////////////////////

// THE FOLLOWING PART WORKS ONLY FOR THE GROUP Z/pZ

Nram:=function(MaxGen,p)

// Given the genus g(C) genus and a prime p
// returns the number of branch points
// of the cover C -> C/Z/p = P1

 x:=2*MaxGen-2;
 M:=[p];
 y:=p*(-2+#M-1/M[1]);
 while x-y gt 0  do
  Append(~M, p);
  z:=#M/p;
  y:=p*(-2+#M-z);
 end while;
return #M;
end function;


PossRami:=function(N,NN)

// It returns a seq of seqs of
// #NN, whos sum of elements
// is less or eq to N

 S:={1..N};
 M:=[];
 i:=1;
  while i le N do
   for h in [1..NN] do
    Append(~M,RestrictedPartitions(i,h,S));
   end for;
   i:=i+1;
  end while;
  K:=[];
  for LL in M do
   for i in [1..#LL] do
    L:=[];
    for j in [1..#LL[i]] do
     L[j]:=LL[i][j];
    end for;
    for k in [#LL[i]+1..NN] do
     L[k]:=0;
    end for;
    Append(~K,L);
   end for;
  end for;
  return K;
end function;


MaybeSur1:=function(s,p)

// Given vectors in H return all possible pairings
// of dim of eigsp. which could give a surface with
// pg=1

 PP:=[];
// all possible pairing
 for i in [1..#s] do
  for j in [i..#s] do
   Append(~PP,[s[i],s[j]]);
  end for;end for;
  GP:=[];
  for i in [1..#PP] do
   n:=0;
   for j in [1..2] do
    for k in [1..p-1] do
     if PP[i][j]`EignSpaces[k] eq 0 then n:=n+1;
    end if;end for;end for;
   if n ge p-2 then Append(~GP, PP[i]);
  end if;end for;
  return GP;
end function;
```

```
TheSur:=function(MaybeSur,p,RF,sol)

 Sur:=[];
 for j in [1..#MaybeSur] do
  for h in [1..p-1] do
   v1:=Rotate(MaybeSur[j][2]'EignSpaces,h);
   if &+[MaybeSur[j][1]'EignSpaces[k]*v1[k]: k in [1..p-1]]
    eq 1 then
     for t in [1..p] do
      v2:=Rotate(MaybeSur[j][2]'FixPoints,t);
      if IsCoercible(Z,Evaluate(sol[1],v2)) then
       w:=rec< RF | EignSpaces := v1, FixPoints:=v2>;
       Include(~Sur,[MaybeSur[j][1],w]);
 end if;end for;end if;end for;end for;
 return Sur;
end function;


/////////////////////////////////////////////////
//
// Spherical Systems of Generators
// for groups of order p or 2p
//
/////////////////////////////////////////////////

GenSys:=function(G,m);

// Given a group G of order n it returns
// m integers whose sum is a multiple of n.

 n:=#G;
 H:=[];
 L:=[];
 M:={};
 for i in [1..n-1] do
  Include(~M,i);
 end for;
 for x in [1..m-1] do
  y:=n*x-(n*x mod n);
  TL:=RestrictedPartitions(y,m,M);
  for j in [1..#TL] do
   Append(~L,Reverse(TL[j]));
  end for;
 end for;
 return L;
end function;

VecRami:=function(H,G,n)

// Given a cyclic group of order p or 2p it returns
// a list of spherical system of generators of size n

 K:=[];
 if IsPrime(#G) then
  g:=G.1;
 else
  g:=G.1*G.2;
 end if;
  for h in H do
   L:=[];
   for i in [1..n] do
    Append(~L,g^h[i]);
   end for;
   if #sub<G | L> eq #G then
    Append(~K, L);
   end if;
  end for;
 return K;
end function;

/////////////////////////////////////////////////
//
// Match the Eigenspaces such that p_g of the surface is 1
//
/////////////////////////////////////////////////

GioCop:=function(H,K,G);
 sum:=0;
 p:=#G;
 for i in [1..p-1] do
  sum:=sum+H[i]*K[p-i];
 end for;
 return sum;
end function;
```

```
// Reordering the Eigenspaces only for group
// of order 2p
//
/////////////////////////////////////////////////
EigReorderingp:=function(V,G,sol)

// The eigenspaces are reordered
// w.r.t. -z, -z^2, -z^3, ...

 p:=Z ! #G;
 L:=[];
 for i in [1..p-1] do
  L[i]:=p*Coefficients(sol[i])[1];
 end for;
 NewV:=[i*0: i in [1..#G-1]];
 for i in [1..p-1] do
  NewV[Z ! L[i]]:=NewV[Z ! L[i]]+V[i];
 end for;
 return NewV;
end function;


EigReordering:=function(V,G,sol)

// The eigenspaces are reordered
// w.r.t. -z, z^2, -z^3, ...

 x:=(#G)/2;
 p:=Z ! x;
 q1:=(p-1)/2;
 q:= Z ! q1;
 L:=[];
 for i in [1..p-1] do
  L[i]:=2*p*Coefficients(sol[p+i])[1];
 end for;
 NewV:=[i*0: i in [1..#G-1]];
// EigS rel to elm of order 2
 NewV[p]:=NewV[1]+V[1];
// EigSs rel to elms of order p
 for i in [1..p-1] do
  if (Z ! L[i]) le p then
   NewV[(Z !L[i])+1]:=NewV[(Z ! L[i])+1]+V[i+1];
  else
   NewV[(Z ! L[i])-1]:=NewV[(Z ! L[i])-1]+V[i+1];
  end if;
 end for;
// EigSs rel to elms of order 2p
 for i in [1..p-1] do
  NewV[Z ! L[i]]:=NewV[Z ! L[i]]+V[p+i];
 end for;
 return NewV;
end function;

/////////////////////////////////////////////////
//
// Fix points reordering
//
/////////////////////////////////////////////////

FixPointsReorderingp:=function(V, G, tmp)

 F:=FiniteField(#G);
 g:=G.1;
 Gelm:=[];
 for i in [1..#G] do
  Append(~Gelm, g^i);
 end for;
 FixPoints:=[i*0: i in [1..#G-1]];
 for a in [1..tmp] do
  x:=F ! Position(Gelm, G ! V[a]);
  y1:=x^-1;
  y:= Z ! y1;
  FixPoints[y]:=(FixPoints[y]+1);
 end for;
 return FixPoints;
end function;


FixPointsReordering:=function(V, G, tmp)

// Needed for the calculation of the dim
// of the eigenspaces

 p1:=#G/2;
 p:= Z !p1;
```

```
 F:=FiniteField(p);
 g:=G.1*G.2;
 Gelm:=[];
// Fixing the one generator G.1*G.2=z we list z,z^2,z^3..
 for i in [1..#G] do
  Append(~Gelm, g^i);
 end for;
 OrdGelm:=[];
 for h in Gelm do
  Append(~OrdGelm, Order(h));
 end for;
 FixPoints:=[i*0: i in [1..#G-1]];
 for a in [1..tmp] do
  if Position(Gelm,V[a]) eq p then
   FixPoints[#G-1]:=FixPoints[#G-1]+p;
  end if;
   if IsEven(Position(Gelm,V[a])) then
    x1:=(Position(Gelm, V[a]))/2;
    x:= F ! x1;
    x2:=x^-1;
    x3:= Z ! x2;
    y:= Z ! p-1+x3;
    FixPoints[y]:=FixPoints[y]+2;
   end if;
    if IsOdd(Position(Gelm,V[a])) then
     if Position(Gelm,V[a]) ne p then
      x:=F ! Position(Gelm, G ! V[a]);
      y1:=x^-1;
      y:= Z ! y1;
      FixPoints[y]:=(FixPoints[y]+1);
     end if;
    end if;
 end for;
 return FixPoints;
end function;

FixPointsReordering2:=function(V, G, tmp)

// The fixpoints are ordered as [-z,-z^2,..][z,z^2,..][z^p]
// V is an ssg, tmp is the number of ramification pts.

 p1:=#G/2;
 p:= Z !p1;
 F:=FiniteField(p);
 g:=G.1*G.2;
 Gelm:=[];
 for i in [1..#G] do
  Append(~Gelm, g^i);
 end for;
 OrdGelm:=[];
 for h in Gelm do
  Append(~OrdGelm, Order(h));
 end for;
 FixPoints:=[i*0: i in [1..#G-1]];
 for a in [1..tmp] do
  if Position(Gelm,V[a]) eq p then
   FixPoints[#G-1]:=FixPoints[#G-1]+p;
  end if;
  if IsEven(Position(Gelm,V[a])) then
   x1:=(Position(Gelm, V[a]))/2;
   x:= F ! x1;
   x2:=x^-1;
   x3:=x2*2;
   x4:= Z ! x3;
   y:= Z ! p-1+x4;
   FixPoints[y]:=FixPoints[y]+2;
  end if;
  if IsOdd(Position(Gelm,V[a])) then
   if Position(Gelm,V[a]) ne p then
    x:=F ! Position(Gelm, G ! V[a]);
    y1:=x^-1;
    y:= Z ! y1;
    FixPoints[y]:=(FixPoints[y]+1);
   end if;
  end if;
 end for;
 return FixPoints;
end function;

//////////////////////////////////////////////////
//
// We couple the curves C_1 and C_2 to return
// potential surfaces of the form T=(C_1 x C_2)/Z/(2p)Z
// such that p_g(T)=1.
//
//////////////////////////////////////////////////
```

```
Surface:=function(GT,GT3,sol,G,t1,t2)

 RF := recformat<SSG, EignSpaces : SeqEnum, FixPoints : SeqEnum >;
 Sur:=[];
 if IsPrime(#G) then
  for i in [1..#GT] do
   W21b:=[];
   for k in [1..#G-1] do
    C:=FixPointsReorderingp(GT[i],G,t1);
    W21b[k]:=Evaluate(sol[k],C);
   end for;
   W21:=EigReorderingp(W21b,G, sol);
   for j in [i..#GT3] do
    W22b:=[];
    for k in [1..#G-1] do
     C:=FixPointsReorderingp(GT3[j],G,t2);
     W22b[k]:=Evaluate(sol[k],C);
    end for;
    W22:=EigReorderingp(W22b,G, sol);
    if GioCop(W21,W22,G) eq 1 then
     s:=[];
     Append(~s, rec< RF | SSG:= GT[i], EignSpaces := W21, FixPoints:=
      FixPointsReorderingp(GT[i],G,t1)>);
     Append(~s, rec< RF | SSG:= GT3[j], EignSpaces := W22, FixPoints:=
      FixPointsReorderingp(GT3[j],G,t2)>);
     Append(~Sur, s);
    end if;
   end for;
  end for;
 else
  for i in [1..#GT] do
   W21b:=[];
   for k in [1..#G-1] do
    C:=FixPointsReordering(GT[i],G,t1);
    W21b[k]:=Evaluate(sol[k],C);
   end for;
   W21:=EigReordering(W21b,G, sol);
   for j in [i..#GT3] do
    W22b:=[];
    for k in [1..#G-1] do
     C:=FixPointsReordering(GT3[j],G,t2);
     W22b[k]:=Evaluate(sol[k],C);
    end for;
    W22:=EigReordering(W22b,G, sol);
    if GioCop(W21,W22,G) eq 1 then
     s:=[];
     Append(~s, rec< RF | SSG:= GT[i], EignSpaces := W21, FixPoints:=
      FixPointsReordering2(GT[i],G,t1)>);
     Append(~s, rec< RF | SSG:= GT3[j], EignSpaces := W22, FixPoints:=
      FixPointsReordering2(GT3[j],G,t2)>);
     Append(~Sur, s);
    end if;
   end for;
  end for;
 end if;
 return Sur;
end function;

//////////////////////////////////////////////////
//
// Singularities check is needed. First we calculate the contributions
// of each singularity to k^2_S, e(S), and \chi(S). We borrowed this part
// of the program from [BP11].
//
//////////////////////////////////////////////////

ContFrac:=function(s)
 CF:=[ ]; r:=1/s;
 while not IsIntegral(r) do
  Append(~CF, Ceiling(r)); r:=1/(Ceiling(r)-r);
 end while;
 return Append(CF, r);
end function;

Nq:=func<cf|#cf eq 1 select cf[1] else cf[1]-1/$$(Remove(cf,1))>;

RatNum:=func<seq|1/Nq(seq)>;

// "Wgt" computes the weight of a sequence, i.e., the sum of its
// entries. It bounds strictly from below B of the corresponding
// singular point.

Wgt:=function(seq)
 w:=0; for i in seq do w+:=i; end for; return w;
end function;
```

```
// The next script computes all rational number
// whose continuous fraction has small weight,
// by listing all sequences (modulo
// "reverse") and storing the corresponding rational number.

RatNumsWithSmallWgt:=function(maxW)
  S:={ }; T:={}; setnums:={RationalField()| };
  for i in [2..maxW] do Include(~S, [i]); end for;
  for i in [1..Floor(maxW/2)-1] do
  for seq in S do
    if #seq eq i then
    if maxW-Wgt(seq) ge 2 then
    for k in [2..maxW-Wgt(seq)] do
     Include(~S,Append(seq, k));
    end for; end if; end if;
  end for; end for;
  for seq in S do
  if Reverse(seq) notin T then Include(~T,seq);
  end if; end for;
  for seq in T do Include(~setnums, RatNum(seq)); end for;
  return setnums;
end function;


// The next two scripts compute the invariants
// B and e of a rational number (i.e., of
// the corresponding singular point).

InvB:=func<r|Wgt(ContFrac(r))+r+RatNum(Reverse(ContFrac(r)))>;

Inve:=func<r|#ContFrac(r)+1-1/Denominator(RationalField()!r)>;

// Here is the invariant k of the basket:

Invk:=func<r|InvB(r)-2*Inve(r)>;

kappa:=function(p)
 KK:=[];
 for i in [1..p-1] do
  KK[i]:=-Invk(i/p);
 end for;
 return KK;
end function;


EuleR:=function(p)
 CC:=[];
 for i in [1..p-1] do
  CC[i]:=Inve(i/p);
 end for;
 return CC;
end function;


///////////////////////////////////////////////
//
// The program for singularities for surfaces with
// group Z/pZ
//
///////////////////////////////////////////////

TypeSing:=function(Sur, p)

 count:=0;
 F:=FiniteField(p);
 KK:=kappa(p);
 CC:=EuleR(p);
 for k in [1..#Sur] do
 L:=[i*0: i in [1..p-1]];
 for i in [1..(p-1)] do
  for j in [1..(p-1)] do
   x:= F ! i;
   y:= F ! j;
   z1:=x*y^-1;
   z:= Z ! z1;
   if z ne 0 then
    L[z]:=L[z]+Sur[k][1]‘FixPoints[i]*Sur[k][2]‘FixPoints[j];
    g1:=&+[Sur[k][1]‘EignSpaces[h]: h in [1..p-1]];
    g2:=&+[Sur[k][2]‘EignSpaces[h]: h in [1..p-1]];
   end if;
  end for;
 end for;
 K2:=8*(g1-1)*(g2-1)/p+&+[L[t]*KK[t]: t in [1..p-1]];
 chi:=4*(g1-1)*(g2-1)/p+&+[L[t]*CC[t]: t in [1..p-1]]+K2;
 if chi eq 24 then
  printf "========================= \n";
  printf"there is a surface with curves %o \n", Sur[k];
  printf"with singularities: \n" ;
  for h in [1..p-1] do
   printf"%o x 1/%o(1,%o) \n", L[h],p,h;
```

```
  end for;
  printf "the minimal resolution has at least %o -1-curves \n", -K2;
  printf "the minimal model has 12*chi = %o \n", chi;
  printf "========================= \n";
  printf "\n";
  count:=count+1;
 end if;
 end for;
 return count;
end function;

///////////////////////////////////////////////
//
// The program for singularities for surfaces with
// group Z/2pZ
//
///////////////////////////////////////////////

Singulp:=function(V,W,p)
 F:=FiniteField(p);
 L:=[i*0: i in [1..p-1]];
 for i in [1..(p-1)] do
  for j in [1..(p-1)] do
   x:= F ! i;
   y:= F ! j;
   z1:=x*y^-1;
   z:= Z ! z1;
   if z ne 0 then
    L[z]:=L[z]+V[i]*W[j];
  end if; end for; end for;
  return L;
end function;

Singul2p:=function(V,W,p)
 F:=FiniteField(p);
 L:=[i*0: i in [1..p-1]];
 for i in [1..(p-1)] do
  for j in [1..(p-1)] do
   x:= F ! 2*i;
   y:= F ! j;
   z1:=x*y^-1;
   z:= Z ! z1;
   if z ne 0 then
    L[z]:=L[z]+V[i]*W[j];
  end if; end for; end for;
  return L;
end function;

Singul2p2:=function(V,W,p)
 F:=FiniteField(p);
 L:=[i*0: i in [1..p-1]];
 for i in [1..(p-1)] do
  for j in [1..(p-1)] do
   x:= F ! 2*i;
   y:= F ! j;
   z1:=x^-1*y;
   z:= Z ! z1;
   if z ne 0 then
    L[z]:=L[z]+V[i]*W[j];
  end if;end for;end for;
  return L;
end function;


///////////////////////////////////////////////
//
// We check if the surfaces given in Sur are potential
// K3 calculating the invariants of (C_1 x C_2)/G
//
///////////////////////////////////////////////

CheckSing:=function(Sur,G)
 x:=#G/2;
 p:= Z ! x;
 count:=0;
 Super:=[];
 Sing:=[];
 os:=#Sing;
 Singolar:=[];
 for k in [1..#Sur] do
  VA:=[];
  WA:=[];
  VB:=[];
  WB:=[];
```

```
      for i in [1..p-1] do
       VA[i]:=Sur[k][1]`FixPoints[i];
       WA[i]:=Sur[k][2]`FixPoints[i];
       VB[i]:=Sur[k][1]`FixPoints[i+p-1];
       WB[i]:=Sur[k][2]`FixPoints[i+p-1];
      end for;

// Calculate the singularities divided in 3 types
// first the ones of the form 1/#G (1,*)
// then the ones of the form 1/p (1,*)
// and in the end of the form 1/2 (1,1)

   Singolar[1]:=Singulp(VA,WA,p);
   Singolar[2]:=[];
   for t in [1..p-1] do
     Append(~Singolar[2], Z ! (Singulp(VB,WB,p)[t]+
     Singul2p2(VA,WB,p)[t]+Singul2p2
(WA,VB,p)[t])/2);
   end for;
   Singolar[3]:=
   [Z ! (Sur[k][1]`FixPoints[#G-1]*Sur[k][2]`FixPoints[#G-1]+
&+[Sur[k][2]`FixPoints[#G-1]*Sur[k][1]`FixPoints[h]:
h in [1..p-1]]+
&+[Sur[k][1]`FixPoints[#G-1]*Sur[k][2]`FixPoints[h]:
h in [1..p-1]])/p];

// We calculate the genera of the curves

   g1:=&+[Sur[k][1]`EignSpaces[h]: h in [1..#G-1]];
   g2:=&+[Sur[k][2]`EignSpaces[h]: h in [1..#G-1]];
   Include(~Sing,Singolar);
   ns:=#Sing;

// Calculate K^2

   kcontrn:=[];
   kcontrp:=[];
   RR:=quo< Z | #G>;
   U1:=[];
   for i in [1..#G-1] do
    x:=RR ! i;
    if IsUnit(x) then
     Append(~U1,x);
    end if;
   end for;
   for i in [1..p-2] do
    if IsOdd(i) then
     kcontrn[i]:=Invk((i)/#G)*Singolar[1][i];
    else
     kcontrn[i]:=Invk((i+p)/#G)*Singolar[1][i];
    end if;
    kcontrp[i]:=Invk(i/p)*Singolar[2][i];
   end for;
   kappa2:=8*(g1-1)*(g2-1)/#G-&+[kcontrn[t]: t in [1..p-2]]-
&+[kcontrp[t]: t in [1..p-2]];

// Calculate e the Euler number

   econtrn:=[];
   econtrp:=[];
   RR:=quo< Z | #G>;
   U:=[];
   for i in [1..#G-1] do
    x:=RR ! i;
    if IsUnit(x) then
     Append(~U,x);
    end if;end for;
   for i in [1..p-1] do
    if IsOdd(i) then
     econtrn[i]:=Inve(i/#G)*Singolar[1][i];
    else
     econtrn[i]:=Inve((i+p)/#G)*Singolar[1][i];
    end if;end for;
   for i in [1..p-1] do
    econtrp[i]:=Inve(i/p)*Singolar[2][i];
   end for;
   econtr12:=Inve(1/2)*Singolar[3][1];
   euler:=4*(g1-1)*(g2-1)/#G+&+[econtrn[t]: t in [1..p-1]]+&
+[econtrp[t]: t in [1..p-1]]+econtr12;

// Calculate \chi

   chi:=(euler+kappa2)/12;

// Only potential K3s survive the two tests below
```

```
      if chi eq 2 then
       if os ne ns then
        Append(~Super,Sur[k]);
        printf "========================= \n";
        printf"there is a surface with curves %o \n", Sur[k];

        printf"with singularities: \n" ;
        for i in [1..p-1] do
         if IsOdd(i) then
          printf"%o x 1/%o(1,%o) \n", Singolar[1][i],#G,i;
         else
          printf"%o x 1/%o(1,%o) \n", Singolar[1][i],#G,(i+p);
        end if;end for;
        for i in [1..p-1] do
         printf"%o x 1/%o(1,%o) \n", Singolar[2][i], p, i;
        end for;
        printf"%o x 1/2(1,1) \n", Singolar[3][1];
        printf "the minimal resolution has at least %o -1-curves \n", -kappa2;
        printf "the minimal model has chi = %o \n", chi;
        printf "========================= \n";
        printf "\n";
        count:=count+1;
       end if; end if;
      os:=#Sing;
     end for;
    return count, Super;
   end function;

/////////////////////////////////////////////
//
// MAIN ROUTINES and COMMANDS
//
/////////////////////////////////////////////
//
// For ANY surfaces with group Z/(p)Z
//
/////////////////////////////////////////////
Surfacesp:=function(G,c)

// Set c=true to perform the calculation of Eigenspaces and max
// genus only.
// Set c=false to get a list of possible K3, in this case the
// algorithm is not optimal and for p >=11 could be extremly slow!

   p:=Z ! #G;
   R:=PolynomialRing(Q,p-1);
   MaxGen,sol:=MaxGenus(G);
   Nrami:=Nram(MaxGen,p);
   printf "========================= \n";
   printf "The max n. of rami is: %o \n", Nrami;
   printf "========================= \n";
   printf "\n";
    if c then return 0;
    end if;
   K:=PossRami(Nrami,p-1);
   printf "========================= \n";
   printf "Starting comb calc fom: %o \n", #K;
   printf "It will be very very slow... \n";
   printf "========================= \n";
   printf "\n";
   S:={1..p-1};
   To:=[];
   t:=#K;
   StS:=SetToSequence(Permutations(S));
   for i in [1..t] do
    for j in [1..#StS] do
     C:=[];
     for k in [1..p-1] do
      C[k]:=K[i][StS[j][k]];
     end for;
     Append(~To,C);
   end for;end for;
   Tot:=SetToSequence(Seqset(To));
   printf "========================= \n";
   printf "Finish transposition \n";
   printf "Starting cyclic permut.... \n";
   printf "========================= \n";
   printf "\n";
   printf "========================= \n";
   printf "The n. of Tot. is: %o \n", #Tot;
   printf "========================= \n";
   printf "\n";
   ops:=0;
```

```
RF := recformat< EignSpaces : SeqEnum, FixPoints : SeqEnum >;
s:=[];
H:=[];
printf "======================= \n";
for i in [1..#Tot] do
 W22:=[];
 for j in [1..p-1] do
  Append(~W22, Evaluate(sol[j],Tot[i]));
 end for;
 OS:=#H;
 if W22[1] eq 1 then
  Include(~H,W22);
  Append(~s, rec< RF | EignSpaces := W22, FixPoints:= Tot[i]>);
  NS:=#H;
  if ops eq 0 then
   if #H eq 1 then
    ops:=ops+1;
   end if;
  end if;
 end if;
end for;
printf "======================= \n";
printf "\n";
printf "======================= \n";
printf"We have %o curves \n", #s;
printf "======================= \n";
printf "\n";
MaybeSur:=MaybeSur1(s,p);
Sur:=TheSur(MaybeSur,p,RF,sol);
printf "\n";
printf "======================= \n";
printf"We have %o possible Surfaces \n", #Sur;
printf "======================= \n";
printf "\n";
c:=TypeSing(Sur, p);
printf "\n";
printf "======================= \n";
printf"We have %o Surfaces \n", c;
printf "======================= \n";
printf "\n";
 return Sur;
end function;


//////////////////////////////////////////////////
//
// For t1-t2 points ramifications surfaces
// with group either Z/(p)Z or Z/(2p)Z
//
//////////////////////////////////////////////////

t1t2PtsSurfaces:=function(G,t1,t2)

 R:=PolynomialRing(Q,#G-1);
 MaxGen,sol:=MaxGenus(G);
 GTT:=GenSys(G,t1);
 GT:=VecRami(GTT,G,t1);
 GTT3:=GenSys(G,t2);
 GT3:=VecRami(GTT3,G,t2);
 Sur:=Surface(GT,GT3,sol,G,t1,t2);
 if IsPrime(#G) then
  count:=TypeSing(Sur,#G);
 else
  count, Super:=CheckSing(Sur,G);
 end if;
printf "\n";
printf "======================= \n";
printf"We have %o possible Surfaces \n", count;
printf "======================= \n";
printf "\n";
 return count;
end function;
```