ON ASYMPTOTIC BOUNDS FOR THE NUMBER OF IRREDUCIBLE COMPONENTS OF THE MODULI SPACE OF SURFACES OF GENERAL TYPE

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Abstract. In this paper we investigate the asymptotic growth of the number of irreducible and connected components of the moduli space of surfaces of general type corresponding to certain families of surfaces isogenous to a higher product. We obtain a higher growth then the previous growth by Manetti [M97].

1. Introduction

It is known that, once two positive integers (y, x) are fixed, the number of irreducible components $\iota(x, y)$ of the moduli space of surfaces of general type with $K^2 = y$ and $\chi = x$ is bounded from above by a function of y. In fact, Catanese proved that the number $\iota^0(y, x)$ of components containing regular surfaces, ie. $q(S) = 0$, has an exponential upper bound in K^2, more precisely [Cat92, p.592] gives the following inequality

$$\iota^0(x, y) \leq y^{77y^2}.$$

There are also some results showing how close one can get to this bound from below. In [M97], for example, Manetti constructed a sequence S_n of simply connected surfaces of general type with $K^2_{S_n} =: y_n$, such that the lower bound for the number $\delta(S_n)$ of $\mathcal{C}^\infty$ inequivalent complex structures on the oriented topological 4-manifold underlying S_n is

$$\delta(S_n) \geq y_n^{\frac{1}{5}\ln y_n}.$$

Using group theoretical methods, we are able to describe the asymptotic growth of the number of irreducible and connected components of the moduli space of surfaces of general type in certain sequences of surfaces. More precisely, we apply the definition and some properties of surfaces isogenous to a product of curves and we reduce the geometric problem of finding connected components into the algebraic one of counting some subfamilies of 2-groups, which can be effectively computed. Similar methods were first applied by Garion and the second author in [GP11], see also [P13]. Our main result is the following.

Theorem 1.1. *Let h be number of connected components containing surfaces isogenous to a product of curves of irregularity $q(S) = q \geq 0$, admitting a group of order 2^{3s} and ramification structure of type $((0|2^{2s+2}), (q|2^{2s-2q+2}))$. Then for $s \to \infty$ we have*

$$h \geq 2^{\frac{2}{9}\left(\ln x_s\right)^3}.$$

In particular, we obtain sequences y_s and $x_s = y_s/8$ with

$$\iota^0(x_s, y_s) \geq y_s^{\frac{2}{13}(\ln y_s)^2}.$$

Let us explain now the way in which this paper is organized.

The next section *Preliminaries* is divided into three parts. In the first part we recall different moduli spaces of surfaces of general type that one can consider and how the number of their irreducible and connected components are related. In the second part we recall the definition and the properties of surfaces isogenous to a higher product and the its associated group theoretical data, the so called *ramification structures*. The third part is purely group theoretical and we

2010 *Mathematics Subject Classification.* 14J10,14J29,20D15,20D25,20H10,30F99.

recall some generating properties of nilpotent groups of Frattini-class 2. In particular we give the asymptotic growth of the number of certain subfamilies of 2-groups.

In the two successive sections we construct infinitely families of regular (respectively irregular) surfaces isogenous to a product associated to nilpotent groups.

In the last section we prove the main theorem.

Acknowledgement The authors acknowledge the travel grant from a DAAD-VIGONI program. They also thank the Hausdorff Center (HIM) in Bonn for the kind hospitality.

Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers. By "*surface*" we mean a projective, non-singular surface S. For such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = \chi(S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic*.

2. Preliminaries

2.1. **The moduli space of surfaces of general type.** It is well known (see [Gie77]) that once are fixed two positive integers x, y there exists a quasiprojective coarse moduli space $\mathcal{M}_{y,x}$ of canonical models of surfaces of general type with $x = \chi(S)$ and $y = K_S^2$. We study the number $\iota(x, y)$ ($\gamma(x, y)$) of irreducible (resp. connected) components of $\mathcal{M}_{y,x}$.

In addition, one can consider different structures on a surface of general type S, for example we can denote by S^{top} the oriented topological 4-manifold underlying S, or with S^{diff} the oriented $\mathcal{C}^\infty$ manifold underlying S, and we can attach to S several integers.

Let $\mathcal{M}^{top}(S)$ be the subspace of $\mathcal{M}_{y,x}$ corresponding to surfaces (orientedly) homeomorphic to S, and $\mathcal{M}^{diff}(S)$ be the subspace corresponding to surfaces diffeomorphic to S, we define:

- $\delta(S)$**:** number of $\mathcal{C}^\infty$ inequivalent complex structures on S^{top}.
- $\gamma(S)$**:** number of connected components of $\mathcal{M}^{top}(S)$.
- $\iota(S)$**:** number of irreducible components of $\mathcal{M}^{top}(S)$.
- $\gamma(x, y)$**:** number of connected components of $\mathcal{M}_{y,x}$.
- $\iota(x, y)$**:** number of irreducible components of $\mathcal{M}_{y,x}$.
- $\iota^0(x, y)$**:** number of irreducible components of $\mathcal{M}_{y,x}$ of regular surfaces.

There are inequalities among the numbers above, the ones we need are the following:

$$\delta(S) \leq \gamma(S) \leq \iota(S) \leq \iota(x, y), \quad \gamma(x, y) \leq \iota(x, y). \tag{1}$$

See also [Cat92]. The union $\mathcal{M}$ over all admissible pairs of invariants (y, x) of these spaces is called the *moduli space of surfaces of general type*.

2.2. **Surfaces isogenous to a product and their moduli.**

Definition 2.1. *A surface S is said to be* isogenous to a higher product of curves *if and only if, equivalently, either:*

(1) *S admits a finite unramified covering which is isomorphic to a product of curves of genera at least two;*
(2) *S is a quotient $(C_1 \times C_2)/G$, where C_1 and C_2 are curves of genus at least two, and G is a finite group acting freely on $C_1 \times C_2$.*

By Proposition 3.11 of [Cat00] the two properties (1) and (2) are equivalent. Using the same notation as in Definition 2.1, let S be a surface isogenous to a product, and $G^\circ := G \cap (Aut(C_1) \times Aut(C_2))$. Then G° acts on the two factors C_1, C_2 and diagonally on the product $C_1 \times C_2$. If G° acts faithfully on both curves, we say that $S = (C_1 \times C_2)/G$ is a *minimal realization*. In [Cat00] it is also proven that any surface isogenous to a product admits a unique minimal realization.

Assumptions I: In the following we always assume:

(1) Any surface S isogenous to a product is given by its unique minimal realization;
(2) $G^\circ = G$, this case is also known as *unmixed type*, see [Cat00].

Under these assumption we have.

Proposition 2.2. [Cat00] *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves, then S is a minimal surface of general type with the following invariants:*

$$\chi(S) = \frac{(g(C_1)-1)(g(C_2)-1)}{|G|}, \quad e(S) = 4\chi(S), \quad K_S^2 = 8\chi(S). \tag{2}$$

The irregularity of these surfaces is computed by

$$q(S) = g(C_1/G) + g(C_2/G). \tag{3}$$

Moreover the fundamental group $\pi_1(S)$ fits in the following short exact sequence of groups

$$1 \longrightarrow \pi_1(C_1) \times \pi_1(C_2) \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1. \tag{4}$$

Among the nice features of surfaces isogenous to a product, one is that they can be obtained in a pure algebraic way. Let us briefly recall how.

Definition 2.3. *Let G be a finite group and let*

$$0 \le g', \qquad 2 \le m_1 \le \cdots \le m_r$$

be integers. A system of generators *for G of* type *$\tau := (g' \mid m_1, ..., m_r)$ is a $(2g'+r)-$tuple of elements of G:*

$$\mathcal{V} = (a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r)$$

such that the following conditions are satisfied:

(1) $\langle a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r \rangle \cong G$.
(2) *$ord(c_i) = m_i$ for all $1 \le i \le r$, denoting by $ord(c)$ the order of c.*
(3) $c_1 \cdot \ldots \cdot c_r \cdot \prod_{i=1}^{g'} [a_i, b_i] = 1$.

*If such a $\mathcal{V}$ exists then G is called $(g' \mid m_1, \dots, m_r)-$*generated.

Moreover, we call the r-tuple $(c_1, \dots, c_r)$ the spherical part *of $\mathcal{V}$ and if $g' = 0$ a system of generators is simply said to be* spherical.

We shall also use the notation , for example, $(g' \mid 2^4, 3^2)$ to indicate the tuple $(g' \mid 2, 2, 2, 2, 3, 3)$.

We have the following reformulation of the Riemann Existence Theorem.

Proposition 2.4. *A finite group G acts as a group of automorphisms of some compact Riemann surface C of genus g if and only if there exist integers $g' \ge 0$ and $m_r \ge m_{r-1} \ge \cdots \ge m_1 \ge 2$ such that G is $(g' \mid m_1, \dots, m_r)-$generated for some system of generators $(a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r)$, and the following Riemann-Hurwitz relation holds:*

$$2g - 2 = |G|(2g' - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})). \tag{5}$$

If this is the case, then g' is the genus of the quotient Riemann surface $C' := C/G$ and the Galois covering $C \to C'$ is branched in r points $p_1, \dots, p_r$ with branching numbers $m_1, \dots, m_r$ respectively. Moreover if $r = 0$ the covering is said to be *unramified* or *étale*.

Definition 2.5. *Two systems of generators $\mathcal{V}_1 := (a_{1,1}, b_{1,1}, \dots, a_{1,g_1'}, b_{1,g_1'}, c_{1,1}, \dots, c_{1,r_1})$ and $\mathcal{V}_2 := (a_{2,1}, b_{2,1}, \dots, a_{2,g_2'}, b_{2,g_2'}, c_{2,1}, \dots, c_{2,r_2})$ of G are said to have* disjoint stabilizers *or simply to be* disjoint*, if:*

$$\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2) = \{1\}, \tag{6}$$

where $\Sigma(\mathcal{V}_i)$ is the set of elements in G that stabilize a point in C,

$$\Sigma(\mathcal{V}_i) := \bigcup_{h \in G} \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{r_i} h \cdot c_{i,k}^j \cdot h^{-1}.$$

We notice that in the above definition only the spherical part of the system of generators plays a rôle.

Remark 2.6. From the above discussion we obtain that the datum of a surface S isogenous to a higher product of curves of unmixed type together with its minimal realization $S = (C_1 \times C_2)/G$ is determined by the datum of a finite group G together with two disjoint systems of generators $\mathcal{V}_1$ and $\mathcal{V}_2$ (for more details see e.g. [BCG06]).

Remark 2.7. The condition of being disjoint ensures that the action of G on the product of the two curves $C_1 \times C_2$ is free.

Indeed the cyclic groups $\langle c_{1,1}\rangle, \ldots, \langle c_{1,r_1}\rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on C_1, whereas $\langle c_{2,1}\rangle, \ldots, \langle c_{2,r_2}\rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on C_2. The singularities of $(C_1 \times C_2)/G$ arise from the points of $C_1 \times C_2$ with non-trivial stabilizer, since the action of G on $C_1 \times C_2$ is diagonal, it follows that the set of all stabilizers for the action of G on $C_1 \times C_2$ is given by $\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2)$.

Definition 2.8. *Let $\tau_i := (g'_i \mid m_{1,i}, \ldots, m_{r_i,i})$ for $i = 1, 2$ be two types. An* (unmixed) ramification structure *of type (τ_1, τ_2) for a finite group G, is a pair $(\mathcal{V}_1, \mathcal{V}_2)$ of disjoint systems of generators of G, whose types are τ_i, and which satisfy:*

$$\mathbb{Z} \ni \frac{|G|(2g'_i - 2 + \sum_{l=1}^{r_i}(1 - \frac{1}{m_{i,l}}))}{2} + 1 \geq 2, \tag{7}$$

for $i = 1, 2$.

Remark 2.9. Note that a group G and a ramification structure determine the main numerical invariants of the surface S. Indeed, by (2) and (5) we obtain:

$$4\chi(S) = |G| \cdot \left(2g'_1 - 2 + \sum_{k=1}^{r_1}(1 - \frac{1}{m_{1,k}})\right) \cdot \left(2g'_2 - 2 + \sum_{k=1}^{r_2}(1 - \frac{1}{m_{2,k}})\right) =: 4\chi(|G|, (\tau_1, \tau_2)). \tag{8}$$

The most important property of surfaces isogenous to a product is their weak rigidity property.

Theorem 2.10. [Cat03b, Theorem 3.3, Weak Rigidity Theorem] *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves. Then every surface with the same*

- *topological Euler number and*
- *fundamental group*

is diffeomorphic to S. The corresponding moduli space $\mathcal{M}^{top}(S) = \mathcal{M}^{diff}(S)$ of surfaces (orientedly) homeomorphic (resp. diffeomorphic) to S is either irreducible and connected or consists of two irreducible connected components exchanged by complex conjugation.

Remark 2.11. Thanks to the Weak Rigidity Theorem, we have that the moduli space of surfaces isogenous to a product of curves with fixed invariants — a finite group G and a type (τ_1, τ_2) — consists of a finite number of irreducible connected components of $\mathcal{M}$. More precisely, let S be a surface isogenous to a product of curves of unmixed type with group G and a pair of disjoint systems of generators of type (τ_1, τ_2). By (8) we have $\chi(S) = \chi(|G|, (\tau_1, \tau_2))$, and consequently, by (2) $K_S^2 = K^2(|G|, (\tau_1, \tau_2)) = 8\chi(S)$, and $e(S) = e(|G|, (\tau_1, \tau_2)) = 4\chi(S)$. Moreover, recall that the fundamental group of S fits into the exact sequence (4) and the subgroup $\pi_1(C_1) \times \pi_1(C_2)$ of $\pi_1(S)$ is unique, see [Cat00].

Let us chose a pair (τ_1, τ_2) of types. Denote by $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ the moduli space of isomorphism classes of surfaces isogenous to a product, which have a minimal realization $(C_1 \times C_2)/G$ that is given by a ramifications structure of type (τ_1, τ_2) for the finite group G. This moduli space is obviously a subset of the moduli space $\mathcal{M}_{K^2(|G|,(\tau_1,\tau_2)),\chi(|G|,(\tau_1,\tau_2))}$. With $y := K^2(n, (\tau_1, \tau_2))$ and $x := \chi(n, (\tau_1, \tau_2))$ we get:

Lemma 2.12. *Given a positive integer n and a pair (τ_1, τ_2) of types, then $\iota(x, y)$ is bounded from below by*

$$\#\{G,\ G \text{ is a group of order } n \text{ with a ramification structure of type } (\tau_1, \tau_2)\}/_{iso}.$$

Proof. It remains to prove that non-isomorphic group lead to distinct irreducible components of the moduli space. Indeed, the fundamental groups of the minimal realizations fit into sequences as given in Prop.2.2(4), with non-isomorphic quotients. On the other hand any isomorphism of the fundamental groups descends to the quotients since the subgroups are preserved thanks to the minimality of the realizations. So our claim follows. $\square$

2.3. **Enumerating p-groups.**

Proposition 2.13. [H60, S65] *If $f(k,p)$ is the number of groups of order p^k, p a prime, and if $A = A(k,p)$ is defined by*

$$f(k,p) = p^{Ak^3}, \tag{9}$$

then

$$\frac{2}{27} - \epsilon_k \leq A \leq \frac{2}{15} - \epsilon_k, \tag{10}$$

where ϵ_k is a positive number, depending only on k, which tends to 0 as k tends to ∞.

We are interested in the constructive part of the proof, where a sufficient number of groups is given, all of them nilpotent of Frattini-class 2, i.e. their Frattini subgroups are central and elementary abelian. Such groups are given by the following presentation. Let r and s be positive integers with $s+r=k$ and $b(i,j)$, $1 \leq i \leq r$, $1 \leq j \leq s$, and $c(i,i',j)$, $1 \leq i < i' \leq r$, $1 \leq j \leq s$, be integers between 0 and $p-1$. Then the relations

$$\begin{array}{lll}
(1) & [g_i, g_{i'}] = h_1^{c(i,i',1)} \cdot \ldots \cdot h_s^{c(i,i',s)}, & 1 \leq i < i' \leq r, \\
(2) & [g_i, h_j] = 1, & 1 \leq i \leq r, 1 \leq j \leq s, \\
(3) & [h_j, h_{j'}] = 1, & 1 \leq j < j' \leq s, \\
(4) & g_i^p = h_1^{b(i,1)} \cdot \ldots \cdot h_s^{b(i,s)} & 1 \leq i \leq r, \\
(5) & h_j^p = 1, & 1 \leq j \leq s,
\end{array}$$

on $g_1, \ldots, g_r$ and $h_1, \ldots h_s$ define a group of order p^k.

Remark 2.14. We can make the following two restrictions which won't change the asymptotic number of $p-$groups considered.

- To prove Proposition 2.13 is enough to consider groups with $r = 2s$.
- We can change (4) to $g_i^p = 1$, and consider only groups which are generated by elements of order p. This means that the $b(i,j)$'s are set to zero. This is allowed since their proportion among all choices is negligible compered to the number of $c(i,i',j)$'s as s goes to infinity.
- We can consider only those groups which are generated by the g_i's only. Indeed, such groups are characterized by the property that the s vectors c_j with entries $c_{i,i',j}$ are linearly independent. So the number of possible choices is
$$\prod_{l=0}^{s-1} (p^{\binom{r}{2}} - p^l).$$
Again the deviation to $p^{\binom{r}{2}s}$ can be subsumed into ϵ_k.

Assuptions II: We assume from now on that:

(1) Let $r = 2s$.
(2) $p = 2$. Nevertheless, what follows can be easily extended to $p > 2$;
(3) All the groups G have a presentation as above with condition (4) changed into $g_i^p = 1$.
(4) All the groups G are generated by the g_i.

As seen in the previous section to give a surface isogenous to a product it is enough to give a finite group G and a ramification structure of G. In this section we give one of the two systems of generators of a ramification structure that we keep fixed in the next two sections. We will complete the ramification structure of G with a second system of generators, once in order to obtain regular surfaces and then to have irregular ones.

We consider the following system of generator of size $2s+2$ for a group of order $2^k = 2^{3s}$.

$$T_1 := (g_1, \dots, g_s, \bar{g}_s, g_{s+1}, \dots, g_{2s}, \bar{g}_{2s}), \tag{11}$$

where $\bar{g}_s = (g_1 \cdots g_s)^{-1}$, $\bar{g}_{2s} = (g_{s+1} \cdots g_{2s})^{-1}$. By construction we have $< T_1 >\cong G$, $g_1 \cdot \ldots \cdot g_s \cdot \bar{g}_s \cdot g_{s+1} \cdot \ldots \cdot g_{2s} \cdot \bar{g}_{2s} = 1_G$, and the orders of each element is 2. This gives an action of a group G of order 2^k on some curve C_1 of genus

$$g(C_1) := 2^{3s-1}\Big(-2 + \sum_{l=1}^{2s+2}(1 - \frac{1}{2})\Big) + 1 = 2^{3s-1}(s-1) + 1. \tag{12}$$

Moreover, $C_1 \to C_1/G \cong \mathbb{P}^1$ branches in $2s+2$ points. In terms of Euler numbers we have:

$$e(C_1) = 2^{3s}\Big(e(\mathbb{P}^1) - \sum_{l=1}^{2s+2}(1 - \frac{1}{2})\Big) = 2^{3s}(1-s).$$

Now we give a criterion to see if two systems of generators are disjoint. Let denote by $H(G) \lhd G$ the subgroup of G generated by the h_j's and $\Phi\colon G \to G/H(G)$.

Lemma 2.15. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two spherical parts of two systems generators $\mathcal{V}_1$ and $\mathcal{V}_2$ of G. Moreover, let $B_i = \{\Phi(x) | x \in \mathcal{T}_i, x \notin \langle h_j \rangle\}$ and $B_i' = \{x | x \in \mathcal{T}_i, x \in \langle h_j \rangle\}$. If $B_1 \cap B_2 = B_1' \cap B_2' = \emptyset$ then $\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2) = 1_G$.*

Proof. Since the order of every element is 2, it is enough to prove that

$$A_1 \cup A_2 := \big(\bigcup_{t \in G} \bigcup_{x_1 \in \mathcal{T}_1} t x_1 t^{-1}\big) \cap \big(\bigcup_{t \in G} \bigcup_{x_2 \in \mathcal{T}_2} t x_2 t^{-1}\big) = \emptyset.$$

Since the kernel of Φ is $H(G)$ and the image is abelian, the images of the two sets A_1 and A_2 are exactly B_1 and B_2. By hypothesis $B_1 \cap B_2 = \emptyset$, so

$$\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2) = \big(\Sigma(\mathcal{V}_1) \cap H\big) \cap \big(\Sigma(\mathcal{V}_2) \cap H\big) = \big(B_1' \cup 1_G\big) \cap \big(B_2' \cup 1_G\big) = 1_G.$$

□

3. Regular Surfaces

As said before, we give the second system of generators of a ramification structure for a group G as above which yields a regular surface isogenous to a product. Let

(13)

$$T_2 := (g_1g_2, g_2g_3, \dots, g_{s-1}g_s, g_sg_2g_3, (g_1g_2g_3)^{-1}, g_{s+1}g_{s+2}, \dots, g_{2s}g_{s+2}g_{s+3}, (g_{s+1}g_{s+2}g_{s+3})^{-1}),$$

One can see that $< T_2 >\cong G$ and by construction the product of the elements in T_2 is 1_G. This yields a second G-Galois cover of $\mathbb{P}^1$ ramified in $2s+2$ points. And again the genus of the curve is

$$g(C_2) = 2^{3s-1}(s-1) + 1. \tag{14}$$

Moreover, the set $B_2 := \{\Phi(x) | x \in T_2\}$ is disjoint from $B_1 := \{\Phi(x) | x \in T_1\}$ so by the Lemma 2.15 the pair (T_1, T_2) is a ramification structure for G. The associated surface isogenous to a product S has irregularity $q(S) = 0$.

This ramification structure can be given to any 2-group as above.

Theorem 3.1. *Let h be number of connected components of regular surfaces isogenous to a product of curves admitting a group of order 2^{3s} and ramification structure of type $((0|2^{2s+2}), (0|2^{2s+2}))$, as above. Then for $s \to \infty$ we have*

$$h \geq 2^{Bs^3}, \tag{15}$$

where $2-\epsilon'_s \leq B \leq \frac{18}{5}-\epsilon'_s$*, and* $\lim_{s\to\infty}\epsilon'=0$*. All these surfaces are regular, i.e.* $q(S)=0$ *and*

$$\chi(S)=2^{3s-2}(s-1)^2. \tag{16}$$

Proof. For fixed group order 2^{3s} the number of groups with this order is 2^{27As^3} by Proposition 2.13. For each of these groups we found a ramification structure (T_1,T_2). By Lemma 2.12, the number of surfaces isogenous to a product associated to those data is at least 2^{Bs^3}, where B is as in the claim. By (12) and (14) the holomorphic Euler characteristic of S is

$$\chi(S)=\frac{(g(C_1)-1)(g(C_2)-1)}{|G|}=2^{3s-2}(s-1)^2.$$

□

4. Irregular Surfaces

Let us consider the irregular case i.e. $q(S)=q>0$. Let $h:=[g_s,g_{2s}]\cdot[g_{s-1},g_{2s-1}]\cdots[g_{s-q+1},g_{2s-q+1}]$ and recall that the commutators are in the center of the group G. Moreover, let

$$T'_2:=\{g_1g_2,g_2g_3,\dots,g_{s-q}g_{s-q+1},g_{s-q+1}g_1,g_{s+1}g_{s+2},\dots,$$
$$g_{2s-q}g_{2s-q+1},g_{2s-q+1}g_{s+1}h\}$$

the spherical part of the generating vector

$$\mathcal{V}_2:=\{T'_2,g_s,g_{2s},g_{s-1},g_{2s-1},\dots g_{s-q+1},g_{2s-q+1}\}$$

It holds $<\mathcal{V}_2>\cong G$ and and by construction the product of the elements in $\mathcal{V}_2$ is 1_G. Moreover, by Proposition 2.4 this yields an action of a group G of order 2^{3s} on some curve C_2 of genus

$$g(C_2):=2^{3s-1}(2q-2+(s-q+1))+1=2^{3s-1}(s+q-1)+1$$

with $g(C_2/G)=q$.

By construction and by Lemma 2.15 T_1 and T'_2 are disjoint and so $(T_1,\mathcal{V}_2)$ is a ramification structure for the groups as above. These data give us a surface isogenous to a product $S:=(C_1\times C_2)/G$, where G is a group of order 2^{3s}, which satisfies the assumption above. The number of these surfaces is at least 2^{27As^3}. Then the following theorem is proven in analogy to Theorem 3.1.

Theorem 4.1. *Let h be number of connected components of the moduli space of surfaces of general type isogenous to a product of curve admitting a group of order 2^{3s} and ramification structure of type $((0|2^{2s+2}),(q|2^{2s-2q+2})$ as above. Then we have*

$$h\geq 2^{Bs^3},$$

where $2-\epsilon'_s\leq B\leq\frac{18}{5}-\epsilon'_s$*, and* $\lim_{s\to\infty}\epsilon'=0$*. All these surfaces are irregular and have irregularity* $q(S)=q$*. Finally, The holomorphic Euler characteristic of S is*

$$\chi(S):=\frac{(g(C_1)-1)(g(C_2)-1)}{|G|}=2^{3s-2}(s-1)(s+q-1).$$

5. Proof of Theorem 1.1

We give the proof for regular surfaces only, the irregular case being analogous.

We start with (16) and we write

$$x_s=2^{3s-2}(s-1)^2.$$

Since x_s is strictly monotonically increasing with s, there is a well defined inverse function $s=s(x)$. From $\log_2 x_s=3s-2+2\log_2(s-1)$ we decuce

$$s(x)=\frac{1}{3}(1+\eta_{x_s})\log_2 x_s$$

with $\eta_{x_s} \to 0$ for $x_s \to \infty$. Substituting $s(x)$ into the inequality (15) we get

$$h \geq 2^{\frac{B}{27}(\log_2 x_s)^3(1+\eta_{x_s})^3}.$$

For x_s large enough this is bounded from below by $2^{\frac{2}{9}(\ln x_s)^3}$, thanks to $0 < 27(\ln 2)^3 < 9$.

We use the identity $x^{f(x)} = e^{f(x)\ln x} = 2^{f(x)\frac{1}{\ln 2}\ln x}$ to derive

$$h \geq x_s^{(\frac{2}{27(\ln 2)^2}(\ln x_s)^2)\frac{1}{\ln 2}\ln x_2} = x_s^{\frac{2}{13}(\ln x_s)^2},$$

thanks to $0 < 27(\ln 2)^2 < 13$. Since y_s is a constant multiple of x_s the asymptotic is the same. This concludes the proof of the theorem.

References

[BCG05] I. Bauer, F. Catanese, F. Grunewald, *Beauville surfaces without real structures.* In: Geometric methods in algebra and number theory, Progr. Math., **235**, Birkhäuser Boston, (2005), 1–42.

[BCG06] I. Bauer, F. Catanese, F. Grunewald, *Chebycheff and Belyi polynomials, dessins d'enfants, Beauville surfaces and group theory.* Mediterr. J. Math. **3**, (2006), 121–146.

[Cat92] F. Catanese, *Chow varieties, Hilbert schemes and moduli spaces of surfaces of general type.* J. Algebraic Geom. **1** (1992), 561–595.

[Cat00] F. Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces.* Amer. J. Math. **122**, (2000), 1–44.

[Cat03b] F, Catanese, *Moduli spaces of surfaces and real structures.* Ann. of Math. **158**, (2003), 577–592.

[CLP11] F. Catanese, M. Lönne, F. Perroni, *Irreducibility of the space of dihedral covers of the projective line of a given numerical type*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, **22**, (2011), 291–309.

[GP11] S. Garion, M. Penegini, *Beauville surfaces, moduli spaces and finite groups* preprint (2011) to appear in Comm. in Algebra.

[Gie77] D. Gieseker, *Global moduli for surfaces of general type.* Invent. Math **43** no. 3, (1977), 233–282.

[H60] G. Higman, *Enumerating p-groups.* Proc. London Math. Soc. (3), **10** (1960), 24–30.

[GP11] S. Garion, M. Penegini *Beauville surfaces, moduli spaces and finite groups* preprint (2011) to appear in Comm. in Algebra.

[M97] M. Manetti, *Iterated double covers and connected components of moduli spaces.* Topology **36**, (1997), 745–764.

[P13] M. Penegini, *Surfaces isogenous to a product of curves, braid groups and mapping class groups* to appear in the proceedings *Beauville surfaces and groups* of the Conference "Beauville Surfaces and Groups", Newcastle University (UK), (2013).

[S65] C.C. Sims, *Enumerating p-groups*, Proc. London Math. Soc. (3), **15**, (1965), 151–166.

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