SHIMURA VARIETIES IN THE TORELLI LOCUS VIA GALOIS COVERINGS OF ELLIPTIC CURVES

PAOLA FREDIANI, MATTEO PENEGINI, PAOLA PORRU

Abstract. We study Shimura subvarieties of A_g obtained from families of Galois coverings $f : C \to C'$ where C' is a smooth complex projective curve of genus $g' \geq 1$ and $g = g(C)$. We give the complete list of all such families that satisfy a simple sufficient condition that ensures that the closure of the image of the family via the Torelli map yields a Shimura subvariety of A_g for $g' = 1, 2$ and for all $g \geq 2, 4$ and for $g' > 2$ and $g \leq 9$. In [13] similar computations were done in the case $g' = 0$. Here we find 6 families of Galois coverings, all with $g' = 1$ and $g = 2, 3, 4$ and we show that these are the only families with $g' = 1$ satisfying this sufficient condition. We show that among these examples two families yield new Shimura subvarieties of A_g, while the other examples arise from certain Shimura subvarieties of A_g already obtained as families of Galois coverings of $\mathbb{P}^1$ in [13]. Finally we prove that if a family satisfies this sufficient condition with $g' \geq 1$, then $g \leq 6g' + 1$.

Contents

1. Introduction

The purpose of this paper is to continue the investigation started in [13] of those special subvarieties of A_g contained in the Torelli locus arising from families of Jacobians of Galois coverings $f : C \to C'$ where C' is a smooth complex projective curve of genus $g' \geq 1$, $g = g(C)$. In [13] the authors systematically studied families of Galois covering of $\mathbb{P}^1$ following the previous work done by Moonen [28] in the cyclic case and initiated in [38, 30, 10, 36] (see also the survey [29, §5]).

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More precisely, denote by A_g the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$, by M_g the moduli space of smooth complex algebraic curves of genus g and by $j\colon \mathsf{M}_g \to \mathsf{A}_g$ the period mapping or Torelli mapping. Set $\mathsf{T}_g^0 := j(\mathsf{M}_g)$ and call it the open Torelli locus. The closure of T_g^0 in A_g is called the *Torelli locus* (see e.g. [29]) and is denoted by T_g.

The expectation formulated by Oort ([31]) is that for large enough genus g there should not exist a positive-dimensional special subvariety Z of A_g, such that $\mathsf{Z} \subset \mathsf{T}_g$ and $\mathsf{Z} \cap \mathsf{T}_g^0 \neq \emptyset$.

One reason for this expectation coming from differential geometry is that a special (or Shimura) subvariety of A_g is totally geodesic with respect to the (orbifold) metric of A_g induced by the symmetric metric on the Siegel space $\mathfrak{H}_g$ of which A_g is a quotient by $Sp(2g, \mathbb{Z})$. One expects the Torelli locus to be very curved and a way of expressing this is to say that it should not contain totally geodesic subvarieties. Important results in this direction were achieved in [16], [11], [40], [20], [21], [4], [15]. In [8] a study of the second fundamental form of the period map allowed to give an upper bound for the possible dimension of a totally geodesic submanifold of A_g contained in the Torelli locus. This study was based on previous work on the second fundamental form of the period map done in [9], [7], [6]. Moreover an important theorem of Moonen [27] says that an algebraic totally geodesic subvariety of A_g is special if and only if it contains a CM point, so the expectation formulated by Oort is both of geometric and arithmetic nature. See [29, §4] for more details.

On the other hand, as we mentioned above, for low genus $g \leq 7$ there are examples of such Z and they are all constructed as families of Jacobians of Galois coverings of the line (see [38, 30, 10, 36, 28], [29, §5] for the abelian Galois coverings, [13] for the non abelian case and for a complete list).

All the examples of families of Galois coverings constructed so far satisfy a sufficient condition to yield a Shimura subvariety that we briefly explain. Consider a Galois covering $f : C \to C' = C/G$, where $G \subset Aut(C)$ is the Galois group, C' is a curve of genus g'. Set $g = g(C)$, then one has a monomorphism of G in the mapping class group $\mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(C))$. The fixed point locus $\mathcal{T}_g^G$ of the action of G on the Teichmüller space $\mathcal{T}_g$ is a complex submanifold of dimension $3g' - 3 + r$ (see section 2). We consider its image M in M_g and then the closure Z of the image of M in A_g via the Torelli morphism.

Set $N := \dim(S^2H^0(C, K_C))^G$, then the condition that we will denote by $(*)$ is that N must be equal to the dimension of Z, that is:

$$(*) \quad N = 3g' - 3 + r.$$

In [8] it is proven that this condition implies that the subvariety Z is totally geodesic and in [13] it is proven that in fact it gives a Shimura subvariety in the case $g' = 0$ and the same proof also works if $g' > 0$ as we remark in section 3. Moonen proved using arithmetic methods that condition $(*)$ is also necessary in the case of cyclic Galois coverings of $\mathbb{P}^1$. Results in this direction can also be found in [26].

In [13] the authors gave the complete list of all the families of Galois coverings of $\mathbb{P}^1$ of genus $g \leq 9$ satisfying condition $(*)$ and hence yielding Shimura subvarieties of A_g contained in the Torelli locus.

In this paper we do the same for Galois coverings of curves of higher genus g' and we find new examples when $g' = 1$. We also prove that if $g' \geq 1$, and the family satisfies $(*)$, then $g \leq 6g' + 1$. This immediately implies that if $g' = 1$ there are no examples satisfying condition $(*)$ for $g \geq 8$. More precisely, we have the following:

Theorem 1.1. *For all $g \geq 2$ and $g' = 1$ there exist exactly 6 positive dimensional families of Galois coverings satisfying condition $(*)$, hence yielding Shimura subvarieties of A_g contained in the Torelli locus.*

Two of the 6 families yield new Shimura subvarieties, while the others yield Shimura subvarieties which have already been obtained as families of Galois coverings of $\mathbb{P}^1$ in [13].

For all $g > 3$ and $g' = 2$ there do not exist positive dimensional families of Galois coverings satisfying condition $(*)$.

For $g \leq 9$ and $g' > 2$ there do not exist positive dimensional families of Galois coverings satisfying condition $(*)$.

Theorem 1.2. *If $g' \geq 1$ and we have a positive dimensional family of Galois coverings $f : C \to C'$ with $g' = g(C')$ and $g = g(C)$ which satisfies condition $(*)$, then $g \leq 6g' + 1$.*

The proof of Theorem 1.1 for $g \leq 9$, $g' \geq 1$ (and extend to $g \leq 13$ for $g' = 2$) is done using the `MAGMA` script that can be found at:
`users.mat.unimi.it/users/penegini/`
`publications/PossGruppigFix_Elliptic_v2.m`

Theorem 1.2 allows us to exclude the existence of any other family satisfying $(*)$ when $g' = 1$ or 2. The proof of Theorem 1.2 uses a result of Xiao ([41]) and an argument analogous to the one used in [35] to give a counterexample to a conjecture of Xiao on the relative irregularity of a fibration of a surface on a curve.

The 6 families with $g' = 1$ satisfying $(*)$, that is $N = r$, are the following:
(1) $g = 2$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 2$.
(2) $g = 3$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 4$.
(3) $g = 3$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 2$.
(4) $g = 3$, $G = \mathbb{Z}/4\mathbb{Z}$, $N = r = 2$.
(5) $g = 3$, $G = Q_8$, $N = r = 1$.
(6) $g = 4$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 3$.

Family (2) and family (6) give two new Shimura subvarieties, while the others yield Shimura subvarieties which have already been obtained as families of Galois coverings of $\mathbb{P}^1$ in [13].

More precisely:

(1) gives the same subvariety as family (26) of Table 2 in [13] (this family was already found in [29]).

(3) gives the same subvariety as family (31) of Table 2 in [13].

(4) gives the same subvariety as family (32) of Table 2 in [13].

(5) gives the same subvariety as family (34) of Table 2 in [13].

All the above families with $g \geq 3$ are not contained in the hyperelliptic locus.

A complete description of the families is given in section 4.

In section 4 we also show with a simple explicit computation that the families we found with `MAGMA` indeed satisfy condition $(*)$, using Eichler trace formula (Theorem (2.8)).

We also notice that in [19] a classification of all the representations of the actions of the possible groups G on the space of holomorphic one forms of a curve of genus $g = 3, 4$ is given and also using their description one can verify that in genus $3, 4$ our families are the only ones satisfying condition $(*)$ if $g' = 1$.

We finally observe that the new family that we find in genus $g = 4$ is very interesting also because it is the same family used by Pirola in [35] to give the above mentioned counterexample to a conjecture of Xiao on the relative irregularity of a fibration of a surface on a curve.

The paper is organised as follows.

In section 2 we recall some basic facts on Galois coverings of curves and we explain the construction of the families.

In section 3 we recall very briefly the definitions and results (mostly without proofs) on special subvarieties of A_g of PEL type and we show how the condition $(*)$ implies that the family yields a Shimura subvariety following [13].

In section 4 we give the explicit description of the new examples of special subvarieties obtained as Galois coverings of a genus 1 curve and we prove Theorem 1.1 and Theorem 1.2.

In section 5 we briefly describe what the `MAGMA` script does and we give the link to the script.

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2. Galois coverings

2.1. Fix a compact Riemann surface Y of genus $g' \geq 0$. Let $t := (t_1, \dots, t_r)$ be an r-tuple of distinct points in Y. Let us set $U_t := Y - \{t_1, \dots, t_r\}$ and choose a base point $t_0 \in U_t$. There exists an isomorphism $\pi_1(U_t, t_0) \cong \Gamma_{g',r} := \langle \alpha_1, \beta_1, \dots \alpha_{g'}, \beta_{g'}, \gamma_1, \dots \gamma_r \mid \prod_1^r \gamma_i \prod_1^{g'} [\alpha_j, \beta_j] = 1 \rangle$ given by the choice of a geometric basis of $\pi_1(U_t, t_0)$ as follows:

$\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}$ are simple loops in $Y - \{t_1, \dots, t_r\}$ which only intersect in t_0, whose homology classes in $H_1(Y, \mathbb{Z})$ form a symplectic basis.

Let $\tilde{\gamma_i}$ be an arc connecting t_0 with t_i contained in $(Y - \{\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}\}) \cup t_0$ and such that for $i \neq j$, $\tilde{\gamma_i}$ and $\tilde{\gamma_j}$ only intersect in t_0. We also assume that $\tilde{\gamma_1}, ... \tilde{\gamma_r}$ stem out of t_0 with distinct tangents following each other in counterclockwise order. The loops $\gamma_1, ..., \gamma_r$ are defined as follows: γ_i starts at t_0, goes along $\tilde{\gamma_i}$ to a point near t_i, makes a small simple loop counterclockwise around t_i and goes back to t_0 following $\tilde{\gamma_i}$.

If $f \colon C \longrightarrow Y$ is a Galois cover with branch locus t, set $V := f^{-1}(U_t)$. Then $f|_V : V \to U_t$ is an unramified Galois covering. Let G denote the group of deck transformations of $f|_V$. Then there is a surjective homomorphism $\pi_1(U_t, t_0) \longrightarrow G$, which is well-defined up to composition by an inner automorphism of G. Since $\Gamma_{g',r} \cong \pi_1(U_t, t_0)$ we get an epimorphism $\theta : \Gamma_{g',r} \to G$. If m_i is the local monodromy around t_i, set $\mathbf{m} = (m_1, \dots, m_r)$.

Definition 2.2. *A* datum *is a triple* $(\mathbf{m}, G, \theta)$*, where* $\mathbf{m} := (m_1, \dots, m_r)$ *is an* r*-tuple of integers* $m_i \geq 2$*,* G *is a finite group and* $\theta : \Gamma_{g',r} \to G$ *is an epimorphism such that* $\theta(\gamma_i)$ *has order* m_i *for each* i*.*

Thus a Galois cover of Y branched over t gives rise – up to some choices – to a datum. The Riemann's existence theorem ensures that the process can be reversed: a branch locus t and a datum determine a covering of Y up to isomorphism (see e.g. [25, Sec. III, Proposition 4.9]). The genus g of the Riemann surface C is given by Riemann-Hurwitz formula:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

We will show that the process can be reversed also in families, namely that to any datum is associated a family of Galois covers of a compact Riemann surface Y of genus g'.

2.3. In fact let $(\mathbf{m}, G, \theta)$ be a datum. Choose a point $[Y, t = (t_1, ..., t_r), \psi]$ in the Teichmüller space $\mathcal{T}_{g',r}$. This means that Y is a compact Riemann surface of genus g', $t = (t_1, ..., t_r)$ is an r-tuple of points in Y such that $t_i \neq t_j$ for $i \neq j$ and $\psi : \pi_1(U_t, t_0) \cong \Gamma_{g',r}$ is an isomorphism, where t_0 is a base point in U_t. Using $\theta \circ \psi$, by the above, we get a G-cover $C_t \to Y$ branched at the points t_i with local monodromies $m_1, \dots, m_r$. This yields a monomorphism of G into the mapping class group $\mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(C_t))$. Denote by $\mathcal{T}_g^G$ the fixed point locus of G on the Teichmüller space $\mathcal{T}_g$. It is a complex submanifold of dimension $3g' - 3 + r$, isomorphic to the Teichmüller space $\mathcal{T}_{g',r}$ (see e.g. [3, 14]). This isomorphism can be described as follows: if (C, φ) is a curve with a marking such that $[(C, \varphi)] \in \mathcal{T}_g^G$, the corresponding point in $\mathcal{T}_{g',r}$ is $[(C/G, \psi, b_1, \dots, b_r)]$, where ψ is the induced marking (see [14]) and $b_1, \dots, b_r$ are the critical values of the projection $C \to C/G$.

We remark that on $\mathcal{T}_g^G$ we have a universal family $\mathcal{C} \to \mathcal{T}_g^G$ of curves with a G–action. It is simply the restriction of the universal family on $\mathcal{T}_g$.

We denote by $\mathsf{M}(\mathbf{m}, G, \theta)$ the image of $\mathcal{T}_g^G$ in M_g. It is an irreducible algebraic subvariety of the same dimension as $\mathcal{T}_g^G \cong \mathcal{T}_{g',r}$, i.e. $3g' - 3 + r$ (see e.g. [3, 14]). Applying the Torelli map to $\mathsf{M}(\mathbf{m}, G, \theta)$ one gets a subset of A_g. We let $\mathsf{Z}(\mathbf{m}, G, \theta)$ denote the closure of this subset in A_g. By the above it is an algebraic subvariety of dimension $3g' - 3 + r$.

2.4. Different data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}, G, \theta')$ may give rise to the same subvariety of M_g. This is related to the choice of the isomorphism $\pi_1(U_t, t_0) \cong \Gamma_{g',r}$. The change from one choice to another can be described using an action of the full mapping class group $\mathrm{Map}_{g',[r]} := \pi_0(\mathrm{Diff}^+(Y - \{t_1, \dots t_r\}))$. This action is described for example in [34, Proposition 1.14].

One has an injection of $\mathrm{Map}_{g',[r]}$ in $\mathrm{Out}^+(\Gamma_{g',r})$ described in [22, Section 4]. Thus we get an action of $\mathrm{Map}_{g',[r]}$ on the set of data $(\mathbf{m}, G, \theta)$ up to inner automorphisms. Also the group $\mathrm{Aut}(G)$ acts on the set of data by $\alpha \cdot (\mathbf{m}, G, \theta) := (\mathbf{m}, G, \alpha \circ \theta)$.

The orbits of the group $\langle \mathrm{Map}_{g',[r]}, \mathrm{Aut}(G)\rangle$–action (*Hurwitz's moves*) are called *Hurwitz equivalence classes*. Data in the same class give rise to the same subvariety $\mathsf{M}(\mathbf{m}, G, \theta)$ and hence to the same subvariety $\mathsf{Z}(\mathbf{m}, G, \theta) \subset \mathsf{A}_g$. For more details see [34, 3, 1].

2.5. Consider a datum $(\mathbf{m}, G, \theta)$ and set $x_i := \theta(\gamma_i)$. If C is a curve with an action of a group G and with datum $(\mathbf{m}, G, \theta)$, then the cyclic subgroups $\langle x_i\rangle$ and their conjugates are the non-trivial stabilizers of the action of G on C.

Near the fixed points the action of the stabilizers can be described in terms of the epimorphism θ, see [17, Theorem 7]. In fact, suppose that an element $\mathsf{g} \in G$ fixes a point $P \in C$ and let m be the order of g. The differential $d\mathsf{g}_P$ acts on T_PC by multiplication by an m-th root of unity $\zeta_P(\mathsf{g})$. The action can be linearized in a neighbourhood of P, i.e. there is a local coordinate z centered in P, such that g acts as $z \mapsto \zeta_P(\mathsf{g})z$. So $\zeta_P(\mathsf{g})$ is a primitive m-th root of unity. (See also [25, Cor. III.3.5 p. 79].)

Denote by $\mathsf{Fix}(\mathsf{g})$ the set of fixed points of g and set $\zeta_m = e^{2\pi i/m}$, $I(m) := \{\nu \in \mathbb{Z} : 1 \le \nu < m, \gcd(\nu, m) = 1\}$. For $\nu \in I(m)$ consider

$$\mathsf{Fix}_\nu(\mathsf{g}) := \{P \in C : \mathbf{g}P = P, \zeta_P(\mathsf{g}) = \zeta_m^{-\nu}\}.$$

Lemma 2.6. *If $G \subseteq \mathrm{Aut}(C)$ and $\mathsf{g} \in G$ has order m, then*

$$|\mathsf{Fix}_\nu(\mathsf{g})| = |C_G(\mathsf{g})| \cdot \sum_{\substack{1\le i\le r,\\ m|m_i,\\ \mathsf{g}\sim_G x_i^{m_i\nu/m}}} \frac{1}{m_i}.$$

(Here $C_G(\mathsf{g})$ denotes the centralizer of g in G and $\sim_G$ denotes the equivalence relation given by conjugation in G.) This lemma follows from [17, Theorem 7], see also [2, Lemma 11.5].

2.7. Given a G-Galois cover $C \to Y$ let $\rho\colon G \longrightarrow \mathrm{GL}(H^0(C, K_C))$ be the representation on holomorphic 1-forms given by $g \mapsto (g^{-1})^*$. Let χ_ρ be the character of ρ. Notice that up to equivalence the representation ρ only depends on the data $(\mathbf{m}, G, \theta)$.

Theorem 2.8. *[Eichler Trace Formula]*

Let g be an automorphism of order $m > 1$ of a Riemann surface C of genus $g > 1$. Then

$$\chi_\rho(\mathsf{g}) = \mathrm{Tr}(\rho(\mathsf{g})) = 1 + \sum_{P\in \mathsf{Fix}(\mathsf{g})} \frac{\overline{\zeta_P(\mathsf{g})}}{1-\overline{\zeta_P(\mathsf{g})}}. \tag{2.1}$$

(See e.g. [12, Thm. V.2.9, p. 264].) Using the previous lemma one gets the following.

Corollary 2.9.

$$\chi_\rho(\mathsf{g}) = 1 + |C_G(\mathsf{g})| \sum_{\nu\in I(m)} \left\{ \sum_{\substack{1\le i\le r,\\ m|m_i,\\ \mathsf{g}\sim_G x_i^{m_i\nu/m}}} \frac{1}{m_i} \right\} \frac{\zeta_m^\nu}{1-\zeta_m^\nu}. \tag{2.2}$$

2.10. Let $\sigma\colon G \to \mathrm{GL}(V)$ be any linear representation of G with character χ_σ. Denote by $S^2\sigma$ the induced representation on S^2V and by $\chi_{S^2\sigma}$ its character. Then for $x \in G$

$$\chi_{S^2\sigma}(x) = \frac{1}{2}\big(\chi_\sigma(x)^2 + \chi_\sigma(x^2)\big). \tag{2.3}$$

(See e.g. [37, Proposition 3]).

We are only interested in the multiplicity N of the trivial representation inside $S^2\rho$. We remark that since the representation ρ only depends on the datum $(\mathbf{m}, G, \theta)$, the same happens for N. Using the orthogonality relations and (2.3), N can be computed as follows:

$$N = (\chi_{S^2\rho}, 1) = \frac{1}{|G|}\sum_{x\in G}\chi_{S^2\rho}(x) = \frac{1}{2|G|}\sum_{x\in G}\big(\chi_\rho(x^2) + \chi_\rho(x)^2\big). \tag{2.4}$$

3. Special subvarieties

3.1. Fix a rank $2g$ lattice Λ and an alternating form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ of type $(1, \dots, 1)$. For F a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, set $\Lambda_F := \Lambda \otimes_{\mathbb{Z}} F$. The Siegel upper half-space is defined as follows [18, Thm. 7.4]:

$$\mathfrak{H}_g := \{J \in \mathrm{GL}(\Lambda_{\mathbb{R}}) : J^2 = -I, J^*Q = Q, Q(x, Jx) > 0, \ \forall x \neq 0\}.$$

The group $\mathrm{Sp}(\Lambda, Q)$ acts on $\mathfrak{H}_g$ by conjugation and $\mathsf{A}_g = \mathrm{Sp}(\Lambda, Q)\backslash\mathfrak{H}_g$. This space has the structure of a smooth algebraic stack and also of a complex analytic orbifold. The orbifold structure is given by the properly discontinuous action of $\mathrm{Sp}(\Lambda, Q)$ on $\mathfrak{H}_g$. Throughout the paper we will work with A_g with this orbifold structure. Denote by A_J the quotient $\Lambda_{\mathbb{R}}/\Lambda$ endowed with the complex structure J and the polarization Q. On $\mathfrak{H}_g$ there is a natural variation of rational Hodge structure, with local system $\mathfrak{H}_g \times \Lambda_{\mathbb{Q}}$ and corresponding to the Hodge decomposition of $\Lambda_{\mathbb{C}}$ in $\pm i$ eigenspaces for J. This descends to a variation of Hodge structure on A_g in the orbifold or stack sense.

3.2. A special subvariety $\mathsf{Z} \subseteq \mathsf{A}_g$ is by definition a Hodge locus of the natural variation of Hodge structure on A_g described above. For the definition of Hodge loci for a variation of Hodge structure we refer to §2.3 in [29]. Special subvarieties contain a dense set of CM points and they are totally geodesic [29, §3.4(b)]. Conversely an algebraic totally geodesic subvariety that contains a CM point is a special subvariety [27, Thm. 4.3]. In this paper we will only deal with the special varieties of PEL type, whose definition is as follows (see [29, §3.9] for more details). Given $J \in \mathfrak{H}_g$, set

$$\mathrm{End}_{\mathbb{Q}}(A_J) := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ\}. \tag{3.1}$$

Fix a point $J_0 \in \mathfrak{H}_g$ and set $D := \mathrm{End}_{\mathbb{Q}}(A_{J_0})$. The *PEL type* special subvariety $\mathsf{Z}(D)$ is defined as the image in A_g of the connected component of the set $\{J \in \mathfrak{H}_g : D \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)\}$ that contains J_0.

Recall the following results proven in [13, Section 3].

Proposition 3.3. *Let $G \subseteq \mathrm{Sp}(\Lambda, E)$ be a finite subgroup. Denote by $\mathfrak{H}_g^G$ the set of points of $\mathfrak{H}_g$ that are fixed by G. Then $\mathfrak{H}_g^G$ is a connected complex submanifold of $\mathfrak{H}_g$.*

Set

$$D_G := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ,\ \forall J \in \mathfrak{H}_g^G\}. \tag{3.2}$$

Lemma 3.4. *If $J \in \mathfrak{H}_g^G$, then $D_G \subseteq \mathrm{End}_{\mathbb{Q}}(A_J)$ and the equality holds for J in a dense subset of $\mathfrak{H}_g^G$.*

Proposition 3.5. *The image of $\mathfrak{H}_g^G$ in A_g coincides with the PEL subvariety $\mathsf{Z}(D_G)$.*

Lemma 3.6. *If $J \in \mathfrak{H}_g^G$, then $\dim \mathfrak{H}_g^G = \dim \mathsf{Z}(D_G) = \dim(S^2\Lambda_{\mathbb{R}})^G$ where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure J.*

Recall that $N = \dim \left(S^2 H^0(C, K_C)\right)^G$ and that $\mathsf{Z}(\mathbf{m}, G, \theta)$ is defined in 2.3.

Theorem 3.7. *Fix a datum $(\mathbf{m}, G, \theta)$ and assume that*

$$N = 3g' - 3 + r. \tag{$*$}$$

Then $\mathsf{Z}(\mathbf{m}, G, \theta)$ is a special subvariety of PEL type of A_g that is contained in T_g and such that $\mathsf{Z}(\mathbf{m}, G, \theta) \cap \mathsf{T}_g^0 \neq \emptyset$.

Proof. Let $\mathcal{C} \to \mathcal{T}_g^G$ be the universal family as in 2.3. For any $t \in \mathcal{T}_g^G$, G acts holomorphically on C_t, so it maps injectively into $\mathrm{Sp}(\Lambda, Q)$, where $\Lambda = H_1(C_t, \mathbb{Z})$ and Q is the intersection form. Denote by G' the image of G in $\mathrm{Sp}(\Lambda, Q)$. It does not depend on t since it is purely topological. Recall that the Siegel upper half-space $\mathfrak{H}_g$ parametrizes complex structures on the real torus $\Lambda_{\mathbb{R}}/\Lambda = H_1(C_t, \mathbb{R})/H_1(C_t, \mathbb{Z})$ which are compatible with the polarization Q. The period map associates to the curve C_t the complex structure J_t on $\Lambda_{\mathbb{R}}$ obtained from the splitting $H^1(C_t, \mathbb{C}) = H^{1,0}(C_t) \oplus H^{0,1}(C_t)$ and the isomorphism $H_1(C_t, \mathbb{R})^*_{\mathbb{C}} = H^1(C_t, \mathbb{C})$. The complex structure J_t is invariant by G', since the group G acts holomorphically on C_t. This shows that $J_t \in \mathfrak{H}_g^{G'}$, so the Jacobian $j(C_t)$ lies in $\mathsf{Z}(D_{G'})$. This shows that $\mathsf{Z}(\mathbf{m}, G, \theta) \subseteq \mathsf{Z}(D_{G'})$. Since $\mathsf{Z}(D_{G'})$ is irreducible (see e.g. Proposition 3.3), to conclude the proof it is enough to check that they have the same dimension. The dimension of $\mathsf{Z}(\mathbf{m}, G, \theta)$ is $3g' - 3 + r$, see 2.3. By Lemma 3.6, if $J \in \mathfrak{H}_g^{G'}$, then $\dim \mathsf{Z}(D_{G'}) = \dim \mathfrak{H}_g^{G'} = \dim(S^2\Lambda_{\mathbb{R}})^{G'}$, where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure J. If J corresponds to the Jacobian of a curve C in the family, then $(S^2\Lambda_{\mathbb{R}})^{G'}$ is isomorphic to the dual of $(S^2H^0(C, K_C))^G$. Thus $\dim \mathsf{Z}(D_{G'}) = N$ and $(*)$ yields the result. $\square$

4. Examples of special subvarieties in the Torelli locus

4.1. In this section we prove Theorem 1.1 and Theorem 1.2. First we give the list of all the families of Galois covers of a curve of genus 1 satisfying property $(*)$, hence giving rise to special subvarieties of A_g. For them we now give a presentation of the Galois group G and an explicit description of a representative of an epimorphism

$$\theta : \Gamma_{1,r} = \langle \alpha, \beta, \gamma_1, ..., \gamma_r \mid \gamma_1 ... \gamma_r \alpha\beta\alpha^{-1}\beta^{-1} = 1\rangle \to G$$

(we use the same notation as in §2.1 and §2.5).

Genus 2

(1) $G = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 = 1\rangle$.
$\mathbf{m} = (2,2)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/2\mathbb{Z}$,
$\theta(\gamma_1) = \theta(\gamma_2) = z$, $\theta(\alpha) = \theta(\beta) = 1$.

Genus 3

(2) $G = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 = 1\rangle$.
$\mathbf{m} = (2,2,2,2)$ $\theta : \Gamma_{1,4} \to \mathbb{Z}/2\mathbb{Z}$,
$\theta(\gamma_i) = z$, $\forall i = 1, ..., 4$, $\theta(\alpha) = \theta(\beta) = 1$.
(3) $G = \mathbb{Z}/3\mathbb{Z} = \langle z \mid z^3 = 1\rangle$.
$\mathbf{m} = (3,3)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/3\mathbb{Z}$,
$\theta(\gamma_1) = z, \theta(\gamma_2) = z^2$, $\theta(\alpha) = \theta(\beta) = 1$.
(4) $G = \mathbb{Z}/4\mathbb{Z} = \langle z \mid z^4 = 1\rangle$.
$\mathbf{m} = (2,2)$ $\theta : \Gamma_{1,2} \to \mathbb{Z}/4\mathbb{Z}$,
$\theta(\gamma_1) = \theta(\gamma_2) = z^2$, $\theta(\alpha) = \theta(\beta) = 1$.
(5) $G = Q_8 = \langle g_1, g_2, g_3 : g_1^2 = g_2^2 = g_3, g_3^2 = 1, g_1^{-1} g_2 g_1 = g_2 g_3\rangle$
$\mathbf{m} = (2)$ $\theta : \Gamma_{1,1} \to Q_8$,
$\theta(\gamma_1) = g_3$, $\theta(\alpha) = g_2$, $\theta(\beta) = g_1$.

Genus 4

(6) $G = \mathbb{Z}/3\mathbb{Z} = \langle z \mid z^3 = 1\rangle$.
$\mathbf{m} = (3,3,3)$ $\theta : \Gamma_{1,3} \to \mathbb{Z}/3\mathbb{Z}$,
$\theta(\gamma_i) = z$, $\forall i = 1, 2, 3$, $\theta(\alpha) = \theta(\beta) = 1$.

4.2. First of all we will show that in each of the above cases, once we fix the group G and the ramification $\mathbf{m}$, all the possible data $(\mathbf{m}, G, \theta)$ belong to the same Hurwitz equivalence class and hence give rise to the same subvariety of A_g. Let us prove this for all the cases, giving the details only for some cases, being the others very similar.

Case (1). The group $\mathbb{Z}/2\mathbb{Z} \cong \langle z\rangle$ has only one element of order 2, this forces $\theta(\gamma_i)$ to be equal to z for $i = 1, 2$. Recall that the action of the mapping class group $\mathrm{Map}_{1,[2]}$ on $\Gamma_{1,2}$ is generated (up to inner automorphism) by the seven moves described in [34, Proposition 1.14] – from where we also borrow the notation. In particular, since $\mathbb{Z}/2\mathbb{Z}$ is abelian, the move induced by $t_{\xi^1_{1,1}}$ is given by $\theta(\alpha) \mapsto \theta(\gamma_1\alpha)$ and the identity on the other generators, while the move induced by $t_{\xi^2_{1,1}}$ is given by $\theta(\beta) \mapsto \theta(\gamma_1\beta)$ and the identity on the other generators. This gives that the systems of generators $\langle\theta(\alpha), \theta(\beta); \theta(\gamma_1), \theta(\gamma_2)\rangle = \langle z, z;\ z, z\rangle$ and $\langle 1, 1;\ z, z\rangle$ are Hurwitz equivalent. In addition the moves t_{δ_1} is given by $\alpha \mapsto \alpha\beta^{-1}$ and the identity on the other generators, and $t_{\tilde{\delta}_1}$ is given by $\beta \mapsto \beta\alpha$ and the identity on the other generators. This moves yields at once that systems of generators $\langle z, z;\ z, z\rangle$, $\langle 1, z;\ z, z\rangle$, $\langle z, 1;\ z, z\rangle$ are Hurwitz equivalent.

The proof for the cases (2) and (4) is the same as the one for case (1) with obvious changes.

Case (3). Since $\mathbb{Z}/3\mathbb{Z}$ is abelian its commutator subgroup is trivial, thus up to automorphisms we can choose $\theta(\gamma_1) = z$ and $\theta(\gamma_2) = z^2$. Then we proceed as in case (1) with obvious changes. Notice that this proof can be adjusted to fit for case (6).

Case (5). Since there is only one element of order 2 in Q_8, $\theta(\gamma_1) = g_3$. Up to simultaneous conjugation the image of the pair (α, β) by θ is one of the following (g_1, g_2), (g_2, g_1), (g_1, g_1g_2), (g_1g_2, g_1), (g_1g_2, g_2) and (g_2, g_1g_2). Using only the moves t_{δ_1} and $t_{\tilde{\delta}_1}$, described above, and the automorphisms of Q_8 we see that all the pairs are equivalent to (g_2, g_1). Therefore all the systems of generators are Hurwitz equivalent to $\langle g_2, g_1; g_3\rangle$. Notice that this proof can be found also in [33, Proposition 5.9]. Indeed, in that article this very covering is used to construct a new surface of general type with $p_g = q = 2$.

4.3. We will now show that the families listed above do in fact verify condition $(*)$ and hence yield special subvarieties of A_g. Nevertheless we will show that only two of them, namely family (2) and (6) give rise to new special subvarieties of A_g, while the others have already been obtained as families of Galois covers of $\mathbb{P}^1$.

Let us explain this. Assume that a family of Galois coverings of genus 1 curves with Galois group G satisfying $(*)$ yields the same Shimura subvariety of A_g of dimension s as one of those obtained via a family of Galois coverings of $\mathbb{P}^1$ satisfying condition $(*)$. Then each covering $\varphi : X \to X/G$ of the family of covers of genus one curves has the property that the curve X also admits an action of a group $K \subset Aut(X)$ such that $X/K \cong \mathbb{P}^1$ and we have: $dim(S^2H^0(K_X)^G) = dim(S^2H^0(K_X)^K) = s$. So each curve X of the family admits an action of a group $\tilde{G} \subset Aut(X)$ containing both G and K such that $X/\tilde{G} \cong \mathbb{P}^1$ and since $S^2H^0(K_X)^{\tilde{G}} \subset S^2H^0(K_X)^K$, also the family of coverings $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ satisfies condition $(*)$ and we have the following commutative diagram:

(4.1)
$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & X/G \arrow[d, "\sigma"] \\ & X/\tilde{G} \cong \mathbb{P}^1 \end{tikzcd}$$

So we can assume that $G \subset K$.

Since all the families of Galois coverings of $\mathbb{P}^1$ satisfying $(*)$ in genus less than 10 have already been found in [13] Table 2, it will suffice to compare our families with the ones listed there.

4.4. Example (2). Clearly θ is an epimorphism. Now we want to show that the sufficient condition $(*)$ is satisfied, so we compute the number N using (2.4). Eichler trace formula (2.2) immediately yields $\chi_\rho(z) = -1$, and since $\chi_\rho(1) = g = 3$ we have $N = 4$, which coincides with the number of critical values, and so with the dimension of the family. This proves that our family of Galois coverings is special.

Since the family just constructed has dimension 4, it has bigger dimension than any possibile family given as Galois covering of $\mathbb{P}^1$ satisfying $(*)$. In fact looking at the table of all possible special varieties presented as Galois coverings of $\mathbb{P}^1$ satisfying $(*)$ of genus $g \leq 9$ we see that none of these has dimension greater than 3 (see Table 2 of [13]). This proves that the family gives a new special subvariety contained in the Torelli locus. It also follows that the family is not contained in the hyperelliptic locus. In fact, if every curve C of the family were hyperelliptic, one could consider the group H generated by G and the hyperelliptic involution σ. The quotient $C/H \cong \mathbb{P}^1$, so we would obtain a family of Galois coverings of $\mathbb{P}^1$ which clearly still verifies condition $(*)$, and this does not exist, as we just pointed out.

4.5. Example (6). Clearly θ is an epimorphism.

Eichler trace formula (2.2) immediately yields $\chi_\rho(z) = \zeta_3$, $\chi_\rho(z^2) = \bar{\zeta}_3$ and since $\chi_\rho(1) = g = 4$, by (2.4) we have $N = 3$, which coincides with the number of critical values, and so with the dimension of the family. This proves that our family is special.

As noted before, to conclude we have to check that the family does not yield the same Shimura subvariety of A_g as one obtained via a family of Galois coverings of $\mathbb{P}^1$ already known to be special. Nonetheless, looking at Table 2 in [13] one checks there are no families of Galois covers of $\mathbb{P}^1$ satisfying condition $(*)$ with dimension greater or equal than 3 admitting $\mathbb{Z}/3\mathbb{Z}$ as a proper subgroup of the Galois group. By (4.3), this proves that the family gives a new special subvariety contained in the Torelli locus. It also follows by the same argument as in the previous example that it is not contained in the hyperelliptic locus.

Remark 4.6. We observe that this family is interesting also for another reason, in fact it is the same family used by Pirola in [35] to construct a counterexample to a conjecture of Xiao on the relative irregularity of a fibration of a surface on a curve.

4.7. Example (1). Clearly θ is an epimorphism.

By Eichler trace formula (2.2) we find $\chi_\rho(z) = 0$, and using (2.4) we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family, therefore our family is special.

We will now show that this family yields the same subvariety in A_g as family (26) in Table 2 of [13] (this family was already found in [29]).

Let us recall the description of this family. It is a family of Galois coverings of $\mathbb{P}^1$ with Galois group $\tilde{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \,|\, x^2 = y^2 = 1, \, xy = yx\rangle$, with ramification data $(2,2,2,2,2)$ and with epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1\rangle \to \tilde{G}$ given by

$$\tilde{\theta}(\delta_1) = x, \; \tilde{\theta}(\delta_2) = x, \; \tilde{\theta}(\delta_3) = x, \; \tilde{\theta}(\delta_4) = y, \; \tilde{\theta}(\delta_5) = xy.$$

We want to prove that for any covering of this family $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$, X also admits an action by a subgroup $G \cong \mathbb{Z}/2\mathbb{Z}$ of $\tilde{G}$ such that the quotient map $\varphi : X \to E \cong X/G$, belongs our family (1) and we have a diagram as in (4.1).

Consider the cyclic subgroup $G \cong \langle y | \, y^2 = 1\rangle < \tilde{G}$.

Looking at ramification data, we see that all stabilizer subgroups of $\tilde{G}$ have order 2 and looking at the epimorphism $\tilde{\theta}$ we see that the stabilizer subgroup associated to the fourth branch point q_4 is $\langle y\rangle = G$. This implies that points in $\psi^{-1}(q_4) = \{p_1, p_2\}$ are critical points for the action of G as well. Moreover they cannot belong to the same fiber with respect to the action of G since every element of G stabilizes both p_1 and p_2. To conclude note that every other stabilizer subgroup for the critical points of ψ does not contain any non trivial element of G, so q_4 is the only branch point of φ. This proves that the map φ has exactly 2 critical values which are the images of p_1 and p_2 by φ and the ramification is $\mathbf{m} = (2, 2)$. So we have a family with the same group and the same ramification as in family (1) and hence by the unicity argument given in 4.2 we conclude that it gives the same special subvariety in A_g as the one given by family (1).

Concluding, the special subvariety given by family (1) gives the same special subvariety obtained as a family of Galois coverings of $\mathbb{P}^1$ via $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ corresponding to family (26) of Table 2 of [13] and already studied in [29].

4.8. Example (3). It is clear that θ is an epimorphism. Using Eichler trace formula we immediately obtain $\chi_\rho(z) = \chi_\rho(z^2) = 0$ and by (2.4) we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family, therefore the family is special.

We claim that this family yields the same Shimura subvariety of A_g as family (31) in Table 2 of [13]. Let us describe this family of Galois coverings of $\mathbb{P}^1$ as in 4.1 of [13]. The Galois group $\tilde{G}$ is isomorphic to the symmetric group S_3, $\tilde{G} = \langle x, y |\, y^2 = x^3 = 1,\, y^{-1}xy = x^2\rangle$, $\mathbf{m} = (2, 2, 2, 2, 3)$ and the epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1\rangle \to \tilde{G}$ is given by

$$\tilde{\theta}(\delta_1) = xy,\ \tilde{\theta}(\delta_2) = x^2y,\ \tilde{\theta}(\delta_3) = y,\ \tilde{\theta}(\delta_4) = xy,\ \tilde{\theta}(\delta_5) = x^2.$$

We will show that every covering $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = \mathbb{Z}/3\mathbb{Z}$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the coverings of our family (3) and we have a factorisation as in (4.1). In fact, set $G = \langle x |\, x^3 = 1\rangle < \tilde{G}$ that is the only cyclic subgroup of order 3 on $\tilde{G}$. Looking at the stabilisers of the action of $\tilde{G}$, we see that the two critical points of ψ in the fibre over the critical value q_5 have both G as stabiliser, hence they are critical points also for the action of φ and they are mapped by φ in two different critical values. All the other critical points of ψ have stabilisers of order 2, hence they are not critical values for the map φ. So the map φ has ramification $(3, 3)$ and by the unicity argument 4.2 we can assume that $\varphi : X \to X/G$ belongs to our family (3).

Concluding, the special subvariety given by family (3) gives the same special subvariety obtained as the family (31) of Galois coverings of $\mathbb{P}^1$ via S_3 found in [13]. This family is not contained in the hyperelliptic locus (see the proof of theorem 5.3 of [13]).

4.9. Example (4). It is clear that θ is an epimorphism. Using Eichler trace formula we immediately obtain $\chi_\rho(z) = \chi_\rho(z^3) = 1$, $\chi_\rho(z^2) = -1$ and by (2.4) we obtain $N = 2$, which coincides with the number of critical values, and so with the dimension of the family, therefore the family is special.

We will now show that this family yields the same Shimura subvariety of A_g as family (32) in Table 2 of [13]. Let us describe this family of Galois coverings of $\mathbb{P}^1$ as in 4.1 of [13].

The Galois group $\tilde{G}$ is isomorphic to the dihedral group D_4, $\tilde{G} = \langle x, y |\, y^2 = x^4 = 1,\, y^{-1}xy = x^3\rangle$, $\mathbf{m} = (2,2,2,2,2)$ and the epimorphism $\tilde{\theta} : \Gamma_{0,5} = \langle \delta_1, ..., \delta_5 \mid \prod_{i=1,...,5} \delta_i = 1\rangle \to \tilde{G}$ is given by

$$\tilde{\theta}(\delta_1) = xy,\ \tilde{\theta}(\delta_2) = x^2y,\ \tilde{\theta}(\delta_3) = x^2,\ \tilde{\theta}(\delta_4) = x^2y,\ \tilde{\theta}(\delta_5) = x^3y.$$

As above we want to show that every covering $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = \mathbb{Z}/4\mathbb{Z}$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the coverings of our family (4) and we have a factorisation as in (4.1).

We can identify $G \cong \langle x |\, x^4 = 1\rangle < \tilde{G}$.

The stabilizer subgroups for the action of $\tilde{G}$ of the critical points over the third branch point q_3 are all given by the center $H = \langle x^2\rangle$ of $\tilde{G}$ which is contained in G. This implies that points in $\psi^{-1}(q_3) = \{p_1, p_2, p_3, p_4\}$ are critical points for the action of G as well.

Morover the four points $p_1, .., p_4$ are partitioned in exactly two orbits for the action of G, hence they give rise to two critical values of the map φ. Finally, observing that the other 4 conjugacy classes of stabilizers for D_4 do not contain nontrivial elements belonging to G, we conclude that the action of $G < D_4$ has ramification data $(2,2)$ and by 4.2 we can assume that it gives a covering belonging to our family (4).

Concluding, the special variety given by family (4) gives the same special variety obtained as the family (32) of Galois coverings of $\mathbb{P}^1$ via D_4 found in [13]. This family is not contained in the hyperelliptic locus (see the proof of theorem 5.3 of [13]).

4.10. Example (5). One easily checks that θ is an epimorphism. Using Eichler trace formula we find that the trace of every non zero element different from g_3 is equal to 1, and that $\chi_\rho(g_3) = -1$. By (2.4) we obtain $N = 1$, which coincides with the number of critical values, and so with the dimension of the family, therefore the family is special.

We will now show that this family yields the same Shimura subvariety of A_g as family (34) in Table 2 of [13]. Let us describe this family of Galois coverings of $\mathbb{P}^1$ as in 4.1 of [13].

The Galois group is

$$\tilde{G} = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong$$

$$\langle y_1, y_2, y_3 |\, y_1^2 = y_2^2 = y_3^4 = 1,\, y_2y_3 = y_3y_2, y_1^{-1}y_2y_1 = y_2y_3^2,\, y_1^{-1}y_3y_1 = y_3\rangle,$$

$\mathbf{m} = (2,2,2,4)$ and the epimorphism

$$\tilde{\theta} : \Gamma_{0,4} = \langle \delta_1, ..., \delta_4 \mid \prod_{i=1,...,4} \delta_i = 1\rangle \to \tilde{G}$$

is given by

$$\tilde{\theta}(\delta_1) = y_1,\ \tilde{\theta}(\delta_2) = y_1y_2y_3^3,\ \tilde{\theta}(\delta_3) = y_2y_3^2,\ \tilde{\theta}(\delta_4) = y_3^3.$$

Observe that the conjugacy classes of the nontrivial elements of $\tilde{G}$ are:

order 2 : $\{y_1, y_3^2y_1\}$, $\{y_2, y_3^2y_2\}$, $\{y_3^2\}$, $\{y_2y_3y_1, y_2y_3^3y_1\}$,

order 4 : $\{y_3\}$, $\{y_3^3\}$ $\{y_2y_3, y_2y_3^3\}$ $\{y_2y_1, y_2y_3^2y_1\}$ $\{y_3y_1, y_3^3y_1\}$, As above we want to show that every covering $\psi : X \to X/\tilde{G} \cong \mathbb{P}^1$ of this family also admits a $G = Q_8$-action such that X/G has genus 1, the map $\varphi : X \to X/G$ is one of the coverings of our family (5) and we have a factorisation as in (4.1).

In order to prove that the factorization holds, first at all we have to check that G is isomorphic to a subgroup of $\tilde{G}$. One easily checks that the following map

$$i : Q_8 \to (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$$
$$g_1 \mapsto y_2 y_3,\ g_2 \mapsto y_2 y_1,\ g_3 \mapsto y_3^2.$$

yields an injective homomorphism that identifies $G = Q_8$ with a proper subgroup of $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$.

As before, to conclude that the two families are in fact the same one, we have to study their stabilizer subgroups. Looking at the epimorphism $\tilde{\theta} : \Gamma_{0,4} \to \tilde{G}$ we see that the stabilizer subgroup associated to the fourth branch point q_4 is the normal subgroup $K := \langle y_3^3 \rangle = \{1, y_3, y_3^2, y_3^3\}$.The subgroup $H = \{1, y_3^2\}$ of G is clearly contained in $K = \langle y_3^3 \rangle$. This implies that points in $\psi^{-1}(q_4) = \{p_1, p_2, p_3, p_4\}$ are critical points for the action of G as well. Up to a permutation of the p_i's, we see that $\tilde{G}$ acts this way on the fiber:

$$p_2 = y_1(p_1) = y_1 y_3(p_1) = y_1 y_3^2(p_1) = y_1 y_3^3(p_1),$$
$$p_3 = y_2(p_1) = y_2 y_3(p_1) = y_2 y_3^2(p_1) = y_2 y_3^3(p_1),$$
$$p_4 = y_2 y_1(p_1) = y_2 y_1 y_3(p_1) = y_2 y_1 y_3^2(p_1) = y_2 y_1 y_3^3(p_1).$$

If we consider the action of G we get:

$$p_2 = y_1 y_3(p_1) = i(g_2^{-1} g_1)(p_1) = y_1 y_3^3(p_1) = i(g_1^{-1} g_2)(p_1),$$
$$p_3 = y_2 y_3(p_1) = i(g_1)(p_1) = y_2 y_3^3(p_1) = i(g_1^{-1})(p_1),$$
$$p_4 = y_2 y_1(p_1) = i(g_2)(p_1) = y_2 y_1 y_3^2(p_1) = i(g_2 g_3)(p_1).$$

So p_1, p_2, p_3, p_4 are 4 ramification points that map to the same critical value for the map φ and they have multiplicity 2. To conclude we have to prove that these are all the critical points of φ. But this is actually true, because none of the stabilizer subgroups of the critical points of ψ that are mapped to the first three critical values includes any subgroup of G. Concluding, by the unicity argument 4.2, the special variety given by the family (5) gives the same special variety obtained as the family (34) of Galois coverings of $\mathbb{P}^1$ via $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ found in [13]. This family is not contained in the hyperelliptic locus (see the proof of theorem 5.3 of [13]).

Finally we prove Theorem 1.2 that we restate here as follows:

Theorem 4.11. *If $g' \geq 1$ and we have a positive dimensional family of Galois coverings $f : C \to C'$ with $g' = g(C')$ and $g = g(C)$ which satisfies condition $(*)$, then $g \leq 6g' + 1$.*

In particular, for $g \geq 8$ (resp. 14) there do not exist positive dimensional families of Galois coverings with $g' = 1$ (resp. 2) and which satisfy condition $(*)$.

Proof. The idea of the proof is the following: if such a family exists, with the same method used by Pirola in section 2 of [35] one constructs a fibration

$S \to B$ of a surface S on a curve B, whose general fibre has genus g and whose relative irregularity is at least $g - g'$. Then we apply Corollary 3 of [41].

In fact, assume that $\mathsf{M} := \mathsf{M}(\mathbf{m}, G, \theta)$ is as usual the variety parametrising elements of such a family for a given datum $(\mathbf{m}, G, \theta)$. Every point $\mathsf{p} \in \mathsf{M}$ corresponds to an isomorphism class of a curve C of genus g admitting G as a subgroup of $Aut(C)$, whose quotient $C' := C/G$ has genus $g' \geq 1$ and whose monodromy is given by θ. Denoting by $f : C \to C'$ the Galois covering, to such a point one can associate the abelian variety $W = J(C)/f^*(J(C'))$, which is isogenous to the Prym variety of the covering. The abelian variety W has a polarisation Θ and we denote by $\mathsf{A}_{g-g'}(\Theta)$ the moduli space of polarised abelian varieties of dimension $g - g'$ with the given type of polarisation. Denote by $\Psi : \mathsf{M} \to \mathsf{A}_{g-g'}(\Theta)$ the map associating to p the polarised variety $[W, \Theta]$. The variety W inherits from C the automorphism group G. The differential of the map Ψ at the point $\mathsf{p} \in \mathsf{M}$ is a map

$$d\Psi_{\mathsf{p}} : H^1(C, T_C)^G \to S^2 H^{0,1}(W)$$

and its image is contained in $(S^2 H^{0,1}(W))^G$, since this is the space of infinitesimal deformations of (W, Θ) that preserve the action of G. So if we denote by $\mathsf{P} \subset \mathsf{A}_{g-g'}(\Theta)$ the image of Ψ , the tangent space $T_{[W,\Theta]}\mathsf{P}$ of P at $[W, \theta]$ is contained in $(S^2 H^{0,1}(W))^G$. The dual of the differential gives a map:

$$d\Psi^*_{\mathsf{p}} : (S^2 H^{1,0}(W))^G \to H^0(C, 2K_C)^G.$$

Observe that $H^0(K_C) = H^0(K_C)^G \oplus H^0(K_C)^-$, where the space of invariants $H^0(K_C)^G \cong H^0(C', K_{C'})$ has dimension g', and the complement $H^0(K_C)^- \cong H^{1,0}(W)$. Therefore we have

$$(S^2 H^0(K_C))^G \cong S^2 H^0(K_{C'}) \oplus (S^2 H^0(K_C)^-)^G \cong \cong S^2 H^0(K_{C'}) \oplus (S^2 H^{1,0}(W))^G \tag{4.2}$$

The dual of the differential $d\Psi^*_{\mathsf{p}} : (S^2 H^0(K_C)^-)^G \to H^0(C, 2K_C)^G$ is given by the multiplication map and since $(S^2 H^0(K_C)^-)^G \subset (S^2 H^0(K_C))^G$ and by our assumption $(*)$ the multiplication map

$$(S^2 H^0(K_C))^G \to H^0(C, 2K_C)^G$$

is an isomorphism, we conclude that $d\Psi^*_{\mathsf{p}}$ is injective. Hence $d\Psi_{\mathsf{p}}$ is surjective and $T_{[W,\Theta]}\mathsf{P} = (S^2 H^{0,1}(W))^G$. By (4.2) its dimension is equal to $N - \frac{g'(g'+1)}{2}$, where $N = \dim(S^2 H^0(K_C))^G$ is the dimension of M, by our condition $(*)$. So for a general point $[W, \Theta] \in \mathsf{P}$, $\dim \Psi^{-1}(W, \Theta) = \frac{g'(g'+1)}{2} \geq 1$, thus we can find a curve $Y \subset \Psi^{-1}(W, \Theta)$ contained in M_g. Denote by $\overline{Y}$ its closure in $\overline{\mathsf{M}}_g$. So we get a family of curves of genus g, $h' : S' \to B'$ such that $\overline{Y}$ is the image of the modular map $B' \to \overline{\mathsf{M}}_g$, $b' \mapsto [h'^{-1}(b')]$. By resolving singularities and taking pullbacks we get a smooth surface S, a smooth curve B and a map $h : S \to B$. Up to a base change we can assume that h has a section η. So if we take the Zariski open subset U of B of points having nonsingular fibres, we can use the section η to take the Abel-Jacobi maps $A_{\eta(t)} : C_t \to J(C_t)$, $t \in U$, compose them with the projections $J(C_t) \to W$ and obtain mappings: $\varphi_t : C_t \to W$, $\forall t \in U$. Using the pull-backs $\varphi_t^* : H^1(W, \mathbb{Q}) \to H^1(C_t, \mathbb{Q})$ we get an injection of $H^1(W, \mathbb{Q})$ in $H^0(B, R^1 h_* \mathbb{Q})$. By the Leray spectral sequence we identify $H^0(B, R^1 h_* \mathbb{Q})$ with the cockernel of the map $h^* : H^1(B, \mathbb{Q}) \to H^1(S, \mathbb{Q})$, thus we have

$$\dim H^1(S, \mathbb{Q}) - \dim H^1(B, \mathbb{Q}) \geq \dim H^1(W, \mathbb{Q}) = 2(g - g').$$

So if we denote by $q = h^0(S, \Omega^1_S)$ and by b the genus of the curve B we have $q - b \geq g - g'$. Since by construction the family is not isotrivial, we can apply Corollary 3 of [41], which says that $q - b \leq \frac{5g+1}{6}$ and so we get $g - g' \leq q - b \leq \frac{5g+1}{6}$, hence $g \leq 6g' + 1$.

Clearly if $g' = 1$ this implies $g \leq 7$.

□

Using the above Theorem, to conclude the proof of Theorem 1.1 it only remains to show that if $g \leq 7$ (resp. 13) and $g' = 1$ (resp. 2) there does not exist any other family satisfying $(*)$ except for the 6 families described above and if $g \leq 9$ and $g' > 1$ there do not exist families satisfying property $(*)$. To do this we use the `MAGMA` script briefly described below.

5. Higher Genus

A slightly modified version of the `MAGMA` script used in [13] enables us to check that the families given is Section 3 are the only ones under the following conditions. The covering curve has genus $g \leq 9$ and the quotient is a curve of genus $g' \geq 1$, moreover for the case $g' = 2$ we extended the calculation up to $g = 13$. By Proposition (4.11) we know that if $g' = 1$ these are all the families satisfying $(*)$. It is not bold to conjecture that these are all also in the case $g' > 1$. The `MAGMA` script that we used is available at:

`users.mat.unimi.it/users/penegini/`
`publications/PossGruppigFix_Elliptic_v2.m`

This script differs from the one in [13] essentially for the fact that it does not return a representative up to Hurwitz equivalence of a datum. But it gives all possible ramification data. This is because the Hurwitz's moves for the data we have found could be easily handled by hand as we have seen in 4.2. In addition, this helped to speed up the finding-example process as well. The other changes in the script are the obvious ones related to the fact that the genus of the base is not 0 anymore. It is important to notice that the `MAGMA` script works perfectly fine for covering curves of genera $g > 9$, we simply did not include other results for time reasons.

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