MULTISCALE MODELING OF MULTI-LAYERED STRUCTURES

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ABSTRACT

A mechanical model based on a novel multiscale approach has been recently formulated in Massabò and Campi, *Meccanica*, 50(4), 2015, to study multi-layered plates with imperfect interfaces and delaminations loaded dynamically. The model couples an equivalent single-layer structural theory and a discrete-layer cohesive-crack model in order to efficiently and accurately describe both the global behaviour and the local perturbations of the fields generated by the inhomogeneous material structure and the presence of interfacial imperfections. The homogenized field equations depend on the global variables only so that problems characterized by a large number of layers and delaminations are conveniently treated and efficient and insightful closed-form solutions can be derived for relevant problems. The model is applied to investigate the effects of the presence of imperfect interfaces on the dynamic characteristics of the plates. Closed form solutions are derived for unidirectionally reinforced wide plates with elastic sliding interfaces. The asymptotic limits, which can be obtained through a perturbation analysis, define the free vibrations of fully-bonded and fully-debonded plates. Changes in the interfacial stiffness strongly affects natural frequencies and modes of vibration: new modes are activated and the cut-off frequency of the second flexural spectrum, which in fully bonded plates is quite large so that the spectrum is usually disregarded, decreases and vanishes for decreasing/vanishing interfacial stiffness.

1 INTRODUCTION

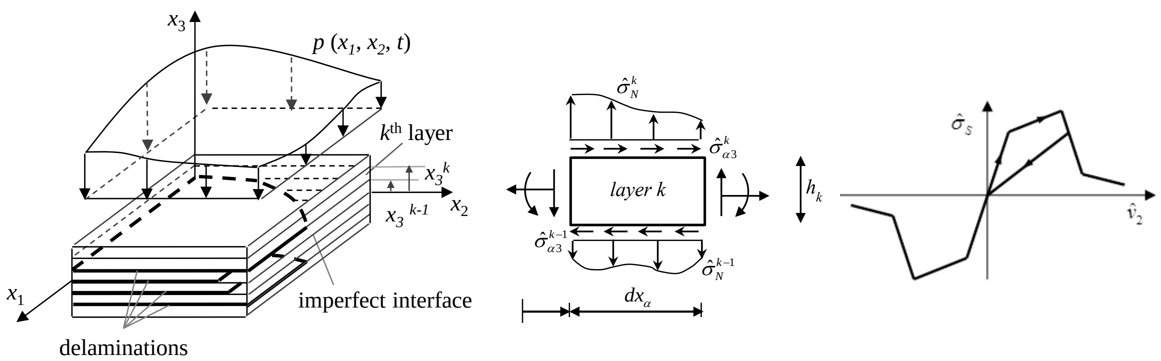
Efficient modeling of the global response of multi-layered composite structures in the elastic regime relies on equivalent single-layer theories, which accurately predict global displacements and stress resultants using a limited number of variables; these theories, however, are unable to describe the local perturbations of the fields, such as zig-zag through-thickness patterns and displacement jumps at the layer interfaces, and arise due to the inhomogeneous structure and the presence of defects. Discrete-layer approaches, which include layer-wise theories and cohesive-crack models for damage evolution, overcome these drawbacks and describe the delaminations as discrete entities; these models, however, depend on a large number of variables, which is related to the number of layers, typically require numerical solutions and are computationally expensive.

A novel multiscale approach, which overcomes the limitations of the discrete-layer models, has been recently proposed by the authors in [1-3] to analyze plates with cohesive interfaces and delaminations. The model couples a classical single-layer theory, which describes the global response of the structure, and a detailed, small-scale model, which is based on a discrete-layer cohesive-crack approach. The coupling is performed by assuming a two length-scales displacement field, which depends on the global variables and on local perturbations. A homogenization technique is then applied to average out the small-scale variables and obtain the macro-scale displacements and the dynamic equilibrium equations for the global variables [1,2]. The model is based on the original theories proposed in [4-6], which have been corrected to make them energetically consistent. The homogenized equilibrium equations have forms similar to those of single-layer theories, depend on the same number of variables and accurately predict local stresses and displacements in plates with arbitrary material structure and number and status of the interfaces, including the relevant limits of fully-bonded and fully-debonded layers (see applications presented in [1-3]). In systems with continuous stationary interfaces, the closed form solution of the equations leads to explicit expressions for displacements and stresses [3], which are useful in the design practice.

In this paper the formulation of the model will be briefly recalled and new results on the problem of the free vibrations of wide plates with continuous purely-elastic sliding interfaces will be presented.

2 MODEL: MULTISCALE APPROACH

The model refers to multi-layered plates with imperfect cohesive interfaces and delaminations (Fig. 1a) characterized by piece-wise linear cohesive-tractions laws (Fig. 1c), which relate the interfacial normal and tangential tractions (Fig.1b) to the relative sliding and opening displacements between adjacent layers. The piece-wise linear laws are used to approximate the different nonlinear mechanisms that may take place at the layer interfaces, e.g. material rupture, cohesive/bridging mechanisms, elastic contact or perfect adhesion. The layers are assumed to be linearly elastic and orthotropic, with principal material directions arbitrarily oriented with respect to the geometrical axes; the mass density is ρ_m .



(a) (b) (c)

Figure 1: (a) Multilayered plate with cohesive interfaces and delaminations. (b) Element of layer k. (c) Piecewise linear cohesive traction law relating the tangential interfacial tractions to the sliding displacement.

In the general formulation of the model, presented in [1] for plates and in [2] for wide plates and beams subjected to dynamic mechanical loading, the interfaces are assumed to be mixed-mode in order to allow the expected sliding and opening displacements at the layer interfaces. In [3] the model has been particularized to problems with sliding only interfaces and extended to treat steady-state thermo-mechanical loading. Preliminary results on the free vibration problem in wide plate and beams can be found in [7,8].

The model couples an equivalent first-order shear first-order normal deformation theory, which describes the global behavior, and a small-scale cohesive-crack model, which describes the local perturbations of the global fields generated by the inhomogeneous structure and the presence of the cohesive interfaces. The coupling is performed by first postulating a small-scale displacement field defined by global displacements which are enriched by local perturbations in the form of zig-zag functions and displacement jumps (Fig. 2). The local variables are then defined in terms of the global variables, through the imposition of continuity conditions for normal and tangential tractions at the layer interfaces which yield the macro-scale displacements. The derivation is presented in [1,2].

Expressions for the macro-scale displacements are presented here for the special case of wide plates deforming in cylindrical bending in the plane $x_2 - x_3$, where the principal material axes of the layers are oriented along the geometrical axes and the interfaces are assumed to be rigid against mode I relative displacements (sliding interfaces). The equations refer to an arbitrary piece of the cohesive traction law, $\hat{\sigma}_S^k = K_S^k \hat{v}_2^k + t_S^k$, with $\hat{v}_2^k(x_2) = {}^{(k+1)}v_2(x_2, x_3 = x_3{}^k) - {}^{(k)}v_2(x_2, x_3 = x_3{}^k)$ the relative sliding displacement between layers k and k+1 at the kth interface with through thickness coordinate $x_3{}^k$ (the upper-scripts on the right indicate the number of the interface and those on the left the layer; layers are numbered from bottom to top) (Fig. 1c). The longitudinal and transverse displacements within the layer k are:

$$
\begin{aligned}
&{}^{(k)}v_2(x_2,x_3,t) = v_{02}(x_2,t) + x_3\varphi_2(x_2,t) + \left[w_0,_2(x_2,t) + \varphi_2(x_2,t)\right] R^k_{S22}(x_3) - \sum_{i=1}^{k-1} t^i_S / K^i_S \\
&{}^{(k)}v_3(x_2,x_3,t) = w_0(x_2,t)
\end{aligned}
\qquad (1)
$$

They depend on the three generalized displacements of the global model, v_{02}, w_0, φ_2, and on the coefficient R^k_{S22} which can be calculated a-priori and is given below:

$$
\begin{aligned}
&R^k_{S22} = R^k_{S22}(x_3) = \sum_{i=1}^{k-1}\left[\Lambda^{(1;i)}_{22}\left(x_3 - x_3{}^i\right) + \Psi^i_{22}\right] \\
&\Lambda^{(i;j)}_{22} = {}^{(i)}C_{55}\left(\frac{1}{{}^{(j+1)}C_{55}} - \frac{1}{{}^{(j)}C_{55}}\right);\ \Psi^i_{22} = \frac{{}^{(i+1)}C_{55}}{K^i_S}\left(1 + \sum_{j=1}^{i}\Lambda^{(1;j)}_{22}\right)
\end{aligned}
\qquad (2)
$$

$R^k_{S22} = 0$ and $t^i_S / K^i_S = 0$ in fully bonded unidirectionally reinforced plates and the model coincides with classical first-order shear deformation theory. Stress and strain components in the layers are derived from the macro-scale displacements using constitutive and compatibility equations [2].

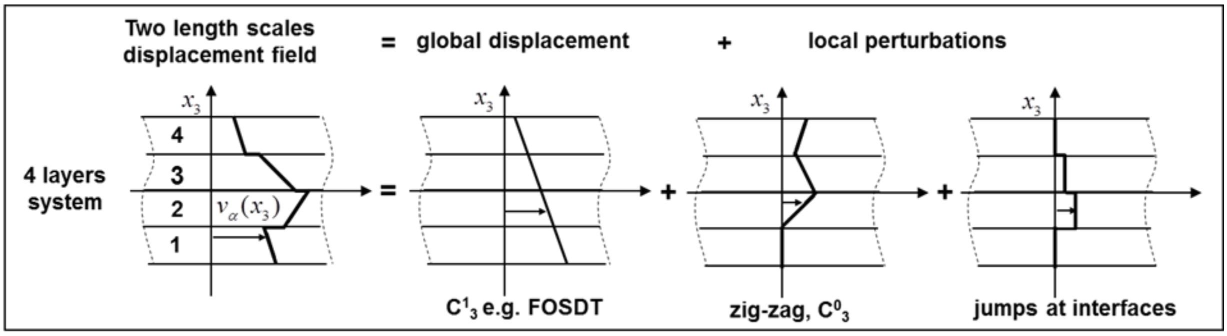


Figure 2: Assumed displacement field: global displacements (first-order shear deformation theory) and local perturbations (zig-zag functions and interfacial jumps)

The application of Hamilton principles using the macro-scale displacements, Eq. (1), yields the homogenized dynamic equilibrium equations [2]. For the free vibration problem, the equations in terms of generalized displacements take the form:

$$
\begin{aligned}
&w_0,_{222222} - a w_0,_{2222} - \left\{ i_1 \ddot{v}_{02},_2 + i_2 \ddot{\varphi}_2,_2 + i_3\left(\ddot{\varphi}_2,_2 + \ddot{w}_0,_{22}\right) + i_0 \ddot{w}_0 \right\} \\
&+ \left\{ i_4 \ddot{v}_{02},_{222} + i_5 \ddot{\varphi}_2,_{222} + i_6\left(\ddot{\varphi}_2,_{222} + \ddot{w}_0,_{2222}\right) + i_7 \ddot{w}_0,_{22} \right\} = 0
\end{aligned}
\qquad (3)
$$

$$
\varphi_{2,2} = -w_0,_{22} + \frac{1}{ac} w_0,_{2222} + \frac{1}{ac}\left\{ i_4 \ddot{v}_{02},_2 + i_5 \ddot{\varphi}_2,_2 + i_6\left(\ddot{\varphi}_2,_2 + \ddot{w}_0,_{22}\right) + i_7 \ddot{w}_0 \right\} \qquad (4)
$$

$$v_{02},_{22} = \frac{C_{22}^1}{C_{22}^0} w_0,_{222} - \frac{C_{22}^1 + C_{22}^{0S}}{C_{22}^0}\frac{1}{ac} w_0,_{22222} + \frac{1}{C_{22}^0}\rho_m \left\{ I_0 \ddot{v}_{02} + I_1 \ddot{\varphi}_2 + I_0^{22S}\left(\ddot{\varphi}_2 + \ddot{w}_0,_2\right)\right\}$$
$$- \frac{C_{22}^1 + C_{22}^{0S}}{C_{22}^0}\frac{1}{ac}\left\{ i_4 \ddot{v}_{02},_{22} + i_5 \ddot{\varphi}_2,_{22} + i_6\left(\ddot{\varphi}_2,_{22} + \ddot{w}_0,_{222}\right) + i_7 \ddot{w}_0,_2 \right\} \quad (5)$$

The various coefficients in the equations depend on the material/geometrical properties, the lay-up and the stiffness of the interfaces; they can be easily calculated a priori and their full expressions are given in [8,9], along with the boundary conditions associated to Eqs. (3-5).

3 FREE VIBRATIONS OF WIDE PLATES WITH ELASTIC INTERFACES

The dynamic equilibrium equations (3-5) have been applied to solve the problem of the free vibrations of a simply supported unidirectionally-reinforced wide plate of length L (Fig. 3); the plate has *n* linearly elastic orthotropic layers, with relevant elastic constants E_L, G_{LT}, and ν_{LT} (*L* and *T* coincide with $x_2 - x_3$), and *n*-1 purely elastic sliding interfaces with interfacial stiffness $K_S^k = K_S$.

Figure 3: Simply supported wide plate with n layers and n-1 imperfect interfaces.

The characteristic equation of the problem is given below in terms of natural vibration frequencies, $\omega_j = \omega_j(k_j)$, with $k_j = j\pi / L$ the wave number and *j* the mode number:

$$\left[\left(\frac{\omega_j}{k_j}\right)^2 - \frac{\bar{E}_L}{\rho_m}\right]\left\{\left(\frac{\omega_j}{k_j}\right)^4 - \left[\left(\frac{\Theta_{2j}}{k_j}\right)^2 + \frac{\bar{E}_L}{\rho_m}\Gamma_j\right]\left(\frac{\omega_j}{k_j}\right)^2 + \frac{\bar{E}_L}{\rho_m}\Gamma_j\left(\frac{\Theta_{1j}}{k_j}\right)^2\right\} = 0 \quad (6)$$

with:

$$\Theta_{1j}^2 = \frac{\bar{E}_L}{\rho_m}\frac{(k_j^2 + a)k_j^2}{\left[k_j^2 + \left(\frac{1}{r^2 k_j^2} - \frac{(n^2-1)}{n}\frac{G_{LT}}{K_S h}\right)a + i\right]}, \qquad \Theta_{2j}^2 = \frac{\bar{E}_L}{\rho_m}\frac{(k_j^2 + ac)k_j^2}{k_j^2 + i}$$

$$\bar{E}_L = E_L / (1 - \nu_{TL}\nu_{LT}), \qquad \Gamma_j = \left[k_j^2 + \left(\frac{1}{r^2 k_j^2} - \frac{(n^2-1)}{n}\frac{G_{LT}}{K_S h}\right)a + i\right]\frac{1}{k_j^2 + i}, \quad (7)$$

$$a = \frac{1}{r^2}\frac{G_{LT}}{\bar{E}_L}\left(K^2 + \frac{n-1}{\left(K_S h/G_{LT}\right)}\right)\frac{n^2}{n^2-1}\left(K_S h/G_{LT}\right)^2$$

$$i = \frac{1}{r^2}n^2\left(1+\frac{2}{n}\left(K_S h/G_{LT}\right)+\frac{1}{n^2-1}\left(K_S h/G_{LT}\right)^2\right), \qquad c = 1+\frac{n^2-1}{n}\frac{1}{\left(K_S h/G_{LT}\right)}$$

with K^2 a shear factor coefficient.

The first root of Eq. (6) defines the natural frequencies, $\omega_j^I(k_j) = \sqrt{\bar{E}_L/\rho_m}k_j$, associated to a spectrum of uniform axial vibrations of the laminate as a whole; they do not depend on the interfacial stiffness and coincides with those of a fully bonded plate. The second and third roots, $\omega_j^{II}(k_j)$ and $\omega_j^{III}(k_j)$, define flexural vibration spectra with frequencies:

$$\omega_j^{II}, \omega_j^{III} = k_j\sqrt{\left(\frac{\Theta_{2j}}{k_j}\right)^2 + \frac{\bar{E}_L}{\rho_m}\Gamma_j}\sqrt{\frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{4\frac{\bar{E}_L}{\rho_m}\Gamma_j\left(\frac{\Theta_{1j}}{k_j}\right)^2}{\left[\left(\frac{\Theta_{2j}}{k_j}\right)^2 + \frac{\bar{E}_L}{\rho_m}\Gamma_j\right]^2}}} \tag{8}$$

and depend on the interfacial stiffness through the coefficients Θ_{1j}, Γ_j.

In the *fully bonded limit*, when $K_S \to \infty$, Eq. (6) modifies into the characteristic equation of Timoshenko beam theory and the dispersion curves of the two flexural spectra in (8) are presented in Fig. 4, which shows $\tilde{\omega}_j = \omega_j r\sqrt{\rho_m/\bar{E}_L}$ versus $\tilde{k}_j = k_j r = j\pi r/L$, with $r = \sqrt{I/h}$ the radius of gyration and I the centroidal moment of inertia. The curves correspond to different values of the relative axial and shear stiffness, $\chi = \bar{E}_L / K^2 G_{LT} = 60,\ 200,\ \infty$ (solid, dashed, dotted lines). The second flexural spectrum, which corresponds to vibrations dominated by shear deformations [9], is characterized by a cutoff frequency for $\tilde{k}_j \to 0$, which is equal to $\tilde{\omega}_{jco}^{III} = 1/\sqrt{\chi} = \sqrt{K^2 G_{LT} / \bar{E}_L}$ and is derived through a perturbation analysis of the problem. In the limit for $\chi \to 0$ the dispersion curve tends to the solution of a thickness-shear beam with $\tilde{\omega}_j^{III} = \tilde{k}_j^k$ [9].

In the *fully debonded limit*, when $K_S \to 0$, a perturbation analysis of Eq. (6) yields the characteristic equation:

$$\left[\left(\frac{\tilde{\omega}_j}{\tilde{k}_j}\right)^2 - 1\right]\left\{\left[\left(\frac{\tilde{\omega}_j}{\tilde{k}_j}\right)^2 - \frac{\tilde{k}_j^2}{n^2+\tilde{k}_j^2}\right]\left[\left(\frac{\tilde{\omega}_j}{\tilde{k}_j}\right)^2 - 1\right]\right\} = 0 \tag{9}$$

The equation highlights the presence of only one flexural spectrum with frequency equal to that of a Rayleigh beam of thickness h/n, with n the number of layers, which is shown in Fig. 5a and is given by:

$$\tilde{\omega}_j^{IIFD} = \sqrt{\frac{\tilde{k}_j^2}{n^2+\tilde{k}_j^2}}\tilde{k}_j \qquad (10)$$

The other two spectra coincide and the natural frequencies are:

$$\tilde{\omega}_j^{IFD} = \tilde{\omega}_j^{IIIFD} = \tilde{k}_j \qquad (11)$$

These spectra are associated to displacement fields characterized by longitudinal vibrations in the absence of transverse displacements. The first is a field of axial vibrations which are uniform in the thickness and coincides with the classical longitudinal vibration mode of a homogeneous beam. The second is characterized by uniform shear deformations (thickness-shear mode [9]) and interfacial sliding and has zero mean value of the longitudinal displacements over the thickness.

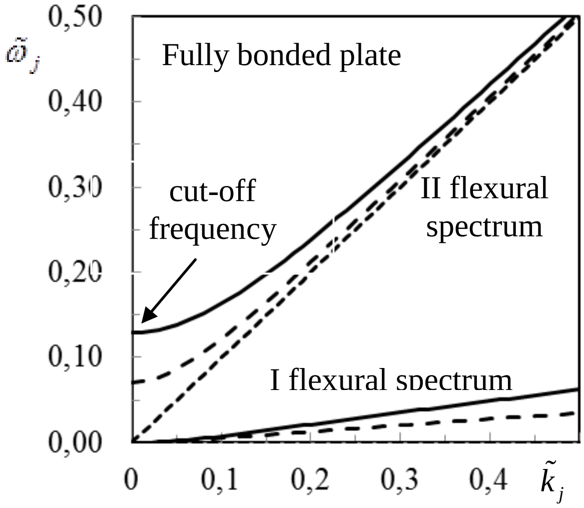
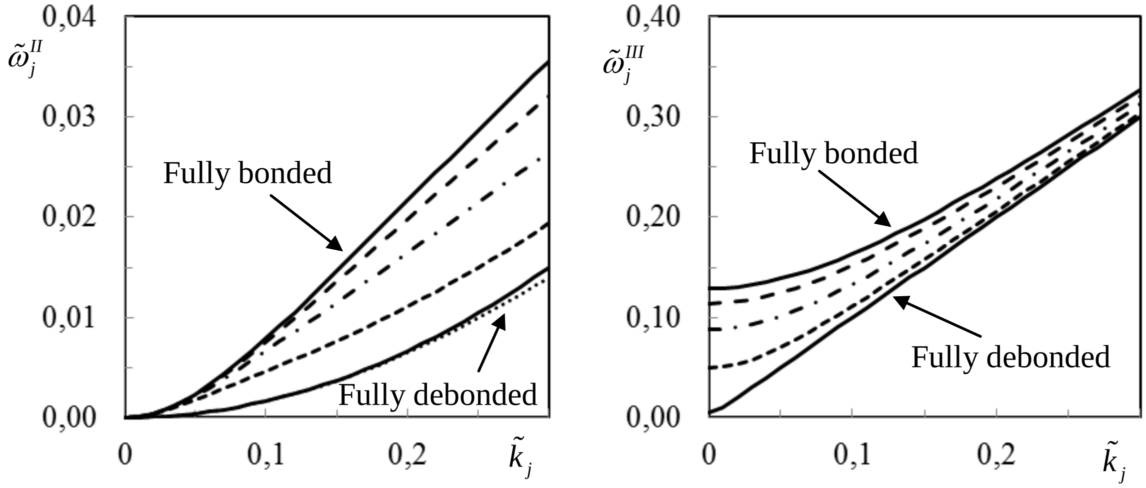


Figure 4. Dispersion curves associated to the flexural spectra of a fully bonded simply supported wide plate (spectra coincide with those of a Timoshenko beam with elastic moduli $E_L = \bar{E}_L$). Solid, dashed and dotted lines correspond to $\chi = \bar{E}_L / K^2 G_{LT} = 60,\ 200,\ \infty$, respectively.

For imperfect interfaces, the natural frequencies of vibration and dispersion curves vary between the two limiting solutions derived above in accordance with the solution, Eq. (8). They are shown in Figs. 5a,b. Results are presented for dimensionless interfacial stiffness $K_S h / \mathrm{G}_{LT} = \infty, 20,\ 5,\ 1,\ 0$, where $K_S h / \mathrm{G}_{LT} = \infty$ defines the *fully bonded limit* (solid upper curves) and $K_S h / \mathrm{G}_{LT} = 0$ the *fully debonded limit* (solid lower curve, Eqs. (10),(11)). The first flexural spectrum, Fig. 5a, is limited from below by the Rayleigh solution for a plate with thickness *h/n*; the dispersion curve is very closed, over the range of wave number examined, to the solution of a Timoshenko plate of the same thickness, which is depicted by the dotted curve (discrepancies are found for larger wave numbers). The dispersion curves associated to the second flexural spectrum highlight the modifications induced by the presence of imperfect interfaces to the cutoff frequency for $\tilde{k}_j \to 0$, which progressively decreases and vanishes on decreasing/vanishing the interfacial stiffness.

4 CONCLUSIONS

The multiscale model formulated in [1-3] and recalled in this paper allows the efficient solution of multi-layered plates with arbitrary layup, and numbers of layers and imperfect interfaces and delaminations subjected to thermo-mechanical loading. This is done through homogenized equilibrium equations which depend on a limited number of variables equal to that of classical equivalent single-layer theories. The accuracy of the approach has been verified with exact analytical solutions [1-3]. In this paper closed-form solutions have been obtained for the natural frequencies and modes of vibration of unidirectionally-reinforced wide plates with elastic sliding interfaces. The solutions highlight the important role played by the imperfections and allow parametric analyses which can be useful for the optimal design of the material/structure systems. New modes of vibration are activated and the cut-off frequency of the second flexural spectrum, which in fully bonded beams is quite large so that the spectrum is usually disregarded in the analyses, decreases and vanishes for decreasing/vanishing interfacial stiffness.



(a) (b)

Figure 5: Dispersion curves associated to the first, (a), and second, (b), flexural spectra of a simply supported wide plate with 6 unidirectionally reinforced layers and sliding interfaces controlled by a linear interfacial traction law with $\hat{\sigma}_S^k = K_S \hat{v}_2^k$, for $k = 1,...,n-1$. Results are presented for $\chi = \bar{E}_L / K^2 C_{55} = 60$ and $K_S h / C_{55} = \infty, 20, 5, 1, 0$ (upper solid, dashed, dash-dot, small dash, lower solid). Note the different scales used for the two spectra.

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