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Non-uniform interpolatory subdivision schemes with improved smoothness

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ABSTRACT

Subdivision schemes are used to generate smooth curves or surfaces by iteratively refining an initial control polygon or mesh. We focus on univariate, linear, binary subdivision schemes, where the vertices of the refined polygon are computed as linear combinations of the current neighbouring vertices. In the classical stationary setting, there are just two such subdivision rules, which are used throughout all subdivision steps to construct the new vertices with even and odd indices, respectively. These schemes are well understood and many tools have been developed for deriving their properties, including the smoothness of the limit curves. For non-stationary schemes, the subdivision rules are not fixed and can be different in each subdivision step. Non-uniform schemes are even more general, as they allow the subdivision rules to be different for every new vertex that is generated by the scheme. The properties of non-stationary and non-uniform schemes are usually derived by relating the scheme to a corresponding stationary scheme and then exploiting the fact that the properties of the stationary scheme carry over under certain proximity conditions. In particular, this approach can be used to show that the limit curves of a non-stationary or non-uniform scheme are *as smooth* as those of a corresponding stationary scheme. In this paper we show that non-uniform subdivision schemes have the potential to generate limit curves that are *smoother* than those of stationary schemes with the same support size of the subdivision rule. For that, we derive interpolatory 2-point and 4-point schemes that generate C^1 and C^2 limit curves, respectively. These values of smoothness exceed the smoothness of classical interpolating schemes with the same support size by one.

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1. Introduction

Given some initial data $f^0 = (f_i^0)_{i\in\mathbb{Z}}$ at level zero with $f_i^0 \in \mathbb{R}$, a linear, uniform, stationary, binary subdivision scheme S generates refined data $f^{k+1} = (f_i^{k+1})_{i\in\mathbb{Z}}$ at level $k+1$ for any $k \in \mathbb{N}_0$ according to the two subdivision rules

$$f_{2i}^{k+1} = \sum_{j\in\mathbb{Z}} f_{i-j}^k a_j \qquad \text{and} \qquad f_{2i+1}^{k+1} = \sum_{j\in\mathbb{Z}} f_{i-j}^k b_j, \tag{1}$$

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with certain weights $a_j, b_j \in \mathbb{R}$ for $j \in \mathbb{Z}$. These rules are often referred to as the *even* and the *odd* subdivision rule, because they determine the refined data with even and odd indices, respectively. In most applications, only a finite number of these weights are non-zero, and we shall adhere to this assumption. The scheme S is called *convergent*, if the sequence $(F_k)_{k\in\mathbb{N}_0}$ of functions $F^k\colon \mathbb{R} \to \mathbb{R}$ that are piecewise linear over $\mathbb{Z}/2^k$ and interpolate f_i^k at the *parameter values* $t_i^k = i/2^k$ for $i \in \mathbb{Z}$ converges uniformly for any initial data f^0 and if the limit function is non-trivial for at least one choice of f^0 (Dyn and Levin, 2002). The function $S^\infty f^0 = \lim_{k\to\infty} F^k$, which is necessarily continuous, is then called the *limit function* of S for f^0. The scheme S can also be applied to initial data consisting of points rather than scalar values, by simply applying the subdivision rules to each coordinate, thus resulting in *limit curves*, and the piecewise linear interpolant F^0 is then called the *control polygon* of the limit curve.

Apart from early work by de Rham (1947), the first subdivision schemes were designed to generate uniform B-spline curves of arbitrary degree (Lane and Riesenfeld, 1980). For these B-spline schemes, the weights a_j and b_j of the subdivision rules in (1) can be derived from the two-scale relations of the uniform B-spline basis functions, and the properties of the limit functions or curves are evident. Later research studied subdivision independently of B-splines, and simple tools are now available for checking the necessary and sufficient conditions for the convergence of S and the smoothness of the limit functions or curves (Cavaretta et al., 1991; Dyn and Levin, 2002), as well as for deriving the degrees of polynomial generation and reproduction (Dyn et al., 2008), which are intimately related to the approximation order of the scheme S.

In order to generate more general classes of limit curves, several modifications of the rules in (1) have been considered. In the *non-stationary* setting, the weights of the rules are allowed to depend on the subdivision level k, and this flexibility can be used to reproduce circles (Jena et al., 2003) or exponential polynomials (Dyn et al., 2003). Circles can as well be generated by *non-linear* subdivision schemes (Sabin and Dodgson, 2005), where the subdivision rules may depend on the data f_i^k and hence vary not only for each subdivision level k, but also for each index i. Besides generating circles and Gaussian functions (Schaefer et al., 2008), non-linear schemes are capable of eliminating shape artefacts (Dyn et al., 2009; Marinov et al., 2005) and creating smooth curves on manifolds and curved surfaces (Noakes, 1998). A third modification are *non-uniform* subdivision schemes (Daubechies et al., 1999; Warren, 1995), which can be used, for example, to generate non-uniform B-splines (Cashman et al., 2007; Goldman and Warren, 1993; Schaefer and Goldman, 2009). For these schemes, the parameter values t_i^k corresponding to the data f_i^k are usually no longer assumed to be uniformly spaced, and consequently the subdivision rules depend on k and i, but they remain linear.

Several tools have been developed for proving the convergence and smoothness of a non-linear, non-stationary, or non-uniform scheme S, including *divided differences* (Daubechies et al., 1999), *asymptotic equivalence* (Dyn and Levin, 1995; Dyn et al., 2014), *asymptotic similarity* (Conti et al., 2015), and *proximity* conditions (Wallner, 2006; Wallner and Dyn, 2005). In a nutshell, the common aim of these techniques is to establish a relation to a corresponding linear, uniform, stationary scheme S' and to show that convergence and smoothness properties of S' are inherited by S. For example, non-uniform corner cutting gives limit curves that are as smooth as Chaikin's quadratic B-spline scheme (Chaikin, 1974; Riesenfeld, 1975), namely C^1, under certain constraints of the cut proportions (Gregory and Qu, 1996), and several variants of the classical interpolatory 4-point scheme (Dubuc, 1986; Dyn et al., 1987) are known to also give C^1 limit functions or curves (Beccari et al., 2007, 2011; Daubechies et al., 1999; Marinov et al., 2005).

In this paper, we show that it is possible to go beyond these results and to increase the smoothness of a linear scheme, if we allow the weights a_j and b_j of the subdivision rules in (1) to depend on k and i. That is, we consider non-uniform schemes, but in contrast to most other non-uniform schemes, we stick to the uniform, dyadic parameter values $t_i^k = i/2^k$. In particular, we focus on schemes that are *interpolatory*, that is, with weights $a_{i,0}^k = 1$ and $a_{i,j}^k = 0$ for $j \neq 0$, independently of k and i, as well as *periodic*, in the sense that the weights $b_{i,j}^k$ repeat at a frequency of 2^k in i, so that the same refinement strategy is applied over each of the initial intervals $[t_m^0, t_{m+1}^0] = [m, m+1]$, $m \in \mathbb{Z}$.

Overall, this means that we consider subdivision schemes that generate the refined data $f^k = (f_i^{k+1})_{i\in\mathbb{Z}}$ at level $k+1$ for any $k \in \mathbb{N}_0$ according to the even, interpolatory subdivision rule

$$f_{2i}^{k+1} = f_i^k \tag{2}$$

and the 2^k odd subdivision rules

$$f_{2i+1}^{k+1} = \sum_{j\in J_l^k} f_{i-j}^k b_{l,j}^k, \tag{3}$$

where $l = i \bmod 2^k$, with certain finite sets $J_i^k \subset \mathbb{Z}$ and weights $b_{i,j}^k \in \mathbb{R}$ for $i \in \{0, 1, \ldots, 2^k - 1\}$, $j \in J_i^k$, and $k \in \mathbb{N}_0$. We further call the subdivision scheme an *n-point* scheme, if the cardinality of all sets J_i^k is at most n.

Within this setting, we first derive two novel interpolatory 2-point schemes with C^1 limit functions, as opposed to the stationary, interpolatory 2-point scheme that simply reproduces the piecewise linear interpolant F^0 of the initial data and hence is only C^0 (Section 2). The main idea behind our construction is inspired by the B-spline schemes, which were designed to converge to uniform B-spline curves, in that we also start with a certain function that we would like the non-uniform subdivision scheme to generate in the limit and then derive the appropriate weights $b_{i,j}^k$ that achieve this. We then

use this approach to derive two interpolatory 4-point schemes with C^2 limit functions (Section 3), and finally discuss how these schemes can be used for generating interpolating curves on surfaces (Section 4).

2. Non-uniform C^1 interpolatory 2-point schemes

Let us start by considering interpolatory 2-point schemes, where $J_i^k = \{-1, 0\}$ for any $k \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2^k - 1\}$ in (3), so that each new data value f_{2i+1}^{k+1} is some linear combination of the neighbouring data values f_i^k and f_{i+1}^k. It follows by induction over $k \in \mathbb{N}_0$ that f_i^k for any $i \in \{2^k m, \dots, 2^k(m+1)\}$ is some linear combination of f_m^0 and f_{m+1}^0, which implies that the whole limit function over the interval $[m, m+1]$ depends only on these two values and is independent of the limit function over the neighbouring intervals $[m-1, m]$ and $[m+1, m+2]$. Therefore, the only chance to generate a limit function that is C^1 at $m \in \mathbb{Z}$ is to force the limit function to match a prescribed derivative at m over both adjacent intervals. The simplest unbiased choice is to set the derivative of the limit function to zero at all $m \in \mathbb{Z}$. Hence, we would like to construct a subdivision scheme that generates, over each interval $[m, m+1]$, a limit function that interpolates the initial data and has vanishing first derivative at the endpoints of this interval.

Without loss of generality, let us focus on the unit interval $[0, 1]$. Our goal now is to determine the weights of the odd subdivision rules in (3), such that the subdivision scheme generates a certain C^1 function $p\colon [0, 1] \to \mathbb{R}$ satisfying

$$\begin{aligned} p(0) &= f_0^0, & p'(0) &= 0, \\ p(1) &= f_1^0, & p'(1) &= 0 \end{aligned} \tag{4}$$

as the limit function over $[0, 1]$.

Theorem 1. *Let S be the non-uniform, periodic, interpolatory 2-point scheme with the odd subdivision rules in (3) and weights*

$$b_i^k = (b_{i,0}^k, b_{i,-1}^k) = (\alpha_i^k, \beta_i^k), \qquad k \in \mathbb{N}_0, \quad i \in \{0, 1, \dots, 2^k - 1\}, \tag{5}$$

which satisfy

$$p(t_{2i+1}^{k+1}) = p(t_i^k)\alpha_i^k + p(t_{i+1}^k)\beta_i^k. \tag{6}$$

Then the scheme S generates data

$$f_i^k = p(t_i^k), \qquad k \in \mathbb{N}_0, \quad i \in \{0, 1, \dots, 2^k\}, \tag{7}$$

is convergent, and the limit functions $f = S^\infty f^0$ are C^1 functions with $f(m) = f_m^0$ and $f'(m) = 0$ for $m \in \mathbb{Z}$.

Proof. We first prove (7) by induction over k and note that the base case ($k = 0$) follows from (4). For the inductive step, we distinguish the even and the odd case. On the one hand, by (2), the induction hypothesis, and the fact that $t_i^k = i/2^k = t_{2i}^{k+1}$, we have

$$f_{2i}^{k+1} = f_i^k = p(t_i^k) = p(t_{2i}^{k+1}).$$

On the other hand, by (3) with the weights in (5), the induction hypothesis, and (6), we have

$$f_{2i+1}^{k+1} = f_i^k \alpha_i^k + f_{i+1}^k \beta_i^k = p(t_i^k)\alpha_i^k + p(t_{i+1}^k)\beta_i^k = p(t_{2i+1}^{k+1}).$$

It then follows from (7) that the functions F^k are piecewise linear interpolants of the function p at the dyadic points t_i^k and as such converge uniformly to p over $[0, 1]$ as $k \to \infty$ (de Boor, 2001). By the periodicity of the scheme, the same applies to all intervals $[m, m+1]$ for $m \in \mathbb{Z}$, thus proving the rest of the statement. □

It remains to understand under which conditions the weights in (5) can always be chosen such that they satisfy condition (6). To this end, let us express p in terms of the initial data f_0^0 and f_1^0 as

$$p(x) = r_0(x) f_0^0 + r_1(x) f_1^0$$

for two *basis functions* $r_0, r_1\colon [0, 1] \to \mathbb{R}$ with

$$\begin{aligned} r_0(0) &= 1, & r_1(0) &= 0, & r_0'(0) &= r_1'(0) = 0, \\ r_0(1) &= 0, & r_1(1) &= 1, & r_0'(1) &= r_1'(1) = 0. \end{aligned}$$

Comparing the coefficients of f_0^0 and f_1^0 on both sides of (6) then yields the linear system

$$\underbrace{\begin{pmatrix} r_0(t_i^k) & r_0(t_{i+1}^k) \\ r_1(t_i^k) & r_1(t_{i+1}^k) \end{pmatrix}}_{=A_i^k} \begin{pmatrix} \alpha_i^k \\ \beta_i^k \end{pmatrix} = \begin{pmatrix} r_0(t_{2i+1}^{k+1}) \\ r_1(t_{2i+1}^{k+1}) \end{pmatrix}. \tag{8}$$

Lemma 2. *If r_0 and r_1 form a partition of unity and r_1 is strictly increasing, then the linear system in (8) has a unique solution for any $k \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2^k - 1\}$, which is given by*

$$\alpha_i^k = \frac{r_1(t_{i+1}^k) - r_1(t_{2i+1}^{k+1})}{r_1(t_{i+1}^k) - r_1(t_i^k)}, \qquad \beta_i^k = \frac{r_1(t_{2i+1}^{k+1}) - r_1(t_i^k)}{r_1(t_{i+1}^k) - r_1(t_i^k)}. \tag{9}$$

These weights α_i^k and β_i^k are both positive and sum to one.

Proof. Using the partition of unity property, we find that $\det(A_i^k) = r_1(t_{i+1}^k) - r_1(t_i^k)$. Moreover, $\det(A_i^k) > 0$, because r_1 is strictly increasing and $t_i^k < t_{i+1}^k$. This implies the invertibility of A_i^k for any $k \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2^k - 1\}$. The formulas in (9) can be derived from (8) after inverting A_i^k explicitly, and α_i^k and β_i^k are positive, because r_1 is strictly increasing and $t_i^k < t_{2i+1}^{k+1} < t_{i+1}^k$. They obviously sum to one. □

Under the assumptions of Lemma 2, it follows that the odd subdivision rules (3) of the corresponding 2-point scheme generate the new data f_{2i+1}^{k+1} as a *convex combination* of the previous data f_i^k and f_{i+1}^k and consequently $f(x) \in [f_m^0, f_{m+1}^0]$ for $x \in [m, m+1]$ and any $m \in \mathbb{Z}$. We shall now discuss two particular choices of basis functions r_0 and r_1 that lead to rather simple expressions of the weights α_i^k and β_i^k in (9).

Example 1. One possibility to satisfy the conditions in (4) is to choose as p the cubic Hermite interpolant, which is given by the basis functions

$$r_0(x) = (1-x)^2(1+2x), \qquad r_1(x) = x^2(3-2x).$$

These functions form a partition of unity and r_1 is strictly increasing, because $r_1'(x) = 6x(1-x) > 0$ for $x \in (0, 1)$. A straightforward calculation reveals that the formulas in (9) can be simplified to

$$(\alpha_i^k, \beta_i^k) = \left(\tfrac{1}{2} + \tfrac{3}{2}\Delta_i^k, \tfrac{1}{2} - \tfrac{3}{2}\Delta_i^k\right), \qquad \Delta_i^k = \frac{2i+1-2^k}{3(2i+1)^2 - 6(2i+1)2^k + 1}$$

in this case. At the endpoints of each interval $[m, m+1]$, these weights converge to

$$\lim_{k\to\infty} (\alpha_0^k, \beta_0^k) = \left(\tfrac{3}{4}, \tfrac{1}{4}\right) \qquad \text{and} \qquad \lim_{k\to\infty} (\alpha_{2^k-1}^k, \beta_{2^k-1}^k) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right),$$

and they converge to

$$\lim_{k\to\infty} (\alpha_{2^{k-1}-1}^k, \beta_{2^{k-1}-1}^k) = \lim_{k\to\infty} (\alpha_{2^{k-1}}^k, \beta_{2^{k-1}}^k) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

next to the centre of the interval.

Because of this limit behaviour, the known tools for analysing the smoothness of non-uniform schemes are not applicable in this case, but it is evident from the construction that the limit functions of this non-uniform, periodic, interpolating 2-point scheme are the cubic C^1 B-splines with duplicated control points $(\dots, f_{-1}^0, f_{-1}^0, f_0^0, f_0^0, f_1^0, f_1^0, \dots)$ and uniform double knots $(\dots, -1, -1, 0, 0, 1, 1, \dots)$.

Example 2. Even simpler weights can be found if we take as p the piecewise quadratic C^1 function given by the basis functions

$$r_0(x) = \begin{cases} 1 - 2x^2, & x \in \left[0, \frac{1}{2}\right], \\ 2(1-x)^2, & x \in \left[\frac{1}{2}, 1\right], \end{cases} \qquad r_1(x) = \begin{cases} 2x^2, & x \in \left[0, \frac{1}{2}\right], \\ 1 - 2(1-x)^2, & x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

for which the weights in (9) turn out to be $(\alpha_0^0, \beta_0^0) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and

$$(\alpha_i^k, \beta_i^k) = \left(\tfrac{1}{2} + \tfrac{1}{4}\Delta_i^k, \tfrac{1}{2} - \tfrac{1}{4}\Delta_i^k\right), \qquad \Delta_i^k = \begin{cases} \dfrac{1}{2i+1}, & i < 2^{k-1}, \\ \dfrac{1}{2i+1-2^{k+1}}, & i \geq 2^{k-1}, \end{cases}$$

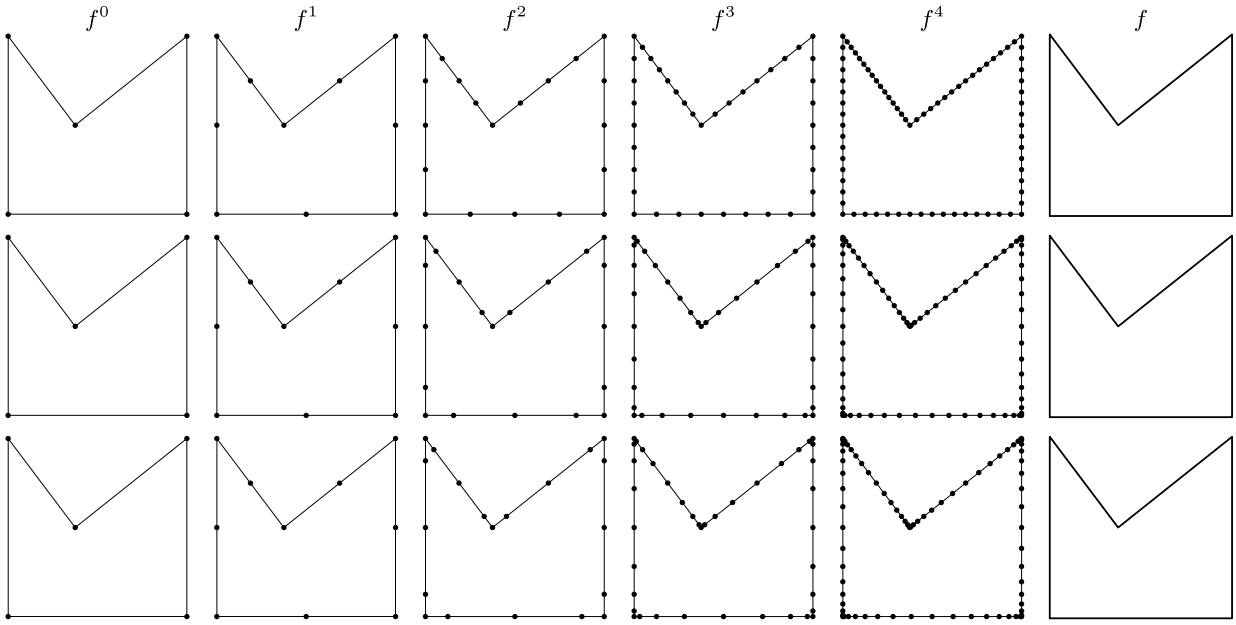

Fig. 1. Initial points, points generated by the first four subdivision steps, and limit curve (from left to right) for the stationary (top) and the non-uniform interpolatory 2-point schemes from Example 1 (middle) and Example 2 (bottom).

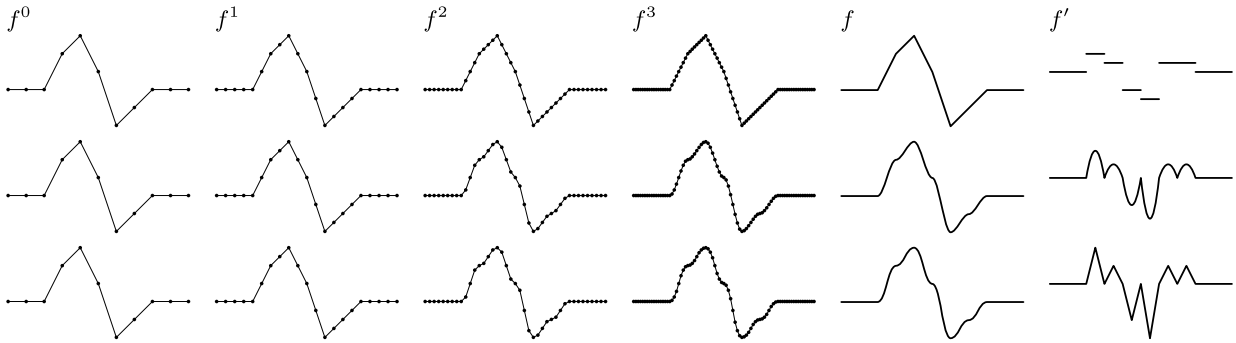

Fig. 2. Initial data, data generated by the first three subdivision steps, limit function, and first derivative of the limit function (from left to right) for the stationary (top) and the non-uniform interpolatory 2-point schemes from Example 1 (middle) and Example 2 (bottom).

for $k > 0$. As in Example 1, these weights converge to $\left(\frac{1}{2}, \frac{1}{2}\right)$ for $i = 2^{k-1} - 1$ and $i = 2^{k-1}$, but here we have $(\alpha_0^k, \beta_0^k) = \left(\frac{3}{4}, \frac{1}{4}\right)$ not only in the limit, but for any $k > 0$, and likewise $(\alpha_{2^k-1}^k, \beta_{2^k-1}^k) = \left(\frac{1}{4}, \frac{3}{4}\right)$.

As in Example 1, the known tools are not able to prove the smoothness of the limit functions generated by this non-uniform, periodic, interpolating 2-point scheme, but it follows from the construction that they are the quadratic C^1 B-splines with duplicated control points $(\ldots, f_{-1}^0, f_{-1}^0, f_0^0, f_0^0, f_1^0, f_1^0, \ldots)$ and uniform knots $\left(\ldots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right) = \mathbb{Z}/2$.

In the geometric setting, the fact that each new point f_{2i+1}^{k+1} is generated as a convex combination of the old points f_i^k and f_{i+1}^k implies that the limit curve has the same shape as the initial control polygon, with apparent C^1 discontinuities at the initial control points. The explanation for this behaviour is that the new points generated by the non-uniform scheme are non-uniformly distributed along the edges of the initial control polygon and clustered towards the endpoints, which differs from the uniform distribution of points given by the classical interpolatory 2-point scheme with the stationary weights $\left(\frac{1}{2}, \frac{1}{2}\right)$ (see Fig. 1). Hence, the limit curve of the non-uniform scheme has a continuous rather than a piecewise constant derivative with respect to the dyadic parameterization, with vanishing first derivative at $m \in \mathbb{Z}$. This fact is more evident in the functional setting, where it leads to flat spots at the integers (see Fig. 2).

We should stress that we do not believe these non-uniform 2-point schemes to be of any practical use, but rather consider them as "academic examples", which showcase that non-uniform schemes have the potential to generate limit functions and curves with higher smoothness than those generated by the corresponding stationary scheme. And it turns out that this potential is not restricted to 2-point schemes.

3. Non-uniform C^2 interpolatory 4-point schemes

For the non-uniform interpolatory 2-point schemes above, we can guarantee the limit functions to be C^1 only by forcing the first derivative to vanish at the integers. The reason for this arguably arbitrary choice is the locality of the subdivision rule, but we can overcome this problem if we enlarge the support of the rules.

Hence, let us now consider the interpolatory 4-point schemes with $J_i^k = \{-2, -1, 0, 1\}$ and weights

$$b_i^k = (b_{i,1}^k, b_{i,0}^k, b_{i,-1}^k, b_{i,-2}^k) = (\alpha_i^k, \beta_i^k, \gamma_i^k, \delta_i^k), \tag{10}$$

for the odd subdivision rules in (3), so that the new data value

$$f_{2i+1}^{k+1} = f_{i-1}^k \alpha_i^k + f_i^k \beta_i^k + f_{i+1}^k \gamma_i^k + f_{i+2}^k \delta_i^k$$

is a linear combination of the neighbouring data values f_{i-1}^k, f_i^k, f_{i+1}^k, and f_{i+2}^k, just as for the classical interpolatory 4-point scheme (Dubuc, 1986), which uses the stationary weights $\left(-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16}\right)$.

As before, we first focus on the unit interval $[0, 1]$, and this time we aim at reproducing in the limit a certain C^2 function $p\colon [0, 1] \to \mathbb{R}$ that satisfies

$$\begin{aligned} p(0) &= f_0^0, & p'(0) &= \tfrac{1}{2}(f_1^0 - f_{-1}^0), & p''(0) &= f_1^0 - 2f_0^0 + f_{-1}^0 \\ p(1) &= f_1^0, & p'(1) &= \tfrac{1}{2}(f_2^0 - f_0^0), & p''(1) &= f_2^0 - 2f_1^0 + f_0^0. \end{aligned} \tag{11}$$

Again, the key trick is to choose the weights in (10) such that

$$p(t_{2i+1}^{k+1}) = p(t_{i-1}^k)\alpha_i^k + p(t_i^k)\beta_i^k + p(t_{i+1}^k)\gamma_i^k + p(t_{i+2}^k)\delta_i^k. \tag{12}$$

Expressing p in terms of the initial data f_{-1}^0, f_0^0, f_1^0, and f_2^0 as

$$p(x) = r_{-1}(x) f_{-1}^0 + r_0(x) f_0^0 + r_1(x) f_1^0 + r_2(x) f_2^0$$

for four basis functions $r_{-1}, r_0, r_1, r_2\colon [0, 1] \to \mathbb{R}$ with

$$\begin{aligned} r_{-1}(0) &= 0, & r_{-1}'(0) &= -\tfrac{1}{2}, & r_{-1}''(0) &= 1, & r_{-1}(1) &= 0, & r_{-1}'(1) &= 0, & r_{-1}''(1) &= 0, \\ r_0(0) &= 1, & r_0'(0) &= 0, & r_0''(0) &= -2, & r_0(1) &= 0, & r_0'(1) &= -\tfrac{1}{2}, & r_0''(1) &= 1, \\ r_1(0) &= 0, & r_1'(0) &= \tfrac{1}{2}, & r_1''(0) &= 1, & r_1(1) &= 1, & r_1'(1) &= 0, & r_1''(1) &= -2, \\ r_2(0) &= 0, & r_2'(0) &= 0, & r_2''(0) &= 0, & r_2(1) &= 0, & r_2'(1) &= \tfrac{1}{2}, & r_2''(1) &= 1, \end{aligned}$$

and comparing the coefficients of $f_{-1}^0, \ldots, f_2^0$ in (12) leads to the linear system

$$\underbrace{\begin{pmatrix} r_{-1}(t_{i-1}^k) & r_{-1}(t_i^k) & r_{-1}(t_{i+1}^k) & r_{-1}(t_{i+2}^k) \\ r_0(t_{i-1}^k) & r_0(t_i^k) & r_0(t_{i+1}^k) & r_0(t_{i+2}^k) \\ r_1(t_{i-1}^k) & r_1(t_i^k) & r_1(t_{i+1}^k) & r_1(t_{i+2}^k) \\ r_2(t_{i-1}^k) & r_2(t_i^k) & r_2(t_{i+1}^k) & r_2(t_{i+2}^k) \end{pmatrix}}_{=B_i^k} \begin{pmatrix} \alpha_i^k \\ \beta_i^k \\ \gamma_i^k \\ \delta_i^k \end{pmatrix} = \begin{pmatrix} r_{-1}(t_{2i+1}^{k+1}) \\ r_0(t_{2i+1}^{k+1}) \\ r_1(t_{2i+1}^{k+1}) \\ r_2(t_{2i+1}^{k+1}) \end{pmatrix}. \tag{13}$$

Lemma 3. *If r_{-1}, r_0, r_1, r_2 reproduce quadratic polynomials in the sense that*

$$\sum_{j=-1}^{2} r_j(x) = 1, \qquad \sum_{j=-1}^{2} r_j(x)\, j = x, \qquad \sum_{j=-1}^{2} r_j(x)\, j^2 = x^2, \tag{14}$$

and the third-order divided difference of r_2 satisfies

$$r_2[t_{i-1}^k, t_i^k, t_{i+1}^k, t_{i+2}^k] \neq 0, \tag{15}$$

for some $k \geq 2$ and $i \in \{1, 2, \ldots, 2^k - 2\}$, then the corresponding linear system in (13) has a unique solution.

Proof. Using (14), we first transform B_i^k into

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=L} \cdot B_i^k = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_{i-1}^k & t_i^k & t_{i+1}^k & t_{i+2}^k \\ (t_{i-1}^k)^2 & (t_i^k)^2 & (t_{i+1}^k)^2 & (t_{i+2}^k)^2 \\ r_2(t_{i-1}^k) & r_2(t_i^k) & r_2(t_{i+1}^k) & r_2(t_{i+2}^k) \end{pmatrix}.$$

We then exploit the fact that the parameter values $t_{i-1}^k, \dots, t_{i+2}^k$ are uniformly spaced with distance 2^{-k}, so that replacing the second, third, and fourth column of this matrix with the first, second, and third order differences of its columns gives the lower triangular matrix

$$\tilde{B}_i^k = L \cdot B_i^k \cdot \underbrace{\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{i-1}^k & 2^{-k} & 0 & 0 \\ (t_{i-1}^k)^2 & * & 2 \cdot 4^{-k} & 0 \\ r_2(t_{i-1}^k) & * & * & \rho \end{pmatrix},$$

where

$$\rho = r_2(t_{i+2}^k) - 3r_2(t_{i+1}^k) + 3r_2(t_i^k) - r_2(t_{i-1}^k) = 6r_2[t_{i-1}^k, t_i^k, t_{i+1}^k, t_{i+2}^k] \cdot 8^{-k}.$$

Since $\det(L) = 2$ and $\det(R) = 1$, we have

$$\det(B_i^k) = \frac{1}{2}\det(\tilde{B}_i^k) = 8^{-k}\rho,$$

hence B_i^k is non-singular, because of (15). □

As in the proof of Theorem 1, we would now like to conclude by induction that the scheme generates data

$$f_i^k = p(t_i^k), \qquad k \in \mathbb{N}_0, \quad i \in \{0, 1, \dots, 2^k\}, \tag{16}$$

which in turn implies that p is the limit curve over $[0, 1]$, and for the even case the arguments are the same. Alas, the inductive step for showing that $f_1^{k+1} = p(t_1^{k+1})$ relies on the identity $f_{-1}^k = p(t_{-1}^k)$, which is not supported by the induction hypothesis for $k > 0$, and likewise for $f_{2^{k+1}-1}^{k+1}$. To fix this problem, let us take a look at the first subdivision step. It is not hard to verify that for $k = 0$ and $i = 0$, the correct weights, which guarantee $f_1^1 = p(\frac{1}{2})$, are

$$b_0^0 = (b_{0,1}^0, b_{0,0}^0, b_{0,-1}^0, b_{0,-2}^0) = (\alpha_0^0, \beta_0^0, \gamma_0^0, \delta_0^0) = \big(r_{-1}(\tfrac{1}{2}), r_0(\tfrac{1}{2}), r_1(\tfrac{1}{2}), r_2(\tfrac{1}{2})\big), \tag{17}$$

generating the data

$$\begin{aligned} f_{-2}^1 &= f_{-1}^0, \\ f_{-1}^1 &= f_{-2}^0\alpha_0^0 + f_{-1}^0\beta_0^0 + f_0^0\gamma_0^0 + f_1^0\delta_0^0, \\ f_0^1 &= f_0^0, \\ f_1^1 &= f_{-1}^0\alpha_0^0 + f_0^0\beta_0^0 + f_1^0\gamma_0^0 + f_2^0\delta_0^0, \\ f_2^1 &= f_1^0. \end{aligned}$$

Therefore, computing f_1^2 in the standard fashion as

$$f_1^2 = f_{-1}^1\alpha_0^1 + f_0^1\beta_0^1 + f_1^1\gamma_0^1 + f_2^1\delta_0^1$$

results in a value that depends on the initial data $f_{-2}^0, \dots, f_2^0$, but in order to have $f_1^2 = p(t_1^2)$, it must not depend on f_{-2}^0. The trick now is to replace the index set $J_0^1 = \{-2, -1, 0, 1\}$ by the set $J_0^1 = \{-2, -1, 0, 2\}$, so that the subdivision rule (3) for f_1^2 becomes

$$\begin{aligned} f_1^2 &= f_{-2}^1\alpha_0^1 + f_0^1\beta_0^1 + f_1^1\gamma_0^1 + f_2^1\delta_0^1 \\ &= f_{-1}^0(\alpha_0^1 + \alpha_0^0\gamma_0^1) + f_0^0(\beta_0^1 + \beta_0^0\gamma_0^1) + f_1^0(\gamma_0^0\gamma_0^1 + \delta_0^1) + f_2^0(\delta_0^0\gamma_0^1). \end{aligned}$$

Comparing the coefficients of $f_{-1}^0, \dots, f_2^0$ with those on the right hand side of

$$p(t_1^2) = r_{-1}(\tfrac{1}{4})f_{-1}^0 + r_0(\tfrac{1}{4})f_0^0 + r_1(\tfrac{1}{4})f_1^0 + r_2(\tfrac{1}{4})f_2^0$$

then reveals that for $k = 1$ and $i = 0$ the correct choice of weights

$$b_0^1 = (b_{0,2}^1, b_{0,1}^1, b_{0,0}^1, b_{0,-1}^1, b_{0,-2}^1) = (\alpha_0^1, 0, \beta_0^1, \gamma_0^1, \delta_0^1) \tag{18a}$$

is

$$\alpha_0^1 = r_{-1}(\tfrac{1}{4}) - \frac{r_{-1}(\frac{1}{2})r_2(\frac{1}{4})}{r_2(\frac{1}{2})}, \quad \beta_0^1 = r_0(\tfrac{1}{4}) - \frac{r_0(\frac{1}{2})r_2(\frac{1}{4})}{r_2(\frac{1}{2})}, \quad \gamma_0^1 = \frac{r_2(\frac{1}{4})}{r_2(\frac{1}{2})}, \quad \delta_0^1 = r_1(\tfrac{1}{4}) - \frac{r_1(\frac{1}{2})r_2(\frac{1}{4})}{r_2(\frac{1}{2})}. \tag{18b}$$

To compute f_3^2, we proceed analogously, using the index set $J_1^1 = \{-3, -1, 0, 1\}$, and it turns out that the correct weights

$$b_1^1 = (b_{1,1}^1, b_{1,0}^1, b_{1,-1}^1, b_{1,-2}^1, b_{1,-3}^1) = (\alpha_1^1, \beta_1^1, \gamma_1^1, 0, \delta_1^1) \tag{19a}$$

are

$$\alpha_1^1 = r_0(\tfrac{3}{4}) - \frac{r_0(\frac{1}{2})r_{-1}(\frac{3}{4})}{r_{-1}(\frac{1}{2})}, \quad \beta_1^1 = \frac{r_{-1}(\frac{3}{4})}{r_{-1}(\frac{1}{2})}, \quad \gamma_1^1 = r_1(\tfrac{3}{4}) - \frac{r_1(\frac{1}{2})r_{-1}(\frac{3}{4})}{r_{-1}(\frac{1}{2})}, \quad \delta_1^1 = r_2(\tfrac{3}{4}) - \frac{r_2(\frac{1}{2})r_{-1}(\frac{3}{4})}{r_{-1}(\frac{1}{2})}. \tag{19b}$$

So far, this ensures the interpolation property in (16) to hold for $k = 0, 1, 2$, but for $k > 2$, we need yet another approach. If we replace again the index set J_0^k for $k \geq 2$ with $\{-2, -1, 0, 2\}$, then we run into the same problem as before, because $f_{-2}^k = f_{-1}^{k-1}$ depends on f_{-2}^0. To resolve this, we replace J_0^k with $\{-3, -2, -1, 0\}$ instead, so that the subdivision rule for f_1^{k+1} becomes

$$f_1^{k+1} = f_0^k\alpha_0^k + f_1^k\beta_0^k + f_2^k\gamma_0^k + f_3^k\delta_0^k.$$

Note that this choice would not work for $k = 1$, because f_3^1 depends on f_3^0. Using the induction hypothesis that $f_j^k = p(t_j^k)$ for $j = 0, 1, 2, 3$, the weights $(\alpha_0^k, \beta_0^k, \gamma_0^k, \delta_0^k)$ that guarantee $f_1^{k+1} = p(t_1^{k+1})$ are then given by the linear system in (13) for $i = 0$, with B_0^k replaced by B_1^k, which still leads to a unique solution under the assumptions of Lemma 3. Similarly, we replace the index set $J_{2^k-1}^k$ with $\{-1, 0, 1, 2\}$ for $k \geq 2$, and the weights $(\alpha_{2^k-1}^k, \beta_{2^k-1}^k, \gamma_{2^k-1}^k, \delta_{2^k-1}^k)$ for computing

$$f_{2^{k+1}-1}^{k+1} = f_{2^k-3}^k\alpha_{2^k-1}^k + f_{2^k-2}^k\beta_{2^k-1}^k + f_{2^k-1}^k\gamma_{2^k-1}^k + f_{2^k}^k\delta_{2^k-1}^k,$$

such that $f_{2^{k+1}-1}^{k+1} = p(t_{2^{k+1}-1}^{k+1})$, are given by (13) for $i = 2^k - 1$, with $B_{2^k-1}^k$ replaced by $B_{2^k-2}^k$.

Based on the discussion above, we are now ready to summarize our construction.

Theorem 4. *Let r_{-1}, r_0, r_1, r_2 be four basis functions that satisfy*

$$r_{-1}(\tfrac{1}{2}) \neq 0, \qquad r_2(\tfrac{1}{2}) \neq 0 \tag{20}$$

as well as the assumptions of Lemma 3 for any $k \geq 2$ and $i \in \{1, 2, \ldots, 2^k - 2\}$. Further let S be the non-uniform, periodic, interpolatory 4-point scheme with the odd subdivision rules in (3) and the weights in (17), (18), and (19) for $k = 0$ and $k = 1$, as well as

$$\begin{aligned} b_0^k &= (b_{0,0}^k, b_{0,-1}^k, b_{0,-2}^k, b_{0,-3}^k) = (\alpha_0^k, \beta_0^k, \gamma_0^k, \delta_0^k) \\ &= \left(r_{-1}(t_1^{k+1}), r_0(t_1^{k+1}), r_1(t_1^{k+1}), r_2(t_1^{k+1})\right) \cdot \left(B_1^k\right)^{-T}, \\ b_i^k &= (b_{i,1}^k, b_{i,0}^k, b_{i,-1}^k, b_{i,-2}^k) = (\alpha_i^k, \beta_i^k, \gamma_i^k, \delta_i^k) \\ &= \left(r_{-1}(t_{2i+1}^{k+1}), r_0(t_{2i+1}^{k+1}), r_1(t_{2i+1}^{k+1}), r_2(t_{2i+1}^{k+1})\right) \cdot \left(B_i^k\right)^{-T}, \qquad i \in \{1, 2 \ldots, 2^k - 2\}, \\ b_{2^k-1}^k &= (b_{2^k-1,2}^k, b_{2^k-1,1}^k, b_{2^k-1,0}^k, b_{2^k-1,-1}^k) = (\alpha_{2^k-1}^k, \beta_{2^k-1}^k, \gamma_{2^k-1}^k, \delta_{2^k-1}^k) \\ &= \left(r_{-1}(t_{2^{k+1}-1}^{k+1}), r_0(t_{2^{k+1}-1}^{k+1}), r_1(t_{2^{k+1}-1}^{k+1}), r_2(t_{2^{k+1}-1}^{k+1})\right) \cdot \left(B_{2^k-2}^k\right)^{-T} \end{aligned}$$

for $k \geq 2$. Then the scheme S generates the data in (16), is convergent, and the limit functions are C^2 functions with $f(m) = f_m^0$, $f'(m) = (f_{m+1}^0 - f_{m-1}^0)/2$, and $f''(m) = f_{m+1}^0 - 2f_m^0 + f_{m-1}^0$ for $m \in \mathbb{Z}$.

As in the previous section, we shall now discuss two particular choices of basis functions r_{-1}, r_0, r_1, r_2, for which rather simple expressions of all weights α_i^k, β_i^k, γ_i^k, δ_i^k can be derived.

Example 3. One possibility to satisfy the conditions in (11) is to choose as p the quintic Hermite interpolant, which is given by the basis functions

$$r_{-1}(x) = x(1-x)^3(-\tfrac{1}{2}-x), \qquad r_0(x) = (1-x)(1+x-\tfrac{9}{2}x^3+3x^4),$$
$$r_1(x) = x(\tfrac{1}{2}+\tfrac{1}{2}x+\tfrac{9}{2}x^2-\tfrac{15}{2}x^3+3x^4), \qquad r_2(x) = x^3(1-x)(-\tfrac{3}{2}+x).$$

It is not hard to verify that these functions satisfy the conditions in (14) and (20) and that

$$r_2[t_{i-1}^k, t_i^k, t_{i+1}^k, t_{i+2}^k] = -4^{-k}\big(10i^2 - 10i(2^k-1) - 5(2^k-1) + \tfrac{3}{2}\cdot 4^k\big),$$

which is equal to zero if and only if

$$i = \tfrac{1}{2}\Big(2^k - 1 \pm \sqrt{\tfrac{2}{5}\cdot 4^k - 1}\Big).$$

For $k=0$, these values are imaginary, and if $k>0$, then 2^k-1 is odd and the root would have to be odd, too, in order for i to be an integer, that is,

$$\sqrt{\tfrac{2}{5}\cdot 4^k - 1} = 2m-1 \quad\Longleftrightarrow\quad \tfrac{2}{5}\cdot 4^k - 1 = 4m^2-4m+1 \quad\Longleftrightarrow\quad 4^k = 5(2m^2-2m+1)$$

for some $m\in\mathbb{Z}$, but this is impossible, because 4^k does not divide by 5.

This asserts that these basis functions satisfy the assumptions of Theorem 4, and the weights of the resulting subdivision scheme turn out to be

$$b_0^0 = \big(-\tfrac{1}{16}, \tfrac{9}{16}, \tfrac{9}{16}, -\tfrac{1}{16}\big), \qquad b_0^1 = \big(-\tfrac{33}{512}, 0, \tfrac{390}{512}, \tfrac{120}{512}, \tfrac{35}{512}\big), \qquad b_1^1 = \big(\tfrac{35}{512}, \tfrac{120}{512}, \tfrac{390}{512}, 0, -\tfrac{33}{512}\big)$$

for $k=0$ and $k=1$, as well as

$$b_0^k = \Bigg(\frac{5}{16}\cdot\frac{113-65\cdot 2^k+6\cdot 4^k}{100-60\cdot 2^k+6\cdot 4^k}, \frac{15}{16}\cdot\frac{87-55\cdot 2^k+6\cdot 4^k}{100-60\cdot 2^k+6\cdot 4^k},$$
$$-\frac{5}{16}\cdot\frac{61-45\cdot 2^k+6\cdot 4^k}{100-60\cdot 2^k+6\cdot 4^k}, \frac{1}{16}\cdot\frac{35-35\cdot 2^k+6\cdot 4^k}{100-60\cdot 2^k+6\cdot 4^k}\Bigg),$$

$$b_i^k = \Bigg(-\frac{1}{16}\cdot\frac{5(7+8i)(7+8i-2^{k+3})+35+3\cdot 4^{k+2}}{5(4+8i)(4+8i-2^{k+3})+80+3\cdot 4^{k+2}}, \quad \frac{9}{16}\cdot\frac{5(5+8i)(5+8i-2^{k+3})+75+3\cdot 4^{k+2}}{5(4+8i)(4+8i-2^{k+3})+80+3\cdot 4^{k+2}},$$
$$\frac{9}{16}\cdot\frac{5(3+8i)(3+8i-2^{k+3})+75+3\cdot 4^{k+2}}{5(4+8i)(4+8i-2^{k+3})+80+3\cdot 4^{k+2}}, -\frac{1}{16}\cdot\frac{5(1+8i)(1+8i-2^{k+3})+35+3\cdot 4^{k+2}}{5(4+8i)(4+8i-2^{k+3})+80+3\cdot 4^{k+2}}\Bigg)$$

for $i\in\{1,2,\dots,2^k-2\}$, and

$$b_{2^k-1}^k = (b_{0,-3}^k, b_{0,-2}^k, b_{0,-1}^k, b_{0,0}^k)$$

for $k\ge 2$. These weights converge to

$$\lim_{k\to\infty} b_0^k = \big(\tfrac{5}{16}, \tfrac{15}{16}, -\tfrac{5}{16}, \tfrac{1}{16}\big) \qquad \text{and} \qquad \lim_{k\to\infty} b_{2^k-1}^k = \big(\tfrac{1}{16}, -\tfrac{5}{16}, \tfrac{15}{16}, \tfrac{5}{16}\big) \tag{21}$$

at the endpoints and to the weights of the classical interpolatory 4-point scheme,

$$\lim_{k\to\infty} b_i^k = \big(-\tfrac{1}{16}, \tfrac{9}{16}, \tfrac{9}{16}, -\tfrac{1}{16}\big),$$

at all interior points with $i\in\{1,2,\dots,2^k-2\}$.

As the limit weights in (21) are the non-symmetric variants of the stationary 4-point weights, in the sense that they can also be derived from locally fitting and evaluating a cubic polynomial, it might be possible to extend one of the known tools to show that the limit functions of this non-uniform, periodic, interpolating 4-point scheme are as smooth as those of the stationary 4-point scheme, namely C^1, but not that they are C^2. In fact, it is not hard to see that the limit functions are the quintic C^2 B-splines with control points

$$\big(\dots, \tfrac{-3f_{-2}+18f_{-1}+5f_0}{20}, \tfrac{5f_{-1}+18f_0-3f_1}{20}, \tfrac{-f_{-1}+22f_0-f_1}{20}, \tfrac{-3f_{-1}+18f_0+5f_1}{20}, \tfrac{5f_0+18f_1-3f_2}{20}, \dots\big)$$

and uniform triple knots $(\dots,-1,-1,-1,0,0,0,1,1,1,\dots)$.

Example 4. Much simpler weights can be found by taking as p the piecewise cubic C^2 function given by the basis functions

$$r_{-1}(x)=\begin{cases} x(-\frac{1}{2}+\frac{1}{2}x+\frac{13}{18}x^2), & x\in\left[0,\frac{1}{4}\right],\\ \frac{1}{36}-\frac{5}{6}x+\frac{11}{6}x^2-\frac{19}{18}x^3, & x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ -\frac{13}{18}(1-x)^3, & x\in\left[\frac{3}{4},1\right],\end{cases}\qquad r_0(x)=\begin{cases} 1-x^2(1+\frac{13}{6}x), & x\in\left[0,\frac{1}{4}\right],\\ \frac{11}{12}+x-5x^2+\frac{19}{6}x^3, & x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ (1-x)(\frac{19}{6}-\frac{29}{6}x+\frac{13}{6}x^2), & x\in\left[\frac{3}{4},1\right],\end{cases}$$

$$r_1(x)=\begin{cases} x(\frac{1}{2}+\frac{1}{2}x+\frac{13}{6}x^2), & x\in\left[0,\frac{1}{4}\right],\\ \frac{1}{12}-\frac{1}{2}x+\frac{9}{2}x^2-\frac{19}{6}x^3, & x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ 1-(1-x)^2(\frac{19}{6}-\frac{13}{6}x), & x\in\left[\frac{3}{4},1\right],\end{cases}\qquad r_2(x)=\begin{cases} -\frac{13}{18}x^3, & x\in\left[0,\frac{1}{4}\right],\\ -\frac{1}{36}+\frac{1}{3}x-\frac{4}{3}x^2+\frac{19}{18}x^3, & x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ (1-x)(\frac{13}{18}-\frac{35}{18}x+\frac{13}{18}x^2), & x\in\left[\frac{3}{4},1\right].\end{cases}$$

A simple calculation reveals that these functions satisfy the conditions in (14) and (20) and that

$$r_2[t_{i-1}^k,t_i^k,t_{i+1}^k,t_{i+2}^k]=\frac{1}{54}\begin{cases} -39, & i=1,2,\dots,2^{k-2}-2,\\ -23, & i=2^{k-2}-1,\\ \ \ 41, & i=2^{k-2},\\ \ \ 57, & i=2^{k-2}+1,2^{k-2}+2,\dots,3\cdot 2^{k-2}-2,\\ \ \ 41, & i=3\cdot 2^{k-2}-1,\\ -23, & i=3\cdot 2^{k-2},\\ -39, & i=3\cdot 2^{k-2}+1,3\cdot 2^{k-2}+2,\dots,2^k-2\end{cases}$$

for any $k\ge 2$. Hence, the assumptions of Theorem 4 are satisfied, and straightforward calculations reveal that the weights of the resulting subdivision scheme are

$$b_0^0=\left(-\tfrac{1}{16},\tfrac{9}{16},\tfrac{9}{16},-\tfrac{1}{16}\right),\qquad b_0^1=\left(-\tfrac{41}{576},0,\tfrac{462}{576},\tfrac{104}{576},\tfrac{51}{576}\right),\qquad b_1^1=\left(\tfrac{51}{576},\tfrac{104}{576},\tfrac{462}{576},0,-\tfrac{41}{576}\right)$$

for $k=0$ and $k=1$, as well as

$$b_0^2=\left(\tfrac{253}{656},\tfrac{471}{656},-\tfrac{61}{656},-\tfrac{7}{656}\right),\qquad b_0^3=\left(\tfrac{99}{368},\tfrac{393}{368},-\tfrac{163}{368},\tfrac{39}{368}\right),\qquad b_0^k=\left(\tfrac{5}{16},\tfrac{15}{16},-\tfrac{5}{16},\tfrac{1}{16}\right),\quad k\ge 4$$

and, by symmetry,

$$b_{2^k-1}^k=(b_{0,-3}^k,b_{0,-2}^k,b_{0,-1}^k,b_{0,0}^k),\quad k\ge 2,$$

for the new points corresponding to the parameter values $t=m\pm 2^{-k-1}$ for $m\in\mathbb{Z}$. Moreover,

$$b_{2^{k-2}}^k=\left(-\tfrac{57}{656},\tfrac{417}{656},\tfrac{321}{656},-\tfrac{25}{656}\right),\quad b_{3\cdot 2^{k-2}-1}^k=(b_{2^{k-2},-2}^k,b_{2^{k-2},-1}^k,b_{2^{k-2},0}^k,b_{2^{k-2},1}^k),\quad k\ge 2, \tag{22a}$$

for the new points at $m+\frac{1}{4}+2^{-k-1}$ and $m+\frac{3}{4}-2^{-k-1}$, and

$$b_{2^{k-2}-1}^k=\left(-\tfrac{7}{368},\tfrac{159}{368},\tfrac{255}{368},-\tfrac{39}{368}\right),\quad b_{3\cdot 2^{k-2}}^k=(b_{2^{k-2}-1,-2}^k,b_{2^{k-2}-1,-1}^k,b_{2^{k-2}-1,0}^k,b_{2^{k-2}-1,1}^k),\quad k\ge 3, \tag{22b}$$

for the new points at $m+\frac{1}{4}-2^{-k-1}$ and $m+\frac{3}{4}+2^{-k-1}$. All other b_i^k are just the classical stationary weights $\left(-\frac{1}{16},\frac{9}{16},\frac{9}{16},-\frac{1}{16}\right)$, because they are used to compute a new point from four old points that lie on a common cubic piece of the limit function.

Because of the special weights in (22), the usual tools for analysing the smoothness of non-uniform schemes cannot be applied to this scheme, but it is clear by construction that the limit functions of this non-uniform, periodic, interpolating 4-point scheme are the cubic C^2 B-splines with control points

$$\left(\dots,\tfrac{-13f_{-2}+109f_{-1}+109f_0-13f_1}{192},\tfrac{7f_{-1}+46f_0-5f_1}{48},\tfrac{-f_{-1}+98f_0-f_1}{96},\tfrac{-5f_{-1}+46f_0+7f_1}{48},\tfrac{-13f_{-1}+109f_0+109f_1-13f_2}{192},\dots\right)$$

and uniform knots $\left(\dots,-1,-\frac{3}{4},-\frac{1}{2},-\frac{1}{4},0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\dots\right)=\mathbb{Z}/4$.

Figs. 3 and 4 show a comparison between the classical interpolatory 4-point scheme and the two non-uniform variants from Examples 3 and 4 in the geometric and the functional setting, respectively. While the shapes of the limit functions for the non-uniform schemes are slightly negatively affected by the prescribed first derivative at the integers, the limit curves look rather pleasing and smoother than the limit curve of the classical scheme, as expected due to the improved smoothness. Note that the limit functions and curves generated by the two non-uniform schemes are visually almost indistinguishable, but the differences are evident for the second derivatives of the limit functions, which are piecewise cubic and piecewise linear, respectively (see Fig. 4).

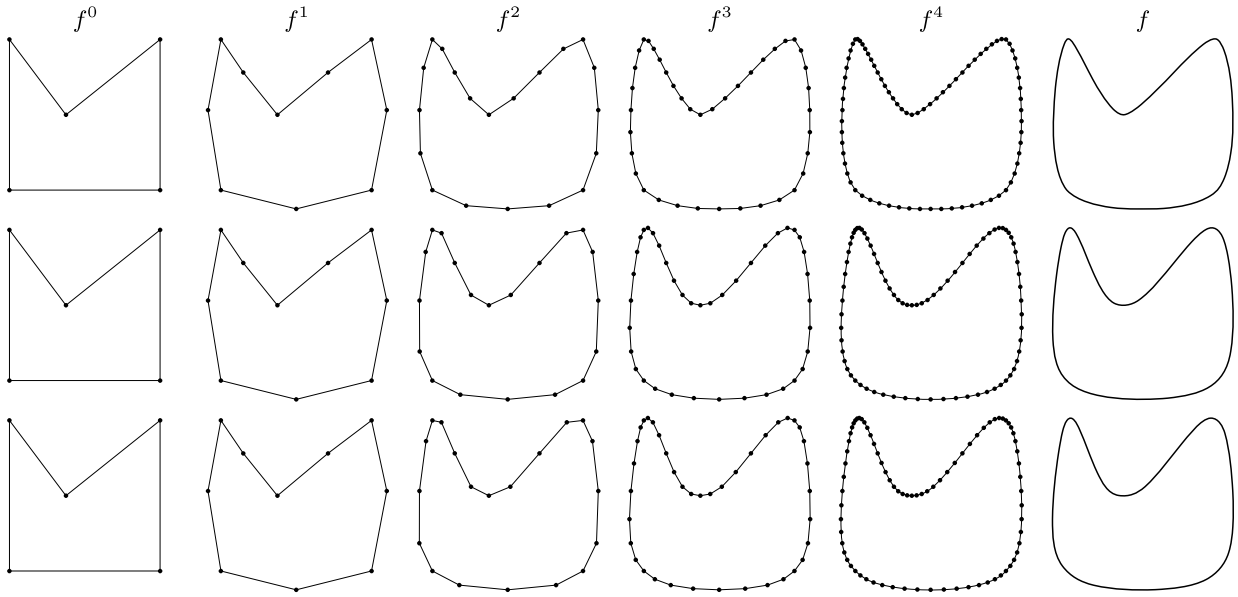

Fig. 3. Initial points, points generated by the first four subdivision steps, and limit curve (from left to right) for the stationary (top) and the non-uniform interpolatory 4-point schemes from Example 3 (middle) and Example 4 (bottom).

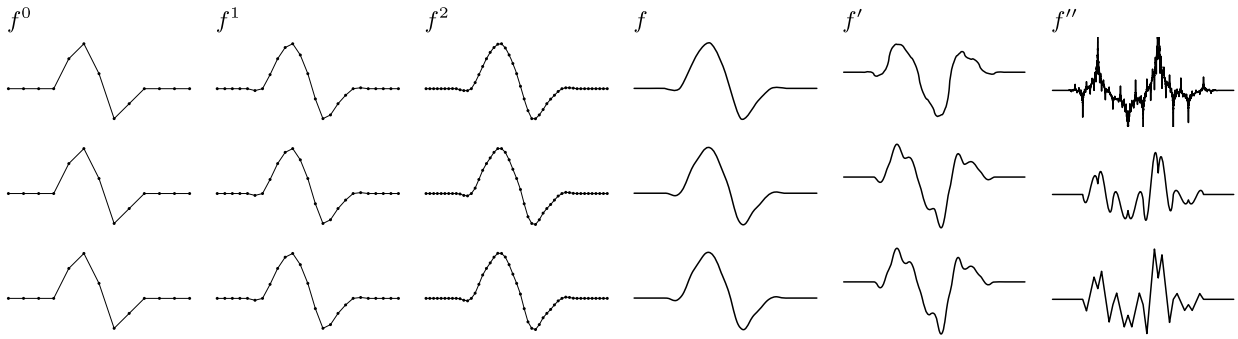

Fig. 4. Initial data, data generated by the first two subdivision steps, limit function, first and second derivative of the limit function (from left to right) for the stationary (top) and the two non-uniform interpolatory 4-point schemes from Example 3 (middle) and Example 4 (bottom).

4. Curves on surfaces

One may argue that the schemes derived above are questionable, because the exact limit curve is known *a priori* and can thus be evaluated directly. However, the same argument would apply to the uniform B-spline schemes (Lane and Riesenfeld, 1980), and the situation changes once we leave the non-Euclidean setting, for example, if we consider subdivision on a manifold surface $\mathcal{M}$. In this case, the usual definition of the affine average of n points $p_1, p_2, \ldots, p_n$ for certain weights w_i that sum to one,

$$\bar{p} = \sum_{i=1}^{n} w_i p_i,$$

is not well-defined anymore, because the points on $\mathcal{M}$ do not form a vector space. Instead, one can define the affine average using the weighted *Fréchet mean*,

$$\bar{p} = \operatorname*{arg\,min}_{p \in \mathcal{M}} \sum_{i=1}^{n} w_i d(p, p_i)^2, \tag{23}$$

where $d\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is the distance function, which measures the length of the *geodesic*, that is, the shortest path between two surface points. If d is the Euclidean distance, then both definitions are equivalent, but in general, the minimizer

$\bar{p}$ in (23) may not be unique and may not depend continuously on the points p_i and the weights w_i (Karcher, 1977; Afsari, 2009). Therefore, even though the limit curves of the non-uniform schemes in Examples 1–4 are B-spline curves and every point on these curves is an affine combination of the initial points f_i^0, evaluating them directly via the Fréchet mean can result in curves that are not even C^0.

If we generate the limit curve instead using a subdivision process, then we are guaranteed to have a sequence of continuous, piecewise geodesic curves, which usually converge to a limit curve that is as smooth as the one in the Euclidean setting (Wallner and Dyn, 2005; Wallner, 2020). Moreover, in order to properly define the subdivision rules for points on surfaces, it suffices to have a generalization of the affine average of two points. In the Euclidean setting, the affine average

$$\mathcal{A}(p,q,t) = (1-t)p + tq \tag{24}$$

of p and q with weights $(1-t)$ and t is simply the unique point $r = \mathcal{A}(p,q,t)$ on the *straight line* through p and q with relative distances $|t|$ and $|1-t|$ to p and q, that is,

$$d(r,p) = |t| \cdot d(p,q), \qquad d(r,q) = |1-t| \cdot d(p,q).$$

Since the equivalent of a straight line on a surface $\mathcal{M}$ is a geodesic, it is then clear how to generalize the average operator in (24) to two points $p, q \in \mathcal{M}$. Note that if $t \in [0,1]$, then $r = \mathcal{A}(p,q,t)$ is a point on the geodesic *between* p and q, while this geodesic needs to be extrapolated beyond p for $t < 0$ and beyond q for $t > 1$. A formal description of this procedure is given by Wallner (2020) for smooth surfaces and by Polthier and Schmies (1998) for surface meshes. The ambiguity of this definition, which may arise whenever the geodesic between p and q is not unique, can be resolved by making an arbitrary but deterministic choice.

With the average operator $\mathcal{A}$ in (24) defined on $\mathcal{M}$, we now follow Schaefer and Goldman (2005) and express the odd subdivision rules (3) of our non-uniform 4-point schemes with $J_l^k = \{j_2, j_1, j_0, j_{-1}\}$ in terms of three 2-point-averages as

$$f_{2i+1}^{k+1} = \mathcal{A}\big(\mathcal{A}(f_{i-j_0}^k, f_{i-j_1}^k, \mu_1), \mathcal{A}(f_{i-j_{-1}}^k, f_{i-j_2}^k, \mu_2), \mu_3\big), \qquad \mu_1 = \frac{\gamma_i^k}{\beta_i^k + \gamma_i^k}, \quad \mu_2 = \frac{\delta_i^k}{\alpha_i^k + \delta_i^k}, \quad \mu_3 = \alpha_i^k + \delta_i^k.$$

For this order of averages, the first two averages are usually geodesic interpolations with $\mu_1, \mu_2 \in (0,1)$ and only the last average requires geodesic extrapolation with a rather small extrapolation factor $-\mu_3 > 0$. For example, this is the case (with $-\mu_3 = \frac{1}{8}$) for almost all subdivision rules of the non-uniform schemes in Examples 3 and 4. Only at the endpoints of each interval, for $k > 1$ and $i \in \{0, 2^k - 1\}$, this order applies extrapolation for the first or the second (or both) averages and interpolation for the third average.

When implementing the scheme from Example 4, we observed that the rather large extrapolations that may occur during the first three subdivision steps (e.g., with factor $-\mu_2 = \frac{41}{10}$ for $k = 1$ and $i = 1$ and $-\mu_1 = \frac{163}{230}$ for $k = 3$ and $i = 0$) can lead to some unexpected artefacts of the limit curve. We therefore propose to circumvent these subdivision steps and to compute instead the data $f^4 = (f_i^4)_{i \in \mathbb{Z}}$ directly as affine combinations of the initial data, using the fact that

$$f_{16i}^4 = f_i^0, \qquad f_{16i+m}^4 = r_{-1}(\tfrac{m}{16}) f_{i-1}^0 + r_0(\tfrac{m}{16}) f_i^0 + r_1(\tfrac{m}{16}) f_{i+1}^0 + r_2(\tfrac{m}{16}) f_{i+2}^0, \quad m \in \{1, 2, \dots, 15\}$$

for any $i \in \mathbb{Z}$. This avoids extrapolation with factors greater than $\frac{1}{8}$ and generally results in artefact-free and presumably smooth curves (see Fig. 5). Proving that these limit curves are C^2 is beyond the scope of this paper and left to future work.

5. Conclusions

The main goal of this paper is to show that non-uniform subdivision schemes can give limit functions and curves that are smoother than those of corresponding stationary subdivision schemes, and the non-uniform schemes described in Examples 1–4 clearly serve as a "proof of concept". However, the theory that leads to these examples is actually more general.

Regarding 2-point schemes, our construction in Section 2 works for any strictly increasing C^1 function $r_1 \colon [0,1] \to \mathbb{R}$ with $r_1(0) = r_1'(0) = 0$, $r_1(1) = 1$, and $r_1'(1) = 0$. Letting $r_0(x) = 1 - r_1(x)$, we then have two basis functions that satisfy the assumptions of Lemma 2, and the limit functions of the non-uniform scheme with the weights in (9) are C^1 by Theorem 1. The support size of the basic limit function of any such scheme is 2, just like for the stationary, interpolatory 2-point scheme, but while the latter reproduces linear functions, our non-linear 2-point schemes reproduce only constant functions, because of the vanishing first derivative at the integers. Consequently, the approximation order is only $O(h)$, compared to $O(h^2)$ for the stationary scheme.

Likewise, our "recipe" for designing non-uniform 4-point schemes in Section 3 works for any C^2 function $r_2 \colon [0,1] \to \mathbb{R}$ with $r_2(0) = r_2'(0) = r_2''(0) = 0$, $r_2(\frac{1}{2}) \notin \{-\frac{1}{8}, 0\}$, $r_2(1) = 0$, $r_2'(1) = \frac{1}{2}$, and $r_2''(1) = 1$, as long as r_2 satisfies (15) for any $k \geq 2$ and $i \in \{1, 2, \dots, 2^k - 2\}$. In this case, we simply define the other three basis functions as

$$r_{-1}(x) = -\tfrac{1}{2}x(1-x) - r_2(x), \qquad r_0(x) = 1 - x^2 + 3r_2(x), \qquad r_1(x) = \tfrac{1}{2}x(1+x) - 3r_2(x),$$

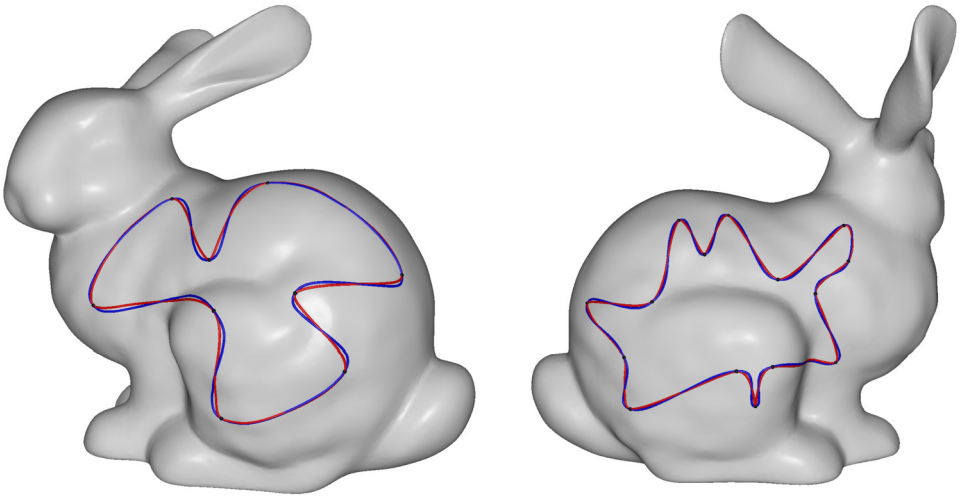

Fig. 5. Initial data (black dots) and interpolating curves on a manifold surface after 8 subdivision steps with the stationary (blue) and the non-uniform (red) 4-point scheme from Example 4. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

which ensures that all assumptions of Lemma 3 and Theorem 4 are satisfied, so that the limit functions of the corresponding scheme are guaranteed to be C^2. The support size of the basic limit function of any such non-uniform scheme is only 4, compared to the support size 6 of the basic limit function of the stationary, interpolatory 4-point scheme, which means that the influence of the initial control points on the shape of the limit curve is more local. Moreover, in the case of an open control polygon with $n+1$ points $f_0^0, f_1^0, \ldots, f_n^0$, the limit curves of these non-uniform schemes are defined for any $t \in [1, n-1]$, while the limit curves of the classical 4-point schemes are defined only for $t \in [2, n-2]$. However, just as for 2-point schemes, the approximation order of the non-uniform 4-point schemes is only $O(h^3)$, compared to $O(h^4)$ for the stationary 4-point scheme, because they reproduce only quadratic functions.

It remains future work to find other non-uniform schemes that do not simply reproduce a known limit function and where the subdivision rules do not depend on the latter, which will require developing new tools for analysing the convergence and smoothness of non-uniform schemes.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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