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Robust Convergence Analysis of Moving-Horizon Estimator for LPV Discrete-Time Systems *

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Abstract: This paper deals with the problem of robust stability analysis of Moving Horizon Estimator (MHE) for Linear Parameter Varying (LPV) systems. The main contribution of the paper lies in the introduction of novel stability analysis tools guaranteeing exponential robust convergence of the MHE under only the incremental Exponential Input-Output-to-State Stability (i-EIOSS) assumption. Indeed, the i-EIOSS property characterizes the detectability of a system, which is a less conservative assumption compared to the observability condition.

Keywords: LPV systems; Moving Horizon Estimator (MHE); stability; incremental input output-to-state stability; robust estimator.

1. INTRODUCTION AND PROBLEM FORMULATION

1.1 Introduction

State estimation is important for real-world applications, such as biomedical systems, robotics, and population dynamic models. Estimation is necessary because the system state variables are used in control design schemes, diagnosis, and monitoring. Estimation is also essential due to the unavailability of the system states in real time. Several estimation methods have been developed in the literature, among others, estimators based on Moving Horizon Estimation (MHE). The MHE reduces the effect of uncertainties by taking advantage of a smoothing estimate update (Alessandri et al., 2008, 2020). Given the same estimator tuning, model, and measurements, MHE provides improved state estimation and greater robustness to poor guesses of the initial state. Moreover, MHE incorporates physical state constraints into optimization and optimizes over a trajectory of states and measurements (Rao et al., 2003; Haseltine and Rawlings, 2005).

Incremental stability extends the notion of asymptotic stability of equilibrium of a nonlinear system to consider the asymptotic behavior of any solution with respect to any other solution. Specifically, any two solutions must eventually asymptotically converge to each other regardless of their initial conditions (Angeli (2002); Sontag (2008)).

Input-to-state stability (ISS) was introduced by Sontag et al. (1989). The importance of ISS is due to the fact that the concept has bridged the gap between input-output and state-space methods. since it is not possible to deal with inputs and outputs separately in general, incremental input/output to-state stability (i-IOSS) was introduced by Sontag and Wang (1995). If a system satisfies the incremental ISS property and the differences between two input signals are small and bounded, then the distance between any two trajectories must eventually be small and independent of initial conditions.

All the above stability notions have motivated the automatic control community to develop a rigorous theoretical framework for the robustness of control and estimation schemes. Sontag and Wang (1997) have shown that systems admitting robust estimators must be i-IOSS. However, these results do not apply to moving-horizon estimators Allan (2020). The use of the notion "i-EIOSS", as defined in Allan et al. (2021); Schiller et al. (2021) in the concept of the MHE, which is a notion of the more general definition of "i-IOSS" using $\mathcal{KL}$ functions, allows getting synthesis conditions of the parameters of the "cost" function that are numerically or analytically easy to verify. This objective is the main motivation for our work.

In this paper, we propose novel robust stability conditions of the MHE by considering only a particular i-IOSS notion, namely the incremental Exponential Input Output-to-State Stability (i-EIOSS) property. To this end, we first introduce new mathematical inequalities, which are exploited to ensure the robust convergence of the MHE without needing the observability condition like in (Alessandri et al., 2008, 2020). The novel stability analysis tools are combined with two new and different prediction techniques to avoid conservative necessary conditions.

1.2 Problem Formulation and Preliminaries

The moving horizon estimation technique (MHE) is a state estimation method that is particularly useful for nonlinear or constrained dynamic systems. Two sources of error are

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present during estimation: the state transition is affected by a disturbance, w, and the measurement process is affected by another disturbance, v. Consider the following LPV discrete-Time system:

$$\begin{cases} x_{t+1} = A(\rho_t)x_t + w_t \\ y_t = C(\rho_t)x_t + v_t \end{cases} \tag{1}$$

where $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the system; $\rho_t \subset \mathcal{X}_\rho \subset \mathbb{R}^n$ with $\mathcal{X}_\rho$ is a compact set; $y_t \in \mathbb{R}^n$ is the output vector and $w_t \in W \subseteq \mathbb{R}^n$ and $v_t \in V \subset \mathbb{R}^n$ are unknown external disturbances. The parameter ρ_t is assumed to be known in real-time. The estimation scheme can be obtained by defining $\hat{x}_{t-N+1|t}, ..., \hat{x}_{t|t}$ as estimates generated by $\hat{x}_{t-N|t}$ through the dynamics

$$\hat{x}_{i+1|t} = A(\rho_i)\hat{x}_{i|t}, \quad i = t-N, ..., t-1.$$

The MHE technique is based on the idea of minimizing a quadratic estimation cost function defined on a sliding window composed of a finite number of time stages. Consider the classic quadratic objective function:

$$J_t^N(\hat{x}_{t-N}) = \mu|\hat{x}_{t-N} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - C(\rho_i)\hat{x}_i|^2 \tag{2}$$

where $\eta \in (0,1)$ and $\mu, \nu > 0$ under the constraints

$$\hat{x}_{t+1} = A(\rho_t)\hat{x}_t. \tag{3}$$

Now we focus on a MHE_N resulting from the minimization of the cost function (2) given by the following:

$$\begin{cases} \hat{x}_{0|t} \in \Big\{ \underset{\hat{x}_0 \in \mathcal{X}}{\operatorname{argmin}}\, J_t^t(\hat{x}_0) \quad \text{s.t. (3) holds for} \quad t = 1, ..., N \Big\} \\ \hat{x}_{t-N|t} \in \Big\{ \underset{\hat{x}_{t-N} \in \mathcal{X}}{\operatorname{argmin}}\, J_t^N(\hat{x}_{t-N}) \quad \text{s.t. (3) holds for} \quad t = N, N+1, \dots \Big\}. \end{cases}$$

For further ease of presentation and for use later in the paper, note that the cost function $J_t^t(\hat{x}_0)$ for $t \le N$ is clearly defined as follows:

$$J_t^t(\hat{x}_0) = \mu|\hat{x}_0 - \bar{x}_0|^2\eta^t + \nu \sum_{i=0}^{t-1} \eta^{t-1-i}|y_i - C(\rho_i)\hat{x}_i|^2, \ \forall\, t \le N. \tag{4}$$

Notice that the prediction equation will be given later in Section 3. Indeed, we will show that depending on the prediction equation, we provide different convergence conditions of the MHE_N. We propose two different prediction steps, where each one requires some specific conditions to ensure robust convergence of the MHE_N.

We start by providing two key definitions used in this paper. We first introduce the following definition of the exponential robust stability of an estimator. For that, we focus only on MHE_N.

Definition 1. An MHE_N is *robustly exponentially stable* (RES) if the following inequality holds:

$$|x_t - \hat{x}_{t|t}| \le \alpha_1|x_0 - \bar{x}_0|\lambda^t + \alpha_2 \sum_{i=0}^{t-1} \lambda^{t-1-i}|v_i|^2 + \alpha_3 \sum_{i=0}^{t-1} \lambda^{t-1-i}|w_i|^2 \tag{5}$$

for some $\lambda \in (0,1)$ and $\alpha_i > 0, \forall i = 1,2,3$. Further, if inequality (5) is satisfied for all $t \ge \ell$, where $\ell \ge 1$ is a natural number, we say that the MHE_N is $\ell-$RES, or the MHE_N is RES in ℓ steps.

Systems in real life are very often affected by noise, perturbations, and errors in observations. Thus it is desirable for a system to be not only stable but input-to-state stable (ISS). ISS is the most useful tool for robust stability analysis of nonlinear systems. In order to establish robust stability of the proposed MHE_N scheme, an appropriate detectability characterization is required. We suppose that system (1) is incrementally exponentially input/output-to-state stable. Incremental stability is a special case of stability, i-IOSS implies IOSS, but the converse does not hold in general (Sontag, 2008). As in the Definition 1, we specify the definition of i-IOSS for the LPV systems. We suppose that system (1) is incrementally exponentially input/output-to-state stable.

Definition 2. System (1) is *incrementally Exponentially Input Output-to-State-Stable* (i-EIOSS) if there exist constants $c_x, c_v, c_w > 0$ and $\eta \in (0,1)$ such that for each pair of initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and each two disturbance sequences $w_t, \tilde{w}_t \in \Omega$, the following holds:

$$|x_t(x_0, w_0^{t-1}) - \tilde{x}_t(\tilde{x}_0, \tilde{w}_0^{t-1})|^2 \le c_x|x_0 - \tilde{x}_0|^2\varrho^t + c_v \sum_{i=0}^{t-1} \varrho^{t-1-i}|y_i(x_0, w_0^{i-1}, v_0^{i-1}) - y_i(\tilde{x}_0, \tilde{w}_0^{i-1}, \tilde{v}_0^{i-1})|^2 + c_w \sum_{i=0}^{t-1} \varrho^{t-1-i}|w_i - \tilde{w}_i|^2 \tag{6}$$

for some $\varrho \in (0,1)$ and positive reals c_x, c_v, and c_w.

Remark 1. Notice that the above definition is general and does not be applied only for states at time t and 0, respectively. It can be applied, for instance, to account for the exponential discount of the error on trajectories between t and $t-\ell$. Especially since for the MHE problem studied here, we will need to apply the definition for $t \ge \ell$, and between t and $t-\ell$, then we will use the following inequality:

$$|x_t(x_{t-\ell}, w_{t-\ell}^{t-1}) - \tilde{x}_t(\tilde{x}_{t-\ell}, \tilde{w}_{t-\ell}^{t-1})|^2 \le c_x|x_{t-\ell} - \tilde{x}_{t-\ell}|^2\varrho^\ell + c_v \sum_{i=t-\ell}^{t-1} \varrho^{t-1-i}\Big|y_i(x_{t-\ell}, w_{t-\ell}^{i-1}, v_{t-\ell}^{i-1}) - y_i(\tilde{x}_{t-\ell}, \tilde{w}_{t-\ell}^{i-1}, \tilde{v}_{t-\ell}^{i-1})\Big|^2 + c_w \sum_{i=t-\ell}^{t-1} \varrho^{t-1-i}|w_i - \tilde{w}_i|^2 \tag{7}$$

which is straightforward from (6). For more details on this inequality, we refer the reader to (Schiller et al., 2021, Eq. (28)) and (Müller, 2017, Definition 2, and Lemma 7) for a more general case.

The above definitions will be exploited in Section 3 for robust stability analysis of the MHE_N. We provide sufficient conditions ensuring the ERS of the MHE_N, according to the Definition 1, by only assuming that system (1) is i-

EIOSS. The observability condition is not necessary with the approach we propose in the paper, which is a significant improvement compared to previous MHE-related results. Also, compared to the elegant and several results by Müller (2017); Knuefer and Müller (2021); Schiller et al. (2022, 2021), we focus only on exponential convergence of the MHE_N with quadratic cost functions; we use new and simple but useful tools for the stability analysis. More importantly, we propose different prediction equations, which play an important role in the stability conditions in terms of required necessary assumptions while ensuring ERS of the MHE_N.

2. PRELIMINARY RESULTS: NEW TOOLS FOR STABILITY ANALYSIS

In this section, we present some key results, presented in two lemmas, that we will exploit in the next section to analyze the robustness of the MHE for the system (1) without observability conditions. The results have been presented in a general framework so that they can be exploited in different cases for different control design problems.

Lemma 1. Let $(u_t)_{t\geq -\ell}$ be a sequence of non-negative real numbers and $\ell \geq 0$ such that

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \forall t \geq \ell,$$

where α and β are scalars such that $\beta \geq 0, 0 < \alpha < 1$. The sequence $(z_t)_{t\geq \ell}$ is non negative (with $z_t = 0, \forall t < \ell$). Then the following inequality holds for any $\kappa \in \mathbb{N}, \kappa \geq 2$:

$$u_t \leq \alpha^{\frac{t}{\kappa\ell}} \max_{-\ell\leq j\leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa\ell}}}\right) \max_{\ell\leq j\leq t}\left(\alpha^{\frac{t-j}{\kappa\ell}} z_j\right). \quad (8)$$

Lemma 1 may be viewed as a general tool for stability analysis, which can be used for several control design problems, such as estimation and control of time-delay systems. It can also be used, in particular, for the MHE problem we handle in this paper. However, to render its use on MHE straightforward, it is more convenient to state the following Lemma 2, which follows from Lemma 1 for a particular case of $(z_t)_{t\geq\ell}$.

Lemma 2. Let $(u_t)_{t\geq -\ell}$ be a nonnegative sequence of real numbers and $\ell \geq 0$ is a natural number such that

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \forall t \geq \ell,$$

with

$$z_t = \sum_{i=t-\ell}^{t-1} \eta^{t-1-i} |d_i|^2 \quad (9)$$

where $\beta \geq 0, 0 < \alpha < 1$, and $(d_j)_{j\geq\ell}$ is any arbitrary bounded sequence with $d_j = 0, \forall j < \ell$, by definition. Then the following inequality holds for any $\kappa \in \mathbb{N}, \kappa \geq 2$:

$$u_t \leq \lambda^t \max_{-\ell\leq j\leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa\ell}}}\right) \sum_{i=0}^{t-1} \lambda^{t-1-i} |d_i|^2 \quad (10)$$

where

$$\lambda \triangleq \max\left(\eta, \alpha^{\frac{1}{\kappa\ell}}\right). \quad (11)$$

Further, if $u_j = u_0$ for $-\ell \leq j \leq 0$, we get

$$u_t \leq u_0 \lambda^t + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa\ell}}}\right) \sum_{i=0}^{t-1} \lambda^{t-1-i} |d_i|^2. \quad (12)$$

Lemma 2 can be used easily in a straightforward way to show robust convergence of the MHE_N is RES according to Definition 1 for all $t \geq N$. Then, Lemma 1 can be used in more general RES definitions, which instead of sum with exponential discount, use *max* operators or $\mathcal{KL}$ functions as in Müller (2017), and the references therein.

3. EXPONENTIAL ROBUST STABILITY ANALYSIS

Now we are ready to tackle the robust stability analysis of the MHE_N. To this end, we focus only on the case $t \geq N$, and then we will deal with $N-$ERS of the MHE_N according to the Definition 1. All the stability conditions provided in the sequel are based on the i-EIOSS requirement without observability assumption. We will show that for N large enough, the MHE_N is $N-$ERS, provided that the system (1) is i-EIOSS. To reduce the size of the moving window, namely N, while ensuring exponential discount in the sense of $N-$ERS, we propose two different prediction steps. The prediction equation of the MHE_N plays an important role since the necessary conditions for $N-$ERS property depend on the prediction.

3.1 Upper bound of the estimation error

Before introducing the different prediction equations, we start by providing an upper bound on the estimation error $e_t \triangleq x_t - \hat{x}_{t|t}$. For that, we will exploit the minimization of the cost function and the i-EIOSS property of system (1). The upper bound on the error e_t depends on the prediction error $\bar{e}_t \triangleq x_t - \bar{x}_t$ (or $\bar{e}_t \triangleq x_{t-N} - \bar{x}_{t-N}$, at time $t-N$). From the definition of minimizer we have $J_t^N(\hat{x}_{t-N|t}) \leq J_t^N(x_{t-N})$,, we get

$$\begin{aligned}
&\mu|\hat{x}_{t-N|t} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - h(\hat{x}_{i|t})|^2 \\
&\quad + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2 \\
&\quad \leq \mu|x_{t-N} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|v_i|^2 \\
&\quad + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2. \quad (13)
\end{aligned}$$

Since we always have

$$\frac{1}{2}|e_{t-N}|^2 \leq |\bar{e}_{t-N}|^2 + \left|\bar{x}_{t-N} - \hat{x}_{t-N|t}\right|^2$$

it follows that

$$\begin{aligned}
&\frac{\mu}{2}|e_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - h(\hat{x}_{i|t})|^2 \\
&\quad + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2 \\
&\quad \leq 2\mu|\bar{e}_{t-N}|^2\eta^N + \nu \sum_{t=k-N}^{t-1} \eta^{t-i-i}|v_i|^2 \\
&\quad + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2.
\end{aligned}$$

Since the system (1) is i-EIOSS according to Definition 2, then by applying inequality (7) with convenient parameters μ, ν, ω, and η such that

$$\begin{cases} \varrho \leq \eta < 1 \\ \mu \geq 2c_x,\ \nu \geq c_v,\ \omega \geq c_w \end{cases}$$

we obtain the following inequality:

$$\begin{aligned} |e_t|^2 \leq & 2\mu \left|\bar{e}_{t-N}\right|^2 \eta^N \\ & + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i} \left|v_i\right|^2 \\ & + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i} \left|w_i\right|^2 . \end{aligned} \tag{14}$$

The inequality (14) is important and provides an appropriate upper bound on the estimation error e_t, however, at this stage, we cannot conclude due to the term $\bar{e}_{t-N}$. The fate of this term depends on the prediction step, namely on how to choose $\bar{x}_{t-N}$. To this end, we propose two solutions, where each one leads to specific conditions ensuring $N-$ERS property of the MHE$_N$. The two prediction techniques are enumerated hereafter:

(1) *The simple prediction (First solution):* This prediction technique uses a very simple equation, which does not involve the dynamics of the system (1).
(2) *The modified standard prediction (Second solution):* This is the standard prediction equation, usually and commonly used in the MHE context.

In the sequel, each solution is detailed in a subsection, where we provide the prediction equation, the resulting upper bound of the estimation error, and the main theorem ensuring the $N-$ERS property of the MHE$_N$ related to the convenient prediction.

Before providing the main theorem, it is worth noticing that due to the definition of the cost function $J_t^t(\hat{x}_0)$, for $t \leq N$, as in (4), we can write the inequality (14) in a unified way for any $t \geq 1$, as follows:

$$\begin{aligned} |e_t|^2 \leq & 2\mu \left|\bar{e}_{t-\min(t,N)}\right|^2 \eta^{\min(t,N)} \\ & + \nu \sum_{i=t-\min(t,N)}^{t-1} \eta^{t-1-i} \left|v_i\right|^2 \\ & + \omega \sum_{i=t-\min(t,N)}^{t-1} \eta^{t-1-i} \left|w_i\right|^2 \end{aligned} \tag{15}$$

which is nothing but the ERS condition (5) for $t \leq N$. Indeed, for $t \leq N$, the previous inequality (15) is reduced to

$$\begin{aligned} |e_t|^2 \leq & 2\mu \left|\bar{e}_0\right|^2 \eta^t \\ & + \nu \sum_{i=0}^{t-1} \eta^{t-1-i} \left|v_i\right|^2 \\ & + \omega \sum_{i=0}^{t-1} \eta^{t-1-i} \left|w_i\right|^2 \end{aligned} \tag{16}$$

It can be obtained by using the same arguments as in (13) and the i-EIOSS property. The objective here is to combine the previous inequality (16) to get a unified and general ERS bound for the estimation error for all $t \geq 1$.

3.2 *The simple prediction*

The natural solution consists in keeping and forcing the term $\bar{e}_{t-N}$ in (14) to be equal to e_{t-N}. To this end, the prediction $\bar{x}_{t-N}$ is obviously determined from the estimate via the following scheme:

$$\bar{x}_{t-N} = \begin{cases} \bar{x}_0, & t = 1, ..., N \\ \hat{x}_{t-N|t-N}, & t \geq N+1. \end{cases} \tag{17}$$

In this case, the inequality (14) becomes as follows:

$$\begin{aligned} |e_t|^2 \leq & 2\mu \left|e_{t-N}\right|^2 \eta^N \\ & + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i} \left|v_i\right|^2 \\ & + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i} \left|w_i\right|^2 . \end{aligned} \tag{18}$$

The above inequality (18) allows concluding on the ERS of the MHE$_N$ for N large enough. Now that the essential stability analysis tools are provided in Section 2, we only need to apply them with particular and convenient parameters. The result is summarized in the following main Theorem 1.

Theorem 1. Assume that system (1) is i-EIOSS according to (6) with the prediction equation (17) and the exponential discount parameter ϱ. Then, the MHE$_N$ is ERS according to the following inequality:

$$\begin{aligned} \left|x_t - \hat{x}_{t|t}\right|^2 \leq & \max(2\mu, 1)|x_0 - \bar{x}_0|^2 \lambda^t \\ & + \frac{\nu}{\left(1 - [2\mu\eta^N]^{\frac{\kappa-1}{\kappa N}}\right)} \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|v_i\right|^2 \\ & + \frac{\omega}{\left(1 - [2\mu\eta^N]^{\frac{\kappa-1}{\kappa N}}\right)} \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|w_i\right|^2 \end{aligned} \tag{19}$$

with the exponential discount parameter

$$\lambda \triangleq \max\left(\eta, \left[2\mu\eta^N\right]^{\frac{1}{\kappa N}}\right),\ \kappa \geq 2 \tag{20}$$

if μ, ν, ω, η, and $N \geq 1$ satisfy the following conditions:

(i) $\varrho \leq \eta < 1$;
(ii) $\mu \geq 2c_x$;
(iii) $\nu \geq c_v$;
(iv) $\omega \geq 2c_w$;
(v) $2\mu\eta^N < 1$, which means that $N \geq 1 + \left\lfloor \frac{\ln(2\mu)}{\ln\left(\frac{1}{\eta}\right)} \right\rfloor$.

Proof. First, notice that conditions (ii)-(iv) guarantee the inequality (18). By applying Lemma 2 on equation (18) with

$$d_i = \begin{bmatrix} \sqrt{\nu}\ v_i \\ \sqrt{\omega}\ w_i \end{bmatrix},\ \alpha = 2\mu\eta^N,\ \beta = 1, \ell = N$$

we get the following bound from (v), i.e. $\alpha = 2\mu\eta^N < 1$, and the definition of λ in (20):

$$\begin{aligned} \left|x_t - x_{t|t}\right|^2 \leq & \left|x_0 - \bar{x}_0\right| \lambda^t \\ & + \frac{\nu}{1 - \alpha^{\frac{\kappa-1}{\kappa N}}} \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|v_i\right|^2 \\ & + \frac{\omega}{1 - \alpha^{\frac{\kappa-1}{\kappa N}}} \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|w_i\right|^2 . \end{aligned} \tag{21}$$

As we can see, Lemma 2 provides (21) for only $t \geq N$. On the other hand, by considering the initial bounds (16) for $t \leq N-1$, and the fact that

$$\nu \leq \frac{\nu}{\left(1 - [2\mu\eta^N]^{\frac{\kappa-1}{\kappa N}}\right)}$$

and

$$\omega \leq \frac{\omega}{\left(1 - [2\mu\eta^N]^{\frac{\kappa-1}{\kappa N}}\right)}$$

the inequality (19) is inferred.

Remark 2. From condition (v) of Theorem 1, we obtain that the window size needs to be sufficiently high, namely

$$N \geq 1 + \left\lfloor -\frac{\ln(2\mu)}{\ln \eta} \right\rfloor$$

where $\lfloor z \rfloor$ denotes the largest integer less than $z \in \Re$. Thus, we deduce that the MHE_N is ERS for any $N \geq 1$ if the parameter μ is chosen such that $\ln(2\mu) \leq 0$, i.e., $\mu \leq 1/2$. Taking into account the condition (ii) of Theorem 1, the necessary condition for the MHE_N to be ERS for any $N \geq 1$ is

$$4c_x \leq 1. \tag{22}$$

Similarly, the ERS of the MHE_N is possible for any $N \geq 2$ if the necessary condition $4c_x \leq \frac{1}{\eta}$ is satisfied. Thus, in general, the necessary condition to design an ERS MHE_N for $N \geq \ell - 1, \ell \in \mathbb{N}, \ell \geq 2$, is given as follows:

$$\frac{1}{4c_x\eta^\ell} \geq 1. \tag{23}$$

Consequently, with the prediction equation (17), we cannot ensure exponential robustness of the MHE_N, with $N \leq \ell - 2$, if (23) is not satisfied.

It is clear that convergence of the MHE_N for any fixed window size, N, depends on the i-EIOSS related parameters, namely c_x and η. On the other hand, the prediction equation (17) does not provide any additional tuning parameter to overcome the limitation related to the necessary condition (23). Hence the interest in seeking new predictions. This is the aim of the next subsections. To this end, first, we start by using the standard prediction equation that we will improve later.

3.3 The modified standard prediction

This subsection is devoted to the MHE_N by using the standard prediction equation with a slight modification.

Now back to the modified standard prediction technique. The modified prediction equation, in this case, is a copy of the original system where the computation of $\bar{x}_{t-N}$ depends on the estimated state at time $t-N-1$, namely $\hat{x}_{t-N-1|t-N-1}$, instead of $\hat{x}_{t-N-1|t-1}$ like with the classical prediction equation. Hence, the prediction of $\bar{x}_{t-N}$ is given by the following equation:

$$\bar{x}_{j+1} = A(\rho_j)\hat{x}_{j|j}. \tag{24}$$

where $j \triangleq t - N - 1$ for the sake of brevity.

With this prediction (24), the term $\bar{e}_{t-N}$ in (14) can be upper bounded as follows:

$$|\bar{e}_{j+1}|^2 \leq 2\left|A(\rho_j)\right|^2 \left| e_j\right|^2 + 2|w_j|^2. \tag{25}$$

By substituting (25) in (14), we obtain

$$\begin{aligned}|e_t|^2 \leq\ & 4\mu\sigma_A \left|e_{t-(N+1)}\right|^2 \eta^N \\ & + \nu \sum_{i=t-(N+1)}^{t-1} \eta^{t-1-i} \left|v_i\right|^2 \\ & + \max(4\mu, \omega) \sum_{i=t-(N+1)}^{t-1} \eta^{t-1-i} \left|w_i\right|^2 .\end{aligned} \tag{26}$$

where

$$\sigma_A \triangleq \sup_{j \geq 0} \left|A(\rho_j)\right|^2 .$$

Then, similarly to Theorem 1, we obtain the following Theorem 2 corresponding to the new bound (26).

Theorem 2. Assume that system (1) is i-EIOSS according to (6) with the prediction equation (24) and the exponential discount parameter ϱ. Then, the MHE_N is ERS according to the following inequality:

$$\begin{aligned}\left|x_t - \hat{x}_{t|t}\right|^2 \leq\ & \max(2\mu, 1)|x_0 - \bar{x}_0|^2 \lambda^t \\ & + \frac{\nu}{\left(1 - [4\mu\,\sigma_A \eta^N]^{\frac{\kappa-1}{\kappa(N+1)}}\right)} \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|v_i\right|^2 \\ & + \left(\frac{\max(4\mu, \omega)}{1 - [4\mu\,\sigma_A \eta^N]^{\frac{\kappa-1}{\kappa(N+1)}}}\right) \sum_{i=0}^{t-1} \lambda^{t-1-i} \left|w_i\right|^2\end{aligned} \tag{27}$$

with the exponential discount parameter

$$\lambda \triangleq \max\left(\eta, \left[4\mu\,\sigma_A \eta^N\right]^{\frac{1}{\kappa(N+1)}}\right),\ \kappa \geq 2 \tag{28}$$

if μ, ν, ω, η, and $N \geq 1$ satisfy the following conditions:

(i) $\varrho \leq \eta < 1$;
(ii) $\mu \geq 2c_x$;
(iii) $\nu \geq c_v$;
(iv) $\omega \geq 2c_w$;
(v) $4\mu\,\sigma_A \eta^N < 1$, which means that

$$N \geq 1 + \left\lfloor \frac{\ln(4\mu\sigma_A)}{\ln\left(\frac{1}{\eta}\right)} \right\rfloor .$$

Proof. Again notice that conditions (ii)-(iv) guarantee the inequality (18). Without expanding the computations, similarly to the proof of Theorem 2, we get (27) by applying Lemma 2 with

$$d_i = \begin{bmatrix} \sqrt{\nu}\ v_i \\ \sqrt{\max(4\mu, \omega)} w_i \end{bmatrix}$$

and

$$\alpha = 4\mu\,\sigma_A \eta^N,\ \beta = 1, \ell = N + 1,$$

we obtain easily (27) since the condition (v) in Theorem 2 allows applying Lemma 2. In addition, as in the proof of Theorem 1, by considering the initial bounds (16) for $t \leq N - 1$, the relation (27) is inferred.

Remark 3. With the prediction equation (24), the new necessary condition for the MHE_N to be ERS for any $N \geq 1$ is $\ln(4\mu\sigma_A) \leq 0$, which means that $4\mu\sigma_A < 1$. Taking into account the condition (ii) of Theorem 1, the necessary condition for the MHE_N to be ERS for any $N \geq 1$ is

$$8c_x\sigma_A \leq 1. \tag{29}$$

Then, the MHE_N is ERS for $N \geq \ell - 1, \ell \in \mathbb{N}, \ell \geq 2$, if the following necessary condition holds:

$$\frac{1}{4c_x\eta^\ell} \geq 2\sigma_A. \tag{30}$$

Compared to (23), robust convergence of the MHE_N can be guaranteed for some systems having $2\sigma_A < 1$, subject to (30), while with the prediction (17) it is not necessarily ensured. On the other hand, for systems having $2\sigma_A > 1$, subject to (30), the prediction equation (17) is better than the modified standard prediction-based equation. Hence, both prediction techniques can be seen as alternative methods. It is for the user to fix which one is more appropriate for the model at hand while ensuring the ERS property of the MHE_N.

4. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed new techniques for analyzing the robust stability of the MHE by assuming only the system to be i-EIOSS. To this end, we first introduced new and general mathematical tools which are used in the context of MHE to guarantee the exponential robust stability of the estimation scheme. Such tools are exploited jointly with the use of two new and different prediction techniques, where each prediction approach leads to specific necessary conditions to be satisfied by the system to ensure the ERS property of the MHE. The two prediction techniques require conservative necessary conditions and may be viewed as alternative methods.

In future work, we aim to explore several new directions in the area of robust convergence of the MHE: first, we will propose a new prediction equation that consists of a copy of the standard prediction with adding a correction term to improve estimation and stability conditions. It is more general and contains an additional decision variable as a tuning parameter, which plays an important role in the design of the MHE scheme ensuring the ERS property. Second, we will work on the proposition of Lyapunov-based methods to investigate the i-EIOSS property jointly with the results of this paper. The objective consists in working on constructive techniques, which can provide a mathematical link between the observability condition and the i-EIOSS property for LPV discrete-time systems, in particular. We also aim to work on applications to biomedical models from estimation and control perspectives viewpoint. However, to tackle this area of applications, we will first work on the extension to nonlinear systems, which is straightforward for the results proposed in this paper, but not obvious to combine with the Lyapunov-based techniques.

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